DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati
Tutorial and practice problems on Single Variable Calculus

MA-101 : Mathematics-I   Tutorial Problem Set - 13   November 20, 2013

PART-A (Tutorial)

1. Let $f : [a, b] \to \mathbb{R}$ be such that $f(x) = 0$ except for a finite number of points $c_1, \ldots, c_n$ in $[a, b]$. Show that $f \in R([a, b])$ and $\int_a^b f(t)dt = 0$. Hence or otherwise prove that if $g \in R([a, b])$ and $h(x) = g(x)$ except for a finite number of points in $[a, b]$ then $h \in R([a, b])$ and $\int_a^b g(t)dt = \int_a^b h(t)dt$.

> **Solution:** Set $M := \max_{1\le j\le n} |f(c_j)|$. Choose $\epsilon > 0$ and set $\delta := \epsilon/nM$. Let $P \in \mathcal{P}([a, b])$ be such that $\|P\| < \delta$ and that each c_i belongs to exactly one subinterval of P. Let $M(c_i)$ and $m(c_i)$ denote the supremum and infimum of f on the subinterval that contains c_i. Then obviously $M(c_i) = \max(f(c_i), 0)$ and $m(c_i) = \min(f(c_i), 0)$. Consequently, $M(c_i) - m(c_i) = |f(c_i)|$. Thus the contribution in $U(P, f) - L(P, f)$ of the subinterval that contains c_i is at most $|f(c_i)|\,\|P\| \le M\|P\|$. Consequently, $U(P, f) - L(P, f) \le nM\|P\| < nM\delta = \epsilon$. This shows that $f \in R([a, b])$.
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> Next, choose $\epsilon > 0$ so small that $[c_i - \delta,\, c_i + \delta]$ are disjoint for all $i = 1, \ldots, n$, where $\delta := \epsilon/2nM$. Then $|\int_a^b f(t)dt| = |\sum_{j=1}^n \int_{c_j-\delta}^{c_j+\delta} f(t)dt| \le \sum_{j=1}^n M2\delta = \epsilon$. This shows that $\int_a^b f(t)dt = 0$.
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> Applying above result to $h - g$, we see that $h - g \in R([a, b])$ and hence $h = (h - g) + g \in R([a, b])$.

2. Let $f : [-1, 1] \to \mathbb{R}$ be given by $f(x) := -1$ when $x < 0$, and $f(x) := 1$ when $x \ge 0$. Show that $f \in \mathcal{R}([-1, 1])$ but f does not have an antiderivative.

3. Let $F : [0, 1] \to \mathbb{R}$ be given by $F(0) := 0$ and $F(x) := x^2 \sin(1/x^2)$ when $x \ne 0$. Show that F is differentiable on $[0, 1]$. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) := F'(x)$. Then F is an antiderivative of f. Show that f is not Riemann integrable. Does this contradict first fundamental theorem?

4. Let p be a real number and $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Show that $\int_a^{a+p} f(t)dt$ has the same value for all a.

> **Solution:** Define $F(x) := \int_x^{x+p} f(t)dt$. Then $F'(x) = f(x + p) - f(x) = 0$ for all $x \in \mathbb{R}$. Hence the result follows.

5. (**Generalized Mean Value Theorem**) Let $f : [a, b] \to \mathbb{R}$ be continuous and $g \in R([a, b])$ be such that $g(x) \ge 0$ for all $x \in [a, b]$. Prove that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Give an example to show that the condition $g(x) \ge 0$ for all $x \in [a, b]$ cannot be dropped.

> **Solution:** Let m and M be the global minimum and global maximum of f, respectively. Since $g(x) \ge 0$, we have $m \le f(x) \le M \Rightarrow mg(x) \le f(x)g(x) \le Mg(x) \Rightarrow m\int_a^b g(t)dt \le \int_a^b f(t)g(t)dt \le M\int_a^b g(t)dt$. If $\int_a^b g(t)dt = 0$ then the result follows. So, suppose that $L := \int_a^b g(t)dt \ne 0$. Then $L > 0$. This shows that $m \le \int_a^b f(t)g(t)dt/L \le M$. Therefore, by the IVT there exists $c \in [a, b]$ such that $f(c) = \int_a^b f(t)g(t)dt/L$. Hence the result follows.
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> Example: Consider $f(x) := x + 2$ and $g(x) := x$ for $x \in [-1, 1]$. Then $\int_{-1}^1 g(x)dx = 0$. But $\int_{-1}^1 f(x)g(x)dx = 2/3 \ne f(c)\int_{-1}^1 g(x)dx = 0$.

6. Examine convergence of the following improper integrals.

$$\text{(i)}\int_0^1 \sin(\tfrac{1}{x})dx; \quad \text{(ii)}\int_0^\infty \tfrac{\sin^2 x}{x^2}dx; \quad \text{(iii)} \int_0^{\pi/2} \tfrac{dx}{\sin(x)}.$$

Solution:

(i) Let $I(\epsilon) := \int_\epsilon^1 \sin(\frac{1}{x})dx$, where $\epsilon > 0$. Substituting $u = 1/x$, we obtain $I(\epsilon) = \int_1^{1/\epsilon} \frac{\sin(u)}{u^2}du$. Since $\int_1^\infty \frac{\sin x}{x^2}dx$ converges absolutely, we conclude that $\lim_{\epsilon\to 0^+} I(\epsilon)$ exists and hence the improper integral converges.

(ii) Note that $\dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ converges. Therefore $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}dx$ converges. Also note that $\displaystyle\int_0^1 \frac{\sin^2 x}{x^2}dx$ exists in the sense of Riemann (because the integrand can be redefined as a continuous function). Thus the improper integral converges.

(iii) Let $f(x) := 1/x$ and $g(x) := 1/\sin(x)$. Then $\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$. Since $\int_0^{\pi/2} \frac{dx}{x}$ diverges, by Limit Comparison Test the improper integral diverges.

7. Prove that the improper integral $\int_1^\infty \frac{\sin(x)}{x^p}dx$ converges but not absolutely for $0 < p \leq 1$.

Solution: By Dirichlet's Test the integral converges. Now observe that $|\sin x| \geq \sin^2 x$ and hence $|\sin(x)/x^p| \geq \sin^2(x)/x^p = \frac{1-\cos(2x)}{2x^p}$. Again by Dirichlet's Test $\int_1^\infty \frac{\cos(2x)}{2x^p}dx$ converges for $p > 0$. Since $\int_1^\infty \frac{dx}{2x^p}$ diverges for $p \leq 1$, we conclude that the integral does not converge absolutely for $0 < p \leq 1$.

8. Find the arc lengths of the following curves:

(i) the cycloid $x = t - \sin t, y = 1 - \cos t, 0 \leq t \leq 2\pi$; $\quad (ii) y = \int_0^x \sqrt{\cos(2t)}\, dt,\ 0 \leq x \leq \pi/4$.

Solution: (i) Length $= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt = \int_0^{2\pi} 2|\sin(t/2)|dt = 4\int_0^\pi |\sin(t)|dt = 8.$

(ii) Length $= \int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx = \int_0^{\pi/4} \sqrt{1+\cos(2x)}\, dx = \sqrt{2}\int_0^{\pi/4} |\cos x|dx = 1.$

9. The cross sections of a certain solid by planes perpendicular to the x-axis are circles with diameter extending from the curve $y = x^2$ to the curve $y = 8 - x^2$. The solid lies between the points of intersections of these two curves. Find the volume of the solid.

Solution: The diameter of the circle at a point x is given by $(8 - x^2) - x^2$, where $x \in [-2, 2]$. So the area $A(x)$ of the cross section at x is given by $A(x) = \pi(4 - x^2)^2$. Thus
Volume $= \int_{-2}^2 \pi(4 - x^2)^2 dx = 2\pi\int_0^2 (4 - x^2)^2 dx = \frac{512\pi}{15}$.

10. Find the volume of the solid generated when the region bounded by the curves $y = 3 - x^2$ and $y = -1$ is revolved about the line $y = -1$, by both Washer Method.

Solution: Washer Method:
Area of washer $= \pi(1 + y)^2 = \pi(1 + (3 - x^2))^2 = \pi(4 - x^2)^2$ so that
Volume $= \int_{-2}^2 \pi(4 - x^2)^2 dx = \frac{512\pi}{15}$.

PART-B (Homework/Practice problems)

1. Evaluate the following limits. (Assume that $f \in \mathcal{R}([a,b])$ and is continuous at x_0.)

   (i) $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} \frac{du}{u+\sqrt{u^2+1}}$; (ii) $\lim_{x\to 0} \frac{1}{x^6}\int_0^{x^2} \frac{t^2 dt}{t^6+1}$; (iii) $\lim_{x\to x_0} \frac{x}{x^2-x_0^2}\int_{x_0}^{x} f(t)dt$.

> **Solution:** (i)By considering antiderivative, it is easy to see that the limit is equal to $1/(x+\sqrt{x^2+1})$.
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> (ii) We have $\int_0^{x^2} \frac{t^2}{t^6+1}dt = \frac{1}{3}\int_0^{x^2} \frac{d(t^3)}{(t^3)^2+1}dt = \frac{1}{3}\tan^{-1}(x^6)$. Hence the given limit is equal to $\frac{1}{3}\lim_{x\to 0}\frac{\tan^{-1}(x^6)}{x^6} = \frac{1}{3}$.
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> (iii) Let $F(x) = \int_{x_0}^{x} f(f)dt$. Then $\frac{x}{x^2-x_0^2}\int_{x_0}^{x} f(t)dt = \frac{x}{x+x_0}\frac{F(x)-F(x_0)}{x-x_0}$. Therefore the required limit is equal to $f(x_0)/2$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $c > 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := \int_{x-c}^{x+c} f(t)dt$. Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.

> **Solution:** It follows that g is differentiable and that $g'(x) = f(x+c) - f(x-c)$.

3. (**Extended Mean Value Theorem**) Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Show that there exists $c \in (a,b)$ such that

$$\int_a^b f(x)dx = f(a)(b-a) + f'(c)\frac{(b-a)^2}{2}.$$

> **Solution:** Define $F(x) := \int_a^x f(t)dt$. Then F' is continuous on $[a,b]$ and $F''(x)$ exists for all $x \in (a,b)$. Hence by Taylor's theorem, we have $F(b) = F(a) + F'(a)(b-a) + F''(c)(b-a)^2/2$ for some $c \in (a,b)$. This gives $\int_a^b f(t)dt = f(a)(b-a) + f'(c)(b-a)^2/2$.

4. Let $f : [0,1] \to \mathbb{R}$ be continuous. If $\int_0^x f(t)dt = \int_x^1 f(t)dt$ for all $x \in [0,1]$ then show that $f(x) = 0$ for all $x \in [0,1]$.

> **Solution:** Differentiating both sides, we have $f(x) = -f(x)$ for all $x \in [0,1]$. Consequently $f(x) = 0$ for all $x \in [0,1]$.

5. Examine convergence of the following improper integrals (here p and q are any real numbers):

   (i) $\int_0^1 x^{p-1}(1-x)^{q-1}dx$; (ii) $\int_0^\infty \frac{xdx}{(1+x)^3}$;

> **Solution:** (i) Set $f(x) := x^{p-1}(1-x)^{q-1}$ and consider $I_1 := \int_0^{1/2} f(x)dx$ and $I_2 := \int_{1/2}^1 f(x)dx$. First, consider I_1. If $p \geq 1$ then setting $f(0) = 0$, it follows that f is Riemann integrable on $[0,1/2]$. So, suppose that $p < 1$ and let $g(x) := 1/x^{1-p}$ for $x \in (0,1/2]$. Then $\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}(1-x)^{q-1} = 1$. Hence by limit comparison test $\int_0^{1/2} f(x)dx$ converges if and only if $\int_0^{1/2} g(x)dx$ converges. But the latter integral converges if and only if $1-p < 1$, that is, $p > 0$. Hence $\int_0^{1/2} f(x)dx$ converges if and only if $p > 0$.

Next consider, I_2. If $q \geq 1$ then setting $f(1) = 0$ it follows that I_2 exists in the sense of Riemann integral. On the other hand, if $q < 1$, then substituting $y = 1 - x$, we have $I_2 = \int_0^{1/2} y^{q-1}(1-y)^{p-1}dy$. Hence by the previous case, I_2 converges if and only if $q > 0$.

This shows that the improper integral converges if and only if $p > 0$ and $q > 0$ and in such a case the value of the integral is known as the beta function and is denoted by $B(p, q)$.

(ii) Note that $\int_0^1 \frac{x}{(1+x)^3}dx$ exists in the sense of Riemann integral. Also note that $x/(1+x)^3 \leq 1/x^2$ for $x \geq 1$ and that $\int_1^\infty \frac{dx}{x^2}$ converges. Hence by comparison test the improper integral converges.

6. Examine whether the following integrals are convergent.
   (a) $\int_0^\infty \sin(x^2)\, dx$
   (b) $\int_0^1 \frac{\log x}{\sqrt{x}}\, dx$

7. Determine all real values of p for which the following integrals are convergent.
   (a) $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx$
   (b) $\int_0^1 (\log \frac{1}{x})^p\, dx$

8. Find the area of the region bounded by the given curves in each of the following cases. $(i)\sqrt{x}+\sqrt{y} = 1, x = 0$ and $y = 0$; $(ii) y = x^4 - 2x^2$ and $y = 2x^2$; $(iii) x = 3y - y^2$ and $x + y = 3$.

Solution: (i) Area $= \int_0^1 y dx = \int_0^1 (1 + x - 2\sqrt{x})dx = 1/6$.
(ii) Area $= 2\int_0^2 (2x^2 - (x^4 - 2x^2))dx = 2\int_0^2 (4x^2 - x^4)dx = 128/15$.
(iii) Area $= \int_1^3 (3y - y^2 - (3 - y))dy = \int_1^3 (4y - y^2 - 3)dy = 4/3$.

9. Let $f(x) = x - x^2$ and $g(x) = ax$. Determine a so that the region above the graph of g and below the graph of f has area 4.5.

Solution: We have $\int_0^{1-a}(x - x^2 - ax)dx = \int_0^{1-a}((1-a)x - x^2)dx = (1-a)^3/6$. By the given condition $(1-a)^3/6 = 4.5$ so that $a = -2$.

10. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

Solution: In the first octant, the sections perpendicular to the y-axis are squares with $0 \leq x \leq \sqrt{a^2 - y^2}$, $0 \leq z \leq \sqrt{a^2 - y^2}$, $0 \leq y \leq q$. Since the squares have sides of length $\sqrt{a^2 - y^2}$, the area of the cross section at y is given by $A(y) := 4(a^2 - y^2)$. Thus we have
Volume $= \int_{-a}^a A(y)dy = 8\int_0^a (a^2 - y^2)dy = \frac{16a^3}{3}$.

11. A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.

Solution: Washer Method:

Volume = Volume of the sphere - Volume generated by revolving an appropriate region (draw the picture) $= \frac{32\pi}{3} - [\int_{-1}^1 \pi x^2 dy - \pi(\sqrt{3})^2 2] = \frac{32\pi}{3} - 2\pi[\int_0^1 (4 - y^2)dy - 3] = 28\pi/3$.

*** End ***