DEPARTMENT OF MATHEMATICS, IIT Guwahati
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Summary of Lectures (Set - II)

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1 The Euclidean Space $\mathbb{R}^n$ and its Subspaces

Definition 1.1. *Let n be a positive integer. Then the space $\mathbb{R}^n$, as defined below, is called the n-dimensional **Euclidean space**. The space $\mathbb{R}^n$ is also a vector space (vector space, in general, will be discussed later).*

$$\mathbb{R}^n = \{[x_1, x_2, \ldots, x_n]^t : x_1, x_2, \ldots, x_n \in \mathbb{R}\}.$$

- Elements of $\mathbb{R}^n$ are called n-vectors or simply **vectors**.
- Note that $[x_1, x_2, \ldots, x_n]^t = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector.
- Sometimes, an element $[x_1, x_2, \ldots, x_n]^t$ of $\mathbb{R}^n$ is also written as a row vector $[x_1, x_2, \ldots, x_n]$ or $(x_1, x_2, \ldots, x_n)$.
- The element $(x_1, x_2, \ldots, x_n)$ is also termed as an n-tuple.
- The vector $[0, 0, \ldots, 0]^t$ of $\mathbb{R}^n$, called the **zero** vector, is denoted by the symbol $\mathbf{0}$.
- If A is an $m \times n$ real matrix (all entries are real numbers) and $\mathbf{b} \in \mathbb{R}^m$, then a solution of $A\mathbf{x} = \mathbf{b}$, if any, is an element of $\mathbb{R}^n$. Note that if A is of size $m \times n$ and if we consider the system $A\mathbf{x} = \mathbf{b}$, then both $\mathbf{x}$ and $\mathbf{b}$ must be regarded as column vectors of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.
- Normally, while discussing a system of linear equations, elements of $\mathbb{R}^n$ are regarded a column vectors. Otherwise, elements of $\mathbb{R}^n$ may be regarded as row vectors just for some conveniences in writing, though there is no hard and fast rules.

Definition 1.2. *Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]^t, \mathbf{v} = [v_1, v_2, \ldots, v_n]^t \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We define*

- $\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]^t$;
- $c\mathbf{u} = [cu_1, cu_2, \ldots, cu_n]^t$;
- $-\mathbf{u} = (-1)\mathbf{u} = [-u_1, -u_2, \ldots, -u_n]^t$; *and*
- $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v} = [u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n]^t$.

The vector $\mathbf{u} + \mathbf{v}$ is called the **vector addition** of $\mathbf{u}$ and $\mathbf{v}$. The vector $c\mathbf{u}$ us called the **scalar multiplication** of c and $\mathbf{u}$.

Some Basic Properties: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition);
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity over scalar addition);
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$, $\quad 0\mathbf{u} = \mathbf{0}$, $\quad \mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, $\quad 1\mathbf{u} = \mathbf{u}$ and $\quad c(d\mathbf{u}) = (cd)\mathbf{u}$.

An Example Revisited: Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}$.

The solutions set for $A\mathbf{x} = \mathbf{0}$ is

$$S_h = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \;:\; s, t \in \mathbb{R} \right\}.$$

Can we describe S_h with a few of the solutions? Can we derive some special properties of solution sets like S_h?

Linear Combination: A vector $\mathbf{v}$ in $\mathbb{R}^n$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ if there exists real numbers $c_1, c_2, \ldots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$. The numbers $c_1, c_2, \ldots, c_k$ are called the **coefficients** of the linear combination.

Example 1.1. *Is the vector $[1, 2, 3]^t$ a linear combination of $[1, 0, 3]^t$ and $[-1, 1, -3]^t$?*

Result 1.1. *A system of linear equations with augmented matrix $[A \mid \mathbf{b}]$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A.*

Span of Vectors: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Then the collection of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the **span** of S (or span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$), and is denoted by $\text{span}(S)$ (or $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$). That is,

$$\text{span}(S) = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k \text{ for some real numbers } c_1, c_2, \ldots, c_k\}.$$

- Convention: $\text{span}(\emptyset) = \{\mathbf{0}\}$.
- If $\text{span}(S) = \mathbb{R}^n$, then S is called a **spanning set** for $\mathbb{R}^n$.
- For example, $\mathbb{R}^2 = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$, where $\mathbf{e}_1 = [1, 0]^t$ and $\mathbf{e}_2 = [0, 1]^t$, since for any $[x, y]^t \in \mathbb{R}^2$ we have $[x, y]^t = x\mathbf{e}_1 + y\mathbf{e}_2$.

Example 1.2. *Let $\mathbf{u} = [1, 2, 3]^t$ and $\mathbf{v} = [-1, 1, -3]^t$. Describe span$(\mathbf{u}, \mathbf{v})$ geometrically.*

Subspace of $\mathbb{R}^n$: Let $S(\neq \emptyset) \subseteq \mathbb{R}^n$. Then S is called a **subspace** of $\mathbb{R}^n$ iff $a\mathbf{u} + b\mathbf{v} \in S$ for every $\mathbf{u}, \mathbf{v} \in S$ and for every $a, b \in \mathbb{R}$.

- For example, $S = \{\mathbf{0}\}$ and $S = \mathbb{R}^n$ are some trivial examples of subspaces of $\mathbb{R}^n$.
- If S is a subspace of $\mathbb{R}^n$ then it is clear that if $\mathbf{u}, \mathbf{v} \in S$ then $0\mathbf{u} + 0\mathbf{v} = \mathbf{0} \in S$, $a\mathbf{u} + 0\mathbf{u} = a\mathbf{u} \in S$ and $1\mathbf{u} + 1\mathbf{v} = \mathbf{u} + \mathbf{v} \in S$.

Example 1.3. *Examine whether the sets $S = \{[x, y, z]^t \in \mathbb{R}^3 : x = y + 1\}$, $T = \{[x, y, z]^t \in \mathbb{R}^3 : x = z^2\}$ and $U = \{[x, y, z]^t \in \mathbb{R}^3 : x = 5y\}$ are subspaces of $\mathbb{R}^3$.*

Example 1.4. *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, v_k\} \subseteq \mathbb{R}^n$. Then span$(S)$ is a subspace of $\mathbb{R}^n$.*

Example 1.5. *Let A be an $m \times n$ matrix. Then $S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the **nullspace** of A.*

Example 1.6. *Let U and V be two subspaces of $\mathbb{R}^n$. Then $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a subspace of $\mathbb{R}^n$.*

If U and V are subspaces of $\mathbb{R}^n$ such that $U \cap V = \{\mathbf{0}\}$, then $U + V$ is called an **internal direct sum**. Notation: $U \oplus V$.

Result 1.2. *Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with n variables. Then exactly one of the following is true:*

1. *there is no solution of the system $A\mathbf{x} = \mathbf{b}$;*
2. *there is a unique solution of the system $A\mathbf{x} = \mathbf{b}$;*
3. *there are infinitely many solutions of the system $A\mathbf{x} = \mathbf{b}$.*

Linear Dependence: A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors (or the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$) in $\mathbb{R}^n$ is (are) said to be **linearly dependent** if there are real numbers $c_1, c_2, \ldots, c_k$, at least one of them is non-zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$.

Linear Independence: The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors (or the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$) in $\mathbb{R}^n$ is (are) said to be **linearly independent** if it is (they are) **not** linearly dependent. That is, if the equation $c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_k\mathbf{v}_k = \mathbf{0}$ implies that $c_1 = c_2 = \ldots = c_k = 0$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

- It is easy to check that a set of vectors containing the $\mathbf{0}$ is always linearly dependent.

Example 1.7. *Let $\mathbf{e}_i \in \mathbb{R}^n$ be the i-th column of the identity matrix I_n. Is $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ linearly independent?*

Example 1.8. *Examine whether the sets $T = \{[1,2,0]^t, [1,1,-1]^t, [1,4,2]^t\}$ and $S = \{[1,4]^t, [-1,2]^t\}$ are linearly dependent.*

Result 1.3. *The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are linearly dependent iff at least one of the vectors can be expressed as a linear combination of the others.*

Linear combinations of rows:

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A. Note that $\mathbf{a}$ is an $1 \times n$ matrix and $\mathbf{a}^T \in \mathbb{R}^n$.
- Note: $c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = [c_1, \ldots, c_m]A$. Thus, for any $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c}^T A$ is a linear combination of rows of A.
- The rows of A are linearly dependent **iff** $\mathbf{c}^T A = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = \mathbf{0}^T$ (zero row) for some nonzero $\mathbf{c} \in \mathbb{R}^m$.
- The rows of A are linearly dependent **iff** $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are linearly dependent, i.e., the columns of A^T are linearly dependent.

Result 1.4. *Suppose $R = RREF(A)$ has a zero row. Then the rows of A are linearly dependent.*

Result 1.5. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$ and let A be the $n \times m$ matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_m]$ with these vectors as its columns. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.*

Result 1.6. *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$ and consider the $m \times n$ matrix $A = \begin{bmatrix} \mathbf{v}_1^t \\ \mathbf{v}_2^t \\ \vdots \\ \mathbf{v}_m^t \end{bmatrix}$. Then S is linearly dependent if and only if rank$(A) < m$.*

Result 1.7. *Any set of m vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$.*

Basis: Let S be a subspace of $\mathbb{R}^n$ and $B \subseteq S$. Then B is said to be a **basis** for S iff B is linearly independent and span$(B) = S$.

- The set $\{1\}$ is a basis for $\mathbb{R}^1$ $(= \mathbb{R})$.
- The set $\{\mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_1 = [1,0]^t$ and $\mathbf{e}_2 = [0,1]^t$, is a basis for $\mathbb{R}^2$.
- In general, if $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots, 0]^t$, where 1 is at the i-th entry and the other entries are zero, then $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$.
- The vectors $\mathbf{e}_i$ (for $i = 1, 2, \ldots, n$) are called the $n \times 1$ standard **unit vector**.
- If $\mathbf{e}_i$ is written as a row vector, then the vectors $\mathbf{e}_i$ (for $i = 1, 2, \ldots, n$) are called the $1 \times n$ standard **unit vector**.

Result 1.8. *For a subspace U, a subset $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq U$ is a basis of U iff every element of U is a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_r$.*

Example 1.9. *Find a basis for the subspace $S = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$, where*

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}.$$

Result 1.9. *Let S be a subspace of $\mathbb{R}^n$. Then S has a basis and any two bases for S have the same number of elements.*

Dimension: Let S be a subspace of $\mathbb{R}^n$. Then the number of elements in a basis for S is called the **dimension**, denoted $\dim(S)$, of S.

- Thus $\dim(\mathbb{R}) = 1,\ \dim(\mathbb{R}^2) = 2$.
- In general, $\dim(\mathbb{R}^n) = n$.

2 Matrices

Let $A = [a_{ij}]$ be an $m \times n$ matrix.

- If $m = n$, then A is called a **square** matrix.
- If A is a square matrix, then the entries a_{ii} are called the **diagonal** entries of A.
- If A is a square matrix and if $a_{ij} = 0$ for all $i \neq j$, then A is called a **diagonal matrix**.
- If an $n \times n$ diagonal matrix has all diagonal entries equal to 1, then it is called the **identity matrix** of size n, and is denoted by I_n (or simply by I).
- An $m \times n$ matrix is called a **zero matrix** of size $m \times n$, denoted $\mathbf{O}_{m\times n}$ (or simply $\mathbf{O}$), if all the entries are equal to 0.
- A matrix B is said to be a **sub matrix** of A if B is obtained by deleting some rows and/or columns of A.
- Recall that the **transpose** A^t of $A = [a_{ij}]$ is defined to be the $n \times m$ matrix $A^t = [b_{ji}]$, where the i-th row of A^t is the i-th column of A for all $i = 1, 2, \ldots, n$, that is $b_{ji} = a_{ij}$ for all i, j.
- The matrix A is said to be **symmetric** if $A^t = A$, and **skew-symmetric** if $A^t = -A$.
- If A is a complex matrix, then $\overline{A} = [\overline{a}_{ij}]$ and $A^* = \overline{A}^t$.
- The matrix A^* is called the **conjugate transpose** of A.
- The (complex) matrix A is said to be **Hermitian** if $A^* = A$, and **skew-Hermitian** if $A^* = -A$.
- A square matrix A is said to be **upper triangular** if $a_{ij} = 0$ for all $i > j$.
- A square matrix A is said to be **lower triangular** if $a_{ij} = 0$ for all $i < j$.
- Let A be an $n \times n$ square matrix. Then we define $A^0 = I_n, A^1 = A$ and $A^2 = AA$.
- In general, if k is a positive integer, we define the power A^k as follows

$$A^k = \underbrace{AA\ldots A}_{k \text{ times}}.$$

It is obvious to see that if A and $\mathbf{O}$ are matrices of the same size, then $A+\mathbf{O} = A = \mathbf{O}+A,\ \ A-\mathbf{O} = A,\ \ \mathbf{O}-A = -A$ and $A - A = \mathbf{O}$.

Result 2.1. *Let A be an $m \times n$ matrix, $\mathbf{e}_i$ an $1 \times m$ standard unit vector, and $\mathbf{e}_j$ an $n \times 1$ standard unit vector. Then $\mathbf{e}_iA$ is the i-th row of A and $A\mathbf{e}_j$ is the j-th column of A.*

Result 2.2. *Let A be a square matrix and let r and s be non-negative integers. Then $A^rA^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.*

Result 2.3. *Let A, B and C be matrices of size $m \times n$, and let $s, r \in \mathbb{R}$. Then*

1. ***Commutative Law:*** $A + B = B + A$.
2. ***Associative Law:*** $(A + B) + C = A + (B + C)$.
3. $1A = A, \quad s(rA) = (sr)A$.
4. $s(A + B) = sA + sB$ *and* $(s + r)A = sA + rA$.

Result 2.4. *Let A, B and C be matrices, and let $s \in \mathbb{R}$. Then*

1. ***Associative Law:*** $(AB)C = A(BC)$, *if the respective matrix products are defined.*
2. ***Distributive Law:*** $A(B + C) = AB + AC, \quad (A + B)C = AC + BC$, *if the respective matrix sum and matrix products are defined.*
3. $s(AB) = (sA)B = A(sB)$, *if the respective matrix products are defined.*
4. $I_m A = A = AI_n$, *if A is of size $m \times n$.*

Result 2.5. *Let A and B be two matrices and $k \in \mathbb{R}$. Then*

1. $(A^t)^t = A, \quad (kA)^t = kA^t$.
2. $(A + B)^t = A^t + B^t$ *if A and B are of the same size.*
3. $(AB)^t = B^t A^t$ *if the matrix product AB is defined.*
4. $(A^r)^t = (A^t)^r$ *for any non-negative integer r.*

Partitioned Matrix: By introducing vertical and horizontal lines into a given matrix, we can partition it into some blocks of smaller sub-matrices. A given matrix can have several partitions possible. For example, three partitions of the matrix A are given below:

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}, A = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right], A = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right], A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ \hline 0 & 0 & 1 & 5 \end{array}\right].$$

- A **block matrix** $A = [A_{ij}]$ is a matrix, where each entry A_{ij} is itself a matrix.
- Thus a partition of a given matrix give us a block matrix.
- For example, the first partition of the previous matrix A is the block matrix $\begin{bmatrix} I & B \\ \mathbf{O} & C \end{bmatrix}$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$.

Result 2.6. *Let A and B be two matrices of sizes $m \times n$ and $n \times r$, respectively.*

1. *If $B = [\mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_r]$, where $\mathbf{b}_i$ is the i-th column of B then $AB = [A\mathbf{b}_1 \mid A\mathbf{b}_2 \mid \ldots \mid A\mathbf{b}_r]$.*
2. *If* $A = \left[\begin{array}{c} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right]$, *where $\mathbf{a}_i$ is the i-th row of A then* $AB = \left[\begin{array}{c} \mathbf{a}_1 B \\ \hline \mathbf{a}_2 B \\ \hline \vdots \\ \hline \mathbf{a}_m B \end{array}\right]$.
3. *If $m = n = r$ and if $A = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \ldots \mid \mathbf{a}_n]$ and* $B = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \vdots \\ \hline \mathbf{b}_n \end{array}\right]$, *where $\mathbf{a}_i$ is the i-th column of A and $\mathbf{b}_i$ is the i-th row of B then $AB = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \ldots + \mathbf{a}_n\mathbf{b}_n$.*

3 The Inverse of a Matrix

Definition 3.1. *An $n \times n$ matrix A is said to be* ***invertible*** *if there exists an $n \times n$ matrix B satisfying $AB = I_n = BA$, and B is called an* ***inverse*** *of A.*

- Note that we can talk of invertibility only for square matrices.
- For example, the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is invertible since

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

- It is easy to see that the zero matrix $\mathbf{O}$ is never invertible.
- The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible.

Result 3.1. *If A is an invertible matrix, then its inverse is unique.*

- We write A^{-1} to denote the inverse of an invertible matrix A.
- That is, if A is invertible then $AA^{-1} = I_n = A^{-1}A$.
- If A is an 1×1 invertible matrix, what is A^{-1}?

Result 3.2. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$ then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = 0$ then A is not invertible.*

Result 3.3. *Let A and B be two invertible matrices of the same size.*

1. *The matrix A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.*
2. *If $c \neq 0$ then cA is also invertible, and $(cA)^{-1} = \frac{1}{c}A^{-1}$.*
3. *The matrix AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.*
4. *The matrix A^t is invertible, and $(A^t)^{-1} = (A^{-1})^t$.*
5. *For any non-negative integer n, the matrix A^n is invertible, and $(A^n)^{-1} = (A^{-1})^n$.*

Elementary Matrices: An **elementary matrix** is a matrix that can be obtained by performing **an** elementary row operation on the identity matrix.

- Since there are three types of elementary row operations, there are three types of corresponding elementary matrices.
- For example, the following are the three types of elementary matrices of size 3.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

- Note that E_1 is obtained by performing $R_2 \leftrightarrow R_3$ on I_3, E_2 is obtained by performing $R_2 \to 5R_2$ on I_3 and E_3 is obtained by performing $R_3 \to R_3 - 2R_1$ on I_3.
- Let A be the 3×3 matrix as given below:

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}.$$

Then we have

$$E_1A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix},\ E_2A = \begin{bmatrix} a & b & c \\ 5x & 5y & 5z \\ p & q & r \end{bmatrix} \text{ and } E_3A = \begin{bmatrix} a & b & c \\ x & y & z \\ p-2a & q-2b & r-2c \end{bmatrix}.$$

- Notice that E_1A is the matrix obtained from A by performing the elementary row operation $R_2 \leftrightarrow R_3$.
- The matrix E_2A is the matrix obtained from A by performing the elementary row operation $R_2 \to 5R_2$.
- The matrix E_3A is the matrix obtained from A by performing the elementary row operation $R_3 \to R_3 - 2R_1$.

Result 3.4.

1. *Let E be an elementary matrix obtained by an elementary row operation on I_n. If the same elementary row operation is performed on an $n \times r$ matrix A, then the resulting matrix is equal to EA.*
2. *The matrix B is row equivalent to A if there are elementary matrices $E_1, E_2, \cdots, E_k$ such that $B = E_k \cdots E_2E_1A$.*

We already know that elementary row operation can be undone or reversed. Applying this fact to the previous elementary matrices E_1, E_2 and E_3, we see that they are invertible.

- Indeed, applying $R_2 \leftrightarrow R_3$ on I_3 we find E_1^{-1}, applying $R_2 \to \frac{1}{5}R_2$ on I_3 we find E_2^{-1}, and applying $R_3 \to R_3+2R_1$ on I_3 we find E_3^{-1}.

We have

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1,\ E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Result 3.5. *Every elementary matrix is invertible, and its inverse is an elementary matrix of the same type.*

Result 3.6 (The Fundamental Theorem of Invertible Matrices: Version I). *Let A be an $n \times n$ matrix. Then the following statements are equivalent.*

1. *A is invertible.*
2. *A^t is invertible.*
3. *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
4. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
6. *The reduced row echelon form of A is I_n.*
7. *The rows of A are linearly independent.*
8. *The columns of A are linearly independent.*
9. *rank$(A) = n$.*
10. *A is a product of elementary matrices.*

Result 3.7. *Let A be a square matrix. If B is a square matrix such that either $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$.*

Result 3.8. *Let A be a square matrix. If a sequence of elementary row operations transforms A to the identity matrix I, then the same sequence of elementary row operations transforms I into A^{-1}.*

Gauss-Jordan Method for Computing Inverse:

Let A be an $n \times n$ matrix.

- Apply elementary row operations on the augmented matrix $[A \mid I_n]$.
- If A is invertible, then $[A \mid I_n]$ will be transformed to $[I_n \mid A^{-1}]$.
- If A is not invertible, then $[A \mid I_n]$ can never be transformed to a matrix of the type $[I_n \mid B]$.

Example 3.1. *Find the inverse of the following matrix A, if it exists:*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$