DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

Tutorial and practice problems on Single Variable Calculus

MA-101 : Mathematics-I Tutorial Problem Set - 9 October 23, 2013

PART-A (Tutorial)

Question 1:

(i) If $\sum x_n$, with $x_n > 0$, is convergent then prove or disprove that $\sum \sqrt{x_n x_{n+1}}$ is convergent.

(ii) Let (x_n) be such that $\lim_{n\to\infty}(n^2 x_n)$ exists. Show that $\sum x_n$ is absolutely convergent.

(iii) If (x_n) is a decreasing sequence of strictly positive numbers and if $\sum x_n$ is convergent then show that $\lim_{n\to\infty}(nx_n) = 0$. Give an example of a divergent series $\sum x_n$ such that (x_n) is decreasing and that $\lim_{n\to\infty}(nx_n) = 0$.

Solution: (i) Note that $\sqrt{x_n x_{n+1}} \leq x_n + x_{n+1}$. Hence, by comparison test, the result follows.

(ii) Suppose that $\lim_{n\to\infty}(n^2 x_n) = L$. Choose r such that $r > |L|$. Then there exists $m \in \mathbb{N}$ such that $|x_n| \leq r/n^2$ for all $n \geq m$. Hence, by comparison test, the result follows.

(iii) Let s_n denote the n-th partial sum of $\sum x_n$. Then $2(s_{2n} - s_n) = 2(x_{n+1} + \cdots + x_{2n}) \geq 2nx_{2n}$. Similarly, $2(s_{2n+1} - s_n) \geq 2(n+1)x_{2n+1} \geq (2n+1)x_{2n+1}$. This shows that $\lim_{n\to\infty}(nx_n) = 0$.
Consider $x_n := 1/(n\log(n))$. Then $\lim_{n\to\infty}(nx_n) = 0$ but $\sum x_n$ is divergent [by Cauchy's condensation test]. ■

Question 2: For x_n given below, test the convergence or divergence of $\sum x_n$.

(i) $x_n := (1 + a^n)^{-1}$, where $a > 0$; (ii) $x_n := n/2^n$; (iii) $x_n := n!/n^n$;

(iv) $x_n := n!/(3.5.7\ldots(2n+1))$; (v) $x_n := \dfrac{1}{(\log(n))^n}, n \geq 2$;

Solution: (i) Note that $x_n = 1/(1 + a^n) < (1/a)^n$. Hence, by comparison test, $\sum x_n$ converges for $a > 1$. For $0 < a \leq 1$, we have $x_n \geq 1/2$. Therefore, by comparison test, $\sum x_n$ diverges for $0 < a \leq 1$. [Alternatively, since x_n does not converge to 0 as $n \to \infty$ - a necessary condition for convergence, $\sum x_n$ diverges for $0 < a \leq 1$.]

(ii) $\lim_{n\to\infty} |x_{n+1}/x_n| = 1/2$. By ratio test, $\sum x_n$ converges.

(iii) Let $\lim_{n\to\infty} |x_{n+1}/x_n| = \lim_{n\to\infty} 1/(1 + 1/n)^n = 1/e < 1$. By ratio test, $\sum x_n$ converges.

(iv) $x_{n+1}/x_n = (n+1)/(2n+3) = (1 + 1/n)/(2 + 3/n) \to 1/2$ as $n \to \infty$. Hence, by ratio test, $\sum x_n$ converges.

(v) $|x_n|^{1/n} = 1/\log(n) \to 0$ as $n \to \infty$. Hence by root test, $\sum x_n$ converges. ■

Question 3:

(i) Suppose that (x_n) is a decreasing sequence such that $x_n > 0$ for $n \in \mathbb{N}$ and that $\lim_{n\to\infty} x_n = 0$. Let s_n and s, respectively, denote the n-th partial sum and the sum of the the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$. Show that $|s - s_n| \leq x_{n+1}$.

(ii) Give an example of a divergent alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ such that $x_n > 0$ and $x_n \to 0$. [Moral: The condition that (x_n) is decreasing cannot be dropped.]

Solution: (i) Notice that $s_{2n+2} - s_{2n} = x_{2n+1} - x_{2n+2} \geq 0$ and $s_{2n+3} - s_{2n+1} = -x_{2n+2} + x_{2n+3} \leq 0$. This shows that (s_{2n}) is increasing and (s_{2n+1}) is decreasing. Since $s_2 \leq s_{2n} \leq s_{2n} + x_{2n+1} = s_{2n+1} \leq s_1$, it follows that $s_{2n} \leq s \leq s_{2n+1}$. Consequently $|s - s_n| \leq |s_n - s_{n+1}| = x_{n+1}$.

(ii) Consider $x_n := 1/n$ for n odd and $x_n := 1/n^2$ for n even. Then (x_n) is not monotonically decreasing but $x_n \to 0$. Let S_n denote the n-th partial sum of the alternating series. Then it follows that $S_{2n} = O_n - E_n$, where $O_n := \displaystyle\sum_{j=1}^{n} \frac{1}{(2j-1)}$ and $E_n := \displaystyle\sum_{j=1}^{n} \frac{1}{(2j)^2}$. Since $O_n \to \infty$ and E_n converges, we conclude that $S_{2n} \to \infty$. Hence the proof. ■

Question 4: For x_n given below, test the series $\sum x_n$ for convergence and for absolute convergence.

(a) $x_n := \dfrac{1}{(n\log(n))^p}, p > 0$; (b) $x_n := \dfrac{1}{n(\log(n))^p}, p > 0$;

(c) $x_n := \dfrac{(-1)^{n+1}}{\log(n+1)}$; (d) $x_n := \dfrac{(-1)^{n+1}\log(n)}{n}$;

Solution: (a) Now $2^n x_{2^n} = 2^n/(2^n\log(2^n))^p = (2^n)^{(1-p)}/(\log 2)^p n^p$. When $p > 1$, we have $2^n x_{2^n} \leq (\log 2)^{-p}/n^p$. This shows that $\sum 2^n x_{2^n}$ converges for $p > 1$. Hence by Cauchy's condensation test $\sum x_n$ converges absolutely for $p > 1$.

For $p = 1$, we have $2^n x_{2^n} = (\log 2)^{-1}/n$. For $0 < p < 1$, we have $2^n x_{2^n} \geq (\log 2)^{-p}/n^p$. This shows that $\sum 2^n x_{2^n}$ diverges for $0 < p \leq 1$. By condensation test, $\sum x_n$ diverges for $0 < p \leq 1$.

(b) We have $2^n x_{2^n} = (\log 2)^{-p}/n^p$. Therefore $\sum x_n$ converges absolutely for $p > 1$ and diverges for $p \leq 1$.

(c) Alternating series. Converges but not absolutely.

(d) Alternating series. Converges but not absolutely. ■

Question 5: Let (x_n) and (y_n) be sequences of positive real numbers. Prove or disprove the following statements.

(i) If $x_{n+1}/x_n < 1$ for all n then $\sum_{n=1}^{\infty} x_n$ is convergent.

(ii) If $\lim_{n\to\infty}(x_n - y_n) = 0$ and $\sum_{n=1}^{\infty} y_n$ is convergent then $\sum_{n=1}^{\infty} x_n$ is convergent.

(iii) If $\lim_{n\to\infty} x_n/y_n = 1$ and $\sum_{n=1}^{\infty} y_n$ is convergent then $\sum_{n=1}^{\infty} x_n$ is convergent.

(iv) The series $\sum_{n=1}^{\infty} x_n$ is convergent $\iff$ $\sum_{n=1}^{\infty} x_n^2$ is convergent.

Solution: (i) False. Consider $x_n = 1/n$. Then $x_{n+1}/x_n = n/(n+1) < 1$ for all $n \in \mathbb{N}$ but $\sum_{n=1}^{\infty} x_n$ is divergent. ■

(ii) False. Consider $x_n := 1/n$ and $y_n := 1/n^2$. Then the conditions are satisfied but $\sum_{n=1}^{\infty} x_n$ diverges.

(iii) True. Follows from limit comparison test.

(iv) False. Consider $x_n := 1/n$. Then $\sum_{n=1}^{\infty} x_n^2$ converges but $\sum_{n=1}^{\infty} x_n$ diverges. However, if $\sum_{n=1}^{\infty} x_n$ converges then $\sum_{n=1}^{\infty} x_n^2$ converges (Why?). ■

PART-B (Homework/Practice problems)

Question 6:

(i) If $\sum x_n$ is a convergent series of nonnegative numbers and if $p > 1$ then prove that $\sum x_n^p$ is convergent.

(ii) Let $x_n > 0$ for $n \in \mathbb{N}$. Show that $\sum x_n$ converges if and only if $\sum \dfrac{x_n}{1 + x_n}$ converges.

Solution: (i) Since $x_n^p/x_n = x_n^{p-1} \to 0$ as $n \to \infty$, by limit comparison test, $\sum_{n=1}^{\infty} x_n^p$ converges.

(ii) We have $0 < \frac{x_n}{1+x_n} < x_n$ for all $n \in \mathbb{N}$. Hence by the comparison test, $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges if $\sum\limits_{n=1}^{\infty} x_n$ converges. Conversely, let $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converge. Then $\frac{x_n}{1+x_n} \to 0$ and so there exists $n_0 \in \mathbb{N}$ such that $\frac{x_n}{1+x_n} < \frac{1}{2}$ for all $n \geq n_0$. This implies that $x_n < 1$ for all $n \geq n_0$, *i.e.* $1 + x_n < 2$ for all $n \geq n_0$ and so $x_n < \frac{2x_n}{1+x_n}$ for all $n \geq n_0$. By the comparison test, we conclude that $\sum\limits_{n=1}^{\infty} x_n$ converges. ■

Question 7: Peculiarity of convergent alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ with $x_n > 0$.

(i) Discuss conditional and absolute convergence of the alternating series $\sum(-1)^{n+1}x_n$ for $x_n := 1/(2n-1)$ and $x_n := 1/n^{3/2}$.

(ii) If p and q are any positive numbers then show that $\sum \dfrac{(-1)^n(\log(n))^p}{n^q}$ is convergent.

(iii) If $\sum_{n=1}^{\infty} x_n = x$ then show that $\sum_{k=1}^{\infty}(x_{2k} + x_{2k-1}) = x$. Is the converse true?

Solution: (i) The series converges for both $x_n = 1/(2n-1)$ and $x_n = 1/n^{3/2}$. The convergence is conditional in the first case whereas the convergence is absolute in the second case.

(ii) Let $f(x) := (\log x)^p/x^q$. Then $f'(x) < 0$ for large enough x. It is easy to see that $x_n := \frac{(-1)^n(\log(n))^p}{n^q}$ converges to 0 as $n \to \infty$. Hence the series converges.

(iii) Let $S_n = \sum_{j=1}^{n} x_j$ and $T_n = \sum_{k=1}^{n}(x_{2k} + x_{2k-1})$. Then $T_n = S_{2n} \longrightarrow x$ as $n \to \infty$. For the converse, consider $x_n = (-1)^n$. ■

Question 8: Let (x_n) and (y_n) be sequences of nonnegative real numbers. Prove or disprove the following statements.

(i) If $\lim_{n\to\infty}(x_{n+1} + \cdots + x_{2n}) = 0$ then $\sum_{n=1}^{\infty} x_n$ is convergent.

(ii) If $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are both convergent then so is $\sum_{n=1}^{\infty} \max(x_n, y_n)$.

(iii) The series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are both convergent $\iff$ $\sum_{n=1}^{\infty} \sqrt{x_n y_n}$ is convergent.

(iv) If $\sum_{n=1}^{\infty} \min(x_n, y_n)$ is convergent then $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are both convergent.

Solution: (i) False. To see this, define $x_n := 1/k$ if $n = 2^k$ for some $k \in \mathbb{N}$, otherwise $x_n = 0$. Then for any n, there is a unique integer k such that $n+1 \leq 2^k \leq 2n$. Obviously $k \to \infty$ as $n \to \infty$. Hence we have $x_{n+1} + \cdots + x_{2n} = 1/k \to 0$ as $n \to \infty$. But $\sum_{n=1}^{\infty} x_n = \sum_{k=1}^{\infty} x_{2^k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges. [Alternatively, consider $x_n := 1/(n\log(n))$. By Cauchy condensation test the series diverges but it satisfies the given condition.]

(ii) True. Note that $\max(x_n, y_n) \leq x_n + y_n$. Hence by comparison test the result follows.

(iii) False. Since $\sqrt{x_n y_n} \leq x_n + y_n$, we conclude that the convergence of $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ imply convergence of $\sum_{n=1}^{\infty} \sqrt{x_n y_n}$. However, the converse is false. To see this, consider $x_n := 1$ when n is odd and $x_n := 0$ when n is even. Similarly, consider $y_n := 0$ when n is odd and $y_n := 1$ when n is even. Then $\sqrt{x_n y_n} = 0$ for all n and hence the series $\sum_{n=1}^{\infty} \sqrt{x_n y_n}$ is convergent. However, neither $\sum_{n=1}^{\infty} x_n$ nor $\sum_{n=1}^{\infty} y_n$ converges.

(iv) False. Consider $x_n := 1/n$ if n is even and $x_n := 1/n^2$ if n is odd. Similarly, consider $y_n := 1/n^2$ if n is even and $y_n := 1/n$ if n is odd. Then $\min(x_n, y_n) = 1/n^2$ for all n. Hence the series $\sum_{n=1}^{\infty} \min(x_n,\, y_n)$ is convergent. However, neither $\sum_{n=1}^{\infty} x_n$ nor $\sum_{n=1}^{\infty} y_n$ converges. ■

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