MA101 Mathematics I

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Plan

- Elementary Matrices
- Inverse of a Matrix
- Inverse of Elementary Matrices
- RREF, GJE
- Rank & Rank Theorem
- Fundamental Theorem of Invertible Matrices
- LU Factorization

Elementary Matrices

Elementary Matrix: An elementary matrix is a matrix that can be obtained by performing an elementary row operation on the identity matrix.

- There are three types of elementary matrices.
- For example, the following are the three types of elementary matrices of size 3.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

- The matrix E_1 is obtained by performing $R_2 \leftrightarrow R_3$ on I_3.
- E_2 is obtained by performing $R_2 \to 5R_2$ on I_3.
- E_3 is obtained by performing $R_3 \to R_3 - 2R_1$ on I_3.

- Let A be the 3×3 matrix as given below:

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}.$$

Then

$$E_1A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}, \quad E_2A = \begin{bmatrix} a & b & c \\ 5x & 5y & 5z \\ p & q & r \end{bmatrix} \text{ and}$$

$$E_3A = \begin{bmatrix} a & b & c \\ x & y & z \\ p-2a & q-2b & r-2c \end{bmatrix}.$$

- The matrix $E_1 A$ is the matrix obtained from A by performing the elementary row operation $R_2 \leftrightarrow R_3$.
- The matrix $E_2 A$ is the matrix obtained from A by performing the elementary row operation $R_2 \to 5R_2$.
- The matrix $E_3 A$ is the matrix obtained from A by performing the elementary row operation $R_3 \to R_3 - 2R_1$.

Result

1. *Let E be an elementary matrix obtained by an elementary row operation on I_n. If the same elementary row operation is performed on an $n \times r$ matrix A, then the resulting matrix is equal to EA.*
2. *The matrix B is row equivalent to A if there are elementary matrices $E_1, E_2, \cdots, E_k$ such that $B = E_k \cdots E_2 E_1 A$.*

The Inverse of a Matrix

Definition

An $n \times n$ matrix A is said to be invertible if there exists a matrix B satisfying $AB = I_n = BA$, and B is called an inverse of A.

(Note: We can talk of invertibility only for square matrices.)

- For example, the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is invertible since

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

- The zero matrix $\mathbf{O}$ is not invertible.
- If A has a zero row, then A is not invertible.

The Inverse of a Matrix

Result

If A is an invertible matrix then its inverse is unique.

- We write A^{-1} for the inverse of A.

Result

If A and B are $n \times n$ invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Result

Let A be an invertible matrix. Then

1. *the matrix A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.*
2. *if $c \neq 0$ then cA is also invertible, and $(cA)^{-1} = \frac{1}{c}A^{-1}$.*
   [Note that $(cA)B = c(AB) = A(cB)$.]
3. *the matrix A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$.*
   [Note that $(AB)^T = B^T A^T$.]
4. *For any non-negative integer k, the k-th power A^k of A is invertible, and $(A^k)^{-1} = (A^{-1})^k$.*

An exercise:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- If $ad - bc \neq 0$ then A is invertible, and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- If $ad - bc = 0$ then A is not invertible.

Elementary Matrices ...

Result

Every elementary matrix is invertible, and its inverse is an elementary matrix of the same type.

- Applying $R_2 \leftrightarrow R_3$ on I_3 we find E_1^{-1}.
- Applying $R_2 \to \frac{1}{5}R_2$ on I_3 we find E_2^{-1}.
- Applying $R_3 \to R_3 + 2R_1$ on I_3 we find E_3^{-1}.
- We have

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Reduced Row Echelon Form: A matrix A is said to be in reduced row echelon form if it satisfies the following properties

1. A is in row echelon form.
2. The leading entry in each non-zero row is a 1.
3. Each column containing a leading 1 has zeros everywhere else.

- The following matrices are in reduced row echelon form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Every matrix A is row equivalent to a matrix B which is in RREF. We will say B to be the RREF of A.
- RREF of A is unique. (Proof later.)

Method of transforming a given matrix to reduced row echelon form: Let A be an $m \times n$ matrix. The following step by step method is used to obtain the reduced row echelon form of A.

(1) Let the i-th column be the left most non-zero column of A. Interchange rows, if necessary, to make the first entry of this column non-zero. Now use elementary row operations to make all the entries below this first entry equal to 0.

(2) Forget the first row and first i columns. Start with the lower $(m-1) \times (n-i)$ sub matrix of the matrix obtained in the first step and proceed as in **Step 1**.

(3) Repeat the above steps until we get a row echelon form.

(4) Now use the leading term in each of the leading column to make (by elementary row operations) all other entries in that column equal to zero. Use this step starting form the rightmost leading column.

(5) Scale all non-zero entries (leading entries) of the matrix obtained in the previous step, by multiplying the rows by suitable constants, to make all the leading entries equal to 1, ending with the unique reduced row echelon form of A.

Example

$$\textit{Find RREF of } A = \begin{bmatrix} 0 & 2 & -4 & 4 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 1 & 7 \\ 2 & 1 & 0 & -3 \end{bmatrix}.$$

$$A \xrightarrow[E_{12}]{} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & -4 & 4 \\ 2 & 2 & 1 & 7 \\ 2 & 1 & 0 & -3 \end{bmatrix} \xrightarrow[E_{31}(-2),\ E_{41}(-2)]{} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & -4 & 4 \\ 0 & 2 & -3 & 7 \\ 0 & 1 & -4 & -3 \end{bmatrix}$$

$$\xrightarrow[E_{32}(-1),\ E_{42}(-1/2)]{} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & -4 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -5 \end{bmatrix} \xrightarrow[E_{43}(2)]{} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & -4 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Gaussian Elimination stops here. Continue with GJE.)

$$\xrightarrow[E_{34}(-3),\ E_{24}(-4)]{} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{13}(-2), E_{23}(4)]{} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{2}(1/2)]{} I_4$$

Example

For $A = \begin{bmatrix} 0 & 2 & -4 & 4 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 1 & 7 \\ 2 & 1 & 0 & -3 \end{bmatrix}$ *find an invertible* P *such that* PA *is in RREF. Is* A *invertible?*

The RREF of A can be obtained by the following sequence of row operations (in that order):

$$E_{12}, E_{31}(-2), E_{41}(-2), E_{32}(-1), E_{42}(-1/2), E_{43}(2), \\ E_{34}(-3),\ E_{24}(-4), E_{13}(-2), E_{23}(4), E_2(1/2).$$

Thus, $PA = I_4$, the RREF of A, where

$$\begin{aligned} P &= E_2(1/2)E_{23}(4)E_{13}(-2)E_{24}(-4)E_{34}(-3)E_{43}(2)E_{42}(-1/2)\times \\ &\quad E_{32}(-1)E_{41}(-2)E_{31}(-2)E_{12} \\ &= \begin{bmatrix} -17 & -36 & -12 & 6 \\ 45 & 96 & 16 & -16 \\ 17/2 & 18 & 6 & -3 \\ -5/2 & -6 & -2 & 1 \end{bmatrix}. \end{aligned}$$

In fact, $P = A^{-1}$, why?

Example

Suppose the RREF of A *is* I*. Is* A *invertible?*

Yes. In that case $E_k \cdots E_2 E_1 A = PA = I$, and therefore, $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$, and A being a product of invertible matrices is invertible. Moreover, $P = A^{-1}$.

Example

Suppose $A_{n\times n}$ *is invertible. What is the RREF of* A*?*

Suppose RREF of A is B. Then $E_k \cdots E_2 E_1 A = PA = B$. Now, $B = PA$ is invertible, and so has no zero row. How many leading 1's it should have? B must be I_n.

Note: If A is invertible, then the RREF of $\left[A \mid I\right]$ is

$$\left[E_k \cdots E_2 E_1 A \mid E_k \cdots E_2 E_1\right] = \left[I \mid A^{-1}\right].$$

So to find A^{-1}, use GJE to $\left[A \mid I\right]$.

Leading and Free Variable:

- Consider the linear system $A\mathbf{x} = \mathbf{b}$ in n variables and m equations.

- Let $[R \mid \mathbf{r}]$ be the reduced row echelon form of the augmented matrix $[A \mid \mathbf{b}]$.

- The variables corresponding to the leading columns in the first n columns of $[R \mid \mathbf{r}]$ are called the leading variables.

- The variables which are not leading are called free variables.

Gauss-Jordan Elimination Method: Use the following steps to solve a system of equations $A\mathbf{x} = \mathbf{b}$.

1. Write the augmented matrix $[A \mid \mathbf{b}]$.

2. Use elementary row operations to transform $[A \mid \mathbf{b}]$ to reduced row echelon form.

3. Use back substitution to solve the equivalent system that corresponds to the reduced row echelon form.

Example

Solve the system
$w - x - y + 2z = 1,\ 2w - 2x - y + 3z = 3,\ -w + x - y = -3$
using Gauss-Jordan elimination method.

Example

Solve the system
$x + y + z = 3,\quad x + 2y + 2z = 5,\ 3x + 4y + 4z = 12$ *using Gauss-Jordan elimination method.*

Example

Solve the system $2y + 3z = 8, 2x + 3y + z = 5, x - y - 2z = -5$.

Here, the augmented matrix,

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right] \overrightarrow{P_1} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 0 & 1 & 2 \end{array}\right],$$

and use back substitution. Or continue till RREF

$$\overrightarrow{P_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right],$$

Conclude, consistent, no free variable, unique solution. Find the solution.

Example

Solve the system $x + y = 3, 2x + 2y = 5$.

Here, the augmented matrix,

$$[A|\mathbf{b}] = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 5 \end{array}\right] \overrightarrow{P_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

Conclude: No solution (inconsistent).

Example

Solve the system

$x - y - z + 2w = 1,\ 2x - 2y - z + 3w = 3,\ -x + y - z = -3.$

Here, the augmented matrix,

$$[A|\mathbf{b}] = \left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \overrightarrow{P_1} \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here, y and w are free variables, $x,\ y$ are leading variables. So, y and w can be given arbitrary values.

Rewrite the equations: $\begin{cases} x - y + w = 2 \\ z - w = 1 \end{cases}$. Ignore the 3rd eqn.

Thus the solutions are: $x = 2 + s - t,\ y = s,\ z = 1 + t,\ w = t,$ where s and t are arbitrary. We write the solutions as

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2+s-t \\ s \\ 1+t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix},\ s, t \in \mathbb{R}.$$

Consider the previous system

$x - y - z + 2w = 1,\ 2x - 2y - z + 3w = 3,\ -x + y - z = -3.$

i.e. $A\mathbf{x} = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} = \mathbf{b}.$

The system $A\mathbf{x} = \mathbf{0}$ is called the corresponding homogeneous system of $A\mathbf{x} = \mathbf{b}$. The set of solutions of $A\mathbf{x} = \mathbf{b}$ is

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix},\ s, t \in \mathbb{R} \right\}.$$

What are the solutions of $A\mathbf{x} = \mathbf{0}$? Indeed,

$$S_h = \left\{ s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix},\ s, t \in \mathbb{R} \right\}.$$

Thus, $S = \mathbf{s} + S_h$, where $\mathbf{s} = [2, 0, 1, 0]^T$ is a solutions of $A\mathbf{x} = \mathbf{b}$.

Result

Suppose the system $A\mathbf{x} = \mathbf{b}$ is consistent with $\mathbf{s}_0$ as one of the solutions. If S_h is the set of solutions of $A\mathbf{x} = \mathbf{0}$, then the set of solutions of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{s}_0 + S_h$.

Three Problems With Similar Proof Structure

Result

RREF of a matrix is unique. Equivalently, if R_1 and R_2 are in RREF and are row equivalent, then $R_1 = R_2$.

Result

Suppose A and B be $m \times n$ matrices. The two systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are equivalent (i.e. have same solutions) <u>if and only if</u> A and B are row equivalent.

Result

Suppose A and B be $m \times n$ matrices, and that the two systems $Ax = b$ and $Bx = c$ are consistent. Then the two systems are equivalent <u>if and only if</u> the matrices $[\ A \mid\ b\ ]$ and $[\ B \mid\ c\ ]$ are row equivalent.

Proof: RREF is unique (Using Induction)

We will apply induction on columns.

Method of Induction

- Let i be an integer.
- Let $P(n)$ be a mathematical statement based on all integers n of the set $\{i, i+1, i+2, \ldots\}$.
- Show that $P(i)$ is true.
- For $k \geq i$, suppose $P(k)$ is true implies that $P(k+1)$ is also true.

Then the statement $P(n)$ is true for all integers of the set $\{i, i+1, i+2, \ldots\}$.

Proof: RREF is unique ...

Base Case Consider a matrix with one column i.e. $A_{m\times 1}$. Observe that its RREF is unique. What are the possibilities? $\mathbf{O}_{m\times 1}$ and e_1. Can you transform $\mathbf{O}_{m\times 1}$ into e_1?

Induction Hypothesis Let any matrix having $k-1$ columns have a unique RREF.

- Consider a matrix $A'_{m\times k-1}$.
- We need to show that $A_{m\times k}$ also has unique RREF, where A is obtained from A' by adding one more column.
- Assume A has two RREF say B and C. Note that because of induction hypothesis the submatrices B_1 and C_1 formed by the first $k-1$ columns of B and C, respectively, are identical. In particular, they have same number of nonzero rows.

Inductive Case (RREF is unique)

- We have proved that the systems $Ax = 0$, $Bx = 0$ and $Cx = 0$ have the same set of solutions. Let the i^{th} row of the k^{th} column be different in B and C i.e. $b_{ik} \neq c_{ik} \Rightarrow b_{ik} - c_{ik} \neq 0$.

$$\begin{bmatrix} & & \\ & & b_{ik} \\ & & \end{bmatrix} \quad \begin{bmatrix} & & \\ & & c_{ik} \\ & & \end{bmatrix}$$

  Let u be an arbitrary solution of $Ax = 0$.
  $\Rightarrow Bu = 0,\ Cu = 0$ and $(B - C)u = 0$
  $\Rightarrow (b_{ik} - c_{ik})x_k = 0 \Rightarrow x_k$ must be 0
  $\Rightarrow x_k$ is not a free variable
  $\Rightarrow$ there must be leading 1 in the k^{th} column of B and C
  $\Rightarrow$ the location of 1 is different in the k^{th} column
  $\Rightarrow$ numbers of nonzero rows in B_1 and C_1 are different,
  a contradiction.

Rank: The rank of a matrix A, denoted rank(A), is the number of non-zero rows in its RREF, i.e. the number of leading 1's, i.e. no. of nonzero rows in any REF of A.

Result

Let A be an $m \times n$ matrix. Then number of free variables in the system $A\mathbf{x} = \mathbf{0}$ is $n -$ rank(A).

Nullity of $A :=$ no. of free variables in $A\mathbf{x} = \mathbf{0}$, denoted by nullity(A). Thus we have (Rank-nullity theorem)

$$\text{rank}(A) + \text{nullity}(A) = n \text{ (no. of columns in } A).$$

Result

If rank$(A) <$ no. of columns in A then the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Result

Let $A\mathbf{x} = \mathbf{b}$ be a system of equations in n variables. Then

1. *it is inconsistent, if $rank(A) < rank([A \mid \mathbf{b}])$;*
2. *it has a unique solution, if $rank(A) = rank([A \mid \mathbf{b}]) = n$; and*
3. *it has infinitely many solutions, if $rank(A) = rank([A \mid \mathbf{b}]) < n$.*

Result (Fundamental Theorem of Invertible Matrices: I)

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. *A is invertible.*
2. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
3. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
4. *The reduced row echelon form of A is I_n.*
5. *A is a product of elementary matrices.*
6. *$rank(A) = n$.*

Example

Let A be a square matrix. If B is a square matrix such that either $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$.

Gauss-Jordan Method for Computing Inverse

Recall

Let A be a square matrix. If a sequence of elementary row operations transforms A to the identity matrix I, then the same sequence of elementary row operations transforms I to A^{-1}.

Let A be an $n \times n$ matrix.

- Apply elementary row operations on the matrix $[A \mid I_n]$.
- If A is invertible, then $[A \mid I_n]$ will be transformed to $[I_n \mid A^{-1}]$.
- If A is not invertible, then $[A \mid I_n]$ can never be transformed to a matrix of the type $[I_n \mid B]$.

Example: Gauss-Jordan method

Let $A := \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$. Then $[A \mid I] \to [I \mid A^{-1}]$ gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Example

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$.

LU Factorization

A lower triangular matrix is said to be unit lower triangular if its all diagonal elements are 1.

Definition

Let A be a square matrix. A factorization of A as $A = LU$, where L is unit lower triangular and U is upper triangular, is called a **LU factorization** *of A.*

Result

If A is a square matrix that can be reduced to row echelon form without using any row interchanges, then A has an LU factorization.

Examples: LU factorization

Let $A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. Then $A = LU$, where

$$L = E_{32}(-1)E_{31}(-1)E_{21}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$U = REF(A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Examples: LU factorization

Let $A := \begin{bmatrix} 2 & 4 & -1 \\ -4 & -5 & 3 \\ 2 & -5 & -4 \end{bmatrix}$. Then $A = LU$, where

$$L = E_{32}(-3)E_{31}(-1)E_{21}(2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix}$$

and

$$U = REF(A) = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Usefulness of LU factorization

- Consider a linear system $A\mathbf{x} = \mathbf{b}$, where the coefficient matrix has an LU factorization $A = LU$.

- We can write $A\mathbf{x} = \mathbf{b}$ as $LU\mathbf{x} = \mathbf{b}$, i.e., $L(U\mathbf{x}) = \mathbf{b}$.

  Define $\mathbf{y} = U\mathbf{x}$. We can now solve for $\mathbf{x}$ in two steps:

  1. Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ by forward substitution.
  2. Solve $U\mathbf{x} = \mathbf{y}$ by back substitution.