DEPARTMENT OF MATHEMATICS, IIT GUWAHATI

MA101: Mathematics I | **Mid Semester Exam** (Maximum Marks: 30)

Date: September 20, 2011 | **Time:** 2 pm - 4 pm

Model Solutions

1. (a) Prove or disprove: If A and B are two matrices of the same size such that the linear system of equations $A\mathbf{x} = \mathbf{a}$ and $B\mathbf{x} = \mathbf{b}$ have the same set of solutions then the matrices $[A \mid \mathbf{a}]$ and $[B \mid \mathbf{b}]$ must be row-equivalent. **2**

Solution: Consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and vectors $\mathbf{a} = \mathbf{b} = [0, 1]^t$. Clearly, the systems $A\mathbf{x} = \mathbf{a}$ and $B\mathbf{x} = \mathbf{b}$ are inconsistent, and hence they have the same solution set $\emptyset$. However, the matrices $[A \mid \mathbf{a}]$ and $[B \mid \mathbf{b}]$, being in different reduced row echelon form, are not row-equivalent. Thus, the given statement is disproved by this counterexample. $\square$

(b) Find all real values of k for which the following system of equations has (i) no solution, (ii) unique solution, and (iii) infinitely many solutions: **3**

$$kx + y + z = 1, \quad x + ky + z = 1, \quad x + y + kz = 1.$$

Solution: The augmented matrix of the given system is

$$[A \mid \mathbf{b}] = \left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ k & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - kR_1} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & 1-k \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & (1-k)(2+k) & 1-k \end{array}\right].$$

If $k = -2$ then $\text{rank}([A \mid \mathbf{b}]) = 3 \neq 2 = \text{rank}(A)$. Hence the system has no solution if $k = -2$.
If $k = 1$ then $\text{rank}([A \mid \mathbf{b}]) = \text{rank}(A) < 3$. Hence the system has infinitely many solutions if $k = 1$.
If $k \neq 1, -2$ then $\text{rank}([A \mid \mathbf{b}]) = \text{rank}(A) = 3$. Hence the system has a unique solution if $k \neq 1, -2$. $\square$

2. (a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three linearly independent vectors in $\mathbb{R}^n$, where $n \geq 3$. For what real values of k, are the vectors $\mathbf{v} - \mathbf{u}, k\mathbf{w} - \mathbf{v}$ and $\mathbf{u} - \mathbf{w}$ linearly independent? **2**

Solution: Let $a(\mathbf{v} - \mathbf{u}) + b(k\mathbf{w} - \mathbf{v}) + c(\mathbf{u} - \mathbf{w}) = \mathbf{0}$ for some $a, b, c \in \mathbb{R}$. Then we have

$$\begin{aligned} & a(\mathbf{v} - \mathbf{u}) + b(k\mathbf{w} - \mathbf{v}) + c(\mathbf{u} - \mathbf{w}) = \mathbf{0} \\ \Rightarrow\ & (c-a)\mathbf{u} + (a-b)\mathbf{v} + (kb-c)\mathbf{w} = \mathbf{0} \\ \Rightarrow\ & c - a = 0,\ a - b = 0,\ kb - c = 0; \quad \text{since } \mathbf{u}, \mathbf{v} \text{ and } \mathbf{w} \text{ are linearly independent} \\ \Rightarrow\ & a = b = c = 0 \ \text{ if } k \neq 1. \end{aligned}$$

Also if $k = 1$, then $1.(\mathbf{v} - \mathbf{u}) + 1.(\mathbf{w} - \mathbf{v}) + 1.(\mathbf{u} - \mathbf{w}) = \mathbf{0}$. Thus the vectors $\mathbf{v} - \mathbf{u}, k\mathbf{w} - \mathbf{v}$ and $\mathbf{u} - \mathbf{w}$ are linearly independent if and only if $k \neq 1$. $\square$

(b) Find a basis for the subspace V, where $V = \{[x_1, x_2, \ldots, x_6]^t \in \mathbb{R}^6 : x_i = 0 \text{ if } i \text{ is even}\}$. $\boxed{3}$

Solution: We have $V = \{[x_1, 0, x_3, 0, x_5, 0]^t : x_1, x_3, x_5 \in \mathbb{R}\}$. Let $\mathbf{e}_1 = [1, 0, 0, 0, 0, 0]^t$, $\mathbf{e}_3 = [0, 0, 1, 0, 0, 0]^t$ and $\mathbf{e}_5 = [0, 0, 0, 0, 1, 0]^t$. Then we have

$$[x_1, 0, x_3, 0, x_5, 0]^t = x_1\mathbf{e}_1 + x_3\mathbf{e}_3 + x_5\mathbf{e}_5.$$

Thus the set $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5\}$ spans V. Also

$$a\mathbf{e}_1 + b\mathbf{e}_3 + c\mathbf{e}_5 = \mathbf{0} \Rightarrow [a, 0, b, 0, c, 0]^t = \mathbf{0} \Rightarrow a = b = c = 0.$$

Thus the set $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5\}$ is also linearly independent. Hence $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5\}$ is a basis for V. $\square$

3. (a) Prove or disprove: There exist 2×2 matrices A and B such that $AB - BA = I_2$. $\boxed{2}$

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then we have

$$\begin{aligned} AB - BA = I_2 &\Rightarrow \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix} - \begin{bmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} bz - cy & (a-d)y + b(w-x) \\ c(x-w) + (d-a)z & cy - bz \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Comparing the diagonal entries on both the sides, we have $bz - cy = 1$ and $cy - bz = 1$. Adding corresponding sides of these two equations, we get $0 = 2$, which is absurd. Hence there **cannot** exist 2×2 matrices A and B such that $AB - BA = I_2$. $\square$

(b) Let A be an invertible matrix with integer entries. Show that A^{-1} has all entries integer if and only if $\det(A) = \pm 1$. $\boxed{3}$

Solution: Since A is an invertible matrix with integer entries, we have that $\det(A)$ is an integer. **First** assume that A^{-1} has all entries integer. Then $\det(A^{-1})$ is also an integer. Now $AA^{-1} = I \Rightarrow \det(A)\det(A^{-1}) = 1$. Hence, each of $\det(A)$ and $\det(A^{-1})$ being integers, we find that $\det(A) = \pm 1$.

Conversely, assume that $\det(A) = \pm 1$. Since entries of A are integers, we find that entries of $\text{adj}(A)$ are also integers. Now $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = (\pm 1)\text{adj}(A)$, which gives that A^{-1} has all entries integer. $\square$

4. Let A be an $n \times n$ real matrix and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for $\mathbb{R}^n$. Show that $\text{rank}(A) = n$ if and only if $\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$. $\boxed{5}$

Solution: Given that A is an $n \times n$ real matrix and $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$. Let $\mathcal{T} = \{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_n\}$.

First assume that $\text{rank}(A) = n$ so that A is an invertible matrix. Let $a_1(A\mathbf{u}_1) + a_2(A\mathbf{u}_2) + \ldots + a_n(A\mathbf{u}_n) = \mathbf{0}$ for some $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Then we have

$$\begin{aligned} & a_1(A\mathbf{u}_1) + a_2(A\mathbf{u}_2) + \ldots + a_n(A\mathbf{u}_n) = \mathbf{0} \\ \Rightarrow\ & A(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n) = \mathbf{0} \\ \Rightarrow\ & a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n = \mathbf{0}, \quad \text{since } A \text{ is invertible} \\ \Rightarrow\ & a_1 = a_2 = \ldots = a_n = 0, \quad \text{since } \mathcal{S} \text{ is linearly independent} \\ \Rightarrow\ & \text{the set } \mathcal{T} \text{ is linearly independent} \\ \Rightarrow\ & \mathcal{T}, \text{ being a linearly independent set of } n \text{ vectors in } \mathbb{R}^n, \text{ is a basis for } \mathbb{R}^n. \end{aligned}$$

Conversely, assume that $\mathcal{T}$ is a basis for $\mathbb{R}^n$. Then for any $\mathbf{b} \in \mathbb{R}^n$, we have $b = a_1(A\mathbf{u}_1) + a_2(A\mathbf{u}_2) + \ldots + a_n(A\mathbf{u}_n)$ for some $a_1, a_2, \ldots, a_n \in \mathbb{R}$. This gives that $\mathbf{b} = A(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n)$ and hence the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$. Thus $\mathbb{R}^n = \{\mathbf{b} : A\mathbf{x} = \mathbf{b} \text{ is consistent}\} = \text{col}(A)$, and hence $n = \dim(\text{col}(A)) = \text{rank}(A)$.

Aliter (proving the contrapositive statements)

First assume that $\mathcal{T}$ is not a basis for $\mathbb{R}^n$. Hence $\mathcal{T}$ is linearly dependent, and so there exists real numbers $a_1, a_2, \ldots, a_n$, not all zero, such that $a_1(A\mathbf{u}_1) + a_2(A\mathbf{u}_2) + \ldots + a_n(A\mathbf{u}_n) = \mathbf{0}$. This gives that $A(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n) = \mathbf{0}$. Then $\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n \neq \mathbf{0}$, since this a non-trivial linear combination of vectors from a basis for $\mathbb{R}^n$. Thus $\mathbf{u}$ is a non-zero solution of the system $A\mathbf{x} = \mathbf{0}$, and hence $\text{rank}(A) < n$.

Conversely, assume that $\text{rank}(A) < n$. Then we can find a non-zero vector $\mathbf{u} \in \mathbb{R}^n$ such that $A\mathbf{u} = \mathbf{0}$. Since $\mathcal{S}$ is a basis for $\mathbb{R}^n$ there exist real numbers $a_1, a_2, \ldots, a_n$, not all zero, such that $\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n$. Then we have $A\mathbf{u} = \mathbf{0} \Rightarrow a_1(A\mathbf{u}_1) + a_2(A\mathbf{u}_2) + \ldots + a_n(A\mathbf{u}_n) = \mathbf{0}$, which gives that the set $\mathcal{T}$ is linearly dependent and hence $\mathcal{T}$ is not a basis for $\mathbb{R}^n$. □

5. (a) Let A be a diagonalizable matrix such that every eigenvalue of A is either 0 or 1. Show that $A^2 = A$. **2**

Solution: Since A is diagonalizable there exists an invertible matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix whose diagonal entries are the eigenvalues of A. Now since every eigenvalue of A is either 0 or 1, we find that $D^2 = D$. Therefore $(P^{-1}AP)(P^{-1}AP) = D^2 = D \Rightarrow P^{-1}A(PP^{-1})AP = D \Rightarrow P^{-1}A^2P = D \Rightarrow A^2 = PDP^{-1} = A$. □

(b) Let λ_1 and λ_2 be two distinct eigenvalues of a matrix A and let $\mathbf{u}_1$ and $\mathbf{u}_2$ be eigenvectors of A corresponding to λ_1 and λ_2, respectively. Show that $\mathbf{u}_1 + \mathbf{u}_2$ is not an eigenvector of A. **3**

Solution: The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, being eigenvectors corresponding to distinct eigenvalues of A, are linearly independent. Suppose, if possible, $\mathbf{u}_1 + \mathbf{u}_2$ is an eigenvector corresponding to an eigenvalue μ of A. Then we have

$$\begin{aligned}
A(\mathbf{u}_1 + \mathbf{u}_2) = \mu(\mathbf{u}_1 + \mathbf{u}_2) &\Rightarrow A\mathbf{u}_1 + A\mathbf{u}_2 = \mu(\mathbf{u}_1 + \mathbf{u}_2) \\
&\Rightarrow \lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 = \mu(\mathbf{u}_1 + \mathbf{u}_2) \\
&\Rightarrow (\lambda_1 - \mu)\mathbf{u}_1 + (\lambda_2 - \mu)\mathbf{u}_2 = 0 \\
&\Rightarrow \lambda_1 - \mu = 0 = \lambda_2 - \mu, \quad \text{since } \mathbf{u}_1 \text{ and } \mathbf{u}_2 \text{ are linearly independent} \\
&\Rightarrow \lambda_1 = \mu = \lambda_2, \text{which contradicts the fact that } \lambda_1 \neq \lambda_2.
\end{aligned}$$

Hence we conclude that $\mathbf{u}_1 + \mathbf{u}_2$ is not an eigenvector of A. □

6. (a) Let W be a subspace of $\mathbb{R}^5$ and $\mathbf{v} \in \mathbb{R}^5$. Suppose that $\mathbf{w}$ and $\mathbf{w}'$ are orthogonal vectors with $\mathbf{w} \in W$ and that $\mathbf{v} = \mathbf{w} + \mathbf{w}'$. Is it necessarily true that $\mathbf{w}' \in W^\perp$? Either prove that it is true or find a counterexample. **2**

Solution: Consider the subspace $W = \{[x, y, 0, 0, 0]^t \in \mathbb{R}^5 : x, y \in \mathbb{R}\}$ of $\mathbb{R}^5$. Consider $\mathbf{v} = [1, 1, 0, 0, 0]^t, \mathbf{w} = [1, 0, 0, 0, 0]^t$ and $\mathbf{w}' = [0, 1, 0, 0, 0]^t$. Then $\mathbf{w} \in W$, $\mathbf{v} = \mathbf{w} + \mathbf{w}'$ and, $\mathbf{w}$ and $\mathbf{w}'$ are orthogonal to each other. However, $\mathbf{w}' \notin W^\perp$ since $\mathbf{w}' \in W$, $\mathbf{w}' \neq \mathbf{0}$ and $W \cap W^\perp = \{\mathbf{0}\}$. □

(b) Find a basis for $M^\perp$, where $M = \{[x, y, z]^t : x = s, y = -s, z = 3s, s \in \mathbb{R}\}$. **3**

Solution: Since $[s, -s, 3s]^t = s[1, -1, 3]^t$, we find that M is spanned by the vector $[1, -1, 3]^t$. Therefore $[x, y, z]^t \in M^\perp$ if and only if $[x, y, z]^t.[1, -1, 3]^t = 0 \iff x - y + 3z = 0 \iff x = y - 3z$. Thus $M^\perp = \{[x, y, z]^t \in \mathbb{R}^3 : x = y - 3z\}$. We have $[y - 3z, y, z]^t = y[1, 1, 0]^t + z[-3, 0, 1]^t$. Thus the set $\{[1, 1, 0]^t, [-3, 0, 1]^t\}$ spans $M^\perp$. Also

$$a[1, 1, 0]^t + b[-3, 0, 1]^t = \mathbf{0} \Rightarrow [a - 3b, a, b]^t = \mathbf{0} \Rightarrow a = b = 0.$$

Thus the set $\{[1, 1, 0]^t, [-3, 0, 1]^t\}$ is linearly independent. Hence the set $\{[1, 1, 0]^t, [-3, 0, 1]^t\}$ is a basis for $M^\perp$. □