Indian Institute of Technology Guwahati

End-semester Examination

MA 101 (Mathematics I)

Maximum Marks : 40

Date : November 26, 2014 **Time :** 1 pm - 4 pm

This question paper contains 10 *questions.*

Attempt as many questions as you can. However, you can score a maximum of 40 *marks.*

No mark will be given for writing only TRUE or FALSE (without justification) in Question 1.

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1. State TRUE or FALSE giving proper justification for each of the following statements. [**3** $\times$ **6**]

   (a) If $x_n = (-1)^n \dfrac{5n \sin^3 n}{3n-2}$ for all $n \in \mathbb{N}$, then no subsequence of the sequence (x_n) is convergent.

   (b) If both the series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ of real numbers are convergent, then the series $\sum\limits_{n=1}^{\infty} x_n y_n$ must be convergent.

   (c) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x) < 3$ for all $x \in \mathbb{Q}$, then it is necessary that $f(x) < 3$ for all $x \in \mathbb{R}$.

   (d) If a differentiable function $f : (1, 2) \to \mathbb{R}$ is strictly increasing, then it is necessary that $f'(x) > 0$ for all $x \in (1, 2)$.

   (e) If $f : [1, 2] \to \mathbb{R}$ is a differentiable function, then the derivative f' must be bounded on $[1, 2]$.

   (f) If $f : [0, 1] \to \mathbb{R}$ is a bounded function such that $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f(\frac{k}{n})$ exists (in $\mathbb{R}$), then f must be Riemann integrable on $[0, 1]$.

2. Let $x_n = 5^n \left(\dfrac{1}{n^3} - \dfrac{1}{n!} \right)$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. [**3**]

3. Examine whether the series $\dfrac{1}{\sqrt{1}} - \dfrac{1}{2} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{4} + \dfrac{1}{\sqrt{5}} - \dfrac{1}{6} + \cdots$ is convergent. [**4**]

4. If $f : [0, 1] \to \mathbb{R}$ is a continuous function, then show that there exist $a, b \in [0, 1]$ such that $a - b = \dfrac{1}{3}$ and $f(a) - f(b) = \dfrac{1}{3}(f(1) - f(0))$. [**3**]

5. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f(x) \neq 0$ for all $x \in [a, b]$. Show that there exists $c \in (a, b)$ such that $\dfrac{f'(c)}{f(c)} = \dfrac{1}{a-c} + \dfrac{1}{b-c}$. [**4**]

(*Continued on the next page*)

6. Using Taylor's theorem, show that $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for all $x \in (0, \pi)$. [**3**]

7. Let $f(x) = \begin{cases} 2x+1 & \text{if } 0 \le x \le 1, \\ 1 & \text{if } 1 < x \le 2, \end{cases}$

   and let $F(x) = \int\limits_0^x f(t)\,dt$ for all $x \in [0,2]$. Examine whether the function $F : [0,2] \to \mathbb{R}$ is differentiable. [**3**]

8. Evaluate: $\lim\limits_{n\to\infty} \left(\frac{1^8 + 3^8 + \cdots + (2n-1)^8}{n^9} \right)$ [**4**]

9. Find the area of the region that is inside the circle $r = 2\cos\theta$ and outside the cardioid $r = 2(1 - \cos\theta)$. [**3**]

10. The region bounded by the parabola $y = x^2 + 1$ and the line $y = x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid. [**3**]

END