PH 101
Tutorial-1
Date: 7/08/2017

1. Express $\hat{x}$ and $\hat{y}$ in terms of $\hat{r}$ and $\hat{\theta}$.

2. The rate of change of acceleration is known as "jerk." Find the direction and magnitude of jerk for a particle moving in a circle of radius R at angular velocity ω. Draw a vector diagram showing the instantaneous position, velocity, acceleration, and jerk.
3. For a smooth ("low jerk") ride, an elevator is programmed to start from rest and accelerate according to

$$a(t) = (a_m/2)[1 - cos(2\pi t/T)] \qquad 0 \le t \le T$$

$$a(t) = -(a_m/2)[1 - cos(2\pi t/T)] \qquad T \le t \le 2T$$

where a_m is the maximum acceleration and $2T$ is the total time for the trip.
(a) Draw sketches of $a(t)$ and the jerk as functions of time.
(b) What is the elevator's maximum speed?
(c) Find an approximate expression for the speed at short times near the start of the ride, $t \ll T$.
(d) What is the distance D covered by the elevator during its trip, which took a total time $2T$?
4. A peaked roof is symmetrical and subtends a right angle, as shown in Fig.1.

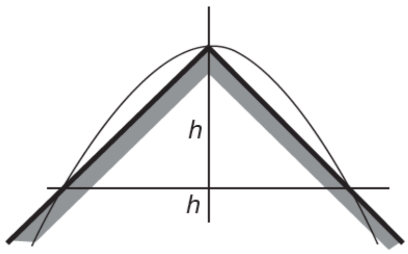

Fig. 1

Standing at a height of distance h below the peak, with what initial speed must a ball be thrown so that it just clears the peak and hits the other side of the roof at the same height?
5. An athlete stands at the peak of a hill that slopes downward uniformly at angle ϕ (Refer to Fig.2). At what angle θ from the horizontal should they throw a rock so that it has the greatest range?

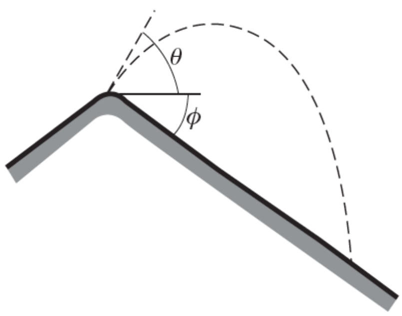

Fig. 2

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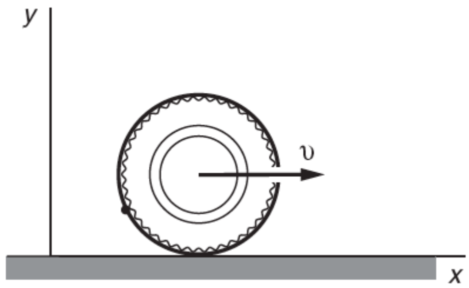
6. A tire of radius R rolls in a straight line without slipping. Its center moves with constant speed V. A small pebble lodged in the tread of the tire touches the road at $t = 0$. Find the pebble's position, velocity, and acceleration as functions of time.



7. A turntable rotates at a constant angular speed ω. An ant crawls directly towards the rim along a radial line at a constant speed b. You observe the ant from above. From your point of view, the ant is moving in a spiral. Write an expression for the velocity and acceleration of the ant in polar coordinates.

Tutorial 1

Solutions

1.

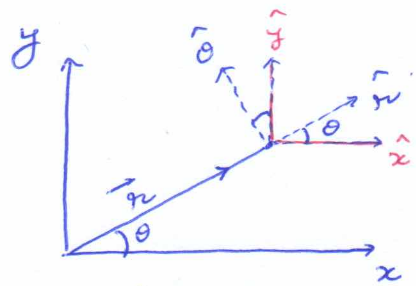


Diagram 1

From the Fig.

$$\hat{r}(\theta) = \hat{x}\cos\theta + \hat{y}\sin\theta$$

$$\hat{\theta}(\theta) = -\hat{x}\sin\theta + \hat{y}\cos\theta$$

Doing algebra or directly from above fig. it is easy to obtain:

$$\boxed{\begin{aligned} \hat{x} &= \hat{r}\cos\theta - \hat{\theta}\sin\theta \\ \hat{y} &= \hat{r}\sin\theta + \hat{\theta}\cos\theta \end{aligned}}$$

2.

$\theta = \omega t$, for uniform motion in a circle, where ~~angl~~ angular speed ω is constant

Now, $\vec{r} = r\hat{r}$; $\vec{v} = r\dot{\theta}\,\hat{\theta} = R\omega\,\hat{\theta}$

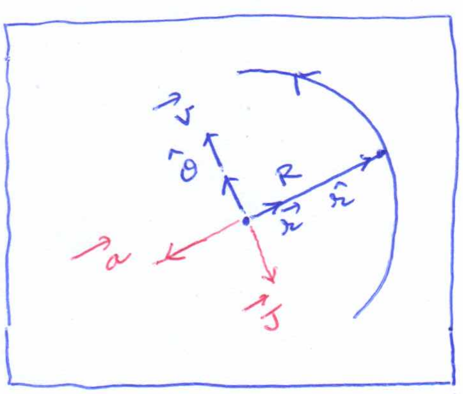
$$\vec{a} = -r\dot{\theta}^2\,\hat{r} = -R\omega^2\,\hat{r}$$

Say, $\vec{J}$ represents jerk.

$$\vec{J} = \frac{d\vec{a}}{dt} \text{ , by definition}$$

$$= -R\omega^2\frac{d\hat{r}}{dt}$$

$$\Rightarrow \boxed{\vec{J} = -R\omega^3\,\hat{\theta}}$$



vector diagram showing instantaneous position, velocity, acceleration and jerk.

3.

Jerk $\vec{J}(t) = \frac{d\vec{a}}{dt}$

$$J(t) = a_m \frac{\pi}{T} \sin\left(\frac{2\pi t}{T}\right) \quad , \quad 0 \leq t \leq T$$

$$= -a_m \frac{\pi}{T} \sin\left(\frac{2\pi t}{T}\right) \quad , \quad T \leq t \leq 2T$$

Let $\vec{v}(t)$ be the speed.

Then,

$$v(t) = v(0) + \int_0^t a(t')\,dt' \quad , \quad 0 \leq t \leq T$$

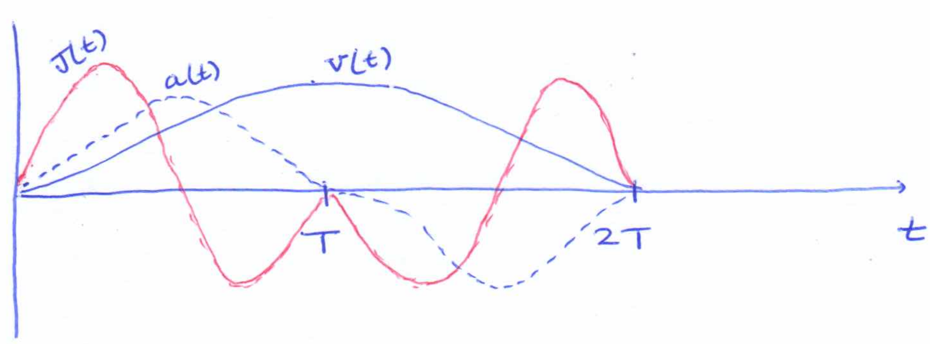
$$= \frac{1}{2} a_m \left[t - \left(\frac{T}{2\pi}\right) \sin\left(\frac{2\pi t}{T}\right) \right]$$

$$v(t) = v(T) + \int_T^t a(t')\,dt' \quad , \quad T \leq t \leq 2T$$

$$= \frac{1}{2} a_m T - \frac{1}{2} a_m \left[(t - T) - \left(\frac{T}{2\pi}\right) \sin\left(\frac{2\pi t}{T}\right) \right]$$

$$= \frac{1}{2} a_m \left[(2T - t) + \left(\frac{T}{2\pi}\right) \sin\left(\frac{2\pi t}{T}\right) \right]$$

(a)



(b) Max^m speed occurs at $t = T$

$$\therefore \quad v_{max} = v(T)$$

$$= \frac{1}{2} a_m T$$

~~(d)~~ (c) For $t << T$, using small angle approximation: $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \cdots$

we obtain:

$$v(t) = \int_0^t a(t')\,dt'$$

$$= \frac{1}{2} a_m \left[t - \left(\frac{T}{2\pi}\right) \sin\left(\frac{2\pi t}{T}\right)\right]$$

$$= \frac{a_m}{2}\left\{ t - \left(\frac{T}{2\pi}\right)\left[\frac{2\pi t}{T} - \frac{1}{3!}\left(\frac{2\pi t}{T}\right)^3 + \cdots\right]\right\}$$

$$\approx \frac{a_m}{2}\,\frac{1}{3!}\left(\frac{2\pi}{T}\right)^2 t^3$$

$$\approx a_m\left(\frac{\pi^2}{3}\right)\left(\frac{t^3}{T^2}\right)$$

(d) Say $x(t)$ is the distance at time t.

$$x(t) = \int v(t')\,dt'$$

where $v(t) = \frac{a_m}{2}\left[t - \left(\frac{T}{2\pi}\right)\sin\left(\frac{2\pi t}{T}\right)\right], \quad 0 \le t \le T$

$v(t) = \frac{a_m}{2}\left[(2T - t) + \left(\frac{T}{2\pi}\right)\sin\left(\frac{2\pi t}{T}\right)\right], \quad T \le t \le 2T$

$$\therefore D = x(2T)$$

$$= \frac{a_m}{2} T^2$$

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4. Say v_0 is the velocity at $t=0$.

Equations of motion are:

$$x = -h + v_{ox} t$$

$$y = v_{oy} t - \frac{1}{2} g t^2$$

$$v_x = v_{ox} \quad , \quad \cancel{v_{oy}}$$

$$v_y = v_{oy} - gt$$

Say at $t = T$, the ball is at the peak where $y = h$ and $v_y = 0$:

$$0 = v_{oy} - gT \Rightarrow T = v_{oy}/g$$

$$h = v_{oy} T - \frac{1}{2} g T^2 = \frac{v_{oy}^2}{g} - \frac{1}{2}\frac{v_{oy}^2}{g}$$

$$\Rightarrow v_{oy} = \sqrt{2gh}$$

At $t = T$, $x = 0 \Rightarrow v_{ox} = \frac{h}{T} = \cancel{\frac{\sqrt{hg}}{2}} \; \sqrt{\frac{gh}{2}}$

Thus,

$$v_0 = \sqrt{v_{ox}^2 + v_{oy}^2}$$

$$= \sqrt{\frac{5}{2}} \sqrt{gh}$$

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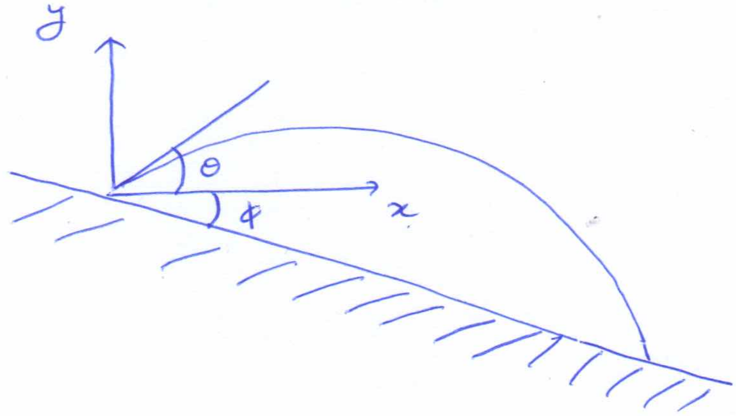
5.



Let v_0 be the initial speed of the rock at angle θ.

$$x = (v_0 \cos\theta)\, t$$
$$y = (v_0 \sin\theta)\, t - \frac{1}{2} g t^2$$

The locus of the hill is

$$y = -x \tan\phi$$

Say the rock land on the hill at time t'.

$$t' = \frac{x}{v_0 \cos\theta}$$

The locus of the hill and the trajectory of the rock intersect at t'.

Thus, $$-x \tan\phi = x \tan\theta - \frac{1}{2}\left(g/v_0^2\right)\left(\frac{x^2}{\cos^2\theta}\right)$$

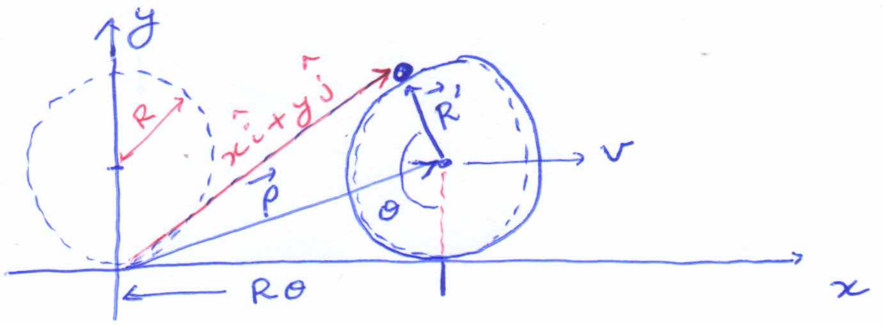
Solving for x and then putting the condition for maximum range $\frac{dx}{d\theta} = 0$ we obtain:

$$\boxed{\theta = \frac{\pi}{4} - \frac{\phi}{2}}$$

//

6.

Say, x, y are the co-ordinates of the pebble measured from the stationary origin. Let $\vec{\rho}$ be the vector from the stationary origin to the center of the rolling tire, and let $\vec{R}'$ be the vector from the center of the tire to the ~~pebble~~. pebble.



$$\vec{\rho} = R\theta\,\hat{i} + R\,\hat{j}$$

$$\vec{R}' = -R\sin\theta\,\hat{i} - R\cos\theta\,\hat{j}$$

From the diagram:

$$x\hat{i} + y\hat{j} = \vec{\rho} + \vec{R}'$$

$$= R\theta\,\hat{i} + R\hat{j} - R\sin\theta\,\hat{i} - R\cos\theta\,\hat{j}$$

$$\Rightarrow \boxed{\begin{aligned} x &= R\theta - R\sin\theta \\ y &= R - R\cos\theta \end{aligned}}$$

$$\therefore \boxed{\begin{aligned} \dot{x} &= R\dot{\theta} - R\cos\theta\,\dot{\theta} \\ \dot{y} &= R\sin\theta\,\dot{\theta} \end{aligned}}$$

The tire is rolling at constant speed without slipping: $\theta = \omega t = \frac{v}{R} t$

Thus,

$$\dot{x} = R\omega - R\omega\cos\theta$$
$$\dot{y} = R\omega\sin\theta$$

$$\ddot{x} = R\omega^2\sin\theta$$
$$\ddot{y} = R\omega^2\cos\theta$$

The pebble on the tire experiences an acceleration

$$a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = R\omega^2 = v^2/R \quad //$$

(message: acceleration measured in the stationary system is the same as measured in the system moving uniformly along with the tire)

7.

Here, given $\dot{r} = b$
$\dot{\theta} = \omega$

Therefore, $r = r_0 + bt$
$\theta = \theta_0 + \omega t$

Note that, $\ddot{\theta} = 0$ and $\ddot{r} = 0$

So,
$$\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = b\hat{r} + bt\omega\hat{\theta}}$$

$$\vec{a} = \hat{r}(\ddot{r} - r\dot{\theta}^2) + \hat{\theta}(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$$\boxed{\vec{a} = -bt\omega^2\hat{r} + 2b\omega\hat{\theta}}$$, taking $r_0 = 0$