PART B

Linear Algebra. Vector Calculus

CHAPTER 10

Vector Integral Calculus. Integral Theorems

10.1 Line Integrals

The concept of a line integral is a simple and natural generalization of a definite integral

$$(1) \qquad \int_a^b f(x)\,dx.$$

Recall that, in (1), we integrate the function $f(x)$, also known as the integrand, from along the x-axis to $x = b$. Now, in a line integral, we shall integrate a given function, also called the **integrand**, along a curve C in space or in the plane. (Hence curve integral would be a better name but line integral is standard).

This requires that we represent the curve C by a parametric representation (as in Sec. 9.5)

$$(2) \qquad \mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad (a \le t \le b).$$

The curve C is called the **path of integration**. Look at Fig. 219a. The path of integration goes from A to B. Thus A: $\mathbf{r}(a)$ is its initial point and B: $\mathbf{r}(b)$ is its terminal point. C is now *oriented*. The direction from A to B, in which t increases is called the positive direction on C. We mark it by an arrow. The points A and B may coincide, as it happens in Fig. 219b. Then C is called a **closed path**.

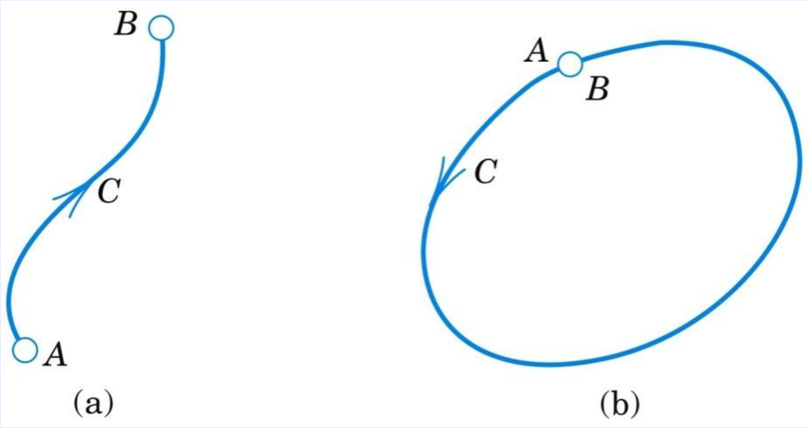

Fig. 219. Oriented curve

C is called a **smooth curve** if it has at each point a unique tangent whose direction varies continuously as we move along C. We note that $\mathbf{r}(t)$ in (2) is differentiable. Its derivative $\mathbf{r}'(t) = d\mathbf{r}/dt$ is continuous and different from the zero vector at every point of C.

General Assumption

In this book, every path of integration of a line integral is assumed to be **piecewise smooth**, *that is, it consists of* ***finitely many*** *smooth curves.*

For example, the boundary curve of a square is piecewise smooth. It consists of four smooth curves or, in this case, line segments which are the four sides of the square.

Definition and Evaluation of Line Integrals

A **line integral** of a vector function $\mathbf{F}(\mathbf{r})$ over a curve C: $\mathbf{r}(t)$ is defined by

(3) $$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) d\mathbf{r} \qquad \mathbf{r}' = \frac{d\mathbf{r}}{dt}$$

where $\mathbf{r}(t)$ is the parametric representation of C as given in (2). (The dot product was defined in Sec. 9.2.) Writing (3) in terms of components, with $d\mathbf{r} = [dx, dy, dz]$ as in Sec. 9.5 and $' = d/dt$, we get

(3′) $$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_C (F_1\, dx + F_2\, dy + F_3\, dz) \\ &= \int_a^b (F_1 x' + F_2 y' + F_3 z')\, dt. \end{aligned}$$

Definition and Evaluation of Line Integrals (continued)

If the path of integration C in (3) is a *closed* curve, then instead of

$$\int_C \quad \text{we also write} \quad \oint_C .$$

Note that the integrand in (3) is a scalar, not a vector, because we take the dot product. Indeed, $\mathbf{F} \bullet \mathbf{r}'/|\mathbf{r}'|$ is the tangential component of $\mathbf{F}$. (For "component" see (11) in Sec. 9.2.)

We see that the integral in (3) on the right is a definite integral of a function of t taken over the interval $a \le t \le b$ on the t-axis in the ***positive*** direction: The direction of increasing t. This definite integral exists for continuous $\mathbf{F}$ and piecewise smooth C, because this makes $\mathbf{F} \bullet \mathbf{r}'$ piecewise continuous.

EXAMPLE 2
Line Integral in Space

The evaluation of line integrals in space is practically the same as it is in the plane. To see this, find the value of (3) when $\mathbf{F}(\mathbf{r}) = [z, x, y] = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and C is the helix (Fig. 221)

(4) $\quad \mathbf{r}(t) = [\cos t, \sin t, 3t] = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3t\mathbf{k} \qquad (0 \le t \le 2\pi).$

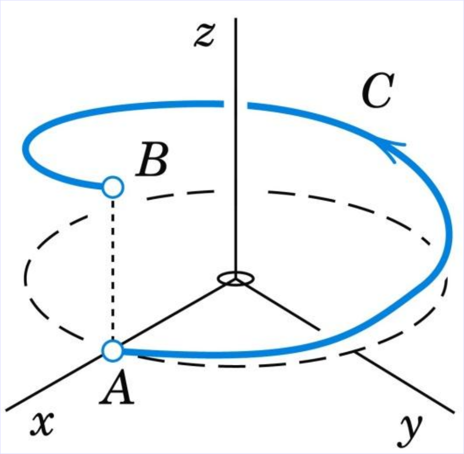

Fig. 221. Example 2

EXAMPLE 2 (continued)
Line Integral in Space

Solution.

From (4) we have $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 3t$. Thus

$$\mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = (3t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) \bullet (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3\mathbf{k}).$$

The dot product is $3t(-\sin t) + \cos^2 t + 3 \sin t$. Hence (3) gives

$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_0^{2\pi} (-3t \sin t + \cos^2 t + 3 \sin t)\, dt$$
$$= 6\pi + \pi + 0 = 7\pi \approx 21.99.$$

Simple general properties of the line integral (3) follow directly from corresponding properties of the definite integral in calculus, namely,

(5a) $$\int_C k\mathbf{F} \bullet d\mathbf{r} = k\int_C \mathbf{F} \bullet d\mathbf{r} \qquad (k \text{ constant})$$

(5b) $$\int_C (\mathbf{F}+\mathbf{G}) \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet d\mathbf{r} + \int_C \mathbf{G} \bullet d\mathbf{r}$$

(5c) $$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} + \int_{C_2} \mathbf{F} \bullet d\mathbf{r} \qquad \text{(Fig. 222)}$$

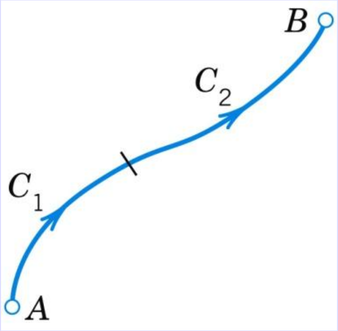

Fig. 222. Formula (5c)

(continued)

where in (5c) the path C is subdivided into two arcs C_1 and C_2 that have the same orientation as C (Fig. 222). In (5b) the orientation of C is the same in all three integrals. If the sense of integration along C is reversed, the value of the integral is multiplied by –1. However, we note the following independence if the sense is preserved.

Theorem 1

Direction-Preserving Parametric Transformations

Any representations of C that give the same positive direction on C also yield the same value of the line integral (3).

Motivation of the Line Integral (3): Work Done by a Force

The work W done by a *constant* force $\mathbf{F}$ in the displacement along a *straight* segment $\mathbf{d}$ is $W = \mathbf{F} \bullet \mathbf{d}$; see Example 2 in Sec. 9.2. This suggests that we define the work W done by a *variable* force $\mathbf{F}$ in the displacement along a curve C: $\mathbf{r}(t)$ as the limit of sums of works done in displacements along small chords of C. We show that this definition amounts to defining W by the line integral (3).

For this we choose points $t_0\ (= a) < t_1 < \dots < t_n\ (= b)$. Then the work ΔW_m done by $\mathbf{F}(\mathbf{r}(t_m))$ in the straight displacement from $\mathbf{r}(t_m)$ to $\mathbf{r}(t_{m+1})$ is

$$\Delta W_m = \mathbf{F}(\mathbf{r}(t_m)) \bullet [\mathbf{r}(t_{m+1}) - \mathbf{r}(t_m)] \approx \mathbf{F}(\mathbf{r}(t_m)) \bullet \mathbf{r}'(t_m)\, \Delta t_m$$
$$(\Delta t_m = \Delta t_{m+1} - t_m).$$

Motivation of the Line Integral (3): Work Done by a Force (continued)

The sum of these n works is $W_n = \Delta W_0 + \ldots + \Delta W_{n-1}$. If we choose points and consider for every n arbitrarily but so that the greatest Δt_m approaches zero as $n \to \infty$, then the limit of W_n as $n \to \infty$ is the line integral (3). This integral exists because of our general assumption that **F** is continuous and C is piecewise smooth; this makes $\mathbf{r}'(t)$ continuous, except at finitely many points where C may have corners or cusps.

EXAMPLE 4

Work Done Equals the Gain in Kinetic Energy

Let **F** be a force, so that (3) is work. Let t be time, so that $d\mathbf{r}/dt = \mathbf{v}$, velocity. Then we can write (3) as

$$W = \int_C \mathbf{F} \bullet d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{v}(t)d\mathbf{r} \tag{6}$$

Now by Newton's second law, that is,
force = mass x acceleration, we get

$$\mathbf{F} = m\mathbf{r}''(t) = m\mathbf{v}'(t),$$

where m is the mass of the body displaced. Substitution into (5) gives [see (11), Sec. 9.4]

$$W = \int_a^b m\mathbf{v}' \bullet \mathbf{v}\, dt = \int_a^b m\left(\frac{\mathbf{v} \bullet \mathbf{v}}{2}\right)' dt = \frac{m}{2}|\mathbf{v}|^2 \Big|_{t=a}^{t=b}.$$

On the right, $m|\mathbf{v}|^2/2$ is the kinetic energy. Hence *the work done equals the gain in kinetic energy*. This is a basic law in mechanics.

Other Forms of Line Integrals

Furthermore, without taking a dot product as in (3) we can obtain a line integral whose value is a vector rather than a scalar, namely,

$$\textbf{(8)}\quad \int_C \mathbf{F}(\mathbf{r})\,dt = \int_a^b \mathbf{F}(\mathbf{r}(t))\,dt = \int_a^b \left[F_1(\mathbf{r}(t)),\ \ F_2(\mathbf{r}(t)),\ \ F_3(\mathbf{r}(t))\right] dt.$$

Path Dependence

Theorem 2

Path Dependence

The line integral (3) generally depends not only on **F** *and on the endpoints A and B of the path, but also on the path itself along which the integral is taken.*

10.2 Path Independence of Line Integrals

$$\textbf{(1)} \quad \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) \qquad (d\mathbf{r} = [dx, \ dy, \ dz])$$

The line integral (1) is said to be **path independent in a domain *D* in space** if for every pair of endpoints *A*, *B* in domain *D*, (1) has the same value for all paths in *D* that begin at *A* and end at *B*. This is illustrated in Fig. 224. (See Sec. 9.6 for "domain.")

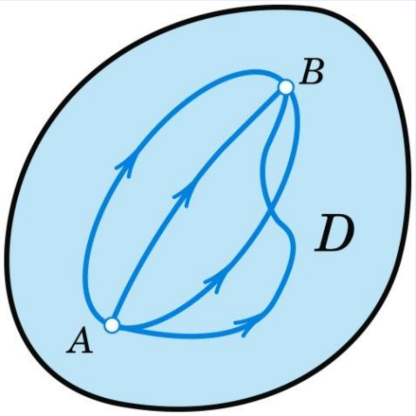

Fig. 224. Path Independence

We shall see that path independence of (1) in a domain D holds if and only if:

(*Theorem* 1) $\mathbf{F} = \operatorname{grad} f$, where grad f is the gradient of f as explained in Sec. 9.7.

(*Theorem* 2) Integration around closed curves C in D always gives 0.

(*Theorem* 3) $\operatorname{curl} \mathbf{F} = \mathbf{0}$, provided D is simply connected, as defined below.

Theorem 1

Path Independence

A line integral (1) *with continuous* F_1, F_2, F_3 *in a domain* D *in space is path independent in* D *if and only if* $\mathbf{F} = [F_1, F_2, F_3]$ *is the gradient of some function* f *in* D,

$$(2) \quad \mathbf{F} = \operatorname{grad} f, \qquad \text{thus,} \qquad F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}.$$

$$(3) \qquad \int_A^B (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A) \qquad [\mathbf{F} = \operatorname{grad} f]$$

is the analog of the usual formula for definite integrals in calculus,

$$\int_a^b g(x)\, dx = G(x)\Big|_a^b = G(b) - G(a) \qquad [G'(x) = g(x)].$$

Formula (3) should be applied whenever a line integral is independent of path.

Potential theory relates to our present discussion if we remember from Sec. 9.7 that when $\mathbf{F} = \text{grad } f$, then f is called a potential of $\mathbf{F}$. Thus the integral (1) is independent of path in D if and only if $\mathbf{F}$ is the gradient of a potential in D.

EXAMPLE 2

Path Independence. Determination of a Potential

Evaluate the integral $I = \int_C (3x^2 dx + 2yz\, dy + y^2 dz)$

from A: (0, 1, 2) to B: (1, 1, 7) by showing that **F** has a potential and applying (3).

Solution. If **F** has a potential f, we should have

$$f_x = F_1 = 3x^2, \quad f_y = F_2 = 2yz, \quad f_z = F_3 = y^2.$$

We show that we can satisfy these conditions. By integration of f_x and differentiation,

$$f = x^3 + g(y, z), \quad f_y = g_y = 2yz, \quad g = y^2 z + h(z), \quad f = x^3 + y^2 z + h(z)$$
$$f_z = y^2 + h' = y^2, \quad h' = 0 \qquad h = 0, \text{ say.}$$

This gives $f(x, y, z) = x^3 + y^3 z$ and by (3),

$$I = f(1, -1, 7) - f(0, 1, 2) = 1 + 7 - (0 + 2) = 6.$$

Path Independence and Integration Around Closed Curves

The simple idea is that two paths with common endpoints (Fig. 225) make up a single closed curve. This gives almost immediately

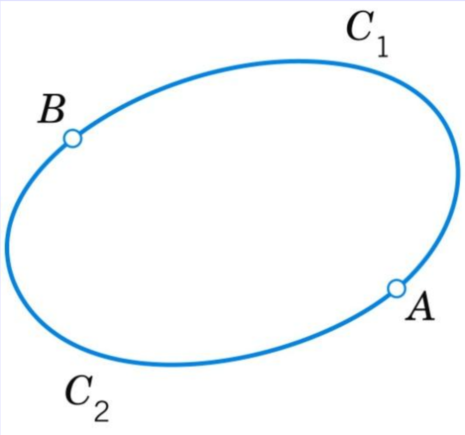

Fig. 225. Proof of Theorem 2

Theorem 2

Path Independence

The integral (1) *is path independent in a domain D if and only if its value around every closed path in D is zero.*

Path Independence and Exactness of Differential Forms

Theorem 1 relates path independence of the line integral (1) to the gradient and Theorem 2 to integration around closed curves. A third idea (leading to Theorems 3* and 3, below) relates path independence to the exactness of the **differential form** or *Pfaffian form*

(4) $$\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$$

under the integral sign in (1). This form (4) is called **exact** in a domain D in space if it is the differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = (\text{grad } f) \bullet d\mathbf{r}$$

of a differentiable function $f(x, y, z)$ everywhere in D,

Path Independence and Exactness of Differential Forms (continued)

that is, if we have

$$\mathbf{F} \bullet d\mathbf{r} = df.$$

Comparing these two formulas, we see that the form (4) is exact if and only if there is a differentiable function $f(x, y, z)$ in D such that everywhere in D,

$$\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$$

(5) $\mathbf{F} = \text{grad}\, f,$ thus, $F_1 = \dfrac{\partial f}{\partial x}, \quad F_2 = \dfrac{\partial f}{\partial y}, \quad F_3 = \dfrac{\partial f}{\partial z}.$

Theorem 3*

Path Independence

The integral (1) *is path independent in a domain* D *in space if and only if the differential form* (4) *has continuous coefficient functions* F_1, F_2, F_3 *and is exact in* D.

Theorem 3

Criterion for Exactness and Path Independence

Let F_1, F_2, F_3 *in the line integral* (1),

$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz),$$

be continuous and have continuous first partial derivatives in a domain D in space. Then:

(a) *If the differential form* (4) *is exact in D—and thus* (1) *is path independent by Theorem 3*—, then in D,*

(6) $$\operatorname{curl} \mathbf{F} = \mathbf{0};$$

Theorem 3

Criterion for Exactness and Path Independence (continued)

in components (see Sec. 9.9)

$$(6') \qquad \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

(b) *If* (6) *holds in D and D is simply connected, then* (4) *is exact in D—and thus* (1) *is path independent by Theorem* 3*.

Line Integral in the Plane.

For
$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C (F_1 dx + F_2 dy)$$
the curl has only one component (the z-component), so that (6′) reduces to the single relation

(6″)
$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

(which also occurs in (5) of Sec. 1.4 on exact ODEs).

10.3 Calculus Review: Double Integrals. *Optional*

In a definite integral (1), Sec. 10.1, we integrate a function $f(x)$ over an interval (a segment) of the x-axis.
In a double integral we integrate a function $f(x, y)$, called the *integrand,* over a closed bounded region R** in the xy-plane, whose boundary curve has a unique tangent at almost every point, but may perhaps have finitely many cusps (such as the vertices of a triangle or rectangle).

A **region R is a domain (Sec. 9.6) plus, perhaps, some or all of its boundary points. R is **closed** if its **boundary** (all its boundary points) are regarded as belonging to R; and R is **bounded** if it can be enclosed in a circle of sufficiently large radius. A **boundary point** P of R is a point (of R or not) such that every disk with center P contains points of R and also points not of R.

Evaluation of Double Integrals by Two Successive Integrations

Double integrals over a region R may be evaluated by two successive integrations. We may integrate first over y and then over x. Then the formula is

$$\textbf{(3)} \qquad \iint_R f(x, y)\, dx\, dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y)\, dy \right] dx \qquad \text{(Fig. 229).}$$

Here $y = g(x)$ and $y = h(x)$ represent the boundary curve of R (see Fig. 229) and, keeping x constant, we integrate $f(x, y)$ over y from $g(x)$ to $h(x)$. The result is a function of x, and we integrate it from $x = a$ to $x = b$ (Fig. 229).

Evaluation of Double Integrals by Two Successive Integrations (continued 1)

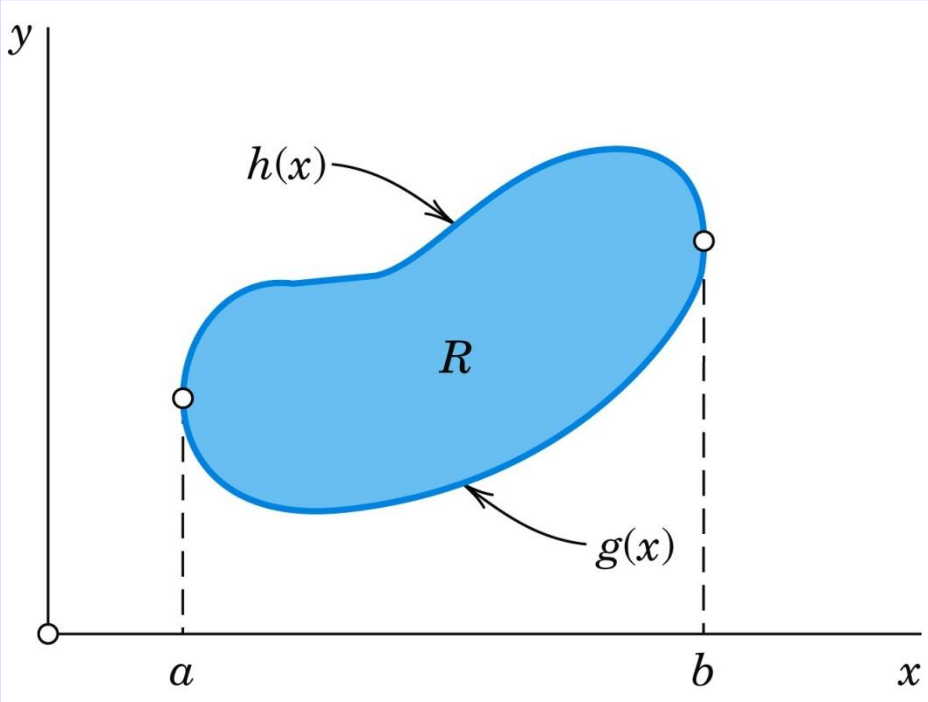

Fig. 229. Evaluation of a double integral

Evaluation of Double Integrals by Two Successive Integrations (continued 2)

Similarly, for integrating first over x and then over y the formula is

(4) $$\iint_R f(x, y)\, dx\, dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x, y)\, dx \right] dy \quad \text{(Fig. 230).}$$

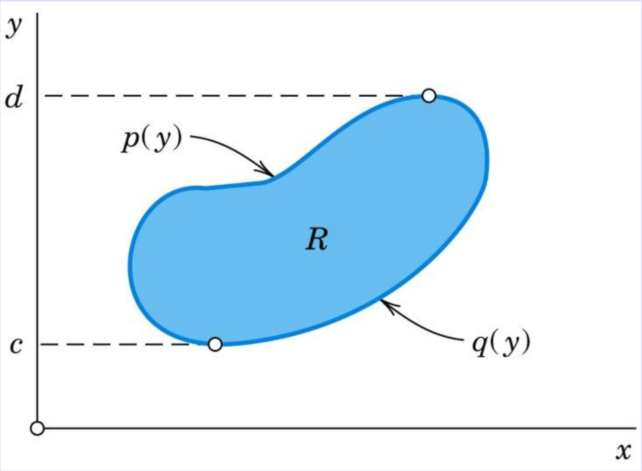

Fig. 230. Evaluation of a double integral

Fig. 230. Evaluation of a double integral

Evaluation of Double Integrals by Two Successive Integrations (continued 3)

The boundary curve of R is now represented by $x = p(y)$ and $x = q(y)$. Treating y as a constant, we first integrate $f(x, y)$ over x from $p(y)$ to $q(y)$ (see Fig. 230) and then the resulting function of y from $y = c$ to $y = d$.

In (3) we assumed that R can be given by inequalities $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$. Similarly in (4) by $c \leq y \leq d$ and $p(y) \leq x \leq q(y)$. If a region R has no such representation, then, in any practical case, it will at least be possible to subdivide R into finitely many portions each of which can be given by those inequalities. Then we integrate $f(x, y)$ over each portion and take the sum of the results. This will give the value of the integral of $f(x, y)$ over the entire region R.

Change of Variables in Double Integrals. Jacobian

The formula for a change of variables in double integrals from x, y to u, v is

(6)
$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv;$$

that is, the integrand is expressed in terms of u and v, and $dx\,dy$ is replaced by $du\,dv$ times the absolute value of the **Jacobian**

(7)
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Change of Variables in Double Integrals. Jacobian (continued)

Here we assume the following. The functions

$$x = x(u, v), \qquad y = y(u, v)$$

effecting the change are continuous and have continuous partial derivatives in some region R^* in the uv-plane such that for every (u, v) in R^* the corresponding point (x, y) lies in R and, conversely, to every (x, y) in R there corresponds one and only one (u, v) in R^*; furthermore, the Jacobian J is either positive throughout R^* or negative throughout R^*.

Change of Variables in Double Integrals. Jacobian (continued)

Of particular practical interest are **polar coordinates** r and θ, which can be introduced by setting $x = r \cos \theta$, $y = r \sin \theta$. Then

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

and

(8) $$\iint_R f(x, y)\, dx\, dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

where R^* is the region in the $r\theta$-plane corresponding to R in the xy-plane.

10.4 Green's Theorem in the Plane

Theorem 1

Green's Theorem in the Plane (Transformation between Double Integrals and Line Integrals)

Let R be a closed bounded region (see Sec. 10.3) *in the xy-plane whose boundary C consists of finitely many smooth curves* (see Sec. 10.1). *Let* $F_1(x, y)$ *and* $F_2(x, y)$ *be functions that are continuous and have continuous partial derivatives* $\partial F_1/\partial y$ *and* $\partial F_1/\partial x$ *everywhere in some domain containing R. Then*

(1) $$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \oint_C \left(F_1 dx + F_2 dy \right).$$

Here we integrate along the entire boundary C of R in such a sense that R is on the left as we advance in the direction of integration (see Fig. 234).

Theorem 1 (continued)

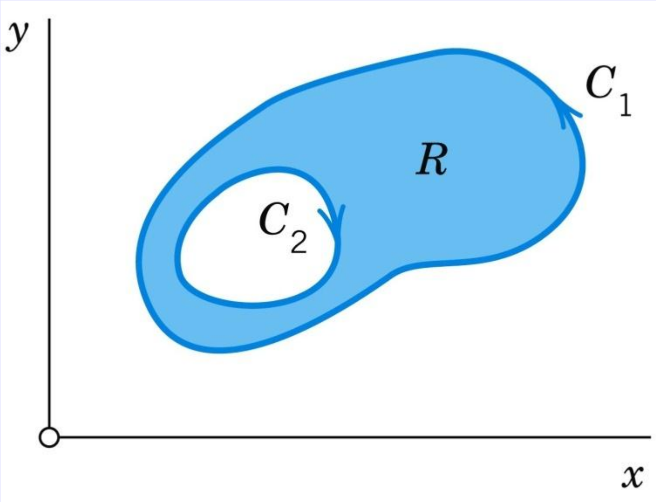

Fig. 234. Region R whose boundary C consists of two parts: C_1 is traversed counterclockwise, while C_2 is traversed clockwise in such a way that R is on the left for both curves

Setting $\mathbf{F} = [F_1, \quad F_2] = F_1\,\mathbf{i} + F_2\,\mathbf{j}$ *and using* (1) *in* Sec. 9.9, we obtain (1) in vectorial form,

(1′) $$\iint_R (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dx\, dy = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Some Applications of Green's Theorem

Green's theorem gives the desired formula relating the Laplacian to the normal derivative,

(9) $$\iint_R \nabla^2 w \, dx \, dy = \oint_C \frac{\partial w}{\partial n} \, ds.$$

For instance, $w = x^2 - y^2$ satisfies Laplace's equation $\nabla^2 w = 0$. Hence its normal derivative integrated over a closed curve must give 0. Can you verify this directly by integration, say, for the square $0 \leq x \leq 1$, $0 \leq y \leq 1$?

10.5 Surfaces for Surface Integrals

Representation of Surfaces

Representations of a surface S in xyz-space are

$$(1) \qquad z = f(x, y) \qquad \text{or} \qquad g(x, y, z) = 0.$$

For example, $z = +\sqrt{a^2 - x^2 - y^2}$ or $x^2 + y^2 + z^2 - a^2 = 0 \quad (z \geq 0)$ represents a hemisphere of radius a and center 0.

Now for *curves* C in line integrals, it was more practical and gave greater flexibility to use a *parametric* representation $\mathbf{r} = \mathbf{r}(t)$, where $a \leq t \leq b$. This is a mapping of the interval $a \leq t \leq b$, located on the t-axis, onto the curve C (actually a portion of it) in xyz-space. It maps every t in that interval onto the point of C with position vector $\mathbf{r}(t)$. See Fig. 241A.

Representation of Surfaces (continued)

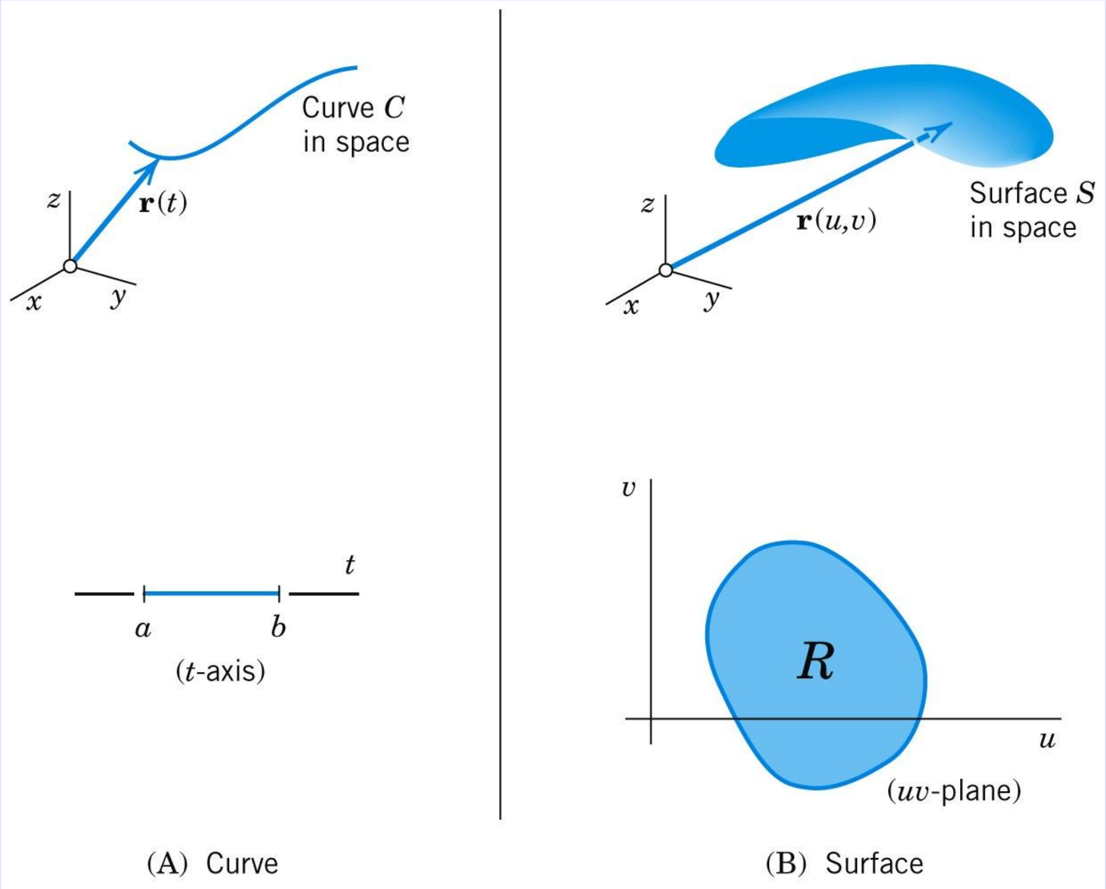

Fig. 241. Parametric representations of a curve and a surface

Representation of Surfaces (continued)

Similarly, for surfaces S in surface integrals, it will often be more practical to use a *parametric* representation. Surfaces are *two*-dimensional. Hence we need *two* parameters,which we call u and v. Thus a **parametric representation** of a surface S in space is of the form

(2) $$\begin{aligned}\mathbf{r}(u, v) &= [x(u, v), y(u, v), z(u, v)] \\ &= x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}\end{aligned}$$

where (u, v) varies in some region R of the uv-plane. This mapping (2) maps every point (u, v) in R onto the point of S with position vector $\mathbf{r}(u, v)$. See Fig. 241B.

EXAMPLE 1
Parametric Representation of a Cylinder

The circular cylinder $x^2 + y^2 = a^2$, $-1 \le z \le 1$, has radius a, height 2, and the z-axis as axis. A parametric representation is

$$\mathbf{r}(u, v) = [a \cos u, a \sin u, v] = a \cos u\, \mathbf{i} + a \sin u\, \mathbf{j} + v\, \mathbf{k} \qquad \text{(Fig. 242).}$$

The components of **r** are $x = a \cos u$, $y = a \sin u$, $z = v$. The parameters u, v vary in the rectangle R: $0 \le u \le 2\pi$, $-1 \le v \le 1$ in the uv-plane. The curves u = const are vertical straight lines. The curves v = const are parallel circles. The point P in Fig. 242 corresponds to $u = \pi/3 = 60°$, $v = 0.7$.

EXAMPLE 1 (continued)
Parametric Representation of a Cylinder

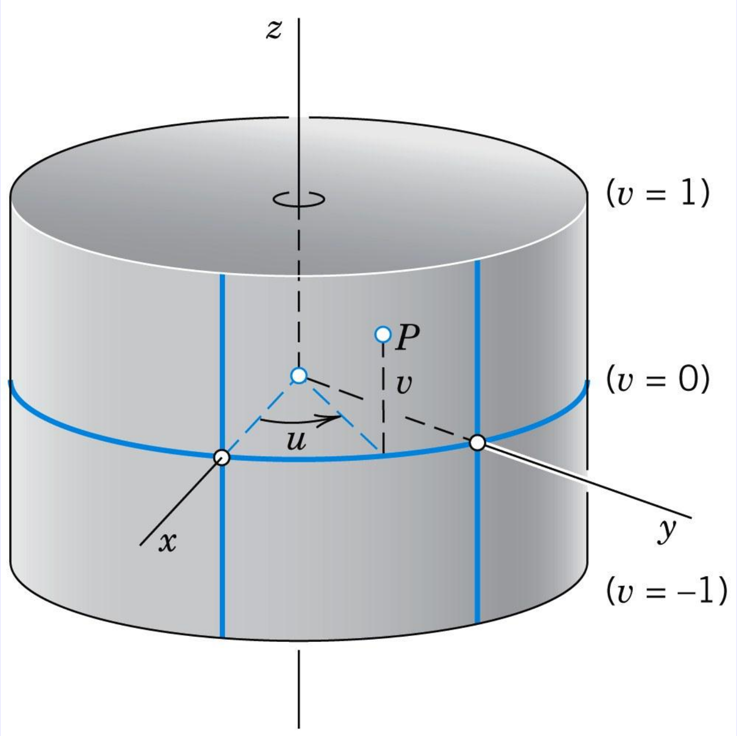

Fig. 242. Parametric representations of a cylinder

Fig. 242. Parametric representations of a cylinder

EXAMPLE 2
Parametric Representation of a Sphere

A sphere $x^2 + y^2 + z^2 = a^2$ can be represented in the form

(3) $\mathbf{r}(u, v) = a \cos v \cos u \,\mathbf{i} + a \cos v \sin u \,\mathbf{j} + a \sin v \,\mathbf{k}$

where the parameters u, v vary in the rectangle R in the uv-plane given by the $0 \le u \le 2\pi$, $-\pi/2 \le v \le \pi/2$. The components of **r** are

$$x = a \cos v \cos u, \qquad y = a \cos v \sin u, \qquad z = a \sin v.$$

The curves u = const and v = const are the "meridians" and "parallels" on S (see Fig. 243). *This representation is used in* ***geography*** *for measuring the latitude and longitude of points on the globe.*

EXAMPLE 2 (continued)

Parametric Representation of a Sphere

Another parametric representation of the sphere also used in mathematics is

$$\mathbf{r}(u, v) = a \cos u \sin v \,\mathbf{i} + a \sin u \sin v \,\mathbf{j} + a \cos v \,\mathbf{k} \tag{3*}$$

where $0 \le u \le 2\pi$, $0 \le v \le \pi$.

EXAMPLE 2 (continued)
Parametric Representation of a Sphere

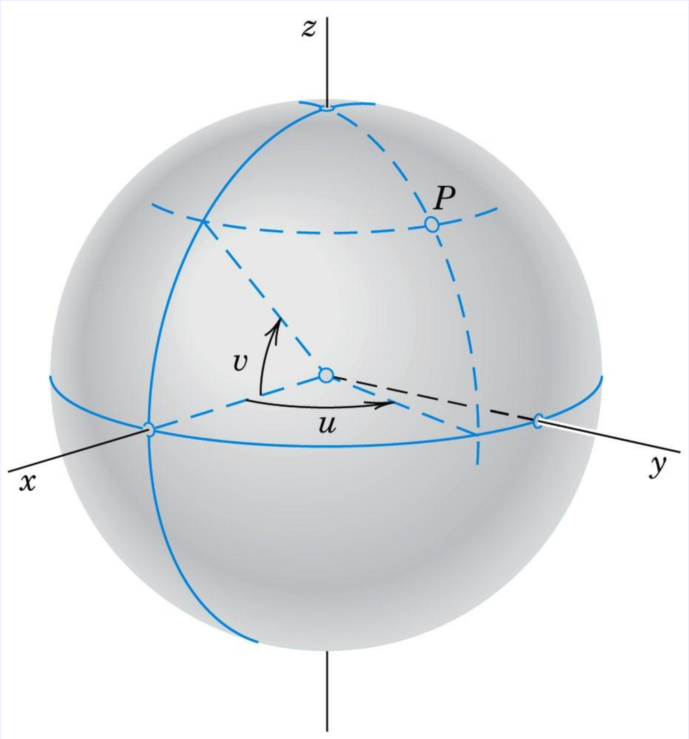

Fig. 243. Parametric representations of a sphere

EXAMPLE 3

Parametric Representation of a Cone

A circular cone $z = \sqrt{x^2 + y^2}$, $0 \le t \le H$
can be represented by

$$\mathbf{r}(u, v) = [u \cos v,\ u \sin v,\ u] = u \cos v\ \mathbf{i} + u \sin v\ \mathbf{j} + u\ \mathbf{k},$$

in components $x = u \cos v$, $y = u \sin v$, $z = u$. The parameters vary in the rectangle R: $0 \le u \le H$, $0 \le v \le 2\pi$. Check that $x^2 + y^2 = z^2$, as it should be. What are the curves u = const and v = const?

Tangent Plane and Surface Normal

Recall from Sec. 9.7 that the tangent vectors of all the curves on a surface S through a point P of S form a plane, called the **tangent plane** of S at P (Fig. 244). Exceptions are points where S has an edge or a cusp (like a cone), so that S cannot have a tangent plane at such a point. Furthermore, a vector perpendicular to the tangent plane is called a **normal vector** of S at P.

Tangent Plane and Surface Normal (continued)

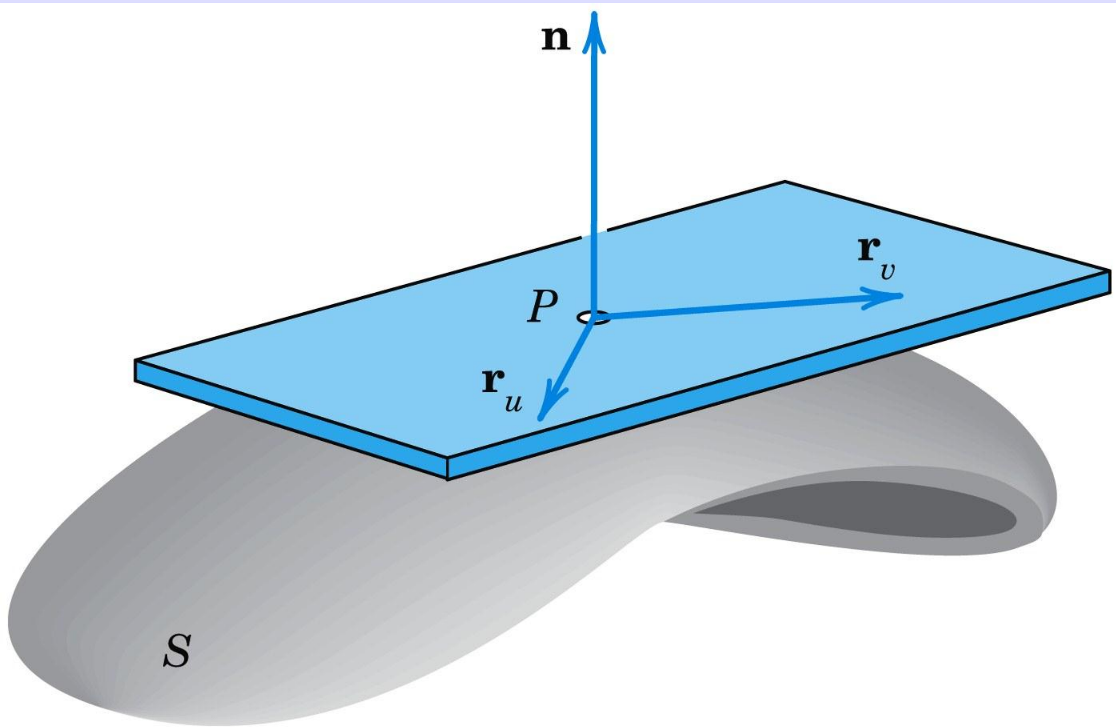

Fig. 244. Tangent plane and normal vector

Tangent Plane and Surface Normal (continued)

Now since S can be given by $\mathbf{r} = \mathbf{r}(u, v)$ in (2), the new idea is that we get a curve C on S by taking a pair of differentiable functions

$$u = u(t), \qquad v = v(t)$$

whose derivatives $u' = du/dt$ and $v' = dv/dt$ are continuous. Then C has the position vector $\tilde{\mathbf{r}}(t) = \mathbf{r}(u(t), v(t))$. By differentiation and the use of the chain rule (Sec. 9.6) we obtain a tangent vector of C on S

$$\tilde{\mathbf{r}}'(t) = \frac{d\tilde{\mathbf{r}}}{dt} = \frac{\partial \mathbf{r}}{\partial u} u' + \frac{\partial \mathbf{r}}{\partial v} v'.$$

Tangent Plane and Surface Normal (continued)

Hence the partial derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$ *at P are tangential to S at P*. We assume that they are linearly independent, which geometrically means that the curves u = const and v = const on S intersect at P at a nonzero angle. Then $\mathbf{r}_u$ and $\mathbf{r}_v$ span the tangent plane of S at P. Hence their cross product gives a **normal vector N** of S at P.

(4) $$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}.$$

The corresponding **unit normal vector n** of S at P is (Fig. 244)

(5) $$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\mathbf{r}_u \times \mathbf{r}_v.$$

Tangent Plane and Surface Normal (continued)

Also, if S is represented by $g(x, y, z) = 0$, then, by Theorem 2 in Sec. 9.7,

(5*)
$$\mathbf{n} = \frac{1}{|\operatorname{grad} g|} \operatorname{grad} g.$$

A surface S is called a **smooth surface** if its surface normal depends continuously on the points of S.

S is called **piecewise smooth** if it consists of finitely many smooth portions.

For instance, a sphere is smooth, and the surface of a cube is piecewise smooth (explain!). We can now summarize our discussion as follows.

Theorem 1

Tangent Plane and Surface Normal

If a surface S is given by (2) *with continuous* $\mathbf{r}_u = \partial\mathbf{r}/\partial u$ *and* $\mathbf{r}_v = \partial\mathbf{r}/\partial v$ *satisfying* (4) *at every point of S, then S has, at every point P, a unique tangent plane passing through P and spanned by* $\mathbf{r}_u$ *and* $\mathbf{r}_v$*, and a unique normal whose direction depends continuously on the points of S. A normal vector is given by* (4) *and the corresponding unit normal vector by* (5). (See Fig. 244.)

EXAMPLE 4

Unit Normal Vector of a Sphere

From (5*) we find that the sphere $x^2 + y^2 + z^2 - a^2 = 0$, has the unit normal vector

$$\mathbf{n}(x, y, z) = \left[\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right] = \frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k}.$$

We see that **n** has the direction of the position vector $[x, y, z]$ of the corresponding point. Is it obvious that this must be the case?

10.6 Surface Integrals

To define a surface integral, we take a surface S, given by a parametric representation as just discussed,

$$(1) \qquad \mathbf{r}(u, v) = [x\,(u, v), y\,(u, v), z\,(u, v)] = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) varies over a region R in the uv-plane. We assume S to be piecewise smooth (Sec. 10.5), so that S has a normal vector

$$(2) \qquad \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \qquad \text{and unit normal vector} \qquad \mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N}$$

at every point (except perhaps for some edges or cusps, as for a cube or cone).

For a given vector function **F** we can now define the **surface integral** over S by

(3) $$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \bullet \mathbf{N}(u, v)\, du\, dv.$$

Here $\mathbf{N} = |\mathbf{N}|\, \mathbf{n}$ by (2), and $|\mathbf{N}| = |\mathbf{r}_u \times \mathbf{r}_v|$ is the area of the parallelogram with sides $\mathbf{r}_u$ and $\mathbf{r}_v$, by the definition of cross product. Hence

(3*) $$\mathbf{n}\, dA = \mathbf{n}\,|\mathbf{N}|\, du\, dv = \mathbf{N}\, du\, dv.$$

And we see that $dA = |\mathbf{N}|\, du\, dv$ is the element of area of S.

Also $\mathbf{F} \bullet \mathbf{n}$ is the normal component of **F**. This integral arises naturally in flow problems, where it gives the **flux** across S when $\mathbf{F} = \rho\mathbf{v}$. We may thus call the surface integral (3) the **flux integral**.

We can write (3) in components, using $\mathbf{F} = [F_1, F_2, F_3]$, $\mathbf{N} = [N_1, N_2, N_3]$, and $\mathbf{n} = [\cos\alpha, \cos\beta, \cos\gamma]$. Here, α, β, γ are the angles between $\mathbf{n}$ and the coordinate axes; indeed, for the angle between $\mathbf{n}$ and $\mathbf{i}$, formula (4) in Sec. 9.2 gives $\cos\alpha = \mathbf{n} \bullet \mathbf{i}/|\mathbf{n}|\,|\mathbf{i}| = \mathbf{n} \bullet \mathbf{i}$, and so on.

We thus obtain from (3)

(4)

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA$$

$$= \iint_S (F_1 N_1 + F_2 N_2 + F_3 N_3)\, du\, dv$$

In (4) we can write $\cos\alpha\, dA = dy\, dz$, $\cos\beta\, dA = dz\, dx$, $\cos\gamma\, dA = dx\, dy$. Then (4) becomes the following integral for the flux:

(5)

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy).$$

EXAMPLE 1
Flux Through a Surface

Compute the flux of water through the parabolic cylinder S: $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$ (Fig. 245) if the velocity vector is $\mathbf{v} = \mathbf{F} = [3z^2, 6, 6xz]$, speed being measured in meters/sec. (Generally, $\mathbf{F} = \rho\mathbf{v}$, but water has the density $\rho = 1 \text{ g/cm}^3 = 1 \text{ ton/m}^3$.)

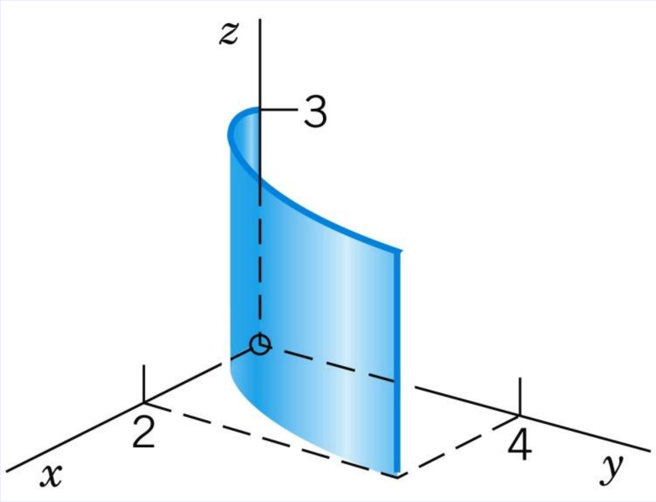

Fig. 245. Surface S in Example 1

EXAMPLE 1 (continued 1) Flux Through a Surface

Solution. Writing $x = u$ and $z = v$, we have $y = x^2 = u^2$. Hence a representation of S is

$$S\colon \mathbf{r} = [u, u^2, v] \qquad (0 \le u \le 2,\ 0 \le v \le 3).$$

By differentiation and by the definition of the cross product,

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [1, 2u, 0] \times [0, 0, 1] = [2u, -1, 0].$$

On S, writing simply $\mathbf{F}(S)$ for $\mathbf{F}[\mathbf{r}(u, v)]$, we have $\mathbf{F}(S) = [3v^2, 6, 6uv]$. Hence $\mathbf{F}(S) \bullet \mathbf{N} = 6uv^2 - 6$. By integration we thus get from (3) the flux

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv = \int_0^3 (3u^2v^2 - 6u)\Big|_{u=0}^{2} dv$$

$$= \int_0^3 (12v^2 - 12)\, dv = (4v^3 - 12v)\Big|_{v=0}^{3}$$

$$= 108 - 36 = 72 \left[\mathrm{m}^3/\mathrm{sec}\right]$$

EXAMPLE 1 (continued 2) Flux Through a Surface

Solution. *(continued)*

or 72,000 liters/sec. Note that the y-component of **F** is positive (equal to 6), so that in Fig. 245 the flow goes from left to right.

Let us confirm this result by (5). Since

$$\mathbf{N} = |\mathbf{N}|\mathbf{n} = |\mathbf{N}|[\cos\alpha,\ \cos\beta,\ \cos\gamma] = [2u, -1, 0] = [2x, -1, 0]$$

we see that $\cos\alpha > 0$, $\cos\beta < 0$, and $\cos\gamma = 0$. Hence the second term of (5) on the right gets a minus sign, and the last term is absent.

EXAMPLE 1 (continued 3) Flux Through a Surface

Solution. *(continued)*

This gives, in agreement with the previous result,

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \int_0^3 \int_0^4 3z^2\, dy\, dz - \int_0^2 \int_0^3 6\, dz\, dx$$

$$= \int_0^3 4(3z^2)\, dz - \int_0^2 6 \cdot 3\, dx = 4 \cdot 3^3 - 6 \cdot 3 \cdot 2 = 72.$$

Orientation of Surfaces

From (3) or (4) we see that the value of the integral depends on the choice of the unit normal vector **n**.
(Instead of **n** we could choose –**n**.) We express this by saying that such an integral is an *integral over an* **oriented surface** *S*, that is, over a surface *S* on which we have chosen one of the two possible unit normal vectors in a continuous fashion.

Orientation of Surfaces

Theorem 1

Change of Orientation in a Surface Integral

The replacement of $\mathbf{n}$ *by* $-\mathbf{n}$ (*hence of* $\mathbf{N}$ *by* $-\mathbf{N}$) *corresponds to the multiplication of the integral in* (3) *or* (4) *by* -1.

Orientation of Smooth Surfaces

A smooth surface S (see Sec. 10.5) is called **orientable** if the positive normal direction, when given at an arbitrary point P_0 of S, can be continued in a unique and continuous way to the entire surface. In many practical applications, the surfaces are smooth and thus orientable.

Orientation of Piecewise Smooth Surfaces

Here the following idea will do it. For a *smooth* orientable surface S with boundary curve C we may associate with each of the two possible orientations of S an orientation of C, as shown in Fig. 247a. Then a *piecewise smooth* surface is called **orientable** if we can orient each smooth piece of S so that along each curve C^* which is a common boundary of two pieces S_1 and S_2 the positive direction of C^* relative to S_1 is opposite to the direction of C^* relative to S_2. See Fig. 247b for two adjacent pieces; note the arrows along C^*.

Orientation of Piecewise Smooth Surfaces (continued)

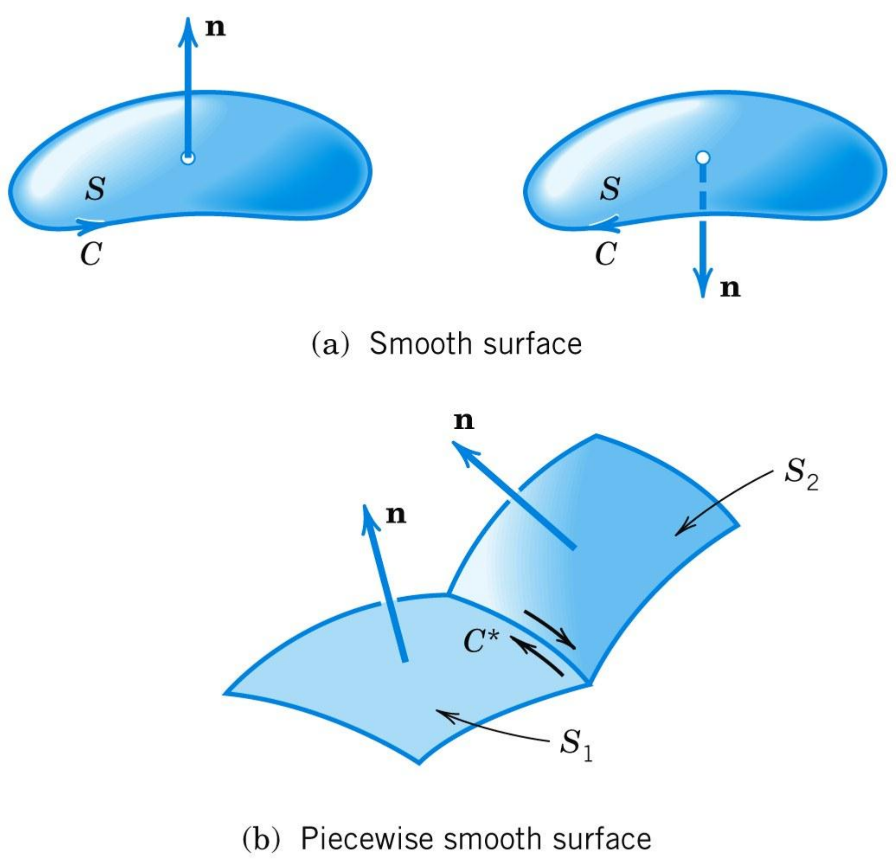

Fig. 247. Orientation of a surface

Theory: Nonorientable Surfaces

A sufficiently small piece of a smooth surface is always orientable. This may not hold for entire surfaces.

A well-known example is the **Möbius strip**, shown in Fig. 248. To make a model, take the rectangular paper in Fig. 248, make a half-twist, and join the short sides together so that A goes onto A, and B onto B. At P_0 take a normal vector pointing, say, to the *left*. Displace it along C to the right (in the lower part of the figure) around the strip until you return to P_0 and see that you get a normal vector pointing to the *right*, opposite to the given one. See also Prob. 17.

Theory: Nonorientable Surfaces (continued)

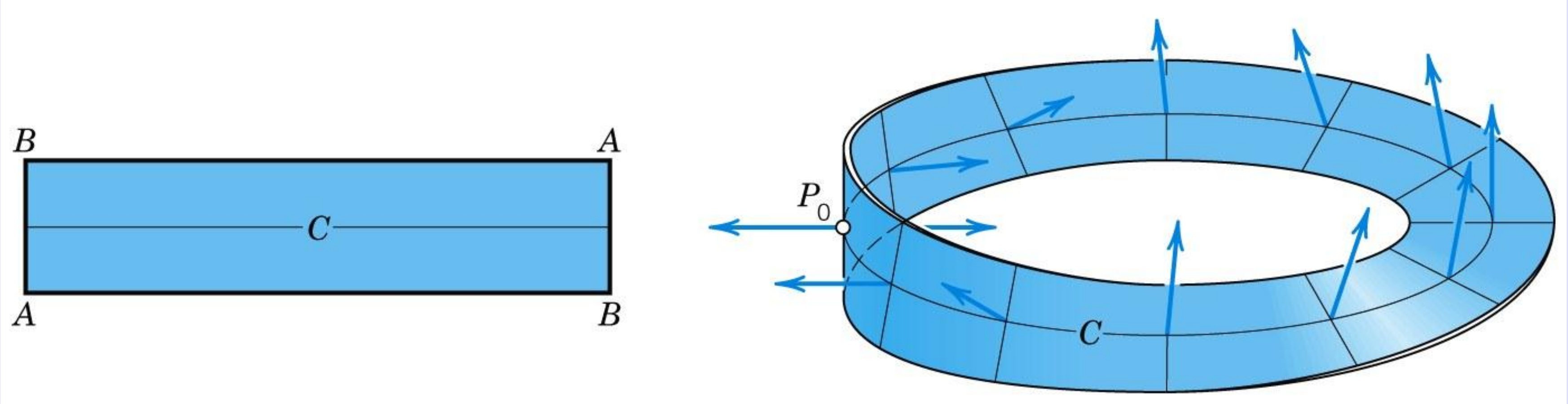

Fig. 248. Möbius strip

Surface Integrals Without Regard to Orientation

Another type of surface integral is

(6) $$\iint_S G(\mathbf{r})\,dA = \iint_R G(\mathbf{r}(u, v))\,|\mathbf{N}(u, v)|\,du\,dv.$$

Here $dA = |\mathbf{N}|\,du\,dv = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$ is the element of area of the surface S represented by (1) and we disregard the orientation.

Surface Integrals Without Regard to Orientation

(continued)

As for applications, if $G(\mathbf{r})$ is the mass density of S, then (6) is the total mass of S. If $G = 1$, then (6) gives the **area** $A(S)$ of S,

(8) $$A(S) = \iint_S dA = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv.$$

Representations $z = f(x, y)$.

If a surface S is given by $z = f(x, y)$, then setting $u = x$, $v = y$, $\mathbf{r} = [u, v, f]$ gives

$$|\mathbf{N}| = |\mathbf{r}_u \times \mathbf{r}_v| = \|[1,\ 0,\ f_u] \times [0,\ 1,\ f_v]\| = \|[-f_u, -f_v, 1]\| = \sqrt{1 + f_u^2 + f_v^2}$$

and, since $f_u = f_x$, $f_v = f_y$, formula (6) becomes

(11) $$\iint_S G(\mathbf{r})\,dA = \iint_{R^*} G(x, y, f(x, y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\; dx\,dy.$$

Representations $z = f(x, y)$. (continued 1)

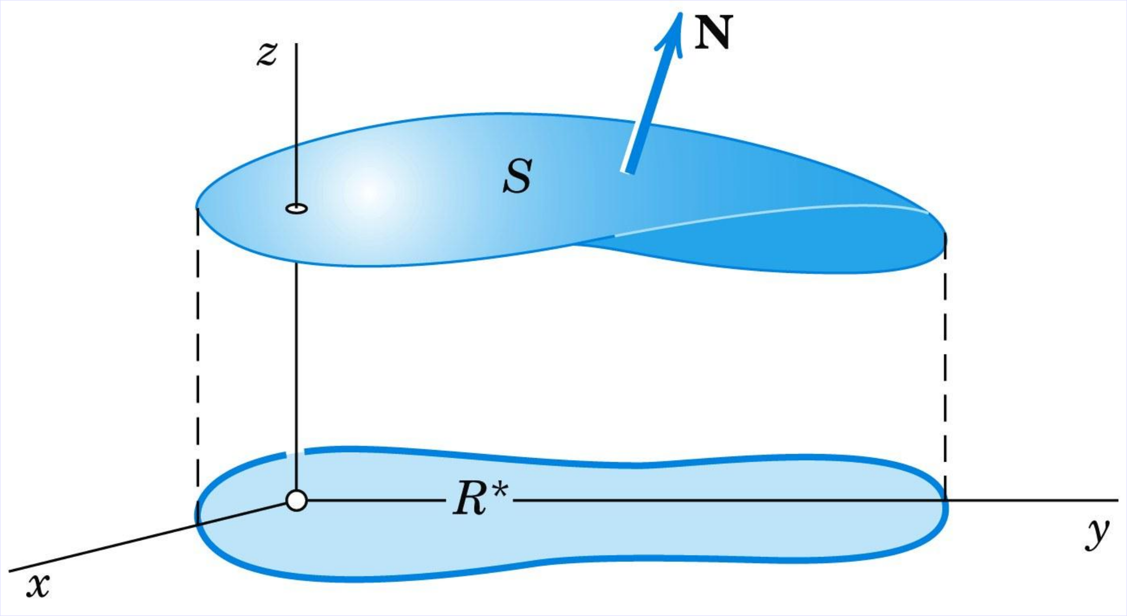

Fig. 250. Formula (11)

Here R^* is the projection of S into the xy-plane (Fig. 250) and the normal vector $\mathbf{N}$ on S points *up*. If it points *down*, the integral on the right is preceded by a minus sign.

Representations $z = f(x, y)$. (continued 2)

From (11) with $G = 1$ we obtain for the **area** $A(S)$ of S: $z = f(x, y)$ the formula

(12)
$$A(S) = \iint_{R^*} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy$$

where R^* is the projection of S into the xy-plane, as before.

10.7 Triple Integrals. Divergence Theorem of Gauss

Divergence Theorem of Gauss

Triple integrals can be transformed into surface integrals over the boundary surface of a region in space and conversely. Such a transformation is of practical interest because one of the two kinds of integral is often simpler than the other. It also helps in establishing fundamental equations in fluid flow, heat conduction, etc., as we shall see. The transformation is done by the *divergence theorem,* which involves the **divergence** of a vector function $\mathbf{F} = [F_1, F_2, F_3] = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, namely,

(1) $$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$ (Sec. 9.8).

Theorem 1

Divergence Theorem of Gauss
(Transformation Between Triple and Surface Integrals)

Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $\mathbf{F}(x, y, z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then

(2)
$$\iiint_T \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \bullet \mathbf{n}\, dA.$$

Theorem 1 (continued)

Divergence Theorem of Gauss
(Transformation Between Triple and Surface Integrals)
(continued)

In components *of* $\mathbf{F} = [F_1, F_2, F_3]$ *and of the outer unit normal vector* $\mathbf{n} = [\cos\alpha,\ \cos\beta,\ \cos\gamma]$ *of* S (as in Fig. 253), *formula* (2) *becomes*

(2*)
$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$
$$= \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA$$
$$= \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy).$$

EXAMPLE 1 Evaluation of a Surface Integral by the Divergence Theorem

Before we prove the theorem, let us show a typical application. Evaluate

$$I = \iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy)$$

where S is the closed surface in Fig. 252 consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular disks $z = 0$ and $z = b$ $(x^2 + y^2 \le a^2)$.

EXAMPLE 1 (continued 1) Evaluation of a Surface Integral by the Divergence Theorem

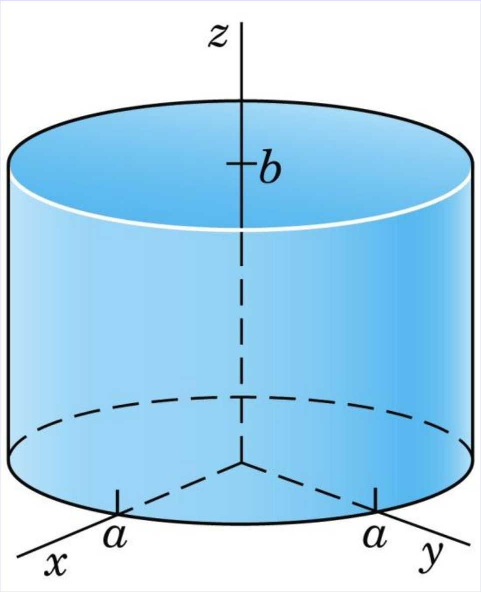

Fig. 252. Surface S in Example 1

Solution.

$F_1 = x^3$, $F_2 = x^2y$, $F_3 = x^2z$. Hence div $\mathbf{F} = 3x^2 + x^2 + x^2 = 5x^2$.

EXAMPLE 1 (continued 2) Evaluation of a Surface Integral by the Divergence Theorem

Solution. *(continued)*

The form of the surface suggests that we introduce polar coordinates r, θ defined by $x = r \cos \theta$, $y = r \sin \theta$ (thus cylindrical coordinates r, θ, z). Then the volume element is $dx\, dy\, dz = r\, dr\, d\theta\, dz$ and we obtain

$$I = \iiint_T 5x^2\, dx\, dy\, dz = \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} (5r^2 \cos^2 \theta)\, r\, dr\, d\theta\, dz$$

$$= 5 \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \frac{a^4}{4} \cos^2 \theta\, d\theta\, dz$$

$$= 5 \int_{z=0}^{b} \frac{a^4 \pi}{4}\, dz = \frac{5\pi}{4}\, a^4 b.$$

Coordinate Invariance of the Divergence.

The divergence (1) is defined in terms of coordinates, but we can use the divergence theorem to show that div **F** has a meaning independent of coordinates.

For this purpose we first note that triple integrals have properties quite similar to those of double integrals in Sec. 10.3. In particular, the **mean value theorem for triple integrals** asserts that for any continuous function $f(x, y, z)$ in a bounded and simply connected region T there is a point Q: (x_0, y_0, z_0) in T such that

$$(9) \quad \iiint_T f(x, y, z)\, dV = f(x_0, y_0, z_0)\, V(T) \qquad (V(T) = \text{volume of } T).$$

Coordinate Invariance of the Divergence. (continued)

In this formula we interchange the two sides, divide by $V(T)$, and set $f = \operatorname{div} \mathbf{F}$. Then by the divergence theorem we obtain for the divergence an integral over the boundary surface $S(T)$ of T,

$$(10) \qquad \operatorname{div} \mathbf{F}(x_0, y_0, z_0) = \frac{1}{V(T)} \iiint_T \operatorname{div} \mathbf{F} \, dV = \frac{1}{V(T)} \iint_{S(T)} \mathbf{F} \bullet \mathbf{n} \, dA.$$

We now choose a point P: (x_1, y_1, z_1) in T and let T shrink down onto P so that the maximum distance $d(T)$ of the points of T from P goes to zero. Then Q: (x_0, y_0, z_0) must approach P.

Hence (10) becomes

(11) $$\operatorname{div} \mathbf{F}(P) = \lim_{d(T)\to 0} \frac{1}{V(T)} \iint_{S(T)} \mathbf{F} \bullet \mathbf{n}\, dA.$$

Theorem 2

Invariance of the Divergence

The divergence of a vector function **F** *with continuous first partial derivatives in a region T is independent of the particular choice of Cartesian coordinates. For any P in T it is given by* (11).

10.8 Further Applications of the Divergence Theorem

EXAMPLE 1

Fluid Flow. Physical Interpretation of the Divergence

From the divergence theorem we may obtain an intuitive interpretation of the divergence of a vector. For this purpose we consider the flow of an incompressible fluid (see Sec. 9.8) of constant density $\rho = 1$ which is **steady**, that is, does not vary with time. Such a flow is determined by the field of its velocity vector $\mathbf{v}(P)$ at any point P.

Let S be the boundary surface of a region T in space, and let $\mathbf{n}$ be the outer unit normal vector of S. Then $\mathbf{v} \bullet \mathbf{n}$ is the normal component of $\mathbf{v}$ in the direction of $\mathbf{n}$, and $|\mathbf{v} \bullet \mathbf{n}\, dA|$ is the mass of fluid *leaving* T (if $\mathbf{v} \bullet \mathbf{n} > 0$ at some P) or *entering* T (if $\mathbf{v} \bullet \mathbf{n} < 0$ at P) per unit time at some point P of S through a small portion ΔS of S of area ΔA.

EXAMPLE 1 (continued 1)

Fluid Flow. Physical Interpretation of the Divergence

Hence the total mass of fluid that flows across S from T to the outside per unit time is given by the surface integral

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dA.$$

Division by the volume V of T gives the **average flow** out of T:

(1)
$$\frac{1}{V}\iint_S \mathbf{v} \cdot \mathbf{n}\, dA.$$

Since the flow is steady and the fluid is incompressible, the amount of fluid flowing outward must be continuously supplied.

EXAMPLE 1 (continued 2)

Fluid Flow. Physical Interpretation of the Divergence

Hence, if the value of the integral (1) is different from zero, there must be **sources** (*positive sources and negative sources, called* **sinks**) in T, that is, points where fluid is produced or disappears.

If we let T shrink down to a fixed point P in T, we obtain from (1) the **source intensity** at P given by the right side of (11) in the last section with $\mathbf{F} \bullet \mathbf{n}$ replaced by $\mathbf{v} \bullet \mathbf{n}$, that is,

$$\text{(2)} \qquad \operatorname{div} \mathbf{v}(P) = \lim_{d(T)\to 0} \frac{1}{V} \iint_{S(T)} \mathbf{v} \bullet \mathbf{n} \, dA.$$

EXAMPLE 1 (continued 3)

Fluid Flow. Physical Interpretation of the Divergence

Hence *the divergence of the velocity vector* **v** *of a steady incompressible flow is the source intensity of the flow at the corresponding point.*

There are no sources in T if and only if div **v** is zero everywhere in T. Then for any closed surface S in T we have

$$\iint_S \mathbf{v} \bullet \mathbf{n}\, dA = 0.$$

EXAMPLE 2

Modeling of Heat Flow. Heat or Diffusion Equation

Physical experiments show that in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient of the temperature. This means that the velocity **v** of the heat flow in a body is of the form

$$\mathbf{v} = -K \operatorname{grad} U \tag{3}$$

where $U(x, y, z, t)$ is temperature, t is time, and K is called the *thermal conductivity* of the body; in ordinary physical circumstances K is a constant. Using this information, set up the mathematical model of heat flow, the so-called **heat equation** or **diffusion equation**.

EXAMPLE 2 (continued 1)

Modeling of Heat Flow. Heat or Diffusion Equation

Solution. Let T be a region in the body bounded by a surface S with outer unit normal vector **n** such that the divergence theorem applies. Then $\mathbf{v} \cdot \mathbf{n}$ is the component of **v** in the direction of **n**, and the amount of heat leaving T per unit time is

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, dA.$$

This expression is obtained similarly to the corresponding surface integral in the last example. Using

$$\operatorname{div}(\operatorname{grad} U) = \nabla^2 U = U_{xx} + U_{yy} + U_{zz}$$

(the Laplacian; see (3) in Sec. 9.8), we have by the divergence theorem and (3) **(continued)**

EXAMPLE 2 (continued 2)

Modeling of Heat Flow. Heat or Diffusion Equation

Solution. **(continued)**

$$\iint_S \mathbf{v} \bullet \mathbf{n}\, dA = -K \iiint_T \operatorname{div}(\operatorname{grad} U)\, dx\, dy\, dz \tag{4}$$

$$= -K \iiint_T \nabla^2 U\, dx\, dy\, dz.$$

On the other hand, the total amount of heat H in T is

$$H = \iiint_T \sigma \rho U\, dx\, dy\, dz$$

where the constant σ is the specific heat of the material of the body and ρ is the density (= mass per unit volume) of the material.

EXAMPLE 2 (continued 3)

Modeling of Heat Flow. Heat or Diffusion Equation

Solution. **(continued)**

Hence the time rate of decrease of H is

$$-\frac{\partial H}{\partial t} = -\iiint_T \sigma\,\rho \frac{\partial U}{\partial t}\; dx\, dy\, dz$$

and this must be equal to the above amount of heat leaving T. From (4) we thus have

$$-\iiint_T \sigma\,\rho \frac{\partial U}{\partial t}\; dx\, dy\, dz = -K \iiint_T \nabla^2 U\; dx\, dy\, dz$$

or

$$\iiint_T \left(\sigma\,\rho \frac{\partial U}{\partial t} - K\,\nabla^2 U \right) dx\, dy\, dz = 0.$$

EXAMPLE 2 (continued 4)

Modeling of Heat Flow. Heat or Diffusion Equation

Solution. **(continued)**

Since this holds for any region T in the body, the integrand (if continuous) must be zero everywhere; that is,

$$\frac{\partial U}{\partial t} = c^2 \nabla^2 U \qquad c^2 = \frac{K}{\sigma\rho} \tag{5}$$

where c^2 is called the *thermal diffusivity* of the material. This partial differential equation is called the **heat equation**. It is the fundamental equation for heat conduction. And our derivation is another impressive demonstration of the great importance of the divergence theorem. Methods for solving heat problems will be shown in Chap. 12.

EXAMPLE 2 (continued 5)
Modeling of Heat Flow. Heat or Diffusion Equation

Solution. **(continued)**

The heat equation is also called the **diffusion equation** because it also models diffusion processes of motions of molecules tending to level off differences in density or pressure in gases or liquids.

If heat flow does not depend on time, it is called **steady-state heat flow**. Then $\partial U/\partial t = 0$, so that (5) reduces to *Laplace's equation* $\nabla^2 U = 0$. We met this equation in Secs. 9.7 and 9.8, and we shall now see that the divergence theorem adds basic insights into the nature of solutions of this equation.

Potential Theory. Harmonic Functions

The theory of solutions of **Laplace's equation**

(6) $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is called **potential theory**. A solution of (6) with ***continuous*** second-order partial derivatives is called a **harmonic function.**

EXAMPLE 3

A Basic Property of Solutions of Laplace's Equation

The integrands in the divergence theorem are div **F** and **F** • **n** (Sec. 10.7). If **F** is the gradient of a scalar function, say, **F** = grad *f*, then div **F** = div (grad *f*) = $\nabla^2 f$, see (3), Sec. 9.8. Also, **F** • **n** = **n** • **F** = **n** • grad *f*. This is the directional derivative of *f* in the outer normal direction of *S*, the boundary surface of the region *T* in the theorem. This derivative is called the (outer) **normal derivative** of *f* and is denoted by $\partial f/\partial n$. Thus the formula in the divergence theorem becomes

(7)
$$\iiint_T \nabla^2 f \, dV = \iint_S \frac{\partial f}{\partial n} \, dA.$$

Theorem 1

A Basic Property of Harmonic Functions

Let $f(x, y, z)$ be a harmonic function in some domain D is space. Let S be any piecewise smooth closed orientable surface in D whose entire region it encloses belongs to D. Then the integral of the normal derivative of f taken over S is zero.

(For "piecewise smooth" see Sec. 10.5.)

Theorem 2

Harmonic Functions

Let $f(x, y, z)$ be harmonic in some domain D and zero at every point of a piecewise smooth closed orientable surface S in D whose entire region T it encloses belongs to D. Then f is identically zero in T.

Theorem 3

Uniqueness Theorem for Laplace's Equation

Let T be a region that satisfies the assumptions of the divergence theorem, and let $f(x, y, z)$ be a harmonic function in a domain D that contains T and its boundary surface S. Then f is uniquely determined in T by its values on S.

Theorem 3*

Uniqueness Theorem for the Dirichlet Problem

If the assumptions in Theorem 3 are satisfied and the Dirichlet problem for the Laplace equation has a solution in T, then this solution is unique.

10.9 Stokes's Theorem

We now introduce another "big" theorem that allows us to transform surface integrals into line integrals and conversely, line integrals into surface integrals. It is called **Stokes's Theorem**, and it generalizes Green's theorem in the plane (see Example 2 below for this immediate observation). Recall from Sec. 9.9 that

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{1}$$

Theorem 1

Stokes's Theorem
(Transformation Between Surface and Line Integrals)

Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple closed curve C. Let $\mathbf{F}(x, y, z)$ *be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then*

$$\iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n} \, dA = \oint_C \mathbf{F} \bullet \mathbf{r}'(s) \, ds. \tag{2}$$

Here **n** *is a unit normal vector of S and, depending on* **n**, *the integration around C is taken in the sense shown in Fig.* 254. *Furthermore,* $\mathbf{r}' = d\mathbf{r}/ds$ *is the unit tangent vector and s the arc length of C.*

Theorem 1 (continued 1)

Stokes's Theorem (continued)
(Transformation Between Surface and Line Integrals)

In components, formula (2) becomes

$$(2^*)\qquad \iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right] du\, dv$$
$$= \oint_{\bar{C}} \left(F_1 dx + F_2 dy + F_3 dz\right).$$

Here, $\mathbf{F} = [F_1, F_2, F_3]$, $\mathbf{N} = [N_1, N_2, N_3]$, $\mathbf{n}\, dA = \mathbf{N}\, du\, dv$, $\mathbf{r}'\, ds = [dx, dy, dz]$, *and* R *is the region with boundary curve* $\bar{C}$ *in the uv-plane corresponding to* S *represented by* $\mathbf{r}(u, v)$.

Theorem 1 (continued 2)

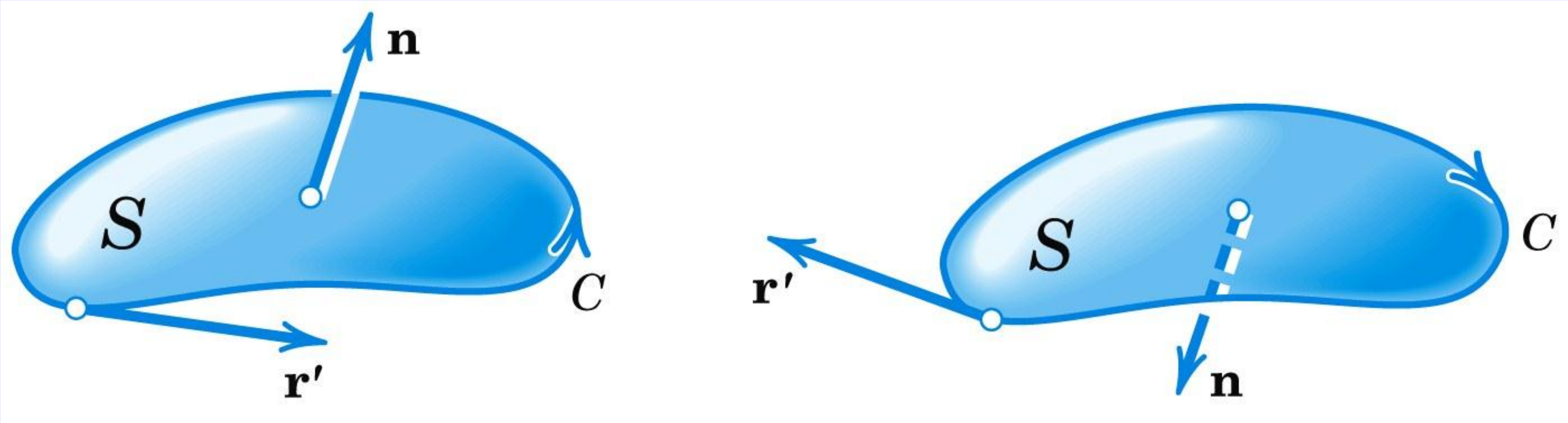

Fig. 254. Stokes's theorem

EXAMPLE 2

Green's Theorem in the Plane as a Special Case of Stokes's Theorem

Let $\mathbf{F} = [F_1, F_2] = F_1\,\mathbf{i} + F_2\,\mathbf{j}$ be a vector function that is continuously differentiable in a domain in the *xy*-plane containing a simply connected bounded closed region S whose boundary C is a piecewise smooth simple closed curve. Then, according to (1),

$$(\text{curl}\ \mathbf{F}) \bullet \mathbf{n} = (\text{curl}\ \mathbf{F}) \bullet \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

EXAMPLE 2 (continued)

Green's Theorem in the Plane as a Special Case of Stokes's Theorem

Hence the formula in Stokes's theorem now takes the form

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C \left(F_1 dx + F_2 dy \right).$$

This shows that Green's theorem in the plane (Sec. 10.4) is a special case of Stokes's theorem (which we needed in the proof of the latter!).

EXAMPLE 3

Evaluation of a Line Integral by Stokes's Theorem

Evaluate $\int_C \mathbf{F} \bullet \mathbf{r}' ds$, where C is the circle $x^2 + y^2 = 4$, $z = -3$, oriented counterclockwise as seen by a person standing at the origin, and, with respect to right-handed Cartesian coordinates,

$$\mathbf{F} = [\, y,\, xz^3,\, -zy^3] = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k}.$$

Solution. As a surface S bounded by C we can take the plane circular disk $x^2 + y^2 \le 4$ in the plane $z = -3$. Then $\mathbf{n}$ in Stokes's theorem points in the positive z-direction; thus $\mathbf{n} = \mathbf{k}$.

EXAMPLE 3 (continued)
Evaluation of a Line Integral by Stokes's Theorem

Solution. *(continued)*

Hence (curl **F**) • **n** is simply the component of curl **F** in the positive z-direction. Since **F** with $z = -3$ has the components $F_1 = y$, $F_2 = -27x$, $F_3 = 3y^3$, we thus obtain

$$(\text{curl}\,\mathbf{F}) \bullet \mathbf{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -27 - 1 = -28.$$

Hence the integral over S in Stokes's theorem equals –28 times the area 4π of the disk S. This yields the answer $-28 \cdot 4\pi = -112\pi \approx -352$. Confirm this by direct calculation, which involves somewhat more work.

EXAMPLE 4

Physical Meaning of the Curl in Fluid Motion. Circulation

Let S_{r_0} be a circular disk of radius r_0 and center P bounded by the circle C_{r_0} (Fig. 257), and let $\mathbf{F}(Q) \equiv \mathbf{F}(x, y, z)$ be a continuously differentiable vector function in a domain containing S_{r_0}.

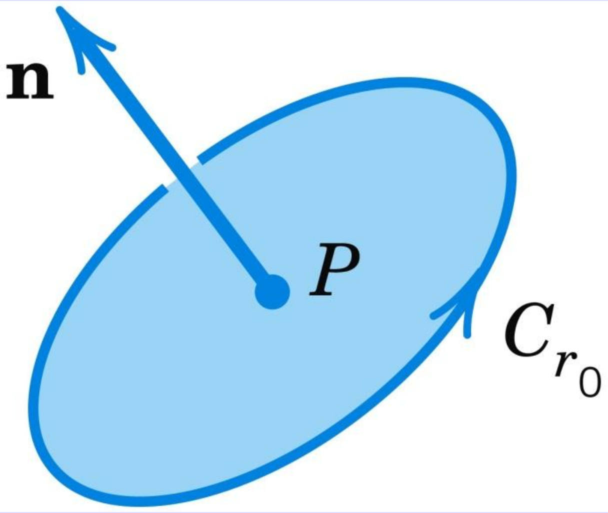

Fig. 257. Example 4

EXAMPLE 4 (continued 1)

Physical Meaning of the Curl in Fluid Motion. Circulation

Then by Stokes's theorem and the mean value theorem for surface integrals (see Sec. 10.6),

$$\oint_{C_{r_0}} \mathbf{F} \bullet \mathbf{r}'(s)\, ds = \iint_{S_{r_0}} (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dA = (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, (P^*) A_{r_0}$$

where A_{r_0} in the area of S_{r_0} and P^* is suitable point of S_{r_0}.

This may be written in the form

$$(\text{curl } \mathbf{F}) \bullet \mathbf{n}\, (P^*) = \frac{1}{A_{r_0}} \oint_{C_{r_0}} \mathbf{F} \bullet \mathbf{r}'\, ds$$

In the case of a fluid motion with velocity vector **F** = **v**, the integral

$$\oint_{C_{r_0}} \mathbf{v} \bullet \mathbf{r}'\, ds$$

is called the circulation of the flow around C_{r_0}.

EXAMPLE 4 (continued 2)

Physical Meaning of the Curl in Fluid Motion. Circulation

It measures the extent to which the corresponding fluid motion is a rotation around the circle C_{r_0}. If we now let r_0 approach zero, we find

$$(8) \qquad (\text{curl } \mathbf{v}) \bullet \mathbf{n}(P) = \lim_{r_0 \to 0} \frac{1}{A_{r_0}} \oint_{C_{r_0}} \mathbf{v} \bullet \mathbf{r}' \, ds;$$

that is, the component of the curl in the positive normal direction can be regarded as the **specific circulation** (circulation per unit area) of the flow in the surface at the corresponding point.

SUMMARY OF CHAPTER 10
Vector Integral Calculus. Integral Theorems

Vector Integral Calculus. Integral Theorems

Chapter 9 extended *differential* calculus to vectors, that is, to vector functions $\mathbf{v}(x, y, z)$ or $\mathbf{v}(t)$. Similarly, Chapter 10 extends *integral* calculus to vector functions. This involves *line integrals* (Sec. 10.1), *double integrals* (Sec. 10.3), *surface integrals* (Sec. 10.6), and *triple integrals* (Sec. 10.7) and the three "big" theorems for transforming these integrals into one another, the theorems of Green (Sec. 10.4), Gauss (Sec. 10.7), and Stokes (Sec. 10.9).

The analog of the definite integral of calculus is the **line integral** (Sec. 10.1)

$$(1)\quad \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz) = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\, dt$$

where C: $\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ $(a \le t \le b)$ is a curve in space (or in the plane). Physically, (1) may represent the work done by a (variable) force in a displacement. Other kinds of line integrals and their applications are also discussed in Sec. 10.1.

Independence of path of a line integral in a domain D means that the integral of a given function over any path C with endpoints P and Q has the same value for all paths from P to Q that lie in D; here P and Q are fixed. An integral (1) is independent of path in D if and only if the differential form $F_1\,dx + F_2\,dy + F_3\,dz$ with continuous F_1, F_2, F_3 is **exact** in D (Sec. 10.2). Also, if curl $\mathbf{F} = \mathbf{0}$, where $\mathbf{F} = [F_1, F_2, F_3]$, has continuous first partial derivatives in a *simply connected* domain D, then the integral (1) is independent of path in D (Sec. 10.2).

Integral Theorems. The formula of **Green's theorem in the plane** (Sec. 10.4)

$$(2) \qquad \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \oint_C \left(F_1 dx + F_2 dy \right)$$

transforms **double integrals** over a region R in the xy-plane into line integrals over the boundary curve C of R and conversely. For other forms of (2) see Sec. 10.4.

Similarly, the formula of the **divergence theorem of Gauss** (Sec. 10.7)

$$(3) \qquad \iiint_T \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \bullet \mathbf{n}\, dA$$

transforms **triple integrals** over a region T in space into surface integrals over the boundary surface S of T, and conversely. Formula (3) implies **Green's formulas**

$$\iiint_T \left(f\nabla^2 g + \nabla f \cdot \nabla g \right) dV = \iint_S f \frac{\partial g}{\partial n}\, dA, \tag{4}$$

$$\iiint_T \left(f\nabla^2 g - g\nabla^2 f \right) dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dA. \tag{5}$$

SUMMARY OF CHAPTER 10
Vector Integral Calculus. Integral Theorems

Finally, the formula of **Stokes's theorem** (Sec. 10.9)

$$\iint_S (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n} \, dA = \oint_C \mathbf{F} \bullet \mathbf{r}'(s) \, ds \qquad (6)$$

transforms **surface integrals** over a surface S into line integrals over the boundary curve C of S and conversely.