CHAPTER 9

Statistical Estimation Theory

ESTIMATION OF PARAMETERS

In the Chapter 8 we saw how sampling theory can be employed to obtain information about samples drawn at random from a known population. From a practical viewpoint, however, it is often more important to be able to infer information about a population from samples drawn from it. Such problems are dealt with in *statistical inference*, which uses principles of sampling theory.

One important problem of statistical inference is the estimation of *population parameters*, or briefly *parameters* (such as population mean and variance), from the corresponding *sample statistics*, or briefly *statistics* (such as sample mean and variance). We consider this problem in this chapter.

UNBIASED ESTIMATES

If the mean of the sampling distribution of a statistic equals the corresponding population parameter, the statistic is called an *unbiased estimator* of the parameter; otherwise, it is called a *biased estimator*. The corresponding values of such statistics are called *unbiased* or *biased* estimates, respectively.

EXAMPLE 1. The mean of the sampling distribution of means $\mu_{\bar{X}}$ is μ, the population mean. Hence the sample mean $\bar{X}$ is an unbiased estimate of the population mean μ.

EXAMPLE 2. The mean of the sampling distribution of variances is

$$\mu_{s^2} = \frac{N-1}{N}\sigma^2$$

where σ^2 is the population variance and N is the sample size (see Table 8.1). Thus the sample variance s^2 is a biased estimate of the population variance σ^2. By using the modified variance

$$\hat{s}^2 = \frac{N}{N-1}s^2$$

we find $\mu_{\hat{s}^2} = \sigma^2$, so that $\hat{s}^2$ is an unbiased estimate of σ^2. However, $\hat{s}$ is a biased estimate of σ.

In the language of expectation (see Chapter 6) we could say that a statistic is unbiased if its expectation equals the corresponding population parameter. Thus $\bar{X}$ and $\hat{s}^2$ are unbiased since $E\{\bar{X}\} = \mu$ and $E\{\hat{s}^2\} = \sigma^2$.

EFFICIENT ESTIMATES

If the sampling distributions of two statistics have the same mean (or expectation), then the statistic with the smaller variance is called an *efficient estimator* of the mean, while the other statistic is called an *inefficient estimator*. The corresponding values of the statistics are called *efficient and unefficient estimates*.

If we consider all possible statistics whose sampling distributions have the same mean, the one with the smallest variance is sometimes called the *most efficient*, or *best*, *estimator* of this mean.

EXAMPLE 3. The sampling distributions of the mean and median both have the same mean, namely, the population mean. However, the variance of the sampling distribution of means is smaller than the variance of the sampling distribution of medians (see Table 8.1). Hence the sample mean gives an efficient estimate of the population mean, while the sample median gives an inefficient estimate of it.

Of all statistics estimating the population mean, the sample mean provides the best (or most efficient) estimate.

In practice, inefficient estimates are often used because of the relative ease with which some of them can be obtained.

POINT ESTIMATES AND INTERVAL ESTIMATES; THEIR RELIABILITY

An estimate of a population parameter given by a single number is called a *point estimate* of the parameter. An estimate of a population parameter given by two numbers between which the parameter may be considered to lie is called an *interval estimate* of the parameter.

Interval estimates indicate the precision, or accuracy, of an estimate and are therefore preferable to point estimates.

EXAMPLE 4. If we say that a distance is measured as 5.28 meters (m), we are giving a point estimate. If, on the other hand, we say that the distance is 5.28 ± 0.03 m (i.e., the distance lies between 5.25 and 5.31 m), we are giving an interval estimate.

A statement of the error (or precision) of an estimate is often called its *reliability*.

CONFIDENCE-INTERVAL ESTIMATES OF POPULATION PARAMETERS

Let μ_S and σ_S be the mean and standard deviation (standard error), respectively, of the sampling distribution of a statistic S. Then if the sampling distribution of S is approximately normal (which as we have seen is true for many statistics if the sample size $N \geq 30$), we can expect to find an actual sample statistic S lying in the intervals $\mu_S - \sigma_S$ to $\mu_S + \sigma_S$, $\mu_S - 2\sigma_S$ to $\mu_S + 2\sigma_S$, or $\mu_S - 3\sigma_S$ to $\mu_S + 3\sigma_S$ about 68.27%, 95.45%, and 99.73% of the time, respectively.

Equivalently, we can expect to find (or we can be *confident* of finding) μ_S in the intervals $S - \sigma_S$ to $S + \sigma_S$, $S - 2\sigma_S$ to $S + 2\sigma_S$, or $S - 3\sigma_S$ to $S + 3\sigma_S$ about 68.27%, 95.45%, and 99.73% of the time, respectively. Because of this, we call these respective intervals the 68.27%, 95.45%, and 99.73% *confidence intervals* for estimating μ_S. The end numbers of these intervals ($S \pm \sigma_S$, $S \pm 2\sigma_S$, and $S \pm 3\sigma_S$) are then called the 68.27%, 95.45%, and 99.73% *confidence limits*, or *fiducial limits*.

Similarly, $S \pm 1.96\sigma_S$ and $S \pm 2.58\sigma_S$ are the 95% and 99% (or 0.95 and 0.99) confidence limits for S. The percentage confidence is often called the *confidence level*. The numbers 1.96, 2.58, etc., in the confidence limits are called *confidence coefficients*, or *critical values*, and are denoted by z_c. From confidence levels we can find confidence coefficients, and vice versa.

Table 9.1 shows the values of z_c corresponding to various confidence levels used in practice. For confidence levels not presented in the table, the values of z_c can be found from the normal-curve area tables (see Appendix II).

Table 9.1

Confidence level	99.73%	99%	98%	96%	95.45%	95%	90%	80%	68.27%	50%
z_c	3.00	2.58	2.33	2.05	2.00	1.96	1.645	1.28	1.00	0.6745

Confidence Intervals for Means

If the statistic S is the sample mean $\bar{X}$, then the 95% and 99% confidence limits for estimating the population mean μ, are given by $\bar{X} \pm 1.96\sigma_{\bar{X}}$ and $\bar{X} \pm 2.58\sigma_{\bar{X}}$, respectively. More generally, the confidence limits are given by $\bar{X} \pm z_c\sigma_{\bar{X}}$, where z_c (which depends on the particular level of confidence desired) can be read from Table 9.1. Using the values of $\sigma_{\bar{X}}$ obtained in Chapter 8, we see that the confidence limits for the population mean are given by

$$\bar{X} \pm z_c \frac{\sigma}{\sqrt{N}} \tag{1}$$

if the sampling is either from an infinite population or with replacement from a finite population, and are given by

$$\bar{X} \pm z_c \frac{\sigma}{\sqrt{N}} \sqrt{\frac{N_p - N}{N_p - 1}} \tag{2}$$

if the sampling is without replacement from a population of finite size N_p.

Generally, the population standard deviation σ is unknown; thus, to obtain the above confidence limits, we use the sample estimate $\hat{s}$ or s. This will prove satisfactory when $N \geq 30$. For $N < 30$, the approximation is poor and small sampling theory must be employed (see Chapter 11).

Confidence Intervals for Proportions

If the statistic S is the proportion of "successes" in a sample of size N drawn from a binomial population in which p is the proportion of successes (i.e., the probability of success), then the confidence limits for p are given by $P \pm z_c\sigma_P$, where P is the proportion of successes in the sample of size N. Using the values of σ_P obtained in Chapter 8, we see that the confidence limits for the population proportion are given by

$$P \pm z_c \sqrt{\frac{pq}{N}} = P \pm z_c \sqrt{\frac{p(1-p)}{N}} \tag{3}$$

if the sampling is either from an infinite population or with replacement from a finite population and are given by

$$P \pm z_c \sqrt{\frac{pq}{N}} \sqrt{\frac{N_p - N}{N_p - 1}} \tag{4}$$

if the sampling is without replacement from a population of finite size N_p.

To compute these confidence limits, we can use the sample estimate P for p, which will generally prove satisfactory if $N \geq 30$. A more exact method for obtaining these confidence limits is given in Problem 9.12.

Confidence Intervals for Differences and Sums

If S_1 and S_2 are two sample statistics with approximately normal sampling distributions, confidence limits for the difference of the population parameters corresponding to S_1 and S_2 are given by

$$S_1 - S_2 \pm z_c \sigma_{S_1 - S_2} = S_1 - S_2 \pm z_c \sqrt{\sigma_{S_1}^2 + \sigma_{S_2}^2} \tag{5}$$

while confidence limits for the sum of the population parameters are given by

$$S_1 + S_2 \pm z_c \sigma_{S_1 + S_2} = S_1 + S_2 \pm z_c \sqrt{\sigma_{S_1}^2 + \sigma_{S_2}^2} \tag{6}$$

provided that the samples are independent (see Chapter 8).

For example, confidence limits for the difference of two population means, in the case where the populations are infinite, are given by

$$\bar{X}_1 - \bar{X}_2 \pm z_c \sigma_{\bar{X}_1 - \bar{X}_2} = \bar{X}_1 - \bar{X}_2 \pm z_c \sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}} \tag{7}$$

where $\bar{X}_1$, σ_1, N_1 and $\bar{X}_2$, σ_2, N_2 are the respective means, standard deviations, and sizes of the two samples drawn from the populations.

Similarly, confidence limits for the difference of two population proportions, where the populations are infinite, are given by

$$P_1 - P_2 \pm z_c \sigma_{P_1 - P_2} = P_1 - P_2 \pm z_c \sqrt{\frac{p_1(1-p_1)}{N_1} + \frac{p_2(1-p_2)}{N_2}} \tag{8}$$

where P_1 and P_2 are the two sample proportions, N_1 and N_2 are the sizes of the two samples drawn from the populations, and p_1 and p_2 are the proportions in the two populations (estimated by P_1 and P_2).

Confidence Intervals for Standard Deviations

The confidence limits for the standard deviation σ of a normally distributed population, as estimated from a sample with standard deviation s, are given by

$$s \pm z_c \sigma_s = s \pm z_c \frac{\sigma}{\sqrt{2N}} \tag{9}$$

using Table 8.1. In computing these confidence limits, we use s or $\hat{s}$ to estimate σ.

PROBABLE ERROR

The 50% confidence limits of the population parameters corresponding to a statistic S are given by $S \pm 0.6745\sigma_S$. The quantity $0.6745\sigma_S$ is known as the *probable error* of the estimate.

Solved Problems

UNBIASED AND EFFICIENT ESTIMATES

9.1 Give an example of estimators (or estimates) that are (*a*) unbiased and efficient, (*b*) unbiased and inefficient, and (*c*) biased and inefficient.

SOLUTION

(*a*) The sample mean $\bar{X}$ and the modified sample variance

$$\hat{s}^2 = \frac{N}{N-1}s^2$$

are two such examples.

(*b*) The sample median and the sample statistic $\frac{1}{2}(Q_1 + Q_3)$, where Q_1 and Q_3 are the lower and upper sample quartiles, are two such examples. Both statistics are unbiased estimates of the population mean, since the mean of their sampling distributions is the population mean.

(*c*) The sample standard deviation s, the modified standard deviation $\hat{s}$, the mean deviation, and the semi-interquartile range are four such examples.

9.2 In a sample of five measurements, the diameter of a sphere was recorded by a scientist as 6.33, 6.37, 6.36, 6.32, and 6.37 centimeters (cm). Determine unbiased and efficient estimates of (*a*) the true mean and (*b*) the true variance.

SOLUTION

(*a*) The unbiased and efficient estimate of the true mean (i.e., the populations mean) is

$$\bar{X} = \frac{\sum X}{N} = \frac{6.33 + 6.37 + 6.36 + 6.32 + 6.37}{5} = 6.35\,\text{cm}$$

(*b*) The unbiased and efficient estimate of the true variance (i.e., the population variance) is

$$\begin{aligned}\hat{s}^2 &= \frac{N}{N-1}s^2 = \frac{\sum(X-\hat{X})^2}{N-1} \\ &= \frac{(6.33-6.35)^2 + (6.37-6.35)^2 + (6.36-6.35)^2 + (6.32-6.35)^2 + (6.37-6.35)^2}{5-1} \\ &= 0.00055\,\text{cm}^2\end{aligned}$$

Note that although $\hat{s} = \sqrt{0.00055} = 0.023\,\text{cm}$ is an estimate of the true standard deviation, this estimate is neither unbiased nor efficient.

9.3 Suppose that the heights of 100 male students at XYZ University represent a random sample of the heights of all 1546 students at the university. Determine unbiased and efficient estimates of (*a*) the true mean and (*b*) the true variance.

SOLUTION

(*a*) From Problem 3.22, the unbiased and efficient estimate of the true mean height is $\bar{X} = 67.45$ inches (in).

(*b*) From Problem 4.17, the unbiased and efficient estimate of the true variance is

$$\hat{s}^2 = \frac{N}{N-1}s^2 = \frac{100}{99}(8.5275) = 8.6136$$

Thus $\hat{s} = \sqrt{8.6136} = 2.93$ in. Note that since N is large, there is essentially no difference between s^2 and $\hat{s}^2$ or between s and $\hat{s}$.

Note that we have not used Sheppard's correction for grouping. To take this into account, we would use $s = 2.79$ in (see Problem 4.21).

9.4 Give an unbiased and inefficient estimate of the true mean diameter of the sphere of Problem 9.2.

SOLUTION

The median is one example of an unbiased and inefficient estimate of the population mean. For the five measurements arranged in order of magnitude, the median is 6.36 cm.

CONFIDENCE INTERVALS FOR MEANS

9.5 Find the (*a*) 95% and (*b*) 99% confidence intervals for estimating the mean height of the XYZ University students in Problem 9.3.

SOLUTION

(*a*) The 95% confidence limits are $\bar{X} \pm 1.96\sigma/\sqrt{N}$. Using $\bar{X} = 67.45$ in, and $\hat{s} = 2.93$ in as an estimate of σ (see Problem 9.3), the confidence limits are $67.45 \pm 1.96(2.93/\sqrt{100})$, or 67.45 ± 0.57 in. Thus the 95% confidence interval for the population mean μ is 66.88 to 68.02 in, which can be denoted by $66.88 < \mu < 68.02$.

We can therefore say that the probability that the population mean height lies between 66.88 and 68.02 in is about 95%, or 0.95. In symbols we write $\Pr\{66.88 < \mu < 68.02\} = 0.95$. This is equivalent to saying that we are 95% *confident* that the population mean (or true mean) lies between 66.88 and 68.02 in.

(*b*) The 99% confidence limits are $\bar{X} \pm 2.58\sigma/\sqrt{N} = \bar{X} \pm 2.58\hat{s}/\sqrt{N} = 67.45 \pm 2.58(2.93/\sqrt{100}) = 67.45 \pm 0.76$ in. Thus the 99% confidence interval for the population mean μ is 66.69 to 68.21 in, which can be denoted by $66.69 < \mu < 68.21$.

In obtaining the above confidence intervals, we assumed that the population was infinite or so large that we could consider conditions to be the same as sampling with replacement. For finite populations where sampling is without replacement, we should use

$$\frac{\sigma}{\sqrt{N}}\sqrt{\frac{N_p - N}{N_p - 1}} \qquad \text{in place of} \qquad \frac{\sigma}{\sqrt{N}}$$

However, we can consider the factor

$$\sqrt{\frac{N_p - N}{N_p - 1}} = \sqrt{\frac{1546 - 100}{1546 - 1}} = 0.967$$

to be essentially 1.0, and thus it need not be used. If it is used, the above confidence limits become 67.45 ± 0.56 in and 67.45 ± 0.73 in, respectively.

9.6 Blaises' Christmas Tree Farm has 5000 trees that are mature and ready to be cut and sold. One-hundred of the trees are randomly selected and their heights measured. The heights in inches are given in Table 9.2. Use MINITAB to set a 95% confidence interval on the mean height of all 5000 trees. If the trees sell for \$2.40 per foot, give a lower and an upper bound on the value of the 5000 trees.

SOLUTION

The MINITAB confidence interval given below indicates that the mean height for the 5000 trees could be as small as 57.24 or as large as 61.20 inches. The total number of inches for all 5000 trees ranges between $(57.24)(5000) = 286{,}2000$ and $(61.20)(5000) = 306{,}000$. If the trees sell for \$2.40 per foot, then cost

Table 9.2

56	61	52	62	63	34	47	35	44	59
70	61	65	51	65	72	55	71	57	75
53	48	55	67	60	60	73	74	43	74
71	53	78	59	56	62	48	65	68	51
73	62	80	53	64	44	67	45	58	48
50	57	72	55	56	62	72	57	49	62
46	61	52	46	72	56	46	48	57	52
54	73	71	70	66	67	58	71	75	50
44	59	56	54	63	43	68	69	55	63
48	49	70	60	67	47	49	69	66	73

per inch is \$0.2. The value of the trees ranges between (286,000)(0.2) = \$57,200 and (306,000)(0.2) = \$61,200 with 95% confidence.

Data Display

```
height
        56   70   53   71   73   50   46   54   44
        48   61   61   48   53   62   57   61   73
        59   49   52   65   55   78   80   72   52
        71   56   70   62   51   67   59   53   55
        46   70   54   60   63   65   60   56   64
        56   72   66   63   67   34   72   60   62
        44   62   56   67   43   47   47   55   73
        48   67   72   46   58   68   49   35   71
        74   65   45   57   48   71   69   69   44
        57   43   68   58   49   57   75   55   66
        59   75   74   51   48   62   52   50   63
        73

MTB > standard deviation cl
```

Column Standard Deviation

```
Standard deviation of height = 10.111
MTB > zinterval 95 percent confidence sd = 10.111 data in cl
```

Confidence Intervals

```
The assumed sigma = 10.1

Variable     N     Mean     StDev    SE Mean     95.0 % CI
height     100    59.22     10.11       1.01   (57.24, 61.20)
```

9.7 In a survey of Catholic priests each priest reported the total number of baptisms, marriages, and funerals conducted during the past calendar year. The responses are given in Table 9.3. Use this data to construct a 95% confidence interval on μ, the mean number of baptisms, marriages,

Table 9.3

32	44	48	35	34	29	31	61	37	41
31	40	44	43	41	40	41	31	42	45
29	40	42	51	16	24	40	52	62	41
32	41	45	24	41	30	42	47	30	46
38	42	26	34	45	58	57	35	62	46

and funerals conducted during the past calendar year per priest for all priests. Construct the interval by use of the confidence interval formula, and also use the `Zinterval` command of MINITAB to find the interval.

SOLUTION

After entering the data from Table 9.3 into column 1 of MINITAB'S worksheet, and naming the column 'number', the mean and standard deviation commands were given.

```
MTB > mean cl
```

Column Mean

```
Mean of Number = 40.261
MTB > standard deviation cl
```

Column Standard Deviation

```
Standard deviation of Number = 9.9895
```

The standard error of the mean is equal to $9.9895/\sqrt{50} = 1.413$, the critical value is 1.96, and the 95% margin of error is $1.96(1.413) = 2.769$. The confidence interval extends from $40.261 - 2.769 = 37.492$ to $40.261 + 2.769 = 43.030$.

The `Zinterval` command produces the following output.

```
MTB > Zinterval 95% confidence sd = 9.9895 data in cl
```

Z Confidence Intervals

```
The assumed sigma = 9.99

Variable    N    Mean     StDev    SE Mean    95.00 % CI
Number     50    40.26    9.99        1.41   (37.49, 43.03)
```

We are 95% confident that the true mean for all priests is between 37.49 and 43.03.

9.8 In measuring reaction time, a psychologist estimates that the standard deviation is 0.05 seconds (s). How large a sample of measurements must he take in order to be (*a*) 95% and (*b*) 99% confident that the error of his estimate will not exceed 0.01 s?

SOLUTION

(*a*) The 95% confidence limits are $\bar{X} \pm 1.96\sigma/\sqrt{N}$, the error of the estimate being $1.96\sigma/\sqrt{N}$. Taking $\sigma = s = 0.05$ s, we see that this error will be equal to 0.01 s if $(1.96)(0.05)/\sqrt{N} = 0.01$; that is, $\sqrt{N} = (1.96)(0.05)/0.01 = 9.8$, or $N = 96.04$. Thus we can be 95% confident that the error of the estimate will be less than 0.01 s if N is 97 or larger.

Another method

$$\frac{(1.96)(0.05)}{\sqrt{N}} \leq 0.01 \quad \text{if} \quad \frac{\sqrt{N}}{(1.96)(0.05)} \geq \frac{1}{0.01} \quad \text{or} \quad \sqrt{N} \geq \frac{(1.96)(0.05)}{0.01} = 9.8$$

Then $N \geq 96.04$, or $N \geq 97$.

(*b*) The 99% confidence limits are $\bar{X} \pm 2.58\sigma/\sqrt{N}$. Then $(2.58)(0.05)/\sqrt{N} = 0.01$, or $N = 166.4$. Thus we can be 99% confident that the error of the estimate will be less than 0.01 s only if N is 167 or larger.

9.9 A random sample of 50 mathematics grades out of a total of 200 showed a mean of 75 and a standard deviation of 10.

(*a*) What are the 95% confidence limits for estimates of the mean of the 200 grades?

(*b*) With what degree of confidence could we say that the mean of all 200 grades is 75 ± 1?

SOLUTION

(*a*) Since the population size is not very large compared with the sample size, we must adjust for it. Then the 95% confidence limits are

$$\bar{X} \pm 1.96\sigma_{\bar{X}} = \bar{X} \pm 1.96 \frac{\sigma}{\sqrt{N}} \sqrt{\frac{N_{\mathrm{p}} - N}{N_{\mathrm{p}} - 1}} = 75 \pm 1.96 \frac{10}{\sqrt{50}} \sqrt{\frac{200 - 50}{200 - 1}} = 75 \pm 2.4$$

(*b*) The confidence limits can be represented by

$$\bar{X} \pm z_c \sigma_{\bar{X}} = \bar{X} \pm z_c \frac{\sigma}{\sqrt{N}} \sqrt{\frac{N_{\mathrm{p}} - N}{N_{\mathrm{p}} - 1}} = 75 \pm z_c \frac{10}{\sqrt{50}} \sqrt{\frac{200 - 50}{200 - 1}} = 75 \pm 1.23 z_c$$

Since this must equal 75 ± 1, we have $1.23z_c = 1$, or $z_c = 0.81$. The area under the normal curve from $z = 0$ to $z = 0.81$ is 0.2910; hence the required degree of confidence is $2(0.2910) = 0.582$, or 58.2%.

CONFIDENCE INTERVALS FOR PROPORTIONS

9.10 A sample poll of 100 voters chosen at random from all voters in a given district indicated that 55% of them were in favor of a particular candidate. Find the (*a*) 95%, (*b*) 99%, and (*c*) 99.73% confidence limits for the proportion of all the voters in favor of this candidate.

SOLUTION

(*a*) The 95% confidence limits for the population p are $P \pm 1.96\sigma_P = P \pm 1.96\sqrt{p(1-p)/N} = 0.55 \pm 1.96\sqrt{(0.55)(0.45)/100} = 0.55 \pm 0.10$, where we have used the sample proportion P to estimate p.

(*b*) The 99% confidence limits for p are $0.55 \pm 2.58\sqrt{(0.55)(0.45)/100} = 0.55 \pm 0.13$.

(*c*) The 99.73% confidence limits for p are $0.55 \pm 3\sqrt{(0.55)(0.45)/100} = 0.55 \pm 0.15$.

9.11 How large a sample of voters should we take in Problem 9.10 in order to be (*a*) 95% and (*b*) 99.73% confident that the candidate will be elected?

SOLUTION

The confidence limits for p are $P \pm z_c\sqrt{p(1-p)/N} = 0.55 \pm z_c\sqrt{(0.55)(0.45)/N} = 0.55 \pm 0.50z_c/\sqrt{N}$, where we have used the estimate $P = p = 0.55$ on the basis of Problem 9.10. Since the candidate will win only if she receives more than 50% of the population's votes, we require that $0.50z_c/\sqrt{N}$ be less than 0.05.

(*a*) For 95% confidence, $0.50z_c/\sqrt{N} = 0.50(1.96)/\sqrt{N} = 0.05$ when $N = 384.2$. Thus N should be at least 385.

(*b*) For 99.73% confidence, $0.50z_c/\sqrt{N} = 0.50(3)/\sqrt{N} = 0.05$ when $N = 900$. Thus N should be at least 901.

Another method

$1.50/\sqrt{N} < 0.05$ when $\sqrt{N}/1.50 > 1/0.05$ or $\sqrt{N} > 1.50/0.05$. Then $\sqrt{N} > 30$ or $N > 900$, so that N should be at least 901.

9.12 A survey is conducted and it is found that 156 out of 500 adult males are smokers. Use the software package STATISTIX to set a 99% confidence interval on p, the proportion in the population of adult males who are smokers. Check the confidence interval by computing it by hand.

SOLUTION

The STATISTIX output is as follows. The 99% confidence interval is shown in bold.

```
One-Sample Proportion Test

Sample Size              500
Successes                156
Proportion           0.31200

Null Hypothesis:     P = 0.5
Alternative Hyp:    P < > 0.5

Difference          -0.18800
Standard Error       0.02072
Z (uncorrected)        -8.41      P   0.0000
Z (corrected)          -8.36      P   0.0000

                     99% Confidence Interval
Uncorrected            (0.25863, 0.36537)
Corrected              (0.25763, 0.36637)
```

We are 99% confident that the true percent of adult male smokers is between 25.9% and 36.5%.

Check:

$$P = 0.312,\ z_c = 2.58,\ \sqrt{\frac{0.312(0.688)}{500}} = 0.0207$$

$P \pm z_c\sqrt{\frac{p(1-p)}{N}}$ or $0.312 \pm 2.58(0.0207)$ or (**0.258**, **0.365**) This is the same as given above by the software package STATISTIX.

9.13 Refer to Problem 9.12. Set a 99% confidence interval on p using the software package MINITAB.

SOLUTION

The 99% confidence interval is shown in bold below. It is the same as the STATISTIX confidence interval shown in Problem 9.12.

```
Sample     X     N     Sample P          99% CI            z-Value      P-Value
1        156   500     0.312000   (0.258629, 0.365371)     -8.41        0.000
```

9.14 A study was undertaken to compare the mean time spent on cell phones by male and female college students per week. Fifty male and 50 female students were selected from Midwestern University and the number of hours per week spent talking on their cell phones determined. The results in hours are shown in Table 9.4. Set a 95% confidence interval on $\mu_1 - \mu_2$ using MINITAB. Check the results by calculating the interval by hand.

Table 9.4

Males					Females				
12	4	11	13	11	11	9	7	10	9
7	9	10	10	7	10	10	7	9	10
7	12	6	9	15	11	8	9	6	11
10	11	12	7	8	10	7	9	12	14
8	9	11	10	9	11	12	12	8	12
10	9	9	7	9	12	9	10	11	7
11	7	10	10	11	12	7	9	8	11
9	12	12	8	13	10	8	13	8	10
9	10	8	11	10	9	9	9	11	9
13	13	9	10	13	9	8	9	12	11

SOLUTION

When both samples exceed 30, the two sample t test and the z test may be used interchangeably since the t distribution and the z distribution are very similar.

```
Two-sample T for males vs females
            N      Mean      StDev     SE Mean
males      50      9.82       2.15        0.30
females    50      9.70       1.78        0.25
Difference = mu (males) - mu (females)
Estimate for difference: 0.120000
95% CI for difference: (-0.663474, 0.903474)
T-Test of difference = 0 (vs not =): T-Value = 0.30 P-Value = 0.762
DF = 98
Both use Pooled StDev = 1.9740
```

According to the MINITAB output, the difference in the population means is between −0.66 and 0.90. There is a good chance that there is no difference in the population means.

Check:

The formula for a 95% confidence interval is $(\bar{x}_1 - \bar{x}_2) \pm z_c\left(\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}\right)$. Substituting, we obtain $0.12 \pm 1.96(0.395)$ and get the answer given by MINITAB.

9.15 Use STATISTIX and SPSS to solve Problem 9.14.

SOLUTION

The STATISTIX solution is given below. Note that the 95% confidence interval is the same as that in Problem 9.14. We shall comment later on why we assume equal variances.

```
Two-Sample T Tests for males vs females

Variable      Mean       N         SD         SE
males       9.8200      50    2.1542    0.3046
females     9.7000      50    1.7757    0.2511
Difference 0.1200

Null Hypothesis: difference = 0
Alternative Hyp: difference < > 0
                                                    95% CI for Difference
Assumption                T       DF        P         Lower        Upper
Equal Variances        0.30     98     0.7618    -0.6635      0.9035
Unequal Variances      0.30     94.6   0.7618    -0.6638      0.9038

Test for Equality           F          DF          P
    of Variances          1.47        49,49       0.0899
```

The SPSS solution is as follows:

Group Statistics

	Sex	N	Mean	Std. Deviation	Std. Error Mean
time	1.00	50	9.7000	1.77569	.25112
	2.00	50	9.8200	2.15416	.30464

Independent Samples Test

		Levene's Test for Equality of Variances		t-test for Equality of Means						
									95% Confidence Interval of the Difference	
		F	Sig.	t	df	Sig. (2-tailed)	Mean Difference	Std. Error Difference	Lower	Upper
time	Equal variances assumed	.898	.346	−.304	98	.762	−.12000	.39480	−.90347	.66347
	Equal variances not assumed			−.304	94.556	.762	−.12000	.39480	−.90383	.66383

9.16 Use SAS to work Problem 9.14. Give the forms of the data files that SAS allows for the analysis.

SOLUTION

The SAS analysis is as follows. The confidence interval is shown in bold at the bottom of the following output.

```
Two Sample t-test for the Means of males and females
Sample Statistics
 Group      N      Mean      Std. Dev.      Std. Error
 ---------------------------------------------------
 males     50      9.82         2.1542          0.3046
 females   50      9.7          1.7757          0.2511

 Null hypothesis: Mean 1- Mean 2 = 0
 Alternative:     Mean 1- Mean 2 ^= 0

 If Variances Are      t statistic     Df        Pr > t
 ------------------------------------------------------
 Equal                   0.304          98       0.7618
 Not Equal               0.304       94.56       0.7618
 95% Confidence Interval for the Difference between Two Means

             Lower Limit       Upper Limit
             -------           -------
                 -0.66            0.90
```

The data file used in the SAS analysis may have the data for males and females in separate columns or the data may consist of hours spent on the cell phone in one column and the sex of the person (Male or Female) in the other column. Male and female may be coded as 1 or 2. The first form consists of 2 columns and 50 rows. The second form consists of 2 columns and 100 rows.

CONFIDENCE INTERVALS FOR STANDARD DEVIATIONS

9.17 A confidence interval for the variance of a population utilizes the chi-square distribution. The $(1-\alpha) \times 100\%$ confidence interval is $\frac{(n-1)S^2}{(\chi^2_{\alpha/2}}) < \sigma^2 < \frac{(n-1)S^2}{(\chi^2_{1-\alpha/2})}$, where n is the sample size, S^2 is the sample variance, $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ come from the chi-square distribution with $(n-1)$ degrees of freedom. Use EXCEL software to find a 99% confidence interval for the variance of twenty 180 ounce containers. The data from twenty containers is shown in Table 9.5.

Table 9.5

181.5	180.8
179.7	182.4
178.7	178.5
183.9	182.2
179.7	180.9
180.6	181.4
180.4	181.4
178.5	180.6
178.8	180.1
181.3	182.2

SOLUTION

The EXCEL worksheet is as follows. The data are in A1:B10. Column D shows the function in column C which gives the values shown.

A	B	C	D
181.5	180.8	2.154211	=VAR(A1:B10)
179.1	182.4	40.93	=19*C1
178.7	178.5	38.58226	=CHIINV(0.005,19)
183.9	182.2	6.843971	=CHIINV(0.995,19)
179.7	180.9		
180.6	181.4	1.06085	=C2/C3
180.4	181.4	5.980446	=C2/C4
178.5	180.6		
178.8	180.1		
181.3	182.2		

The following is a 99% confidence interval for $\sigma^2 : (1.06085 < \sigma^2 < 5.980446)$.The following is a 99% confidence interval for $\sigma : (1.03, 2.45)$.

Note that =VAR(A1:B10) gives S^2, =CHIINV(0.005,19) is the chi-square value with area 0.005 to its right, and =CHIINV(0.995,19) is the chi-square value with area 0.995 to its right. In all cases, the chi-square distribution has 19 degrees of freedom.

9.18 When comparing the variance of one population with the variance of another population, the following $(1-\alpha) \times 100\%$ confidence interval may be used:

$$\frac{S_1^2}{S_2^2} \cdot \frac{1}{F_{\alpha/2(\nu_1,\nu2)}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2} F_{\alpha/2(\nu_2,\nu_1)},$$

where n_1 and n_2 are the two sample sizes, S_1^2 and S_2^2 are the two sample variances, $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ are the numerator and denominator degrees of freedom for the F distribution and the F values are from the F distribution. Table 9.6 gives the number of e-mails sent per week by employees at two different companies.

Set a 95% confidence interval for $\dfrac{\sigma_1}{\sigma_2}$.

Table 9.6

Company 1	Company 2
81	99
104	100
115	104
111	98
85	103
121	113
95	95
112	107
100	98
117	95
113	101
109	109
101	99
	93
	105

SOLUTION

The EXCEL worksheet is shown below. Column D shows the function in column C which gives the values shown. The two sample variances are computed in C1 and C2. The F values are computed in C3 and C4. The end points of the confidence interval for the ratio of the variances are computed in C5 and C6. We see that a 95% confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is (1.568, 15.334). A 95% confidence interval for $\frac{\sigma_1}{\sigma_2}$ is (1.252, 3.916). Note that =FINV(0.025,12,14) is the point associated with the F distribution with $\nu_1 = 12$ and $\nu_2 = 14$ degrees of freedom so that 0.025 of the area is to the right of that point.

A	B	C	D
Company 1	Company 2	148.5769231	=VAR(A2:A14)
81	99	31.06666667	=VAR(B2:B16)
104	100	3.050154789	=FINV(0.025,12,14)
115	104	3.2062117	=FINV(0.025,14,12)
111	98	1.567959436	=(C1/C2)/C3
85	103	15.33376832	=(C1/C2)*C4
121	113		
95	95	1.25218187	=SQRT(C5)
112	107	3.915835584	=SQRT(C6)
100	98		
117	95		
113	101		
109	109		
101	99		
	93		
	105		

PROBABLE ERROR

9.19 The voltages of 50 batteries of the same type have a mean of 18.2 volts (V) and a standard deviation of 0.5 V. Find (a) the probable error of the mean and (b) the 50% confidence limits.

SOLUTION

(a)

$$\text{Probable error of the mean} = 0.674\sigma_{\bar{X}} = 0.6745\frac{\sigma}{\sqrt{N}} = 0.6745\frac{\hat{s}}{\sqrt{N}}$$

$$= 0.6745\frac{s}{\sqrt{N-1}} = 0.6745\frac{0.5}{\sqrt{49}} = 0.048\text{ V}$$

Note that if the standard deviation of 0.5 V is computed as $\hat{s}$, the probable error is $0.6745(0.5/\sqrt{50}) = 0.048$ also, so that either estimate can be used if N is large enough.

(b) The 50% confidence limits are 18 ± 0.048 V.

9.20 A measurement was recorded as 216.480 grams (g) with a probable error of 0.272 g. What are the 95% confidence limits for the measurement?

SOLUTION

The probable error is $0.272 = 0.6745\sigma_{\bar{X}}$, or $\sigma_{\bar{X}} = 0.272/0.6745$. Thus the 95% confidence limits are $\bar{X} \pm 1.96\sigma_{\bar{X}} = 216.480 \pm 1.96(0.272/0.6745) = 216.480 \pm 0.790$ g.

Supplementary Problems

UNBIASED AND EFFICIENT ESTIMATES

9.21 Measurements of a sample of masses were determined to be 8.3, 10.6, 9.7, 8.8, 10.2, and 9.4 kilograms (kg), respectively. Determine unbiased and efficient estimates of (*a*) the population mean and (*b*) the population variance, and (*c*) compare the sample standard deviation with the estimated population standard deviation.

9.22 A sample of 10 television tubes produced by a company showed a mean lifetime of 1200 hours (h) and a standard deviation of 100 h. Estimate (*a*) the mean and (*b*) the standard deviation of the population of all television tubes produced by this company.

9.23 (*a*) Work Problem 9.22 if the same results are obtained for 30, 50, and 100 television tubes.

(*b*) What can you conclude about the relation between sample standard deviations and estimates of population standard deviations for different sample sizes?

CONFIDENCE INTERVALS FOR MEANS

9.24 The mean and standard deviation of the maximum loads supported by 60 cables (see Problem 3.59) are given by 11.09 tons and 0.73 ton, respectively. Find the (*a*) 95% and (*b*) 99% confidence limits for the mean of the maximum loads of all cables produced by the company.

9.25 The mean and standard deviation of the diameters of a sample of 250 rivet heads manufactured by a company are 0.72642 in and 0.00058 in, respectively (see Problem 3.61). Find the (*a*) 99%, (*b*) 98%, (*c*) 95%, and (*d*) 90% confidence limits for the mean diameter of all the rivet heads manufactured by the company.

9.26 Find (*a*) the 50% confidence limits and (*b*) the probable error for the mean diameters in Problem 9.25.

9.27 If the standard deviation of the lifetimes of television tubes is estimated to be 100 h, how large a sample must we take in order to be (*a*) 95%, (*b*) 90%, (*c*) 99%, and (*d*) 99.73% confident that the error in the estimated mean lifetime will not exceed 20 h?

9.28 A group of 50 internet shoppers were asked how much they spent per year on the Internet. Their responses are shown in Table 9.7.

Find an 80% confidence interval for μ, the mean amount spent by all internet shoppers, using the equations found in Chapter 9 as well as statistical software.

Table 9.7

418	379	77	212	378
363	434	348	245	341
331	356	423	330	247
351	151	220	383	257
307	297	448	391	210
158	310	331	348	124
523	356	210	364	406
331	364	352	299	221
466	150	282	221	432
366	195	96	219	202

9.29 A company has 500 cables. A test of 40 cables selected at random showed a mean breaking strength of 2400 pounds (lb) and a standard deviation of 150 lb.

(*a*) What are the 95% and 99% confidence limits for estimating the mean breaking strength of the remaining 460 cables?

(*b*) With what degree of confidence could we say that the mean breaking strength of the remaining 460 cables is 2400 ± 35 lb?

CONFIDENCE INTERVALS FOR PROPORTIONS

9.30 An urn contains an unknown proportion of red and white marbles. A random sample of 60 marbles selected with replacement from the urn showed that 70% were red. Find the (*a*) 95%, (*b*) 99%, and (*c*) 99.73% confidence limits for the actual proportion of red marbles in the urn.

9.31 A poll of 1000 individuals over the age of 65 years was taken to determine the percent of the population in this age group who had an Internet connection. It was found that 387 of the 1000 had an internet connection. Using the equations in the book as well as statistical software, find a 97.5% confidence interval for p.

9.32 It is believed that an election will result in a very close vote between two candidates. What is the least number of voters that one should poll in order to be (*a*) 80%, (*b*) 90%, (*c*) 95%, and (*d*) 99% confident of a decision in favor of either one of the candidates?

CONFIDENCE INTERVALS FOR DIFFERENCES AND SUMS

9.33 Of two similar groups of patients, A and B, consisting of 50 and 100 individuals, respectively, the first was given a new type of sleeping pill and the second was given a conventional type. For the patients in group A, the mean number of hours of sleep was 7.82 with a standard deviation of 0.24 h. For the patients in group B, the mean number of hours of sleep was 6.75 with a standard deviation of 0.30 h. Find the (*a*) 95% and (*b*) 99% confidence limits for the difference in the mean number of hours of sleep induced by the two types of sleeping pills.

9.34 A study was conducted to compare the mean lifetimes of males with the mean lifetimes of females. Random samples were collected from the obituary pages and the data in Table 9.8 were collected.

Table 9.8

Males					Females				
85	53	100	49	65	64	93	82	71	77
60	51	61	83	65	64	60	75	87	60
55	99	56	55	55	61	84	91	61	85
90	72	62	69	59	105	90	59	86	62
49	72	58	60	68	71	99	98	54	94
90	74	85	80	77	98	61	108	79	50
62	65	81	55	71	66	74	60	90	95
78	49	78	80	75	81	86	65	86	81
53	82	109	87	78	92	77	82	86	79
72	104	70	31	50	91	93	63	93	53

Using the above data, the equations in the book, and statistical software, set an 85% confidence interval on $\mu_{MALE} - \mu_{FEMALE}$.

9.35 Two areas of the country are compared as to the percent of teenagers who have at least one cavity. One area fluoridates its water and the other doesn't. The sample from the non-fluoridated area finds that 425 out of 1000 have at least one cavity. The sample from the fluoridated area finds that 376 out of 1000 have at least one cavity. Set a 99% confidence interval on the difference in percents using the equations in the book as well as statistical software.

CONFIDENCE INTERVALS FOR STANDARD DEVIATIONS

9.36 The standard deviation of the breaking strengths of 100 cables tested by a company was 180 lb. Find the (*a*) 95%, (*b*) 99%, and (*c*) 99.73% confidence limits for the standard deviation of all cables produced by the company.

9.37 Solve Problem 9.17 using SAS.

9.38 Solve Problem 9.18 using SAS.