1 Inner Product and Gram-Schmidt Process

Let $\mathbb{K}$ denote $\mathbb{C}$ or $\mathbb{R}$. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{K}^n$.

- The **dot product** or the standard **inner product** $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^* \mathbf{v} = [\overline{u_1} \quad \cdots \quad \overline{u_n}] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \overline{u_1}v_1 + \overline{u_2}v_2 + \ldots + \overline{u_n}v_n = \sum \overline{u_i}v_i.$$

- Sometimes, the notation $\langle \mathbf{u}, \mathbf{v} \rangle$ is also used to denote the inner product of $\mathbf{u}$ and $\mathbf{v}$.
- The **length** or **norm** $\|\mathbf{u}\|$ of $\mathbf{u}$ is defined by

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}.\mathbf{u}} = \sqrt{|u_1|^2 + |u_2|^2 + \ldots + |u_n|^2}.$$

- Observe that $\|\alpha \mathbf{u}\| = |\alpha| \, \|\mathbf{u}\|$.
- The vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthogonal** if $\mathbf{u}.\mathbf{v} = 0$.
- A vector $\mathbf{u}$ is called an **unit** vector if $\|\mathbf{u}\| = 1$.

Result 1.1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{K}^n$ and $c \in \mathbb{K}$. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^* \mathbf{v} = (\mathbf{v}^* \mathbf{u})^* = \overline{\mathbf{v} \cdot \mathbf{u}}$;
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u}^*(\mathbf{v} + \mathbf{w}) = \mathbf{u}^*\mathbf{v} + \mathbf{u}^*\mathbf{w} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$;
3. $(c\mathbf{u}) \cdot \mathbf{v} = (c\mathbf{u})^*\mathbf{v} = \sum \overline{cu_i}v_i = \overline{c} \sum \overline{u_i}v_i = \overline{c}(\mathbf{u} \cdot \mathbf{v})$;
4. $\mathbf{u} \cdot (c\mathbf{v}) = \mathbf{u}^*(c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$;
5. $\mathbf{u} \cdot \mathbf{u} = \sum |c_i|^2 \geq 0$. The equality holds iff each each $u_i = 0$, that is, iff $\mathbf{u} = 0$.

Orthogonal Set: A set $S \subseteq \mathbb{K}^n$ is said to be an **orthogonal set** if $\mathbf{v} \cdot \mathbf{w} = 0$ holds for each $\mathbf{v}, \mathbf{w} \in S$, $\mathbf{v} \neq \mathbf{w}$.

- The set $\{[2, 1, -1]^t, [0, 1, 1]^t, [1, -1, 1]^t\}$ is an orthogonal set in $\mathbb{R}^3$.
- An orthogonal set can contain a zero.
- Let S be an orthogonal set and $\mathbf{u}, \mathbf{v} \in S$. Put $\mathbf{x} = 2\mathbf{u} + (1 - i)\mathbf{v}$. Then

$$\mathbf{u} \cdot \mathbf{x} = \mathbf{u}.(2\mathbf{u} + (1 - i)\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{u}) + (1 - i)(\mathbf{u} \cdot \mathbf{v}) = 2\|u\|^2.$$

  Similarly $\mathbf{v} \cdot \mathbf{x} = (1 - i)\|v\|^2$.

Result 1.2. *If S is an orthogonal set of non-zero vectors, then S is linearly independent.*

Proof. Suppose that S is linearly dependent. Then there exist $\mathbf{v_1}, \ldots, \mathbf{v_k} \in S$ such that $\sum_{i=1}^k c_i \mathbf{v_i} = \mathbf{0}$, where some c_i are nonzero. But then (see previous item)

$$c_i \|v_i\|^2 = \mathbf{v_i} \cdot \sum_{i=1}^k c_i \mathbf{v_i} = \mathbf{v_i} \cdot \mathbf{0} = 0.$$

As $\|\mathbf{v_i}\| \neq 0$, we get $c_i = 0$. And this holds for each i, a contradiction. □

Orthogonal Basis: An **orthogonal basis** for a subspace W is a basis for W that is an orthogonal set.

- The set $\{\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\}$ is an orthogonal basis for $\mathbb{R}^3$. Take $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$. Find a, b, c such that $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

  Ans. $a = \frac{\mathbf{u} \cdot \mathbf{x}}{\|\mathbf{u}\|^2} = \frac{1}{3}$, $b = \frac{\mathbf{v} \cdot \mathbf{x}}{\|\mathbf{v}\|^2} = 1$ and $c = \frac{\mathbf{w} \cdot \mathbf{x}}{\|\mathbf{w}\|^2} = \frac{1}{3}$.

- The set $\{\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\}$ is an orthogonal basis for $\mathbb{C}^3$. Take $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix} \in \mathbb{C}^3$. Find a, b, c such that $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

  Ans. $a = \frac{\mathbf{u} \cdot \mathbf{x}}{\|\mathbf{u}\|^2} = \frac{3-i}{6}$, $b = \frac{\mathbf{v} \cdot \mathbf{x}}{\|\mathbf{v}\|^2} = \frac{1+i}{2}$ and $c = \frac{\mathbf{w} \cdot \mathbf{x}}{\|\mathbf{w}\|^2} = \frac{i}{3}$.

Result 1.3. *Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal basis for a subspace W and let $\mathbf{w} \in W$. Then the unique scalars $c_1, c_2, \ldots, c_k$ such that $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$ are given by*

$$c_i = \frac{\mathbf{v}_i.\mathbf{w}}{\mathbf{v}_i.\mathbf{v}_i} \quad \textit{for} \quad i = 1, 2, \ldots, k.$$

Proof. Follows from the evaluation that $\mathbf{v_i} \cdot \mathbf{w} = c_i \|\mathbf{v_i}\|^2$. □

Orthonormal Set: A set of vectors is said to be an **orthonormal set** if it is an orthogonal set of unit vectors.

Orthonormal Basis: An **orthonormal basis** of a subspace W is a basis of W that is an orthonormal set.

Result 1.4. *Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be an orthonormal basis for a subspace W and let $\mathbf{w} \in W$. Then*

$$\mathbf{w} = (\mathbf{u}_1.\mathbf{w})\mathbf{u}_1 + (\mathbf{u}_2.\mathbf{w})\mathbf{u}_2 + \ldots + (\mathbf{u}_k.\mathbf{w})\mathbf{u}_k,$$

and this representation is unique.

Proof. Follows from the previous result. □

Orthogonal Complement: Let W be a subspace of $\mathbb{K}^n$.

- A vector $\mathbf{v} \in \mathbb{K}^n$ is said to be **orthogonal** to W if $\mathbf{v}$ is orthogonal to every vector in W.
- The orthogonal complement of W, denoted $W^\perp$ (called W-perp), is defined as

$$W^\perp = \{\mathbf{v} \in \mathbb{K}^n : \mathbf{v}.\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

- In $\mathbb{R}^3$, take $W = \{\mathbf{e_1}\}$. Then $W^\perp = yz$-plane.
- In $\mathbb{R}^3$, take $W = \{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$. Then $W^\perp = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$.
- In $\mathbb{R}^3$, take $W = \{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$. Then $W^\perp = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$.
- In $\mathbb{R}^3$, take $W = \text{SPAN}\{\mathbf{e_1}\}$. Then $W^\perp = yz$-plane.
- In $\mathbb{R}^3$, take $W = \text{SPAN}\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$. Then $W^\perp = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$.
- In $\mathbb{C}^3$, take $W = \text{SPAN}\{\begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ 2i \\ 3 \end{bmatrix}\}$. Then $W^\perp = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} 1 & 1 & -i \\ 1 & -2i & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$.
- Let W be a subspace. Then $W^\perp$ is also a subspace.

  Proof. Just show that if $\mathbf{x}, \mathbf{y} \in W^\perp$, then $\mathbf{x} + c\mathbf{y} \in W^\perp$. □

- $W \cap W^\perp = \{\mathbf{0}\}$.

  Proof. If $\mathbf{v} \in W \cap W^\perp$, then $\mathbf{v} \cdot \mathbf{v} = 0$. Hence $\mathbf{v} = \mathbf{0}$. □

- If $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is a basis for W, then $\mathbf{v} \in W^\perp$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, 2, \ldots, k$.

  Proof. Suppose that $\mathbf{v} \cdot \mathbf{w_i} = 0$, for each i. Let $\mathbf{w} \in W$. Then $\mathbf{w} = \sum a_i \mathbf{w_i}$. Hence $\mathbf{v} \cdot \mathbf{w} = \sum a_i(\mathbf{v} \cdot \mathbf{w_i}) = 0$.

  Conversely, if $\mathbf{v} \cdot \mathbf{w} = 0$ for each $\mathbf{w} \in W$, then in particular, $\mathbf{v} \cdot \mathbf{w_i} = 0$ for each i. □

- Let W be a subspace of $\mathbb{K}^n$ with a basis $\{\mathbf{w_1} = \begin{bmatrix} w_{11} \\ \vdots \\ w_{1n} \end{bmatrix}, \ldots, \mathbf{w_k} = \begin{bmatrix} w_{k1} \\ \vdots \\ w_{kn} \end{bmatrix}\}$. Form the matrix A by taking $\mathbf{w_i}$ as the i-th column. Then W is nothing but $\textsc{col}\, A$ and so

$$\begin{aligned}
(\textsc{col}\, A)^{\perp} = W^{\perp} &= \{\mathbf{v} \;:\; \mathbf{v} \cdot \mathbf{w} = 0 \text{ for each } \mathbf{w} \in W\} \\
&= \{\mathbf{v} \;:\; \mathbf{w} \cdot \mathbf{v} = 0 \text{ for each } \mathbf{w} \in W\} \\
&= \{\mathbf{v} \;:\; \mathbf{w_i} \cdot \mathbf{v} = 0 \text{ for each } i = 1, \ldots, k\} \\
&= \{\mathbf{v} \;:\; \mathbf{w_i}^* \mathbf{v} = 0 \text{ for each } i = 1, \ldots, k\} \\
&= \left\{ \mathbf{v} \;:\; \begin{bmatrix} \overline{w_{11}} & \cdots & \overline{w_{1n}} \\ & \vdots & \\ \overline{w_{k1}} & \cdots & \overline{w_{kn}} \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \\
&= \textsc{null}\, A^*.
\end{aligned}$$

  In particular, we have,

  - $W^{\perp}$ has dimension $n-k$.
  - $(\textsc{row}\, \overline{A})^{\perp} = (\textsc{col}\, A^*)^{\perp}$ (as $\textsc{row}\, A = \textsc{col}\, A^t$) $= \textsc{null}\, A$.

- Let W be a subspace of $\mathbb{K}^n$ with a basis $\{\mathbf{w_1} = \begin{bmatrix} w_{11} \\ \vdots \\ w_{1n} \end{bmatrix}, \ldots, \mathbf{w_k} = \begin{bmatrix} w_{k1} \\ \vdots \\ w_{kn} \end{bmatrix}\}$. Form the matrix A by taking $\mathbf{w_i}$ as the i-th column. Note that $\textsc{rank}\, A = k$. Hence there is an invertible matrix T such that $TA = \textsc{rref}\, A = \begin{bmatrix} I_k \\ \mathbf{O} \end{bmatrix}$. Consider the last $n-k$ rows of T. These rows are linearly independent, as they are part of an invertible matrix. Notice that the matrix product of such a row with the vectors $\mathbf{w_i}$ is 0. Then the last $n-k$ columns of T^* will belong to $W^{\perp}$. In view of the previous item, they will form a basis of $W^{\perp}$.

- **Orthogonal Decomposition Theorem** Let W be a subspace of $\mathbb{K}^n$ of dimension k. Then $W \oplus W^{\perp} = \mathbb{K}^n$.

  Proof. We already know that the dimension of $W^{\perp}$ is $n-k$. Let $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ be a basis for W and $\{\mathbf{v_{k+1}}, \ldots, \mathbf{v_n}\}$ be a basis for $W^{\perp}$. Then $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is linearly independent. Indeed, if $\sum a_i \mathbf{v_i} = \mathbf{0}$, then we have $\sum_{i=1}^{k} a_i \mathbf{v_i} + \sum_{i=k+1}^{n} a_i \mathbf{v_i} = \mathbf{0}$. That is, $\sum_{i=1}^{k} a_i \mathbf{v_i} = -\sum_{i=k+1}^{n} a_i \mathbf{v_i} \in W \cap W^{\perp}$. Hence $\sum_{i=1}^{k} a_i \mathbf{v_i} = \sum_{i=k+1}^{n} a_i \mathbf{v_i} = 0$. Hence each $a_i = 0$, as $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ and $\{\mathbf{v_{k+1}}, \ldots, \mathbf{v_n}\}$ are bases. ☐

- Let $W \subseteq \mathbb{K}^n$. Then $\textsc{span}\, W \subseteq (W^{\perp})^{\perp}$.

  Proof. Note that $(W^{\perp})^{\perp}$ contains those elements of $\mathbb{K}^n$ which are orthogonal to $W^{\perp}$. In particular, as each element w of W is orthogonal to $W^{\perp}$, we see that $w \in (W^{\perp})^{\perp}$. As $W \subseteq (W^{\perp})^{\perp}$ and the later is a subspace, we see that $\textsc{span}\, W \subseteq (W^{\perp})^{\perp}$. ☐

- Let W be a subspace of $\mathbb{K}^n$ of dimension k. Then $(W^{\perp})^{\perp} = W$.

  Proof. As $\textsc{dim}\, W = k$, by a previous item, we know that $\textsc{dim}\, W^{\perp} = n-k$. Hence $\textsc{dim}(W^{\perp})^{\perp} = n-(n-k) = k$. By the preceding item, $W \subseteq (W^{\perp})^{\perp}$. But as both have the same dimension, they must be the same. ☐

- Let A be an $m \times n$ matrix. Then $(\textsc{col}\, A)^{\perp} = \textsc{null}\, A^*$, $(\textsc{row}\, \overline{A})^{\perp} = \textsc{null}\, A$, and $\textsc{row}\, \overline{A} = (\textsc{null}\, A)^{\perp}$.

  Proof. We already have proved the first two. The third one follows from the second by using the preceding item. ☐

Example 1.1. Consider $W = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x+y+z = 0\}$. A basis for W is $\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\}$. A basis for $W^{\perp}$ is $\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$.

- Notice that if $\mathbf{v} \in \mathbb{R}^3$ satisfies

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_1' = \mathbf{w}_2 + \mathbf{w}_2', \text{ where } \mathbf{w}_1, \mathbf{w}_2 \in W, \mathbf{w}_1', \mathbf{w}_2' \in W^{\perp};$$

  then $\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{w}_2' - \mathbf{w}_1' \in W \cap W^{\perp}$. Hence $\mathbf{w}_1 = \mathbf{w}_2$ and $\mathbf{w}_1' = \mathbf{w}_2'$.

- Thus $\mathbf{v}$ can be written as a sum of a vector in W and a vector in $W^{\perp}$ uniquely.

- Consider $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find (somehow) $\mathbf{w} \in W$ and $\mathbf{w}' \in W^{\perp}$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}'$. This $\mathbf{w}$ is called the orthogonal projection of $\mathbf{v}$ on W.

- Let W be a subspace of $\mathbb{K}^n$ and $\mathbf{v} \in \mathbb{K}^n$. Then the **orthogonal projection of v on** W is the unique vector $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}'$, for some $\mathbf{w}' \in W^\perp$. In other words, it is the unique vector $\mathbf{w} \in W$ such that $\mathbf{v} - \mathbf{w} \in W^\perp$. Again in words, it is the unique vector $\mathbf{w} \in W$ such that $(\mathbf{v} - \mathbf{w})$ is orthogonal to $\mathbf{w}$. We denote this vector $\mathbf{w}$ by $\mathrm{proj}_W(\mathbf{v})$. The vector $\mathbf{v} - \mathrm{proj}_W(\mathbf{v})$ which denotes the perpendicular from $\mathbf{v}$ to W is denoted by $\mathrm{perp}_W(\mathbf{v})$.
- Let W be a subspace of $\mathbb{K}^n$ and $\mathbf{v} \in \mathbb{K}^n$. If we have an orthonormal basis $\{\mathbf{w_1}, \ldots, \mathbf{w_k}\}$ of W, then $\mathrm{proj}_W(\mathbf{v})$ can be easily computed. In fact
$$\mathrm{proj}_W(\mathbf{v}) = (\mathbf{w_1} \cdot \mathbf{v})\mathbf{w_1} + \cdots + (\mathbf{w_k} \cdot \mathbf{v})\mathbf{w_k} = \sum_{i=1}^{k} (\mathbf{w_i} \cdot \mathbf{v})\mathbf{w_i}.$$
If the basis is orthogonal, then
$$\mathrm{proj}_W(\mathbf{v}) = (\mathbf{w_1} \cdot \mathbf{v})\frac{\mathbf{w_1}}{\|\mathbf{w_1}\|^2} + \cdots + (\mathbf{w_k} \cdot \mathbf{v})\frac{\mathbf{w_k}}{\|\mathbf{w_k}\|^2} = \sum_{i=1}^{k} (\mathbf{w_i} \cdot \mathbf{v})\frac{\mathbf{w_i}}{\|\mathbf{w_i}\|^2}.$$
- For the previous example, we have $\{\begin{bmatrix}-1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\}$ is an orthogonal basis. Hence
$$\mathrm{proj}_W(\begin{bmatrix}1\\2\\3\end{bmatrix}) = \frac{1}{6}(\begin{bmatrix}-1\\2\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix})\begin{bmatrix}-1\\2\\1\end{bmatrix} + \frac{1}{2}(\begin{bmatrix}1\\0\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix})\begin{bmatrix}1\\0\\-1\end{bmatrix} = 0\begin{bmatrix}-1\\2\\-1\end{bmatrix} - \begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}.$$
Thus,
$$\mathrm{perp}_W(\begin{bmatrix}1\\2\\3\end{bmatrix}) = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}.$$
- Let W be a subspace of $\mathbb{K}^n$ with an orthonormal basis $\{\mathbf{w_1}, \ldots, \mathbf{w_k}\}$ and $\mathbf{v} \in \mathbb{K}^n$. Put $W_i = \mathrm{SPAN}(\mathbf{w_i})$. Then
$$\mathrm{proj}_W(\mathbf{v}) = \mathrm{proj}_{W_1}(\mathbf{v}) + \ldots + \mathrm{proj}_{W_k}(\mathbf{v}).$$

Example 1.2. *Let A and B be two $m \times n$ matrices and let the linear systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution space. Show that the matrix A is row equivalent to B.*
Ans. *We have $\mathrm{NULL}\, A = \mathrm{NULL}\, B$. Hence $(\mathrm{NULL}\, A)^\perp = (\mathrm{NULL}\, B)^\perp$, that is $\mathrm{ROW}\, \overline{A} = \mathrm{ROW}\, \overline{B}$. Hence $\mathrm{ROW}\, A = \mathrm{ROW}\, B$. Let $k = \mathrm{DIM\,ROW}\, A$. Let A' be the matrix obtained by taking the first k rows of RREF A and B' be the matrix obtained by taking the first k rows of RREF B. As rows of A' are linear combinations of rows of B', we have $A' = SB'$ for some $k \times k$ matrix S. Note that $k = \mathrm{RANK}\, A' = \mathrm{RANK}(SB') \le \mathrm{RANK}\, S, \mathrm{RANK}\, B'$ and hence $k \le \mathrm{RANK}\, S$. That is, S is invertible. Hence A' is row equivalent to B'. Hence RREF A is row equivalent to RREF B, that is, they are equal.* $\square$

Example 1.3. *Let A and B be two $m \times n$ matrices and let the* ***consistent*** *linear systems $A\mathbf{x} = \mathbf{c}$ and $B\mathbf{x} = \mathbf{d}$ have the same solution set. Show that the matrix A is row equivalent to B.*
Ans. *If $A\mathbf{x} = \mathbf{c}$ and $B\mathbf{x} = \mathbf{d}$ have the same solution set, then $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set.*

Example 1.4. *Let W be the subspace of $\mathbb{R}^5$ spanned by the vectors $\mathbf{w}_1 = [1, -3, 5, 0, 5]^t, \mathbf{w}_2 = [-1, 1, 2, -2, 3]^t$ and $\mathbf{w}_3 = [0, -1, 4, -1, 5]^t$. Find a basis for $W^\perp$.*

Example 1.5. *Let S be a subspace of $\mathbb{C}^n$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ form a basis for $S^\perp$. Consider the $k \times n$ matrix A whose i-th row is $\mathbf{v}_i^*$. Show that $S = \mathrm{NULL}\, A$.*
Ans. *Note that $\mathbf{v_i}^*\mathbf{w} = 0$ for each $\mathbf{w} \in S$. Hence $S \subseteq \mathrm{NULL}\, A$. Furthermore, $\mathrm{RANK}\, A = k = \mathrm{DIM}\, S^\perp$. Hence $\mathrm{DIM}\, S = n - k = \mathrm{DIM\,NULL}\, A$, so that $S = \mathrm{NULL}\, A$.* $\square$

Result 1.5 (The Gram-Schmidt Process). *Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a basis for a subspace W of $\mathbb{C}^n$ and define*

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1, & W_1 &= span(\mathbf{x}_1);\\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{v}_1.\mathbf{x}_2}{\mathbf{v}_1.\mathbf{v}_1}\right)\mathbf{v}_1, & W_2 &= span(\mathbf{x}_1, \mathbf{x}_2);\\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{v}_1.\mathbf{x}_3}{\mathbf{v}_1.\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2.\mathbf{x}_3}{\mathbf{v}_2.\mathbf{v}_2}\right)\mathbf{v}_2, & W_3 &= span(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3);\\
&\vdots\vdots & &\vdots\vdots\\
&\vdots\vdots & &\vdots\vdots\\
\mathbf{v}_k &= \mathbf{x}_k - \left(\frac{\mathbf{v}_1.\mathbf{x}_k}{\mathbf{v}_1.\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2.\mathbf{x}_k}{\mathbf{v}_2.\mathbf{v}_2}\right)\mathbf{v}_2 - \ldots - \left(\frac{\mathbf{v}_{k-1}.\mathbf{x}_k}{\mathbf{v}_{k-1}.\mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}, & W_k &= span(\mathbf{x}_1, \ldots, \mathbf{x}_k).
\end{aligned}$$

Then for each $i = 1, 2, \ldots, k$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. In particular, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for W.

Example 1.6. *Apply the Gram-Schmidt process to find an **orthonormal** basis of the subspace spanned by $\mathbf{u} = [1, -1, 1]^t$, $\mathbf{v} = [0, 3, -3]^t$ and $\mathbf{w} = [3, 2, 2]^t$.*

- Given a set of vectors S, we can use Gram-Schmidt process to check its linear dependency.
- We can find an orthonormal basis B for span(S).
- The vectors in S corresponding to the elements of B are linearly independent.
- The **angle** θ between $\mathbf{u}$ and $\mathbf{v}$, $(\mathbf{u}, \mathbf{v} \in \mathbb{R}^n)$, is defined by

$$\cos\theta = \frac{\mathbf{u}.\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}, \theta \in [0, \pi].$$

Orthogonal Matrix: An $n \times n$ matrix Q whose columns form an orthonormal set ($i.e.$, $QQ^t = I = Q^tQ$) is called an **orthogonal matrix**.