1–6 CALCULATION OF THE DIVERGENCE

Find div $\mathbf{v}$ and its value at P.

1. $\mathbf{v} = [x^2, 4y^2, 9z^2]$, $P: (-1, 0, \frac{1}{2}]$
2. $\mathbf{v} = [0, \cos xyz, \sin xyz]$, $P: (2, \frac{1}{2}\pi, 0]$
3. $\mathbf{v} = (x^2 + y^2)^{-1}[x, y]$
4. $\mathbf{v} = [v_1(y, z), v_2(z, x), v_3(x, y)]$, $P: (3, 1, -1)]$
5. $\mathbf{v} = x^2y^2z^2[x, y, z]$, $P: (3, -1, 4)$
6. $\mathbf{v} = (x^2 + y^2 + z^2)^{-3/2}[x, y, z]$
7. For what v_3 is $\mathbf{v} = [e^x \cos y, e^x \sin y, v_3]$ solenoidal?
8. Let $\mathbf{v} = [x, y, v_3]$. Find a v_3 such that **(a)** div $\mathbf{v} > 0$ everywhere, **(b)** div $\mathbf{v} > 0$ if $|z| < 1$ and div $\mathbf{v} < 0$ if $|z| > 1$.

9. **PROJECT. Useful Formulas for the Divergence.** Prove
   (a) $\text{div}\,(k\mathbf{v}) = k\,\text{div}\,\mathbf{v}$ (k constant)
   (b) $\text{div}\,(f\mathbf{v}) = f\,\text{div}\,\mathbf{v} + \mathbf{v} \cdot \nabla f$
   (c) $\text{div}\,(f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$
   (d) $\text{div}\,(f\nabla g) - \text{div}\,(g\nabla f) = f\nabla^2 g - g\nabla^2 f$
   Verify (b) for $f = e^{xyz}$ and $\mathbf{v} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$. Obtain the answer to Prob. 6 from (b). Verify (c) for $f = x^2 - y^2$ and $g = e^{x+y}$. Give examples of your own for which (a)–(d) are advantageous.
10. **CAS EXPERIMENT. Visualizing the Divergence.** Graph the given velocity field $\mathbf{v}$ of a fluid flow in a square centered at the origin with sides parallel to the coordinate axes. Recall that the divergence measures outflow minus inflow. By looking at the flow near the sides of the square, can you see whether div $\mathbf{v}$ must be positive or negative or may perhaps be zero? Then calculate div $\mathbf{v}$. First do the given flows and then do some of your own. Enjoy it.
    (a) $\mathbf{v} = \mathbf{i}$
    (b) $\mathbf{v} = x\mathbf{i}$
    (c) $\mathbf{v} = x\mathbf{i} - y\mathbf{j}$
    (d) $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$
    (e) $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$
    (f) $\mathbf{v} = (x^2 + y^2)^{-1}(-y\mathbf{i} + x\mathbf{j})$
11. **Incompressible flow.** Show that the flow with velocity vector $\mathbf{v} = y\mathbf{i}$ is incompressible. Show that the particles that at time $t = 0$ are in the cube whose faces are portions of the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ occupy at $t = 1$ the volume 1.
12. **Compressible flow.** Consider the flow with velocity vector $\mathbf{v} = x\mathbf{i}$. Show that the individual particles have the position vectors $\mathbf{r}(t) = c_1e^t\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ with constant c_1, c_2, c_3. Show that the particles that at $t = 0$ are in the cube of Prob. 11 at $t = 1$ occupy the volume e.
13. **Rotational flow.** The velocity vector $\mathbf{v}(x, y, z)$ of an incompressible fluid rotating in a cylindrical vessel is of the form $\mathbf{v} = \mathbf{w} \times \mathbf{r}$, where $\mathbf{w}$ is the (constant) rotation vector; see Example 5 in Sec. 9.3. Show that div $\mathbf{v} = 0$. Is this plausible because of our present Example 2?
14. Does div $\mathbf{u}$ = div $\mathbf{v}$ imply $\mathbf{u} = \mathbf{v}$ or $\mathbf{u} = \mathbf{v} + \mathbf{k}$ ($\mathbf{k}$ constant)? Give reason.

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Calculate $\nabla^2 f$ by Eq. (3). Check by direct differentiation. Indicate when (3) is simpler. Show the details of your work.

15. $f = \cos^2 x + \sin^2 y$
16. $f = e^{xyz}$
17. $f = \ln(x^2 + y^2)$
18. $f = z - \sqrt{x^2 + y^2}$
19. $f = 1/(x^2 + y^2 + z^2)$
20. $f = e^{2x} \cosh 2y$