PART B

Linear Algebra. Vector Calculus

CHAPTER 9

Vector Differential Calculus. Grad, Div, Curl

9.1 Vectors in 2-Space and 3-Space

In engineering, physics, mathematics, and other areas we encounter two kinds of quantities. They are scalars and vectors.

A **scalar** is a quantity that is determined by its magnitude. It takes on a numerical value, i.e., a number. Examples of scalars are time, temperature, length, distance, speed, density, energy, and voltage.

In contrast, a **vector** is a quantity that has both magnitude and direction. We can say that a vector is an ***arrow*** or a ***directed line segment***. For example, a velocity vector has length or magnitude, which is speed, and direction, which indicates the direction of motion.

Typical examples of vectors are displacement, velocity, and force, see Fig. 164 as an illustration.

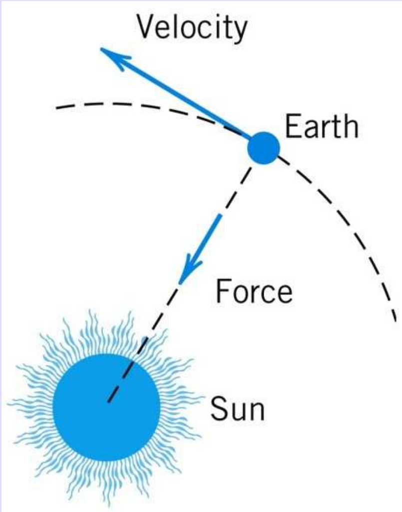


Fig. 164. Force and velocity

More formally, we have the following. We denote vectors by lowercase boldface letters **a**, **b**, **v**, etc. In handwriting you may use arrows, for instance, $\vec{a}$ (in place of **a**), $\vec{b}$, etc.

A vector (arrow) has a tail, called its **initial point**, and a tip, called its **terminal point**. This is motivated in the **translation** (displacement without rotation) of the triangle in Fig. 165, where the initial point P of the vector $\mathbf{a}$ is the original position of a point, and the terminal point Q is the terminal position of that point, its position *after* the translation. The length of the arrow equals the distance between P and Q. This is called the **length** (or *magnitude*) of the vector $\mathbf{a}$ and is denoted by $|\mathbf{a}|$. Another name for *length* is **norm** (or *Euclidean norm*).

A vector of length 1 is called a **unit vector**.

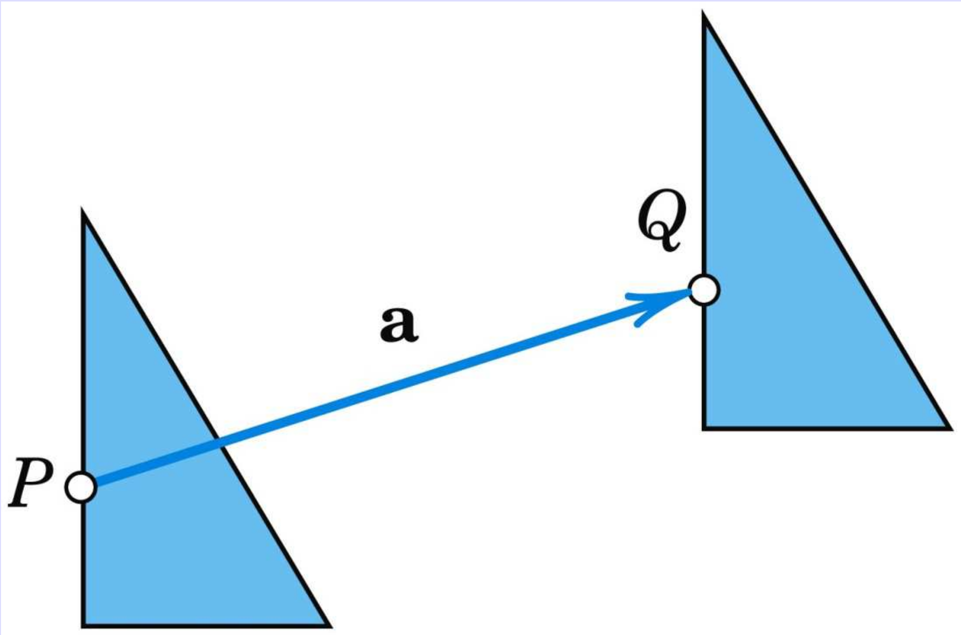


Fig. 165. Translation

Definition

Equality of Vectors

Two vectors **a** and **b** are equal, written **a** = **b**, if they have the same length and the same direction [as explained in Fig. 166; in particular, note (B)]. Hence a vector can be arbitrarily translated; that is, its initial point can be chosen arbitrarily.

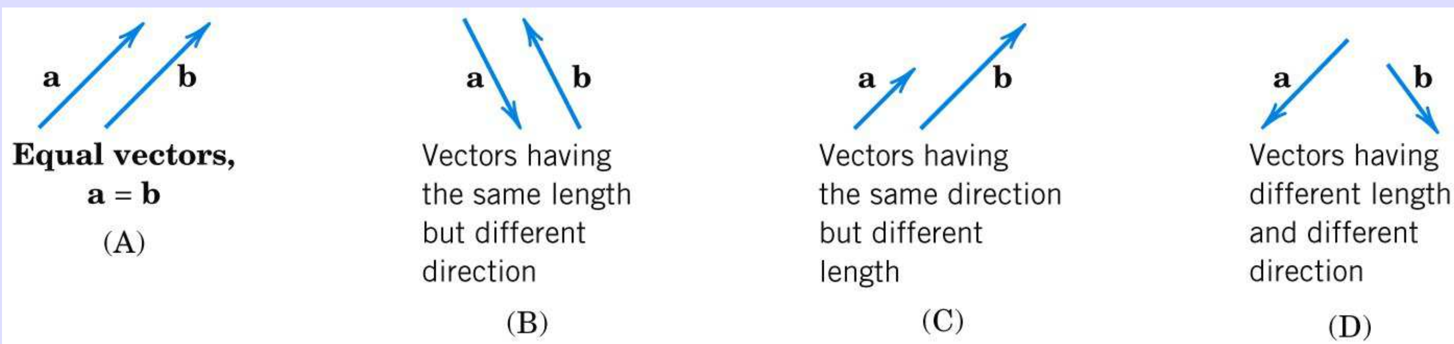


Fig. 166. (A) Equal vectors. (B)–(D) Different vectors

Components of a Vector

We choose an xyz **Cartesian coordinate system** in space (Fig. 167), that is, a usual rectangular coordinate system with the same scale of measurement on the three mutually perpendicular coordinate axes. Let **a** be a given vector with initial point P: (x_1, y_1, z_1) and terminal point Q: (x_2, y_2, z_2). Then the three coordinate differences

(1) $$a_1 = x_2 - x_1, \; a_2 = y_2 - y_1, \; a_3 = z_2 - z_1$$

are called the **components** of the vector **a** with respect to that coordinate system, and we write simply $\mathbf{a} = [a_1, a_2, a_3]$. See Fig. 168.

Components of a Vector (continued)

The **length** $|\mathbf{a}|$ of **a** can now readily be expressed in terms of components because from (1) and the Pythagorean theorem we have

(2) $$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

A Cartesian coordinate system being given, the **position vector r** of a point A: (x, y, z) is the vector with the origin (0, 0, 0) as the initial point and A as the terminal point (see Fig. 169). Thus in components, $\mathbf{r} = [x, y, z]$. This can be seen directly from (1) with $x_1 = y_1 = z_1 = 0$.

(See next slide for Figures 167, 168 and 169.)

Components of a Vector (continued)

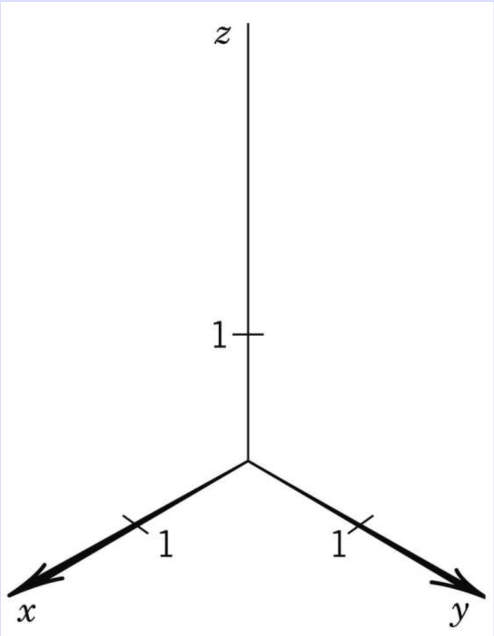


Fig. 167. Cartesian coordinate system

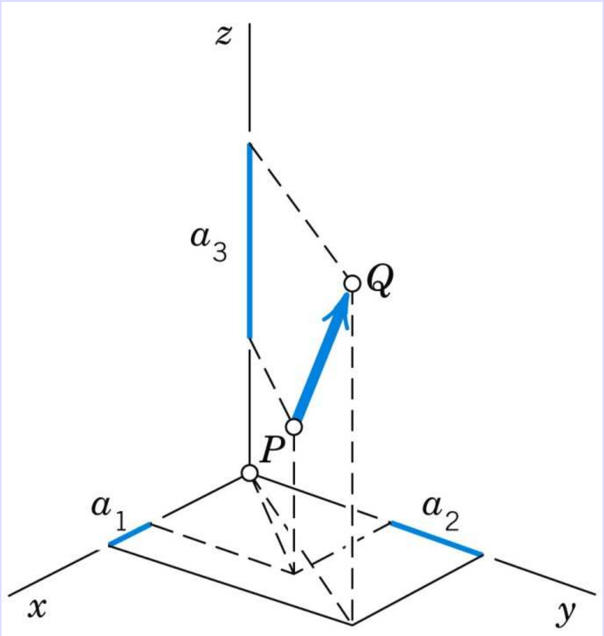


Fig. 168. Components of a vector

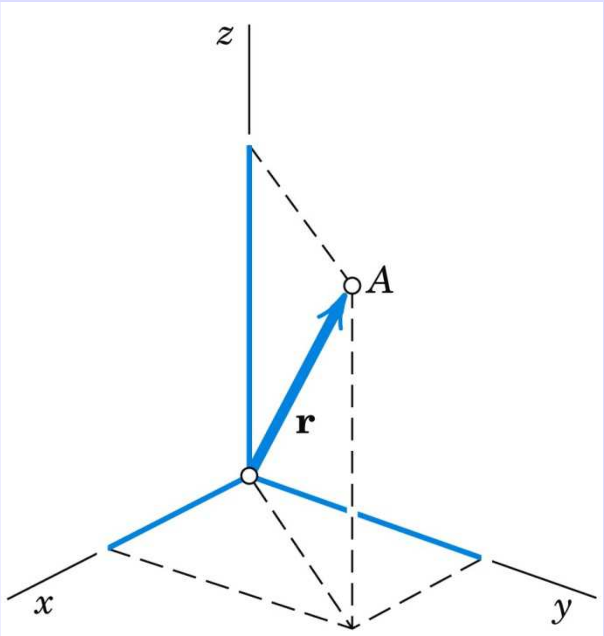


Fig. 169. Position vector **r** of a point A: (x, y, z)

Theorem 1

Vectors as Ordered Triples of Real Numbers

A fixed Cartesian coordinate system being given, each vector is uniquely determined by its ordered triple of corresponding components. Conversely, to each ordered triple of real numbers (a_1, a_2, a_3) *there corresponds precisely one vector* $\mathbf{a} = [a_1, a_2, a_3]$, *with* (0, 0, 0) *corresponding to the* **zero vector 0**, *which has length* 0 *and no direction.*

Hence a vector equation $\mathbf{a} = \mathbf{b}$ *is equivalent to the three equations* $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$ *for the components.*

Vector Addition, Scalar Multiplication

Definition

Addition of Vectors

The **sum a + b** of two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ is obtained by adding the corresponding components,

(3) $$\mathbf{a} + \mathbf{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3].$$

Geometrically, place the vectors as in Fig. 170 (the initial point of **b** at the terminal point of **a**); then **a** + **b** is the vector drawn from the initial point of **a** to the terminal point of **b**.

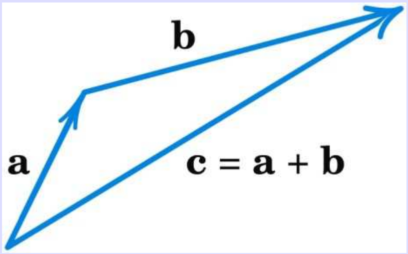


Fig. 170. Vector addition

For forces, this addition is the parallelogram law by which we obtain the **resultant** of two forces in mechanics. See Fig. 171.

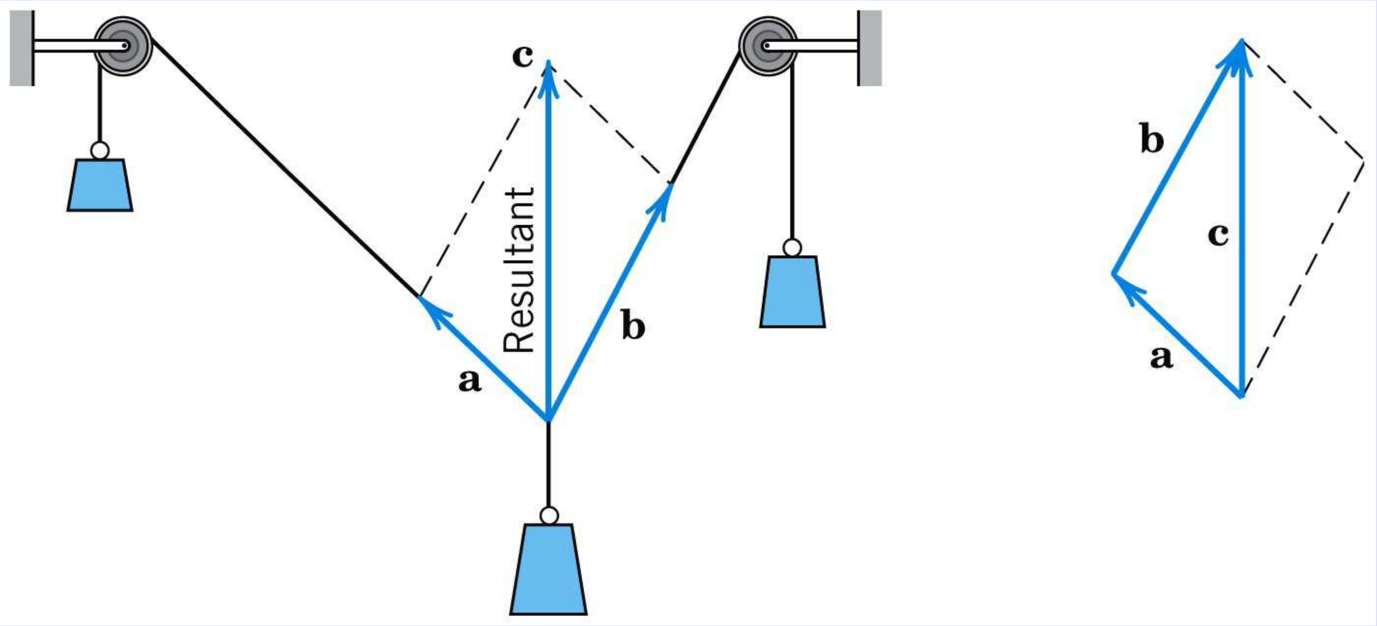


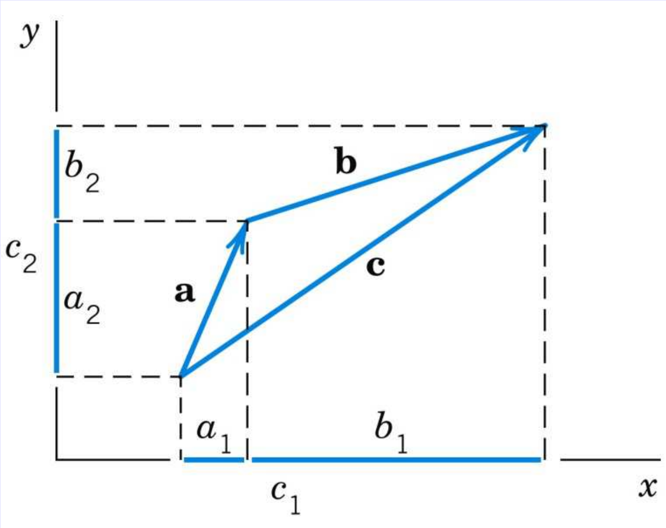
Fig. 171. Resultant of two forces (**parallelogram law**)

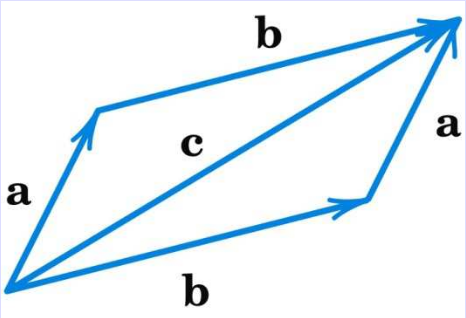
Figure 172 ***(next slide)*** shows (for the plane) that the "algebraic" way and the "geometric way" of vector addition give the same vector.

Basic Properties of Vector Addition. Familiar laws for real numbers give immediately

$$
\textbf{(4)}\quad
\begin{aligned}
&\text{(a)} && \mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} && (\textit{Commutativity})\\
&\text{(b)} && (\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}) && (\textit{Associativity})\\
&\text{(c)} && \mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a} && \\
&\text{(d)} && \mathbf{a}+(-\mathbf{a})=\mathbf{0}. &&
\end{aligned}
$$

Properties (a) and (b) are verified geometrically in Figs. 173 and 174 ***(next slide)*** . Furthermore, $-\mathbf{a}$ denotes the vector having the length $|\mathbf{a}|$ and the direction opposite to that of $\mathbf{a}$.





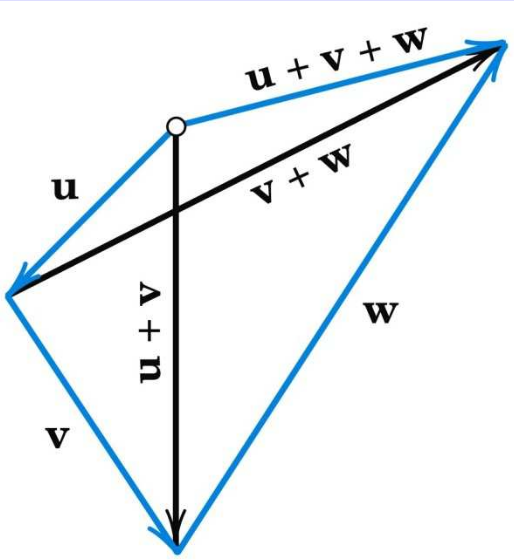


Fig. 172. Vector addition **Fig. 173.** Cummutativity of vector addition **Fig. 174.** Associativity of vector addition

In (4b) we may simply write $\mathbf{u} + \mathbf{v} + \mathbf{w}$, and similarly for sums of more than three vectors. Instead of $\mathbf{a} + \mathbf{a}$ we also write $2\mathbf{a}$, and so on. This (and the notation $-\mathbf{a}$ used just before) motivates defining the second algebraic operation for vectors as follows.

Definition

Scalar Multiplication (Multiplication by a Number)

The product $c\mathbf{a}$ of any vector $\mathbf{a} = [a_1, a_2, a_3]$ and any scalar c (real number c) is the vector obtained by multiplying each component of $\mathbf{a}$ by c,

(5) $$c\mathbf{a} = [ca_1, ca_2, ca_3]$$

Geometrically, if $\mathbf{a} \neq \mathbf{0}$, then $c\mathbf{a}$ with $c > 0$ has the direction of $\mathbf{a}$ and with $c < 0$ the direction opposite to $\mathbf{a}$. In any case, the length of $c\mathbf{a}$ is $|c\mathbf{a}| = |c|\,|\mathbf{a}|$ and $c\mathbf{a} = \mathbf{0}$ if $\mathbf{a} = \mathbf{0}$ or $c = 0$ (or both). (See Fig. 175.)

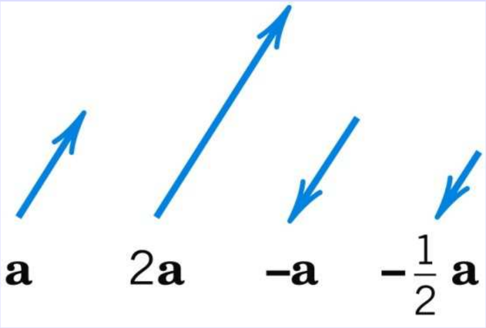


Fig. 175. Scalar multiplication [multiplication of vectors by scalars (numbers)]

Basic Properties of Scalar Multiplication. From the definitions we obtain directly

$$
\textbf{(6)}\qquad
\begin{aligned}
&\text{(a)}\quad c(\mathbf{a}+\mathbf{b}) = c\mathbf{a}+c\mathbf{b}\\
&\text{(b)}\quad (c+k)\mathbf{a} = c\mathbf{a}+k\mathbf{a}\\
&\text{(c)}\quad c(k\mathbf{a}) = (ck)\mathbf{a} \qquad \text{(written } ck\mathbf{a}\text{)}\\
&\text{(d)}\quad 1\mathbf{a} = \mathbf{a}.
\end{aligned}
$$

You may prove that (4) and (6) imply for any vector **a**

$$
\textbf{(7)} \qquad \begin{aligned} &\text{(a)} \qquad 0\mathbf{a} = \mathbf{0} \\ &\text{(b)} \quad (-1)\mathbf{a} = -\mathbf{a} \end{aligned}
$$

Instead of $\mathbf{b} + (-\mathbf{a})$ we simply write $\mathbf{b} - \mathbf{a}$ (Fig. 176).

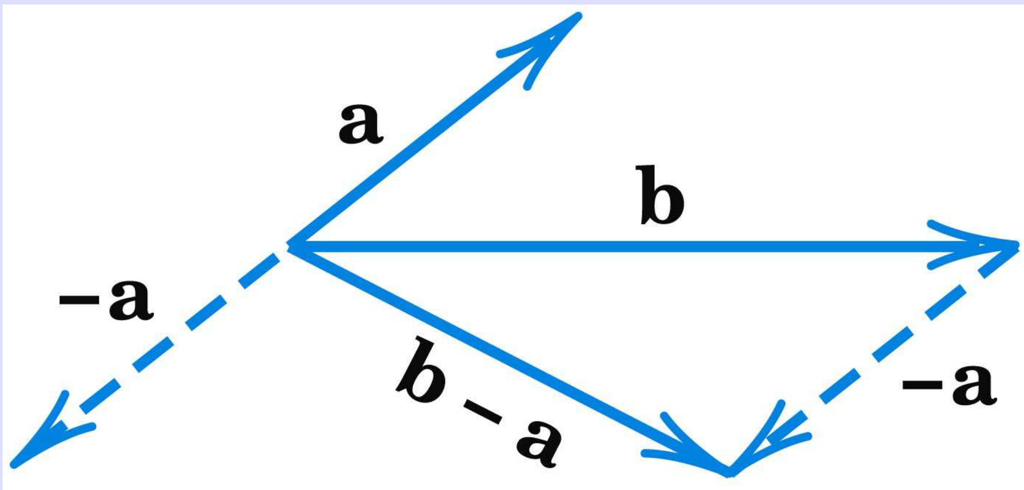


Fig. 176. Difference of vectors

Unit Vectors i, j, k. Besides $\mathbf{a} = [a_1, a_2, a_3]$ another popular way of writing vectors is

(8) $$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

In this representation, **i**, **j**, **k** are the unit vectors in the positive directions of the axes of a Cartesian coordinate system (Fig. 177). Hence, in components,

(9) $$\mathbf{i} = [1, 0, 0], \quad \mathbf{j} = [0, 1, 0], \quad \mathbf{k} = [0, 0, 1]$$

and the right side of (8) is a sum of three vectors parallel to the three axes.

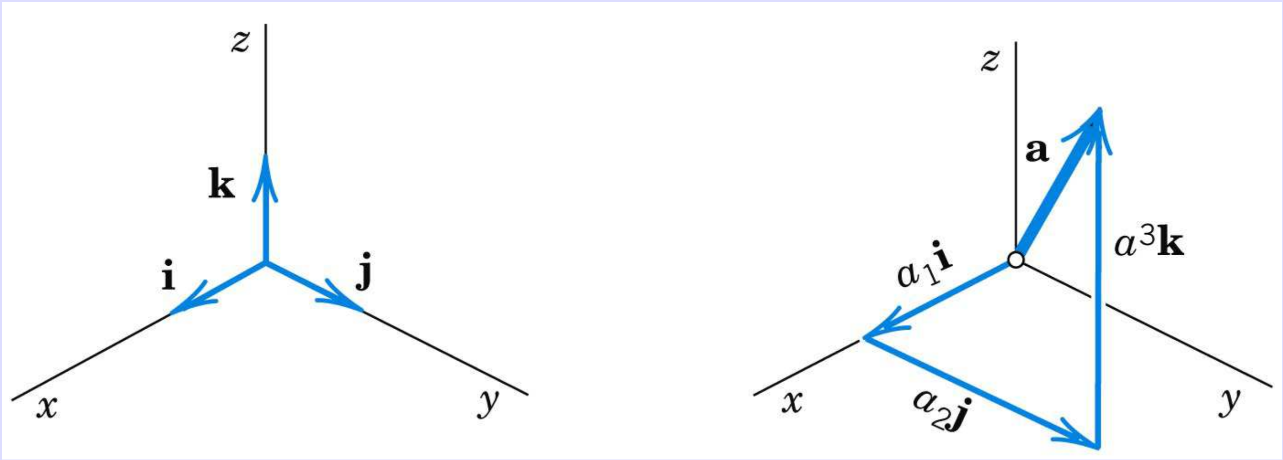


Fig. 177. The unit vectors i, j, k and the representation (8)

All the vectors **a** = $[a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ (with real numbers as components) form the **real vector space** R^3 with the two *algebraic operations* of vector addition and scalar multiplication as just defined. R^3 has **dimension** 3. The triple of vectors **i, j, k** is called a **standard basis** of R^3. Given a Cartesian coordinate system, the representation (8) of a given vector is unique.

9.2 Inner Product (Dot Product)

Orthogonality

The inner product or dot product can be motivated by calculating work done by a constant force, determining components of forces, or other applications. It involves the length of vectors and the angle between them. The inner product is a kind of multiplication of two vectors, defined in such a way that the outcome is a scalar. Indeed, another term for inner product is scalar product, a term we shall not use here. The definition of the inner product is as follows.

Definition

Inner Product (Dot Product) of Vectors

The **inner product** or **dot product** $\mathbf{a} \cdot \mathbf{b}$ (read "**a** dot **b**") of two vectors **a** and **b** is the product of their lengths times the cosine of their angle (see Fig. 178),

(1)
$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos \gamma && \text{if} \quad \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0} \\ \mathbf{a} \cdot \mathbf{b} &= 0 && \text{if} \quad \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0}. \end{aligned}$$

The angle γ, $0 \leq \gamma \leq \pi$, between **a** and **b** is measured when the initial points of the vectors coincide, as in Fig. 178. In components, $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and

(2)
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The second line in (1) is needed because γ is undefined when $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

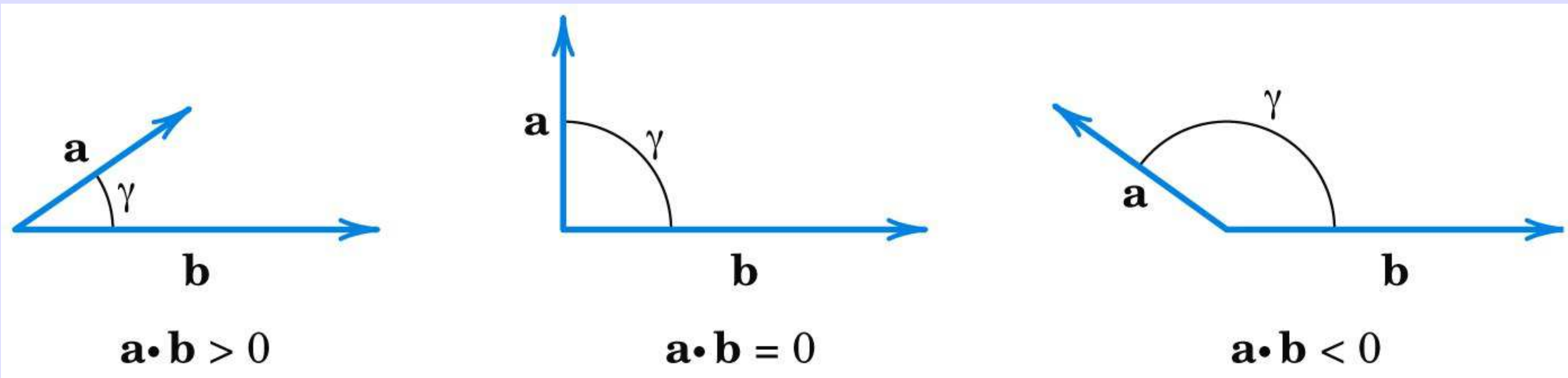


Fig. 178. Angle between vectors and value of inner product

Orthogonality.

Since the cosine in (1) may be positive, 0, or negative, so may be the inner product (Fig. 178, *previous slide*). The case that the inner product is zero is of particular practical interest and suggests the following concept.

A vector **a** is called **orthogonal** to a vector **b** if **a** • **b** = 0. Then **b** is also orthogonal to **a**, and we call **a** and **b** **orthogonal vectors**. Clearly, this happens for nonzero vectors if and only if cos $\gamma = 0$; thus $\gamma = \pi/2$ (90°). This proves the important following theorem.

Theorem 1

Orthogonality Criterion

The inner product of two nonzero vectors is 0 *if and only if these vectors are perpendicular.*

Length and Angle.

Equation (1) with $\mathbf{b} = \mathbf{a}$ gives $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2$. Hence

(3) $$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

From (3) and (1) we obtain for the angle between two nonzero vectors

(4) $$\cos \gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}}\sqrt{\mathbf{b} \cdot \mathbf{b}}}.$$

From the definition we see that the inner product has the following properties. For any vectors **a**, **b**, **c** and scalars q_1, q_2,

$$
\textbf{(5)}\quad
\begin{array}{lcr}
\text{(a)} & (q_1\mathbf{a} + q_2\mathbf{b}) \cdot \mathbf{c} = q_1\mathbf{a} \cdot \mathbf{c} + q_1\mathbf{b} \cdot \mathbf{c} & \textit{(Linearity)} \\
\text{(b)} & \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} & \textit{(Symmetry)} \\
\text{(c)} & \left.\begin{array}{c} \mathbf{a} \cdot \mathbf{a} \geqq 0 \\ \mathbf{a} \cdot \mathbf{a} = 0 \quad \text{if and only if} \quad \mathbf{a} = \mathbf{0} \end{array}\right\} & \textit{(Positive-definiteness)}.
\end{array}
$$

Hence *dot multiplication is commutative* as shown by (5b). Furthermore, it is *distributive with respect to vector addition*. This follows from (5a) with $q_1 = 1$ and $q_2 = 1$:

(5a*) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$ (*Distributivity*).

Furthermore, from (1) and $|\cos \gamma| \leq 1$ we see that

(6) $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$ (*Cauchy–Schwarz inequality*).

Using this and (3), you may prove (see Prob. 16)

(7) $$|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$$ **(Triangle inequality)**.

Geometrically, (7) with < says that one side of a triangle must be shorter than the other two sides together; this motivates the name of (7).

A simple direct calculation with inner products shows that

(8) $$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$$ (*Parallelogram equality*).

Applications of Inner Products

EXAMPLE 2

Work Done by a Force Expressed as an Inner Product

This is a major application. It concerns a body on which a *constant* force $\mathbf{p}$ acts. (For a *variable* force, see Sec. 10.1.) Let the body be given a displacement $\mathbf{d}$. Then the work done by $\mathbf{p}$ in the displacement is defined as

(9) $$W = |\mathbf{p}||\mathbf{d}| \cos \alpha = \mathbf{p} \bullet \mathbf{d},$$

that is, magnitude $|\mathbf{p}|$ of the force times length $|\mathbf{d}|$ of the displacement times the cosine of the angle α between $\mathbf{p}$ and $\mathbf{d}$ (Fig. 179).

EXAMPLE 2 (continued)
Work Done by a Force Expressed as an Inner Product

(continued) If $\alpha < 90°$, as in Fig. 179, then $W > 0$. If **p** and **d** are orthogonal, then the work is zero (why?). If $\alpha > 90°$, then $W < 0$, which means that in the displacement one has to do work against the force. For example, think of swimming across a river at some angle α against the current.

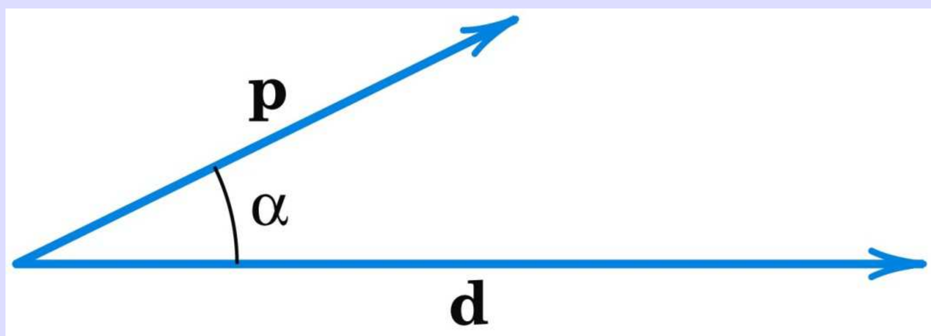


Fig. 179. Work done by a force

EXAMPLE 3

Component of a Force in a Given Direction

What force in the rope in Fig. 180 will hold a car of 5000 lb in equilibrium if the ramp makes an angle of 25° with the horizontal?

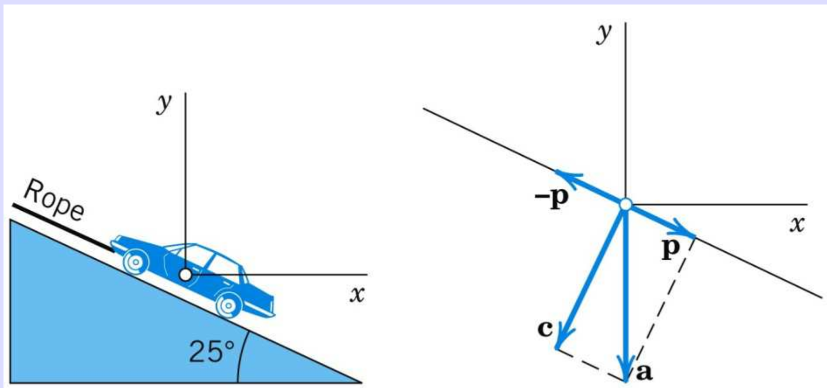


Fig. 180. Example 3

Solution. Introducing coordinates as shown, the weight is $\mathbf{a} = [0, -5000]$ because this force points downward, in the negative y-direction.

EXAMPLE 3 (continued)
Component of a Force in a Given Direction

Solution. *(continued 1)*

We have to represent $\mathbf{a}$ as a sum (resultant) of two forces, $\mathbf{a} = \mathbf{c} + \mathbf{p}$, where $\mathbf{c}$ is the force the car exerts on the ramp, which is of no interest to us, and $\mathbf{p}$ is parallel to the rope. A vector in the direction of the rope is (see Fig. 180)

$$\mathbf{b} = [-1, \tan 25°] = [-1, 0.46631], \text{ thus } |\mathbf{b}| = 1.10338,$$

The direction of the unit vector $\mathbf{u}$ is opposite to the direction of the rope so that

$$\mathbf{u} = -\frac{1}{|\mathbf{b}|}\mathbf{b} = [0.90631, \ -0.42262].$$

EXAMPLE 3 (continued)
Component of a Force in a Given Direction

Solution. *(continued 2)*

Since $|\mathbf{u}| = 1$ and $\cos\gamma > 0$, we see that we can write our result as

$$|\mathbf{p}| = (|\mathbf{a}|\cos\gamma)|\mathbf{u}| = \mathbf{a} \bullet \mathbf{u} = -\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|} = \frac{5000 \cdot 0.46631}{1.10338} = 2113 \text{ [lb]}.$$

We can also note that $\gamma = 90° - 25° = 65°$ is the angle between **a** and **p** so that

$$|\mathbf{p}| = |\mathbf{a}|\cos\gamma = 5000\cos 65° = 2113 \text{ [lb]}.$$

Answer: About 2100 lb.

Example 3 is typical of applications that deal with the **component** or **projection** *of a vector* **a** *in the direction of a vector* **b**$(\neq \mathbf{0})$. If we denote by p the length of the orthogonal projection of **a** on a straight line l parallel to **b** as shown in Fig. 181, then

$$p = |\mathbf{a}| \cos \gamma. \tag{10}$$

Here p is taken with the plus sign if $p\mathbf{b}$ has the direction of **b** and with the minus sign if $p\mathbf{b}$ has the direction opposite to **b**.

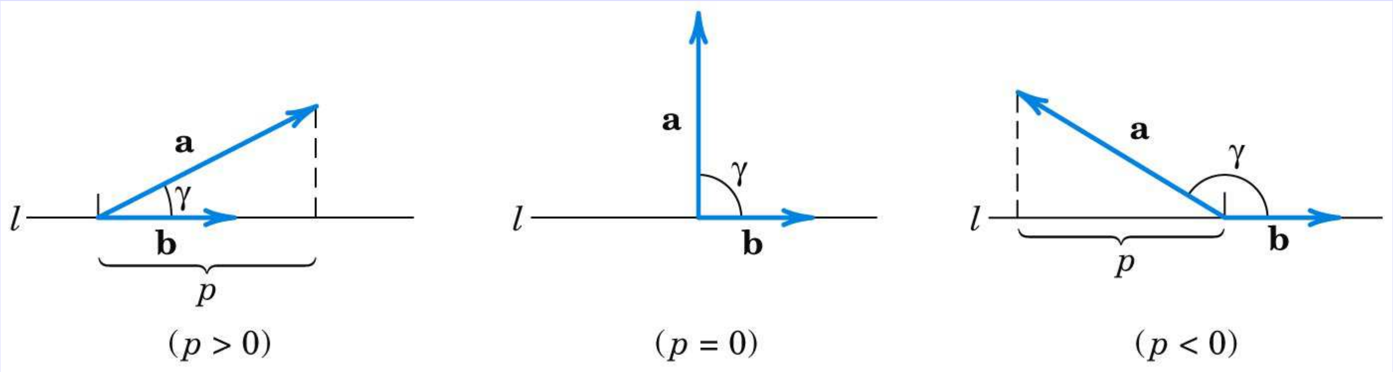


Fig. 181. Component of a vector **a** in the direction of a vector **b**

Multiplying (10) by $|\mathbf{b}|/|\mathbf{b}| = 1$, we have $\mathbf{a} \cdot \mathbf{b}$ in the numerator and thus

(11) $$p = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \qquad (\mathbf{b} \neq \mathbf{0}).$$

If $\mathbf{b}$ is a unit vector, as it is often used for fixing a direction, then (11) simply gives

(12) $$p = \mathbf{a} \cdot \mathbf{b} \qquad (|\mathbf{b}| = 1).$$

Figure 182 shows the projection p of $\mathbf{a}$ in the direction of $\mathbf{b}$ (as in Fig. 181) and the projection $q = |\mathbf{b}|\cos\gamma$ of $\mathbf{b}$ in the direction of $\mathbf{a}$.

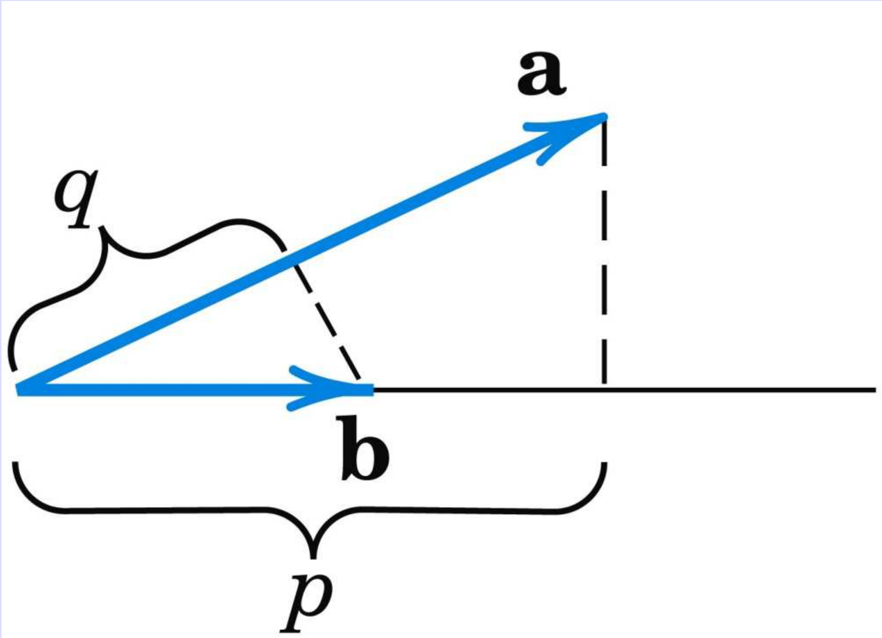


Fig. 182. Projections p of $\mathbf{a}$ on $\mathbf{b}$ and q of $\mathbf{b}$ on $\mathbf{a}$

EXAMPLE 4 Orthonormal Basis

By definition, an *orthonormal basis* for 3-space is a basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ consisting of orthogonal unit vectors. It has the great advantage that the determination of the coefficients in representations $\mathbf{v} = l_1\mathbf{a} + l_2\mathbf{b} + l_3\mathbf{c}$ of a given vector $\mathbf{v}$ is very simple. We claim that $l_1 = \mathbf{a} \cdot \mathbf{v}$, $l_2 = \mathbf{b} \cdot \mathbf{v}$, $l_3 = \mathbf{c} \cdot \mathbf{v}$. Indeed, this follows simply by taking the inner products of the representation with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, respectively, and using the orthonormality of the basis,

$$\mathbf{a} \cdot \mathbf{v} = l_1\mathbf{a} \cdot \mathbf{a} + l_2\mathbf{a} \cdot \mathbf{b} + l_3\mathbf{a} \cdot \mathbf{c} = l_1, \text{ etc.}$$

For example, the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in (8), Sec. 9.1, associated with a Cartesian coordinate system form an orthonormal basis, called the **standard basis** with respect to the given coordinate system.

9.3 Vector Product (Cross Product)

Definition

Vector Product (Cross Product, Outer Product) of Vectors

The **vector product** or **cross product** $\mathbf{a} \times \mathbf{b}$ (read "**a** cross **b**") of two vectors **a** and **b** is the vector **v** denoted by

$$\mathbf{v} = \mathbf{a} \times \mathbf{b}$$

I. If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then we define $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

II. If both vectors are nonzero vectors, then vector **v** has the length

$$|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin \gamma, \tag{1}$$

where γ is the angle between **a** and **b** as in Sec. 9.2.

Definition (continued)

Vector Product (Cross Product, Outer Product) of Vectors (continued 1)

Furthermore, by design, **a** and **b** form the sides of a parallelogram on a plane in space. The parallelogram is shaded in blue in Fig. 185. The area of this blue parallelogram is precisely given by Eq. (1), so that the length $|\mathbf{v}|$ of the vector **v** is equal to the area of that parallelogram.

III. If **a** and **b** lie in the same straight line, i.e., **a** and **b** have the same or opposite directions, then γ is 0° or 180° so that $\sin\gamma = 0$. In that case $|\mathbf{v}| = 0$ so that $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Definition (continued)

Vector Product (Cross Product, Outer Product) of Vectors (continued 2)

IV. If cases I and III do not occur, then **v** is a nonzero vector. The direction of $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b** such that **a**, **b**, **v**—precisely in this order (!)—form a right-handed triple as shown in Figs. 185–187 and explained below.

Another term for vector product is outer product.

Remark. Note that I and III completely characterize the exceptional case when the cross product is equal to the zero vector, and II and IV the regular case where the cross product is perpendicular to two vectors.

Just as we did with the dot product, we would also like to express the cross product in components. Let $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$. Then $\mathbf{v} = [v_1, v_2, v_3] = \mathbf{a} \times \mathbf{b}$ has the components

(2) $$v_1 = a_2b_3 - a_3b_2, \quad v_2 = a_3b_1 - a_1b_3, \quad v_3 = a_1b_2 - a_2b_1.$$

Here the Cartesian coordinate system is *right-handed*, as explained below (see also Fig. 188). (For a left-handed system, each component of $\mathbf{v}$ must be multiplied by -1. Derivation of (2) in App. 4.)

Right-Handed Triple.
A triple of vectors **a**, **b**, **v** is *right-handed* if the vectors in the given order assume the same sort of orientation as the thumb, index finger, and middle finger of the right hand when these are held as in Fig. 186. We may also say that if **a** is rotated into the direction of **b** through the angle $\gamma (< \pi)$, then **v** advances in the same direction as a right-handed screw would if turned in the same way (Fig. 187).

Right-Handed Triple. (continued)

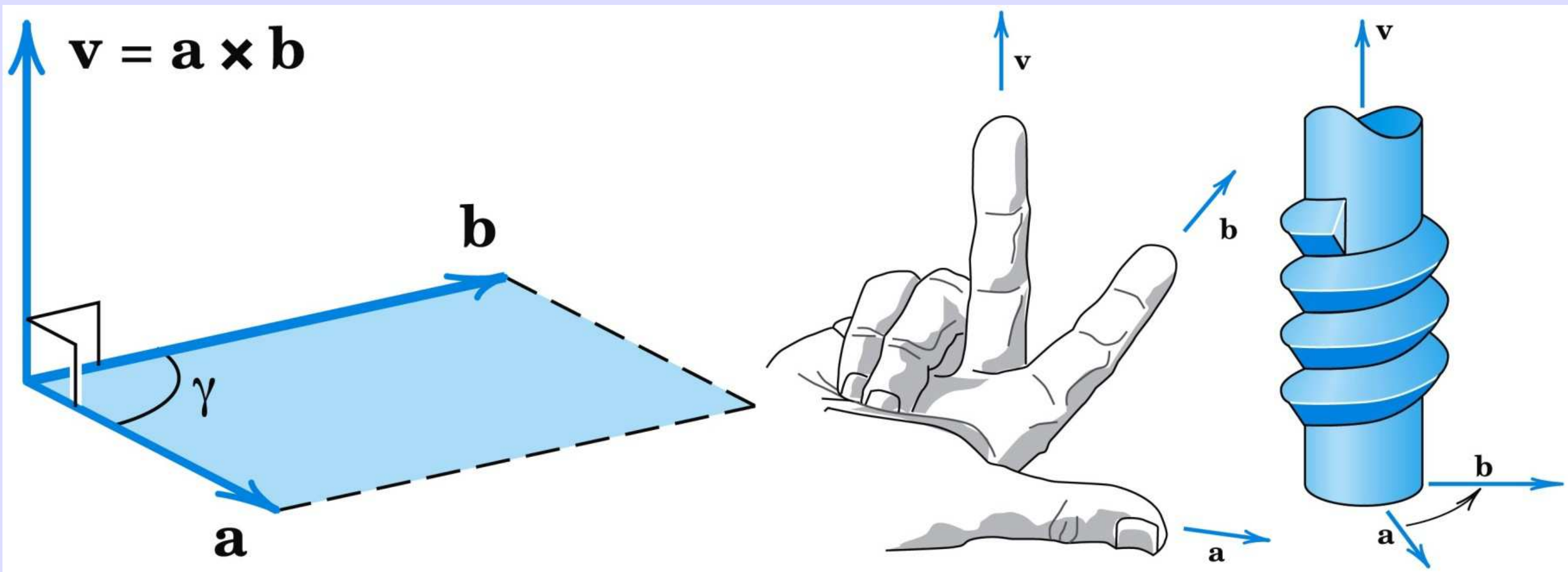


Fig. 185. Vector product

Fig. 186. Right-handed triple of vectors **a, b, v**

Fig. 187. Right-handed screw

Right-Handed Cartesian Coordinate System.

The system is called **right-handed** if the corresponding unit vectors **i**, **j**, **k** in the positive directions of the axes (see Sec. 9.1) form a right-handed triple as in Fig. 188a. The system is called **left-handed** if the sense of **k** is reversed, as in Fig. 188b. In applications, we prefer right-handed systems.

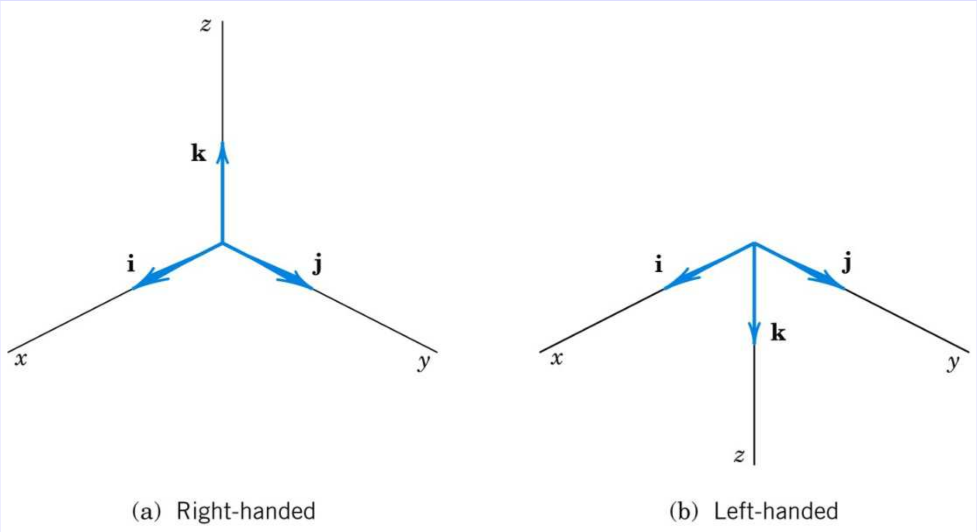


Fig. 188. The two types of Cartesian coordinate systems

How to Memorize (2).

If you know second- and third-order determinants, you see that (2) can be written

$$(2^*)\quad v_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad v_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = +\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \qquad v_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

and $\mathbf{v} = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k}$ is the expansion of the following symbolic determinant by its first row. (We call the determinant "symbolic" because the first row consists of vectors rather than of numbers.)

$$(2^{**})\qquad \mathbf{v} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.$$

For a left-handed system the determinant has a minus sign in front.

EXAMPLE 2

Vector Products of the Standard Basis Vectors

$$
\begin{aligned}
&\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j} \\
&\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.
\end{aligned} \tag{3}
$$

Theorem 1

General Properties of Vector Products

(a) *For every scalar l,*

$$(l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}). \tag{4}$$

(b) *Cross multiplication is distributive with respect to vector addition; that is,*

$$\begin{aligned} &(\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}), \\ &(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}). \end{aligned} \tag{5}$$

(c) *Cross multiplication is* ***not commutative*** *but* ***anticommutative;*** *that is,*

$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \qquad \text{(Fig. 189)}. \tag{6}$$

Theorem 1 (continued)

General Properties of Vector Products (continued)

(d) *Cross multiplication is* ***not associative****; that is, in general,*

(7) $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

so that the parentheses cannot be omitted.

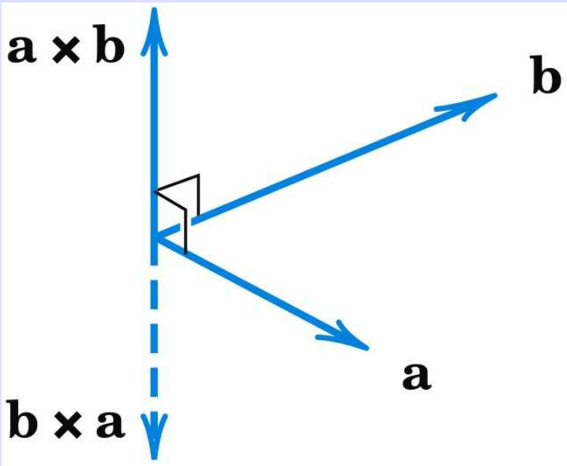


Fig. 189. Anticommutativity of cross multiplication

EXAMPLE 3 Moment of a Force

In mechanics the moment m of a force $\mathbf{p}$ about a point Q is defined as the product $m = |\mathbf{p}|d$, where d is the (perpendicular) distance between Q and the line of action L of $\mathbf{p}$ (Fig. 190). If $\mathbf{r}$ is the vector from Q to any point A on L, then $d = |\mathbf{r}| \sin \gamma$, as shown in Fig. 190, and

$$m = |\mathbf{r}||\mathbf{p}| \sin \gamma.$$

Since γ is the angle between $\mathbf{r}$ and $\mathbf{p}$, we see from (1) that $m = |\mathbf{r} \times \mathbf{p}|$.

EXAMPLE 3 (continued) Moment of a Force

The vector

$$m = \mathbf{r} \times \mathbf{p} \tag{8}$$

is called the **moment vector** or **vector moment** of **p** about Q. Its magnitude is m. If $\mathbf{m} \neq \mathbf{0}$, its direction is that of the axis of the rotation about Q that **p** has the tendency to produce. This axis is perpendicular to both **r** and **p**.

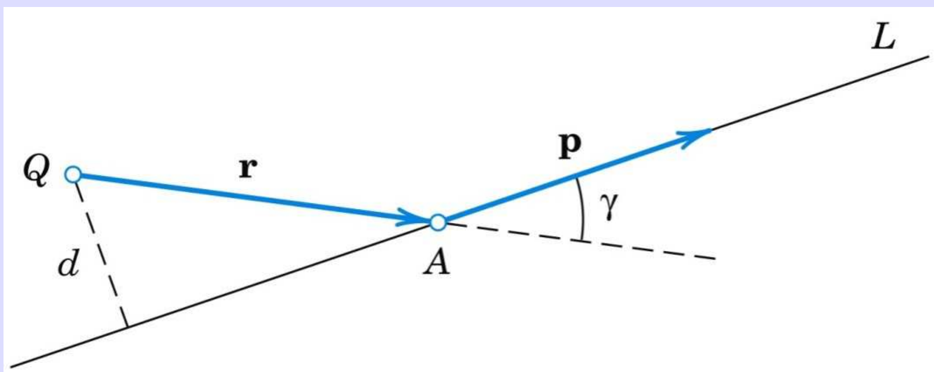


Fig. 190. Moment of a force **p**

EXAMPLE 5 Velocity of a Rotating Body

A rotation of a rigid body B in space can be simply and uniquely described by a vector $\mathbf{w}$ as follows. The direction of $\mathbf{w}$ is that of the axis of rotation and such that the rotation appears clockwise if one looks from the initial point of $\mathbf{w}$ to its terminal point. The length of $\mathbf{w}$ is equal to the **angular speed** $\omega(>0)$ of the rotation, that is, the linear (or tangential) speed of a point of B divided by its distance from the axis of rotation.

Let P be any point of B and d its distance from the axis. Then P has the speed ωd. Let $\mathbf{r}$ be the position vector of P referred to a coordinate system with origin 0 on the axis of rotation.

EXAMPLE 5 (continued)
Velocity of a Rotating Body

(continued 1)

Then $d = |\mathbf{r}| \sin \gamma$, where γ is the angle between $\mathbf{w}$ and $\mathbf{r}$. Therefore,

$$\omega d = |\mathbf{w}|\,|\mathbf{r}| \sin \gamma = |\mathbf{w} \times \mathbf{r}|.$$

From this and the definition of vector product we see that the velocity vector $\mathbf{v}$ of P can be represented in the form (Fig. 192)

(9) $$\mathbf{v} = \mathbf{w} \times \mathbf{r}.$$

This simple formula is useful for determining $\mathbf{v}$ at any point of B.

EXAMPLE 5 (continued)
Velocity of a Rotating Body

(continued 2)

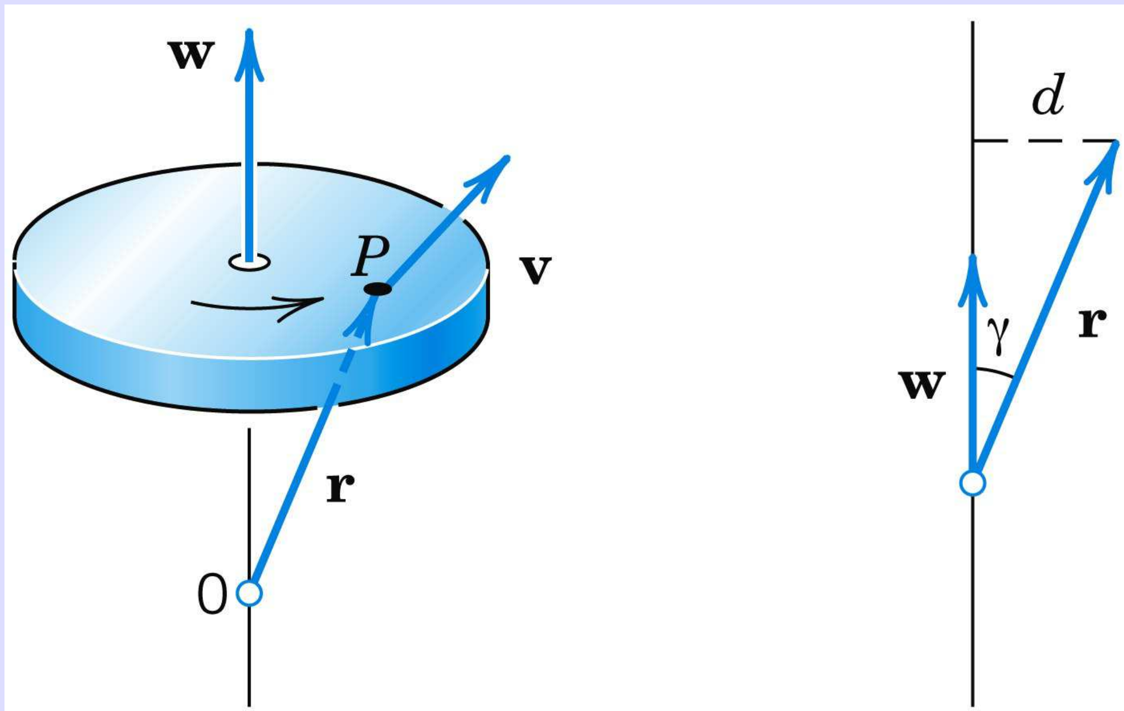


Fig. 192. Rotation of a rigid body

Scalar Triple Product

Certain products of vectors, having three or more factors, occur in applications. The most important of these products is the scalar triple product or mixed product of three vectors **a**, **b**, **c**.

(10*) $$(\mathbf{a}\quad \mathbf{b}\quad \mathbf{c}) = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}).$$

The scalar triple product is indeed a scalar since (10*) involves a dot product, which in turn is a scalar. We want to express the scalar triple product in components and as a third order determinant. To this end, let $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and $\mathbf{c} = [c_1, c_2, c_3]$.

Scalar Triple Product *(continued 1)*

Also set $(\mathbf{b} \times \mathbf{c}) = \mathbf{v} = [v_1, v_2, v_3]$. Then from the dot product in components [formula (2) in Sec. 9.2] and from (2*) with **b** and **c** instead of **a** and **b** we first obtain

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \bullet \mathbf{v} = a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Scalar Triple Product *(continued 2)*

The sum on the right is the expansion of a third-order determinant by its first row. Thus we obtain the desired formula for the scalar triple product, that is,

(10)
$$(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The most important properties of the scalar triple product are as follows.

Theorem 2

Properties and Applications of Scalar Triple Products

(a) *In* (10) *the dot and cross can be interchanged:*

(11) $$(\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}) = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}.$$

(b) Geometric interpretation. *The absolute value* $|(\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c})|$ *of* (10) *is the volume of the parallelepiped* (oblique box) *with* **a**, **b**, **c** *as edge vectors* (Fig. 193).

(c) Linear independence. *Three vectors in* R^3 *are linearly independent if and only if their scalar triple product is not zero.*

PROOF

(a) Dot multiplication is commutative, so that by (10)

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

From this we obtain the determinant in (10) by interchanging Rows 1 and 2 and in the result Rows 2 and 3. But this does not change the value of the determinant because each interchange produces a factor −1, and (−1)(−1) = 1. This proves (11).

(b) The volume of that box equals the height $h = |\mathbf{a}|\,|\cos\gamma|$ (Fig. 193) times the area of the base, which is the area $|\mathbf{b} \times \mathbf{c}|$ of the parallelogram with sides **b** and **c**. Hence the volume is

$$|\mathbf{a}|\,|\mathbf{b} \times \mathbf{c}|\,|\cos\gamma| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \qquad \text{(Fig. 193)}$$

as given by the absolute value of (11).

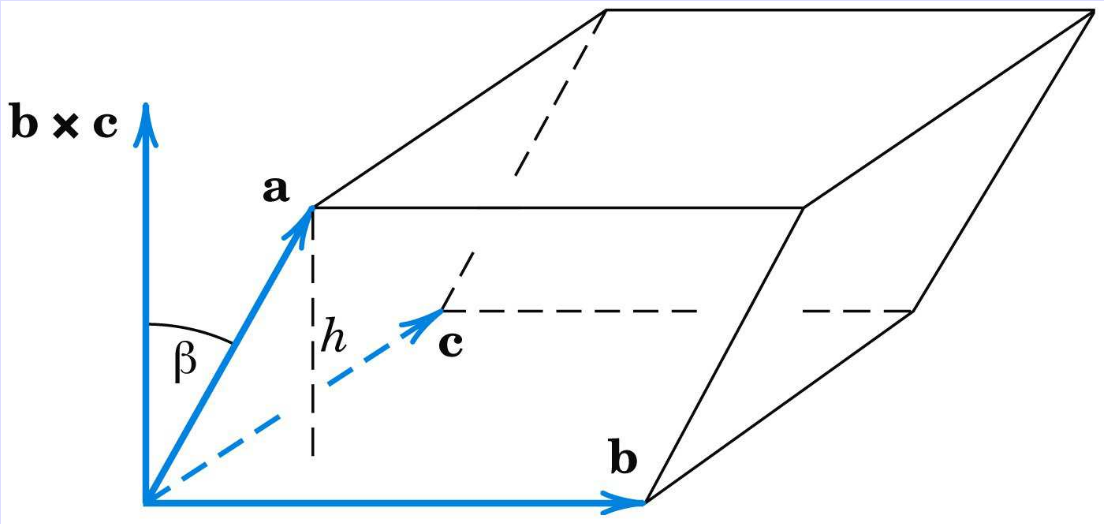


Fig. 193. Geometric interpretation of a scalar triple product

(c) Three nonzero vectors, whose initial points coincide, are linearly independent if and only if the vectors do not lie in the same plane nor lie on the same straight line.

This happens if and only if the triple product in (b) is not zero, so that the independence criterion follows. (The case of one of the vectors being the zero vector is trivial.)

9.4 Vector and Scalar Functions and Their Fields. Vector Calculus: Derivatives

Our discussion of vector calculus begins with identifying the two types of functions on which it operates. Let P be any point in a domain of definition. Typical domains in applications are three-dimensional, or a surface or a curve in space. Then we define a **vector function v**, whose values are vectors, that is,

$$\mathbf{v} = \mathbf{v}(P) = [v_1(P), v_2(P), v_3(P)]$$

that depends on points P in space. We say that a vector function defines a **vector field** in a domain of definition. Typical domains were just mentioned.

Examples of vector fields are the field of tangent vectors of a curve (shown in Fig. 195), normal vectors of a surface (Fig. 196), and velocity field of a rotating body (Fig. 197). Note that vector functions may also depend on time t or on some other parameters.

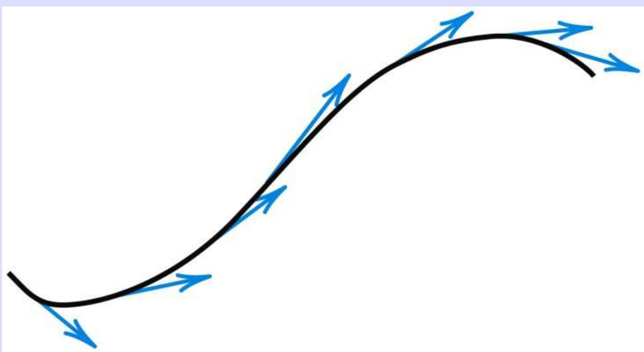


Fig. 195. Field of tangent vectors of a curve

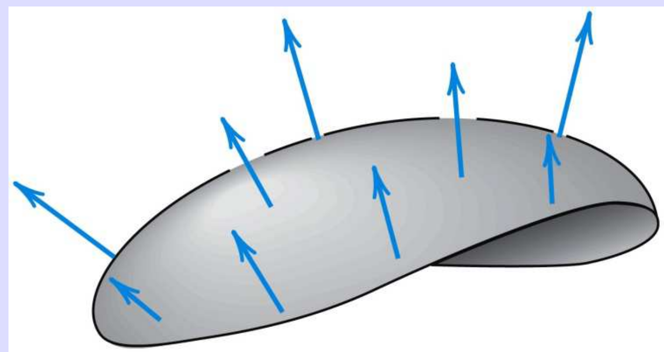


Fig. 196. Field of normal vectors of a surface

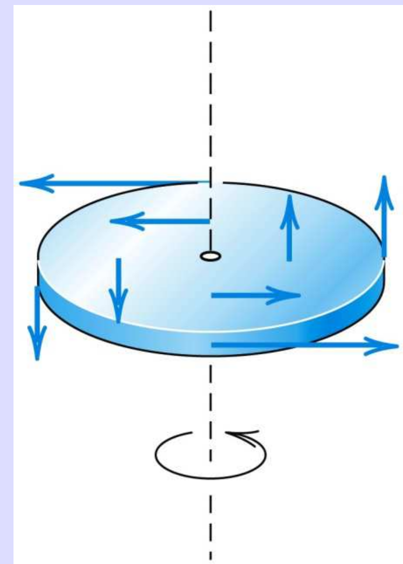


Fig. 197. Velocity field of a rotating body

Similarly, we define a **scalar function** f, whose values are scalars, that is,

$$f = f(P)$$

that depends on P. We say that a scalar function defines a scalar field in that three-dimensional domain or surface or curve in space. Two representative examples of scalar fields are the temperature field of a body and the pressure field of the air in Earth's atmosphere. Note that scalar functions may also depend on some parameter such as time t.

EXAMPLE 1

Scalar Function (Euclidean Distance in Space)

The distance $f(P)$ of any point P from a fixed point P_0 in space is a scalar function whose domain of definition is the whole space. $f(P)$ defines a scalar field in space. If we introduce a Cartesian coordinate system and P_0 has the coordinates x_0, y_0, z_0, then f is given by the well-known formula

$$f(P) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

where x, y, z are the coordinates of P.

EXAMPLE 1 (continued)

Scalar Function (Euclidean Distance in Space)

If we replace the given Cartesian coordinate system with another such system by translating and rotating the given system, then the values of the coordinates of P and P_0 will in general change, but $f(P)$ will have the same value as before. Hence $f(P)$ is a scalar function. The direction cosines of the straight line through P and P_0 are not scalars because their values depend on the choice of the coordinate system.

Vector Calculus

Convergence.

An infinite sequence of vectors $\mathbf{a}_{(n)}$, $n = 1, 2, \ldots$, is said to **converge** if there is a vector $\mathbf{a}$ such that

(4) $$\lim_{n\to\infty} \left| \mathbf{a}_{(n)} - \mathbf{a} \right| = 0.$$

$\mathbf{a}$ is called the **limit vector** of that sequence, and we write

(5) $$\lim_{n\to\infty} \mathbf{a}_{(n)} = \mathbf{a}.$$

Vector Calculus

Convergence. (continued 1)
If the vectors are given in Cartesian coordinates, then this sequence of vectors converges to **a** if and only if the three sequences of components of the vectors converge to the corresponding components of **a**.

Similarly, a vector function $\mathbf{v}(t)$ of a real variable t is said to have the **limit** $\boldsymbol{l}$ as t approaches t_0, if $\mathbf{v}(t)$ is defined in some neighborhood of t_0 (possibly except at t_0) and

(6) $$\lim_{t \to t_0} |\mathbf{v}(t) - \boldsymbol{l}| = 0.$$

Vector Calculus

Convergence. **(continued 2)**

Then we write

$$\lim_{t \to t_0} \mathbf{v}(t) = \boldsymbol{l}. \tag{7}$$

Here, a *neighborhood* of t_0 is an interval (segment) on the t-axis containing t_0 as an interior point (not as an endpoint).

Continuity. A vector function $\mathbf{v}(t)$ is said to be **continuous** at $t = t_0$ if it is defined in some neighborhood of t_0 (including at t_0 itself!) and

(8) $$\lim_{t \to t_0} \mathbf{v}(t) = \mathbf{v}(t_0).$$

If we introduce a Cartesian coordinate system, we may write

$$\mathbf{v}(t) = \left[v_1(t),\ v_2(t),\ v_3(t) \right] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}.$$

Then $\mathbf{v}(t)$ is continuous at t_0 if and only if its three components are continuous at t_0.

Definition

Derivative of a Vector Function

A vector function $\mathbf{v}(t)$ is said to be **differentiable** at a point t if the following limit exists:

(9)
$$\mathbf{v}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}.$$

This vector $\mathbf{v}'(t)$ is called the **derivative** of $\mathbf{v}(t)$. See Fig. 199.

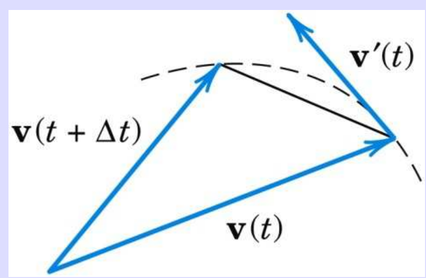


Fig. 199. Derivative of a vector function

In components with respect to a given Cartesian coordinate system,

(10) $$\mathbf{v}'(t) = \left[v_1'(t),\ v_2'(t),\ v_3'(t)\right].$$

Hence the derivative* $\mathbf{v}'(t)$ *is obtained by differentiating each component separately.

For instance, if $\mathbf{v} = [t, t^2, 0]$, then $\mathbf{v}' = [1, 2t, 0]$.

Equation (10) follows from (9) and conversely because (9) is a "vector form" of the usual formula of calculus by which the derivative of a function of a single variable is defined. [The curve in Fig. 199 is the locus of the terminal points representing $\mathbf{v}(t)$ for values of the independent variable in some interval containing t and $t + \Delta t$ in (9)]. It follows that the familiar differentiation rules continue to hold for differentiating vector functions, for instance

$$(c\mathbf{v})' = c\mathbf{v}' \qquad (c \text{ constant}),$$

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

and in particular

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' \tag{11}$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}' \tag{12}$$

$$(\mathbf{u} \quad \mathbf{v} \quad \mathbf{w})' = (\mathbf{u}' \quad \mathbf{v} \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v}' \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}'). \tag{13}$$

Partial Derivatives of a Vector Function

Suppose that the components of a vector function

$$\mathbf{v} = [v_1, \ v_2, \ v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

are differentiable functions of n variables $t_1, \ldots, t_n$. Then the **partial derivative** of $\mathbf{v}$ with respect to t_m is denoted by $\partial\mathbf{v}/\partial t_m$ and is defined as the vector function

$$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$

Similarly, second partial derivatives are and so on.

$$\frac{\partial^2 \mathbf{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\mathbf{k}.$$

9.5 Curves. Arc Length. Curvature. Torsion

The application of vector calculus to geometry is a field known as **differential geometry**.
Bodies that move in space form paths that may be represented by curves C. This and other applications show the need for **parametric representations** of C with **parameter** t, which may denote time or something else (see Fig. 200).

A typical parametric representation is given by

$$\textbf{(1)} \qquad \mathbf{r}(t) = [x(t), \quad y(t), \quad z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

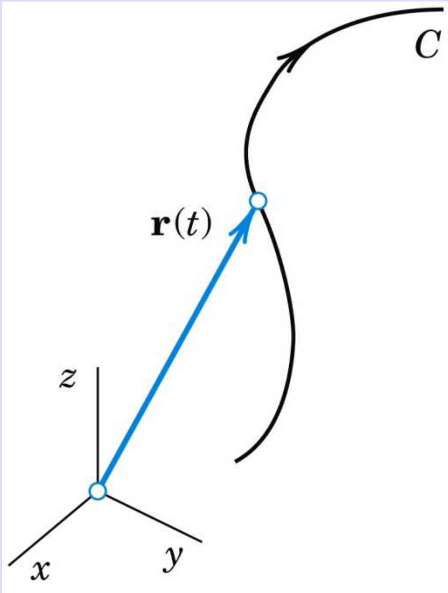


Fig. 200. Parametric representation of a curve

Here t is the parameter and x, y, z are Cartesian coordinates, that is, the usual rectangular coordinates as shown in Sec. 9.1.

To each value $t = t_0$, there corresponds a point of C with position vector $\mathbf{r}(t_0)$ whose coordinates are $x(t_0)$, $y(t_0)$, $z(t_0)$. This is illustrated in Figs. 201 and 202.

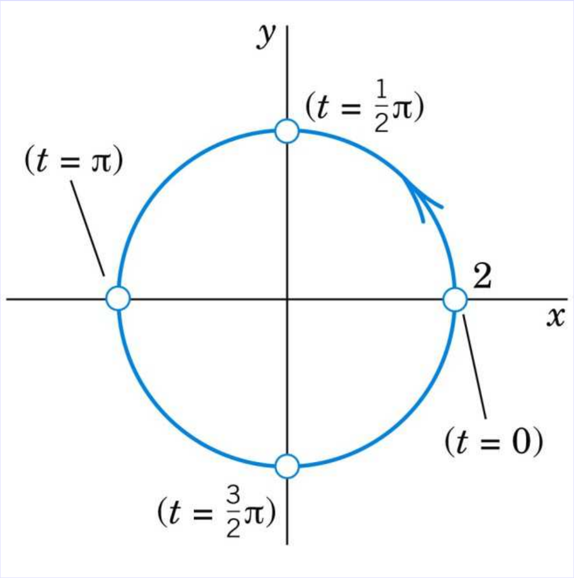


Fig. 201. Circle in Example 1

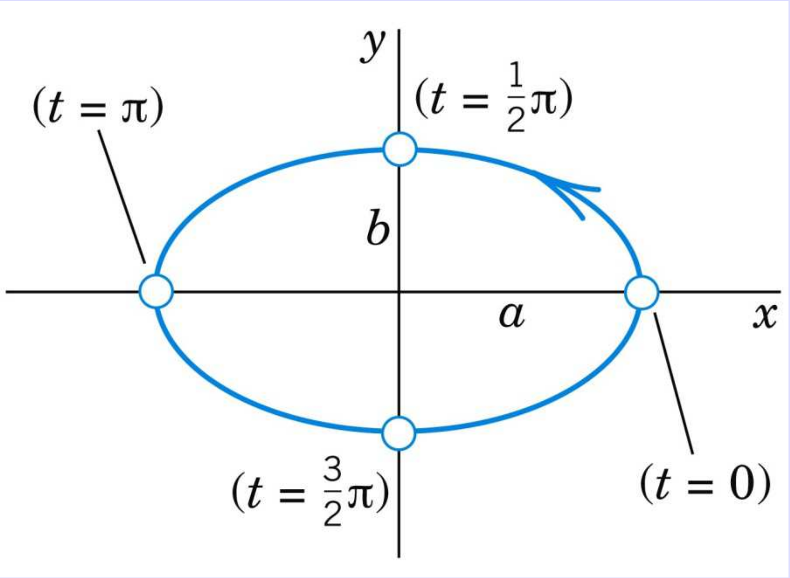


Fig. 202. Ellipse in Example 2

The use of parametric representations has key advantages over other representations that involve projections into the *xy*-plane and *xz*-plane or involve a pair of equations with y or with z as independent variable. The projections look like this:

$$y = f(x), \qquad z = g(x). \tag{2}$$

The advantages of using (1) instead of (2) are that, in (1), the coordinates x, y, z all play an equal role, that is, all three coordinates are dependent variables. Moreover, the parametric representation (1) induces an orientation on C. This means that as we increase t, we travel along the curve C in a certain direction. The sense of increasing t is called the positive sense on C. The sense of decreasing t is then called the negative sense on C, given by (1).

EXAMPLE 3 Straight Line

A straight line L through a point A with position vector $\mathbf{a}$ in the direction of a constant vector $\mathbf{b}$ (see Fig. 203) can be represented parametrically in the form

$$\mathbf{r}(t) = \mathbf{a} + t\mathbf{b} = [a_1 + tb_1, a_2 + tb_2, a_3 + tb_3]. \tag{4}$$

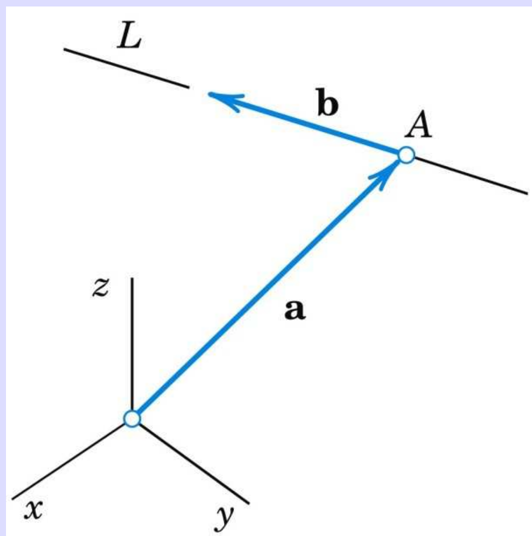


Fig. 203. Parametric representation of a straight line

A **plane curve** is a curve that lies in a plane in space. A curve that is not plane is called a **twisted curve**.

A **simple curve** is a curve without **multiple points**, that is, without points at which the curve intersects or touches itself. Circle and helix are simple curves. Figure 206 shows curves that are not simple.

An **arc** of a curve is the portion between any two points of the curve. For simplicity, we say "curve" for curves as well as for arcs.

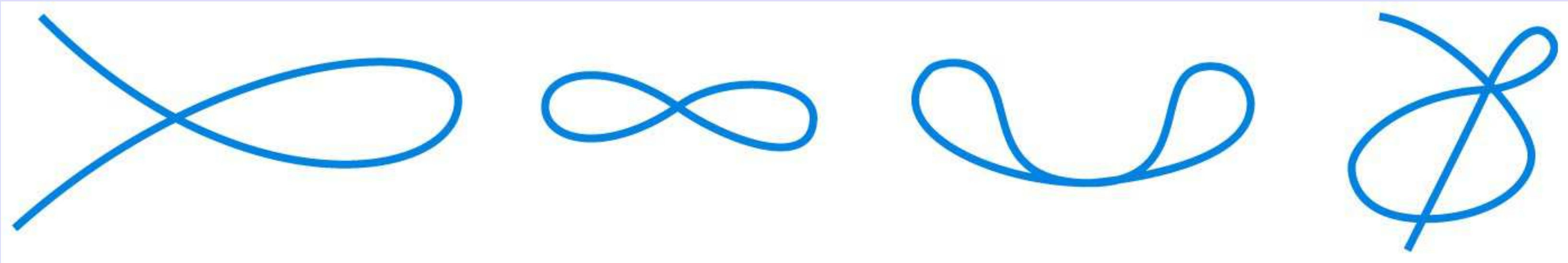


Fig. 206. Curves with multiple points

Tangent to a Curve

The next idea is the approximation of a curve by straight lines, leading to tangents and to a definition of length. Tangents are straight lines touching a curve. The **tangent** to a simple curve C at a point P of C is the limiting position of a straight line L through P and a point Q of C as Q approaches P along C. See Fig. 207.

Let us formalize this concept. If C is given by $\mathbf{r}(t)$, and P and Q correspond to t and $t + \Delta t$, then a vector in the direction of L is

$$(6) \qquad \frac{1}{\Delta t}\left[\mathbf{r}(t+\Delta t)-\mathbf{r}(t)\right].$$

Tangent to a Curve (continued)

In the limit this vector becomes the derivative

(7)
$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)\right],$$

provided $\mathbf{r}(t)$ is differentiable, as we shall assume from now on. If $\mathbf{r}'(t) \neq \mathbf{0}$ we call $\mathbf{r}'(t)$ a **tangent vector** of C at P because it has the direction of the tangent. The corresponding unit vector is the **unit tangent vector** (see Fig. 207)

(8)
$$\mathbf{u} = \frac{1}{|\mathbf{r}'|}\mathbf{r}'.$$

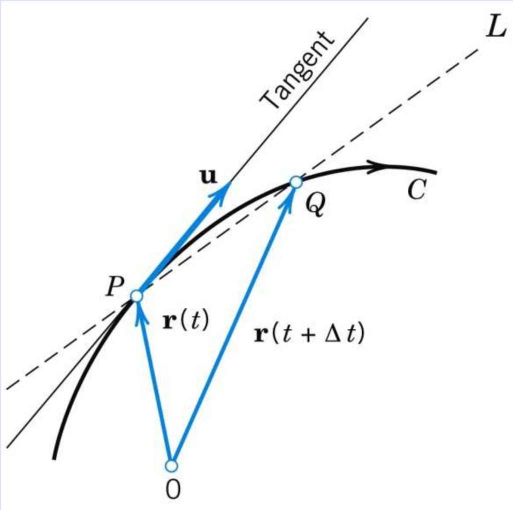
Note that both $\mathbf{r}'$ and $\mathbf{u}$ point in the direction of increasing t. Hence their sense depends on the orientation of C. It is reversed if we reverse the orientation.

Tangent to a Curve (continued)

It is now easy to see that the **tangent** to C at P is given by

$$\mathbf{q}(w) = \mathbf{r} + w\mathbf{r}' \qquad \text{(Fig. 208)}. \tag{9}$$

This is the sum of the position vector **r** of P and a multiple of the tangent vector $\mathbf{r}'$ of C at P. Both vectors depend on P. The variable w is the parameter in (9).



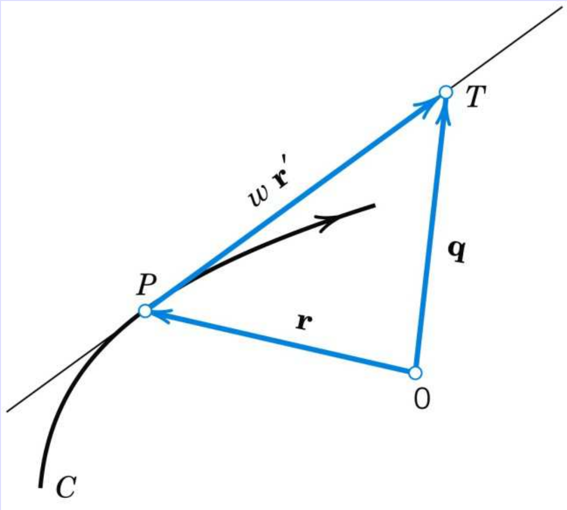


Fig. 207. Tangent to a curve **Fig. 208.** Formula (9) for the tangent to a curve

Length of a Curve

If $\mathbf{r}(t)$ has a continuous derivative $\mathbf{r}'$, it can be shown that the sequence $l_1, l_2, \ldots$ has a limit, which is independent of the particular choice of the representation of C and of the choice of subdivisions. This limit is given by the integral

(10)
$$l = \int_a^b \sqrt{\mathbf{r}' \bullet \mathbf{r}'}\, dt \qquad \left(\mathbf{r}' = \frac{d\mathbf{r}}{dt}\right).$$

l is called the **length** of C, and C is called **rectifiable**.

Arc Length s of a Curve

The length (10) of a curve C is a constant, a positive number. But if we replace the fixed b in (10) with a variable t, the integral becomes a function of t, denoted by $s(t)$ and called the *arc length function* or simply the **arc length** of C. Thus

$$\textbf{(11)} \qquad s(t) = \int_a^t \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, d\tilde{t} \qquad \left(\mathbf{r}' = \frac{d\mathbf{r}}{d\tilde{t}}\right)$$

Here the variable of integration is denoted by $\tilde{t}$ because t is now used in the upper limit.

Arc Length *s* of a Curve

Linear Element *ds*.

It is customary to write

(13*) $$d\mathbf{r} = [dx, dy, dz] = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

and

(13**)** $$ds^2 = d\mathbf{r} \bullet d\mathbf{r} = dx^2 + dy^2 + dz^2.$$

ds is called the **linear element** of C.

Arc Length *s* of a Curve (continued)

Arc Length as Parameter.

The use of s in (1) instead of an arbitrary t simplifies various formulas. For the unit tangent vector (8) we simply obtain

$$\mathbf{u}(s) = \mathbf{r}'(s). \tag{14}$$

Indeed, $|\mathbf{r}'(s)| = (ds/ds) = 1$ in (12) shows that $\mathbf{r}'(s)$ is a unit vector.

Curves in Mechanics. Velocity. Acceleration

Curves play a basic role in mechanics, where they may serve as paths of moving bodies. Then such a curve C should be represented by a parametric representation $\mathbf{r}(t)$ with **time** t as parameter. The tangent vector (7) of C is then called the **velocity vector v** because, being tangent, it points in the instantaneous direction of motion and its length gives the **speed** $|\mathbf{v}| = |\mathbf{r}'| = \sqrt{\mathbf{r}' \bullet \mathbf{r}'} = ds/dt$ see (12). The second derivative of $\mathbf{r}(t)$ is called the **acceleration vector** and is denoted by **a**. Its length $|\mathbf{a}|$ is called the **acceleration** of the motion. Thus

$$(16) \qquad \mathbf{v}(t) = \mathbf{r}'(t), \qquad \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

Tangential and Normal Acceleration.
Whereas the velocity vector is always tangent to the path of motion, the acceleration vector will generally have another direction. We can split the acceleration vector into two directional components, that is,

$$\mathbf{a} = \mathbf{a}_{\tan} + \mathbf{a}_{\text{norm}}, \tag{17}$$

where the **tangential acceleration vector** $\mathbf{a}_{\tan}$ is tangent to the path (or, sometimes, $\mathbf{0}$) and the **normal acceleration vector** $\mathbf{a}_{\text{norm}}$ is normal (perpendicular) to the path (or, sometimes, $\mathbf{0}$).

Now the length $|\mathbf{a}_{\tan}|$ is the absolute value of the projection of **a** in the direction of **v**, given by (11) in Sec. 9.2 with $\mathbf{b} = \mathbf{v}$; that is, $|\mathbf{a}_{\tan}| = |\mathbf{a} \cdot \mathbf{v}|/|\mathbf{v}|$. Hence $\mathbf{a}_{\tan}$ is this expression times the unit vector $(1/|\mathbf{v}|)\mathbf{v}$ in the direction of **v**, that is,

$$\text{(18*)} \qquad \mathbf{a}_{\tan} = \frac{\mathbf{a} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}. \qquad \text{Also,} \qquad \mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\tan}.$$

EXAMPLE 7 Centripetal Acceleration. Centrifugal Force

The vector function

$$\mathbf{r}(t) = [R \cos \omega t, R \sin \omega t] = R \cos \omega t\, \mathbf{i} + R \sin \omega t\, \mathbf{j}$$

(Fig. 210)

(with fixed **i** and **j**) represents a circle C of radius R with center at the origin of the xy-plane and describes the motion of a small body B counterclockwise around the circle. Differentiation gives the velocity vector

$$\mathbf{v} = \mathbf{r}' = [-R\omega \sin \omega t, R\omega \cos \omega t] = -R\omega \sin \omega t\, \mathbf{i} + R\omega \cos \omega t\, \mathbf{j}$$

(Fig. 210)

v is tangent to C.

EXAMPLE 7 (continued)
Centripetal Acceleration. Centrifugal Force

Its magnitude, the speed, is

$$|\mathbf{v}| = |\mathbf{r}'| = \sqrt{\mathbf{r}' \cdot \mathbf{r}'} = R\omega.$$

Hence it is constant. The speed divided by the distance R from the center is called the **angular speed**. It equals ω, so that it is constant, too. Differentiating the velocity vector, we obtain the acceleration vector

$$\text{(19)} \qquad \begin{aligned} \mathbf{a} = \mathbf{v}' &= [-R\omega^2 \cos \omega t, -R\omega^2 \sin \omega t] \\ &= -R\omega^2 \cos \omega t \, \mathbf{i} - R\omega^2 \sin \omega t \, \mathbf{j}. \end{aligned}$$

EXAMPLE 7 (continued)
Centripetal Acceleration. Centrifugal Force

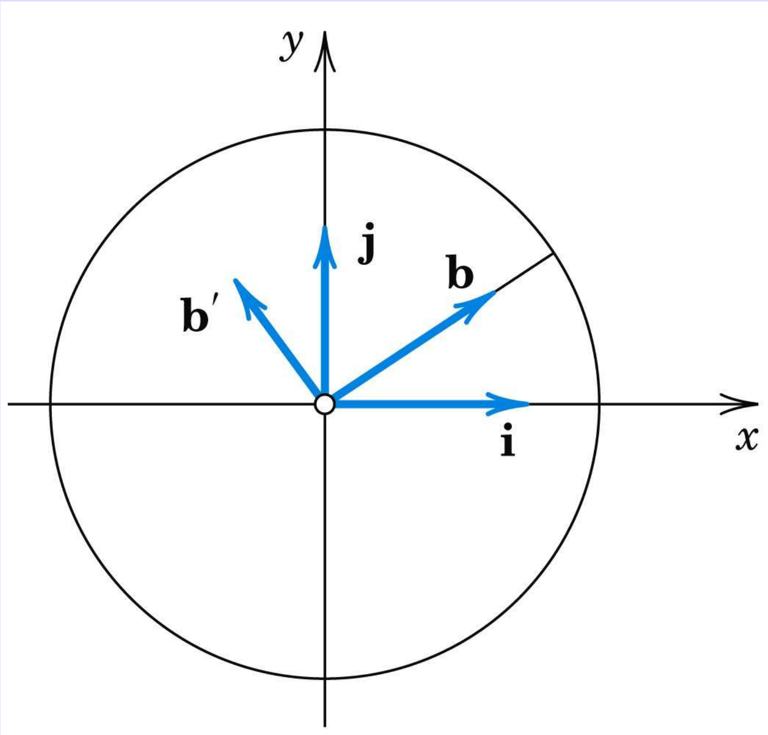


Fig. 210. Centripetal acceleration **a**

EXAMPLE 7 (continued)
Centripetal Acceleration. Centrifugal Force

This shows that $\mathbf{a} = -\omega^2\mathbf{r}$ (Fig. 210), so that there is an acceleration toward the center, called the **centripetal acceleration** of the motion. It occurs because the velocity vector is changing direction at a constant rate. Its magnitude is constant, $|\mathbf{a}| = \omega^2|\mathbf{r}| = \omega^2 R$. Multiplying **a** by the mass m of B, we get the **centripetal force** $m\mathbf{a}$. The opposite vector $-m\mathbf{a}$ is called the **centrifugal force**. At each instant these two forces are in equilibrium.

We see that in this motion the acceleration vector is normal (perpendicular) to C; hence there is no tangential acceleration.

EXAMPLE 8
Superposition of Rotations. Coriolis Acceleration

A projectile is moving with constant speed along a meridian of the rotating Earth in Fig. 211. Find its acceleration.

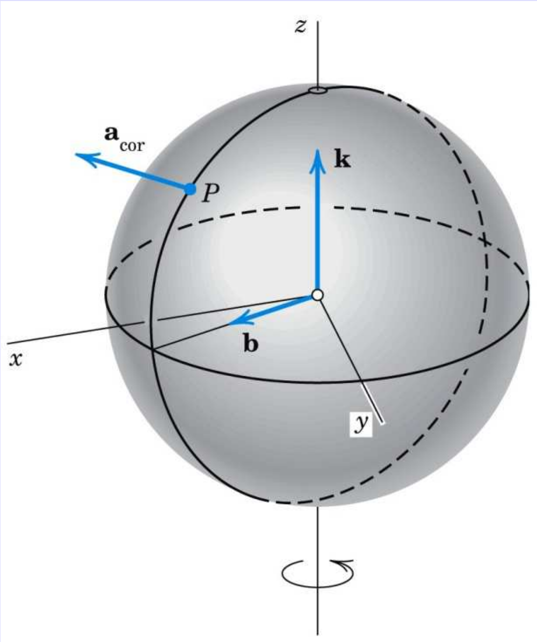


Fig. 211. Example 8. Superposition of two rotations

EXAMPLE 8 (continued)
Superposition of Rotations. Coriolis Acceleration

Solution. Let x, y, z be a fixed Cartesian coordinate system in space, with unit vectors **i**, **j**, **k** in the directions of the axes. Let the Earth, together with a unit vector **b**, be rotating about the z-axis with angular speed $\omega > 0$ (see Example 7). Since **b** is rotating together with the Earth, it is of the form

$$\mathbf{b}(t) = \cos \omega t \, \mathbf{i} + \sin \omega t \, \mathbf{j}.$$

Let the projectile be moving on the meridian whose plane is spanned by **b** and **k** (Fig. 211) with constant angular speed $\omega > 0$. Then its position vector in terms of **b** and **k** is

$$\mathbf{r}(t) = R \cos \gamma t \, \mathbf{b}(t) + R \sin \gamma t \, \mathbf{k}$$

(R = Radius of the Earth).

EXAMPLE 8 (continued)

Superposition of Rotations. Coriolis Acceleration

Solution. *(continued 1)*

We have finished setting up the model. Next, we apply vector calculus to obtain the desired acceleration of the projectile. Our result will be unexpected—and highly relevant for air and space travel. The first and second derivatives of **b** with respect to t are

$$\begin{aligned} \mathbf{b}'(t) &= -\omega \sin \omega t\,\mathbf{i} + \omega \cos \omega t\,\mathbf{j} \\ \mathbf{b}''(t) &= -\omega^2 \cos \omega t\,\mathbf{i} - \omega^2 \sin \omega t\,\mathbf{j} = -\omega^2 \mathbf{b}(t). \end{aligned} \tag{20}$$

EXAMPLE 8 (continued)
Superposition of Rotations. Coriolis Acceleration

Solution. *(continued 2)*

The first and second derivatives of $\mathbf{r}(t)$ with respect to t are

$$\mathbf{v} = \mathbf{r}'(t) = R\cos\gamma t\,\mathbf{b}' - \gamma R\sin\gamma t\,\mathbf{b} + \gamma R\cos\gamma t\,\mathbf{k}$$

$$\begin{aligned}
(21)\qquad \mathbf{a} &= \mathbf{v}'(t) \\
&= R\cos\gamma t\,\mathbf{b}'' - 2\gamma R\sin\gamma t\,\mathbf{b}' - \gamma^2 R\cos\gamma t\,\mathbf{b} - \gamma^2 R\sin\gamma t\,\mathbf{k} \\
&= R\cos\gamma t\,\mathbf{b}'' - 2\gamma R\sin\gamma t\,\mathbf{b}' - \gamma^2\mathbf{r}.
\end{aligned}$$

By analogy with Example 7 and because of $\mathbf{b}'' = -\omega^2\mathbf{b}$ in (20) we conclude that the first term in $\mathbf{a}$ (involving ω in $\mathbf{b}''$!) is the centripetal acceleration due to the rotation of the Earth.

EXAMPLE 8 (continued)
Superposition of Rotations. Coriolis Acceleration

Solution. *(continued 3)*

Similarly, the third term in the last line (involving γ!) is the centripetal acceleration due to the motion of the projectile on the meridian M of the rotating Earth.

The second, unexpected term $-2\gamma R \sin \gamma t\, \mathbf{b}'$ in $\mathbf{a}$ is called the **Coriolis acceleration** (Fig. 211) and is due to the interaction of the two rotations.

EXAMPLE 8 (continued)
Superposition of Rotations. Coriolis Acceleration

Solution. *(continued 4)*

On the Northern Hemisphere, $\sin \gamma t > 0$ (for $t > 0$; also $\gamma > 0$ by assumption), so that $\mathbf{a}_{\text{cor}}$ has the direction of $-\mathbf{b}'$, that is, opposite to the rotation of the Earth. $|\mathbf{a}_{\text{cor}}|$ is maximum at the North Pole and zero at the equator. The projectile B of mass m_0 experiences a force $-m_0\mathbf{a}_{\text{cor}}$ opposite to $m_0\mathbf{a}_{\text{cor}}$, which tends to let B deviate from M to the right (and in the Southern Hemisphere, where $\sin \gamma t < 0$, to the left). This deviation has been observed for missiles, rockets, shells, and atmospheric airflow.

9.6 Calculus Review: Functions of Several Variables.

Optional

Chain Rules

Theorem 1

Chain Rule

Let $w = f(x, y, z)$ be continuous and have continuous first partial derivatives in a domain D in xyz-space. Let $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ be functions that are continuous and have first partial derivatives in a domain B in the uv-plane, where B is such that for every point (u, v) in B, the corresponding point $[x(u, v), y(u, v), z(u, v)]$ lies in D. See Fig. 213.

Chain Rules

Theorem 1 (continued)

Chain Rule (continued 1)

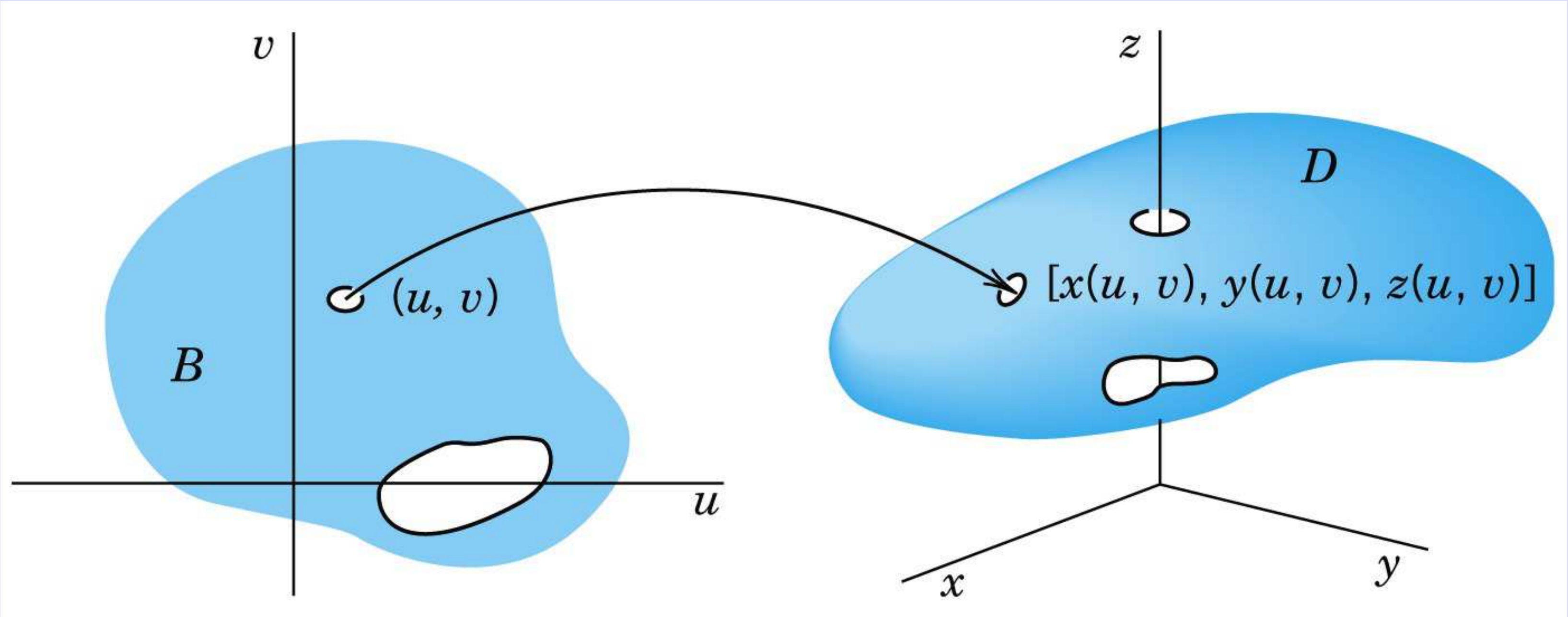


Fig. 213. Notations in Theorem 1

Chain Rules

Theorem 1 (continued)

Chain Rule (continued 2)

Then the function

$$w = f(x(u, v), y(u, v), z(u, v))$$

is defined in B, has first partial derivatives with respect to u and v in B, and

$$\text{(1)}\quad \begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}. \end{aligned}$$

In this theorem, a **domain** D is an open connected point set in xyz-space, where "connected" means that any two points of D can be joined by a broken line of finitely many linear segments all of whose points belong to D. "Open" means that every point P of D has a neighborhood (a little ball with center P) all of whose points belong to D. For example, the interior of a cube or of an ellipsoid (the solid without the boundary surface) is a domain.

In calculus, x, y, z are often called the **intermediate variables**, in contrast with the **independent variables** u, v and the **dependent variable** w.

Special Cases of Practical Interest

If $w = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$ as before, then (1) becomes

(2)
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}.$$

Special Cases of Practical Interest (continued 1)

If $w = f(x, y, z)$ and $x = x(t)$, $y = y(t)$, $z = z(t)$, then (1) gives

(3) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

If $w = f(x, y)$ and $x = x(t)$, $y = y(t)$, then (3) reduces to

(4) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Finally, the simplest case $w = f(x)$, $x = x(t)$ gives

(5) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt}.$$

Mean Value Theorems

Theorem 2

Mean Value Theorem

Let $f(x, y, z)$ be continuous and have continuous first partial derivatives in a domain D in xyz-space. Let P_0: (x_0, y_0, z_0) and P: $(x_0 + h, y_0 + k, z_0 + l)$ be points in D such that the straight line segment P_0P joining these points lies entirely in D. Then

(7) $$f(x_0 + h, y_0 + k, z_0 + l) - f(x_0, y_0, z_0) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z},$$

the partial derivatives being evaluated at a suitable point of that segment.

9.7 Gradient of a Scalar Field. Directional Derivative

Definition 1

Gradient

The setting is that we are given a scalar function $f(x, y, z)$ that is defined and differentiable in a domain in 3-space with Cartesian coordinates x, y, z. We denote the **gradient** of that function by grad f or ∇f (read **nabla** f).

Then the gradient of $f(x, y, z)$ is defined as the vector function

(1) $$\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The notation ∇f is suggested by the *differential operator* ∇ (read **nabla** f) defined by

(1*)
$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Directional Derivative

Definition 2

Directional Derivative

The directional derivative $D_{\mathbf{b}}f$ or df/ds of a function $f(x, y, z)$ at a point P in the direction of a vector $\mathbf{b}$ is defined by (see Fig. 215)

$$D_{\mathbf{b}}f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}. \tag{2}$$

Here Q is a variable point on the straight line L in the direction of $\mathbf{b}$, and $|s|$ is the distance between P and Q. Also, $s > 0$ if Q lies in the direction of $\mathbf{b}$ (as in Fig. 215), $s < 0$ if Q lies in the direction of $-\mathbf{b}$, and $s = 0$ if $Q = P$.

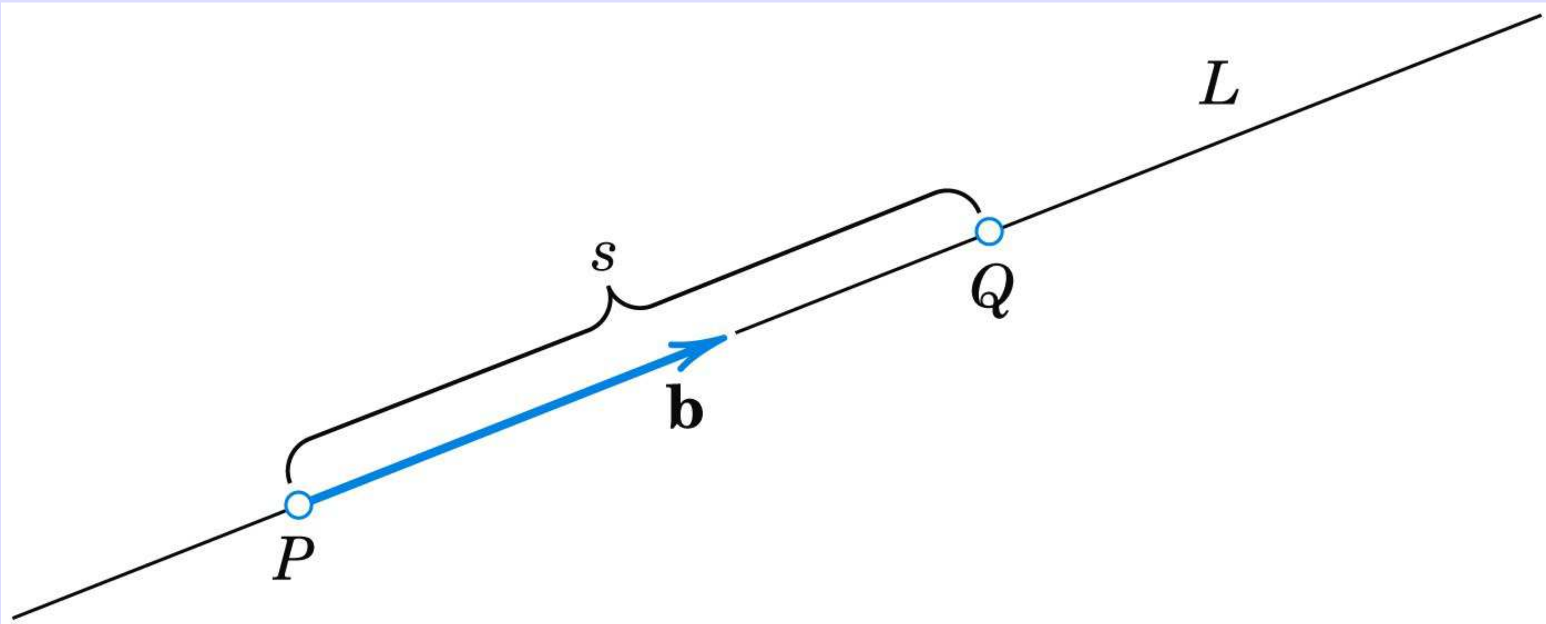


Fig. 215. Directional derivative

The line L is given by

$$(3)\quad \mathbf{r}(s) = x(s)\,\mathbf{i} + y(s)\,\mathbf{j} + z(s)\,\mathbf{k} = \mathbf{p}_0 + s\mathbf{b} \qquad (|\mathbf{b}| = 1)$$

where $\mathbf{p}_0$ is the position vector of P.

Hence, assuming that f has continuous partial derivatives and applying the chain rule [formula (3) in the previous section], we obtain

$$(4) \qquad D_{\mathbf{b}} f = \frac{df}{ds} = \frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial z} z'$$

where primes denote derivatives with respect to s (which are taken at $s = 0$). But here, differentiating (3) gives $\mathbf{r}' = x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k} = \mathbf{b}$.

Hence (4) is simply the inner product of grad f and **b** [see (2), Sec. 9.2]; that is,

(5)
$$D_{\mathbf{b}}f = \frac{df}{ds} = \mathbf{b} \bullet \operatorname{grad} f$$

ATTENTION! If the direction is given by a vector **a** of any length ($\neq 0$), then

(5*)
$$D_{\mathbf{a}}f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|}\mathbf{a} \bullet \operatorname{grad} f.$$

Gradient Is a Vector. Maximum Increase

Theorem 1

Use of Gradient: Direction of Maximum Increase

Let $f(P) = f(x, y, z)$ be a scalar function having continuous first partial derivatives in some domain B in space. Then grad f *exists in B and is a vector, that is, its length and direction are independent of the particular choice of Cartesian coordinates. If* grad $f(P) \neq \mathbf{0}$ *at some point P, it has the direction of maximum increase of f at P.*

Gradient as Surface Normal Vector

Theorem 2

Gradient as Surface Normal Vector

Let be a differentiable scalar function in space. Let $f(x, y, z) = c =$ const represent a surface S. Then if the gradient of f at a point P of S is not the zero vector, it is a normal vector of S at P.

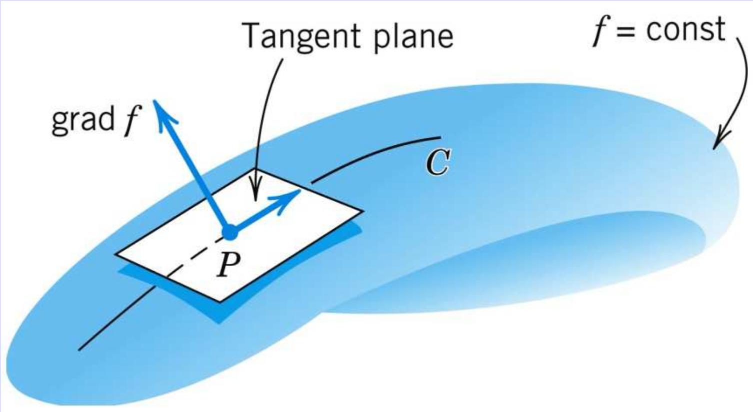


Fig. 216. Gradient as surface normal vector

EXAMPLE 2 Gradient as Surface Normal Vector. Cone

Find a unit normal vector **n** of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point P: (1, 0, 2).

Solution.

The cone is the level surface $f = 0$ of $f(x, y, z) = 4(x^2 + y^2) - z^2$. Thus (Fig. 217)

$$\operatorname{grad} f = [8x, 8y, -2z], \qquad \operatorname{grad} f(P) = [8, 0, 4]$$

$$\mathbf{n} = \frac{1}{|\operatorname{grad} f(P)|} \operatorname{grad} f(P) = \left[\frac{2}{\sqrt{5}},\ 0,\ -\frac{1}{\sqrt{5}}\right].$$

n points downward since it has a negative z-component. The other unit normal vector of the cone at P is $-\mathbf{n}$.

EXAMPLE 2 Gradient as Surface Normal Vector. Cone

Solution. *(continued)*

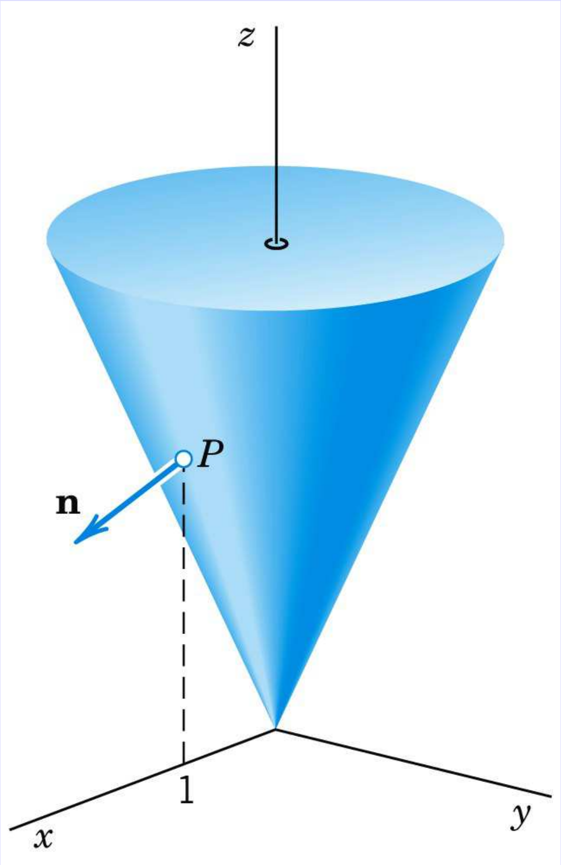


Fig. 217. Cone and unit normal vector **n**

Vector Fields That Are Gradients of Scalar Fields ("Potentials")

At the beginning of this section we mentioned that some vector fields have the advantage that they can be obtained from scalar fields, which can be worked with more easily. Such a vector field is given by a vector function $\mathbf{v}(P)$, which is obtained as the gradient of a scalar function, say, $\mathbf{v}(P) = \text{grad } f(P)$. The function is called a *potential function* or a **potential** of $\mathbf{v}(P)$. Such a $\mathbf{v}(P)$ and the corresponding vector field are called **conservative** because in such a vector field, energy is conserved; that is, no energy is lost (or gained) in displacing a body (or a charge in the case of an electrical field) from a point P to another point in the field and back to P. We show this in Sec. 10.2.

Vector Fields That Are Gradients of Scalar Fields ("Potentials")

(continued)

Conservative fields play a central role in physics and engineering. A basic application concerns the gravitational force (see Example 3 in Sec. 9.4) and we show that it has a potential which satisfies Laplace's equation, the most important partial differential equation in physics and its applications.

Theorem 3

Gravitational Field. Laplace's Equation

The force of attraction

$$\mathbf{p} = -\frac{c}{r^3}\mathbf{r} = -c\left[\frac{x-x_0}{r^3},\ \frac{y-y_0}{r^3},\ \frac{z-z_0}{r^3}\right] \tag{8}$$

between two particles at points P_0: (x_0, y_0, z_0) and P: (x, y, z) (as given by Newton's law of gravitation) *has the potential* $f(x, y, z) = c/r$, *where* r (> 0) *is the distance between* P_0 *and* P.

Theorem 3 (continued)

Gravitational Field. Laplace's Equation

Thus $\mathbf{p} = \text{grad } f = \text{grad } (c/r)$. *This potential f is a solution of* **Laplace's equation**

(9)
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

[$\nabla^2 f$ (read *nabla squared f*) is called the **Laplacian** of f.]

9.8 Divergence of a Vector Field

To begin, let $\mathbf{v}(x, y, z)$ be a differentiable vector function, where x, y, z are Cartesian coordinates, and let v_1, v_2, v_3 be the components of $\mathbf{v}$. Then the function

(1) $$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the **divergence** of $\mathbf{v}$ or the *divergence of the vector field defined by* $\mathbf{v}$. For example, if

$$\mathbf{v} = [3xz, 2xy, -yz^2] = 3xz\mathbf{i} + 2xy\,\mathbf{j} - yz^2\mathbf{k},$$

then $\qquad \operatorname{div} \mathbf{v} = 3z + 2x - 2yz.$

Another common notation for the divergence is

$$\begin{aligned}\operatorname{div} \mathbf{v} &= \nabla \bullet \mathbf{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial \mathbf{y}}, \frac{\partial}{\partial \mathbf{z}}\right] \bullet [v_1, v_2, v_3] \\ &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \bullet (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},\end{aligned}$$

with the understanding that the "product" $(\partial/\partial x)v_1$ in the dot product means the partial derivative $\partial v_1/\partial x$, etc. This is a convenient notation, but nothing more. Note that $\nabla \bullet \mathbf{v}$ means the scalar div **v**, whereas ∇f means the vector grad f defined in Sec. 9.7.

Theorem 1

Invariance of the Divergence

The divergence div **v** *is a scalar function, that is, its values depend only on the points in space* (*and, of course, on* **v**) *but not on the choice of the coordinates in* (1), *so that with respect to other Cartesian coordinates* x^*, y^*, z^* *and corresponding components* v_1^*, v_2^*, v_3^* *of* **v**,

$$\text{div}\,\mathbf{v} = \frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} + \frac{\partial v_3^*}{\partial z^*}. \tag{2}$$

Let $f(x, y, z)$ be a twice differentiable scalar function. Then its gradient exists,

$$\mathbf{v} = \operatorname{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and we can differentiate once more, the first component with respect to x, the second with respect to y, the third with respect to z, and then form the divergence,

$$\operatorname{div} \mathbf{v} = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Hence we have the basic result that *the divergence of the gradient is the Laplacian* (Sec. 9.7),

(3) $$\operatorname{div}(\operatorname{grad} f) = \nabla^2 f.$$

EXAMPLE 2 Flow of a Compressible Fluid. Physical Meaning of the Divergence

We consider the motion of a fluid in a region R having no **sources** or **sinks** in R, that is, no points at which fluid is produced or disappears. The concept of **fluid state** is meant to cover also gases and vapors. Fluids in the restricted sense, or liquids, such as water or oil, have very small compressibility, which can be neglected in many problems. In contrast, gases and vapors have high compressibility. Their density ρ (= mass per unit volume) depends on the coordinates x, y, z in space and may also depend on time t. We assume that our fluid is compressible. We consider the flow through a rectangular box B of small edges Δx, Δy, Δz parallel to the coordinate axes as shown in Fig. 218.

EXAMPLE 2 (continued) Flow of a Compressible Fluid. Physical Meaning of the Divergence

(continued 1)

(Here Δ is a standard notation for small quantities and, of course, has nothing to do with the notation for the Laplacian in (11) of Sec. 9.7.)

The box B has the volume $\Delta V = \Delta x + \Delta y + \Delta z$.

Let $\mathbf{v} = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be the velocity vector of the motion. We set

$$\mathbf{u} = \rho\mathbf{v} = [u_1, u_2, u_3] = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \tag{4}$$

and assume that $\mathbf{u}$ and $\mathbf{v}$ are continuously differentiable vector functions of x, y, z, and t, that is, they have first partial derivatives which are continuous.

EXAMPLE 2 (continued) Flow of a Compressible Fluid. Physical Meaning of the Divergence

(continued 2) Let us calculate the change in the mass included in B by considering the **flux** across the boundary, that is, the total loss of mass leaving B per unit time. Consider the flow through the left of the three faces of B that are visible in Fig. 218, whose area is $\Delta x \Delta z$. Since the vectors $v_1\mathbf{i}$ and $v_3\mathbf{k}$ are parallel to that face, the components v_1 and v_2 of $\mathbf{v}$ contribute nothing to this flow. Hence the mass of fluid entering through that face during a short time interval Δt is given approximately by

$$(\rho v_2)_y \, \Delta x \, \Delta z \, \Delta t = (u_2)_y \, \Delta x \, \Delta z \, \Delta t,$$

where the subscript y indicates that this expression refers to the left face.

EXAMPLE 2 (continued) Flow of a Compressible Fluid. Physical Meaning of the Divergence

(continued 3)

The mass of fluid leaving the box B through the opposite face during the same time interval is approximately $(u_2)_{y+\Delta y}\,\Delta x\,\Delta z\,\Delta t$, where the subscript $y + \Delta y$ indicates that this expression refers to the right face (which is not visible in Fig. 218). The difference

$$\Delta u_2\;\Delta x\;\Delta y\;\Delta t = \frac{\Delta u_2}{\Delta y}\Delta V\;\Delta t \qquad \left[\Delta u_2 = (u_2)_{y+\Delta y} - (u_2)_y\right]$$

is the approximate loss of mass. Two similar expressions are obtained by considering the other two pairs of parallel faces of B.

EXAMPLE 2 (continued) Flow of a Compressible Fluid. Physical Meaning of the Divergence

(continued 4)

If we add these three expressions, we find that the total loss of mass in B during the time interval Δt is approximately

$$\left(\frac{\Delta u_1}{\Delta x}+\frac{\Delta u_2}{\Delta y}+\frac{\Delta u_3}{\Delta z}\right)\Delta V\,\Delta t,$$

where

$$\Delta u_1=(u_1)_{x+\Delta x}-(u_1)_x \quad \text{and} \quad \Delta u_3=(u_3)_{z+\Delta z}-(u_3)_z.$$

This loss of mass in B is caused by the time rate of change of the density and is thus equal to

$$-\frac{\partial\rho}{\partial t}\Delta V\,\Delta t.$$

EXAMPLE 2 (continued) Flow of a Compressible Fluid. Physical Meaning of the Divergence

(continued 5)

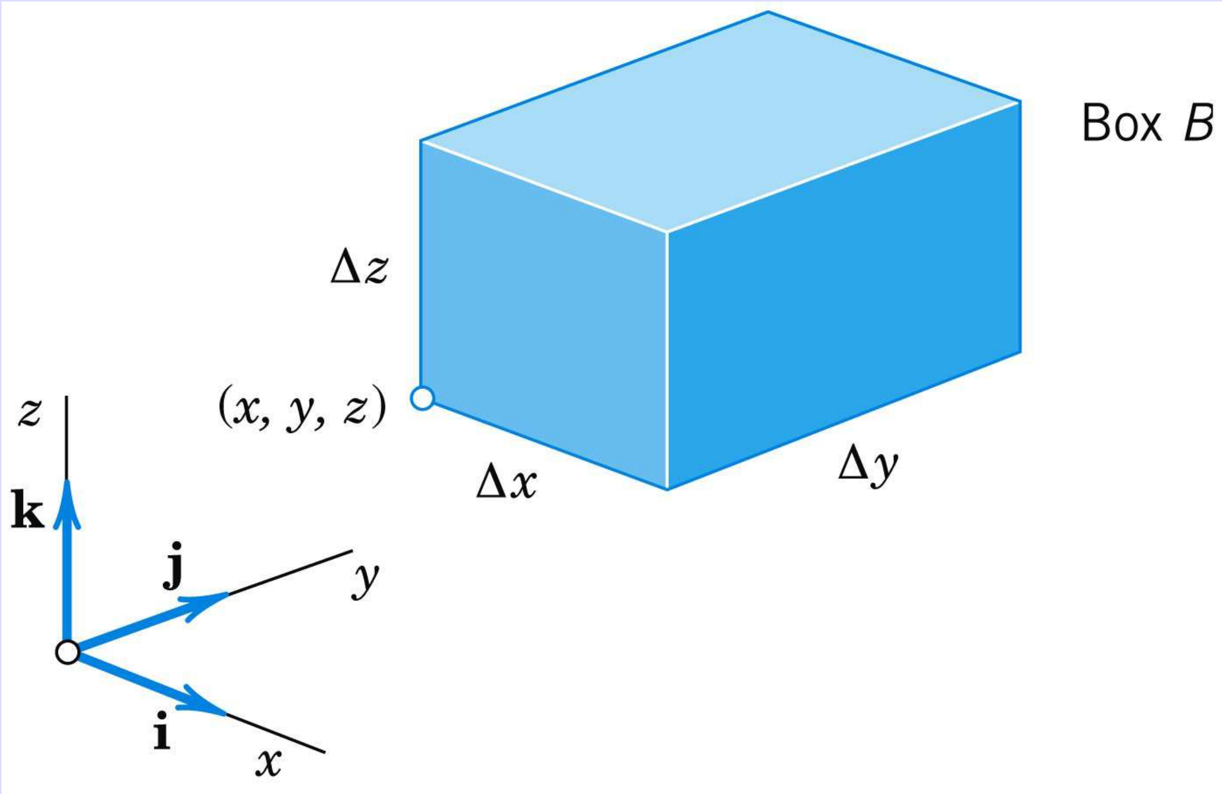


Fig. 218. Physical interpretation of the divergence

EXAMPLE 2 (continued) Flow of a Compressible Fluid. Physical Meaning of the Divergence

(continued 6)

If we equate both expressions, divide the resulting equation by $\Delta V\,\Delta t$, and let Δx, Δy, Δz, and Δt approach zero, then we obtain

$$\operatorname{div}\mathbf{u} = \operatorname{div}(\rho\mathbf{v}) = -\frac{\partial\rho}{\partial t}$$

or

(5)
$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0.$$

This important relation is called the *condition for the conservation of mass* or the **continuity equation** *of a compressible fluid flow.*

EXAMPLE 2 (continued) Flow of a Compressible Fluid. Physical Meaning of the Divergence

(continued 7)

If the flow is **steady**, that is, independent of time, then $\partial\rho/\partial t = 0$ and the continuity equation is

$$\operatorname{div}(\rho\mathbf{v}) = 0. \tag{6}$$

If the density is constant, so that the fluid is incompressible, then equation (6) becomes

$$\operatorname{div}\mathbf{v} = 0. \tag{7}$$

This relation is known as the **condition of incompressibility**. It expresses the fact that the balance of outflow and inflow for a given volume element is zero at any time. Clearly, the assumption that the flow has no sources or sinks in R is essential to our argument. $\mathbf{v}$ is also referred to as **solenoidal**.

EXAMPLE 2 (continued) Flow of a Compressible Fluid. Physical Meaning of the Divergence

(continued 8)

From this discussion you should conclude and remember that, roughly speaking, ***the divergence measures outflow minus inflow***.

Comment. The **divergence theorem** of Gauss, an integral theorem involving the divergence, follows in the next chapter (Sec. 10.7).

9.9 Curl of a Vector Field

Let $\mathbf{v}(x, y, z) = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k}$ be a differentiable vector function of the Cartesian coordinates x, y, z. Then the **curl** *of the vector function* $\mathbf{v}$ or *of the vector field given by* $\mathbf{v}$ is defined by the "symbolic" determinant

$$\text{(1)} \qquad \operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

This is the formula when x, y, z are *right-handed*. If they are *left-handed*, the determinant has a minus sign in front.

Instead of curl **v** one also uses the notation rot **v**. This is suggested by "rotation," an application explored in Example 2. Note that curl **v** is a vector, as shown in Theorem 3.

EXAMPLE 2

Rotation of a Rigid Body. Relation to the Curl

We have seen in Example 5, Sec. 9.3, that a rotation of a rigid body B about a fixed axis in space can be described by a vector $\mathbf{w}$ of magnitude ω in the direction of the axis of rotation, where $\omega(>0)$ is the angular speed of the rotation, and $\mathbf{w}$ is directed so that the rotation appears clockwise if we look in the direction of $\mathbf{w}$.

EXAMPLE 2 (continued 1)
Rotation of a Rigid Body. Relation to the Curl

According to (9), Sec. 9.3, the velocity field of the rotation can be represented in the form

$$\mathbf{v} = \mathbf{w} \times \mathbf{r}$$

where $\mathbf{r}$ is the position vector of a moving point with respect to a Cartesian coordinate system *having the origin on the axis of rotation*. Let us choose right-handed Cartesian coordinates such that the axis of rotation is the z-axis. Then (see Example 2 in Sec. 9.4)

$$\mathbf{w} = [0, 0, \omega] = \omega\mathbf{k}, \qquad \mathbf{v} = \mathbf{w} \times \mathbf{r} = [-\omega y,\ \omega x,\ 0] = -\omega y\mathbf{i} + \omega x\mathbf{j}.$$

EXAMPLE 2 (continued 2)

Rotation of a Rigid Body. Relation to the Curl

Hence

$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = \begin{bmatrix} 0, & 0, & 2\omega \end{bmatrix} = 2\omega \mathbf{k} = 2\mathbf{w}.$$

This proves the following theorem.

Theorem 1

Rotating Body and Curl

The curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude equals twice the angular speed of the rotation.

Theorem 2

Grad, Div, Curl

Gradient fields are ***irrotational***. *That is, if a continuously differentiable vector function is the gradient of a scalar function f, then its curl is the zero vector,*

(2) $$\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}.$$

Furthermore, the divergence of the curl of a twice continuously differentiable vector function **v** *is zero,*

(3) $$\operatorname{div}(\operatorname{curl} \mathbf{v}) = 0.$$

Theorem 3

Invariance of the Curl

curl $\mathbf{v}$ is a vector. It has a length and a direction that are independent of the particular choice of a Cartesian coordinate system in space.

SUMMARY OF CHAPTER 9

Vector Differential Calculus. Grad, Div, Curl

SUMMARY OF CHAPTER 9
Vector Differential Calculus. Grad, Div, Curl

All vectors of the form $\mathbf{a} = [a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ constitute the **real vector space** R^3 with componentwise vector addition

$$(1) \qquad [a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

and componentwise scalar multiplication (c a scalar, a real number)

$$(2) \qquad c[a_1, a_2, a_3] = [ca_1, ca_2, ca_3] \qquad \text{(Sec. 9.1).}$$

For instance, the *resultant* of forces **a** and **b** is the sum **a** + **b**.

The **inner product** or **dot product** of two vectors is defined by

$$(3) \qquad \mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3. \qquad \text{(Sec. 9.2)}$$

where γ is the angle between **a** and **b**.

This gives for the **norm** or **length** $|\mathbf{a}|$ of **a**

(4) $$|\mathbf{a}| = \sqrt{\mathbf{a} \bullet \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

as well as a formula for γ. If $\mathbf{a} \bullet \mathbf{b} = 0$, we call **a** and **b** **orthogonal**. The dot product is suggested by the *work* $W = \mathbf{p} \bullet \mathbf{d}$ done by a force **p** in a displacement **d**.

The **vector product** or **cross product** $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is a vector of length

(5) $$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin \gamma$$ (Sec. 9.3)

and perpendicular to both **a** and **b** such that **a**, **b**, **v** form a *right-handed* triple.

In terms of components with respect to right-handed coordinates,

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} \qquad (6)$$

(Sec. 9.3).

The vector product is suggested, for instance, by moments of forces or by rotations.

CAUTION!
This multiplication is *anti*commutative, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and is *not* associative.

An (oblique) box with edges **a**, **b**, **c** has volume equal to the absolute value of the **scalar triple product**

(7) $$(\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}) = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}.$$

Sections 9.4–9.9 extend differential calculus to vector functions

$$\mathbf{v}(t) = [v_1(t),\ v_2(t),\ v_3(t)] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}$$

and to vector functions of more than one variable (see below). The derivative of $\mathbf{v}(t)$ is

(8) $$\mathbf{v}'(t) = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = [v_1',\ v_2',\ v_3'] = v_1'\mathbf{i} + v_2'\mathbf{j} + v_3'\mathbf{k}.$$

SUMMARY OF CHAPTER 9
Vector Differential Calculus. Grad, Div, Curl

Differentiation rules are as in calculus. They imply (Sec. 9.4)

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' \qquad (\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'.$$

Curves C in space represented by the position vector $\mathbf{r}(t)$ have $\mathbf{r}'(t)$ as a **tangent vector** (the **velocity** in mechanics when t is time), $\mathbf{r}'(s)$ (s arc length, Sec. 9.5) as the *unit tangent vector,* and $|\mathbf{r}''(s)| = \kappa$ as the *curvature* (the *acceleration* in mechanics).

Vector functions $\mathbf{v}(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$ represent vector fields in space.

Partial derivatives with respect to the Cartesian coordinates x, y, z are obtained componentwise, for instance,

$$\frac{\partial \mathbf{v}}{\partial x} = \left[\frac{\partial v_1}{\partial x}, \frac{\partial v_2}{\partial x}, \frac{\partial v_3}{\partial x}\right] = \frac{\partial v_1}{\partial x}\mathbf{i} + \frac{\partial v_2}{\partial x}\mathbf{j} + \frac{\partial v_3}{\partial x}\mathbf{k} \qquad \text{(Sec. 9.6).}$$

The **gradient** of a scalar function is

$$\text{(9)} \qquad \operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \qquad \text{(Sec. 9.7).}$$

The **directional derivative** of f in the direction of a vector **a** is

$$\text{(10)} \qquad D_{\mathbf{a}} f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|}\mathbf{a} \bullet \nabla f \qquad \text{(Sec. 9.7).}$$

SUMMARY OF CHAPTER 9
Vector Differential Calculus. Grad, Div, Curl

The **divergence** of a vector function **v** is

(11) $$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$ (Sec. 9.8).

The **curl** of **v** is

(12) $$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$ (Sec. 9.9)

or minus the determinant if the coordinates are left-handed.

Some basic formulas for grad, div, curl are (Secs. 9.7–9.9)

$$\nabla(fg) = f\nabla g + g\nabla f$$
(13)
$$\nabla(f/g) = (1/g^2)(g\nabla f - f\nabla g)$$

$$\operatorname{div}(f\mathbf{v}) = f\operatorname{div}\mathbf{v} + \mathbf{v}\cdot\nabla f$$
(14)
$$\operatorname{div}(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$$

$$\nabla^2 f = \operatorname{div}(\nabla f)$$
(15)
$$\nabla^2(fg) = g\nabla^2 f + 2\nabla f\cdot\nabla g + f\nabla^2 g$$

(16)
$$\operatorname{curl}(f\mathbf{v}) = \nabla f \times \mathbf{v} + f \operatorname{curl} \mathbf{v}$$
$$\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{u} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}$$

(17)
$$\operatorname{curl}(\nabla f) = \mathbf{0}$$
$$\operatorname{div}(\operatorname{curl} \mathbf{v}) = 0.$$

For grad, div, curl, and ∇^2 in **curvilinear coordinates** see App. A3.4.