PROBLEM SET 10.5

1–8 PARAMETRIC SURFACE REPRESENTATION

Familiarize yourself with parametric representations of important surfaces by deriving a representation (1), by finding the **parameter curves** (curves u = const and v = const) of the surface and a normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ of the surface. Show the details of your work.

1. xy-plane $\mathbf{r}(u, v) = (u,\quad v)$ (thus $u\mathbf{i} + v\mathbf{j}$; similarly in Probs. 2–8).
2. xy-plane in polar coordinates $\mathbf{r}(u, v) = [u \cos v,\quad u \sin v]$ (thus $u = r$, $v = \theta$)
3. Cone $\mathbf{r}(u, v) = [u \cos v,\quad u \sin v,\quad cu]$
4. Elliptic cylinder $\mathbf{r}(u, v) = [a \cos v,\quad b \sin v,\quad u]$
5. Paraboloid of revolution $\mathbf{r}(u, v) = [u \cos v,\quad u \sin v,\quad u^2]$
6. Helicoid $\mathbf{r}(u, v) = [u \cos v,\quad u \sin v,\quad v]$. Explain the name.
7. Ellipsoid $\mathbf{r}(u, v) = [a \cos v \cos u,\quad b \cos v \sin u,\quad c \sin v]$
8. Hyperbolic paraboloid $\mathbf{r}(u, v) = [au \cosh v,\quad bu \sinh v,\quad u^2]$

9. **CAS EXPERIMENT. Graphing Surfaces, Dependence on a, b, c.** Graph the surfaces in Probs. 3–8. In Prob. 6 generalize the surface by introducing parameters a, b. Then find out in Probs. 4 and 6–8 how the shape of the surfaces depends on a, b, c.

10. **Orthogonal parameter curves** u = const and v = const on $\mathbf{r}(u, v)$ occur if and only if $\mathbf{r}_u \cdot \mathbf{r}_v = 0$. Give examples. Prove it.

11. **Satisfying (4).** Represent the paraboloid in Prob. 5 so that $\tilde{\mathbf{N}}(0, 0) \neq \mathbf{0}$ and show $\tilde{\mathbf{N}}$.

12. **Condition (4).** Find the points in Probs. 1–8 at which (4) $\mathbf{N} \neq \mathbf{0}$ does not hold. Indicate whether this results from the shape of the surface or from the choice of the representation.

13. **Representation $z = f(x, y)$.** Show that $z = f(x, y)$ or $g = z - f(x, y) = 0$ can be written ($f_u = \partial f/\partial u$, etc.)

$$\text{(6)}\qquad \mathbf{r}(u, v) = [u,\quad v,\quad f(u, v)] \quad \text{and} \quad \mathbf{N} = \operatorname{grad} g = [-f_u,\quad -f_v,\quad 1].$$

14–19 DERIVE A PARAMETRIC REPRESENTATION

Find a normal vector. The answer gives *one* representation; there are many. Sketch the surface and parameter curves.

14. Plane $4x + 3y + 2z = 12$
15. Cylinder of revolution $(x - 2)^2 + (y + 1)^2 = 25$
16. Ellipsoid $x^2 + y^2 + \frac{1}{9} z^2 = 1$
17. Sphere $x^2 + (y + 2.8)^2 + (z - 3.2)^2 = 2.25$
18. Elliptic cone $z = \sqrt{x^2 + 4y^2}$
19. Hyperbolic cylinder $x^2 - y^2 = 1$
20. **PROJECT. Tangent Planes $T(P)$** will be less important in our work, but you should know how to represent them.

    (a) If S: $\mathbf{r}(u, v)$, then $T(P)$: $(\mathbf{r}^* - \mathbf{r}\quad \mathbf{r}_u\quad \mathbf{r}_v) = 0$ (a scalar triple product) or
    $$\mathbf{r}^*(p, q) = \mathbf{r}(P) + p\mathbf{r}_u(P) + q\mathbf{r}_v(P).$$
    (b) If S: $g(x, y, z) = 0$, then
    $$T(P): (\mathbf{r}^* - \mathbf{r}(P)) \cdot \nabla g = 0.$$
    (c) If S: $z = f(x, y)$, then
    $$T(P): z^* - z = (x^* - x)f_x(P) + (y^* - y)f_y(P).$$
    Interpret (a)–(c) geometrically. Give two examples for (a), two for (b), and two for (c).