A Survey of Black Hole Uniqueness Theorems.

by

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I hereby declare that I am the sole author of this project. This is a true copy of the project, including any required final revisions, as accepted by my examiners.

I understand that my project may be made electronically available to the public.

Abstract

The purpose of this project is to understand how a black hole is defined in the premise of general relativity and to investigate the current results on the uniqueness of black holes. In this essay, the mathematical concepts required to define a black hole are presented. The Schwarzschild and the Kerr black hole solutions are investigated.

Several black hole uniqueness theorems are presented in this report, including Israel's theorem and the Carter-Robinson theorem. Modern approaches to these uniqueness theorems using non-linear sigma models are briefly discussed. Higher dimensional black holes, in particular the Myers-Perry black hole and the Black Ring solution is described. Investigating the current state of the uniqueness theorems of black holes is the main objective of this report.

Acknowledgment

I would like to express my gratitude towards my supervisor, Prof. Ruxandra Moraru, whose kindness and guidance helped me realize my dream of pursuing graduate studies and I must thank her for getting me to where I am today.

I would also like to thank Prof. Spiro Karigiannis for being a wonderful mentor and lending me his valuable time whenever I had to ask him any questions regarding geometry or physics.

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Dedication

To my wonderful family: Ma, Papa, Santona, Dipto and the lovely Yang.

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Chapter 1

Introduction

Einstein's theory of General Relativity - currently the most widely accepted theory of gravity - allows for the existence of remarkable objects known as black holes. The purpose of this project is to understand how a black hole is defined in the premise of general relativity and to investigate the current results on the uniqueness of black holes.

1.1 Motivation

The early works of Chandrashekhar [4] and Oppenheimer et. al. [5] establish the fact that the death of a massive star, having at least 1.4 times the solar mass, will result in a catastrophic collapse under the star's own own gravitational force to form a point-like source (i.e., a singularity) of gravitational field. Since there are several such massive stars observed today, and if general relativity were to be a successful theory of gravity, then it must admit singularities arising from the death of these stars in the possible structure of our universe.

However, both our physical observations as well as the mathematical structure of manifolds used in general relativity demand a certain level of smoothness in our models of space and time. Therefore, one has to be careful in introducing singular sources of gravitational attraction, and this is precisely where black holes step in.

After more than half a century of laborious work in mathematical rigor, researchers realized that the best way to incorporate singularities in general relativity is through regions of space-time known as black hole regions. As it will be discussed in chapter 5, a black hole region essentially shields any singularity inside the region by not allowing matter to escape from within the region.

A large part of this report will be focused on defining a black hole region and related concepts. Once a black hole has been defined, however, there are several interesting properties of black holes that can be talked about. For example, it is quite interesting to discuss the process through which black holes are formed, both from a mathematical and a physical point of view. On the other hand, it may also be enthralling to converse on the philosophical implications of black holes, and what end may an unfortunate traveler meet through a fall into a black hole region. Equally engaging are the topics on the physics of black holes: from quantum mechanical fluctuations at event horizons to the thermodynamical laws of black holes. Unfortunately, this essay will not cover any of these topics.

Instead, the later part of this project has been devoted towards understanding the uniqueness theorems regarding black holes. This is a vast topic on which a very active line of research is being pursued at the moment. There are standard results, such as the Carter-Robinson theorem, that are looked into in this essay. Moreover, here we also look into the more recent efforts towards improving such standard results, and the challenges that are brought forth in extending the known uniqueness results into higher dimensional theories of gravity. Overall, this report hopes to present an entertaining journey through the subject, and may the readers be drawn into studying black holes themselves.

1.2 Organization

The report has been designed to be a self-sufficient resource for readers versed in the topics of differential geometry. Therefore, chapter 2 starts from the roots of general relativity. The purpose of chapter 2, titled 'From Geometry to Space-Time', is to guide the reader through the basic concepts of differential geometry such as manifolds and tensors, as well as to introduce the core ideas underlying the theory of general relativity. A key motif of this chapter is to declare the assumptions that are implied in this report when we talk about space-time, which hopefully serves to reduce any confusion in the later chapters.

Chapter 3 introduces one of the earliest known solutions to the Einstein equations. Fortunately, this resulting space-time, known as the Schwarzschild space-time, contains a black hole region and therefore acts as a precursor to much of the theoretical concepts introduced later in the essay. The Schwarzschild space-time is surveyed briefly in this chapter, and the readers are shown a glimpse of the concepts one encounter when dealing with black holes.

Comparatively, chapter 4 is straight-forward. It introduces three key concepts from general relativity - causality, geodesic congruences and asymptotic flatness - that are required to study black holes. The important energy conditions are also discussed in this chapter. The purpose of this chapter is to serve as a platform to build the later chapters on.

The concept of a black hole is finally introduced rigorously in chapter 5. As an added reward, the Kerr space-time is also discussed here. This gives an opportunity for the reader to reflect upon the subtleties involved in defining a black hole and perhaps also to appreciate one of the most interesting black hole solutions known to date. Introducing the Kerr black hole in this chapter also sets up really nicely to jump into the uniqueness theorems in the proceeding chapter.

Chapter 6 discusses the uniqueness of black holes. The chapter starts by establishing the Schwarzschild solution as the unique spherically symmetric solution to Einstein's equations. Consequently, Birkhoff's theorem is proved in this chapter. Israel's theorem regarding the uniqueness of Schwarzschild solution is stated here as well, and an outline of Robinson's proof to Israel's theorem is presented. The rest of the chapter is then devoted towards building up the stage to introduce the celebrated Carter-Robinson theorem. This theorem is only stated here, and the proof is deferred to the source. The end of chapter 6 briefly introduces the reader to the modern viewpoint on the uniqueness of black holes. Concepts such as the topological censorship theorem and harmonic maps are mentioned to give the reader an idea on the direction that modern research in this field is headed towards.

The penultimate chapter, chapter 7, flirts with the concept of a black hole in higher dimensions. The well known example of the Myers-Perry black hole is discussed here. Finally, some special attention is given towards $4 + 1$-dimensional black holes, where the introduction of the Black Ring solution throws off balance our hopes of extending the uniqueness results from $3 + 1$-dimensions to higher dimensions. There is perhaps a new approach or point of view required to achieve uniqueness of black holes in higher dimensions, and this is the note on which chapter 7 ends.

The overall picture is once again discussed in the final chapter, and the report concludes with the hope that the journey through this essay has served the readers their purpose.

Chapter 2

From Geometry to Space-Time

> Newton forgive me,
> You found the only way which in your age was just about possible
> for a man of highest thought and creative power.
>
> Albert Einstein

Besting General Relativity comes in two flavors: understanding the physical role of gravity and mastering the mathematics of differential geometry. Initial theories about gravity, such as the one by Sir Isaac Newton, did not seem to require a great devotion towards non-Euclidean geometry. However, the more recent viewpoint on gravity, due to Albert Einstein, relies precisely in linking gravitational phenomena to the geometric structure of the world we live in. Modern experiments have, to a large extent, supported Einstein's theory and hence differential geometry has ensconced itself in the field of theoretical physics.

Therefore, this chapter is committed towards providing a brief overview of geometry that governs the physics of relativity. It has been assumed that the average reader of this document is familiar with the concepts of differential geometry. Consequently, the amount of definitions, detailed derivations, and proofs of statements have been left at a bare minimum. The details may be found in the relevant references.

The current chapter opens with an account of the basic elements of differential geometry. Then it presents the concept of a Lorentzian metric tensor, perhaps most important structure imposed on a manifold in the theory of relativity. Finally all these concepts are tied together to gravitation by introducing the basic tenets of general relativity. The end goal of this chapter is to establish the meaning of the term ′space-time′ as used in this document.

2.1 Manifolds and Tensors

Most of the observations about large scale objects in physics seem to indicate that the space and time around us are ′continuous′. The mathematically precise way to describe this type of structure is through the concept of manifolds.

Definition 2.1.1. *A topological space $\mathcal{M}$ is called a topological* **manifold of dimension** *n (with boundary) if it is* Hausdorff, second countable *and every point of $\mathcal{M}$ has a neighborhood that is* homeomorphic *to an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$-the upper half plane, resp.).*

Roughly speaking, a two-dimensional manifold then represents any physical structure that can be built by gluing together pieces of paper such that

i. a single piece of paper does not fold over to over-lap itself and

ii. there are no ′kinks′ and ′corners′ in the structure.

Clearly, $\mathbb{R}^n$ is an example of a manifold, and so is a cylinder, a sphere, or even the Möbius strip.

Often the conditions on a manifold being Hausdorff and second countable are thought of as peripheral to the definition. However, they have been included here as an integral part of the definition for two reasons:

1. the properties of being Hausdorff and second countable make sure that a manifold is a 'nicely behaved' mathematical object [6]. For instance, in Hausdorff spaces a convergent sequence has a unique limit. On the other hand, second countability leads to the existence of partitions of unity on a manifold, which in turn guarantees the existence of (many) Riemannian metrics on it.

2. these properties can be related to certain physical aspects of nature that we expect to be true. For example, a single observer in a non-Hausdorff universe may split into two (i.e., his/her world-line may bifurcate) at some parts of the universe, which is a highly nonsensical physical scenario [1].

Nevertheless, there are still known examples of exact solutions to Einstein's field equations in general relativity that are non-Hausdorff, such as the analytic extension of the Taub-NUT space-time, where such bifurcations do not occur (see [7] for the derivation of this result and [1] for general discussions on the extension of Taub-NUT space-time). Such solutions are often studied (e.g. in [8]) to investigate whether one may relax the the Hausdorff requirement on a space-time model [1]. However, for the purposes of this article, it will always be assumed that the space-time models are indeed manifolds as defined above.

Differentiable structures on the manifold can be defined in the usual manner (see [6] or [1] for the details). The manifolds discussed in this essay are always assumed to be smooth unless otherwise stated. Coordinate charts are, in general, denoted as (x^i), where the index i runs from 0 to $n-1$ for an n-dimensional manifold. In fact, throughout the paper all indexes will run from 0 to $n-1$ for objects on an n dimensional manifold and the Einstein's summation convention of summing over repeated indexes will be maintained. Any exception to the rules will be clearly stated.

2.1.1 Tensorial Notation

First, we consider all possible curves passing through a point p on an n-manifold $\mathcal{M}$. Then the collection of all tangent vectors at p forms an n-dimensional vector space called the **tangent space** of $\mathcal{M}$ at p, denoted by $T_p(\mathcal{M})$. Given a coordinate chart (x^i) around the point p in $\mathcal{M}$, the tangent space $T_p(\mathcal{M})$ is assumed to adopt the coordinate vectors

$$\left\{ \frac{\partial}{\partial x^0}\bigg|_p, \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^{n-1}}\bigg|_p \right\}$$

as a basis. The **cotangent space** $T_p^*(\mathcal{M})$ at point p is understood here as the dual space of $T_p(\mathcal{M})$ and has dual basis

$$\left\{ dx^0\big|_p, dx^1\big|_p, \ldots, dx^{n-1}\big|_p \right\}.$$

The (r, s)-**tensor space** at p is given by

$$T_p(\mathcal{M})^r_s = \underbrace{T_p(\mathcal{M}) \otimes \cdots \otimes T_p(\mathcal{M})}_{r-\text{times}} \otimes \underbrace{T_p^*(\mathcal{M}) \otimes \cdots \otimes T_p^*(\mathcal{M})}_{s-\text{times}}.$$

The (r, s)-tensor bundle on $\mathcal{M}$ can be formed in the usual way. It will always be assumed that all the computation being carried out in the tensor bundle is done in a coordinate frame whenever a local coordinate system is given, unless explicitly stated otherwise.

Therefore, a vector field will often be prescribed by identifying its components v^i, a co-vector field ω by its components ω_i and an (r, s)-tensor field is completely determined by its components $\tau^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$ in a given coordinate basis. All tensor fields discussed in this essay are assumed to be smooth unless it is mentioned otherwise.

Here, the usual notational convention of general relativity will be followed for tensors (see [1, 9, 10] or [2] for example):

i. The same letter would be used if the a tensor has been derived from an old one via contraction of indexes, such as

$$R_{ab} = R^c{}_{acb}.$$

ii. Usually the metric tensor (to be discussed in greater detail in the next section) will be specified by g_{ij}, while the components of its inverse are denoted g^{ij}. The notation η_{ij} will be used to denote the Minkowski metric. The Kronecker delta symbol $\delta^a{}_c$ will be used to indicate the components of the identity matrix, and hence

$$g^{ij} g_{jk} = \delta^i{}_k$$

iii. Since every manifold discussed here comes with a metric, the musical isomorphism [11] will often be used to 'raise' and 'lower' indexes in the following manner:

$$T^a{}_b{}^c = g^{kc} T^a{}_{bk} \quad \text{and} \quad W_b{}^c = g_{kb} W^{kc}$$

Notice that the same letter has been used to denote both the tensors when indexes have been raised or lowered.

iv. It is often convenient to introduce the following notations

$$T_{(a_1 \ldots a_k)} = \frac{1}{k!} \sum_{\sigma \in S_k} T_{a_{\sigma(1)} \ldots a_{\sigma(k)}} \tag{2.1.1}$$

$$T_{[a_1 \ldots a_k]} = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) T_{a_{\sigma(1)} \ldots a_{\sigma(k)}} \tag{2.1.2}$$

where σ is a permutation of $(1\ 2\ \cdots\ k)$ and S_k is the group of all such permutations. Similar definitions would apply for the upper indexes as well [2], for example,

$$T^{(ab)c}{}_{[de]} = \frac{1}{4} \left[T^{abc}{}_{de} + T^{bac}{}_{de} - T^{abc}{}_{ed} - T^{bac}{}_{ed} \right].$$

v. Suppose (x^i) is a given coordinate system around a point p such that in the coordinate basis the tensor τ has components $\tau^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$. Let (y^j) be another set of coordinates around the same point p. In the new coordinate basis, say τ has components $\tau^{c_1 c_2 \ldots c_r}{}_{e_1 e_2 \ldots e_s}$. Moreover, suppose we define

$$A^c{}_a = \frac{\partial y^c}{\partial x^a} \quad \text{and} B^b{}_e = \frac{\partial x^b}{\partial y^e}$$

then note that $A^c{}_a B^a{}_e = \delta^c{}_e$, where $\delta^c{}_e$ is the Kronecker delta. Then the components of τ satisfy the relation

$$\tau^{c_1 c_2 \ldots c_r}{}_{e_1 e_2 \ldots e_s} = A^{c_1}{}_{a_1} A^{c_2}{}_{a_2} \cdots A^{c_r}{}_{a_r} \tau^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} B^{b_1}{}_{e_1} B^{b_2}{}_{e_2} \cdots B^{b_s}{}_{e_s} \tag{2.1.3}$$

There are a few other geometric objects that will be used in this report. However, these will be introduced in the proceeding sections.

2.2 The Metric Tensor

In general relativity, each of the manifolds always carries an additional structure - a metric - on it. The theory of relativity relies heavily on studying metrics that satisfy a system of non-linear partial differential equations known as the Einstein field equations. In this section, the general properties of a Lorentzian metric will be studied.

Definition 2.2.1. *A* **metric***, almost always denoted by the letter 'g', is given by a symmetric, non-degenerate* $(0, 2)$ *tensor field on a manifold* $\mathcal{M}$*. This means that at a point p in* $\mathcal{M}$

$$g : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$$

is a bilinear map. Moreover, it must satisfy

i. For all v_1, v_2 *in* $T_p\mathcal{M}$*,*

$$g(v_1, v_2) = g(v_2, v_1)$$

ii. For a given vector v_1 *in* $T_p\mathcal{M}$

$$g(v, v_1) = 0 \textit{ for all } v \in T_p\mathcal{M}$$

if and only if $v_1 = 0$*.*

At a point p in the manifold $\mathcal{M}$, the metric assigns a **norm** to every vector $v \in T_p\mathcal{M}$ given by

$$\|v\| = \left(\left|g(v,v)\right|\right)^{\frac{1}{2}}, \tag{2.2.1}$$

and a vector v_1 is said to be **orthogonal** to another vector v_2 if

$$g(v_1, v_2) = 0.$$

Since it is a $(0,2)$ tensor, in a coordinate basis the metric is often written as

$$g = g_{ij}dx^i dx^j$$

or as

$$ds^2 = g_{ij}dx^i dx^j.$$

The non-degeneracy of the metric ensures that there is a $(2,0)$ tensor field on $\mathcal{M}$ with components g^{ij} in the same coordinate basis such that

$$g^{ij}g_{jk} = \delta^i{}_k.$$

Given a metric g, the corresponding matrix (g_{ij}) at any point p is a real symmetric matrix, and hence can always be orthogonally diagonalized. In fact, there exists a basis $\{v_1, \ldots, v_n\}$ of $T_p\mathcal{M}$ such that $g(v_a, v_b) = \pm\delta^a{}_b$ for $a, b = 1, \ldots, n$ (see [1, 2]). The **signature** of g at p is the number of positive eigenvalues of the matrix (g_{ij}) at p, minus the number of negative ones.

There happens to be a very rich geometric theory for a positive definite metric (i.e., **Riemannian metric**), where the signature is n at each point, on an n-manifold (see [11, 12] or [13] for an exposition to this theory). In fact, due to the second countability of a manifold, there exist many such metrics on a manifold. However, general relativity uses a metric whose signature is $(n-2)$, called a **Lorentzian metric**. The existence criteria for such a metric will be studied next.

Definition 2.2.2. *Suppose $\mathcal{M}$ is a smooth manifold. Then* **line element field** *on $\mathcal{M}$ refers to an assignment of a pair of equal and opposite vectors $(X, -X)$ at each point p of $\mathcal{M}$.*

Therefore, in essence, a non-vanishing line element field determines an unoriented one-dimensional subspace of $T_p\mathcal{M}$ at each point p in $\mathcal{M}$[1]. The presence of a non-vanishing line element field on $\mathcal{M}$ leads us to the following result due to Steenrod [14].

Lemma 2.2.3. *A smooth manifold $\mathcal{M}$ admits a smooth Lorentzian metric if and only if it admits a non-vanishing smooth line element field.*

Proof. The proof will be developed following the discussion in [1]. Let $\hat{g}$ be any Riemannian metric on $\mathcal{M}$. Then if such a line element field $(X, -X)$ exists on $\mathcal{M}$, one can define a metric g by:

$$g(Y, Z) = \hat{g}(Y, Z) - 2\frac{\hat{g}(X, Y)\hat{g}(X, Z)}{\hat{g}(X, X)} \tag{2.2.2}$$

at each point p for Y, Z in $T_p\mathcal{M}$.

Then it is quite easy to see that $g(X, X) = -\hat{g}(X, X)$. Clearly, as $\hat{g}$ is positive definite, this indicates that g is not positive definite. Now, suppose that Y is a non-zero vector such that $\hat{g}(Y, X) = 0$, that is, Y is orthogonal to X at p with respect to the Riemannian metric. Then

$$g(Y, X) = \hat{g}(Y, X) - 2\frac{\hat{g}(X, Y)\hat{g}(X, X)}{\hat{g}(X, X)} = 0,$$

thus Y is orthogonal to X with respect to g as well. Also notice that in this case

$$g(Y, Y) = \hat{g}(Y, Y) - 2\frac{\hat{g}(X, Y)\hat{g}(X, Y)}{\hat{g}(X, X)} = \hat{g}(Y, Y).$$

Therefore, this implies that g is indeed an indefinite metric.

Now assume that there exists a line element field $(X, -X)$ on an n-manifold $\mathcal{M}$. Then at every point p on the manifold, by virtue of the Gram-Schmidt process, there

[1] The author must thank Prof. Karigiannis for clarifying this definition.

exists a set of non-zero vectors $\{Y_1, Y_2, \dots, Y_{n-1}\}$ such that for each $i = 1, \dots, n-1$

$$\hat{g}(X, Y_i) = 0.$$

Consequently the set $\{X, Y_1, Y_2, \dots, Y_{n-1}\}$ forms a basis of $T_p\mathcal{M}$. Note that for a metric g given by equation (2.2.2), this means that

$$g(X, X) = -\hat{g}(X, X) \quad \text{and} \quad g(Y_i, Y_i) = \hat{g}(Y_i, Y_i)$$

for some Riemannian metric $\hat{g}$. Then, by rescaling the vectors if necessary, in this basis the matrix (g_{ij}) takes the form

$$(g_{ij}) = diag(-1, +1, +1, \dots, +1).$$

Hence clearly the metric g has signature $(n-2)$ and is thus Lorentzian .

Conversely, suppose g is a smooth Lorentzian metric on $\mathcal{M}$. Then its components g_{ij} must also vary smoothly in a local coordinate system. Consequently, the eigenvector, say X, corresponding to the negative eigenvalue of (g_{ij}) at each point is determined up to a sign and a normalization. The sign of X can be chosen at each point so that collectively it can form a non-vanishing smooth vector field. Moreover, the same applies for $-X$, and hence $(X, -X)$ forms a non-vanishing, smooth line element field. □

Remark 2.2.4. A few remarks are in order:

1. The construction of the Lorentzian metric in the above proof used an arbitrary Riemannian metric $\hat{g}$. However, as discussed earlier, there are in fact many choices of Riemannian metrics on a manifold. As $\hat{g}$ is not unique, there are actually many Lorentzian metrics on $\mathcal{M}$ if there is one.

2. Hawking and Ellis [1] state that any non-compact manifold admits a line element field, and that a manifold can be a reasonable model of space-time only if it is non-compact. Therefore, henceforth the discussion will be restricted to non-compact manifolds only.

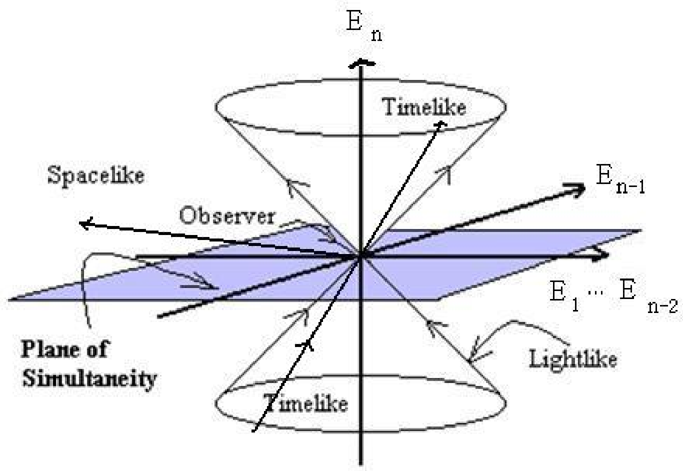

Figure 2.1: The light-cone of a Lorentzian metric (adapted from Figure 8 in [1]).

3. However, from now on, the word 'manifold' will be used to indicate a **Lorentzian manifold**: not just the set $\mathcal{M}$ itself, but actually a pair $(\mathcal{M}, g)$, where g is a Lorentzian metric. Hence, if $g \neq g'$, then the manifold $(\mathcal{M}, g)$ will be considered different from $(\mathcal{M}, g')$.

With a Lorentzian metric g on $\mathcal{M}$, the non-zero vectors at a point p can be divided into three classes

Definition 2.2.5. *A non-zero vector $X \in T_p\mathcal{M}$ is said to be*

(a) **time-like** $\iff g(X, X) < 0,$

(b) **null** *or* **light-like** $\iff g(X, X) = 0$ *and*

(c) **space-like** $\iff g(X, X) > 0.$

The null vectors can be seen to form a double cone in $T_p\mathcal{M}$, known as the **light cone**, which separates the time-like vectors from space-like vectors. The distinction between these classes can be clearly seen in figure 2.1.

Definition 2.2.6. *A curve $\gamma : J \to \mathcal{M}$, where $J \subset \mathbb{R}$, is said to be*

(a) **time-like** *if its tangent vector at each point is time-like;*

(b) **null** *or a* **light ray** *if its tangent vector at each point is null and*

(c) **space-like** *if its tangent vector at each point is space-like.*

Suppose γ is a space-like curve. Then its arc-length s, also known as the "**proper length**", is given by

$$s = \int_0^t (g(X, X))^{\frac{1}{2}} dt.$$

On the other hand, if γ is time-like, then its arc-length is defined in the following way

$$\tau = \int_0^t (-g(X, X))^{\frac{1}{2}} dt,$$

and τ is called the "**proper time**".

2.2.1 Connection and Curvature

Apart from the metric on a manifold another structure, known as a connection, may be introduced on a manifold. The general definition of a connection and the details of how it works are not very relevant to this essay, and hence will be deferred to [1] or [11]. A smooth connection is determined on a coordinate chart U by n^3 smooth functions $\Gamma^c{}_{ab}$. These functions $\Gamma^c{}_{ab}$ are known as **connection coefficients** or **Christoffel symbols**.

Given a vector field Y on a coordinate system (x^i), and connection coefficients $\Gamma^c{}_{ab}$, one may define a new $(1, 1)$-tensor called the **covariant derivative** of Y, denoted by its components

$$Y^c{}_{;b} = \frac{\partial Y^c}{\partial x^b} + \Gamma^c{}_{ab} Y^a. \tag{2.2.3}$$

Similarly, one can define the components of a covariant derivative of an (r,s) tensor to be

$$\begin{aligned} T^{a_1\ldots a_r}{}_{b_1\ldots b_s;c} = & \frac{\partial T^{a_1\ldots a_r}{}_{b_1\ldots b_s}}{\partial x^c} \\ & + \Gamma^{a_1}{}_{cj} T^{j a_2\ldots a_r}{}_{b_1\ldots b_s} + \ldots + \Gamma^{a_r}{}_{cj} T^{a_1\ldots a_{r-1} j}{}_{b_1\ldots b_s} \\ & - \Gamma^{j}{}_{cb_1} T^{a_1\ldots a_r}{}_{j b_2\ldots b_s} - \ldots - \Gamma^{j}{}_{cb_s} T^{a_1\ldots a_r}{}_{b_1\ldots b_{s-1} j} \end{aligned} \tag{2.2.4}$$

It is now time to introduce the **Levi-Civita connection**. For a given metric g on a manifold $\mathcal{M}$, the Levi-Civita connection is the unique torsion-free connection such that the covariant derivative of g is zero (see [1, 11]). In this case the Christoffel symbols in a coordinate system (x^i) are entirely determined by the components of the metric:

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{ce} \left(\frac{\partial g_{eb}}{\partial x^a} + \frac{\partial g_{ea}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^e} \right). \tag{2.2.5}$$

From now on, the Lorentzian manifolds $(\mathcal{M}, g)$ will always use the Levi-Civita connection for covariant differentiation.

Definition 2.2.7. *Let (x^i) be a coordinate system on M. Then define*

i. the **Riemann curvature tensor** *by its components*

$$R^a{}_{bcd} = \frac{\partial \Gamma^a{}_{db}}{\partial x^c} - \frac{\partial \Gamma^a{}_{cb}}{\partial x^d} + \Gamma^a{}_{cf}\Gamma^f{}_{db} - \Gamma^a{}_{df}\Gamma^f{}_{cb} \tag{2.2.6}$$

ii. the **Ricci tensor** *by its components*

$$R_{bd} = R^a{}_{bad} \tag{2.2.7}$$

iii. and the **scalar curvature** *to be*

$$R = R^a{}_a. \tag{2.2.8}$$

The quantities defined above are extremely important in general relativity. In particular, the Riemann curvature tensor contains important information about the geometry of a manifold. In short, they provide a method of distinguishing the manifolds from each other by means of their geometry. However, in relativity it is the geometry of the manifold that determines the physical implications of a gravitational field, and hence the Riemann tensor is a key tool in studying different types of gravitational effects as well. More about the geometric aspects of the Riemann tensor can be found in [11], and its physical implications in relativity texts such as [1, 10, 2, 15].

2.2.2 Geodesics

Finally, the concept of geodesics also needs to be introduced. Let $(t, x^1, \ldots, x^{n-1})$ be a coordinate system on a Lorentzian manifold $\mathcal{M}$. Let $\gamma(t)$ be a smooth curve on $\mathcal{M}$ parametrized by the coordinate t. Let $\dot{\gamma}(t_0)$ be the tangent vector to γ at $\gamma(t_0)$, that is, it is a vector in $T_{\gamma(t_0)}\mathcal{M}$. Let X be a smooth vector field on $\mathcal{M}$ such that at any point on the curve γ, we have $X = \dot{\gamma}$. Then the curve $\gamma(t)$ is said to be a **geodesic curve** if

$$X^a{}_{;b}X^b = \lambda X^a$$

along γ for some smooth function λ on $\mathcal{M}$.

For such a curve, one can re-parametrize (see [1, 2]) the curve with some parameter $v(t)$ such that along the curve

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}v^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}v}\frac{\mathrm{d}x^c}{\mathrm{d}v} = 0. \tag{2.2.9}$$

The equation (2.2.9) is usually known as the **geodesic equation**. The parameter v is called an **affine parameter**. The geodesic equation says that $\gamma(v)$ has zero acceleration. In fact, any curve that satisfies the geodesic equation is a geodesic curve.

Note that the geodesic equation depends on the Christoffel symbols, and hence in general, the particular metric on the manifold. This suggest that the geodesics on two manifolds with distinct geometries, that is, the curvature quantities defined in definition 2.2.7 are different for these manifolds, will not be the same.

Given a point p on a manifold $\mathcal{M}$, and a vector $V \in T_p\mathcal{M}$, there exists a unique geodesic $\gamma(t)$ through p such that V lies tangent to γ at p (see Theorem 4.10 in [11]). Let q be another point in $\mathcal{M}$. Then it can also be shown that, locally, a curve with endpoints p and q extremizes its arc-length if and only if it is a geodesic [2]. Note that this is different from the case when g is a Riemannian metric, where the arc-length is always locally minimized. As discussed later, these two properties make geodesics extremely important in general relativity, while the following theorem quoted from ([16], page 134) adds to its relevance in the theory:

Theorem 2.2.8. *Let $\mathcal{M}$ be a smooth manifold with a smooth connection. Then for any point p in $\mathcal{M}$, there is a neighborhood U of p which is* **convex***; i.e., for any two points in U, there is a unique geodesic curve which joins the two points and lies in U.*

The above theorem will allow us to examine the causality relations (introduced in chapter 4) between two points in $\mathcal{M}$.

2.3 Space-Time and Beyond

Newton theorized gravity to be a force. Physically, a force acts on an object to change its momentum, which, for objects with constant mass, corresponds to acceleration. Therefore in the older picture of gravity, bodies under gravitational influence will accelerate towards the direction of the force. In this formulation one may define a gravitational field as a conservative vector field in the direction of the force, and may thus also define a scalar potential that gives rise to the field. This conforms to our daily experience with gravity: things seem to accelerate as they fall towards the earth - a source of gravity. In fact, this theory also turns out to be very accurate in scientific experiments. Thus, Newton's view on gravity remained unchallenged for around 300 years. However, there are some small discrepancies between the theory and data from certain experiments, such as that of the precession of the perihelion of Mercury (see [17] for details), that this theory cannot reconcile.

In contrast to that, Einstein's version of gravity seems, at first, to be quite bizarre. It does not look at all like the old theory, nor does it seem intuitive. In this new formulation gravity is no longer a force. This means that objects free-falling onto the earth, i.e., when they are *under the influence of gravity only*, are not accelerating!

Instead, Einstein considers the universe to have a geometrical structure, and gravity to be the result of living in a world of non-Euclidean geometry. This theory proposes that the universe is a four dimensional Lorentzian manifold, and free particles (i.e, not influenced by a force) travel along geodesics (recall that geodesics are non-accelerating curves). However, a massive object can distort the geometry of its neighboring regions. It is usually assumed that a distant observer is not influenced by this local change in geometry. Hence to him it may seem that free particles, even though they still travel along geodesics, near a massive object are influenced by some 'force' due to the different nature of the geodesics in the near and the far region.

So Einstein's general relativity may be summarized by the **equivalence principle**. This principle states that experiments in a sufficiently small freely falling laboratory, over sufficiently short time, give results that are indistinguishable from those of the same experiments in an inertial frame in empty space [17]. In other words, locally the effects of gravity cannot be distinguished from the effect that non-Euclidean geometry would have on free-falling observers.

Einstein's theory of gravity can be shown to reduce to Newtonian gravity for weak gravitational fields [2]. Hence the predictions of general relativity in the appropriate limit agree with those of Newtonian gravity. However, in case of strong gravitational sources, Einstein's theory triumphs over Newton's when it comes to experimental verification. For instance, data for the anomalous motion of Mercury matches with the predictions of general relativity. Moreover, the theory of relativity has also provided new predictions, some of which have been verified through experiment. Consequently, Einstein's theory of general relativity has proved superior to Newtonian theory of gravity in explaining the structure of the universe.

2.3.1 The Structure of Space-Time

In this section, the concept of space-time will be explored. The aim here is to determine the conditions under which a manifold shall be called space-time for the purposes of this essay. However, the term 'space-time' does not quite have a precise canonical definition in the literature of general relativity and so the description given here might not agree with the ones by other authors.

In general relativity, the basic structure is a connected four dimensional manifold $(\mathcal{M}, g)$ with a Lorentzian metric and each point on a manifold is called an **event**. Two manifolds that are isometric are considered to be same physically, and will thus be considered as the same physical structure. On such a manifold, use the Ricci tensor and the scalar curvature to define the **Einstein tensor**

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}. \tag{2.3.1}$$

The first condition to call a Lorentzian manifold (M, g) as a space-time is that the metric must satisfy the **Einstein Field Equations**

$$G_{ab} = 8\pi T_{ab} \tag{2.3.2}$$

where T_{ab} is called the **stress tensor** or the **energy momentum tensor**, and represents the presence of matter on the space-time.

Matter in space-time is often modeled as a fluid (see [9]) and then T_{ab} would indeed represent the stress-energy tensor of the fluid as defined in continuum mechanics. That is, in some coordinate frame, the component T^{ab} denotes the flux of the a-component of momentum carried by the fluid across a hypersurface of constant x^b. A detailed explanation of what the stress-energy tensor is, and how it is related to general relativity, can be found in [10].

Due to the physics of fluids, the stress-energy tensor is restricted to be a symmetric tensor

$$T_{ab} = T_{ba}$$

and must satisfy the conservation of energy-momentum, formulated as

$$T^{ab}{}_{;b} = 0. \tag{2.3.3}$$

Since matter is considered to be the source of gravity, then the T_{ab} tensor provides the energy conditions that the space-time must respect. Since the left hand side of equation (2.3.2) refers to the curvature of the manifold, it appears as if the presence of matter dictates the space-time to be curved. On the other hand, as discussed earlier, the curvature of the space-time governs the motion of free-falling matter through geodesics. This interplay of the role of matter and geometry is perhaps best summarized by the memorable quote of J. A. Wheeler:

> Matter tells space how to curve.
>
> Space tells matter how to move.

However, in this essay, all the examples discussed will assume that the space-time is devoid of matter, and hence $T_{ab} = 0$. Thus the metric needs to satisfy $G_{ab} = 0$. But by taking the trace on both sides of $G_{ab} = 0$, one obtains $R = 0$ and hence the vacuum Einstein equation reduces to

$$R_{ab} = 0. \tag{2.3.4}$$

Equation (2.3.4) is called the **Vacuum Einstein Equation**, and a manifold whose metric satisfies this condition is known as Ricci flat. Non-zero stress-energy tensors will be discussed in chapter 5 to derive important properties of black holes.

Next, recall that the Lorentzian metric on the manifold leads to the presence of a light cone in $T_p\mathcal{M}$ at each point p in $\mathcal{M}$ (see figure 2.1). Now at each point p, one half of the double cone would be designated as a **future light cone**, while the other

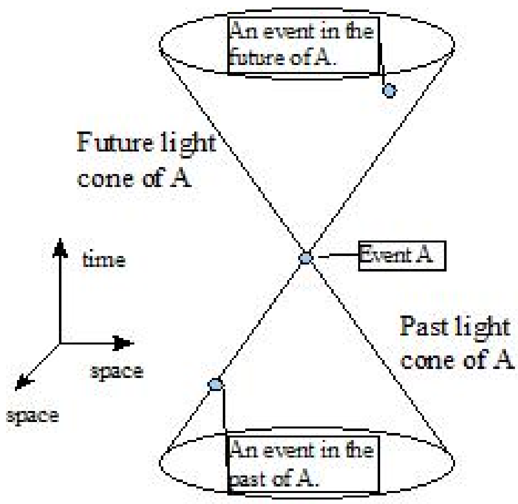

Figure 2.2: The future and past of a point A on the space-time $\mathcal{M}$.

would be called a **past light cone** (figure 2.2). If this designation can be made in a continuous, well-defined manner for all points on the manifold then $\mathcal{M}$ is called **time orientable**. This is an important desired property on a space-time due to the following lemma (see [2] for the proof):

Lemma 2.3.1. *Let $(\mathcal{M}, g)$ be time orientable. Then there exists a (highly non-unique) non-vanishing time-like vector field t^a on $\mathcal{M}$. Conversely, if a continuous, non-vanishing time-like vector field can be chosen on $\mathcal{M}$, then $(\mathcal{M}, g)$ is time orientable.*

Once we have such a time-like vector field t^a, there is now a systematic way to designate the future and the past light cone. At a given point $p \in \mathcal{M}$, simply take the half of the light cone that contain t^a to be the future, and the other half to be the past light cone at p.

Finally, time-like curves are considered to be paths that matter is allowed to travel on. Then, to avoid time-travel paradoxes in the theory, it is desirable that no matter should be able to come back to an event after the first visit. Therefore, there should be no self-intersecting time-like curves, neither should there be any closed time-like loops in a space-time. However, there are known solutions, such as

Gödel's universe [1], of the Einstein Field Equations that do admit closed time-like curves. To prevent the presence of such curves (even under slight perturbations of the metric) on the manifold, a space-time will be required to be 'stably causal'. This is defined with the help of the following lemma:

Lemma 2.3.2. *Let $(\mathcal{M}, g)$ be a Lorentzian 4-manifold. Let t^a be a time-like vector a point $p \in \mathcal{M}$, and define a tensor $\tilde{g}$ by*

$$\tilde{g}_{ab} = g_{ab} - t_a t_b.$$

Then $\tilde{g}_{ab}$ is also a Lorentz metric at p. Furthermore, the light cone due to $\tilde{g}$ is strictly larger than that due to g at p.

Proof. Suppose $\left\{t^a, e^a_i\right\}, i = 2, \ldots, n$, be a frame where

$$g(t^a, t^b) = g_{ab} t^a t^b = t_b t^b = -1,$$

and

$$g_{ab} e^a_i e^b_j = \delta_{ij}, \quad g_{ab} t^a e^b_i = 0.$$

Now define:

$$\tilde{g}_{ab} = g_{ab} - t_a t_b.$$

Then we get

$$\begin{aligned} \tilde{g}_{ab} t^a t^b &= g_{ab} t^a t^b - t_a t_b t^a t^b \\ &= -1 - (-1)(-1) = -2, \end{aligned}$$

while

$$\begin{aligned} \tilde{g}_{ab} e^a_i e^b_j = g_{ab} e^a_i e^b_j - t_a t_b e^a_i e^b_j \\ = \delta_{ij} - g_{ac} t^c e^a_i t_b e^b_j = \delta_{ij}. \end{aligned}$$

Hence the frame $\left\{\frac{1}{\sqrt{2}} t^a, e^a_i\right\}, i = 2, \ldots, n$ puts $\tilde{g} = diag(-1, +1, \ldots, +1)$. Therefore $\tilde{g}$ is also Lorentzian.

Moreover, suppose $t^a + \varepsilon^j e^a_j$ is a time-like vector for g. Then we get

$$\begin{aligned}(t^a + \varepsilon^j e^a_j)\tilde{g}_{ab}(t^b + \varepsilon^j e^b_j) &= (t^a + \varepsilon^j e^a_j)g_{ab}(t^b + \varepsilon^j e^b_j) - (t^a + \varepsilon^j e^a_j)t_a t_b(t^b + \varepsilon^j e^b_j) \\ &= (t^a + \varepsilon^j e^a_j)g_{ab}(t^b + \varepsilon^j e^b_j) - 1 < 0,\end{aligned}$$

so $(t^a + \varepsilon^j e^a_j)$ is also a time-like vector for $\tilde{g}$. Moreover, it is now quite obvious that the light cone due to $\tilde{g}$ is strictly larger, i.e., there are some vectors that are time-like for $\tilde{g}$ but are not time-like for g. □

Definition 2.3.3. *Assume the notation as above. Then a manifold* $(\mathcal{M}, g)$ *is said to be* **stably causal** *if there exists a continuous non-vanishing time-like vector field* t^a *such that the manifold* $(\mathcal{M}, \tilde{g})$ *contains no closed time-like curves.*

Therefore, to summarize, for most part of this project the word **space-time** will refer to a connected, time-orientable, stably-causal, four-dimensional smooth Lorentzian manifold $(\mathcal{M}, g)$ such that the metric g satisfies the Einstein field equations (2.3.2). Using what has been developed so far, from the next chapter onwards, this article will be devoted towards analyzing black-holes, initially in four dimensions, but later on in higher dimensions as well.

Chapter 3

The Schwarzschild Black Hole: an Introduction

> As you see, the war treated me kindly enough, in spite of the heavy gunfire, to allow me to get away from it all and take this walk in the land of your ideas.
>
> Karl Schwarzschild (1916)
> in a letter to A. Einstein

This chapter will survey one of the earliest known solutions to Einstein's vacuum field equations. German mathematician Karl Schwarzschild found this solution only a few months after Einstein published his vacuum field equations. The Schwarzschild solution is perhaps the most important known solution as well.

It describes the external gravitational effect due to a spherically symmetric source of gravity and can accurately describe the motion of planets in our solar system, predict the bending of light and time delay effects due to gravity, etc. However, the aspect of the Schwarzschild solution that will be inspected here is the presence of an object called a black hole in the space-time. However, the concept of a black hole will not be rigorously developed in this chapter. It is intended to be a precursor to the later chapters where the theory of black holes will be introduced.

3.1 The Schwarzschild Metric

The Schwarzschild solution was initially developed to study the gravitational effect outside a spherical star due to the mass of the star. Since the solution is only valid in the empty region outside the star, the vacuum equations (2.3.4) do apply. Thus, initially consider the space-time outside a star of mass m. Assume it has a coordinate patch (t, r, θ, ϕ), such that for $r > 2m$ (outside the star), the Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.1.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Physically, very far away from the star (at infinity), an observer would measure changes in t as time past, changes in r as radial displacement from the center of the star, and $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$ as angular coordinates.

Now one may get rid of the idea of a star and instead consider m to be a parameter. Then equation (3.1.1) still provides a valid metric on the space-time, but only in the region where $r > 2m$. However at $r = 2m$, the given expression for the metric becomes degenerate, and hence cannot be used past this hypersurface. To obtain a valid expression for the metric in the region where $r \leq 2m$, one needs a new set of coordinates in which the metric will admit a non-degenerate coordinate expression.

The Eddington-Finklestein coordinates is an example of a coordinate system that allows the metric to be defined in the region $0 < r \leq 2m$. This is a new set of coordinates (v, r, θ, ϕ) such that on the overlap with the previous coordinates,

$(r > 2m)$, we define

$$v = t + r^*, \quad r = r, \quad \theta = \theta, \quad \phi = \phi,$$

where $r^* = r + 2m \ln \left|\frac{r}{2m} - 1\right|$. Then it can be shown that the new coordinate expression for the metric on the overlap is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2. \tag{3.1.2}$$

Note that in this coordinate system, when $r = 2m$, the matrix for the metric tensor is of the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4m^2 & 0 \\ 0 & 0 & 0 & 4m^2 \sin^2\theta \end{pmatrix}$$

and is hence well-defined and non-degenerate for $\theta \neq 0, \pi$. However, the surfaces where $\theta = 0$ and $\theta = \pi$ present coordinate singularities of the Schwarzschild metric analogous to those for the round metric on a unit 2-sphere. Thus, these degeneracies may simply be taken care of in the same way it is done on a 2-sphere. Once the coordinate singularities at $r = 2m$ is removed, the metric can then be defined smoothly for all $r > 0$.

However, it is possible to conjure up another set of coordinates that cover a larger region of the manifold than that covered by the Eddington-Finklestein coordinates. The Schwarzschild metric can be smoothly extended into this new region as shown in the next section.

3.1.1 The Kruscal-Szekeres Extension

Define a new coordinate system (u, v, θ, ϕ), known as null coordinates, such that on the overlap with the older coordinates (t, r, θ, ϕ), the change of coordinates is given by

$$u = t - r^*, \quad v = t + r^*,$$
$$\theta = \theta, \quad \phi = \phi.$$

where once again, $r^* = r + 2m \ln\left|\frac{r}{2m} - 1\right|$. Note that $\frac{\mathrm{d}r^*}{\mathrm{d}r} = \left(1 - \frac{2m}{r}\right)^{-1}$.

Then the coordinate expression for the metric becomes

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \\ &= \left(1 - \frac{2m}{r}\right)\left[-dt^2 + \left(1 - \frac{2m}{r}\right)^{-2} dr^2\right] + r^2 d\Omega^2 \\ &= \left(1 - \frac{2m}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2 \\ &= \left(1 - \frac{2m}{r}\right)(-dt + dr^*)(dt + dr^*) + r^2 d\Omega^2 \end{aligned}$$

and hence

$$ds^2 = -\left(1 - \frac{2m}{r}\right) du dv + r^2 d\Omega^2 \tag{3.1.3}$$

where r is now understood as a function of u and v, implicitly defined as:

$$r + 2m \ln\left|\frac{r}{2m} - 1\right| = \frac{v - u}{2}.$$

Note that the above can be solved for $\left(1 - \frac{2m}{r}\right)$ in the following manner:

$$\begin{aligned} r + 2m \ln\left|\frac{r}{2m} - 1\right| &= \frac{v-u}{2} \\ \implies ln\,|r/2m - 1| &= \frac{v-u}{4m} - \frac{r}{2m} \\ \implies (r/2m - 1) &= \exp\left(\frac{v-u}{4m}\right)\exp\left(-\frac{r}{2m}\right) \\ \implies \left(1 - \frac{2m}{r}\right) &= \frac{2m}{r}\exp\left(-\frac{r}{2m}\right)\exp\left(\frac{v-u}{4m}\right). \end{aligned}$$

Therefore, one may rewrite the expression for the metric in (3.1.3) as

$$ds^2 = -\frac{2m}{r} e^{\left(-\frac{r}{2m}\right)} e^{\left(\frac{v-u}{4m}\right)} du dv + r^2 d\Omega^2. \tag{3.1.4}$$

Next, define another new coordinate system (U, V, θ, ϕ), called the Kruscal-Szekeres coordinates, such that on the overlap with the null coordinates, the change of coordinates is given by

$$U = \exp(-u/4m), \quad V = \exp(v/4m),$$
$$\theta = \theta, \quad \phi = \phi.$$

Also define the function $r(U, V)$, given implicitly by

$$e^{(r/2m)}\left(\frac{r}{2m} - 1\right) = UV.$$

Thus on the appropriate overlap with the Schwarzschild coordinates (t, r, θ, ϕ), this function would coincide with coordinate function r.

Then one may obtain the new coordinate expression for the metric as

$$ds^2 = -\frac{32m^3}{r}e^{-(r/2m)}dUdV + r^2 d\Omega^2. \tag{3.1.5}$$

One will immediately notice that the expression for the metric in (3.1.5) is well-defined at $r = 2m$. What is perhaps not so evident is that this provides a way to cover a greater part of the Schwarzschild space-time than before. To accomplish this, yet another coordinate system needs to be defined.

Define (T, X, θ, ϕ) to be a coordinate system on the space-time such that on the overlap with the (U, V, θ, ϕ) coordinates, the change of coordinates is given by:

$$T = \frac{U+V}{2}, \quad X = \frac{V-U}{2},$$
$$\theta = \theta, \quad \phi = \phi$$

and define $r(T, X)$ implicitly by

$$e^{(r/2m)}\left(\frac{r}{2m} - 1\right) = X^2 - T^2.$$

Then the metric in the new coordinate basis becomes:

$$ds^2 = \frac{32m^3}{r}e^{-(r/2m)}(-dT^2 + dX^2) + r^2 d\Omega^2 \tag{3.1.6}$$

Note that the metric is once again not well-defined at $r = 0$. In fact, computation of various scalar quantities obtained from the Riemann curvature tensor of the Schwarzschild metric reveals [10] that the scalar

$$R^{abcd}R_{abcd} = 48\frac{m^2}{r^6}.$$

Thus as $r \to 0$, the above scalar diverges. Therefore, $r = 0$ is considered to be a real singularity, and the space-time cannot be extended in a continuous manner across the hypersurface $r = 0$.

Consequently, in this new coordinate system, although both T and X run from $-\infty$ to ∞, but the values of X and T such that $r(X, T) \leq 0$ are not considered to be in the space-time. However, the mere presence of a curvature singularity in the space-time does not always result in the presence of a black hole (black holes will be defined in chapter 5). For example, the big bang singularity of the Friedmann-Robertson-Walker metric does not imply a black hole cosmology (see [2]).

To recognize the presence of a black hole in the Schwarzschild geometry, one needs to describe the Schwarzschild space-time in the (T, X, θ, ϕ) coordinates. This will be done in the next section.

3.2 Surveying the Space-Time

In the figure 3.1 below, the Schwarzschild space-time has been plotted the X and T axes (angular dependence has been suppressed). The positive T direction will be referred to as the ’future’, and it will be assumed that all moving objects always travel towards the future. The lines $X = T$ and $X = -T$ correspond to $U = 0$ and $V = 0$ respectively in the Kruscal-Szekeres coordinate system. Perhaps more importantly, these lines also correspond to the surface $r = 2m$. Thus these lines are used to ‘split up’ the space-time into four regions as shown.

Another aspect of this space-time is that the surfaces $r = const.$, $r \neq 2m$ are

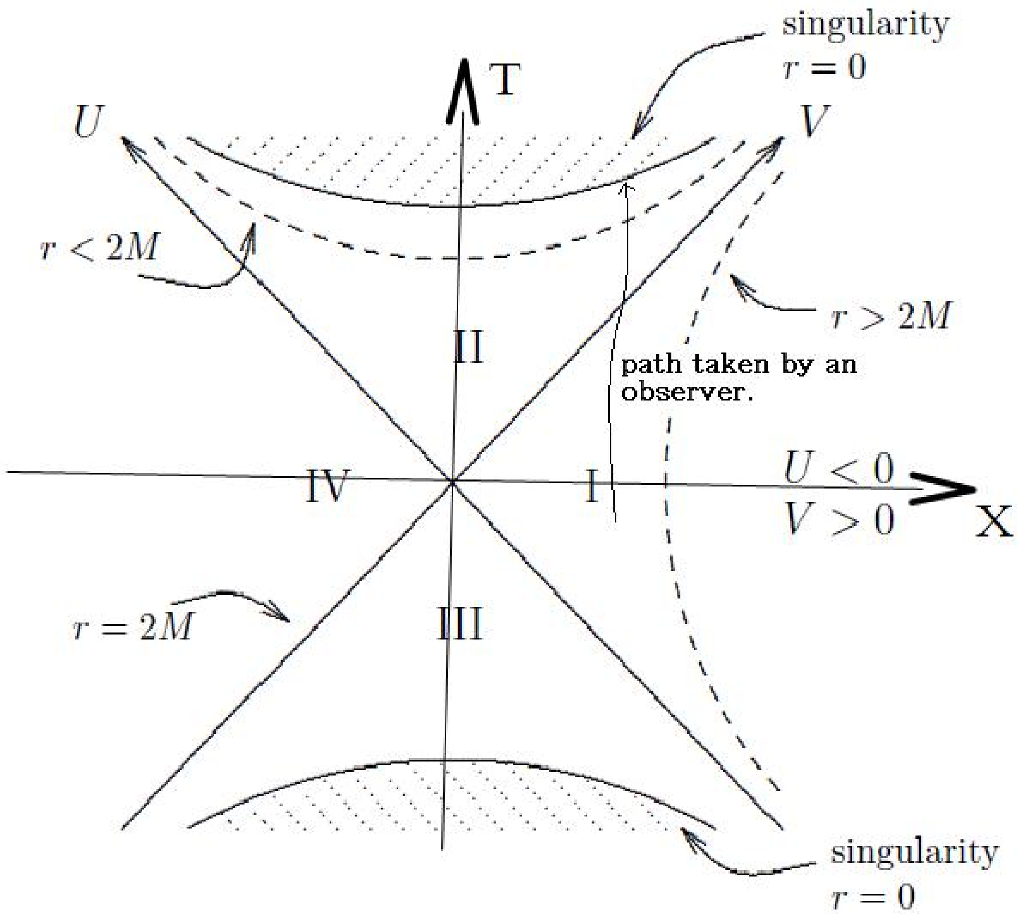

Figure 3.1: Schwarzschild Space-time in Kruscal-Szekeres coordinates (adapted from [2]).

drawn as hyperbolas in the given diagram. From the picture, one can easily notice curves of constant r have different characteristics for $r > 2m$ and for $r < 2m$. Thus $r = 2m$ are the asymptotes of the family of hyperbolas $r = constant$ and is indeed a special surface.

From the expression for the metric in (3.1.6), it is easy to notice that at each point on the manifold, in these coordinates, the light cones are bound by lines that make $45°$ angle to the X-axis. Hence the surface $r = 2m$ acts as a gigantic light cone on the manifold. Then, it is easy to see in the given picture that the time-like curves can only have tangent lines whose slope is greater than 1.

Hence if a massive observer starts from a point in region I and travels along a non space-like curve into region II, then he will not be able to travel back to region I. Moreover, once the observer enters region II, if he continues to travel along a non space-like curve, he is sure to end up on the surface $r = 0$. A similar situation applies to an observer starting his journey in region III. Notice also that an observer from region I would never meet one from region III unless they are both inside region II. On the other hand, a material object that starts from within region II would always stay in region II and eventually end up on the singularity at $r = 0$.

Therefore, one gets the idea that neither matter, nor light can escape from region II. This notion provides a first idea of a black hole. Moreover, the surface $r = 2m$ seems to act as a point of no return. This will later develop into the idea of an event horizon. These concepts will be rigorously defined in chapter 5.

On a curious note, we also observe that anything starting its journey from within region IV would always escape into either region I or region III. Moreover, nothing can go back into region IV along a non space-like curve. Thus, region IV acts as a source of light and matter, and is called a white hole.

3.2.1 Penrose-Carter diagram

In this section, we will use conformal compactification to represent the entire Schwarzschild space-time by a finite region. This will provide a view of the entire space-time at once, as well as introduce new terms that will be developed further in the next chapter.

The compactification of the space-time is carried out by defining two new functions:

$$\tilde{U} = \arctan U \quad \tilde{V} = \arctan V.$$

These new functions ensure that as U and V run off to infinity, $\tilde{U}$ and $\tilde{V}$ stay within the range

$$-\pi/2 \leq \tilde{U}, \tilde{V} \leq \pi/2.$$

Also notice that when with $\tilde{U} = 0$ or $\tilde{V} = 0$ then $r = 2m$. On the other hand, using the trigonometric identity

$$\tilde{U} - \tilde{V} = \arctan U - \arctan V = \arctan\left(\frac{U - V}{1 + UV}\right),$$

we get that when $\tilde{U} - \tilde{V} = \pm\frac{pi}{2}$, then

$$\begin{aligned} \arctan\left(\frac{U - V}{1 + UV}\right) &= \pm\frac{pi}{2} \\ \implies 1 + UV &= 0 \\ \implies 1 + e^{(r/2m)}\left(\frac{r}{2m} - 1\right) &= 0. \end{aligned}$$

The above equation has a solution if and only if $r = 0$. Therefore the lines $\tilde{U} - \tilde{V} = \pm\pi/2$ correspond to the surface $r = 0$.

Then, one may define

$$\tilde{T} = \frac{\tilde{U} + \tilde{V}}{2}, \quad \tilde{X} = \frac{\tilde{V} - \tilde{U}}{2}$$

and draw and label figure 3.2 depicting the compactified Schwarzschild space-time. In the picture, the following objects have been introduced:

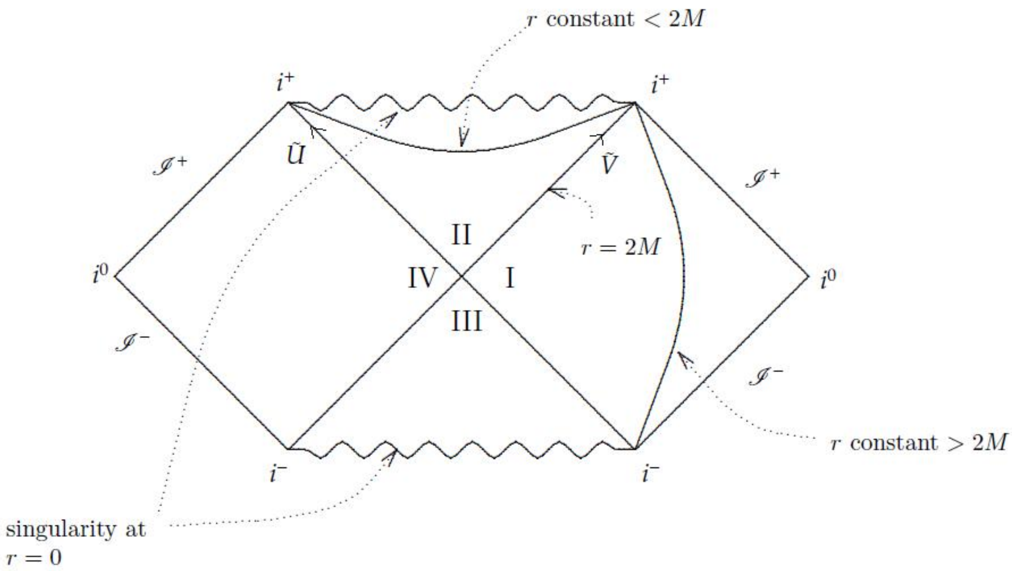

Figure 3.2: The Penrose-Carter diagram of the Schwarzschild space-time (adapted from [3])

- $\mathscr{I}^+$ - the **future null infinity**: contains the future endpoints of all null geodesics that do not enter region II. For the Schwarzschild space-time, this hypersurface is located along $\tilde{U} = \pi/2$ and also along $\tilde{V} = \pi/2$.

- $\mathscr{I}^-$ - the **past null infinity**: contains the past endpoints of all null geodesics that do not originate from region IV. For the Schwarzschild space-time, this hypersurface is located along $\tilde{U} = -\pi/2$ and also along $\tilde{V} = -\pi/2$.

- i^0 - **space-like infinity**: contains end points of all space-like geodesics. For the Schwarzschild space-time, this surface is located at the intersection of $\tilde{U} = \pi/2$ and $\tilde{V} = \pi/2$.

- i^+ - **future time-like infinity**: contains future end points of all time-like geodesics that do not terminate at $r = 0$. These surfaces are located at the intersection of $\tilde{U} = \pi/2$ and $\tilde{V} = 0$ and also at the intersection of $\tilde{U} = 0$ and $\tilde{V} = \pi/2$.

- i^- - **past time-like infinity**: contains past end points of geodesics that do not originate at $r = 0$. These surfaces are located at the intersection of $\tilde{U} = -\pi/2$ and $\tilde{V} = 0$ and also at the intersection of $\tilde{U} = 0$ and $\tilde{V} = -\pi/2$.

Once again, these objects will be rigorously defined in chapters 4 and 5. Nevertheless, in this chapter, all the important concepts required to define a black hole has been touched upon. The next few chapters will provide a rigorous mathematical description of a black hole, and only then will it be really possible to comprehend and appreciate the ideas that have been introduced here.

Chapter 4

Causality, Congruences and Asymptotic Flatness

> The law of causality, I believe, like much that passes muster among philosophers, is a relic of a bygone age, surviving, like the monarchy, only because it is erroneously supposed to do no harm.
>
> Bertrand Russell

The current chapter is intended to serve as a platform towards defining and understanding black holes. Three very important topics will be introduced here: causality, congruences, and the notion of asymptotic flatness. The causal structure of space-time is introduced to conform with our experiences with time as well as cause and effect. On the other hand, geodesic congruences are very useful tools in studying the global properties of space-time. In particular, such congruences will be used in this essay to prove some very important theorems about black holes. Finally, asymptotic flatness is a concept introduced to ensure that gravitational effects of isolated massive bodies remain 'local', that is, the effect of gravity should weaken as one goes further away from the source.

Each of these topics is quite enormous, and to be honest, deserve an individual chapter. However, for brevity's sake, only the bare minimum of the concepts involved will be introduced here. The goal is to set up a stage that is enough to launch an investigation into black holes, but not any more than that.

4.1 The Causal Structure of Space-Time

Two aspects of the causal structure of space-time have already been introduced by insisting that a space-time be time orientable and stably causal. Some other very basic definitions and properties will be introduced here. Most of the definitions here are taken directly from [2].

Since the space-time is time orientable, then at any point p one half of the light cone is designated as 'future' and the other as 'past'. A time-like or null vector in $T_p\mathcal{M}$ lying in the "future half" of the light cone will be called **future directed**, with one in the "past half" would be **past directed**.

Definition 4.1.1. *1. A smooth curve $\gamma(t)$ is said to be a **future directed time-like curve** if at each point on the curve the tangent vector is a future directed time-like vector. Denote Γ^+ for the set of all such curves on $\mathcal{M}$.*

*2. A smooth curve $\gamma(t)$ is said to be a **future directed causal curve** if at each point on the curve the tangent vector is a future directed time-like or null vector. Denote Λ^+ to indicate the set of all such curves on $\mathcal{M}$.*

Past directed time-like and null curves are defined analogously.

Definition 4.1.2. *1. The* **chronological future** *of a point $p \in \mathcal{M}$ is defined as:*

$$I^+(p) = \{q \in \mathcal{M} | \text{ there exists a } \gamma(t) \in \Gamma^+ \text{ with } \gamma(0) = p, \gamma(1) = q\}.$$

Moreover, for any subset $S \subset \mathcal{M}$ define

$$I^+(S) = \bigcup_{p \in S} I^+(p).$$

Chronological pasts $I^-(p)$ and $I^-(S)$ are defined analogously.

2. *The* **causal future** *of a point $p \in \mathcal{M}$ is defined as:*

$$J^+(p) = \{q \in \mathcal{M}|\ \textit{there exists a}\ \gamma(t) \in \Lambda^+\ \textit{with}\ \gamma(0) = p, \gamma(1) = q\}.$$

Moreover, for any subset $S \subset \mathcal{M}$ define

$$J^+(S) = \bigcup_{p \in S} J^+(p).$$

Causal pasts $J^-(p)$ and $J^-(S)$ are defined analogously.

Intuitively, from the picture of the light-cone in Minkowski space-time, one expects some relation to exist between the region of space covered by $J^+(p)$ and $I^+(p)$ for some point p in $\mathcal{M}$. In fact, it can be argued that, if $q \in J^+(p) - I^+(p)$, then any causal curve connecting p to q must be a null geodesic (see [2]). Moreover, for any set $S \subset \mathcal{M}$, it follows that $\overline{J^+(S)} = \overline{I^+(S)}$ [2].

Now $I^+(p)$ is always an open set for any p in $\mathcal{M}$ as one can perform a sufficiently small deformation of the endpoint of a time-like curve while preserving the time-like nature of the curve (the proof of this can be found in [2]), and hence $I^+(S)$ is open as well. On the other hand, $J^+(p)$ may or may not be closed (see [2]). Also recall that the *boundary* ∂S of a set S is defined to be the difference between its closure and interior. As $\overline{J^+(S)} = \overline{I^+(S)}$, we get the relationship $\partial I = \partial J$, that is, the boundary of chronological and causal futures of a set are always equal.

The sets Γ^+ and Λ^+, defined above, contain piece-wise differentiable curves $\gamma : \mathbf{E} \to \mathcal{M}$ parametrized by, say, $t \in \mathbf{E}$, where $\mathbf{E}$ is some real interval. These curves may or may not have endpoints on $\mathcal{M}$. To be more precise, a point p is said to be a **future endpoint** of $\gamma(t) \in \Lambda^+$ if for every neighborhood U of p there is a $t \in \mathbf{E}$ such that $\gamma(t_1) \in U$ for every $t_1 \geq t$ in $\mathbf{E}$. Graphically, this means that the path traced by γ would 'stop' at p, though the endpoint p need not lie on the curve γ itself.

On the other hand, a curve $\gamma(t) \in \Lambda^+$ is said to be **future inextendible** if it has no future end point. Past endpoints and inextendibility are defined in a similar manner. Note that if p is an endpoint of a curve γ, usually it may be extended beyond p by concatenating it with another curve λ that starts at p. This extension may not be differentiable at p, but on a smooth manifold one would expect every curve to be extendible in a continuous way so that no endpoints exist. Unfortunately, sometimes the space-time actually does not allow such extensions at particular points, and this idea will be developed into the notion of a singularity later.

Next, a set $S \subset \mathcal{M}$ is said to be **achronal** if no two points, say p and q, on S can be joined by a time-like curve. In other words, $q \notin I^+(p)$ for any pair of points $p, q \in S$, and in such a case, $I^+(S) \cap S = \emptyset$. Physically, two observers on an achronal set would will only be able to send messages to each other via light rays, but not by any material media. Achronal surfaces seem to play in important role in specifying initial conditions for the evolution of a physical system in general relativity.

Definition 4.1.3. *Let S be a closed, achronal set. Let $\Upsilon^-(p)$ denote the set of past inextendible curves $\gamma \in \Lambda^-$ that pass through a point p in $\mathcal{M}$. Then the **future** **domain of dependence** of S is defined to be*

$$D^+(S) = \{p \in \mathcal{M} | \text{ every } \gamma \in \Upsilon^-(p) \text{ intersects } S\}.$$

Hence, according to the postulates of relativity, any signal received by some point $p \in D^+(S)$ must have visited S some time in the past. Therefore, if all the 'initial conditions' of a deterministic system is known on S, then the behavior of that system can be predicted at any point $p \in D^+(S)$.

The past domain of dependence of S, denoted $D^-(S)$ is defined similarly by using $\Upsilon^+(p)$, for future inextendible curves through p, instead. Again, the knowledge of the initial conditions on S allows one to retrodict an earlier state of the system at some $p \in D^-(S)$. However, one is often interested in extrapolating the available data to capture both the past and future behavior of a system. Consequently, the **domain of dependence** of S is defined as

$$D(S) = D^+(S) \cup D^-(S)$$

and the initial conditions on S provide knowledge of the state of the system at any $p \in D(S)$.

Hence, it is very desirable for a space-time $\mathcal{M}$ to contain a closed, achronal set Σ, such that $D(\Sigma) = \mathcal{M}$. Such a set Σ is called a **Cauchy surface**, and a space-time which possesses a Cauchy surface Σ is said to be **globally hyperbolic**. Every Cauchy surface Σ can be shown to be an embedded $(n-1)$-dimensional C^0 submanifold of $\mathcal{M}$, and may be thought of as 'an instant of time' throughout the entire space-time (see [2]). Usually, it will be assumed in this essay that space-time is globally hyperbolic. However, this assumption will be explicitly stated when being used. The following theorem from [2] (proven in [1]) gives a glimpse of the advantages to a space-time being globally hyperbolic.

Theorem 4.1.4. *Suppose $(\mathcal{M}, g)$ is a connected, smooth, Lorentzian manifold. If $(\mathcal{M}, g)$ is globally hyperbolic, then it is stably causal, and hence a space-time. Furthermore, a global time function, f, can be chosen such that each surface of constant f is a Cauchy surface. Thus, topologically, $\mathcal{M}$ has the structure of $\mathbb{R} \times \Sigma$, where Σ denotes any Cauchy surface.*

Note that the validity of the above theorem relies on the fact that for a globally hyperbolic space-time, any two Cauchy surfaces, say Σ and Σ' are homeomorphic. This follows from lemma 2.3.1, that for a time orientable manifold, there exists a nowhere vanishing time-like vector field t^a on $\mathcal{M}$. Then each integral curve of t^a intersects both Σ and Σ' at exactly one point on each surface. Then the required homeomorphism between Σ and Σ' may be constructed by mapping the points of Σ to the points on Σ' via the integral curves connecting these points.

The very basic properties of the causal structure of space-time have been introduced above. These concepts are important to understand the evolution of the universe and other cosmological aspects of general relativity. They also lead the way towards the definition of black hole regions as well as determining properties of black holes.

4.2 Geodesic Congruences

As mentioned in chapter 2, matter present in a space-time is often modeled as fluids. The motion of fluids is usually described by flow lines (also known as stream-lines): the integral curve of the velocity field of the fluid. In the case of matter, if there are no external forces present (remember, gravity is *not* a force), then these flow lines are locally described by geodesics. Since it is assumed that the matter particles composing the fluid are ubiquitous, then there exists at least one flow line through each point of the manifold. This motivates the following definition:

Definition 4.2.1. *Let $\mathcal{M}$ be a manifold, and U be an open subset of $\mathcal{M}$. A congruence in U is a family of curves such that through each $p \in U$ there passes precisely one curve in this family.*

Clearly, the flow lines of free-falling matter then constitute a congruence of time-like geodesics in $\mathcal{M}$, known as a **geodesic congruence** for short. Note that it will be assumed that the velocity of the matter field forms a smooth vector field, and hence the corresponding geodesic congruence is also smooth. It will also be assumed, without loss of generality, that the geodesic curves in the congruence are unit speed, and hence the velocity field is given by a vector field ξ^a such that

$$\xi^a \xi_a = -1. \tag{4.2.1}$$

Such time-like geodesics will first be studied in a two dimensional setting to get the feel of the 'shear', 'twist' and 'expansion' of such congruences. Then the corresponding definitions will be generalized for higher dimensions. These concepts will become very valuable for deriving some properties of black holes.

4.2.1 Kinematics of a two-dimensional Deformable Medium

Assume that $\gamma_s(t)$ describes a geodesic congruence in $\mathbb{R}^2$ modeling some fluid flow. Note that all the curves in the family are parametrized by the same parameter. Let the vector field describing the tangent vectors of this congruence be given by ξ^a in some coordinate basis. Suppose there are two members of the congruence that are 'close enough', that is, there is a small, local displacement vector field X^a whose norm measures how far apart these curves are at some given value of t. For a given time t, the rate of change of the displacement vector may be written as (see [3]):

$$\frac{\mathrm{d}X^a}{\mathrm{d}t} = B^a{}_b(t)X^b + O(\|X^a\|^2) \tag{4.2.2}$$

for some tensor $B^a{}_b$, whose time-dependence is related to the flow of the fluid.

Note that for short time intervals, the linear approximation of the deviation vector may be written as

$$X^a(t + \Delta t) = X^a(t) + B^a{}_b(t)X^b(t)\Delta t + O(\Delta t^2)$$

and hence the effect of $B^a{}_b$ may be studied.

1. **Expansion**: Suppose the manifold is $\mathbb{R}^2$ with polar coordinates. Consider the deviation vector at time t to be given by

   $$X^a(t) = r(t)\begin{pmatrix}\cos\phi(t)\\ \sin\phi(t)\end{pmatrix}.$$

   Also, suppose $B^a{}_b$ is proportional to the identity matrix, given by

   $$B^a{}_b(t) = \begin{pmatrix}\frac{1}{2}\theta & 0\\ 0 & \frac{1}{2}\theta\end{pmatrix}.$$

   Then notice that

   $$B^a{}_b(t)X^b(t) = \frac{1}{2}r(t)\theta\Delta t\begin{pmatrix}\cos\phi(t)\\ \sin\phi(t)\end{pmatrix}$$

   and thus, to the first order of Δt,

   $$X^a(t + \Delta t) = \left(r(t) + \frac{1}{2}r(t)\theta\Delta t\right)\begin{pmatrix}\cos\phi(t)\\ \sin\phi(t)\end{pmatrix}.$$

Now suppose $r(t)$ represents the radius of a disk O centered at the origin. Then the above equation implies the the members of the congruence going through O at time t would occupy a larger disk O', of radius $r' = r(t) + \frac{1}{2}r(t)\theta\Delta t$ at time $t + \Delta t$. Hence, this gives the picture of the geodesics in the congruence moving away from each other.

Moreover, the difference in area covered by O and O' is given by (to the first order of Δt)

$$\begin{aligned} \Delta A &= \pi(r'^2 - r(t)^2) = \pi\theta r(t)^2\Delta t \\ \implies \theta &= \frac{1}{\pi r(t)^2}\frac{\Delta A}{\Delta t} \end{aligned}$$

and hence θ provides a measure of the rate of change of area covered by the geodesic congruence relative to the current area. In other words, θ is the fractional change of area per unit time, also known as the **expansion parameter**.

2. **Shear**: Suppose that the manifold and the deviation vector is described as above. This time, let the tensor $B^a{}_b$ be given by

$$B^a{}_b(t) = \begin{pmatrix} \sigma & \kappa \\ \kappa & -\sigma \end{pmatrix},$$

that is, $B^a{}_b$ is symmetric and trace-free. Then by linear approximation to the first order, obtain

$$X^a(t + \Delta t) - X^a(t) = B^a{}_b(t)X^b(t)\Delta t = r(t)\Delta t \begin{pmatrix} \sigma\cos\phi + \kappa\sin\phi \\ -\sigma\sin\phi + \kappa\cos\phi. \end{pmatrix}$$

Let $\begin{pmatrix} x & y \end{pmatrix}$ denote the Cartesian coordinates on $\mathcal{M}$. Then for a fixed t and $r(t)$, setting

$$\begin{pmatrix} x & y \end{pmatrix} = r(t)\begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix}$$

provides the parametric description of a circle as ϕ varies, while setting

$$\begin{pmatrix} x & y \end{pmatrix} = r(t)\Delta t\begin{pmatrix} \sigma\cos\phi + \kappa\sin\phi & -\sigma\sin\phi + \kappa\cos\phi \end{pmatrix}$$

describes, in general, an ellipse.

Therefore, this form of the $B^a{}_b$ tensor leads to a shearing of the surface covered by the geodesic congruence, and hence σ and κ are called **shear parameters**.

3. **Rotation**: Finally, assume that $B^a{}_b$ is antisymmetric, given by

$$B^a{}_b(t) = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

Then for the above displacement vector, to the first order,

$$\begin{aligned} X^a(t+\Delta t) &= X^a(t) + B^a{}_b(t) X^b(t) \Delta t \\ &= r(t) \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} + r(t)\omega\Delta t \begin{pmatrix} \sin\phi & -\cos\phi \end{pmatrix} \\ &= r(t) \begin{pmatrix} \cos\phi + (\omega\Delta t)\sin\phi \\ \sin\phi - (\omega\Delta t)\cos\phi \end{pmatrix} \\ &= r(t) \begin{pmatrix} \cos(\phi - \omega\Delta t) \\ \sin(\phi - \omega\Delta t). \end{pmatrix} \end{aligned}$$

Clearly, the new displacement vector $X^a(t+\Delta t)$ is simply the old one rotated by an angle of $\omega\Delta t$, and therefore represents a twist in the congruence. The parameter ω is called the **rotation parameter**.

4. **General Case**: In general, for the $2D$ case, $B^a{}_b$, is a 2×2 matrix. Hence, it has a trace, and one may easily obtain an antisymmetric and a trace-less symmetric matrix from $B^a{}_b$ as shown in the next section. Then overall, the matrix $B^a{}_b$ may be written as

$$B^a{}_b = \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix} + \begin{pmatrix} \sigma & \kappa \\ \kappa & -\sigma \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

and hence it is a linear combination of expansion, shear and twist.

4.2.2 Geodesic Deviation

Now suppose $\mathcal{M}$ is a space-time. Let $\gamma_s(t)$ be a congruence of time-like geodesics in $\mathcal{M}$ with normalized tangent vectors ξ^a, and let $W_p = span\left\{\xi^a_p\right\}$ be a one-dimensional subspace of $T_p\mathcal{M}$ consisting of the tangent vectors to the geodesic $\gamma_s(t)$ passing through each p in $\mathcal{M}$. Suppose the union of the curves $\gamma_s(t)$ form a two-dimensional submanifold S of $\mathcal{M}$ such that s and t may be chosen as coordinates of S. Then, one may define a vector field $X^a = \left(\frac{\partial}{\partial s}\right)^a$ on $\mathcal{M}$. The vector field X^a may be chosen such that it is always orthogonal to ξ^a (in the Riemannian sense)[1], and hence X^a is the orthogonal deviation vector for the given geodesic congruence.

Now define the tensor field B_{ab} by

$$B_{ab} = \nabla_b \xi_a = \xi_{a;b}. \tag{4.2.3}$$

Since the tangent vectors are normalized and each $\gamma_s(t)$ is a geodesic, therefore

$$\begin{aligned} B_{ab}\xi^a &= \nabla_b \xi_a \xi^a = \nabla_b(-1) = 0 \\ B_{ab}\xi^b &= \nabla_b \xi_a \xi^b = \xi_{a;b}\xi^b = 0 \quad \text{(geodesic equation)}. \end{aligned}$$

Moreover, as X^a and ξ^b are coordinate vector fields on S, then $\mathcal{L}_\xi X^a = 0$, and using the fact that the connection is torsion free, one computes the rate of change of X^a along the family $\gamma_s(t)$ to be given by

$$\xi^b \nabla_b X^a = \mathcal{L}_\xi X^a + X^b \nabla_b \xi^a = X^b \nabla_b \xi^a = B^a{}_b X^b. \tag{4.2.4}$$

Thus, if $B^a{}_b = 0$ then X^b is parallelly transported along $\gamma_s(t)$, otherwise $B^a{}_b$ gives an indication of the amount of change that X^b would go through as it is moved along the family. In other words, it measures *geodesic deviation*. This is exactly analogous to the two-dimensional case, and hence encourages the next few definitions.

Definition 4.2.2. *A* $(0, 2)$*-tensor field, called the* **spatial metric**, *is given by*

$$h_{ab} = g_{ab} + \xi_a \xi_b.$$

[1] This is actually due to a "gauge freedom" in choosing X^a. The details can be found in [2].

Then define, for the congruence, the **expansion** θ *as*

$$\theta = B^{ab}h_{ab}$$

and the **shear** σ_{ab} *to be*

$$\sigma_{ab} = \frac{1}{2}(B_{ab} + B_{ba}) - \frac{1}{(n-1)}\theta h_{ab}$$

as well as the **twist** ω_{ab} *as*

$$\omega_{ab} = \frac{1}{2}(B_{ab} - B_{ba})$$

where n *is the dimension of* $\mathcal{M}$*, so that one may write*

$$B_{ab} = \frac{1}{(n-1)}\theta h_{ab} + \sigma_{ab} + \omega_{ab}. \tag{4.2.5}$$

The interpretation of θ, σ_{ab} and ω_{ab} is exactly as it was in the $2D$ case. That is, θ measures the average expansion of the infinitesimally nearby geodesics, ω_{ab} measures their rotation while σ_{ab} measures their shear. Also note that

$$\begin{aligned}
\sigma_{ab}\xi^b &= \frac{1}{2}\left(B_{ab}\xi^b + B_{ba}\xi^b\right) - \frac{1}{(n-1)}\theta h_{ab}\xi^b \\
&= -\frac{1}{(n-1)}\theta h_{ab}\xi^b = -\frac{1}{(n-1)}\theta(g_{ab} + \xi_a\xi_b)\xi^b \\
&= -\frac{1}{(n-1)}\theta(g_{ab}\xi^b + \xi_a\xi_b\xi^b) = -\frac{1}{(n-1)}\theta(\xi_a - \xi_a) = 0
\end{aligned}$$

and similarly

$$\omega_{ab}\xi^b = \frac{1}{2}\left(B_{ab}\xi^b - B_{ba}\xi^b\right) = 0.$$

Thus, since ξ^a is time-like, the tensors B_{ab}, σ_{ab} and ω_{ab} are often called 'purely spatial'.

4.2.3 Energy Conditions

In this section, the concept of energy conditions will be introduced. The energy conditions are statements regarding the nature of matter that is allowed to occupy the space-time. These conditions are defined according to the material properties observed in nature, and are hence usually accepted to be physical. These conditions have deep implications on both the geometry of space-time as well as the properties of black holes.

Suppose that our universe is modeled by a space-time $\mathcal{M}$ containing matter whose stress-energy tensor is T_{ab}. Moreover, assume that the matter is given by a fluid, whose flow lines form a time-like geodesic congruence and has velocity vector ξ^a. Then it should not be too hard to assume the first energy condition:

Condition 1 (Weak Energy Condition)**.** *For all time-like unit vector ξ^a, the stress-energy tensor satisfies*

$$T_{ab}\xi^a\xi^b \geq 0. \tag{4.2.6}$$

This condition states that energy cannot be negative, a reasonable assumption from our day-to-day experience. Next, we introduce a stricter condition on energy that relates to how moving observers will perceive energy:

Condition 2 (Dominant Energy Condition)**.** *For any future-directed, unit time-like vector ξ^a, the quantity $-T^a{}_b\xi^b$ should be a future-directed time-like or null vector.*

In short, this condition seems to say that the speed of energy flow of matter is always less than the speed of light (see [2]). The Dominant Energy Condition implies the Weak Energy Condition.

Finally, it seems physically reasonable to insist that stresses of matter will not be larger than the energy content. This introduces the Strong Energy Condition, independent of the other two conditions, which becomes relevant when studying singularities in space-time.

Condition 3 (Strong Energy Condition)**.** *For any normalized time-like vector ξ^a,*

$$T_{ab}\xi^a\xi^b \geq -\frac{1}{2}T \tag{4.2.7}$$

and is equivalent to

$$R_{ab}\xi^a\xi^b \geq 0. \tag{4.2.8}$$

There is another way of quantifying these energy conditions. It involves diagonalizing T_{ab}, which is possible as T_{ab} is a symmetric bilinear quadratic form. Therefore, we shall assume that in some local orthonormal frame[2] $(\hat{e}^0_a, \hat{e}^1_a, \hat{e}^2_a, \hat{e}^3_a)$ the stress tensor is expressed as

$$T_{ab} = \rho\hat{e}^0_a\hat{e}^0_b + p_1\hat{e}^1_a\hat{e}^1_b + p_2\hat{e}^2_a\hat{e}^2_b + p_3\hat{e}^3_a\hat{e}^3_b$$

that is, $T = diag(\rho, p_1, p_2, p_3)$. Then ρ is interpreted as the rest energy of matter, while p_1, p_2 and p_3 are called principal pressures.

In this case, the energy conditions become:

1. Weak Energy Condition $\iff$ $\quad \rho \geq 0$ and $\rho + p_i \geq 0$ for $i = 1, 2, 3$.

   Proof. Suppose X^a is a unit time-like vector, given by

   $$X^a = \xi^0\hat{e}^{0a} + \xi^1\hat{e}^{1a} + \xi^2\hat{e}^{2a} + \xi^3\hat{e}^{3a}. \tag{I}$$

   Then we get

   $$T_{ab}X^aX^b = \rho(\xi^0)^2 + p_1(\xi^1)^2 + p_2(\xi^2)^2 + p_3(\xi^3)^2. \tag{II}$$

   Also, as X^a is a time-like vector, we have:

   $$-(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = -1. \tag{III}$$

   Then, first, we notice that by setting $\xi^1 = \xi^2 = \xi^3 = 0$ and $\xi^0 = 1$, we get

   $$T_{ab}\hat{e}^{0a}\hat{e}^{0b} = \rho.$$

[2] Since g_{ab} is not positive definite, it is not always possible to find such a frame for diagonalizing a generic symmetric bilinear form [2]. Nevertheless, most reasonable models of matter does admit a local orthonormal frame in which T_{ab} is diagonalized [2], and hence it may be safe to assume the existence of such a frame for our purposes.

Clearly, since $\hat{e}^{0a}$ is a unit time-like vector, if T_{ab} satisfies the weak energy condition, then $\rho \geq 0$.

Next, suppose we set $\xi^2 = \xi^3 = 0, \xi^1 \neq 0$ and $-(\xi^0)^2 + (\xi^1)^2 = -1$. Then we get

$$\begin{aligned} T_{ab}X^aX^b &= \rho(\xi^0)^2 + p_1(\xi^1)^2 \\ &= \rho + (\xi^1)^2 [\rho + p_1] . \end{aligned}$$

If the weak energy is satisfied, then $T_{ab}X^aX^b \geq 0$ and hence

$$\rho + (\xi^1)^2 [\rho + p_1] \geq 0$$

for any ξ^1. This is possible only if $\rho + p_1 \geq 0$.

A similar analysis can be performed on ξ^2 and ξ^3 to obtain that the conditions $\rho \geq 0$ and $\rho + p_i \geq 0$, for all i, are necessarily satisfied by the weak energy condition.

Next, for sufficiency, assume that the weak energy condition does not hold for T_{ab}, but $\rho \geq 0$ and $\rho + p_i \geq 0$ holds for all $i = 1, 2, 3$. Then, for some time-like vector X^a, we have

$$T_{ab}X^aX^b < 0.$$

Using the same notation as above, we know that X^a will satisfy equations I and III. Then by using equation II, we get:

$$\rho(\xi^0)^2 + p_1(\xi^1)^2 + p_2(\xi^2)^2 + p_3(\xi^3)^2 < 0.$$

Now by simple algebraic manipulation, we may write this as

$$\begin{aligned} \rho(\xi^0)^2 + p_1(\xi^1)^2 &+ \left[\rho(\xi^1)^2 - \rho(\xi^1)^2\right] \\ + p_2(\xi^2)^2 &+ \left[\rho(\xi^2)^2 - \rho(\xi^2)^2\right] \\ + p_3(\xi^3)^2 &+ \left[\rho(\xi^3)^2 - \rho(\xi^3)^2\right] < 0, \end{aligned}$$

which gives us

$$\rho\left[(\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2\right] + [\rho + p_1] (\xi^1)^2 + [\rho + p_2] (\xi^2)^2 + [\rho + p_3] (\xi^3)^2 < 0.$$

Obviously, equation III and our hypothesis tell us that the left hand side of the above inequality must be non-negative, a contradiction to the obtained expression. Hence the given conditions are sufficient as well. □

2. Dominant Energy Condition $\iff \quad \rho \geq |p_i|$ for $i = 1, 2, 3$.

Proof. Let us use the same notation for T_{ab} and X^a as in the previous proof. Let us compute

$$\begin{aligned}(-T^c{}_d X^d) g_{ca} (-T^a{}_b X^b) &= T_{ad} T^a{}_b X^d X^b \\ &= \left[-\rho(\xi^0)\hat{e}^0_a + p_1(\xi^1)\hat{e}^1_a + p_2(\xi^2)\hat{e}^2_a + p_3(\xi^3)\hat{e}^3_a\right] \\ &\times \left[-\rho(\xi^0)\hat{e}^{0a} + p_1(\xi^1)\hat{e}^{1a} + p_2(\xi^2)\hat{e}^{2a} + p_3(\xi^3)\hat{e}^{3a}\right] \\ &= -(\rho)^2(\xi^0)^2 + (p_1)^2(\xi^1)^2 + (p_2)^2(\xi^2)^2 + (p_3)^2(\xi^3)^2.\end{aligned}$$

Suppose T_{ab} satisfies the dominant energy condition. Let us choose the vector X^a such that $\xi^2 = 0$ and $\xi^3 = 0$. Then we are left with

$$-(\xi^0)^2 + (\xi^1)^2 < 0$$

since X^a is time-like. Then, if the dominant energy condition is satisfied, we need that

$$\begin{aligned} -(\xi^0)^2\rho^2 + (p_1)^2(\xi^1)^2 &\leq 0 \\ \Longrightarrow \; -\rho^2(\xi^0)^2 + \rho^2(\xi^1)^2 - (\xi^1)^2\rho^2 + (\xi^1)^2(p_1)^2 &\leq 0 \\ \Longrightarrow \; \rho^2\left[-(\xi^0)^2 + (\xi^1)^2\right] - (\xi^1)^2\left[\rho^2 - (p_1)^2\right] &\leq 0. \end{aligned}$$

Now, the first term on the right hand side of above inequality is negative as X^a is time-like. So, if the inequality is to hold for arbitrary large values of $(\xi^1)^2$, we need $\rho^2 - (p_1)^2 \geq 0$ and hence we get

$$\rho \geq |p_1|.$$

Similar analysis can be done for p_2 and p_3 to obtain the required necessary condition $\rho \geq |p_i|$, for all i, to be satisfied by the dominant energy condition.

On the other hand, suppose $\rho \geq |p_i|$, for $i = 1, 2, 3$. Let X^a be an arbitrary future-directed time-like vector.

Claim 1. *The vector $-T^a{}_b X^b$ is a time-like or null vector.*

Proof. From our previous computation, we have

$$\begin{aligned}
\left\| -T^a{}_b X^b \right\| &= -(\rho)^2(\xi^0)^2 + (p_1)^2(\xi^1)^2 + (p_2)^2(\xi^2)^2 + (p_3)^2(\xi^3)^2 \\
&= -(\rho)^2(\xi^0)^2 + \left[(\xi^1)^2\rho^2 - (\xi^1)^2\rho^2\right] + (p_1)^2(\xi^1)^2 \\
&\quad + \left[(\xi^2)^2\rho^2 - (\xi^2)^2\rho^2\right] + (p_2)^2(\xi^2)^2 + \left[(\xi^3)^2\rho^2 - (\xi^3)^2\rho^2\right] + (p_3)^2(\xi^3)^2 \\
&= (\rho)^2\left[-(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2\right] \\
&\quad + \left[(p_1)^2 - \rho^2\right](\xi^1)^2 + \left[(p_2)^2 - \rho^2\right](\xi^2)^2 + \left[(p_3)^2 - \rho^2\right](\xi^3)^2,
\end{aligned}$$

which is obviously non-positive. □

Claim 2. *The vector $-T^a{}_b X^b$ is future-directed.*

Proof. The proof follows from the observation that the inner product between two future-directed non-space-like vectors is non-positive. Then note that

$$-T^a{}_b X^b g_{ac} X^c = -T_{cb} X^b X^c.$$

However, by hypothesis, we get that $\rho + p_i \geq 0$, for all i, and hence T_{ab} satisfies the weak energy condition, so $T_{ab}X^a X^b \geq 0$. Thus we get our desired result. □

Since X^a was arbitrary, therefore T_{ab} satisfies the dominant energy condition. □

3. Strong Energy Condition $\iff$ $\rho + \sum_{i=1}^{3} p_i \geq 0$ and $\rho + p_i \geq 0$ for $i = 1, 2, 3$.

Proof. Assume same notation as in the proof above. Then a simple computation yields:

$$T_{ab}X^aX^b + \frac{1}{2}T^a{}_a = \rho(\xi^0)^2 + p_1(\xi^1)^2 + p_2(\xi^2)^2 + p_3(\xi^3)^2 + \frac{1}{2}(-\rho + p_1 + p_2 + p_3).$$

Suppose we choose $\xi^1 = \xi^2 = \xi^3 = 0$ and $\xi^0 = 1$. In this case, we get that if the strong energy condition is satisfied, then

$$\rho + p_1 + p_2 + p_3 \geq 0.$$

Next, choose $\xi^2 = \xi^3 = 0, \xi^1 \neq 0$ and set $(\xi^0)^2 = 1 + (\xi^1)^2$. If the strong energy condition is satisfied then

$$\begin{aligned} 1 + (\xi^1)^2\rho + (\xi^1)^2 p_1 + \frac{1}{2}(-\rho + p_1 + p_2 + p_3) \geq 0 \\ \implies \rho + p_1 + p_2 + p_3 \geq -(\xi^1)^2(\rho + p_1). \end{aligned}$$

Since the above inequality must hold for all values of (ξ^1), therefore $\rho + p_1 \geq 0$, otherwise the strong energy condition will be violated for some large value of (ξ^1).

Similar results also hold to give $\rho + p_2 \geq 0$ and $\rho + p_3 \geq 0$, and therefore the strong energy condition iimplies that $\rho + p_1 + p_2 + p_3 \geq 0$ and $\rho + p_i \geq 0$, for all i.

On the contrary, suppose $\rho + \sum_{i=1}^3 p_i \geq 0$ and $\rho + p_i \geq 0$ for $i = 1, 2, 3$. Then for an arbitrary time-like unit vector X^a, we get:

$$\begin{aligned} T_{ab}X^aX^b + \frac{1}{2}T^a{}_a &= \rho(\xi^0)^2 + p_1(\xi^1)^2 + p_2(\xi^2)^2 + p_3(\xi^3)^2 + \frac{1}{2}(-\rho + p_1 + p_2 + p_3) \\ &= \rho\left[1 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2\right] \\ &\quad + p_1(\xi^1)^2 + p_2(\xi^2)^2 + p_3(\xi^3)^2 + \frac{1}{2}(-\rho + p_1 + p_2 + p_3) \\ &= \frac{1}{2}(\rho + p_1 + p_2 + p_3) + (\xi^1)^2\left[\rho + p_1\right] + (\xi^2)^2\left[\rho + p_2\right] + (\xi^3)^2\left[\rho + p_3\right]. \end{aligned}$$

Clearly the last expression is greater than zero, and hence the strong energy condition holds. □

4.3 Asymptotic Flatness

In this section the notion of *asymptotic flatness* will be introduced. To be more specific, asymptotic flatness is merely a part of a very rich theory of the asymptotic nature of space-time. The premise of the theory is to study the behavior of space-time far away from gravitational sources - such as the stability (e.g. [18]) and the symmetry (e.g. [19]) of these regions - and it has often led to uniqueness theorems regarding the structure of space-time. However, the focus of this essay is to introduce black holes, and hence only the necessary concept of asymptotic flatness will be discussed.

The first approach towards the asymptotic structure of space-time has been pioneered by Bondi et. al. (see [20]) and Sachs (see [21]) for studying gravitational waves. Their results were then formulated and extended by Penrose (see [22] for the article, or [23] for a comparison between the two formulations). However, these approaches are relevant only in the null regime, while in the spatial regime the theory was developed by Arnowitt et. al. (e.g. [24]) and then extended by Geroch [25]. These notions were then unified to obtain the structure of space-time at null and spatial infinity by Ashteker and Hansen in [19], and this is the approach followed in this article.

In order to appreciate the importance of the asymptotic structure of the universe, perhaps it is best to start from the Newtonian theory of gravity. In Newton's eyes, gravity is a force whose strength is inversely proportional to the square of the distance from the source. Thus, going further away from an isolated massive object, one expects the effect of gravity to decrease in a fashion so that 'infinitely far' away from the source, the presence of the massive object cannot be felt through gravity.

This nature of gravity seems to be upheld by experiment, and thus one expects space-time from general relativity to support this feature as well. Hence, one defines an **isolated system** to be one for which space-time becomes similar to Minkowski space-time as one moves away from the source. Consequently, one has to discuss the notion of being 'far away' and also of being 'flat' (i.e., $g_{ab} \to \eta_{ab}$) in such far away regions.

However, there does not seem to be a straightforward way to indicate the falloff rates of the curvature of the space-time metric g_{ab} with respect to an inherent notion of distance: there simply is no such notion. In particular, there may not be a global coordinate system available to define a radial coordinate r from the source, and so one cannot simply write down expressions such as $g_{ab} = \eta_{ab} + O(1/r)$ to indicate that g_{ab} will approach the flat metric as $r \to \infty$. Therefore, it is better to first define the notion of being far away, and then insist the flatness of space-time in a coordinate independent manner.

The key idea here will be to define a set of "points at infinity", and then bring this "infinity" to a finite distance by means of conformal transformations. Then the asymptotic structure of the gravitational field is explored by studying the local geometric properties at points representing infinities. Asymptotic flatness will then be imposed as a list of conditions on the manifold and the metric at infinity. This approach avoids the problems of coordinate dependence and introduces a well-defined notion of asymptotic flatness in general relativity.

4.3.1 Definition of Asymptotic Flatness

As noted earlier, there have been several approaches towards studying the asymptotic structure of space-time, and this resulted in "asymptotically flat" being defined in a number of mutually inconsistent ways. Here the goal would be to avoid such ambiguity. Initially the concept of an "asymptotically simple" space-time will be adopted from Hawking and Ellis [1]. Often this is taken to be the definition of being asymptotically flat (see [26] for example). However, for the purposes of this essay, asymptotic flatness will be delineated as it is done in Wald [2], following the work of Ashteker and Hansen (see [19] and chapter 2 from volume 2 of [27]).

Definition 4.3.1. *A space-time* $(\mathcal{M}, g)$ *is* **asymptotically simple** *if there exists another manifold* $(\tilde{\mathcal{M}}, \tilde{g})$ *such that*

i. $\mathcal{M}$ *is an open submanifold of* $\tilde{\mathcal{M}}$ *with smooth boundary* $\partial\mathcal{M}$*,*

ii. *there exists a real, non-negative function* Ω *on* $\tilde{\mathcal{M}}$ *such that* $\tilde{g}_{ab} = \Omega^2 g_{ab}$ *on* $\mathcal{M}$*,* $\Omega = 0$ *and* $\tilde{\nabla}_a\Omega \neq 0$ *on* $\partial\mathcal{M}$*,*

iii. *Every null geodesic has two end points on* $\partial\mathcal{M}$ *(i.e., none of the end points are in* $\mathcal{M}$*),*

iv. $R_{ab} = 0$ *near* $\partial\mathcal{M}$*.*

Then $(\mathcal{M}, g)$ *is called the* **physical space-time** *and* $(\tilde{\mathcal{M}}, \tilde{g})$ *is the* **unphysical space-time***.*

Note that the above definition already captures most of the required conditions for the physical space-time to be asymptotically flat. In fact, the purpose of the boundary $\partial\mathcal{M}$ is to serve as an "infinity", and it can be shown that a null geodesic on $\mathcal{M}$ can never reach $\partial\mathcal{M}$ for finite parameter values (see [26]). Using the metric $\tilde{g}$, one may now define $J^{\pm}(\partial\mathcal{M})$, and locally $J^{\pm}(\partial\mathcal{M}) \cap \mathcal{M}$ tells us whether $\mathcal{M}$ is to the past or the future of $\partial\mathcal{M}$.

Definition 4.3.2. *If* $\mathcal{M}$ *lies to the past of* $\partial\mathcal{M}$*, then the future end points of null geodesics form a null hypersurface* $\mathscr{I}^+$ *known as the* **future null infinity***, while if* $\mathcal{M}$ *lies to the future of* $\partial\mathcal{M}$ *then the analogous hypersurface is the* **past null infinity** $\mathscr{I}^-$*.*

Clearly, condition iv. captures the idea that the space-time shall be "flat" sufficiently far away from the source. Moreover, the above definition may be weakened to allow for black holes, where there may be some null geodesics that never escape to infinity.

Definition 4.3.3. $(\mathcal{M}, g)$ *is* **weakly asymptotically simple** *if there exists an asymptotically simple space-time* $(\mathcal{M}', g')$ *and a neighborhood* U' *of* $\partial\mathcal{M}'$ *in* $\mathcal{M}'$ *such that* $U' \cap \mathcal{M}'$ *is isometric to an open subspace* U *of* $\mathcal{M}$*.*

In addition to that, it can be shown that an asymptotically simple space-time is globally hyperbolic [2]. All known examples of space-time that are expected to be asymptotically flat are indeed asymptotically simple. Nevertheless, our precise definition of asymptotic flatness will be adopted from Wald [2].

Definition 4.3.4. *Let $(\mathcal{M}, g)$ be a vacuum space-time (i.e., $R_{ab} = 0$). Then $(\mathcal{M}, g)$ is called* **asymptotically flat at spatial and null infinity** *if the following are satisfied:*

1. *There exists an unphysical space-time $(\tilde{\mathcal{M}}, \tilde{g})$ with $\tilde{g}_{ab}$ smooth everywhere except possibly at a point i^0, where it is $C^{>0}$ (defined below).*

2. *There exists a conformal diffeomorphism $\psi : \mathcal{M} \to \psi(\mathcal{M})$, where $\psi(\mathcal{M}) \subset \tilde{\mathcal{M}}$, such that*
$$\tilde{g}_{ab} = \Omega^2 \psi^* g_{ab}$$
for some positive function Ω on $\psi(\mathcal{M})$ (but Ω is non-negative on $\tilde{\mathcal{M}}$). In fact, since ψ is a diffeomorphism, from now on $\psi(\mathcal{M})$ shall be identified with $\mathcal{M}$, and also $\mathcal{M}$ has a boundary $\partial\mathcal{M}$ in $\tilde{\mathcal{M}}$.

3. *$\overline{J^+(i^0)} \cup \overline{J^-(i^0)} = \tilde{\mathcal{M}} - \mathcal{M}$. Hence i^0 is space-like related to all points in $\mathcal{M}$.*

4. *Let $\mathscr{I}^{\pm} = \partial J^{\pm}(i^0) - i^0$. Then require $\partial\mathcal{M} = i^0 \cup \mathscr{I}^+ \cup \mathscr{I}^-$.*

5. *There exists an open neighborhood V of $\partial\mathcal{M}$ such that $(V, \tilde{g})$ is strongly causal (defined below).*

6. *Ω can be extended to a function on all of $\tilde{\mathcal{M}}$ which is C^2 at i^0 and smooth everywhere else.*

7. *On $\mathscr{I}^+$ and $\mathscr{I}^-$, we have $\Omega = 0$ and $\tilde{\nabla}_a\Omega \neq 0$.*

8. *Require $\Omega(i^0) = 0$, $lim_{i^0}\tilde{\nabla}_a\Omega = 0$ and $lim_{i^0}\tilde{\nabla}_a\tilde{\nabla}_b\Omega = 2\tilde{g}_{ab}(i^0)$.*

9. *The map of null directions at i^0 into the space of integral curves of $\eta^a = \tilde{g}^{ab}\tilde{\nabla}_a\Omega$ on $\mathscr{I}^+$ and $\mathscr{I}^-$ is a diffeomorphism.*

10. *For a smooth function ω on $\tilde{\mathcal{M}} - i^0$ with $\omega > 0$ on $\mathcal{M} \cup \mathscr{I}^+ \cup \mathscr{I}^-$ which satisfies $\tilde{\nabla}_a(\omega^4 \eta^a) = 0$ on $\mathscr{I}^+ \cup \mathscr{I}^-$, the vector field $\omega^{-1}\eta^a$ is complete on $\mathscr{I}^+ \cup \mathscr{I}^-$.*

Remark 4.3.5 (Discussion on definition 4.3.4). The definition of asymptotic flatness needs to be analyzed further. This will be done in a condition-by-condition basis:

1. The first condition insists on the existence of a larger manifold $\tilde{\mathcal{M}}$ than the physical space-time $\mathcal{M}$, with a special point i^0 in $\tilde{\mathcal{M}}$. As it was done for the Schwarzschild space-time in chapter 3, i^0 shall be called a **space-like infinity**, or **spatial infinity** for short.

   Meaning of $C^{>0}$: Suppose that (x^a) is a smooth coordinate system with origin at i^0. Then define "radial function" ρ by

   $$\rho^2 = \sum_{\mu=0}^{n-1} (x^\mu)^2$$

   where n is the dimension of $\tilde{\mathcal{M}}$ (usually, $n = 4$). Then define $(n-1)$ angular functions θ^α in terms of (x^a) such that:

   $$\begin{aligned} x^0 &= \rho \cos(\theta^1) \sin(\theta^2) \sin(\theta^3) \cdots \sin(\theta^{n-1}) \\ x^1 &= \rho \sin(\theta^1) \sin(\theta^2) \sin(\theta^3) \cdots \sin(\theta^{n-1}) \\ x^2 &= \rho \cos(\theta^2) \sin(\theta^3) \cdots \sin(\theta^{n-1}) \\ x^3 &= \rho \sin(\theta^2) \sin(\theta^3) \cdots \sin(\theta^{n-1}) \\ &\vdots \\ x^{n-1} &= \rho \cos(\theta^{n-1}), \end{aligned}$$

   that is, $(\rho, \theta^1, \ldots, \theta^{n-1})$ forms a set of higher dimensional spherical coordinates corresponding to the coordinates (x^a) on $\tilde{\mathcal{M}}$.

   A function f is said to have **regular direction dependent limit** at i^0 if the following properties are satisfied:

i. For each C^1 curve γ ending at i^0, the limit of f along γ exists at i^0. Furthermore, the value of this limit depends only on the tangent directions to γ at i^0.

ii. Define $F(\theta^\alpha) = \lim_{i^0} f$, where the limit is taken along a curve whose tangent direction at i^0 is characterized by θ^α. Require F to be a smooth function on $\rho = constant$.

iii. Along every C^1 curve ending at i^0, require that for all $n \geq 1$

$$\lim_{i^0} \frac{\partial^n f}{\partial(\theta^\alpha)^n} = \frac{\partial^n F}{\partial(\theta^\alpha)^n} \quad \text{and} \quad \lim_{i^0} \rho^n \frac{\partial^n f}{\partial \rho^n} = 0.$$

Then by insisting $\tilde{g}_{ab}$ to be $C^{>0}$ at i^0 it is required that $\tilde{g}_{ab}$ be C^0 at i^0 and all the first partial derivatives of the components of $\tilde{g}_{ab}$ in some smooth chart covering i^0 have regular direction dependent limits at i^0.

Note that in this case $\tilde{\mathcal{M}}$ is said to be equipped with a $C^{>1}$ *differential structure* at i^0. This means that given a family of smooth coordinate charts at p, there exists a subfamily of charts such that the second derivatives of coordinate functions in any one chart with respect to the coordinate functions in any other chart admit regular direction dependent limits at i^0 where both charts are from this subfamily. Note that a smooth manifold is $C^{>n}$ for all n (see chapter 2, volume 2 from [27]). This awkward differential structure is needed to ensure that the connection components admit direction dependent limits at i^0, a necessity for mass to be defined in the space-time (see [19]).

2. The conformal map defined in condition 2. allows one to bring the spatial infinity i^0 to a finite distance from any point on $\mathcal{M}$.

3. Condition 3. merely justifies the nomenclature of i^0. In particular, that there cannot be a non space-like curve joining a point of $\mathcal{M}$ to i^0.

4. $\mathscr{I}^+$ and $\mathscr{I}^-$ are two disjoint sets. $\mathscr{I}^+$ lies to the future of i^0, while $\mathscr{I}^-$ lies to the past of i^0.

5. A space-time $(V, \tilde{g})$ is said to be **strongly causal** if for all $p \in V$ and every neighborhood U of p, there exists a neighborhood U' of p, where $U' \subset U$, such that no causal curve intersects U' more that once. Hence by requiring there to be a strongly causal neighborhood of $\partial\mathcal{M}$, it guarantees that there will be no closed time-like curves in $\partial\mathcal{M}$ and that the physics of the region will behave appropriately.

6. Condition 6. asserts that Ω is well behaved near i^0.

7. Condition 7. insists that $\mathscr{I}^+$ and $\mathscr{I}^-$ lie on the hypersurface $\Omega = 0$. Moreover, since $g_{ab} = \frac{1}{\Omega}\tilde{g}_{ab}$ on $\mathcal{M}$, this indicates that an "infinite amount of stretching" is involved in going from $\tilde{g}_{ab}$ to g_{ab} near $\mathscr{I}^\pm$, and hence these surfaces represent a region "infinitely far" away from any point on $\mathcal{M}$.

8. $\Omega(i^0) = 0$ also supports the idea that i^0 is at infinity with respect to $\mathcal{M}$. Moreover, the other conditions imply that the physical metric approaches flatness as one approaches infinity. One may also notice that these conditions indicate a similarity between i^0 and the vertex of a light cone that appears at each point Minkowski space. Further details of this can be found in [19].

9. This is a technical condition. It can be shown that η^a is a null vector field on $\mathscr{I}^\pm$ and that $\mathscr{I}^\pm$ are null hypersurfaces (see chapter 2, volume 2 of [27]), that is, the normal to such a surface is null. In particular, if one considers the hypersurface given by $\Omega = 0$ on $\tilde{\mathcal{M}}$, then this simply relates to the fact that the null directions at i^0 are in one-to-one correspondence with the integral curves of η^a.

10. This is also a technical condition and essentially requires all of $\mathscr{I}^\pm$ to be present in $(\tilde{\mathcal{M}}, \tilde{g})$, that is, the integral curves of η^a to be complete. Note that a weakly asymptotically simple space-time satisfies all the conditions of definition 4.3.4 except for condition 10 (see [2] and references therein).

It is important to note that the unphysical space-time $(\tilde{\mathcal{M}}, \tilde{g})$ for a space-time $(\mathcal{M}, g)$ is not unique. If $(\tilde{\mathcal{M}}, \tilde{g})$ satisfies all the properties in definition 4.3.4 with conformal factor Ω, then so does $(\tilde{\mathcal{M}}, \omega^2\tilde{g})$ with conformal factor $\omega\Omega$, where ω is a strictly positive function, and is smooth everywhere except possibly at i^0, where it is required to be $C^{>0}$, and satisfies $\omega(i^0) = 1$ (see [2] or [27]). Thus there is a freedom of choice for the unphysical metric $\tilde{g}$.

Chapter 5

Black Holes in Space-Time

> Why it is that of all the billions and billions of strange objects in the Cosmos - novas, quasars, pulsars, black holes - you are beyond doubt the strangest?
>
> Walker Percy
> (American Novelist)

Most of the preliminary background required to define a black hole has now been established. Therefore, the goal in this chapter will be to define a black hole region, and discuss the known properties of the region.

5.1 Black Hole Regions and Event Horizon

Suppose $(\mathcal{M}, g)$ is an asymptotically flat space-time with a corresponding unphysical space-time $(\tilde{\mathcal{M}}, \tilde{g})$, and hence the objects i^0, $\mathscr{I}^+$ and $\mathscr{I}^-$ are defined as in section 4.3.1. In accordance with chapter 3, the surfaces $\mathscr{I}^+$ and $\mathscr{I}^-$ shall be called **future null infinity** and **past null infinity** respectively. The discussion of Hawking and Ellis [1] on causal boundaries in an asymptotically simple space-time states that all the inextendible null geodesics in $\mathcal{M}$ would have a future end point on $\mathscr{I}^+$

and a past endpoint on $\mathscr{I}^-$, while there would be two points i^+ and i^- in $\tilde{\mathcal{M}}$ for the future and past endpoints of inextendible time-like geodesics in $\mathcal{M}$.

On the other hand, in a weakly asymptotically simple space-time, it is possible to have null geodesics that do not end on $\mathscr{I}^+$ in the future. In such a case, it is interpreted that these geodesics never 'escape' off to infinity. For example, in the case of a Schwarzschild space-time, the geodesics inside region II of figure 3.1 never reach $\mathscr{I}^+$ or i^+. This motivates the definition of black hole regions to be the part of the manifold from which no null geodesic would escape to $\mathscr{I}^+$.

Definition 5.1.1. *Let $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ be defined as above. Then $(\mathcal{M}, g)$ is said to be* ***strongly asymptotically predictable*** *if there is an open region*

$$\tilde{V} \subset \tilde{\mathcal{M}} \text{ with } \overline{\mathcal{M} \cap J^-(\mathscr{I}^+)} \subset \tilde{V}$$

such that $(\tilde{V}, \tilde{g})$ is globally hyperbolic.

Proposition 1. *Using the same notation as in definition 5.1.1, it can be inferred that $(\mathcal{M} \cap \tilde{V}, g)$ is a globally hyperbolic region of $\mathcal{M}$.*

Proof. By definition 4.3.4, we have

$$\mathcal{M} = \tilde{\mathcal{M}} - [J^+(i^0) \cup J^-(i^0)]$$

and hence a Cauchy surface, say Σ, for $(\tilde{V}, \tilde{g})$ which passes through i^0, $\Sigma \cap \mathcal{M}$ will also be a Cauchy surface for $(\mathcal{M} \cap \tilde{V}, \tilde{g})$. However, $\tilde{g}$ and g are conformal metrics, and hence on $\mathcal{M}$, they have the same causal structure. Therefore Σ will also be a Cauchy surface for $(\mathcal{M} \cap \tilde{V}, g)$, and thus it is globally hyperbolic. $\square$

Definition 5.1.2. *A strongly asymptotically predictable space-time is said to contain a* **black hole** *if $\mathcal{M}$ is not contained in $J^-(\mathscr{I}^+)$. The* **black hole region***, $\mathcal{B}$, of such a space-time is defined to be*

$$\mathcal{B} = [\mathcal{M} - J^-(\mathscr{I}^+)] \tag{5.1.1}$$

and the boundary of $\mathcal{B}$ in $\mathcal{M}$

$$\mathcal{H} = \partial\mathcal{B} = \partial J^-(\mathscr{I}^+) \cap \mathcal{M} \tag{5.1.2}$$

is called the **event horizon**.

Remark 5.1.3. The concept of a black hole often comes hand in hand with space-time singularities. Singularities are not desirable objects in general relativity as there is no current consensus on what governs the physics of a singularity.

Therefore, the black hole region defined in definition 5.1.2 is carefully designed to avoid the aspect of 'naked' singularities: any singularity present inside the black hole region will not be visible from the 'infinities'. A black hole therefore hides any singularity within it by means of an event horizon. Singularities which are not shielded thus are called **naked**.

The motivation for such a refined definition for a black hole comes from expecting the Cosmic Censorship Conjecture to be valid. There is no proof for the conjecture available yet, but it is widely believed, due to physical reasons, to hold true. The details of this conjecture are out of scope for this essay, and the interested reader will be deferred to Wald's [2] fantastic section on the topic.

Thus, finally, a black hole region has been defined! An example of such a region would be region II from the Kruscal-Szekeres diagram of the Schwarzschild space-time (see figure 3.1).

Throughout the rest of this section, it will be assumed that $(\mathcal{M}, g)$ is an asymptotically predictable space-time such that $\overline{\mathcal{M} \cap J^-(\mathscr{I}^+)}$ is contained in a globally hyperbolic region $\tilde{V}$. Suppose there exists some black hole region $\mathcal{B} = M - J^-(\mathscr{I}^+)$ in the space-time. If Σ is a Cauchy surface of $\tilde{V}$, then it may or may not intersect $\mathcal{B}$. It it does, then $\Sigma \cap \mathcal{B}$ will be regarded as the black hole region at time Σ, and each connected component, denoted $B(t)$, of $\Sigma \cap \mathcal{B}$ will be called a black hole at time Σ. Then on can prove the following theorem.

Theorem 5.1.4. *Let $(\mathcal{M}, g)$ be as above, and let Σ_1 and Σ_2 be Cauchy surfaces for $\tilde{V}$ with $\Sigma_2 \subset I^+(\Sigma_1)$, that is, Σ_2 is in the future of Σ_1. Let B_1 be a non-empty connected component of $\mathcal{B} \cap \Sigma_1$, then $J^+(B_1) \cap \Sigma_2$ is a connected subset of $\mathcal{B} \cap \Sigma_2$.*

This theorem asserts that if a black hole exists at some time, then it will never bifurcate into more than one black hole.

Proof. Given $\Sigma_2 \subset I^+(\Sigma_1)$ and that Σ_2 and Σ_1 are Cauchy surfaces, then clearly $J^+(B_1) \subset \mathcal{B}$ implies $J^+(B_1) \cap \Sigma_2 \subset B \cap \Sigma_2$. Moreover, $J^+(B_1) \cap \Sigma_2$ is non-empty as Σ_2 lies to the future of Σ_1. Now it is required to show that $J^+(B_1) \cap \Sigma_2$ is indeed connected.

Suppose, for the sake of contradiction, that $J^+(B_1) \cap \Sigma_2$ is not connected. Then there exist disjoint open sets U and U' contained in Σ_2 such that

$$U \cap J^+(B_1) \neq \emptyset \quad \text{and} \quad U' \cap J^+(B_1) \neq \emptyset,$$

and such that $J^+(B_1) \cap \Sigma_2 \subset U \cup U'$.

Then, equivalently, one may deduce

$$B_1 \cap I^-(U) \neq \emptyset \quad \text{and} \quad B_1 \cap I^-(U') \neq \emptyset.$$

However, as $I^-(U)$ and $I^-(U')$ are open and B_1 is connected, then this means that there exists some $p \in B_1 \cap I^-(U) \cap I^-(U')$. Then one can divide the future directed time-like geodesics from p into two classes: whether they intersect Σ_2 in U or in U'. Since there is a unique future directed time-like vector in $T_p\mathcal{M}$ for each such geodesic, then clearly this divides the time-like vectors at p into two non-empty disjoint open sets. However, this contradicts the fact the interior of the future light cone at $T_p\mathcal{M}$ is connected!

Then one is left to conclude that $I^-(U) \cap \Sigma_1$ and $I^-(U') \cap \Sigma_2$ are disjoint open subsets of Σ_1, but leads to a contradiction since B_1 is assumed to be connected.

Hence $J^+(B_1) \cap \Sigma_2$ must be connected. □

5.2 Overview of the Kerr Space-Time

Roy Kerr discovered his famous solution in 1963 as an example of an algebraically special metric that produces the gravitational field of a spinning mass [28]. At first, it was expected to model space-time exterior to a massive object with nonzero angular momenta, however, no satisfactory interior solutions have been found that matches with the Kerr solution at the surface of the object (see [29] or [30]). Nevertheless, it was then quickly realized that this solution can represent the final state of gravitational collapse of a rotating body and is thus linked to a rotating black hole. For a more historical account, from Kerr's perspective, see [31].

The structure of Kerr space-time was studied extensively by Boyer and Lindquist in [32], and they established a set of "Schwarzschild-Like" coordinates, called the Boyer-Lindquist (BL) coordinates, that provides perhaps the best physical depiction of Kerr's solution. In this coordinate system, labeled (t, r, θ, ϕ), the metric is given by

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Lambda}\right)dt^2 - \left(\frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Lambda}\right)dtd\phi \\ &+\left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Lambda}\right)\sin^2\theta d\phi^2 + \frac{\Lambda}{\Delta}dr^2 + \Lambda d\theta^2 \end{aligned} \tag{5.2.1}$$

where

$$\Lambda = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 - 2mr + a^2.$$

Notice here that m and a are two parameters and thus the metric represents a family of solutions, each member having a particular value of a and m. These parameters have physical meaning as well: m is the total mass parameter and a can be shown to be the total angular momentum per unit mass parameter for the space-time (see [3]).

It may immediately be deduced that the vectors $\left(\frac{\partial}{\partial t}\right)$ and $\left(\frac{\partial}{\partial\phi}\right)$ are Killing vector fields, and thus there is a rotational symmetry of the space-time about an axis.

Moreover resulting space-time can be shown to be asymptotically flat, such that $r \to \infty$ can be regarded as "infinity" with respect to the manifold in the sense described in chapter 4. Then $\left(\frac{\partial}{\partial t}\right)$ is time-like for large r and as discussed later in chapter 6, this means that the space-time is stationary.

Just as it was for the Schwarzschild space-time, this set of coordinates can be interpreted in terms of physical observables at infinity. Then, the t component for an observer will give his time, while θ and ϕ components will measure the polar and azimuthal angles with respect to the axis of symmetry. In addition to that, for $a = 0$, the given space-time reduces to exactly the Schwarzschild solution.

The given form of the metric can easily be seen to be singular at $\Lambda = 0$ as well as at $\Delta = 0$. In fact, when $\Lambda = 0$, it can be shown that the Weyl scalar

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48m^2(r^2 - a^2\cos^2\theta)(\Lambda^4 - 16a^2r^2\cos^2\theta)}{\Lambda^{12}} \tag{5.2.2}$$

blows up. Hence

$$\Lambda = r^2 + a^2\cos^2\theta = 0$$

indeed represents a physical singularity. Note that this occurs for $a > 0$ when

$$r = 0 \quad \text{and} \quad \cos\theta = 0 \implies \theta = \frac{\pi}{2}.$$

Therefore, unlike the Schwarzschild case, the singularity is now a two-dimensional surface, and $r = 0$ is not entirely singular.

On the other hand, when $\Delta = 0$, one can solve for r to get

$$r_\pm = m \pm \sqrt{m^2 - a^2}. \tag{5.2.3}$$

Now, if $a^2 > m^2$, there is no real value of r that causes Δ to be zero. Unfortunately, for these values of the parameters, it can be shown that the space-time contains naked singularities, which violates our assumption of the manifold being asymptotically predictable. Hence, only the cases for which $a^2 \leq m^2$ will be considered. Notice that for $a = 0$, the solution for $\Delta = 0$ reduces to $r = 0$ and

$r = 2m$, corresponding to the singular regions of the Schwarzschild coordinates in that space-time. For $a^2 = m^2$, on the other hand, there is only one value of r that satisfies $\Delta = 0$, and this also leads to a naked singularity.

In the case when $0 < a^2 < m^2$, one gets two distinct solutions to $\Delta = 0$ given by r_+ and r_-. Since $r_+ > r_-$, it will be assumed that the (BL) coordinates are well-defined for $r > r_+$. However, as discussed in [33], it can be shown that all curvature components and curvature invariants remain finite as $r \to r_+$. Furthermore, as $a \to 0$, one obtains the smooth limit $r_+ \to 2m$, that is, r_+ approaches the Schwartzschild event horizon in this limiting case. Hence, it is tempting to believe that the surface $r = r_+$ indeed represents an event horizon in Kerr space-time.

Nevertheless, this fact is best demonstrated by explicitly constructing a coordinate system that extends the (BL) coordinate patch, and hence the metric, into regions past the surface $r = r_+$. The new set of coordinates (T, r, θ, Φ) can be expressed in terms of the (BL) coordinates on the overlap:

$$T = t + 2m \int \frac{r}{\Delta} dr; \quad \Phi = -\phi - \int \frac{a}{\Delta} dr$$
$$r = r; \quad \theta = \theta.$$

In this new set of coordinates, the Kerr metric is denoted by

$$ds^2 = -\, dt^2 + dr^2 + 2a \sin^2\theta dr d\Phi + (r^2 + a^2\cos^2\theta) d\theta^2 \tag{5.2.4}$$
$$+\, (r^2 + a^2)\sin^2\theta d\Phi^2 + \frac{2mr}{\Lambda}(dt + dr + a\sin^2\theta d\Phi)^2 \tag{5.2.5}$$

and it is easy to observe that the metric is well-defined at $r = M + \sqrt{M^2 - a^2}$ for $a \leq M$. Therefore the condition $\Delta = 0$ merely indicates a coordinate singularity in the (BL) coordinates.

But how does one know that $r = r_+$ depicts an event horizon? Well, strictly speaking, one will need to study all the geodesics that arise in the Kerr space-time, and find a surface that divides the space-time into two distinct regions, say I and II, such that an observer may cross the surface along causal curves from region I to II, but not vice versa. Such a surface then indeed signifies an event horizon. However, thanks to the Strong Rigidity Theorem, it is enough to study the behavior of the Killing vector fields of the space-time to obtain the knowledge of the horizon.

As discussed earlier, the Kerr metric admits two different Killing vector fields, expressed as $\left(\frac{\partial}{\partial t}\right)$ and $\left(\frac{\partial}{\partial \phi}\right)$ in the (BL) coordinates. Then one can show that there exists a Killing vector field given by

$$\xi^\alpha \partial_\alpha = \left(\frac{\partial}{\partial t}\right) + \Omega \left(\frac{\partial}{\partial \phi}\right)$$

that becomes null at $r_+ = m + \sqrt{m^2 - a^2}$, and hence r_+ is a Killing horizon. Then, by the Strong Rigidity Theorem discussed in chapter 6, $r = r_+$ is also the event horizon for a black hole that resides inside this region.

Also, note that $\left(\frac{\partial}{\partial t}\right)$ becomes null when the g_{tt} component of the metric becomes zero, that is, when

$$r^2 - 2mr + a^2 \cos^2 \theta = 0 \tag{5.2.6}$$

which has a unique solution for $r > r_+$ given by:

$$r_{sl}(\theta) = m + \sqrt{m^2 - a^2 \cos^2 \theta}$$

which does not coincide with r_+ unless $\cos^2 \theta = 1$. This solution $r_{sl}(\theta)$ is known as the **static limit** for the space-time. It corresponds to the surface $r = r_{sl}(\theta)$ such that for $r_+ < r < r_{sl}(\theta)$, no observer will be able to remain stationary with respect to some other zero momentum observer at infinity. It is interesting to note that such a surface may only occur at the event horizon for the Schwarzschild space-time, while in the Kerr space-time it may occur before the event horizon has been reached.

Along the same note, one can observe the angular motion of an observer. Suppose an observer is traveling with velocity u^α and his trajectory $(t, r(t), \theta(t), \phi(t))$ is parametrized by the coordinate t. Then one can define his angular momentum to be

$$L = u_\phi \left(\frac{\partial}{\partial \phi}\right).$$

while his angular velocity is given by $\omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$. Now, for an observer with zero angular momentum, in flat space-time, one expects his angular velocity to be zero as well. However, in Kerr space-time, it can be shown that even if $L = 0$ for some observer, as the radial coordinate r decreases, there observer gains a non-zero angular velocity ω with respect to infinity. This aspect of rotational motion for an inertial observer is called **frame dragging**, and is perhaps the most direct physical manifestation of the angular momentum of the gravitational source.

Other than being a really interesting space-time to study, the Kerr solution also plays a big role in the uniqueness theorems of black holes. We will study the uniqueness properties of black holes in the next chapter.

Chapter 6

The Uniqueness of Black Holes

> There is a curious parallel between the histories of black holes and continental drift. Evidence for both was already non-ignorable by 1916, but both ideas were stopped in their tracks for half a century by a resistance bordering on the irrational.
>
> Werner Israel

In this chapter, some of the most important results on the uniqueness of black holes will be introduced. The Schwarzchild solution will be established as the unique spherically symmetric stationary black hole. Birkhoff's theorem will be investigated in this chapter and the proof of Israel's theorem will be outlined. The later parts of this chapter will be devoted towards discussing the Carter-Robinson theorem, although no formal proof will be present here. This chapter ends with a brief overview of recent results and new techniques used towards proving a new generation of uniqueness theorems.

6.1 Uniqueness

The abstract concept of a black hole has been introduced in chapter 5. As seen earlier, the causal structure of a space-time is essential to determine whether it contains a black hole or not. There are many examples of space-times that do not contain a black hole region, while there are also numerous examples of space-time geometries that do include such a region. Clearly, then, the existence of black holes are subject to the geometry of the particular solutions of the Einstein field equations; but what about the uniqueness of such solution? That is, given two space-times containing black hole regions, how much can be said about the properties they have in common?

One of the earliest results regarding the uniqueness of black holes comes from Werner Israel's investigation in 1967, where he claims that under certain regularity conditions, a broad class of static, asymptotically flat, non-rotating vacuum black hole solutions must be diffeomorphic to Schwartzschild's spherically symmetric vacuum solution. This has since stirred up a general interest in the topic of uniqueness of black holes and similar results have been proved for static electro-vac space-times containing rotating black holes as well.

Since black hole regions can be dynamic and evolving, the resulting space-time is evolving as well. This makes it very difficult to compare two black hole solutions at a given time, as they may be at different stages of evolution. Consequently, it makes better sense to speak of uniqueness of black holes once the dust has settled down, that is, once the space-time is stationary. This is why most of the current results for uniqueness of black holes refers to 'static' or 'stationary' space-times, and the precise definition of these words will be introduced in this section.

Furthermore, the goal of this section is to discuss some of the important theorems regarding the uniqueness of black holes in four dimensions and provide the reader with some insight into the current state of affairs regarding this topic. In particular, in this section Birkhoff's theorem will be proven to convince the reader that Israel's claim is indeed true. Moreover, this section will lead up to much celebrated result of Carter and Robinson regarding uniqueness of rotating, charged black holes, though a detailed proof will not be present.

6.1.1 Uniqueness of Schwarzschild solution

Over the last four decades, the mathematical foundations for the study of black hole uniqueness has been carefully laid down by the works of many distinguished physicists including Sir Roger Penrose, Stephen Hawking, Werner Israel, David Robinson, Brandon Carter and Piotr Chrusciel , just to name a few (for a brief historical survey of the topic, please see [30] and references therein). These efforts set up the rigorous definition of black holes and establish the fact that there is a sense of uniqueness of the corresponding space-times containing certain types of black holes.

As mentioned earlier, most uniqueness results only deal with the final state of a black hole. This means that the space-time containing the black hole is stationary in the following sense (slightly modified from [1]):

Definition 6.1.1. *A strongly asymptotically predictable space-time* (M, g) *is said to be* **stationary** *if there exists an isometry* $\theta_t : \mathcal{M} \to \mathcal{M}$ *whose corresponding Killing vector* K *is time-like near* $\mathscr{I}^+$ *and* $\mathscr{I}^-$*.*

In other words, a black hole space-time is called **stationary** if it admits a Killing field which is time-like outside of the black hole region. In a stationary space-time, with corresponding K as defined above, near 'infinity' a stationary space-time takes the structure of a trivial bundle $\Sigma \times \mathbb{R}$, where the base Σ is a space-like hypersurface with Riemannian metric h. It is possible to express the metric g near $\mathscr{I}^\pm$ in the following way (see [15]):

$$g = -\psi^2(dt + a)^2 + h \tag{6.1.1}$$

where ψ is a scalar function, a is a 1−form and h is a Riemannian metric while $t \in \mathbb{R}$ corresponds to a time coordinate such that $K = \frac{\partial}{\partial t}$. From now on, whenever a stationary space-time is being discussed, the letter K will be reserved for the corresponding Killing vector field.

Definition 6.1.2. *A stationary space-time is said to be* **static** *if the curl of K, given by $K_{a;b}K_c\varepsilon^{abcd}$ (see [1]), is zero everywhere. In such a case equation* (6.1.1) *can be written as:*

$$g = -\psi^2 dt^2 + h. \tag{6.1.2}$$

There are a couple of interesting properties of a static space-time. First, it is time-invariant, that is, such a space-time is invariant under $t \to -t$. Moreover, the vector spaces orthogonal to K are globally integrable [15], and if K is time-like, then it is orthogonal to 3−dimensional space-like surfaces.

One may now check that the Schwarzschild solution presented in chapter 3 is an example of a static space-time. As a matter of fact, in the Schwarzschild case, K remains time-like up until the event horizon, whence it becomes null and lies tangent (as well as normal with respect to the Lorentzian metric) to the horizon. This, in fact, is true for any black hole solution [15].

As stated earlier, the purpose of this section is to prove that all smooth, spherically symmetric vacuum space-times must be the Schwarzschild solution. In particular, the Schwarzschild space-time is **spherically symmetric** in the following manner (adapted from [15]):

Definition 6.1.3. *Suppose $(\mathcal{M}, g)$ is a stationary space-time such that near $\mathscr{I}^{\pm}$, one may write express M as $\Sigma \times \mathbb{R}$ as above. Then the space-time is said to be* **spherically symmetric** *if for each constant $t \in \mathbb{R}$:*

i. The corresponding base manifold Σ_t can be represented as the exterior $\mathbb{R}^3 - B_t$ of a ball B_t of $\mathbb{R}^3$ centered at the origin O.

ii. The $3-$*manifold* (Σ_t, h_t) *is spherically symmetric in the usual sense. Then in* $\mathbb{R}^3 - B_t$ *the metric* h_t *reads in standard polar coordinates as:*

$$h_t = e^{\lambda(r,t)}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2).$$

K

iii. For each t*, the length* $g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)$ *and that of the representative of the projection,* $h_t\left(\pi\left(\frac{\partial}{\partial t}\right), \pi\left(\frac{\partial}{\partial t}\right)\right)$*, are invariant under the group of rotations that preserve* Σ.

The fact that these space-times are spherically symmetric allows a convenient result through the following lemma[1]:

Lemma 6.1.4. *A spherically symmetric space-time* $(\mathcal{M}, g)$ *admits a metric of the form:*

$$g = -e^{\nu(t,r)}dt^2 + e^{\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \tag{6.1.3}$$

Proof. In general, due to its symmetries, $(\mathcal{M}, g)$ is expected to contain a metric of the form

$$g = -a^2(r,t)dt^2 + 2b(r,t)a(r,t)dtdr + e^{\lambda}(t,r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{6.1.4}$$

where a, b, f, λ are functions of r and t, and $a, b > 0$.

Now define an $1-$form $\omega = adt - bdr$ and insist that by introducing an integrating factor, there exists an exact $1-$form

$$d\tau = e^{-2f(t,r)}(adt - bdr).$$

Note that such an integrating factor exists since the above equation can be thought of in terms of a first order differential equation in the variable τ. Now notice that:

$$\begin{aligned} -a^2dt^2 + 2badrdt &= -(e^{2f}d\tau + bdr)^2 + (2badr)\frac{(e^{2f}d\tau + bdr)}{a} \\ &= -e^{4f}d\tau^2 - 2e^{2f}bd\tau dr - b^2dr^2 + 2be^{2f}drd\tau + 2b^2dr^2 \\ &= -e^{4f}d\tau^2. \end{aligned}$$

Thus by relabeling τ to be t and $\nu = 4f$, the lemma holds. ∎

[1] I would like to thank Prof. Charbonneau for pointing out the importance of this lemma

The above result then leads to the following theorem, which can be considered as the first uniqueness result in this article:

Theorem 6.1.5. *A smooth spherically symmetric solution is a solution of the vacuum Einstein equations if and only if it is the Schwarzschild metric*

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.1.5}$$

in the Schwarzschild coordinates.

Proof (Sketch). Note that the definition of being spherically symmetric already incorporates the fact that the space-time is stationary. Now considering the standard coordinates (τ, r, θ, ϕ) identified with (x^0, x^1, x^2, x^3) respectively, one writes down the metric as

$$g = -e^{\nu(\tau,r)}d\tau^2 + e^{\lambda(\tau,r)}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

and computes the Christoffel symbols and consequently, the Ricci curvature tensor. The result for these computations are provided in [15], and one also easily compute them using mathematical softwares such as Maple^{TM}.

Three of the non-zero components of the Ricci tensor are:

$$R_{10} = r^{-1}\partial_\tau\lambda, \quad R_{22} = -e^{-\lambda}\left(1 + \frac{r}{2}(\partial_r\nu - \partial_r\lambda)\right), \quad R_{33} = \sin^2\theta R_{22}.$$

Since the space-time is a solution of the vacuum Einstein equations, then it must satisfy

$$R_{\alpha\beta} = 0,$$

and hence from $R_{10} = 0$, λ is independent of τ. Consequently, from $R_{22} = 0$, $\partial_r\nu$ is also independent of τ, that is:

$$\partial_\tau\partial_r\nu = 0 \implies \nu(\tau, r) = h(r) + f(\tau).$$

From the other non-zero components of the Ricci tensor, one obtains the relation (see [15])

$$\partial_r\nu + \partial_r\lambda = 0$$

and hence $R_{22} = 0$ now becomes:

$$
\begin{aligned}
-e^{-\lambda}(1 - r\partial_r\lambda) + 1 &= 0 \\
\implies \partial_r(e^{-\lambda}) + \frac{e^{-\lambda}}{r} &= \frac{1}{r}
\end{aligned}
$$

Using an integrating factor:

$$
\begin{aligned}
&\implies \partial_r\left(re^{-\lambda}\right) = r \\
&\implies e^{-\lambda} = 1 + \frac{A}{r} \quad \text{for some constant } A.
\end{aligned}
$$

and since $\partial_r\nu = h'(r)$, one also obtains

$$
e^h = B\left(1 + \frac{A}{r}\right)
$$

for some arbitrary constant B.

Now set

$$
t = \int e^{\frac{1}{2}f(\tau)}d\tau.
$$

Then $dt = e^{\frac{1}{2}f(\tau)}d\tau$ and one gets

$$
e^{\nu}d\tau^2 = e^{(h+f)}e^{(-f)}dt^2 = e^{h(r)}dt^2
$$

Therefore, it is possible to set $B = 1$ by rescaling t, and then by denoting $A = -2m$, one immediately obtains the desired result in terms of (t, r, θ, ϕ). □

Since it has been noted that the Schwarzschild space-time is also static, one immediately gets the following corollary, known as the **Birkhoff theorem** in the literature.

Corollary 6.1.6 (Birkhoff)**.** *A smooth spherically symmetric metric solution of the vacuum Einstein equations is necessarily static.*

Note that the theorem and corollary above are actually quite general. There is no mention of a black hole in either of the statements. However, by realizing the fact that uncharged, non-rotating black holes contained in vacuum space-time satisfy all the conditions for theorem 6.1.5 if they are spherically symmetric, one can easily regard the result as a uniqueness theorem.

Nevertheless the above version of the uniqueness of the Schwarzschild black hole is not a very big surprise. The vast number of symmetries of the space-time, coupled with the uniqueness of spherically symmetric gravitational fields from Newtonian mechanics, can easily make someone anticipate such a result. Therefore, the next step is remove the assumption that the space-time is spherically symmetric, but keep the weaker assumption that the space-time due to a black hole is static.

As a matter of fact, this is exactly where Israel's theorem begins. Werner Israel has shown [34] such a static space-time, along with certain other assumptions, has to be Schwarzschild's spherically symmetric vacuum solution. Hawking, as well as Robinson, has since shown that some of Israel's assumptions are redundant or unnecessary. Eventually, Robinson improved Israel's theorem into the following form [35]:

Theorem 6.1.7 (Israel)**.** *A static asymptotically predictable space-time corresponding to a vacuum solution of Einstein's field equations must be a positive mass Schwarzschild solution if past and future event horizons exist and intersect in a connected compact space-like two-surface $H(t)$.*

Remark 6.1.8 (Discussions on theorem 6.1.7)*.* One of the original assumptions made by Israel asks the two surface $H(t)$ to have the topology of a two-sphere. Since then, Hawking has proven (see Proposition 9.3.2 in [1]) that each connected component of the event horizon at a given time in a stationary, asymptotically predictable space-time is homeomorphic to a two-sphere. More will be said about this proposition later.

The assumption that $H(t)$ is connected refers to the fact that there is only one black hole present at time t (recall that the time coordinate is defined by means of a Cauchy surface). This is a very common assumption made in such uniqueness theorems. It is also implicitly assumed in the above theorem that the space-time metric is analytic.

Additionally, it is interesting to note that theorem 6.1.7 provides, to some extent, a converse to Birkhoff's result. This is due to the fact that the Schwarzschild space-time is also spherically symmetric.

Proof (Outline). Robinson [35] provides a very simple proof of theorem 6.1.7, which is outlined below. The proof stems from a result of Carter that the metric in the simply connected static region exterior to event horizons can be written in terms of coordinates (t, x^1, x^2, x^3) as

$$ds^2 = -V^2(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j$$

where i, j, k runs from 1 to 3, g_{ij} is a Riemannian metric on surface Σ orthogonal to the Killing vector K and $K^\alpha K_\alpha = -V^2$, α runs from 0 to 3.

The vacuum Einstein field equations then take the form:

$$R_{ij} + V^{-1}V_{;ab} = 0$$
$$g^{ij}V_{;ab} = 0.$$

One then defines a function

$$W := -\frac{1}{2}K^{[\alpha;\beta]}K_{[\beta;\alpha]} = g^{ij}V_{,a}V_{,b}$$

that is positive and constant, say $W = W_0 > 0$ at $H(t)$.

The proof then follows by considering the Cotton tensor

$$R_{ijk} = R_{ij;k} - R_{ik;j} + \frac{1}{4}(g_{ik}R_{;j} - g_{ij}R_{;k})$$

to construct the scalar $R_{ijk}R^{ijk}$ and use the Einstein field equations and W to obtain a set of inequalities that are compatible if and only if the space-time corresponds to the Schwarzschild solution.

The details of this proof can be found in [35]. □

Following the above result, one may now try to remove more of the symmetries that have been imposed on the manifold and see what other uniqueness results can be obtained. Perhaps the next logical step is to remove the assumption that a space-time containing a black hole is static, and insist that it only needs to be stationary. Currently, the uniqueness results for such cases also insist on an additional condition of axis symmetry. This is the subject of the Carter-Robinson theorem discussed in the next section.

6.1.2 The Carter-Robinson Theorem

This section will present an overview of the mathematical tools necessary to approach non spherically symmetric black holes. The discussions here include the Strong Rigidity theorem and the assumptions necessary to state the Carter-Robinson (CR) theorem for axis-symmetric black holes. The goal of this section is to introduce the concept of surface gravity, and finally to state the CR theorem.

The most basic difference between the CR theorem and theorem 6.1.7 due to Israel is in the assumption on the symmetry of the manifold. In the case of CR, one considers axis-symmetric space-times.

Definition 6.1.9. *A sliced space-time, i.e.* $(\mathcal{M} = \Sigma \times \mathbb{R}, g)$*, is called* **axis-symmetric** *if there exists in* $\mathcal{M}$ *a line* A*, called the* **axis***, and a group* $S^1 \equiv SO(2)$ *of isometries of* $(\mathcal{M}, g)$ *leaving invariant the manifolds* $\Sigma \times \{t\}$ *and leaving* $A \times \{t\}$ *fixed point-wise.*

Consequently, there is a corresponding Killing vector field, denoted M from henceforth, that generates these isometries. Then it is possible to have a space-time that is both stationary and axis-symmetric, but where the axis of symmetry is not invariant under the time-like isometries. One then defines the following:

Definition 6.1.10. *A space-time which is both stationary and axis-symmetric is called a*

stationary axis-symmetric *space-time if and only if its two Killing vector fields commute.*

The CR theorem states the uniqueness of such stationary axis-symmetric space-times that contain a black hole. There is an inherent assumption that the event horizon of such a black hole is connected and at a given time has the topology of a $2-$sphere.

Proposition 2 (Hawking)**.** *Suppose the Strong Energy Condition is satisfied in a space-time containing a black hole. Then each connected component of the horizon $H(t)$ in a stationary, asymptotically predictable space is homeomorphic to a two-sphere.*

Proof (Outline). Since $H(t)$ is a two-dimensional space-like surface, there are two null vectors Y_1 and Y_2 normal to $H(t)$, with normalization

$$Y_1{}^a Y_{2a} = -1.$$

The induced metric on $H(t)$ is then

$$h_{ab} = g_{ab} + Y_{1a}Y_{2b} + Y_{2a}Y_{1b}$$

where g_{ab} is the metric on the space-time.

Define a family of surfaces $F(s, w)$ by moving each point of $H(t)$ along the null geodesic curve with tangent vector $Y_2{}^a$ along a parameter distance w. Thus, one inspects how $H(t)$ behaves if is deformed slightly outwards into the region $J^-(\mathscr{I}^+)$.

Now by considering the expansion θ of outgoing (i.e., moving into the region $J^-(\mathscr{I}^+)$) null geodesics orthogonal to $H(t)$, one obtains

$$\frac{\mathrm{d}\theta}{\mathrm{d}w} = (Y_1{}^a{}_{;b}h^b{}_a)_{;c}Y_2{}^c$$

which on the horizon leads to an expression

$$\left.\frac{\mathrm{d}\theta'}{\mathrm{d}w'}\right|_{w=0} = p_{b;d}h^{bd} + y_{;bd}h^{bd} - R_{ac}Y_1{}^aY_1{}^c + R_{adcb}Y_1{}^dY_2{}^cY_2{}^aY_1{}^b + p'^a p'_a \tag{6.1.6}$$

where $p^a := -h^{ba} Y_{2c;b} {Y_1}^c$ and there has been a rescaling such that ${Y_1}' = e^y Y_1$ and ${Y_2}' = e^{-y} Y_2$, causing $p'^a = p^a + h^{ab} y_{;b}$.

The proof in [1] discusses that the sign of the right hand side for equation (6.1.6) is then determined by that of the integral of

$$(-R_{ac} {Y_1}^a {Y_1}^c + R_{adcb} {Y_1}^d {Y_2}^c {Y_2}^a {Y_1}^b)$$

over $H(t)$. This integral can be evaluated in terms of an integral of the scalar curvature $\hat{R}$ of $H(t)$ using the Gauss-Codacci equations. Also, by the Gauss-Bonnet theorem:

$$\int_{H(t)} \hat{R} d\hat{S} = 2\pi\chi$$

where $d\hat{S}$ is the surface area element of $H(t)$ and χ is the Euler number of $H(t)$. Thus

$$\int_{H(t)} (-R_{ac} {Y_1}^a {Y_1}^c + R_{adcb} {Y_1}^d {Y_2}^c {Y_2}^a {Y_1}^b) d\hat{S} = -\pi\chi + E$$

where E is a positive number due to the Strong Energy Condition. Then clearly, the only way that the right hand side can be negative is if $H(t)$ is a sphere. If this is negative, then the right hand side of equation (6.1.6) will also be negative.

The details of why the quantity $\frac{\mathrm{d}\theta'}{\mathrm{d}w'}$ has to be negative, and other details left out in this proof, are given in the proof of Proposition 9.3.2 in [1]. The punchline of the theorem remains the statement about the topology of the event horizon.

□

There is a considerable amount of concern about how to generalize the above result to higher dimensions. The problem lies in the fact that the Gauss-Bonnet theorem played a crucial role in the proof when integrating over the two-dimensional surface $H(t)$. However, in higher dimensions, $H(t)$ is no longer two-dimensional and using the generalized Gauss-Bonnet theorem in higher dimensions is not as helpful. Nevertheless in four dimensions, proposition 2 leads the way to much greater results such as the uniqueness theorems discussed here.

Moreover, Hawking also introduced the Strong Rigidity Theorem that relates the event horizon of a black hole to the locally defined concept of a Killing horizon. The following statement of the theorem 6.1.11 has been taken verbose from [15]. A proof for this theorem will not be attempted here.

Theorem 6.1.11 (Strong Rigidity Theorem)**.** *Let $(\mathcal{M}, g)$ be an analytic, asymptotically flat black hole space-time which is stationary (but not static) and which satisfies the vacuum Einstein field equations. Assume that there is a component H of the black hole horizon which is diffeomorphic to $\Sigma \times \mathbb{R}$ for compact Σ and which has an analytic embedding of Σ in H which is transverse to the stationarity Killing field K. Then the space-time admits a Killing field L which is tangent to the null geodesic generators of H and everywhere independent of K.*

Remark 6.1.12. The following remarks about theorem 6.1.11 are in order:

1. In the case of static space-times, K must be tangent to the horizon H, and it has to be null at H.

2. A null hypersurface whose null generators coincide with the orbits of an one parameter group of isometries is called a **Killing Horizon**. In other words, a Killing horizon for a given Killing vector field (KVF) is the subset of the space-time where the KVF becomes null. Such a horizon is a type of ruled surface, generated by the null geodesics corresponding to the KVF. This is well defined as the norm of a Killing vector is preserved if it is parallel transported along its flow[2].

3. According to the Strong Rigidity Theorem (SRT), the event horizon of a stationary black hole is the Killing horizon of a Killing vector L. The horizon is called **rotating** if this KVF does not coincide with K. Thus stationary, but not static, space-times are rotating.

[2]Thanks to Prof. Karigiannis for helping me reconfirm this result.

4. Since L is independent of K, and the linear combination of two Killing vectors is always another Killing vector, then there is a two dimensional vector space $span(K, L)$ containing KVFs everywhere in $\mathcal{M}$. In the case of a stationary axis-symmetric space-time, the KVF responsible for axis-symmetry, denoted M, is usually given by

$$L^a = K^a + \Omega M^a$$

where the non-zero constant Ω is the **angular velocity** of the horizon.

5. The theorem 6.1.11 was introduced by Hawking but Chrusciel observed an error [36] in Hawking's proof of the SRT and has since improved the result. Chrusciel and Wald [37] has also provided a more complete - i.e., requiring fewer assumptions about the analyticity of the metric and the horizon - proof of proposition 2, justifying the spherical topology of the event horizon of a black hole using a recent result known as the topological censorship theorem. Neither of these proofs will be attempted here.

Finally, to state the CR theorem, the concept of surface gravity will be required. Suppose X is a KVF and

$$N(X) = \{g(X, X) = 0, X \neq 0\}$$

is Killing horizon associated with X. The **surface gravity** κ of a Killing horizon is defined by the formula [15]

$$d(g(X, X)) = -2\kappa\tilde{X} \tag{6.1.7}$$

where d is the exterior derivative and $\tilde{X} = g_{\mu\nu}X^{\nu}dx^{\mu}$ for some coordinate system (x^{μ}). An equivalent expression for the surface gravity is given by

$$(X^{\alpha}X_{\alpha})_{;\beta} = -2\kappa X_{\beta}. \tag{6.1.8}$$

The surface gravity on each connected component of the horizon of a stationary black hole, given by

$$(L^{\alpha}L_{\alpha})_{;\beta} = -2\kappa L_{\beta} \tag{6.1.9}$$

is constant (see [3] for details). A connected component of the horizon is called non-degenerate if the surface gravity is non-zero and degenerate otherwise.

Finally, all the required concepts have been introduced so that the titular theorem (adapted from [15]) may now be stated:

Theorem 6.1.13 (Carter-Robinson)**.** *Asymptotically flat, stationary, axis-symmetric space-time solutions of the vacuum Einstein equations which have a regular and connected event horizon, with a non-zero surface gravity, are uniquely specified by two parameters, the mass m and the angular momentum a, where $a^2 < m^2$, if these space-times are smooth outside the horizon.*

Since the Kerr metric satisfies all the requirements of the CR theorem for a given a and m, with $a^2 < m^2$, then the Kerr family of solutions are the only such stationary axis-symmetric black holes. A complete proof for this uniqueness result is spread over several articles by Carter and Robinson (see [38, 39] and [40]), and involves some new concepts and geometric results that have not been discussed here. Therefore, there will be no effort to prove theorem 6.1.13.

6.2 Afterword on Uniqueness: Recent Techniques and Results

The Carter-Robinson Theorem (theorem 6.1.13) has been a significant achievement in the theory of uniqueness of black holes. This theorem does indeed generalize Israel's theorem (theorem 6.1.7) and provide encouragement towards further research in the field. Moreover, the CR result seems to be stable, in the sense that small perturbations of the Kerr spaces still preserves uniqueness (see recent work by Alexakis et. al. [41] for the precise notion). However, as discussed by Robinson [30], the original proof was quite complicated. At the end of 1970's, it was expected that the uniqueness theorems could be extended to incorporate the case of a single charged black hole, but it was not evident how to do so.

In this section we will discuss some of the more recent techniques that have been able to improve the CR theorem to include charged black holes. These new ideas have also provided a new outlook on the older results and have enabled mathematically more rigorous proofs of proposition 2 and theorem 6.1.11. The discussion in this section will be brief, and much of the details surrounding the concepts and theorems will be referred to the source documents.

There is a known family of solutions of the Einstein equations (2.3.2) that contain a single rotating, charged black hole. A member of this family, called the **Kerr-Newman family**, has the space-time metric given by:

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Lambda}\right)dt^2 - \left(\frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Lambda}\right)dtd\phi \\ &+\left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Lambda}\right)\sin^2\theta d\phi^2 + \frac{\Lambda}{\Delta}dr^2 + \Lambda d\theta^2 \end{aligned} \tag{6.2.1}$$

where

$$\Lambda = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 - 2mr + a^2 + e^2.$$

Here e is a parameter of the family that refers to an electric charge. Hence the space-time also admits an electromagnetic potential:

$$A = -\frac{er}{\Lambda}[dt - a\sin^2\theta d\phi]. \tag{6.2.2}$$

Note that the resulting space-time no longer satisfies the vacuum Einstein field equations. In fact, it is an example of an **electro-vac** space-time and the corresponding Einstein Field Equations are known as Einstein-Maxwell equations. Such space-times will not be considered in detail here, but discussions about them can be found in standard texts such as [2].

Bunting and Mazur (see [30, 42] and references therein) managed to prove that the only possible stationary and axis-symmetric black hole solutions of the Einstein-Maxwell equations satisfying some reasonable boundary conditions (details given in [42]) are the Kerr-Newman solutions subject to the constraint $a^2 + e^2 < m^2$. This means that such black holes are completely characterized by three parameters only: the total mass m, the total angular momentum per unit mass a, and the total electric charge e. This result provides, to some extent, a justification of Wheeler's famous conjecture that "black holes have no hair".

Bunting and Mazur's approach to uniqueness employed harmonic maps and sigma models. Harmonic maps are defined as follows (adapted from [42]):

Definition 6.2.1. *Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be two Lorentzian manifolds with metrics g and h respectively. Suppose $\mathcal{M}$ has local coordinates (x^a) and $\mathcal{N}$ has coordinates (y^a). Then on these coordinate charts, suppose $v : \mathcal{M} \to \mathcal{N}$ is a map that can be written as $y^b = v^b(x^a)$. Then v is called a* **harmonic map** *if the* **energy functional**

$$E[v] = \frac{1}{2} \int_{\mathcal{M}} g^{ab} \tilde{h}_{cd} \tilde{y}^c{}_{;a} \tilde{y}^d{}_{;b} \sqrt{|g|} dx,$$

where $\tilde{h}$ and $\tilde{y}^c$ refer to the pull-backs of h and y^c by v, is stable under small deformations of v. This means that the Euler-Lagrange equations for the functional E are satisfied at v.

Harmonic maps from a manifold $\mathcal{M}$ to a coset space $\mathcal{N} = G/H$, where G is a connected Lie group and H is a closed sub-group of G, is known as **nonlinear sigma models**. The details of how such sigma models can be used to prove uniqueness of the Kerr-Newman family can be found in [42]. Nevertheless, this approach also manages to generalize the Carter-Robinson theorem and provides better understanding of Robinson's proof for theorem 6.1.13 [30].

During the 1990's the framework in which the uniqueness theorems are proved underwent a major overhaul. In particular, Piotr Chrusciel noted several mathematical shortcomings in the earlier works of Hawking et. al. (see [43] for a review on this topic) and led the way (along with Robert Wald) towards a more rigorous theory of classification of static and stationary space-times. One theorem, titled "topological censorship", in particular, needs to be mentioned here.

Recall that the energy conditions on a space-time were introduced in chapter 4. A slightly weaker energy condition than those already introduced, known as the **null energy condition**, requires that $T_{ab}k^a k^b \geq 0$, for all null vectors k^a. The null energy condition is implied by each of the other energy conditions introduced so far: the weak energy condition, the strong energy condition and the dominant energy condition [44].

Now suppose C is an inextendible null geodesic with affine parameter λ and corresponding tangent vector k^a. Then an even weaker form of energy condition, known as the **averaged null energy condition** (ANEC), is said to be satisfied if the integral

$$\int_C T_{ab}k^a k^b d\lambda \geq 0,$$

for every inextendible null geodesic C in the space-time.

Finally, suppose γ_0 is a time-like curve with past end point in $\mathscr{I}^-$ and future end point in $\mathscr{I}^+$ such that γ_0 lies in a neighborhood U of $\mathscr{I} := \overline{J^+(i^0)} \cup \overline{J^-(i^0)}$ using the notation from definition 4.3.4. Then concepts from algebraic topology can be used to define what it means for another curve γ to be deformable to γ_0 rel $\mathscr{I}$ (see any standard text, such as [45], on algebraic topology for details). Then the topological censorship theorem [44] is stated as follows:

Theorem 6.2.2 (**Topological Censorship Theorem**). *If an asymptotically flat, globally hyperbolic space-time* $(\mathcal{M}, g)$ *satisfies the averaged null energy condition, then every causal curve from* $\mathscr{I}^-$ *to* $\mathscr{I}^+$ *is deformable to* γ_0 *rel* $\mathscr{I}$.

As mentioned in the previous section, Chrusciel and Wald [37] were able to improve the results for horizon topology of black holes (proposition 2) as well the rigidity theorem (theorem 6.1.11) using theorem 6.2.2. Moreover, this theorem also led to the study of black holes in space-time containing other types of matter than just electric and magnetic fields.

The recent works on black holes go well beyond the Einstein-Maxwell solutions and the Kerr-Newman family. In these cases the picture is not so simple. For example, it is possible to have static black holes without spherical symmetry, or rotating black holes that are not just characterized by their mass, angular momentum and global charges. An excellent introduction to such topics, and related results on uniqueness, is provided by Huesler in [46]. There have been numerous uniqueness results discovered for such "exotic" black holes recently, and the field of study is indeed very active in the present day.

Chapter 7

Black Holes in Higher Dimensions

> So I will point to three directions: up-down, left-right and front-back. Point to me where the fourth direction lies.
>
> Dr. Richard Epp (paraphrased).

This chapter will serve as brief introduction to the cutting-edge research in black holes. The motivation for black holes in higher dimensions is derived from efforts to unify gravity (which is not really a force in relativity, but one in Newtonian mechanics) with the other forces of nature. A lot of the times these efforts have led to physical theories, such as string theory, being described in more than four dimensions. It seems that studying black holes in higher dimensions (i.e., dimensions greater than four) is important in obtaining a full understanding of these theories.

The first section of this chapter will be devoted to bookkeeping. Here the reader will be reminded of the how a black hole is defined in the space-time setting, and the minute differences that occur in higher dimensions will the discussed. The second section will describe the Myers-Perry black hole - the earliest known black hole solution in higher dimensions. Finally, the third section will focus on five dimensional black holes. In particular, the Reall-Emperan solution of black rings will be briefly discussed, along with the challenges this example brings to the uniqueness theorem in higher dimensions.

7.1 Higher Dimensional Space-Times

The **space-time** that will now be considered will contain an $(N+1)-$dimensional manifold $\mathcal{M}$, where $N \geq 3$, with a Lorentzian metric g. Here N refers to the fact that the metric will have N positive eigenvalues. Then the concept of a light-cone can still be applied to $T_p\mathcal{M}$, for all $p \in \mathcal{M}$. Thus it is then possible to extend the definition of a manifold being time-orientable and stably-causal in $(N+1)$ dimensions in the most natural way.

Therefore, the space-time will still be assumed to be **connected**, **time-orientable** and **stably-causal**. Moreover, the metric g will be required to solve the Einstein equations:

$$G_{ab} = 8\pi T_{ab}$$

where G_{ab} is defined through the curvature of the manifold and T_{ab} is the stress-energy tensor defined in analogous fashion to the four dimensional case. For the rest of this article, it will be assumed that $T_{ab} = 0$.

Next, the concept of asymptotic flatness will be revisited. Along the lines of definition 4.3.4, this will be defined as (adapted from [47]):

Definition 7.1.1. *An* $(N+1)$*-dimensional physical space-time* $(\mathcal{M}, g)$ *will be said to be* **asymptotically flat** *at spatial infinity* i^0 *if there exists an unphysical space-time* $(\tilde{\mathcal{M}}, \tilde{g})$*, where* $\tilde{g}$ *is* $C^{>N-3}$ *at* i^0 *(see remark 4.3.5 for the idea, and [19] or [47] for details on the definition of* $C^{>n}$*), and there is an embedding of* $\mathcal{M}$ *into* $\tilde{\mathcal{M}}$ *satisfying the following conditions:*

i. $\overline{J^+(i^0)} \cup \overline{J^-(i^0)} = \tilde{M} - M$.

ii. There exists a non-negative function Ω *on* $\tilde{\mathcal{M}}$ *that is* C^2 *at* i^0 *such that*

$$\tilde{g} = \Omega^2 g$$

on M and

$$lim_{i^0} \tilde{\nabla}_a \tilde{\nabla}_b \Omega = 2\tilde{g}_{ab}$$

and

$$lim_{i^0} \tilde{\nabla}_a \Omega = 0$$

on $\tilde{\mathcal{M}}$.

The above definition has recently been used by Tanabe et. al. [47] to study the asymptotic structure of space-time in higher dimensions. Even though [47] does not make use of the null infinities $\mathscr{I}^+$ and $\mathscr{I}^-$ in the definition of asymptotic flatness, nevertheless, there should be no problem to define these infinities in the same manner as definition 4.3.4.

Finally, one can also assume that the space-time is strongly asymptotically predictable and simply define the black hole region to be $\mathcal{B} = [\mathcal{M} - J^-(\mathscr{I}^+)]$ and its horizon to be $\mathcal{H} = \partial\mathcal{B}$ as it was in four dimensions.

7.2 The Myers-Perry Solution

The first example of a higher dimensional space-time containing a spinning black hole has been found by Robert Myers and Malcolm Perry [48]. In fact, their solution provides a family of black holes very similar to the Kerr family in $(N+1)$-dimensions for any $N \geq 3$. These Myers-Perry (MP) solutions will be described in this section.

7.2.1 Static Black Holes in $(N+1)$-dimensions

First, let us consider the case of non-rotating black hole. As in four dimensions, the space-time is assumed to be static, (hyper)spherically symmetric (defined in an analogous way to definition 6.1.3), and the black holes are assumed to have a topology of $\mathbb{R}^2 \times \mathbb{S}^{N-1}$. The idea of a space-time being static has been defined in definition 6.1.2, but now we will present a modern definition in $(N+1)$-dimensions (adapted from [15]):

Definition 7.2.1. 1. *Recall that a black hole space-time is called* **stationary** *if it admits a Killing field which is time-like outside of the black hole region and preserves the event horizon. This requires $\mathcal{M}$ to admit a group $G_1 \equiv \mathbb{R}$ of isometries with the time-like Killing vector field K near $\mathscr{I}$.*

2. *Also, recall the definition of an axis-symmetric space-time from definition 6.1.9. This introduces a group $\mathbb{S}^1$ of isometries of $(\mathcal{M} = \Sigma \times \mathbb{R}, g)$ corresponding to a Killing vector field M. Then a* **stationary axis-symmetric** *$(N+1)$-dimensional space-time is invariant under an Abelian group $G_2 \equiv \mathbb{R} \times \mathbb{S}^1$. It is the union of the product $A \times \mathbb{R}$, where A is the axis - hence a space-like line - and $\mathbb{R}$ can be considered to be a time-like line in $\mathcal{M}$, with a principal fiber bundle $\mathcal{M} - \{A \times \mathbb{R}\}$. The Abelian group G_2 acts on $(\mathcal{M}, g)$ by time-like and space-like isometries. The base*

$$S := [\mathcal{M} - \{A \times \mathbb{R}\}]/G_2$$

is an $(N-1)$-dimensional Lorentzian manifold. The axis A is invariant under the $\mathbb{S}^1$ isometries.

3. *A stationary axis-symmetric space-time is* **circular** *if the 2-planes orthogonal to the two Killing fields are* **integrable**, *i.e., they are tangent to $2-$surfaces. Then the manifold is locally a product of $V \times U$, where V is a 2-surface and U is an open subset of the base S and lies orthogonal to V.*

4. *A stationary circular space-time is called* **static** *if the time-like orbits and the space-like orbits of the respective one-parameter isometry groups $\mathbb{R}$ and $\mathbb{S}^1$ are orthogonal.*

In a very similar effort to the proof of theorem 6.1.5, Tangherlini [49] showed that for static, spherically symmetric space-times (i.e., surfaces of constant t are formed from concentric spheres), the metric that solves the vacuum Einstein equations is given by:

$$ds^2 = -\left(1 - \frac{C}{r^{N-2}}\right)dt^2 + \left(1 - \frac{C}{r^{N-2}}\right)^{-1} dr^2 + r^2 d\Omega^2_{N-1} \tag{7.2.1}$$

where r is a radial coordinate, and $d\Omega^2_{N-1}$ is the metric on the unit $(N-1)$-sphere. The parameter C is a constant of integration, and for our purposes, consider $C > 0$. Then the hypersurface $r = C^{1/(N-2)}$ is a horizon. If one sets

$$M = \frac{C(N-1)\omega_{N-1}}{16\pi}$$

to be the mass parameter, where

$$\omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

is the surface area of a unit $(N-1)$-sphere, then the space-time behaves exactly in the same manner as the Schwarzschild space-time. Therefore, this generalizes the Schwarzschild space-time in $(N+1)$ dimensions.

Uniqueness results corresponding to Israel's theorem (theorem 6.1.7) for static black-holes in higher dimensions have recently been established by Gibbons et. al. (see [50]) using σ-models. These say that the only possible asymptotically flat static vacuum black hole is the $(N+1)$-dimensional Schwarzschild-Tangherlini solution.

7.2.2 Spinning Black Holes

In 1986, Myers and Perry found an analogue to the Kerr metric in higher dimensions. Their approach consists of considering, as an ansatz, the metric that solves vacuum

Einstein equations to take the form:

$$
\begin{aligned}
ds^2 = & -\beta^2(r,\rho)(du + a\sin^2\theta d\phi^2)^2 + 2(du + a\sin^2\theta d\phi)(dr + a\sin^\theta d\phi) \\
& + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) + r^2\cos^2\theta d\Omega^2_{N-3}
\end{aligned} \tag{7.2.2}
$$

where $(u, r, \theta, \phi, \ldots)$ is a local coordinate system, $(\theta, \phi, \ldots)$ being the angular coordinates, $0 \leq \theta \leq \pi$ and $0 \leq \phi, \ldots < 2\pi$, and $\rho^2 = r^2 + a^2\cos^2\theta$. Once again, the parameter a refers to the total angular momentum per unit mass.

Note that for $N = 3$, it is presumed that $d\Omega^2_{N-3} = 0$, so the last term in equation (7.2.2) drops out. Then it can be shown that for $\beta^2 = 1 - \frac{2mr}{\rho^2}$, the above metric is exactly the Kerr solution. In fact, Myers and Perry worked out in detail (see appendix B in [48]) that for $N \geq 3$, the metric (7.2.2) is a solution of the vacuum Einstein equation if

$$
\beta^2 = 1 - \frac{2m}{r^{N-4}\rho^2}.
$$

where m is again the total mass parameter.

The above metric can be written in the familiar form of the Kerr solution (5.2.1) by considering the following "Boyer-Lindquist"-like coordinates:

$$
\begin{aligned}
dt &= du - \frac{(r^2 + a^2)}{r^2 + a^2 - 2mr^{4-N}}dr, \\
d\varphi &= d\phi + \frac{a}{r^2 + a^2 - 2mr^{4-N}}dr
\end{aligned}
$$

while the other coordinate components remain unchanged. The new form of the metric is then given by

$$
\begin{aligned}
ds^2 = & -dt^2 + \sin^2\theta(r^2 + a^2)d\varphi^2 + \Delta(dt + a\sin^2\theta d\varphi^2)^2 \\
& + \Psi dr^2 + \rho^2 d\theta^2 + r^2\cos^2\theta d\Omega^2_{N-3}
\end{aligned} \tag{7.2.3}
$$

where

$$
\begin{aligned}
\Delta &= \frac{2m}{r^{N-4}\rho^2} \\
\Psi &= \frac{r^{N-4}\rho^2}{r^{N-4}(r^2 + a^2) - 2m}.
\end{aligned}
$$

It follows that there are event horizons that occur when $\Psi^{-1} = 0$, that is, when

$$r^{N-4}(r^2 + a^2) - 2m = 0. \tag{7.2.4}$$

Remarkably, the above equation always has a solution for $N > 4$ if $m > 0$. This can be seen with the help of the Intermediate Value Theorem: as $r \to \infty$, $\Psi^{-1} \to 1 > 0$ while as $r \to 0, \Psi^{-1} \to -2mr^{4-N}\rho^{-2} < 0$, and as Ψ^{-1} is continuous for $\rho^2 \neq 0$, then there must be some intermediate r at which Ψ^{-1} crosses zero. This is in stark contrast to the cases $N = 3$ and $N = 4$, where a horizon exists if $m^2 \geq a^2$ and $2m \geq a^2$ respectively. A detailed analysis of the singularities and horizons is given in [48].

7.3 End of Black Hole Uniqueness

The title of this section has been derived from an essay [51] of the same name published by Roberto Emparan and Harvey Reall in 2002. Without going into the details, this section will summarize their article and speculate on how some uniqueness results may yet be recovered.

As noted in the last section, the Myers-Perry solution provides a spinning black hole in (4 + 1)-dimensions with $2m \geq a^2$. As a matter of fact, there are two distinct angular momentum components, denoted a_1 and a_2, associated with the Myers-Perry solution in 4 + 1 dimensions (details in [48]). The total angular momentum parameter a is then given by

$$a^2 = a_1^2 + a_2^2 + 2\,|a_1 a_2|\,, \tag{7.3.1}$$

and hence the existence of a horizon requires

$$2m \geq a_1^2 + a_2^2 + 2\,|a_1 a_2|\,.$$

Myers and Perry duly noted that the event horizon topology at a given time in the above case is that of $\mathbb{S}^3$. Nevertheless, they pointed out that it may be possible to obtain a black hole solution where the corresponding event horizon topology would be $\mathbb{S}^2 \times \mathbb{S}^1$, but expected such solutions to be (thermodynamically) unstable (see discussion in [48]).

In 2002, Emparan and Reall [52] came up with a fascinating class of stationary, vacuum black hole solutions in 4 + 1 dimensional space-time. These solutions are known as the "**Black Ring**" solutions. These black holes indeed have an event horizon topology of $\mathbb{S}^2 \times \mathbb{S}^1$ (the "ring" refers to the $\mathbb{S}^1$ part). The space-time is obtained by gluing together two coordinate patches, in each of which the metric takes the form:

$$\begin{aligned} g = &-\frac{F(x)}{F(y)}\left(dt + \sqrt{\frac{\nu}{\xi_F}}\frac{\xi_1 - y}{A}d\psi\right)^2 \\ &+ \frac{F(y)}{A^2(x-y)^2}\left[-F(x)\left(\frac{dy^2}{G(y)} + \frac{G(y)}{F(y)}d\psi^2\right)\right. \\ &\left.+ F(y)\left(\frac{ds^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2\right)\right], \end{aligned} \tag{7.3.2}$$

where $A > 0$, ν and ξ_F are constants, and

$$\begin{aligned} F(\xi) &= 1 - \frac{\xi}{\xi_F}, \\ G(\xi) &= 1 - \xi^2 + \nu\xi^3 = \nu(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3). \end{aligned}$$

The constant ν is chosen to satisfy $0 < \nu \leq \nu_* = \frac{2}{3\sqrt{3}}$, which ensures that the roots of $G(\xi)$ given by ξ_1, ξ_2, ξ_3 are real and satisfy

$$-1 < \xi_1 < 0 < \xi_2 < \xi_3 < \frac{1}{\nu}.$$

A detailed description of this metric and its relation to black holes is given in [52].

As it turns out, one can define a total mass m and angular momentum per unit mass, $\tilde{a}$, corresponding to the metric in equation (7.3.2). Emparan and Reall [51] then argues that if one considers a black hole in 4 + 1 dimensions of mass m, and sets $a_1 = \tilde{a}, a_2 = 0$ for $2m < a_1^2$ in the Myers-Perry metric, one obtains two distinct possibilities for the black hole type: the "black ring" and the Myers-Perry black hole. This seems to drown the hopes of proving an analogue of the Carter-Robinson theorem that satisfies the Wheeler conjecture that black holes have no hair.

Finally, the black ring solution is also an interesting counter-example to the spherical topology theorem from four dimensions. Therefore, it seems that extending the uniqueness results to higher dimensions is a non-trivial problem. The question is left open and the research is very much active in this field. There are speculations that perhaps fixing the event horizon topology will provide us the required uniqueness [53], but there is no definite answer available at the moment.

Chapter 8

Summary and Conclusions

Black holes are certainly very exciting objects to study. They seem to provide a unique challenge in every step towards comprehending their nature. As we have seen through chapters 2 to 5, it is quite difficult to even define black holes without running into trouble with other known physical phenomenon. Similarly, chapters 6 and 7 give us an impression of how difficult it is to characterize black holes in a manner that can easily be extended to higher dimensions.

Ironically, black holes have been regarded as very simple objects for quite some time. After all, as we saw in chapter 6, a black hole is essentially identified by only three parameters: its charge, mass and angular momentum. This is precisely what makes black holes so fascinating. On one hand, we have a highly complicated picture, involving possibly a singularity in space-time and certainly quantum gravitational phenomenon, while on the other, we have a well understood model of space-time depending only on a handful of parameters. Surely understanding black holes will reap high rewards.

Throughout this essay, we have built up the tools to declare a theorem of the following nature:

Theorem 8.0.1. *Let $\mathcal{M}$ be a 'good' vacuum space-time that contains an isolated black hole* *region, then $\mathcal{M}$ is diffeomorphically isometric to the Kerr space-time.*

The holy grail of this project has been to understand what surmounts a 'good' space-time. To establish our eventual result, the Carter-Robinson theorem, we kept on adding different characteristics to the manifolds that we have been studying to reach a consensus on the definition of a 'good' space-time.

This essay starts off by saying that our space-time needs to be a four-dimensional, connected, non-compact manifold with a Lorentzian metric that solves the Einstein equations. Then the space-time is restricted to be time-orientable and stably causal to satisfy our physical intuitions. Very soon, it is also required to be asymptotically flat and globally hyperbolic. Only then is it possible to define a black hole region.

Next, the focus changes to the symmetries of the space-time. The first uniqueness result presented here requires spherical symmetry. This is then weakened to the requirement of a static space-time, and the Schwarzschild space-time remains the unique solution to such a space-time.

Once stationary axis-symmetric space-times are introduced, a couple of Hawking's theorems on the event horizon of black holes come in handy to reach the point where we can state the Carter-Robinson theorem. There are several other assumptions that need to be made on the space-time and the event horizon, but eventually we get to the results we have been chasing after.

Then, just to make things more interesting, a two-headed snake pops up. Its first head introduces us to the modern approach towards obtaining uniqueness results for black holes using fancy jargons such as sigma models and harmonic maps. These current approaches are no longer restricted to vacuum space-times, but can easily deal with charge and some other exotic fields. These new ideas make room for some exciting research into general relativity.

The other head takes us on a trip through higher dimensional space-times, where we find out that our four dimensional uniqueness results can no longer be applied. This certainly opens up a new avenue of speculations and expectations, but this is truly uncharted territory.

This leaves the author to a very likely conclusion that we need find a better interpretation of the uniqueness results: we need to take a look at their statements, their proofs and perhaps also think about alternate methods of proving the result. Perhaps we also need to be open to the idea that there is something fundamentally different about higher dimensional gravity, and may be uniqueness in these settings is simply not meant to be.

Finally, the author wants to thank the readers for their patience and effort in reading this report. Hopefully everyone has gained something useful by reading through the essay.

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