CHAPTER 6

Elementary Probability Theory

DEFINITIONS OF PROBABILITY

Classic Definition

Suppose that an event E can happen in h ways out of a total of n possible equally likely ways. Then the probability of occurrence of the event (called its *success*) is denoted by

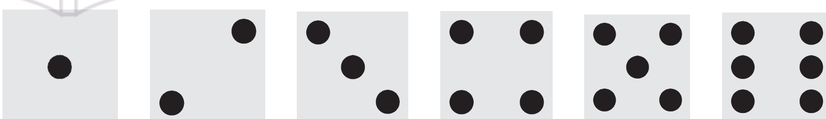
$$p = \Pr\{E\} = \frac{h}{n}$$

The probability of nonoccurrence of the event (called its *failure*) is denoted by

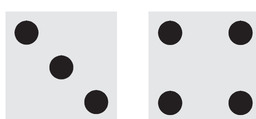
$$q = \Pr\{\text{not } E\} = \frac{n-h}{n} = 1 - \frac{h}{n} = 1 - p = 1 - \Pr\{E\}$$

Thus $p + q = 1$, or $\Pr\{E\} + \Pr\{\text{not } E\} = 1$. The event "not E" is sometimes denoted by $\bar{E}$, $\tilde{E}$, or $\sim E$.

EXAMPLE 1. When a die is tossed, there are 6 equally possible ways in which the die can fall:



The event E, that a 3 or 4 turns up, is:



and the probability of E is $\Pr\{E\} = 2/6$ or $1/3$. The probability of not getting a 3 or 4 (i.e., getting a 1, 2, 5, or 6) is $\Pr\{\bar{E}\} = 1 - \Pr\{E\} = 2/3$.

Note that the probability of an event is a number between 0 and 1. If the event cannot occur, its probability is 0. If it must occur (i.e., its occurrence is *certain*), its probability is 1.

If p is the probability that an event will occur, the *odds* in favor of its happening are $p : q$ (read "p to q"); the odds against its happening are $q : p$. Thus the odds against a 3 or 4 in a single toss of a fair die are $q : p = \frac{2}{3} : \frac{1}{3} = 2 : 1$ (i.e., 2 to 1).

Relative-Frequency Definition

The classic definition of probability has a disadvantage in that the words "equally likely" are vague. In fact, since these words seem to be synonymous with "equally probable," the definition is *circular* because we are essentially defining probability in terms of itself. For this reason, a statistical definition of probability has been advocated by some people. According to this the estimated probability, or *empirical probability*, of an event is taken to be the *relative frequency* of occurrence of the event when the number of observations is very large. The probability itself is the *limit* of the relative frequency as the number of observations increases indefinitely.

EXAMPLE 2. If 1000 tosses of a coin result in 529 heads, the relative frequency of heads is $529/1000 = 0.529$. If another 1000 tosses results in 493 heads, the relative frequency in the total of 2000 tosses is $(529 + 493)/2000 = 0.511$. According to the statistical definition, by continuing in this manner we should ultimately get closer and closer to a number that represents the probability of a head in a single toss of the coin. From the results so far presented, this should be 0.5 to one significant figure. To obtain more significant figures, further observations must be made.

The statistical definition, although useful in practice, has difficulties from a mathematical point of view, since an actual limiting number may not really exist. For this reason, modern probability theory has been developed *axiomatically*; that is, the theory leaves the concept of probability undefined, much the same as *point* and *line* are undefined in geometry.

CONDITIONAL PROBABILITY; INDEPENDENT AND DEPENDENT EVENTS

If E_1 and E_2 are two events, the probability that E_2 occurs given that E_1 has occurred is denoted by $\Pr\{E_2|E_1\}$, or $\Pr\{E_2 \text{ given } E_1\}$, and is called the *conditional probability* of E_2 given that E_1 has occurred.

If the occurrence or nonoccurrence of E_1 does not affect the probability of occurrence of E_2, then $\Pr\{E_2|E_1\} = \Pr\{E_2\}$ and we say that E_1 and E_2 are *independent events*; otherwise, they are *dependent events*.

If we denote by E_1E_2 the event that "both E_1 and E_2 occur," sometimes called a *compound event*, then

$$\Pr\{E_1E_2\} = \Pr\{E_1\}\Pr\{E_2|E_1\} \tag{1}$$

In particular,

$$\Pr\{E_1E_2\} = \Pr\{E_1\}\ \Pr\{E_2\} \qquad \text{for independent events} \tag{2}$$

For three events E_1, E_2, and E_3, we have

$$\Pr\{E_1E_2E_3\} = \Pr\{E_1\}\ \Pr\{E_2|E_1\}\ \Pr\{E_3|E_1E_2\} \tag{3}$$

That is, the probability of occurrence of E_1, E_2, and E_3 is equal to (the probability of E_1) $\times$ (the probability of E_2 given that E_1 has occurred) $\times$ (the probability of E_3 given that both E_1 and E_2 have occurred). In particular,

$$\Pr\{E_1E_2E_3\} = \Pr\{E_1\}\ \Pr\{E_2\}\ \Pr\{E_3\} \qquad \text{for independent events} \tag{4}$$

In general, if $E_1, E_2, E_3, \ldots, E_n$ are n independent events having respective probabilities $p_1, p_2, p_3, \ldots, p_n$, then the probability of occurrence of E_1 and E_2 and E_3 and $\cdots E_n$ is $p_1p_2p_3 \cdots p_n$.

EXAMPLE 3. Let E_1 and E_2 be the events "heads on fifth toss" and "heads on sixth toss" of a coin, respectively. Then E_1 and E_2 are independent events, and thus the probability of heads on both the fifth and sixth tosses is (assuming the coin to be fair)

$$\Pr\{E_1E_2\} = \Pr\{E_1\}\ \Pr\{E_2\} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

EXAMPLE 4. If the probability that A will be alive in 20 years is 0.7 and the probability that B will be alive in 20 years is 0.5, then the probability that they will both be alive in 20 years is $(0.7)(0.5) = 0.35$.

EXAMPLE 5. Suppose that a box contains 3 white balls and 2 black balls. Let E_1 be the event "first ball drawn is black" and E_2 the event "second ball drawn is black," where the balls are not replaced after being drawn. Here E_1 and E_2 are dependent events.

The probability that the first ball drawn is black is $\Pr\{E_1\} = 2/(3+2) = \frac{2}{5}$. The probability that the second ball drawn is black, given that the first ball drawn was black, is $\Pr\{E_2|E_1\} = 1/(3+1) = \frac{1}{4}$. Thus the probability that both balls drawn are black is

$$\Pr\{E_1E_2\} = \Pr\{E_1\}\,\Pr\{E_2|E_1\} = \frac{2}{5}\cdot\frac{1}{4} = \frac{1}{10}$$

MUTUALLY EXCLUSIVE EVENTS

Two or more events are called *mutually exclusive* if the occurrence of any one of them excludes the occurrence of the others. Thus if E_1 and E_2 are mutually exclusive events, then $\Pr\{E_1E_2\} = 0$.

If $E_1 + E_2$ denotes the event that "either E_1 or E_2 or both occur," then

$$\Pr\{E_1 + E_2\} = \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1E_2\} \tag{5}$$

In particular,

$$\Pr\{E_1 + E_2\} = \Pr\{E_1\} + \Pr\{E_2\} \qquad \text{for mutually exclusive events} \tag{6}$$

As an extension of this, if $E_1, E_2, \ldots, E_n$ are n mutually exclusive events having respective probabilities of occurrence $p_1, p_2, \ldots, p_n$, then the probability of occurrence of either E_1 or E_2 or $\cdots E_n$ is $p_1 + p_2 + \cdots + p_n$.

Result (5) can also be generalized to three or more mutually exclusive events.

EXAMPLE 6. If E_1 is the event "drawing an ace from a deck of cards" and E_2 is the event "drawing a king," then $\Pr\{E_1\} = \frac{4}{52} = \frac{1}{13}$ and $\Pr\{E_2\} = \frac{4}{52} = \frac{1}{13}$. The probability of drawing either an ace or a king in a single draw is

$$\Pr\{E_1 + E_2\} = \Pr\{E_1\} + \Pr\{E_2\} = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

since both an ace and a king cannot be drawn in a single draw and are thus mutually exclusive events (Fig. 6-1).

A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠

Fig. 6-1 E_1 is the event "drawing an ace" and E_2 is the event "drawing a king."

Note that E_1 and E_2 have no outcomes in common. They are mutually exclusive.

EXAMPLE 7. If E_1 is the event "drawing an ace" from a deck of cards and E_2 is the event "drawing a spade," then E_1 and E_2 are not mutually exclusive since the ace of spades can be drawn (Fig. 6-2). Thus the probability of drawing either an ace or a spade or both is

$$\Pr\{E_1 + E_2\} = \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1E_2\} = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠

Fig. 6-2 E_1 is the event "drawing an ace" and E_2 is the event "drawing a spade."

Note that the event "E_1 and E_2" consisting of those outcomes in both events is the ace of spades.

PROBABILITY DISTRIBUTIONS

Discrete

If a variable X can assume a discrete set of values $X_1, X_2, \ldots, X_K$ with respective probabilities $p_1, p_2, \ldots, p_K$, where $p_1 + p_2 + \cdots + p_K = 1$, we say that a *discrete probability distribution* for X has been defined. The function $p(X)$, which has the respective values $p_1, p_2, \ldots, p_K$ for $X = X_1, X_2, \ldots, X_K$, is called the *probability function*, or *frequency function*, of X. Because X can assume certain values with given probabilities, it is often called a *discrete random variable*. A random variable is also known as a *chance variable* or *stochastic variable*.

EXAMPLE 8. Let a pair of fair dice be tossed and let X denote the sum of the points obtained. Then the probability distribution is as shown in Table 6.1. For example, the probability of getting sum 5 is $\frac{4}{36} = \frac{1}{9}$; thus in 900 tosses of the dice we would expect 100 tosses to give the sum 5.

Note that this is analogous to a relative-frequency distribution with probabilities replacing the relative frequencies. Thus we can think of probability distributions as theoretical or ideal limiting

Table 6.1

X	2	3	4	5	6	7	8	9	10	11	12
$p(X)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

forms of relative-frequency distributions when the number of observations made is very large. For this reason, we can think of probability distributions as being distributions of *populations*, whereas relative-frequency distributions are distributions of *samples* drawn from this population.

The probability distribution can be represented graphically by plotting $p(X)$ against X, just as for relative-frequency distributions (see Problem 6.11).

By cumulating probabilities, we obtain *cumulative probability distributions*, which are analogous to cumulative relative-frequency distributions. The function associated with this distribution is sometimes called a *distribution function*.

The distribution in Table 6.1 can be built using EXCEL. The following portion of an EXCEL worksheet is built by entering Die1 into A1, Die2 into B1, and Sum into C1. The 36 outcomes on the dice are entered into A2:B37. In C2, =SUM(A2:B2) is entered and a click-and-drag is performed from C2 to C37. By noting that the sum 2 occurs once, the sum 3 occurs twice, etc., the probability distribution in Table 6.1 is formed.

Die1	Die2	Sum
1	1	2
1	2	3
1	3	4
1	4	5
1	5	6
1	6	7
2	1	3
2	2	4
2	3	5
2	4	6
2	5	7

Die1	Die2	Sum
2	6	8
3	1	4
3	2	5
3	3	6
3	4	7
3	5	8
3	6	9
4	1	5
4	2	6
4	3	7
4	4	8
4	5	9
4	6	10
5	1	6
5	2	7
5	3	8
5	4	9
5	5	10
5	6	11
6	1	7
6	2	8
6	3	9
6	4	10
6	5	11
6	6	12

Continuous

The above ideas can be extended to the case where the variable X may assume a continuous set of values. The relative-frequency polygon of a sample becomes, in the theoretical or limiting case of a population, a continuous curve (such as shown in Fig. 6-3) whose equation is $Y = p(X)$. The total area under this curve bounded by the X axis is equal to 1, and the area under the curve between lines $X = a$ and $X = b$ (shaded in Fig. 6-3) gives the probability that X lies between a and b, which can be denoted by $\Pr\{a < X < b\}$.

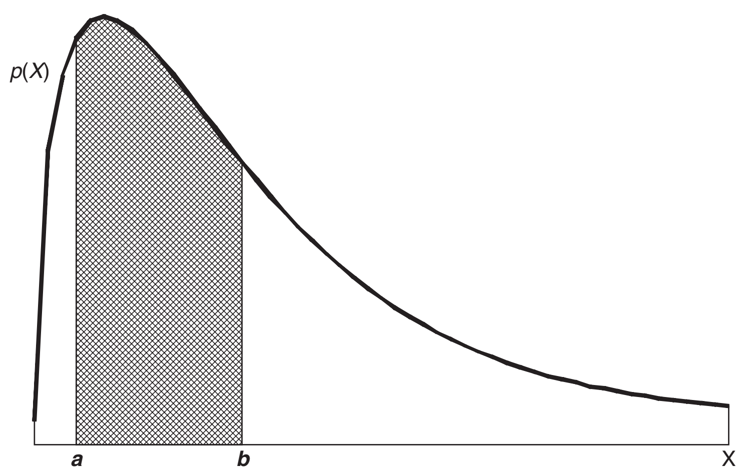


Fig. 6-3 $\Pr\{a<X<b\}$ is shown as the cross-hatched area under the density function.

We call $p(X)$ a *probability density function*, or briefly a *density function*, and when such a function is given we say that a *continuous probability distribution* for X has been defined. The variable X is then often called a *continuous random variable*.

As in the discrete case, we can define cumulative probability distributions and the associated distribution functions.

MATHEMATICAL EXPECTATION

If p is the probability that a person will receive a sum of money S, the *mathematical expectation* (or simply the *expectation*) is defined as pS.

EXAMPLE 9. Find $E(X)$ for the distribution of the sum of the dice given in Table 6.1. The distribution is given in the following EXCEL printout. The distribution is given in A2:B12 where the $p(X)$ values have been converted to their decimal equivalents. In C2, the expression =A2*B2 is entered and a click-and-drag is performed from C2 to C12. In C13, the expression =Sum(C2:C12) gives the mathematical expectation which equals 7.

X	p(X)	XP(X)
2	0.027778	0.055556
3	0.055556	0.166667
4	0.083333	0.333333
5	0.111111	0.555556
6	0.138889	0.833333
7	0.166667	1.166667
8	0.138889	1.111111
9	0.111111	1
10	0.083333	0.833333
11	0.055556	0.611111
12	0.027778	0.333333
		7

The concept of expectation is easily extended. If X denotes a discrete random variable that can assume the values $X_1, X_2, \ldots, X_K$ with respective probabilities $p_1, p_2, \ldots, p_K$, where $p_1 + p_2 + \cdots + p_K = 1$, the *mathematical expectation* of X (or simply the *expectation* of X), denoted by $E(X)$, is defined as

$$E(X) = p_1X_1 + p_2X_2 + \cdots + p_KX_K = \sum_{j=1}^{K} p_jX_j = \sum pX \tag{7}$$

If the probabilities p_j in this expectation are replaced with the relative frequencies f_j/N, where $N = \sum f_j$, the expectation reduces to $(\sum fX)/N$, which is the arithmetic mean $\bar{X}$ of a sample of size N in which $X_1, X_2, \ldots, X_K$ appear with these relative frequencies. As N gets larger and larger, the relative frequencies f_j/N approach the probabilities p_j. Thus we are led to the interpretation that $E(X)$ represents the mean of the population from which the sample is drawn. If we call m the sample mean, we can denote the population mean by the corresponding Greek letter μ (mu).

Expectation can also be defined for continuous random variables, but the definition requires the use of calculus.

RELATION BETWEEN POPULATION, SAMPLE MEAN, AND VARIANCE

If we select a sample of size N at random from a population (i.e., we assume that all such samples are equally probable), then it is possible to show that the *expected value of the sample mean m is the population mean μ*.

It does not follow, however, that the expected value of any quantity computed from a sample is the corresponding population quantity. For example, the expected value of the sample variance as we have defined it is not the population variance, but $(N-1)/N$ times this variance. This is why some statisticians choose to define the sample variance as our variance multiplied by $N/(N-1)$.

COMBINATORIAL ANALYSIS

In obtaining probabilities of complex events, an enumeration of cases is often difficult, tedious, or both. To facilitate the labor involved, use is made of basic principles studied in a subject called *combinatorial analysis*.

Fundamental Principle

If an event can happen in any one of n_1 ways, and if when this has occurred another event can happen in any one of n_2 ways, then the number of ways in which both events can happen in the specified order is $n_1 n_2$.

EXAMPLE 10. The numbers 0 through 5 are entered into A1 through A6 in EXCEL and =FACT(A1) is entered into B1 and a click-and-drag is performed from B1 through B6. Then the chart wizard is used to plot the points. The function =FACT(n) is the same as $n!$. For $n = 0$, 1, 2, 3, 4, and 5, =FACT(n) equals 1, 1, 2, 6, 24, and 120. Figure 6.4 was generated by the chart wizard in EXCEL.

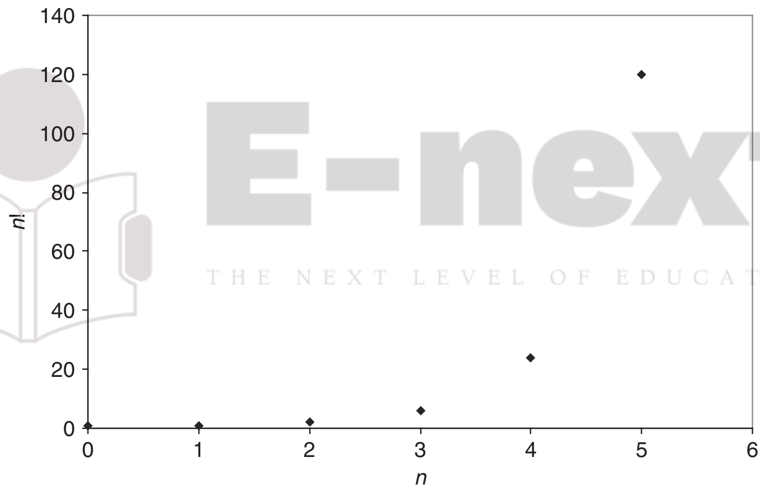


Fig. 6-4 Chart for $n!$ generated by EXCEL.

EXAMPLE 11. The number of permutations of the letters a, b, and c taken two at a time is ${}_3P_2 = 3 \cdot 2 = 6$. These are ab, ba, ac, ca, bc, and cb.

The number of permutations of n objects consisting of groups of which n_1 are alike, n_2 are alike, $\cdots$ is

$$\frac{n!}{n_1!\,n_2!\cdots} \qquad \text{where } n = n_1 + n_2 + \cdots \tag{10}$$

EXAMPLE 12. The number of permutations of letters in the word *statistics* is

$$\frac{10!}{3!\,3!\,1!\,2!\,1!} = 50{,}400$$

since there are 3s's, 3t's, 1a, 2i's, and 1c.

COMBINATIONS

A combination of n different objects taken r at a time is a selection of r out of the n objects, with no attention given to the order of arrangement. The number of combinations of n objects taken r at a time is denoted by the symbol $\binom{n}{r}$ and is given by

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!} \tag{11}$$

EXAMPLE 13. The number of combinations of the letters a, b, and c taken two at a time is

$$\binom{3}{2} = \frac{3 \cdot 2}{2!} = 3$$

These are ab, ac, and bc. Note that ab is the same combination as ba, but not the same permutation.

The number of combinations of 3 things taken 2 at a time is given by the EXCEL command = COMBIN(3,2) which gives the number 3.

STIRLING'S APPROXIMATION TO *n*!

When n is large, a direct evaluation of $n!$ is impractical. In such cases, use is made of an approximate formula developed by James Stirling:

$$n! \approx \sqrt{2\pi n}\, n^n e^{-n} \tag{12}$$

where $e = 2.71828\cdots$ is the natural base of logarithms (see Problem 6.31).

RELATION OF PROBABILITY TO POINT SET THEORY

As shown in Fig. 6-5, a *Venn diagram* represents all possible outcomes of an *experiment* as a rectangle, called the *sample space*, S. Events are represented as four-sided figures or circles inside the sample space. If S contains only a finite number of points, then with each point we can associate a nonnegative number, called *probability*, such that the sum of all numbers corresponding to all points in S add to 1. An event is a set (or collection) of points in S, such as indicated by E_1 and E_2 in Fig. 6-5.

EULER OR VENN DIAGRAMS AND PROBABILITY

The event $E_1 + E_2$ is the set of points that are *either in* E_1 *or* E_2 *or both*, while the event E_1E_2 is the set of points *common to both* E_1 *and* E_2. Thus the probability of an event such as E_1 is the sum of the probabilities associated with all points contained in the set E_1. Similarly, the probability of $E_1 + E_2$, denoted by $\Pr\{E_1 + E_2\}$, is the sum of the probabilities associated with all points contained in the set $E_1 + E_2$. If E_1 and E_2 have no points in common (i.e., the events are mutually exclusive), then $\Pr\{E_1 + E_2\} = \Pr\{E_1\} + \Pr\{E_2\}$. If they have points in common, then $\Pr\{E_1 + E_2\} = \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1E_2\}$.

The set $E_1 + E_2$ is sometimes denoted by $E_1 \cup E_2$ and is called the *union* of the two sets. The set E_1E_2 is sometimes denoted by $E_1 \cap E_2$ and is called the *intersection* of the two sets. Extensions to more than two sets can be made; thus instead of $E_1 + E_2 + E_3$ and $E_1E_2E_3$, we could use the notations $E_1 \cup E_2 \cup E_3$ and $E_1 \cap E_2 \cap E_3$, respectively.

The symbol ϕ (the Greek letter *phi*) is sometimes used to denote a set with no points in it, called the *null set*. The probability associated with an event corresponding to this set is zero (i.e., $\Pr\{\phi\} = 0$). If E_1 and E_2 have no points in common, we can write $E_1E_2 = \phi$, which means that the corresponding

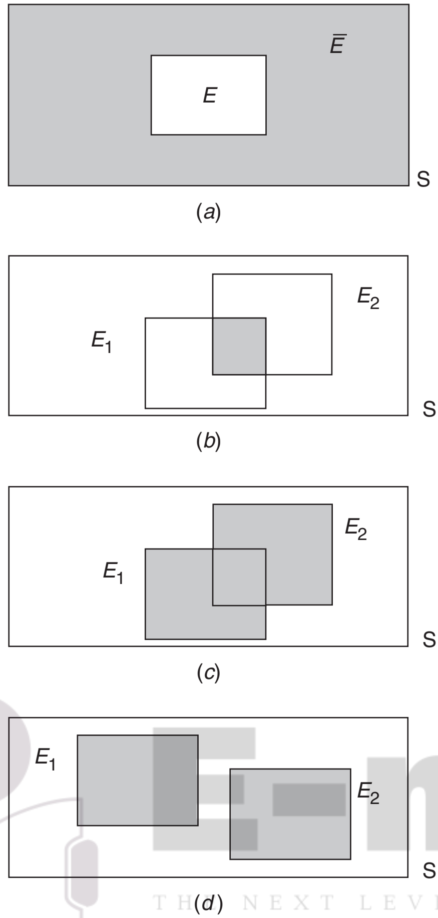


Fig. 6-5 Operations on events. (*a*) Complement of event E shown in gray, written as $\bar{E}$; (*b*) Intersection of events E_1 and E_2 shown in gray, written as $E_1 \cap E_2$; (*c*) Union of events E_1 and E_2 shown in gray, written as $E_1 \cup E_2$; (*d*) Mutually exclusive events E_1 and E_2, that is $E_1 \cap E_2 = \phi$.

events are mutually exclusive, whereby $\Pr\{E_1 E_2\} = 0$.

With this modern approach, a random variable is a function defined at each point of the sample space. For example, in Problem 6.37 the random variable is the sum of the coordinates of each point.

In the case where S has an infinite number of points, the above ideas can be extended by using concepts of calculus.

EXAMPLE 14. An experiment consists of rolling a pair of dice. Event E_1 is the event that a 7 occurs, i. e., the sum on the dice is 7. Event E_2 is the event that an odd number occurs on die 1. Sample space S, events E_1 and E_2 are shown. Find $\Pr\{E_1\}$, $\Pr\{E_2\}$, $\Pr\{E_1 \cap E_2\}$ and $\Pr(E_1 \cup E_2)$. E_1, E_2, and S are shown in separate panels of the MINITAB output shown in Fig. 6-6.

$$P(E_1) = 6/36 = 1/6 \quad P(E_2) = 18/36 = 1/2 \quad P(E_1 \cap E_2) = 3/36 = 1/12$$
$$P(E_1 \cup E_2) = 6/36 + 18/36 - 3/36 = 21/36.$$

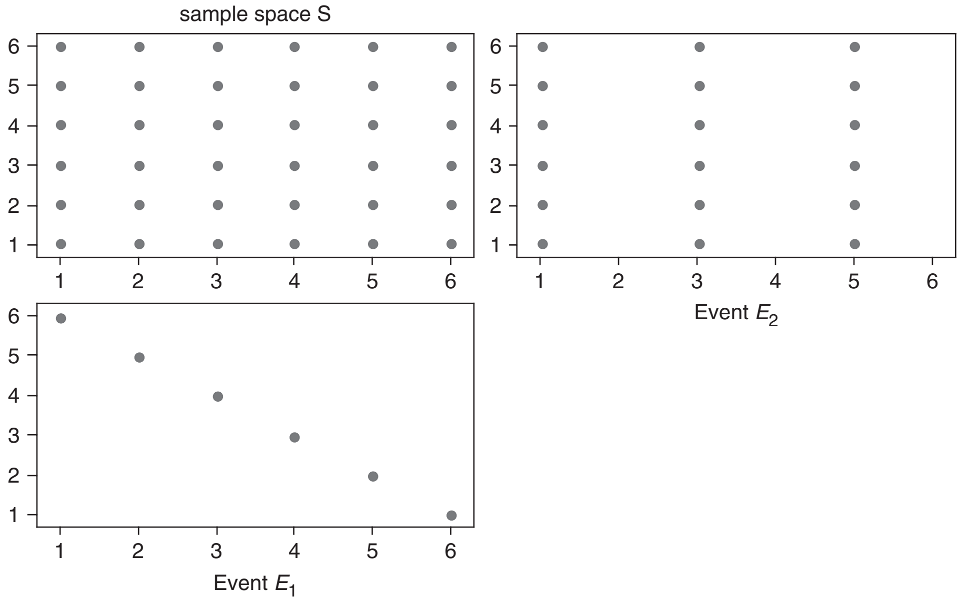


Fig. 6-6 MINITAB output for Example 14.

Solved Problems

FUNDAMENTAL RULES OF PROBABILITY

6.1 Determine the probability p, or an estimate of it, for each of the following events:

(*a*) An odd number appears in a single toss of a fair die.

(*b*) At least one head appears in two tosses of a fair coin.

(*c*) An ace, 10 of diamonds, or 2 of spades appears in drawing a single card from a well-shuffled ordinary deck of 52 cards.

(*d*) The sum 7 appears in a single toss of a pair of fair dice.

(*e*) A tail appears in the next toss of a coin if out of 100 tosses 56 were heads.

SOLUTION

(*a*) Out of six possible equally likely cases, three cases (where the die comes up 1, 3, or 5) are favorable to the event. Thus $p = \frac{3}{6} = \frac{1}{2}$.

(*b*) If H denotes "head" and T denotes "tail," the two tosses can lead to four cases: HH, HT, TH, and TT, all equally likely. Only the first three cases are favorable to the event. Thus $p = \frac{3}{4}$.

(*c*) The event can occur in six ways (ace of spades, ace of hearts, ace of clubs, ace of diamonds, 10 of diamonds, and 2 of spades) out of 52 equally likely cases. Thus $p = \frac{6}{52} = \frac{3}{26}$.

(*d*) Each of the six faces of one die can be associated with each of the six faces of the other die, so that the total number of cases that can arise, all equally likely, is $6 \cdot 6 = 36$. These can be denoted by $(1, 1)$, $(2, 1)$, $(3, 1), \ldots, (6, 6)$.

There are six ways of obtaining the sum 7, denoted by $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$, and $(6, 1)$. Thus $p = \frac{6}{36} = \frac{1}{6}$.

(*e*) Since $100 - 56 = 44$ tails were obtained in 100 tosses, the *estimated* (or *empirical*) *probability* of a tail is the relative frequency $44/100 = 0.44$.

6.2 An experiment consists of tossing a coin and a die. If E_1 is the event that "head" comes up in tossing the coin and E_2 is the event that "3 or 6" comes up in tossing the die, state in words the meaning of each of the following:

(*a*) $\bar{E}_1$ (*c*) E_1E_2 (*e*) $\Pr\{E_1|E_2\}$

(*b*) $\bar{E}_2$ (*d*) $\Pr\{E_1\bar{E}_2\}$ (*f*) $\Pr\{\bar{E}_1 + \bar{E}_2\}$

SOLUTION

(*a*) Tails on the coin and anything on the die

(*b*) 1, 2, 4, or 5 on the die and anything on the coin

(*c*) Heads on the coin and 3 or 6 on the die

(*d*) Probability of heads on the coin and 1, 2, 4, or 5 on the die

(*e*) Probability of heads on the coin, given that a 3 or 6 has come up on the die

(*f*) Probability of tails on the coin or 1, 2, 4, or 5 on the die, or both

6.3 A ball is drawn at random from a box containing 6 red balls, 4 white balls, and 5 blue balls. Determine the probability that the ball drawn is (*a*) red, (*b*) white, (*c*) blue, (*d*) not red, and (*e*) red or white.

SOLUTION

Let R, W, and B denote the events of drawing a red ball, white ball, and blue ball, respectively. Then:

(*a*) $$\Pr\{R\} = \frac{\text{ways of choosing a red ball}}{\text{total ways of choosing a ball}} = \frac{6}{6+4+5} = \frac{6}{15} = \frac{2}{5}$$

(*b*) $$\Pr\{W\} = \frac{4}{6+4+5} = \frac{4}{15}$$

(*c*) $$\Pr\{B\} = \frac{5}{6+4+5} = \frac{5}{15} = \frac{1}{3}$$

(*d*) $$\Pr\{\bar{R}\} = 1 - \Pr\{R\} = 1 - \frac{2}{5} = \frac{3}{5} \qquad \text{by part } (a)$$

(*e*) $$\Pr\{R+W\} = \frac{\text{ways of choosing a red or white ball}}{\text{total ways of choosing a ball}} = \frac{6+4}{6+4+5} = \frac{10}{15} = \frac{2}{3}$$

Another method

$$\Pr\{R+W\} = \Pr\{\bar{B}\} = 1 - \Pr\{B\} = 1 - \frac{1}{3} = \frac{2}{3} \qquad \text{by part } (c)$$

Note that $\Pr\{R+W\} = \Pr\{R\} + \Pr\{W\}$ (i.e., $\frac{2}{3} = \frac{2}{5} + \frac{4}{15}$). This is an illustration of the general rule $\Pr\{E_1 + E_2\} = \Pr\{E_1\} + \Pr\{E_2\}$ that is true for *mutually exclusive* events E_1 and E_2.

6.4 A fair die is tossed twice. Find the probability of getting a 4, 5, or 6 on the first toss and a 1, 2, 3, or 4 on the second toss.

SOLUTION

Let E_1 = event "4, 5, or 6" on the first toss, and let E_2 = event "1, 2, 3, or 4" on the second toss. Each of the six ways in which the die can fall on the first toss can be associated with each of the six ways in which it can fall on the second toss, a total of $6 \cdot 6 = 36$ ways, all equally likely. Each of the three ways in which E_1 can occur can be associated with each of the four ways in which E_2 can occur, to give $3 \cdot 4 = 12$ ways in which both E_1 and E_2, or E_1E_2 occur. Thus $\Pr\{E_1E_2\} = 12/36 = 1/3$.

Note that $\Pr\{E_1E_2\} = \Pr\{E_1\}\ \Pr\{E_2\}$ (i.e., $\frac{1}{3} = \frac{3}{6} \cdot \frac{4}{6}$) is valid for the *independent events* E_1 and E_2.

6.5 Two cards are drawn from a well-shuffled ordinary deck of 52 cards. Find the probability that they are both aces if the first card is (*a*) replaced and (*b*) not replaced.

SOLUTION

Let E_1 = event "ace" on the first draw, and let E_2 = event "ace" on the second draw.

(*a*) If the first card is replaced, E_1 and E_2 are independent events. Thus Pr{both cards drawn are aces} $= \Pr\{E_1 E_2\} = \Pr\{E_1\}\,\Pr\{E_2\} = (\frac{4}{52})(\frac{4}{52}) = \frac{1}{169}$.

(*b*) The first card can be drawn in any one of 52 ways, and the second card can be drawn in any one of 51 ways since the first card is not replaced. Thus both cards can be drawn in $52 \cdot 51$ ways, all equally likely.

There are four ways in which E_1 can occur and three ways in which E_2 can occur, so that both E_1 and E_2, or $E_1 E_2$, can occur in $4 \cdot 3$ ways. Thus $\Pr\{E_1 E_2\} = (4 \cdot 3)/(52 \cdot 51) = \frac{1}{221}$.

Note that $\Pr\{E_2 | E_1\}$ = Pr{second card is an ace given that first card is an ace} $= \frac{3}{51}$. Thus our result is an illustration of the general rule that $\Pr\{E_1 E_2\} = \Pr\{E_1\}\,\Pr\{E_2 | E_1\}$ when E_1 and E_2 are dependent events.

6.6 Three balls are drawn successively from the box of Problem 6.3. Find the probability that they are drawn in the order red, white, and blue if each ball is (*a*) replaced and (*b*) not replaced.

SOLUTION

Let R = event "red" on the first draw, W = event "white" on the second draw, and B = event "blue" on the third draw. We require $\Pr\{RWB\}$.

(*a*) If each ball is replaced, then R, W, and B are independent events and

$$\Pr\{RWB\} = \Pr\{R\}\,\Pr\{W\}\,\Pr\{B\} = \left(\frac{6}{6+4+5}\right)\left(\frac{4}{6+4+5}\right)\left(\frac{5}{6+4+5}\right) = \left(\frac{6}{15}\right)\left(\frac{4}{15}\right)\left(\frac{5}{15}\right) = \frac{8}{225}$$

(*b*) If each ball is not replaced, then R, W, and B are dependent events and

$$\Pr\{RWB\} = \Pr\{R\}\,\Pr\{W|R\}\,\Pr\{B|WR\} = \left(\frac{6}{6+4+5}\right)\left(\frac{4}{5+4+5}\right)\left(\frac{5}{5+3+5}\right)$$
$$= \left(\frac{6}{15}\right)\left(\frac{4}{14}\right)\left(\frac{5}{13}\right) = \frac{4}{91}$$

where $\Pr\{B|WR\}$ is the conditional probability of getting a blue ball if a white and red ball have already been chosen.

6.7 Find the probability of a 4 turning up at least once in two tosses of a fair die.

SOLUTION

Let E_1 = event "4" on the first toss, E_2 = event "4" on the second toss, and $E_1 + E_2$ = event "4" on the first toss or "4" on the second toss or both = event that at least one 4 turns up. We require $\Pr\{E_1 + E_2\}$.

First method

The total number or equally likely ways in which both dice can fall is $6 \cdot 6 = 36$. Also,

$$\begin{aligned}
&\text{Number of ways in which } E_1 \text{ occurs but not } E_2 = 5\\
&\text{Number of ways in which } E_2 \text{ occurs but not } E_1 = 5\\
&\text{Number of ways in which both } E_1 \text{ and } E_2 \text{ occur} = 1
\end{aligned}$$

Thus the number of ways in which at least one of the events E_1 or E_2 occurs is $5 + 5 + 1 = 11$, and thus $\Pr\{E_1 + E_2\} = \frac{11}{36}$.

Second method

Since E_1 and E_2 are not mutually exclusive, $\Pr\{E_1 + E_2\} = \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1 E_2\}$. Also, since E_1 and E_2 are independent, $\Pr\{E_1 E_2\} = \Pr\{E_1\}\Pr\{E_2\}$. Thus $\Pr\{E_1 + E_2\} = \Pr\{E_1\} + \Pr\{E_2\} - \Pr\{E_1\}\Pr\{E_2\} = \frac{1}{6} + \frac{1}{6} - (\frac{1}{6})(\frac{1}{6}) = \frac{11}{36}$.

Third method

$$\Pr\{\text{at least one 4 comes up}\} + \Pr\{\text{no 4 comes up}\} = 1$$

Thus
$$\begin{aligned}\Pr\{\text{at least one 4 comes up}\} &= 1 - \Pr\{\text{no 4 comes up}\} \\ &= 1 - \Pr\{\text{no 4 on first toss and no 4 on second toss}\} \\ &= 1 - \Pr\{\bar{E}_1\bar{E}_2\} = 1 - \Pr\{\bar{E}_1\}\Pr\{\bar{E}_2\} \\ &= 1 - (\tfrac{5}{6})(\tfrac{5}{6}) = \tfrac{11}{36}\end{aligned}$$

6.8 One bag contains 4 white balls and 2 black balls; another contains 3 white balls and 5 black balls. If one ball is drawn from each bag, find the probability that (*a*) both are white, (*b*) both are black, and (*c*) one is white and one is black.

SOLUTION

Let W_1 = event "white" ball from the first bag, and let W_2 = event "white" ball from the second bag.

(*a*) $$\Pr\{W_1W_2\} = \Pr\{W_1\}\Pr\{W_2\} = \left(\frac{4}{4+2}\right)\left(\frac{3}{3+5}\right) = \frac{1}{4}$$

(*b*) $$\Pr\{\bar{W}_1\bar{W}_2\} = \Pr\{\bar{W}_1\}\Pr\{\bar{W}_2\} = \left(\frac{2}{4+2}\right)\left(\frac{5}{3+5}\right) = \frac{5}{24}$$

(*c*) The event "one is white and one is black" is the same as the event "either the first is white and the second is black *or* the first is black and the second is white"; that is, $W_1\bar{W}_2 + \bar{W}_1W_2$. Since events $W_1\bar{W}_2$ and $\bar{W}_1W_2$ are mutually exclusive, we have

$$\begin{aligned}\Pr\{W_1\bar{W}_2 + \bar{W}_1W_2\} &= \Pr\{W_1\bar{W}_2\} + \Pr\{\bar{W}_1W_2\} \\ &= \Pr\{W_1\}\Pr\{\bar{W}_2\} + \Pr\{\bar{W}_1\}\Pr\{W_2\} \\ &= \left(\frac{4}{4+2}\right)\left(\frac{5}{3+5}\right) + \left(\frac{2}{4+2}\right)\left(\frac{3}{3+5}\right) = \frac{13}{24}\end{aligned}$$

Another method

The required probability is $1 - \Pr\{W_1W_2\} - \Pr\{\bar{W}_1\bar{W}_2\} = 1 - \frac{1}{4} - \frac{5}{24} = \frac{13}{24}$.

6.9 *A* and *B* play 12 games of chess, of which 6 are won by *A*, 4 are won by *B*, and 2 end in a draw. They agree to play a match consisting of 3 games. Find the probability that (*a*) *A* wins all 3 games, (*b*) 2 games end in a draw, (*c*) *A* and *B* win alternately, and (*d*) *B* wins at least 1 game.

SOLUTION

Let A_1, A_2, and A_3 denote the events "*A* wins" in the first, second, and third games, respectively; let B_1, B_2, and B_3 denote the events "*B* wins" in the first, second, and third games, respectively; and let, D_1, D_2, and D_3 denote the events "there is a draw" in the first, second, and third games, respectively.

On the basis of their past experience (empirical probability), we shall assume that $\Pr\{A \text{ wins any one game}\} = \frac{6}{12} = \frac{1}{2}$, that $\Pr\{B \text{ wins any one game}\} = \frac{4}{12} = \frac{1}{3}$, and that $\Pr\{\text{any one game ends in a draw}\} = \frac{2}{12} = \frac{1}{6}$.

(*a*) $$\Pr\{A \text{ wins all 3 games}\} = \Pr\{A_1A_2A_3\} = \Pr\{A_1\}\Pr\{A_2\}\Pr\{A_3\} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

assuming that the results of each game are independent of the results of any others, which appears to be justifiable (unless, of course, the players happen to be *psychologically influenced* by the other one's winning or losing).

(*b*) Pr{2 games end in a draw} = Pr{1st and 2nd *or* 1st and 3rd *or* 2nd and 3rd games end in a draw}

$$= \Pr\{D_1 D_2 \bar{D}_3\} + \Pr\{D_1 \bar{D}_2 D_3\} + \Pr\{\bar{D}_1 D_2 D_3\}$$
$$= \Pr\{D_1\}\Pr\{D_2\}\Pr\{\bar{D}_3\} + \Pr\{D_1\}\Pr\{\bar{D}_2\}\Pr\{D_3\} + \Pr\{\bar{D}_1\}\Pr\{D_2\}\Pr\{D_3\}$$
$$= \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{15}{216} = \frac{5}{72}$$

(*c*) Pr{*A* and *B* win alternately} = Pr{*A* wins then *B* wins then *A* wins *or* *B* wins then *A* wins then *B* wins}

$$= \Pr\{A_1 B_2 A_3 + B_1 A_2 B_3\} = \Pr\{A_1 B_2 A_3\} + \Pr\{B_1 A_2 B_3\}$$
$$= \Pr\{A_1\}\Pr\{B_2\}\Pr\{A_3\} + \Pr\{B_1\}\Pr\{A_2\}\Pr\{B_3\}$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{5}{36}$$

(*d*) Pr{*B* wins at least 1 game} = 1 − Pr{*B* wins no game}

$$= 1 - \Pr\{\bar{B}_1 \bar{B}_2 \bar{B}_3\} = 1 - \Pr\{\bar{B}_1\}\Pr\{\bar{B}_2\}\Pr\{\bar{B}_3\}$$
$$= 1 - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{19}{27}$$

PROBABILITY DISTRIBUTIONS

6.10 Find the probability of boys and girls in families with three children, assuming equal probabilities for boys and girls.

SOLUTION

Let B = event "boy in the family," and let G = event "girl in the family." Thus according to the assumption of equal probabilities, $\Pr\{B\} = \Pr\{G\} = \frac{1}{2}$. In families of three children the following mutually exclusive events can occur with the corresponding indicated probabilities:

(*a*) Three boys (*BBB*):

$$\Pr\{BBB\} = \Pr\{B\}\Pr\{B\}\Pr\{B\} = \frac{1}{8}$$

Here we assume that the birth of a boy is not influenced in any manner by the fact that a previous child was also a boy, that is, we assume that the events are independent.

(*b*) Three girls (*GGG*): As in part (*a*) or by symmetry,

$$\Pr\{GGG\} = \frac{1}{8}$$

(*c*) Two boys and one girl ($BBG + BGB + GBB$):

$$\Pr\{BBG + BGB + GBB\} = \Pr\{BBG\} + \Pr\{BGB\} + \Pr\{GBB\}$$
$$= \Pr\{B\}\Pr\{B\}\Pr\{G\} + \Pr\{B\}\Pr\{G\}\Pr\{B\} + \Pr\{G\}\Pr\{B\}\Pr\{B\}$$
$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

(*d*) Two girls and one boy ($GGB + GBG + BGG$): As in part (*c*) or by symmetry, the probability is $\frac{3}{8}$.

If we call X the *random variable* showing the number of boys in families with three children, the probability distribution is as shown in Table 6.2.

Table 6.2

Number of boys X	0	1	2	3
Probability $p(X)$	1/8	3/8	3/8	1/8

6.11 Graph the distribution of Problem 6.10.

SOLUTION

The graph can be represented either as in Fig. 6-7 or Fig. 6-8. Note that the sum of the areas of the rectangles in Fig. 6-8 is 1; in this figure, called a *probability histogram*, we are considering X as a continuous variable even though it is actually discrete, a procedure that is often found useful. Figure 6-7, on the other hand, is used when one does not wish to consider the variable as continuous.

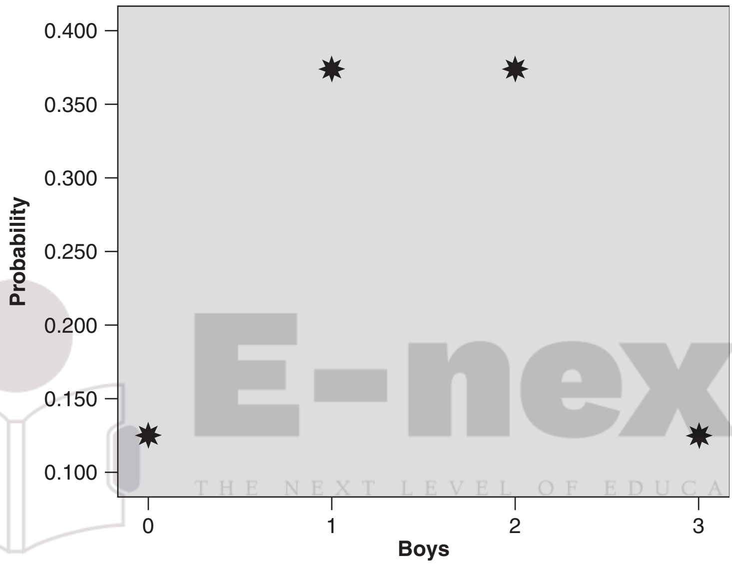


Fig. 6-7 SPSS plot of probability distribution.

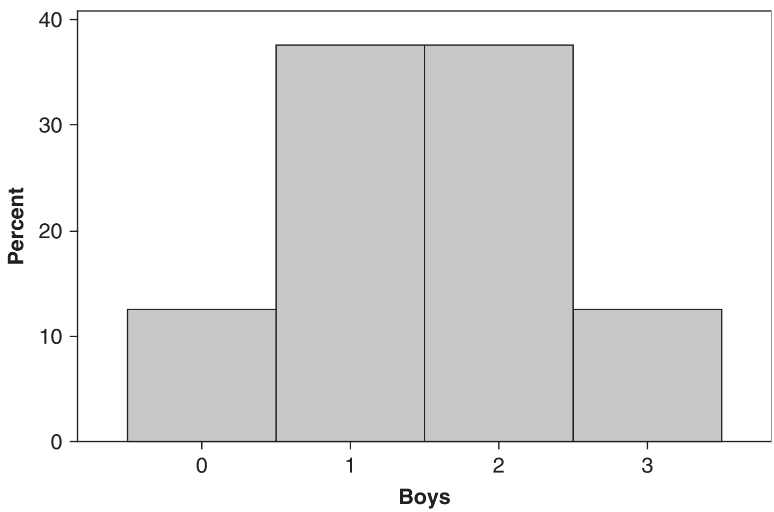


Fig. 6-8 MINITAB probability histogram.

6.12 A continuous random variable X, having values only between 0 and 5, has a density function given by $p(X) = \begin{cases} 0.2, & 0 < X < 5 \\ 0, & \text{otherwise} \end{cases}$. The graph is shown in Fig. 6-9.

(*a*) Verify that it is a density function.

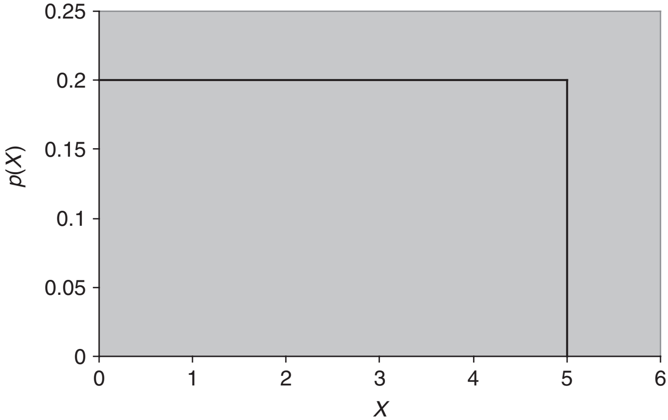


Fig. 6-9 Probability density function for the variable X.

(*b*) Find and graph $\Pr\{2.5 < X < 4.0\}$.

SOLUTION

(*a*) The function $p(X)$ is always ≥ 0 and the total area under the graph of $p(X)$ is $5 \times 0.2 = 1$ since it is rectangular in shape and has width 0.2 and length 5 (see Fig. 6-9).

(*b*) The probability $\Pr\{2.5 < X < 4.0\}$ is shown in Fig. 6-10.

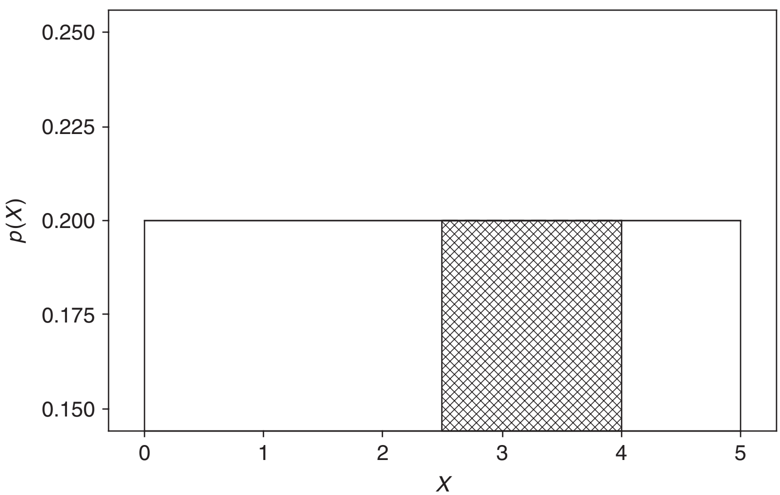


Fig. 6-10 The probability $\Pr(2.5 < X < 4.0)$ is shown as the cross-hatched area.

The rectangular area, $\Pr\{2.5 < X < 4.0\}$, is $(4 - 2.5) \times 0.2 = 0.3$.

6.13 If a man purchases a raffle ticket, he can win a first prize of \$5000 or a second prize of \$2000 with probabilities 0.001 and 0.003. What should be a fair price to pay for the ticket?

SOLUTION

His expectation is (\$5000)(0.001) + (\$2000)(0.003) = \$5 + \$6 = \$11, which is a fair price to pay.

6.14 In a given business venture a lady can make a profit of \$300 with probability 0.6 or take a loss of \$100 with probability 0.4. Determine her expectation.

SOLUTION

Her expectation is (\$300)(0.6) + (−\$100)(0.4) = \$180 − \$40 = \$140.

6.15 Find (*a*) $E(X)$, (*b*) $E(X^2)$, and (*c*) $E[(X-\bar{X})^2)]$ for the probability distribution shown in Table 6.3.
(*d*) Give the EXCEL solution to parts (*a*), (*b*), and (*c*).

Table 6.3

X	8	12	16	20	24
$p(X)$	1/8	1/6	3/8	1/4	1/12

SOLUTION

(*a*) $E(X)=\sum Xp(X)=(8)(\frac{1}{8})+(12)(\frac{1}{6})+(16)(\frac{3}{8})+(20)(\frac{1}{4})+(24)(\frac{1}{12})=16$; this represents the *mean* of the distribution.

(*b*) $E(X^2)=\sum X^2p(X)=(8)^2(\frac{1}{8})+(12)^2(\frac{1}{6})+(16)^2(\frac{3}{8})+(20)^2(\frac{1}{4})+(24)^2(\frac{1}{12})=276$; this represents the *second moment* about the origin zero.

(*c*) $E[(X-\bar{X})^2]=\sum(X-\bar{X})^2p(X)=(8-16)^2(\frac{1}{8})+(12-16)^2(\frac{1}{6})+(16-16)^2(\frac{3}{8})+(20-16)^2(\frac{1}{4})+$ $(24-16)^2(\frac{1}{12})=20$; this represents the *variance* of the distribution.

(*d*) The labels are entered into A1:E1 as shown. The X values and the probability values are entered into A2:B6. The expected values of X are computed in C2:C7. The expected value is given in C7. The second moment about the origin is computed in D2:D7. The second moment is given in D7. The variance is computed in E2:E7. The variance is given in E7.

A	B	C	D	E
X	P(X)	Xp(X)	X^2p(X)	(X − E(X))^2*p(X)
8	0.125	1	8	8
12	0.166667	2	24	2.666666667
16	0.375	6	96	0
20	0.25	5	100	4
24	0.083333	2	48	5.333333333
		16	276	20

6.16 A bag contains 2 white balls and 3 black balls. Each of four persons, A, B, C, and D, in the order named, draws one ball and does not replace it. The first to draw a white ball receives \$10. Determine the expectations of A, B, C, and D.

SOLUTION

Since only 3 black balls are present, one person must win on his or her first attempt. Denote by A, B, C, and D the events "A wins," "B wins," "C wins," and "D wins," respectively.

$$\Pr\{A \text{ wins}\} = \Pr\{A\} = \frac{2}{3+2} = \frac{2}{5}$$

Thus A's expectation $= \frac{2}{5}(\$10) = \$4$.

$$\Pr\{A \text{ loses and } B \text{ wins}\} = \Pr\{\bar{A}B\} = \Pr\{\bar{A}\}\ \Pr\{B|\bar{A}\} = \left(\frac{3}{5}\right)\left(\frac{2}{4}\right) = \frac{3}{10}$$

Thus B's expectation $= \$3$.

$$\Pr\{A \text{ and } B \text{ lose and } C \text{ wins}\} = \Pr\{\bar{A}\bar{B}C\} = \Pr\{\bar{A}\}\ \Pr\{\bar{B}|\bar{A}\}\ \Pr\{C\bar{A}\bar{B}\} = \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{5}$$

Thus C's expectation $= \$2$.

$$\begin{aligned}\Pr\{A, B, \text{ and } C \text{ lose and } D \text{ wins}\} &= \Pr\{\bar{A}\bar{B}\bar{C}D\} \\ &= \Pr\{\bar{A}\}\ \Pr\{\bar{B}|\bar{A}\}\ \Pr\{\bar{C}|\bar{A}\bar{B}\}\ \Pr\{D|\bar{A}\bar{B}\bar{C}\} \\ &= \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{3}\right)\left(\frac{1}{1}\right) = \frac{1}{10}\end{aligned}$$

Thus D's expectation $= \$1$.
Check: $\$4 + \$3 + \$2 + \$1 = \$10$, and $\frac{2}{5} + \frac{3}{10} + \frac{1}{5} + \frac{1}{10} = 1$.

PERMUTATIONS

6.17 In how many ways can 5 differently colored marbles be arranged in a row?

SOLUTION

We must arrange the 5 marbles in 5 positions: $- - - - -$. The first position can be occupied by any one of 5 marbles (i.e., there are 5 ways of filling the first position). When this has been done, there are 4 ways of filling the second position. Then there are 3 ways of filling the third position, 2 ways of filling the fourth position, and finally only 1 way of filling the last position. Therefore:

$$\text{Number of arrangements of 5 marbles in a row} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$$

In general,

$$\text{Number of arrangements of } n \text{ different objects in a row} = n(n-1)(n-2)\cdots 1 = n!$$

This is also called the number of *permutations* of n different objects taken n at a time and is denoted by ${}_nP_n$.

6.18 In how many ways can 10 people be seated on a bench if only 4 seats are available?

SOLUTION

The first seat can be filled in any one of 10 ways, and when this has been done there are 9 ways of filling the second seat, 8 ways of filling the third seat, and 7 ways of filling the fourth seat. Therefore:

$$\text{Number of arrangements of 10 people taken 4 at a time} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$$

In general,

$$\text{Number of arrangements of } n \text{ different objects taken } r \text{ at a time} = n(n-1)\cdots(n-r+1)$$

This is also called the number of *permutations* of n different objects taken r at a time and is denoted by ${}_nP_r$, $P(n, r)$ or $P_{n,r}$. Note that when $r = n$, ${}_nP_n = n!$, as in Problem 6.17.

6.19 Evaluate (*a*) ${}_8P_3$, (*b*) ${}_6P_4$, (*c*) ${}_{15}P_1$, and ${}_3P_3$, and (*e*) parts (*a*) through (*d*) using EXCEL.

SOLUTION

(*a*) ${}_8P_3 = 8 \cdot 7 \cdot 6 = 336$, (*b*) ${}_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3 = 360$, (*c*) ${}_{15}P_1 = 15$, and (*d*) ${}_3P_3 = 3 \cdot 2 \cdot 1 = 6$,

(*e*) =PERMUT(8, 3) = 336 = PERMUT(6, 4) = 360
=PERMUT(15, 1) = 15 = PERMUT(15, 1) = 6

6.20 It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

SOLUTION

The men may be seated in ${}_5P_5$ ways and the women in ${}_4P_4$ ways; each arrangement of the men may be associated with each arrangement of the women. Hence the required number of arrangements is ${}_5P_5 \cdot {}_4P_4 = 5!4! = (120)(24) = 2880$.

6.21 How many four-digit numbers can be formed with the 10 digits 0, 1, 2, 3, ..., 9, if (*a*) repetitions are allowed, (*b*) repetitions are not allowed, and (*c*) the last digit must be zero and repetitions are not allowed?

SOLUTION

(*a*) The first digit can be any one of 9 (since 0 is not allowed). The second, third and fourth digits can be any one of 10. Then $9 \cdot 10 \cdot 10 \cdot 10 = 9000$ numbers can be formed.

(*b*) The first digit can be any one of 9 (any one but 0).
The second digit can be any one of 9 (any but that used for the first digit).
The third digit can be any one of 8 (any but those used for the first two digits).
The fourth digit can be any one of 7 (any but those used for the first three digits).
Thus $9 \cdot 9 \cdot 8 \cdot 7 = 4536$ numbers can be formed.

Another method

The first digit can be any one of 9 and the remaining three can be chosen in ${}_9P_3$ ways. Thus $9 \cdot {}_9P_3 = 9 \cdot 9 \cdot 8 \cdot 7 = 4536$ numbers can be formed.

(*c*) The first digit can be chosen in 9 ways, the second in 8 ways, and the third in 7 ways. Thus $9 \cdot 8 \cdot 7 = 504$ numbers can be formed.

Another method

The first digit can be chosen in 9 ways and the next two digits in ${}_9P_2$ ways. Thus $9 \cdot {}_8P_2 = 9 \cdot 8 \cdot 7 = 504$ numbers can be found.

6.22 Four different mathematics books, 6 different physics books, and 2 different chemistry books are to be arranged on a shelf. How many different arrangements are possible if (*a*) the books in each particular subject must all stand together and (*b*) only the mathematics books must stand together?

SOLUTION

(*a*) The mathematics books can be arranged among themselves in ${}_4P_4 = 4!$ ways, the physics books in ${}_6P_6 = 6!$ ways, the chemistry books in ${}_2P_2 = 2!$ ways, and the three groups in ${}_3P_3 = 3!$ ways. Thus the required number of arrangements $= 4!\,6!\,2!\,3! = 207{,}360$.

(*b*) Consider the 4 mathematics books as one big book. Then we have 9 books that can be arranged in ${}_9P_9 = 9!$ ways. In all of these ways the mathematics books are together. But the mathematics books can be arranged among themselves in ${}_4P_4 = 4!$ ways. Thus the required number of arrangements $= 9!\,4! = 8{,}709{,}120$.

6.23 Five red marbles, 2 white marbles, and 3 blue marbles are arranged in a row. If all the marbles of the same color are not distinguishable from each other, how many different arrangements are possible? Use the EXCEL function =MULTINOMIAL to evaluate the expression.

SOLUTION

Assume that there are P different arrangements. Multiplying P by the numbers of ways of arranging (*a*) the 5 red marbles among themselves, (*b*) the 2 white marbles among themselves, and (*c*) the 3 blue marbles among themselves (i.e., multiplying P by $5!\,2!\,3!$), we obtain the number of ways of arranging the 10 marbles if they are distinguishable (i.e., 10!). Thus

$$(5!2!3!)P = 10! \qquad \text{and} \qquad P = \frac{10!}{5!2!3!}$$

In general, the number of different arrangements of n objects of which n_1 are alike, n_2 are alike, $\ldots, n_k$ are alike is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

where $n_1 + n_2 + \cdots + n_k = n$.

The EXCEL function =MULTINOMIAL(5,2,3) gives the value 2520.

6.24 In how many ways can 7 people be seated at a round table if (*a*) they can sit anywhere and (*b*) 2 particular people must not sit next to each other?

SOLUTION

(*a*) Let 1 of them be seated anywhere. Then the remaining 6 people can be seated in $6! = 720$ ways, which is the total number of ways of arranging the 7 people in a circle.

(*b*) Consider the 2 particular people as 1 person. Then there are 6 people altogether and they can be arranged in 5! ways. But the 2 people considered as 1 can be arranged among themselves in 2! ways. Thus the number of ways of arranging 6 people at a round table with 2 particular people sitting together $= 5!2! = 240$.

Then using part (*a*), the total number of ways in which 6 people can be seated at a round table so that the 2 particular people do not sit together $= 720 - 240 = 480$ ways.

COMBINATIONS

6.25 In how many ways can 10 objects be split into two groups containing 4 and 6 objects, respectively.

SOLUTION

This is the same as the number of arrangements of 10 objects of which 4 objects are alike and 6 other objects are alike. By Problem 6.23, this is

$$\frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$$

The problem is equivalent to finding the number of selections of 4 out of 10 objects (or 6 out of 10 objects), the order of selection being immaterial.

In general the number of selections of r out of n objects, called the number of *combinations* of n things taken r at a time, is denoted by $\binom{n}{r}$ and is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{{}_nP_r}{r!}$$

6.26 Evaluate (*a*) $\binom{7}{4}$, (*b*) $\binom{6}{5}$, (*c*) $\binom{4}{4}$, and (*d*) parts (*a*) through (*c*) using EXCEL.

SOLUTION

(*a*) $$\binom{7}{4} = \frac{7!}{4!\,3!} = \frac{7\cdot 6\cdot 5\cdot 4}{4!} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35$$

(*b*) $$\binom{6}{5} = \frac{6!}{5!\,1!} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{5!} = 6 \qquad \text{or} \qquad \binom{6}{5} = \binom{6}{1} = 6$$

(*c*) $\binom{4}{4}$ is the number of selections of 4 objects taken all at a time, and there is only one such selection; thus $\binom{4}{4} = 1$, Note that formally

$$\binom{4}{4} = \frac{4!}{4!\,0!} = 1$$

if we *define* $0! = 1$.

(*d*) = COMBIN(7, 4) gives 35, = COMBIN(6, 5) gives 6, and = COMBIN(4, 4) gives 1.

6.27 In how many ways can a committee of 5 people be chosen out of 9 people?

SOLUTION

$$\binom{9}{5} = \frac{9!}{5!\,4!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{5!} = 126$$

6.28 Out of 5 mathematicians and 7 physicists, a committee consisting of 2 mathematicians and 3 physicists is to be formed. In how many ways can this be done if (*a*) any mathematician and any physicist can be included, (*b*) one particular physicist must be on the committee, and (*c*) two particular mathematicians cannot be on the committee?

SOLUTION

(*a*) Two mathematicians out of 5 can be selected in $\binom{5}{2}$ ways, and 3 physicists out of 7 can be selected in $\binom{7}{3}$ ways. The total number of possible selections is

$$\binom{5}{2}\cdot\binom{7}{3} = 10\cdot 35 = 350$$

(*b*) Two mathematicians out of 5 can be selected in $\binom{5}{2}$ ways, and 2 additional physicists out of 6 can be selected in $\binom{6}{2}$ ways. The total number of possible selections is

$$\binom{5}{2}\cdot\binom{6}{2} = 10\cdot 15 = 150$$

(*c*) Two mathematicians out of 3 can be selected in $\binom{3}{2}$ ways, and 3 physicists out of 7 can be selected in $\binom{7}{3}$ ways. The total number possible selections is

$$\binom{3}{2}\cdot\binom{7}{3} = 3\cdot 35 = 105$$

6.29 A girl has 5 flowers, each of a different variety. How many different bouquets can she form?

SOLUTION

Each flower can be dealt with in 2 ways: It can be chosen or not chosen. Since each of the 2 ways of dealing with a flower is associated with 2 ways of dealing with each of the other flowers, the number of ways of dealing with the 5 flowers $= 2^5$. But these 2^5 ways include the case in which no flower is chosen. Hence the required number of bouquets $= 2^5 - 1 = 31$.

Another method

She can select either 1 out of 5 flowers, 2 out of 5 flowers, ..., 5 out of 5 flowers. Thus the required number of bouquets is

$$\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 5 + 10 + 10 + 5 + 1 = 31$$

In general, for any positive integer n,

$$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n - 1$$

6.30 From 7 consonants and 5 vowels, how many words can be formed consisting of 4 different consonants and 3 different vowels? The words need not have meaning.

SOLUTION

The 4 different consonants can be selected in $\binom{7}{4}$ ways, the 3 different vowels can be selected in $\binom{5}{3}$ ways, and the resulting 7 different letters (4 consonants and 3 vowels) can then be arranged among themselves in ${}_7P_7 = 7!$ ways. Thus the number of words is

$$\binom{7}{4} \cdot \binom{5}{3} \cdot 7! = 35 \cdot 10 \cdot 5040 = 1{,}764{,}000$$

STIRLING'S APPROXIMATION TO $n!$

6.31 Evaluate 50!.

SOLUTION

For large n, we have $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$; thus

$$50! \approx \sqrt{2\pi(50)}\, 50^{50} e^{-50} = S$$

To evaluate S, use logarithms to the base 10. Thus

$$\begin{aligned}\log S = \log(\sqrt{100\pi}\, 50^{50} e^{-50}) &= \tfrac{1}{2}\log 100 + \tfrac{1}{2}\log \pi + 50\log 50 - 50\log e \\ &= \tfrac{1}{2}\log 100 + \tfrac{1}{2}\log 3.142 + 50\log 50 - 50\log 2.718 \\ &= \tfrac{1}{2}(2) + \tfrac{1}{2}(0.4972) + 50(1.6990) - 50(0.4343) = 64.4846\end{aligned}$$

from which $S = 3.05 \times 10^{64}$, a number that has 65 digits.

PROBABILITY AND COMBINATORIAL ANALYSIS

6.32 A box contains 8 red, 3 white, and 9 blue balls. If 3 balls are drawn at random, determine the probability that (*a*) all 3 are red, (*b*) all 3 are white, (*c*) 2 are red and 1 is white, (*d*) at least 1 is white, (*e*) 1 of each color is drawn, and (*f*) the balls are drawn in the order red, white, blue.

SOLUTION

(*a*) **First method**

Let R_1, R_2, and R_3 denote the events "red ball on first draw," "red ball on second draw," and "red ball on third draw," respectively. Then $R_1R_2R_3$ denotes the event that all 3 balls drawn are red.

$$\Pr\{R_1R_2R_3\} = \Pr\{R_1\}\Pr\{R_2|R_1\}\Pr\{R_3|R_1R_2\} = \left(\frac{8}{20}\right)\left(\frac{7}{19}\right)\left(\frac{6}{18}\right) = \frac{14}{285}$$

Second method

$$\text{Required probability} = \frac{\text{number of selections of 3 out of 8 red balls}}{\text{number of selections of 3 out of 20 balls}} = \frac{\binom{8}{3}}{\binom{20}{3}} = \frac{14}{285}$$

(*b*) Using the second method of part (*a*),

$$\Pr\{\text{all 3 are white}\} = \frac{\binom{3}{3}}{\binom{20}{3}} = \frac{1}{1140}$$

The first method of part (*a*) can also be used.

(*c*) $$\Pr\{\text{2 are red and 1 is white}\} = \frac{\begin{pmatrix}\text{selections of 2 out}\\ \text{of 8 red balls}\end{pmatrix}\begin{pmatrix}\text{selections of 1 out}\\ \text{of 3 white balls}\end{pmatrix}}{\text{number of selections of 3 out of 20 balls}} = \frac{\binom{8}{2}\binom{3}{1}}{\binom{20}{3}} = \frac{7}{95}$$

(*d*) $$\Pr\{\text{none is white}\} = \frac{\binom{17}{3}}{\binom{20}{3}} = \frac{34}{57} \qquad \text{so} \qquad \Pr\{\text{at least 1 is white}\} = 1 - \frac{34}{57} = \frac{23}{57}$$

(*e*) $$\Pr\{\text{1 of each color is drawn}\} = \frac{\binom{8}{1}\binom{3}{1}\binom{9}{1}}{\binom{20}{3}} = \frac{18}{95}$$

(*f*) Using part (*e*),

$$\Pr\{\text{balls drawn in order red, white, blue}\} = \frac{1}{3!}\Pr\{\text{1 of each color is drawn}\} = \frac{1}{6}\left(\frac{18}{95}\right) = \frac{3}{95}$$

Another method

$$\Pr\{R_1W_2B_2\} = \Pr\{R_1\}\Pr\{W_2|R_1\}\Pr\{B_3|R_1W_2\} = \left(\frac{8}{20}\right)\left(\frac{3}{19}\right)\left(\frac{9}{18}\right) = \frac{3}{95}$$

6.33 Five cards are drawn from a pack of 52 well-shuffled cards. Find the probability that (*a*) 4 are aces; (*b*) 4 are aces and 1 is a king; (*c*) 3 are 10's and 2 are jacks; (*d*) a 9, 10, jack, queen, and king are obtained in any order; (*e*) 3 are of any one suit and 2 are of another; and (*f*) at least 1 ace is obtained.

SOLUTION

(*a*) $$\Pr\{\text{4 aces}\} = \binom{4}{4}\cdot\frac{\binom{48}{1}}{\binom{52}{5}} = \frac{1}{54{,}145}$$

(*b*) $$\Pr\{\text{4 aces and 1 king}\} = \binom{4}{4}\cdot\frac{\binom{4}{1}}{\binom{52}{5}} = \frac{1}{649{,}740}$$

(c) $$\Pr\{3 \text{ are 10's and 2 are jacks}\} = \binom{4}{3} \cdot \frac{\binom{4}{2}}{\binom{52}{5}} = \frac{1}{108{,}290}$$

(d) $$\Pr\{9,\ 10,\ \text{jack, queen, king in any order}\} = \frac{\binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} \cdot \binom{4}{1}}{\binom{52}{5}} = \frac{64}{162{,}435}$$

(e) Since there are 4 ways of choosing the first suit and 3 ways of choosing the second suit,

$$\Pr\{3 \text{ of any one suit, } 2 \text{ of another}\} = \frac{4\binom{13}{3} \cdot 3\binom{13}{2}}{\binom{52}{5}} = \frac{429}{4165}$$

(f) $$\Pr\{\text{no ace}\} = \frac{\binom{48}{5}}{\binom{52}{5}} = \frac{35{,}673}{54{,}145} \quad \text{and} \quad \Pr\{\text{at least 1 ace}\} = 1 - \frac{35{,}673}{54{,}145} = \frac{18{,}482}{54{,}145}$$

6.34 Determine the probability of three 6's in five tosses of a fair die.

SOLUTION

Let the tosses of the die be represented by the 5 spaces – – – – –. In each space we will have either the events 6 or non-6 ($\bar{6}$); for example, three 6's and two non-6's can occur as $6\ 6\ \bar{6}\ 6\ \bar{6}$ or as $6\ \bar{6}\ 6\ \bar{6}\ 6$, etc.

Now the probability of an event such as $6\ 6\ \bar{6}\ 6\ \bar{6}$ is

$$\Pr\{6\,6\,\bar{6}\,6\,\bar{6}\} = \Pr\{6\}\Pr\{6\}\ \Pr\{\bar{6}\}\ \Pr\{6\}\ \Pr\{\bar{6}\} = \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{5}{6} = \left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

Similarly, $\Pr\{6\,\bar{6}\,6\,\bar{6}\,6\} = (\frac{1}{6})^3(\frac{5}{6})^2$, etc., for all events in which three 6's and two non-6's occur. But there are $\binom{5}{3} = 10$ such events, and these events are mutually exclusive; hence the required probability is

$$\Pr\{6\,6\,\bar{6}\,6\,\bar{6} \text{ or } 6\,\bar{6}\,6\,\bar{6}\,6 \text{ or etc.}\} = \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = \frac{125}{3888}$$

In general, if $p = \Pr\{E\}$ and $q = \Pr\{\bar{E}\}$, then by using the same reasoning as given above, the probability of getting exactly X E's in N trials is $\binom{N}{X}p^X q^{N-X}$.

6.35 A factory finds that, on average, 20% of the bolts produced by a given machine will be defective for certain specified requirements. If 10 bolts are selected at random from the day's production of this machine, find the probability (a) that exactly 2 will be defective, (b) that 2 or more will be defective, and (c) that more than 5 will be defective.

SOLUTION

(a) Using reasoning similar to that of Problem 6.34,

$$\Pr\{2 \text{ defective bolts}\} = \binom{10}{2}(0.2)^2(0.8)^8 = 45(0.04)(0.1678) = 0.3020$$

(b) $$\begin{aligned}\Pr\{2 \text{ or more defective bolts}\} &= 1 - \Pr\{0 \text{ defective bolts}\} - \Pr\{1 \text{ defective bolt}\}\\ &= 1 - \binom{10}{0}(0.2)^0(0.8)^{10} - \binom{10}{1}(0.2)^1(0.8)^9\\ &= 1 - (0.8)^{10} - 10(0.2)(0.8)^9\\ &= 1 - 0.1074 - 0.2684 = 0.6242\end{aligned}$$

(*c*) $$\begin{aligned}\Pr\{\text{more than 5 defective bolts}\} &= \Pr\{\text{6 defective bolts}\} + \Pr\{\text{7 defective bolts}\}\\ &\quad + \Pr\{\text{8 defective bolts}\} + \Pr\{\text{9 defective bolts}\}\\ &\quad + \Pr\{\text{10 defective bolts}\}\\ &= \binom{10}{6}(0.2)^6(0.8)^4 + \binom{10}{7}(0.2)^7(0.8)^3 + \binom{10}{8}(0.2)^8(0.8)^2\\ &\quad + \binom{10}{9}(0.2)^9(0.8) + \binom{10}{10}(0.2)^{10}\\ &= 0.00637\end{aligned}$$

6.36 If 1000 samples of 10 bolts each were taken in Problem 6.35, in how many of these samples would we expect to find (*a*) exactly 2 defective bolts, (*b*) 2 or more defective bolts, and (*c*) more than 5 defective bolts?

SOLUTION

(*a*) Expected number = (1000)(0.3020) = 302, by Problem 6.35(*a*).

(*b*) Expected number = (1000)(0.6242) = 624, by Problem 6.35(*b*).

(*c*) Expected number = (1000)(0.00637) = 6, by Problem 6.35(*c*).

EULER OR VENN DIAGRAMS AND PROBABILITY

6.37 Figure 6-11 shows how to represent the sample space for tossing a fair coin 4 times and event E_1 that exactly two heads and two tails occurred and event E_2 that the coin showed the same thing on the first and last toss. This is one way to represent Venn diagrams and events in a computer worksheet.

	sample	space		event E1	event E2
h	h	h	h		Y
h	h	h	t		
h	h	t	h		Y
h	h	t	t	X	
h	t	h	h		Y
h	t	h	t	X	
h	t	t	h	X	Y
h	t	t	t		
t	h	h	h		
t	h	h	t	X	Y
t	h	t	h	X	
t	h	t	t		Y
t	t	h	h	X	
t	t	h	t		Y
t	t	t	h		
t	t	t	t		Y

Fig. 6-11 EXCEL display of sample space and events E_1 and E_2.

The outcomes in E_1 have an X beside them under event E_1 and the outcomes in E_2 have a Y beside them under event E_2.

(*a*) Give the outcomes in $E_1 \cap E_2$ and $E_1 \cup E_2$.

(*b*) Give the probabilities $\Pr\{E_1 \cap E_2\}$ and $\Pr\{E_1 \cup E_2\}$.

SOLUTION

(*a*) The outcomes in $E_1 \cap E_2$ have an X and a Y beside them. Therefore $E_1 \cap E_2$ consists of outcomes htth and thht. The outcomes in $E_1 \cup E_2$ have an X, Y, or X and Y beside them. The outcomes in $E_1 \cup E_2$ are: hhhh, hhth, hhtt, hthh, htth, thht, thth, thtt, tthh, ttht, and tttt.

(*b*) $\Pr\{E_1 \cap E_2\} = 2/16 = 1/8$ or 0.125. $\Pr\{E_1 \cup E_2\} = 12/16 = 3/4$ or 0.75.

6.38 Using a sample space and Venn diagrams, show that:

(*a*) $\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}$

(*b*) $\Pr\{A \cup B \cup C\} = \Pr\{A\} + \Pr\{B\} + \Pr\{C\} - \Pr\{A \cap B\} - \Pr\{B \cap C\} -$
$\Pr\{A \cap C\} + \Pr\{A \cap B \cap C\}$

SOLUTION

(*a*) The nonmutually exclusive union $A \cup B$ is expressible as the mutually exclusive union of $A \cap \bar{B}$, $B \cap \bar{A}$, and $A \cap B$.

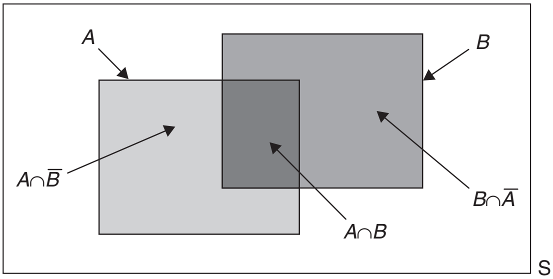


Fig. 6-12 A union expressed as a disjoint union.

$$\Pr\{A \cup B\} = \Pr\{A \cap \bar{B}\} + \Pr\{B \cap \bar{A}\} + \Pr\{A \cap B\}$$

Now add and subtract $\Pr\{A \cap B\}$ from the right-hand side of this equation.

$$\Pr\{A \cup B\} = \Pr\{A \cap \bar{B}) + \Pr\{B \cap \bar{A}\} + \Pr\{A \cap B\} + [\Pr\{A \cap B\} - \Pr\{A \cap B\}]$$

Rearrange this equation as follows:

$$\Pr\{A \cup B\} = [\Pr\{A \cap \bar{B}) + \Pr\{A \cap B\}] + [\Pr\{B \cap \bar{A}\} + Pr\{A \cap B\}] - \Pr\{A \cap B\}$$
$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}$$

(*b*) In Fig. 6-13, event A is composed of regions 1, 2, 3, and 6, event B is composed of 1, 3, 4, and 7, and

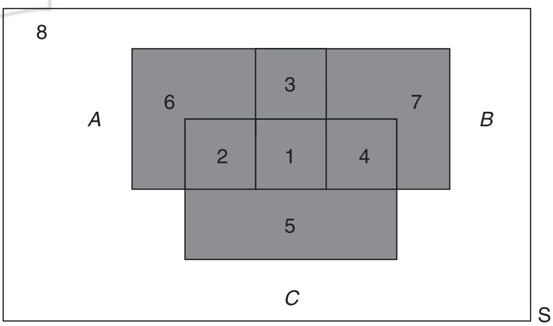


Fig. 6-13 The union of three nonmutually exclusive events, $A \cup B \cup C$.

event C is composed of regions 1, 2, 4, and 5.

The sample space in Fig. 6-13 is made up of 8 mutually exclusive regions. The eight regions are described as follows: region 1 is $A \cap B \cap C$, region 2 is $A \cap C \cap \bar{B}$, region 3 is $A \cap B \cap \bar{C}$, region 4 is $\bar{A} \cap C \cap B$, region 5 is $\bar{A} \cap C \cap \bar{B}$, region 6 is $A \cap \bar{C} \cap \bar{B}$, region 7 is $\bar{A} \cap \bar{C} \cap B$, and region 8 is $\bar{A} \cap \bar{C} \cap \bar{B}$.

The probability $\Pr\{A \cup B \cup C\}$ may be expressed as the probability of the 7 mutually exclusive regions that make up $A \cup B \cup C$ as follows:

$$\Pr\{A \cap B \cap C\} + \Pr\{A \cap C \cap \bar{B}\} + \Pr\{A \cap B \cap \bar{C}\} + \Pr\{\bar{A} \cap C \cap B\}$$
$$+ \Pr\{\mathrm{A} \cap C \cap \bar{B}\} + \Pr\{A \cap \bar{C} \cap \bar{B}\} + \Pr\{\bar{A} \cap \bar{C} \cap B\}$$

Each part of this equation may be re-expressed and the whole simplified to the result:

$$\Pr\{A \cup B \cup C\} = \Pr\{A\} + \Pr\{B\} + \Pr\{C\} - \Pr\{A \cap B\} - \Pr\{B \cap C\} - \Pr\{A \cap C\} + \Pr\{A \cap B \cap C\}$$

For example, $\Pr\{\bar{A} \cap C \cap \bar{B}\}$ is expressible as:

$$\Pr\{C\} - \Pr\{A \cap C\} - Pr\{B \cap C\} + \Pr\{A \cap B \cap C\}$$

6.39 In a survey of 500 adults were asked the three-part question (1) Do you own a cell phone, (2) Do you own an ipod, and (3) Do you have an internet connection? The results of the survey were as follows (no one answered no to all three parts):

cell phone	329	cell phone and ipod	83
ipod	186	cell phone and internet connection	217
internet connection	295	ipod and internet connection	63

Give the probabilities of the following events:

(*a*) answered yes to all three parts, (*b*) had a cell phone but not an internet connection, (*c*) had an ipod but not a cell phone, (*d*) had an internet connection but not an ipod, (*e*) had a cell phone or an internet connection but not an ipod and, (*f*) had a cell phone but not an ipod or an internet connection.

SOLUTION

Event A is that the respondent had a cell phone, event B is that the respondent had an ipod, and event C is that the respondent had an internet connection.

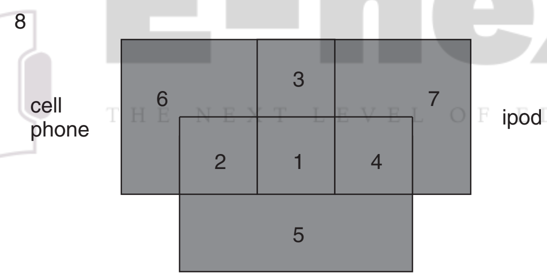


Fig. 6-14 Venn diagram for Problem 6.39.

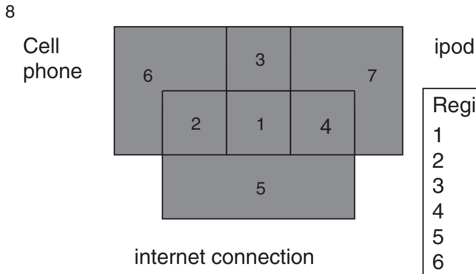
(*a*) The probability that everyone is in the union is 1 since no one answered no to all three parts. $\Pr\{A \cup B \cup C\}$ is given by the following expression:

$$\Pr\{A\} + \Pr\{B\} + \Pr\{C\} - \Pr\{A \cap B\} - \Pr\{B \cap C\} - \Pr\{A \cap C\} + \Pr\{A \cap B \cap C\}$$

$$1 = 329/500 + 186/500 + 295/500 - 83/500 - 63/500 - 217/500 + \Pr\{A \cap B \cap C\}$$

Solving for $\Pr\{A \cap B \cap C\}$, we obtain $1 - 447/500$ or $53/500 = 0.106$.

Before answering the other parts, it is convenient to fill in the regions in Fig. 6-14 as shown in Fig. 6-15. The number in region 2 is the number in region $A \cap C$ minus the number in region 1 or $217 - 53 = 164$. The number in region 3 is the number in region $A \cap B$ minus the number in region 1 or $83 - 53 = 30$. The number in region 4 is the number in region $B \cap C$ minus the number in region 1 or $63 - 53 = 10$. The number in region 5 is the number in region C minus the number in regions 1, 2, and 4 or $295 - 53 - 164 - 10 = 68$. The number in region 6 is the number in region A minus the number in regions 1, 2, and 3 or $329 - 53 - 164 - 30 = 82$. The number in region 7 is $186 - 53 - 30 - 10 = 93$.



Region	number
1	53
2	164
3	30
4	10
5	68
6	82
7	93
Total	500

Fig. 6-15 A, B, and C broken into mutually exclusive regions.

(*b*) regions 3 and 6 or $30 + 82 = 112$ and the probability is $112/500 = 0.224$.

(*c*) regions 4 and 7 or $10 + 93 = 103$ and the probability is $103/500 = 0.206$

(*d*) regions 2 and 5 or $164 + 68 = 232$ and the probability is $232/500 = 0.464$.

(*e*) regions 2, 5, or 6 or $164 + 68 + 82 = 314$ and the probability is $314/500 = 0.628$.

(*f*) region 6 or 82 and the probability is $82/500 = 0.164$.

Supplementary Problems

FUNDAMENTAL RULES OF PROBABILITY

6.40 Determine the probability p, or an estimate of it, for each of the following events:

(*a*) A king, ace, jack of clubs, or queen of diamonds appears in drawing a single card from a well-shuffled ordinary deck of cards.

(*b*) The sum 8 appears in a single toss of a pair of fair dice.

(*c*) A nondefective bolt will be found if out of 600 bolts already examined, 12 were defective.

(*d*) A 7 or 11 comes up in a single toss of a pair of fair dice.

(*e*) At least one head appears in three tosses of a fair coin.

6.41 An experiment consists of drawing three cards in succession from a well-shuffled ordinary deck of cards. Let E_1 be the event "king" on the first draw, E_2 the event "king" on the second draw, and E_3 the event "king" on the third draw. State in words the meaning of each of the following:

(*a*) $\Pr\{E_1\bar{E}_2\}$ (*c*) $\bar{E}_1 + \bar{E}_2$ (*e*) $\bar{E}_1\bar{E}_2\bar{E}_3$

(*b*) $\Pr\{E_1 + E_2\}$ (*d*) $\Pr\{E_3|E_1\bar{E}_2\}$ (*f*) $\Pr\{E_1E_2 + \bar{E}_2E_3\}$

6.42 A ball is drawn at random from a box containing 10 red, 30 white, 20 blue, and 15 orange marbles. Find the probability that the ball drawn is (*a*) orange or red, (*b*) not red or blue, (*c*) not blue, (*d*) white, and (*e*) red, white, or blue.

6.43 Two marbles are drawn in succession from the box of Problem 6.42, replacement being made after each drawing. Find the probability that (*a*) both are white, (*b*) the first is red and the second is white, (*c*) neither is orange, (*d*) they are either red or white or both (red and white), (*e*) the second is not blue, (*f*) the first is orange, (*g*) at least one is blue, (*h*) at most one is red, (*i*) the first is white but the second is not, and (*j*) only one is red.

6.44 Work Problem 6.43 if there is no replacement after each drawing.

6.45 Find the probability of scoring a total of 7 points (*a*) once, (*b*) at least once, and (*c*) twice in two tosses of a pair of fair dice.

6.46 Two cards are drawn successively from an ordinary deck of 52 well-shuffled cards. Find the probability that (*a*) the first card is not a 10 of clubs or an ace, (*b*) the first card is an ace but the second is not, (*c*) at least one card is a diamond, (*d*) the cards are not of the same suit, (*e*) not more than one card is a picture card (jack, queen, king), (*f*) the second card is not a picture card, (*g*) the second card is not a picture card given that the first was a picture card, and (*h*) the cards are picture cards or spades or both.

6.47 A box contains 9 tickets numbered from 1 to 9 inclusive. If 3 tickets are drawn from the box one at a time, find the probability that they are alternately either (1) odd, even, odd or (2) even, odd, even.

6.48 The odds in favor of A winning a game of chess against B are $3:2$. If three games are to be played, what are the odds (*a*) in favor of A's winning at least two games out of the three and (*b*) against A losing the first two games to B?

6.49 A purse contains 2 silver coins and 4 copper coins, and a second purse contains 4 silver coins and 3 copper coins. If a coin is selected at random from one of the two purses, what is the probability that it is a silver coin?

6.50 The probability that a man will be alive in 25 years is $\frac{3}{5}$, and the probability that his wife will be alive in 25 years is $\frac{2}{3}$. Find the probability that (*a*) both will be alive, (*b*) only the man will be alive, (*c*) only the wife will be alive, and (*d*) at least one will be alive.

6.51 Out of 800 families with 4 children each, what percentage would be expected to have (*a*) 2 boys and 2 girls, (*b*) at least 1 boy, (*c*) no girls, and (*d*) at most 2 girls? Assume equal probabilities for boys and girls.

PROBABILITY DISTRIBUTIONS

6.52 If X is the random variable showing the number of boys in families with 4 children (see Problem 6.51), (*a*) construct a table showing the probability distribution of X and (*b*) represent the distribution in part (*a*) graphically.

6.53 A continuous random variable X that can assume values only between $X = 2$ and 8 inclusive has a density function given by $a(X + 3)$, where a is a constant. (*a*) Calculate a. Find (*b*) $\Pr\{3 < X < 5\}$, (*c*) $\Pr\{X \geq 4\}$, and (*d*) $\Pr\{|X - 5| < 0.5\}$.

6.54 Three marbles are drawn without replacement from an urn containing 4 red and 6 white marbles. If X is a random variable that denotes the total number of red marbles drawn, (*a*) construct a table showing the probability distribution of X and (*b*) graph the distribution.

6.55 (*a*) Three dice are rolled and $X =$ the sum on the three upturned faces. Give the probability distribution for X. (*b*) Find $\Pr\{7 \leq X \leq 11)$.

MATHEMATICAL EXPECTATION

6.56 What is a fair price to pay to enter a game in which one can win \$25 with probability 0.2 and \$10 with probability 0.4?

6.57 If it rains, an umbrella salesman can earn \$30 per day. If it is fair, he can lose \$6 per day. What is his expectation if the probability of rain is 0.3?

6.58 A and B play a game in which they toss a fair coin 3 times. The one obtaining heads first wins the game. If A tosses the coin first and if the total value of the stakes is \$20, how much should be contributed by each in order that the game be considered fair?

6.59 Find (*a*) $E(X)$, (*b*) $E(X^2)$, (*c*) $E[(X - \bar{X})^2]$, and (*d*) $E(X^3)$ for the probability distribution of Table 6.4.

Table 6.4

X	-10	-20	30
$p(X)$	$1/5$	$3/10$	$1/2$

6.60 Referring to Problem 6.54, find the (*a*) mean, (*b*) variance, and (*c*) standard deviation of the distribution of X, and interpret your results.

6.61 A random variable assumes the value 1 with probability p, and 0 with probability $q = 1 - p$. Prove that (*a*) $E(X) = p$ and (*b*) $E[(X - \bar{X})^2] = pq$.

6.62 Prove that (*a*) $E(2X + 3) = 2E(X) + 3$ and (*b*) $E[(X - \bar{X})^2] = E(X^2) - [E(X)]^2$.

6.63 In Problem 6.55, find the expected value of X.

PERMUTATIONS

6.64 Evaluate (*a*) ${}_4P_2$, (*b*) ${}_7P_5$, and (*c*) ${}_{10}P_3$. Give the EXCEL function for parts (*a*), (*b*), and (*c*).

6.65 For what value of n is ${}_{n+1}P_3 = {}_nP_4$?

6.66 In how many ways can 5 people be seated on a sofa if there are only 3 seats available?

6.67 In how many ways can 7 books be arranged on a shelf if (*a*) any arrangement is possible, (*b*) 3 particular books must always stand together, and (*c*) 2 particular books must occupy the ends?

6.68 How many numbers consisting of five different digits each can be made from the digits $1, 2, 3, \ldots, 9$ if (*a*) the numbers must be odd and (*b*) the first two digits of each number are even?

6.69 Solve Problem 6.68 if repetitions of the digits are allowed.

6.70 How many different three-digit numbers can be made with three 4's, four 2's, and two 3's?

6.71 In how many ways can 3 men and 3 women be seated at a round table if (*a*) no restriction is imposed, (*b*) 2 particular women must not sit together, and (*c*) each woman is to be between 2 men?

COMBINATIONS

6.72 Evaluate (*a*) $\binom{7}{3}$, (*b*) $\binom{8}{4}$, and (*c*) $\binom{10}{8}$. Give the EXCEL function for parts (*a*), (*b*), and (*c*).

6.73 For what value of n does $3\binom{n+1}{3} = 7\binom{n}{2}$?

6.74 In how many ways can 6 questions be selected out of 10?

6.75 How many different committees of 3 men and 4 women can be formed from 8 men and 6 women?

6.76 In how many ways can 2 men, 4 women, 3 boys, and 3 girls be selected from 6 men, 8 women, 4 boys, and 5 girls if (*a*) no restrictions are imposed and (*b*) a particular man and woman must be selected?

6.77 In how many ways can a group of 10 people be divided into (*a*) two groups consisting of 7 and 3 people and (*b*) three groups consisting of 4, 3, and 2 people?

6.78 From 5 statisticians and 6 economists a committee consisting of 3 statisticians and 2 economists is to be formed. How many different committees can be formed if (*a*) no restrictions are imposed, (*b*) 2 particular statisticians must be on the committee, and (*c*) 1 particular economist cannot be on the committee?

6.79 Find the number of (*a*) combinations and (*b*) permutations of four letters each that can be made from the letters of the word *Tennessee*?

6.80 Prove that $1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n} = 0$.

STIRLING'S APPROXIMATION TO *n*!

6.81 In how many ways can 30 individuals be selected out of 100?

6.82 Show that $\binom{2n}{n} = 2^{2n}/\sqrt{\pi n}$ approximately, for large values of n.

MISCELLANEOUS PROBLEMS

6.83 Three cards are drawn from a deck of 52 cards. Find the probability that (*a*) two are jacks and one is a king, (*b*) all cards are of one suit, (*c*) all cards are of different suits, and (*d*) at least two aces are drawn.

6.84 Find the probability of at least two 7's in four tosses of a pair of dice.

6.85 If 10% of the rivets produced by a machine are defective, what is the probability that out of 5 rivets chosen at random (*a*) none will be defective, (*b*) 1 will be defective, and (*c*) at least 2 will be defective?

6.86 (*a*) Set up a sample space for the outcomes of 2 tosses of a fair coin, using 1 to represent "heads" and 0 to represent "tails."

(*b*) From this sample space determine the probability of at least one head.

(*c*) Can you set up a sample space for the outcomes of 3 tosses of a coin? If so, determine with the aid of it the probability of at most two heads?

6.87 A sample poll of 200 voters revealed the following information concerning three candidates (A, B, and C) of a certain party who were running for three different offices:

28 in favor of both A and B	122 in favor of B or C but not A
98 in favor of A or B but not C	64 in favor of C but not A or B
42 in favor of B but not A or C	14 in favor of A and C but not B

How many of the voters were in favor of (*a*) all three candidates, (*b*) A irrespective of B or C, (*c*) B irrespective of A or C, (*d*) C irrespective of A or B, (*e*) A and B but not C, and (*f*) only one of the candidates?

6.88 (*a*) Prove that for any events E_1 and E_2, $\Pr\{E_1 + E_2\} \le \Pr\{E_1\} + \Pr\{E_2\}$.

(*b*) Generalize the result in part (*a*).

6.89 Let E_1, E_2, and E_3 be three different events, at least one of which is known to have occurred. Suppose that any of these events can result in another event A, which is also known to have occurred. If all the probabilities $\Pr\{E_1\}$, $\Pr\{E_2\}$, $\Pr\{E_3\}$ and $\Pr\{A|E_1\}$, $\Pr\{A|E_2\}$, $\Pr\{A|E_3\}$ are assumed known, prove that

$$\Pr\{E_1|A\} = \frac{\Pr\{E_1\}\ \Pr\{A|E_1\}}{\Pr\{E_1\}\ \Pr\{A|E_1\} + \Pr\{E_2\}\ \Pr\{A|E_2\} + \Pr\{E_3\}\ \Pr\{A|E_3\}}$$

with similar results for $\Pr\{E_2|A\}$ and $\Pr\{E_3|A\}$. This is known as *Bayes' rule* or *theorem*. It is useful in computing probabilities of various *hypotheses* E_1, E_2, or E_3 that have resulted in the event A. The result can be generalized.

6.90 Each of three identical jewelry boxes has two drawers. In each drawer of the first box there is a gold watch. In each drawer of the second box there is a silver watch. In one drawer of the third box there is a gold watch, while in the other drawer there is a silver watch. If we select a box at random, open one of the drawers, and find it to contain a silver watch, what is the probability that the other drawer has the gold watch? [*Hint:* Apply Problem 6.89.]

6.91 Find the probability of winning a state lottery in which one is required to choose six of the numbers 1, 2, $3, \ldots, 40$ in any order.

6.92 Work Problem 6.91 if one is required to choose (*a*) five, (*b*) four, and (*c*) three of the numbers.

6.93 In the game of poker, five cards from a standard deck of 52 cards are dealt to each player. Determine the odds against the player receiving:

(*a*) A royal flush (the ace, king, queen, jack, and 10 of the same suit)

(*b*) A straight flush (any five cards in sequence and of the same suit, such as the 3, 4, 5, 6, and 7 of spades)

(*c*) Four of a kind (such as four 7's)

(*d*) A full house (three of one kind and two of another, such as three kings and two 10's)

6.94 A and B decide to meet between 3 and 4 P.M. but agree that each should wait no longer than 10 minutes for the other. Determine the probability that they meet.

6.95 Two points are chosen at random on a line segment whose length is $a > 0$. Find the probability that the three line segments thus formed can be the sides of a triangle.

6.96 A regular tetrahedron consists of four sides. Each is equally likely to be the one facing down when the tetrahedron is tossed and comes to rest. The numbers 1, 2, 3, and 4 are on the four faces. Three regular tetrahedrons are tossed upon a table. Let $X =$ the sum of the three faces that are facing down. Give the probability distribution for X.

6.97 In Problem 6.96, find the expected value of X.

6.98 In a survey of a group of people it was found that 25% were smokers and drinkers, 10% were smokers but not drinkers, and 35% were drinkers but not smokers. What percent in the survey were either smokers or drinkers or both?

6.99 Acme electronics manufactures MP3 players at three locations. The plant at Omaha manufactures 50% of the MP3 players and 1% are defective. The plant at Memphis manufactures 30% and 2% at that plant are defective. The plant at Fort Meyers manufactures 20% and 3% at that plant are defective. If an MP3 player is selected at random, what is the probability that it is defective?

6.100 Refer to Problem 6.99. An MP3 player is found to be defective. What is the probability that it was manufactured in Fort Meyers?