MA101 Mathematics I

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Slides 6

Plan

- Determinant of Matrices
- Eigenvalues and Eigenvectors

Determinant of a matrix

Recall:

- $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}.$
- $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Definition

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$. The determinant of A is defined by

$\det(A) = a_{11}$, if $n = 1$, i.e., $A = [a_{11}]$. For $n \geq 2$:

$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \ldots + (-1)^{1+n}a_{1n}\det(A_{1n})$,

where A_{ij} be the submatrix of A obtained by deleting the i-th row and the j-th column of A.

Note that $\det(A) \in \mathbb{F}$.

Properties of Determinant

Let A be a square matrix.

1. If B is obtained by interchanging two rows of A, then $\det(B) = -\det(A)$.
2. If A has a zero row, then $\det(A) = 0$.
3. If A has two identical rows, then $\det(A) = 0$.
4. If B is obtained by multiplying a row of A by a scalar α, then $\det(B) = \alpha \det(A)$.
5. If the matrices A, B and C are identical except that one of the rows of C is the sum of the corresponding rows of A and B, then $\det(C) = \det(A) + \det(B)$.
6. If B is obtained by adding a multiple of one row of A to another row, then $\det(B) = \det(A)$.
7. $\det(\alpha A) = \alpha^n \det A$.

Example

$$\det\begin{bmatrix}1&5&0&0\\2&0&8&0\\3&6&9&0\\4&7&10&1\end{bmatrix} = 1\cdot\det\begin{bmatrix}0&8&0\\6&9&0\\7&10&1\end{bmatrix} - 5\cdot\det\begin{bmatrix}2&8&0\\3&9&0\\4&10&1\end{bmatrix} = -18.$$

or

$$= \det\begin{bmatrix}1&5&0&0\\0&-10&8&0\\0&-9&9&0\\0&-13&10&1\end{bmatrix}\begin{matrix}E_{21}(-2)\\E_{31}(-3)\\E_{41}(-4)\end{matrix} = 18\cdot\det\begin{bmatrix}1&5&0&0\\0&5&-4&0\\0&1&-1&0\\0&-13&10&1\end{bmatrix}\begin{matrix}E_2(-\frac{1}{2})\\E_3(-\frac{1}{9})\end{matrix}$$

$$= 18\cdot\det\begin{bmatrix}1&5&0&0\\0&0&1&0\\0&1&-1&0\\0&0&-3&1\end{bmatrix}\begin{matrix}E_{23}(-5)\\E_{43}(13)\end{matrix} = 18\cdot(-1)\cdot\det\begin{bmatrix}1&5&0&0\\0&1&-1&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{matrix}E_{43}(3)\\E_{23}\end{matrix}$$

$$= 18.$$

Result

The determinant of a diagonal, upper or lower triangular matrix is the product of its diagonal entries.

Result

Let $A \in \mathcal{M}_n(\mathbb{F})$. Then for any $i, j \in \{1, \dots, n\}$

$$\begin{aligned} det(A) &= (-1)^{i+1}a_{i1}A_{i1} + (-1)^{i+2}a_{i2}A_{i2} + \dots + (-1)^{i+n}A_{in}C_{in} \\ &\qquad \textit{(Expansion along the i-th row)} \\ &= (-1)^{1+j}a_{1j}A_{1j} + (-1)^{2+j}a_{2j}A_{2j} + \dots + (-1)^{n+j}a_{nj}A_{nj} \\ &\qquad \textit{(Expansion along the j-th column).} \end{aligned}$$

Result

- *For any $A, B \in \mathcal{M}_n(\mathbb{F})$,* $\det(AB) = \det(A)\det(B)$.
- *Determinants of elementary matrices:*
  $\det E_{ij} = -1, \quad \det E_i(\alpha) = \alpha, \quad \det E_{ij}(\alpha) = 1.$
- *For any elementary matrix E,* $\det(E) = \det(E^T)$.

Two important properties

Suppose $A \in \mathcal{M}_n$. Now, $A = E_k \cdots E_1 \cdot \text{rref}(A)$ for some elementary E_i. We have

$$\det(A) = \det(E_k) \cdots \det(E_1) \det(\text{rref}(A)).$$

Thus,

$\det(A) \neq 0$ iff $\det(\text{rref}(A)) \neq 0$ iff $\text{rref}(A) = I_n$ iff A is invertible.

Definition

A matrix $A \in \mathcal{M}_n$ is said to be singular or non-singular according as $\det(A) = 0$ or $\det(A) \neq 0$.

Result

A matrix $A \in \mathcal{M}_n$ is invertible if and only if it is non-singular.

Result

For any matrix $A \in \mathcal{M}_n$, $\det(A) = \det(A^T)$.

Optional: For some additional material, you may consult books or the write up on determinant available at MA101 webpage.

Eigenvalues and Eigenvectors

Suppose $\mathbb{V}$ is a vector space and $T : \mathbb{V} \to \mathbb{V}$ is a linear transformation. Suppose $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{V}$ and $\lambda_i \in \mathbb{F}$ be such that $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$. Then, T is nicely described by

$$T(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n) = \lambda_1 \alpha_1 \mathbf{v}_1 + \cdots + \lambda_n \alpha_n \mathbf{v}_n.$$

Remark

- $[T\mathbf{v}_i]_B = \alpha[\mathbf{v}_i]_B$.
- $[T]_B = \big[[T(\mathbf{v}_1)]_B, \ldots, [T(\mathbf{v}_n)]_B\big] = diag\{\lambda_1, \ldots, \lambda_n\}$.

Definition

Suppose $\mathbb{V}$ is a vector space over $\mathbb{F}$ and $T : \mathbb{V} \to \mathbb{V}$ is a linear transformation. Suppose there is $\lambda \in \mathbb{F}$ and $\mathbf{0} \neq \mathbf{v} \in \mathbb{V}$ such that $T(\mathbf{v}) = \lambda \mathbf{v}$. Then, λ is called an eigenvalue of T and $\mathbf{v}$ an eigenvector of T corresponding to λ.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Then, 4 is an eigenvalue of T with $[1, 1]^t$ as a corresponding eigenvector.

Definition

Let $A \in \mathcal{M}_n(\mathbb{F})$. If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{F}$ and $\mathbf{0} \neq \mathbf{x} \in \mathbb{F}^n$, then λ is called an eigenvalue of A and $\mathbf{x}$ an eigenvector of A corresponding to λ.

Remark

- Suppose $\dim(\mathbb{V}) = n$, and $T : \mathbb{V} \to \mathbb{V}$ is an LT. Then $T(\mathbf{v}) = \lambda\mathbf{v}$ iff $[T(\mathbf{v})]_B = \lambda[\mathbf{v}]_B$ for any ordered basis B of $\mathbb{V}$. This gives the motivation for studying eigenvales/eigenvectors of square matrices.

Definition

Let λ be an eigenvalue of a matrix A. Then

$$E_\lambda = \{\mathbf{x} \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

is called the eigenspace of λ. Note that E_λ consists of all eigenvectors of A corresponding to λ, together with the zero vector.

Result

- *$E_\lambda = null(A - \lambda I)$, and therfore E_λ is a subspace of $\mathbb{C}^n$.*
- *λ is an eigenvalue of A iff $det(A - \lambda I) = 0$.*

Definition

Let A be an $n \times n$ matrix. Then

- $\det(A - \lambda I)$ is called the characteristic polynomial of A.
- $\det(A - \lambda I) = 0$ is called the characteristic equation of A.

Result

- *The eigenvalues of A are the zeroes of the characteristic polynomial $\det(A-\lambda I)$ of A.*
- *The eigenvalues of a triangular matrix are its diagonal entries.*
- *A is invertible iff 0 is not an eigenvalue of A.*

Example

What are the eigenvalues of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and their eigenspaces?

We have

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} = 0 \quad \Leftrightarrow \quad \lambda = 4, -2,$$

$$E_4 = \mathrm{null}(A-4I) = \left\{ \mathbf{x} : \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \mathbf{x} = \mathbf{0} \right\} = \mathrm{span}([1,1]^T),$$

$$E_{(-2)} = \mathrm{null}(A+2I) = \left\{ \mathbf{x} : \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \mathbf{x} = \mathbf{0} \right\} = \mathrm{span}([1,-1]^T),$$

To find eigenvalues and bases for eigenspaces of A

1. Compute the char. poly. $\det(A-\lambda I)$.
2. Solve the char. eqn. $\det(A-\lambda I) = 0$ for eigenvalues λ.
3. For each eigenvalue λ, find $\mathrm{null}(A-\lambda I) = E_\lambda$.
4. Find a basis for each eigenspace.

Example

A real matrix may have complex eigenvalues and complex eigenvectors. No wonder, every real matrix is also a complex matrix. Consider $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The char. poly. of A is λ^2+1 and so the eigenvalues are $\pm i$. Corresponding to the eigenvalues $\pm i$, it has eigenvectors $[1, i]^T$ and $[i, 1]^T$, respectively. The eigenspaces E_i and $E_{(-i)}$ are one dimensional.

Example

What are the eigenvalues and the corresponding eigenspaces of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$? Eigenvalues are $1, 1, 1$ (1 appearing three times, multiplicity three). Moreover,

$$E_1 = \text{null}\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

We have $\dim(E_1) = 1$.

Definition

Let λ be an eigenvalue of a matrix A.

- The algebraic multiplicity of λ is the multiplicity of λ as a root of the characteristic polynomial of A.
- The geometric multiplicity of λ is the dimension of E_λ.

For an eigenvalue, the algebraic and geometric multiplicities can be different. See previous example.

Result

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

PROOF. Suppose that $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of A, $\mathbf{x}_i$ is an eigenvector of A corresponding to λ_i. Suppose, if possible, $\mathbf{x}_i$ are not LI. Then there is $k > 1$ such that $\{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$ is LI, but $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is not. Let $\mathbf{x}_k = \alpha_1\mathbf{x}_1 + \cdots + \alpha_{k-1}\mathbf{x}_{k-1}$. Then,

$$A\mathbf{x}_k = \lambda_1\alpha_1\mathbf{x}_1 + \cdots + \lambda_{k-1}\alpha_{k-1}\mathbf{x}_{k-1}.$$

Therefore,
$$\begin{aligned} \mathbf{0} &= \lambda_k\mathbf{x}_k - A\mathbf{x}_k \\ &= (\lambda_k - \lambda_1)\alpha_1\mathbf{x}_1 + \cdots + (\lambda_k - \lambda_{k-1})\alpha_{k-1}\mathbf{x}_{k-1}. \end{aligned}$$

Since $\{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$ is LI,

$$(\lambda_k - \lambda_1)\alpha_1 = \cdots = (\lambda_k - \lambda_{k-1})\alpha_{k-1} = 0,$$

i.e., $\alpha_1 = \cdots = \alpha_{k-1} = 0$ (as λ_i are distinct), i.e. $\mathbf{x}_k = \mathbf{0}$, a contradiction. ∎

Cayley-Hamilton Theorem

Let $p(\lambda)$ be the characteristic polynomial of a matrix A. Then $p(A) = \mathbf{0}$, the zero matrix.

[This is a beautiful and useful theorem. However, we omit the proof.]

Exercise

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

- Verify that char. poly. of A is $\lambda^2 - 2\lambda - 3$.
- Verify that $A^2 - 2A - 3I = \mathbf{0}$.
- Use the fact $A^2 = 2A + 3I$ to compute A^5 without computing any matrix multiplication.
- Argue that A is invertible using its char. poly. Since $3I = A^2 - 2A$, we should have $A^{-1} = \frac{1}{3}\left[A - 2I\right]$. Verify.

The Fundamental Theorem of Invertible Matrices: Version III

For $A \in \mathcal{M}_n(\mathbb{F})$, $(\mathbb{F} = \mathbb{R}$ or $\mathbb{C})$ the following are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{F}^n$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. The reduced row echelen form of A is I_n.
5. A is a product of elementary matrices.
6. $\text{rank}(A) = n$.
7. $\text{nullity}(A) = 0$.
8. The column vectors of A are linearly independent.
9. The column vectors of A span $\mathbb{F}^n$.
10. The column vectors of A form a basis for $\mathbb{F}^n$.
11. The row vectors of A are linearly independent.
12. The row vectors of A span $\mathbb{F}^n$.
13. The row vectors of A form a basis for $\mathbb{F}^n$.

The Fundamental Theorem ... Contdd.

14. The map $T : \mathbb{F}^n \to \mathbb{F}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is one-one.
15. The map $T : \mathbb{F}^n \to \mathbb{F}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is onto.
16. The map $T : \mathbb{F}^n \to \mathbb{F}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is a bijection.
17. $\det A \neq 0$, i.e., A is non-singular.
18. 0 is not an eigenvalue of A.

Exercise

- Find the eigenvalues and the corresponding eigenspaces of the following matrices:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Exercise

- Let A be a matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.
  - For any positive integer n, show that λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.
  - If A is invertible, then show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$.
  - If A is invertible then show that for any integer n, λ^{-n} is an eigenvalue of A^{-n} with corresponding eigenvector $\mathbf{x}$.
- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be eigenvectors of a matrix A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, respectively. Let $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$. Show that for any positive integer k,

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \ldots + c_m\lambda_m^k\mathbf{v}_m.$$
