(b) Write a program for calculating the A_m's in Example 1 and extend the table to $m = 15$. Verify numerically that $\alpha_m \approx (m - \frac{1}{4})\pi$ and compute the error for $m = 1, \cdots, 10$.

(c) Graph the initial deflection $f(r)$ in Example 1 as well as the first three partial sums of the series. Comment on accuracy.

(d) Compute the radii of the nodal lines of u_2, u_3, u_4 when $R = 1$. How do these values compare to those of the nodes of the vibrating string of length 1? Can you establish any empirical laws by experimentation with further u_m?

13. Frequency. What happens to the frequency of an eigenfunction of a drum if you double the tension?

14. Size of a drum. A small drum should have a higher fundamental frequency than a large one, tension and density being the same. How does this follow from our formulas?

15. Tension. Find a formula for the tension required to produce a desired fundamental frequency f_1 of a drum.

16. Why is $A_1 + A_2 + \cdots = 1$ in Example 1? Compute the first few partial sums until you get 3-digit accuracy. What does this problem mean in the field of music?

17. Nodal lines. Is it possible that for fixed c and R two or more u_m [see (16)] with different nodal lines correspond to the same eigenvalue? (Give a reason.)

18. Nonzero initial velocity is more of theoretical interest because it is difficult to obtain experimentally. Show that for (17) to satisfy (9b) we must have

$$\textbf{(21)} \qquad B_m = K_m \int_0^R r g(r) J_0(\alpha_m r/R)\, dr$$

where $K_m = 2/(c\alpha_m R)J_1^2(\alpha_m)$.

VIBRATIONS OF A CIRCULAR MEMBRANE DEPENDING ON BOTH r AND θ

19. (Separations) Show that substitution of $u = F(r, \theta)G(t)$ into the wave equation (6), that is,

$$(22) \qquad u_{tt} = c^2\left(u_{rr} + \frac{1}{r}\, u_r + \frac{1}{r_2}\, u_{\theta\theta}\right),$$

gives an ODE and a PDE

$$(23) \qquad \ddot{G} + \lambda^2 G = 0, \qquad \text{where } \lambda = ck,$$

$$(24) \qquad F_{rr} + \frac{1}{r}\, F_r + \frac{1}{r^2}\, F_{\theta\theta} + k^2 F = 0.$$

Show that the PDE can now be separated by substituting $F = W(r)Q(\theta)$, giving

$$(25) \qquad Q'' + n^2 Q = 0,$$

$$(26) \qquad r^2 W'' + rW' + (k^2 r^2 - n^2)W = 0.$$

20 Periodicity. Show that $Q(\theta)$ must be periodic with period 2π and, therefore, $n = 0, 1, 2, \cdots$ in (25) and (26). Show that this yields the solutions $Q_n = \cos n\theta$, $Q_n^* = \sin n\theta$, $W_n = J_n(kr)$, $n = 0, 1, \cdots$.

21. Boundary condition. Show that the boundary condition

$$(27) \qquad u(R, \theta, t) = 0$$

leads to $k = k_{mn} = \alpha_{mn}/R$, where $s = \alpha_{nm}$ is the mth positive zero of $J_n(s)$.

22. Solutions depending on both r and θ. Show that solutions of (22) satisfying (27) are (see Fig. 310)

$$(28) \qquad \begin{aligned} u_{nm} &= (A_{nm} \cos ck_{nm}t + B_{nm} \sin ck_{nm}t) \\ &\quad \times J_n(k_{nm}r) \cos n\theta \\ u_{nm}^* &= (A_{nm}^* \cos ck_{nm}t + B_{nm}^* \sin ck_{nm}t) \\ &\quad \times J_n(k_{nm}r) \sin n\theta \end{aligned}$$

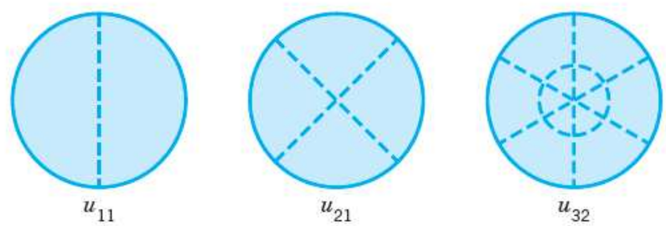

Fig. 310. Nodal lines of some of the solutions (28)

23. Initial condition. Show that $u_t(r, \theta, 0) = 0$ gives $B_{nm} = 0$, $B_{nm}^* = 0$ in (28).

24. Show that $u_{0m}^* = 0$ and u_{0m} is identical with (16) in this section.

25. Semicircular membrane. Show that u_{11} represents the fundamental mode of a semicircular membrane and find the corresponding frequency when $c^2 = 1$ and $R = 1$.