EE 101
Electrical Sciences

Department of Electronics & Electrical Engineering

EE 101

Circuit Differential Equations
RL RC Circuits

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Operational Impedance

- When a time varying voltage e is applied to a simple resistor R, the current is given by Ohm's law as:

$$i = \frac{e}{R} \qquad (1)$$

- If the voltage is applied to a simple inductor, the relation between e and i is given by Faraday's law as

$$e = L\frac{di}{dt} \Rightarrow e = pLi \quad \text{or} \quad i = \frac{e}{pL} \qquad (2)$$

where, p = the differential operator d / dt

- When voltage e is applied to a simple capacitor the pertinent relationship is

$$C\frac{de}{dt} = i \quad \Rightarrow \quad Cpe = i \Rightarrow \quad i = \frac{e}{1/pC} \qquad (3)$$

- The use of the differential operator permits the differential equation to be written as an algebraic expression, where pL plays a role for inductance that is the same as R for resistance. The inductor is said to posses an *operational impedance* of pL.

Operational Impedance

- From the analogy the capacitor can be said to present to the forcing function ***e*** an operational impedance of *1/pC.*
- The term operational impedance can be applied to single elements or to several elements in series. Consider the network shown in Fig. 1. By KVL, the describing differential equation for the circuit is:

$$e = Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt \qquad (4)$$

In terms of the ***p*** operator the expression becomes

$$e = Ri + pLi + \frac{1}{pC}i$$

$$\Rightarrow \quad e = \left(R + pL + \frac{1}{pC}\right)i = Z(p)i \qquad (5)$$

(5)

where, $Z(p) = R + pL + \dfrac{1}{pC}$

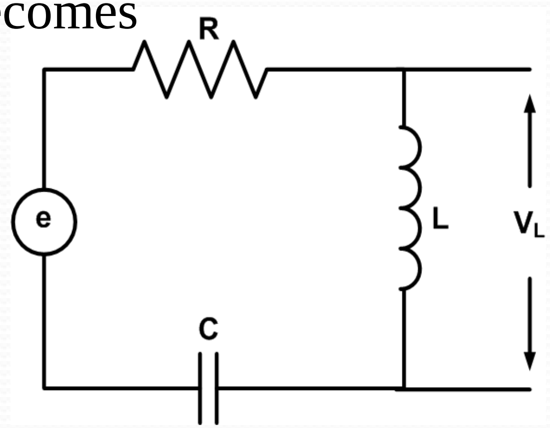
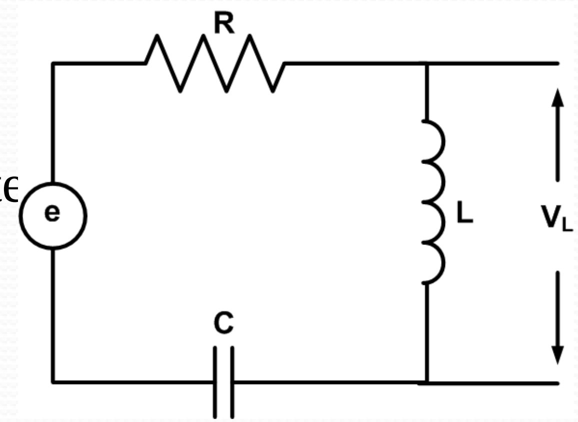


Fig.1: Circuit to illustrate equivalency between p operator algebraic equation and differential equation

Operational Impedance

- There are situations when it is required to relate an electrical quantity in one part of the circuit to an appropriate electrical quantity in another part of the circuit.
- In the network of Fig. 1, it may be required to express the ratio of voltage appearing across the inductor to the source voltage.
- If v_L is used to denote the voltage developed across the inductor, the desired result can then be writte



$$\frac{v_L}{e} = \frac{pL}{R + pL + 1/pC} \qquad (6)$$

- The expression (6) is commonly called the ***transfer function***. The transfer function relates two quantities that are identified at different terminal pairs.
- The transfer functions may be used to relate ratios of voltages, ratios of currents, ratios of current to voltage, ratios of voltage to current provided that the quantities are not identified at the same terminal pairs.

General Formulation of Circuit Differential Equation

- Consider the network shown in Fig. 2, where we like to relate v_o to e_2.
- Writing the node equation at node ***a*** using operational impedances:

$$\frac{v_a}{1+2/p}+\frac{v_a}{2+1+2p}+\frac{v_a-e_2}{1}$$

$$\Rightarrow \quad v_a=\frac{(2+p)(3+2p)}{4p^2+11p+8}e_2$$

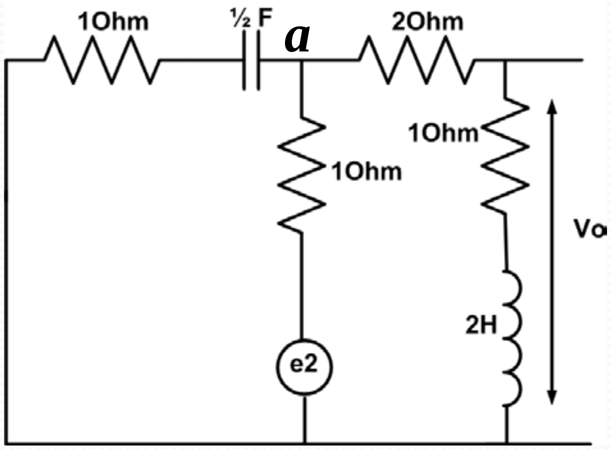


Fig. 2: An RLC network

- Then, $v_0 \quad =\dfrac{1+2p}{3+2p}v_a$

$$=\frac{2p^2+5p+2}{4p^2+11p+8}e_2$$

General Formulation of Circuit Differential Equation

- In the form of differential equation, the voltage v_o is related to e_2 by

$$4\frac{d^2v_0}{dt^2}+11\frac{dv_0}{dt}+8v_0=2\frac{d^2e_2}{dt^2}+5\frac{de_2}{dt}+2e_2 \qquad (7)$$

- Eq. 7 is often referred to as an *equilibrium equation*. It relates a linear combinations of the response function and its derivatives to linear combinations of the source function and its derivatives.

- The operator version of eq. 7 is

$$v_0=\frac{2p^2+5p+2}{4p^2+11p+8}e_2 \qquad (8)$$

is more generally written as

$$v_0=G(p)e_2 \qquad (9)$$

where,

$G(p)=$ operational network function

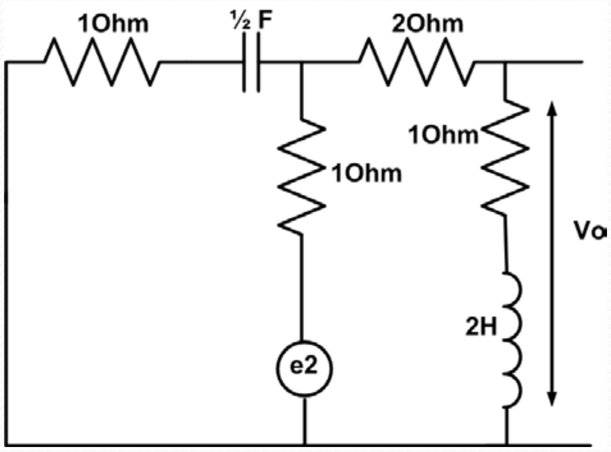


Fig. 2: An RLC network

General Formulation of Circuit Differential Equation

- For any network, the following generalization holds true

$$response\ function = (network\ function) \times `(applied\ source\ function)$$

$$y(t) = G(p)f(t) = \frac{N(p)}{D(p)} f(t) \qquad (10)$$

where

$y(t)$ = response function such as current or voltage in a particular part of the circuit

$G(p)$= total network function

$N(p)$=numerator polynomial of network function G(p)

$D(p)$= denominator polynomial of the network function G(p)

$f(t)$= applied voltage source function

- The eq. 10 is generally written as

$$D(p)y(t) = N(p)f(t) \qquad (11)$$

Example 1

- Write the expression for the operational impedance for each branch of the network shown in Fig.3 and determine the expression for the driving point impedance appearing at the terminal a-d.

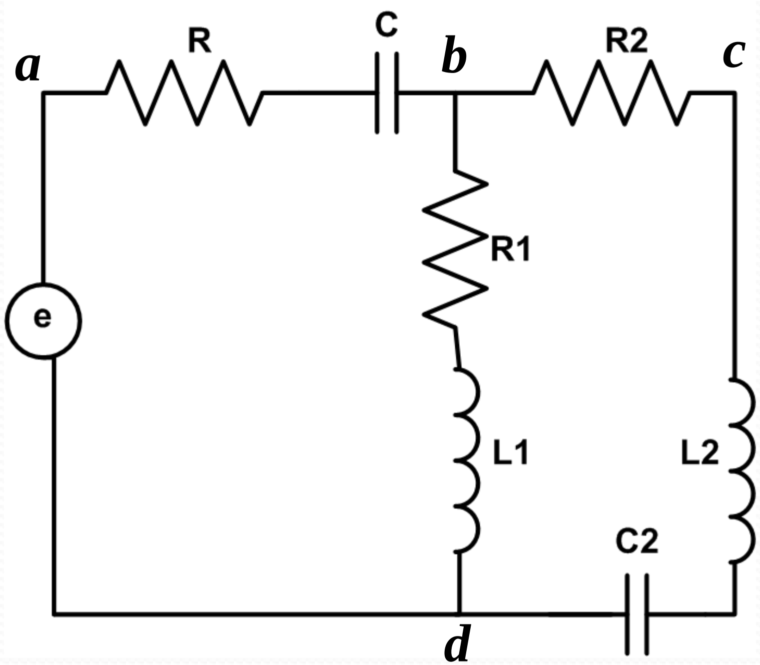


Fig.3: Circuit for the example 1

Solution 1

$$Z_{ab} = R + \frac{1}{pC}$$

$$Z_{bd} = R_1 + pL_1$$

$$Z_{bcd} = R_2 + pL_2 + \frac{1}{pC_2}$$

Driving point impedance

$$Z_{ad} = Z_{ab} + \frac{Z_{bd} Z_{bcd}}{Z_{bd} + Z_{bcd}}$$

The Steady State (Forced or Particular) Solution

- The general response equation of a network is

$$D(p)y(t) = N(p)f(t) \qquad (1.1)$$

- If *f(t)* is an applied source function, the *y(t)* is said to be a solution to the homogeneous linear differential equation.
- If *f(t)* is a sinusoidal function, then *y(t)* must also be sinusoidal. In general *y(t)* will differ from the source function in amplitude and phase and it will continue to exist in the circuit so long as the source function remains applied.
- This is the reason why *f(t)* is referred to as the *forced solution*. It is also referred to as *steady state solution* because it remains in the circuit long after the transient terms disappear.
- Another commonly used description for this part of the solution is the term *particular solution*. This name is due to the fact that the form of *y(t)* is particularized to the nature of the source function. When the source function is an exponential, the particular solution will likewise be exponential.

The response to constant sources

- The response to a constant source is developed for the network shown in Fig. 4. It is required to find the battery current $i_f(t)$ due to the source voltage E.
- The general procedure to obtain the solution is:
  - Obtain the equilibrium equation by applying circuit theory to the configuration of circuit elements described, in terms of their Operational impedances.

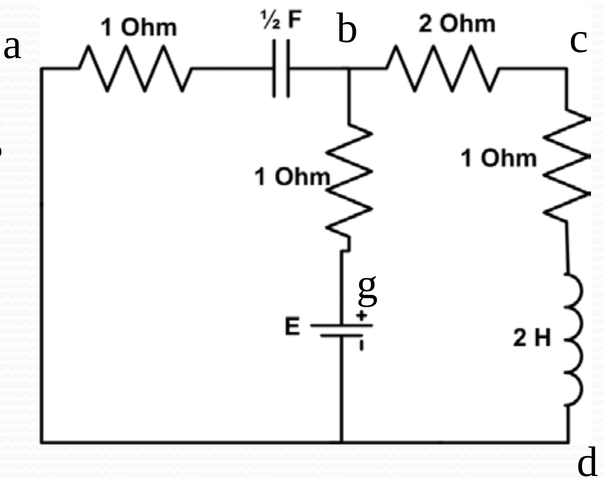


Fig. 4: The network 1

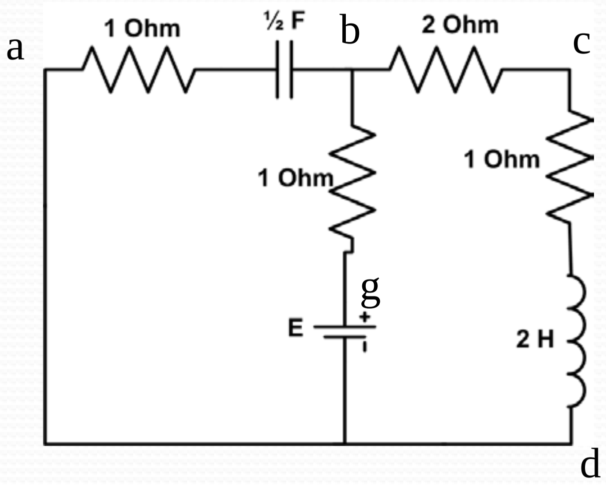
  - Put the resulting expression in the form of eq.11. Then examine the right side of this equation to determine the exact form needed by the forced solution to qualify as a solution to the equilibrium equation.
  - Finally, solve for the unknown quantities.
- To find the expression for the current $i_f(t)$ in the circuit shown in Fig.4, it is required to find the impedance Z_{gd}.

The response to constant sources

- The operation impedance across the terminals g-d is:

$$Z_{gd}(p) = Z_{bg} + \frac{Z_{ab}Z_{bcd}}{Z_{ab} + Z_{bcd}}$$

$$= 1 + \frac{\frac{p+2}{2}(3+2p)}{\frac{p+2}{p} + 3 + 2p} = \frac{4p^2 + 11p + 8}{2p^2 + 4p + 2} \qquad (12)$$



- The current $i_f(t)$ is given by

$$i_f(t) = \frac{E}{Z_{gd}} = \frac{2p^2 + 4p + 2}{4p^2 + 11p + 8}$$

$$\Rightarrow (4p^2 + 11p + 8)i_f(t) = (2p^2 + 4p + 2)E \qquad (13)$$

- The corresponding differential equation is

$$4\frac{d^2 i_f}{dt^2} + 11\frac{di_f}{dt} + 8i_f = 2E \qquad (14)$$

The response to constant sources

- Examination of the equilibrium equation (eq. 13) reveals that because there are no derivative terms on the right side of the equation, the derivative terms of the response function must be identically equal to zero.
- Hence, the solution for $i_f(t)$ is

$$i_f(t) = I_0 \qquad (15)$$

- Substituting the solution in eq. (15) into eq. (13) gives the forced or the steady state solution.

$$I_0 = \frac{E}{4} \qquad (16)$$

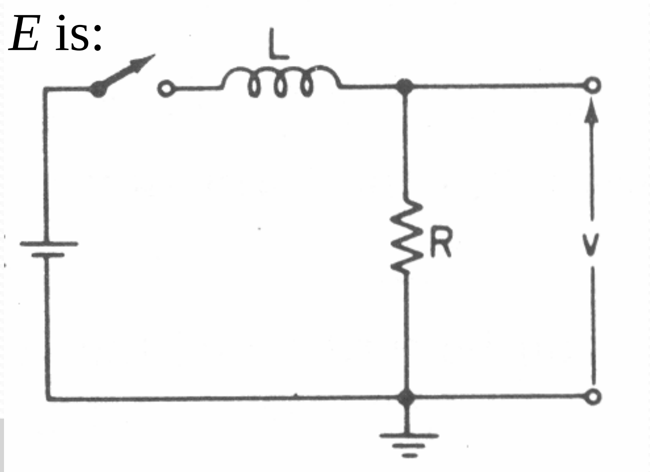
- The examination of the network given in Fig. 4 reveals that this result is entirely expected. The current $i_f(t)$ finds a closed path only in loop *gbcdg*. The left hand part of the loop is open circuit in the steady state due to presence of the capacitor.
- The voltage across the 2-H inductor is zero in steady state. Hence, the current is just a d-c current of magnitude E/4.

The Natural Response (Transient Solution – RL Circuit)

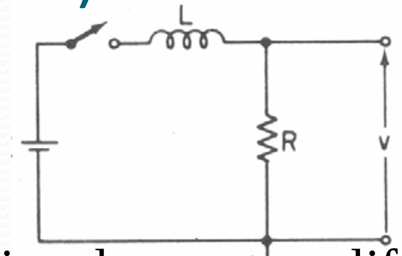
- The forced solution is the solution that is found to exist when the circuit has settled down in its response to the disturbing effect of an applied source function. This forced solution is also known as *steady state solution*.
- The forced or the steady state solution always satisfies the defining differential equation but in general it *is not a valid a solution over the entire time domain*.
- To illustrate the point consider the circuit shown in Fig. 5. In this network it is desired to find the complete solution for the voltage *v* appearing across the resistor for all time *t* after the switch is closed.
- The equation that relates *v* to the source voltage *E* is:

$$\frac{v}{E} = \frac{R}{R + pL} = \frac{1}{1 + p(L/R)} \qquad (17)$$

$$\Rightarrow \frac{L}{R}\frac{dv}{dt} + v = E \qquad (18)$$



The Natural Response (Transient Solution)



- The forced or the steady state solution of eq. 18 is

$$v_f = E$$

- The solution in eq. 19 satisfies eq. 18, however this solution does not qualify as a solution of the governing differential equation in the period immediately following the closing of the switch at time $t=0^+$
- At time $t=0^+$ the current is zero because of the presence of the inductor. Hence *v*, which is *iR*, is also zero. Hence, the solution in eq. 19 cannot be taken to be a complete description at $t=0^+$.
- In essence, then, there arises, in this time period, $t=0^+$, immediately following the application of the source function, a need to add a *complementary function* to the forced solution.
- This complementary function will disappear as steady state is reached. It is the purpose of the *complementary function* to provide a smooth transition from the initial state of the response in the presence of *energy storing* elements to the final state.

The Natural Response (Transient Solution)

- In Fig. 5, the complementary function (natural response) is obtained by solving equation 18

$$\frac{L}{R}\frac{dv}{dt} + v = E \qquad (18)$$

- The solution to the eq. 18 should satisfy that function, i.e, *v* and its derivative, *dv/dt*, must be of the same form to make the left hand side equal to the right hand side.

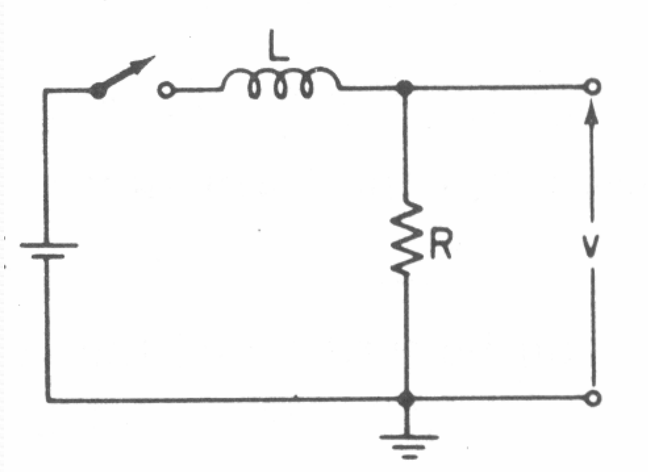
- The plausible solution for eq. 18 is

$$v = Ke^{st} \qquad (19)$$

- Substitution of *v* from eq. 19 into eq. 18 gives

$$\left((L/R)s + 1\right)Ke^{st} = 0 \qquad (20)$$

$$\Rightarrow \quad s = -\left(R/L\right) \qquad (21)$$



The Natural Response (Transient Solution)

- Hence, the complementary solution becomes

$$v = Ke^{-(R/L)t} \qquad (22)$$

- The complete solution of the original governing differential equation (18) is

$$v = E + Ke^{-(R/L)t} \qquad (23)$$

- The quantity K is found from the initial condition which requires that v to be zero at $t=0^+$

$$0 = E + K \Rightarrow K = -E \qquad (24)$$

- Hence, the complete solution is

$$v = E(1 - e^{-(R/L)t}) \qquad (25)$$

- A graph of eq. 25 is shown in Fig. 6. The rapidity with which this transition takes place is entirely dependent upon the value *s*. In eq. 25 when R is large (or L is small) the transition occurs quickly because the transient term dies out in little time.

The Complete (Transient+ Steady State) Solution

- The graph of eq. 25 is shown in Fig. 6. The rapidity with which this transition takes place is entirely dependent upon the value *s*. In eq. 25 when *R* is large (or *L* is small) the transition occurs quickly because the transient term dies out in little time.

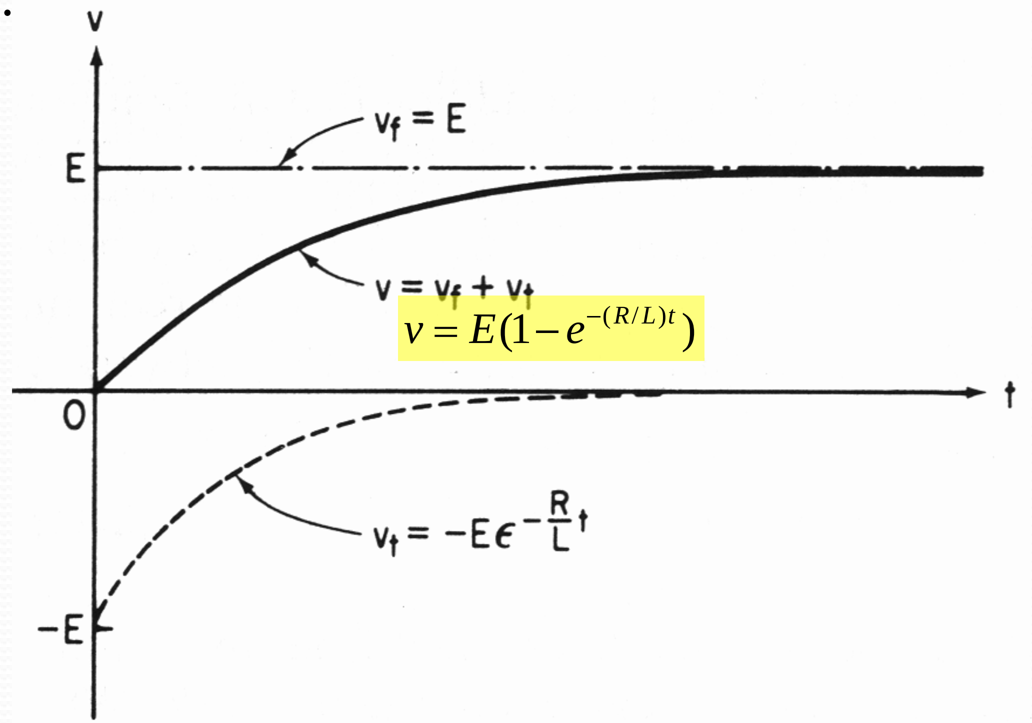


Fig. 6: Complete solution of voltage across the resistor in network 5

Homework: (RC Circuit)
Investigate the voltage across a resistor in an RC circuit after switching a dc source.

The Response to a Sinusoidal Source

- Consider the network shown in Fig.1
- The equilibrium equation for this network is

$$i_f(t) = \frac{1}{Z_{ad}} e_1(t) = \left(\frac{2p^2 + 4p}{4p^2 + 11p + 8} \right) 10\sin(3t) \qquad (1)$$

- To find the solution the Euler's relation can be used. The sinusoidal source in Euler's form can be written as:

$$\begin{aligned} 10\sin(3t) &= \mathrm{Im}(10e^{j3t}) \\ &= \mathrm{Im}(10\cos(3t) + j10\sin(3t)) \qquad (2) \end{aligned}$$

where

Im = imaginary part of the expression

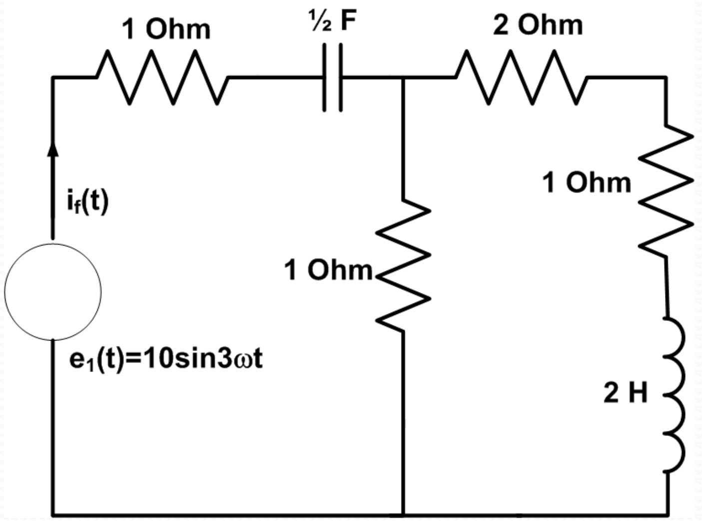


Fig.1: Network with sinusoidal source

The Response to a Sinusoidal Source

- The solution of eq. 1 is obtained as

$$i_f^*(t) = \left|\frac{2p^2+4p}{4p^2+11p+8}\right|_{p=s} 10e^{j3t} = \left|\frac{2s^2+4s}{4s^2+11s+8}\right|_{s=j3} 10e^{j3t} = \left|\frac{2(j3)^2+4(j3)}{4(j3)^2+11(j3)+8}\right| 10e^{j3t}$$

$$= \left|\frac{-18+j12}{-28+j33}\right| 10e^{j3t} = \frac{21.6}{43.3}\frac{\angle 146.3^o}{130.3^o} 10e^{j3t} = 5\angle 16^o e^{j3t} = 5e^{j(3t+16^o)} \qquad (3)$$

- The actual solution corresponding to the applied sinusoidal function is obtained as the imaginary part of eq.3

$$i_f(t) = im(i_f^*(t)) = im(5e^{j(3t+16^o)}) = 5\sin(3t+16^o) \qquad (4)$$

- By expanding the sine term in eq. 4:

$$i_f(t) = 5\sin(3t+16^o)$$

$$= 5\sin(3t)\cos(16^o) + 5\cos(3t)\sin(16^o) = 4.8\sin(3t) + 1.38\cos(3t) \qquad (5)$$

- Solutions for cosine inputs can be similarly obtained.

[Exercise: Input : $e_1(t) = 10\cos(3t)$, Output: $i_f(t) = -1.38\sin(3t) + 4.8\cos(3t)$]

Complete Response of RL Circuit to Sinusoidal Input

- Consider the network shown in Fig. 2. A cosine voltage source is considered.
- When the switch is closed, the governing differential equation for the circuit becomes

$$E_m \cos(\omega t) = Ri + L\frac{di}{dt} \qquad (6)$$

- In the operator form the response is

$$i = \left[\frac{1}{R + pL}\right] E_m \cos(\omega t) \qquad (7)$$

- Using the Euler's relation, it can be written as

$$i_f^* = \left[\frac{1}{R + pL}\right] E_m e^{j\omega t} \qquad (8)$$

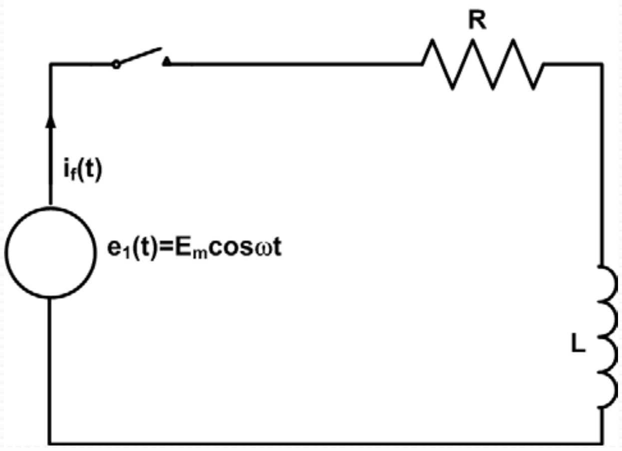


Fig.2: The RL network

- The solution is obtained as

$$i_f^* = \left[\frac{1}{R + pL}\right]_{p=s} E_m e^{j\omega t} = \left[\frac{1}{R + sL}\right]_{s=j\omega} E_m e^{j\omega t} = \left[\frac{1}{R + j\omega L}\right] E_m e^{j\omega t}$$

Complete Response of RL Circuit to Sinusoidal Input

- The eq.8 can be written as

$$i_f^* = \frac{E_m}{R + j\omega L} e^{j\omega t} = \frac{E_m}{Z\angle\theta} e^{j\omega t} = \frac{E_m}{Z} e^{-j\theta} e^{j\omega t} = \frac{E_m}{Z} e^{j(\omega t - \theta)} \qquad (9)$$

where $Z = \sqrt{R^2 + \omega^2 L^2}$, and $\theta = \tan^{-1}\frac{\omega L}{R}$

- Substituting eq. 9 into eq. 8, the forced solution or the steady state solution is obtained as

$$i_f = \mathrm{Re}\left[\frac{E_m}{Z} e^{j(\omega t - \theta)}\right] = \frac{E_m}{Z}\cos(\omega t - \theta) \qquad (10)$$

- The forced solution (eq. 10) has the same form as the forcing function but it differs in amplitude and phase.

Complete Response of RL Circuit to Sinusoidal Input

- To find the transient component of the solution, the characteristic equation is obtained first. The characteristics equation for eq. 19 is

$$R + pL = R + sL = 0 \qquad (11)$$

And the root is: $s = -\frac{R}{L}$

- The transient solution is

$$i_t = Ke^{st} \qquad (12)$$

substituting for s from eq.11:

$$i_t = Ke^{-(R/L)t} \qquad (13)$$

- The complete solution is obtained as

$$i = i_f + i_t = \frac{E_m}{Z}\cos(\omega t - \theta) + Ke^{-(R/L)t} \qquad (14)$$

Complete Response of RL Circuit to Sinusoidal Input

- The presence of the inductor means that $i(0^+)=0$. Inserting this result into eq. 14 at $t=0^+$ gives the value of K as

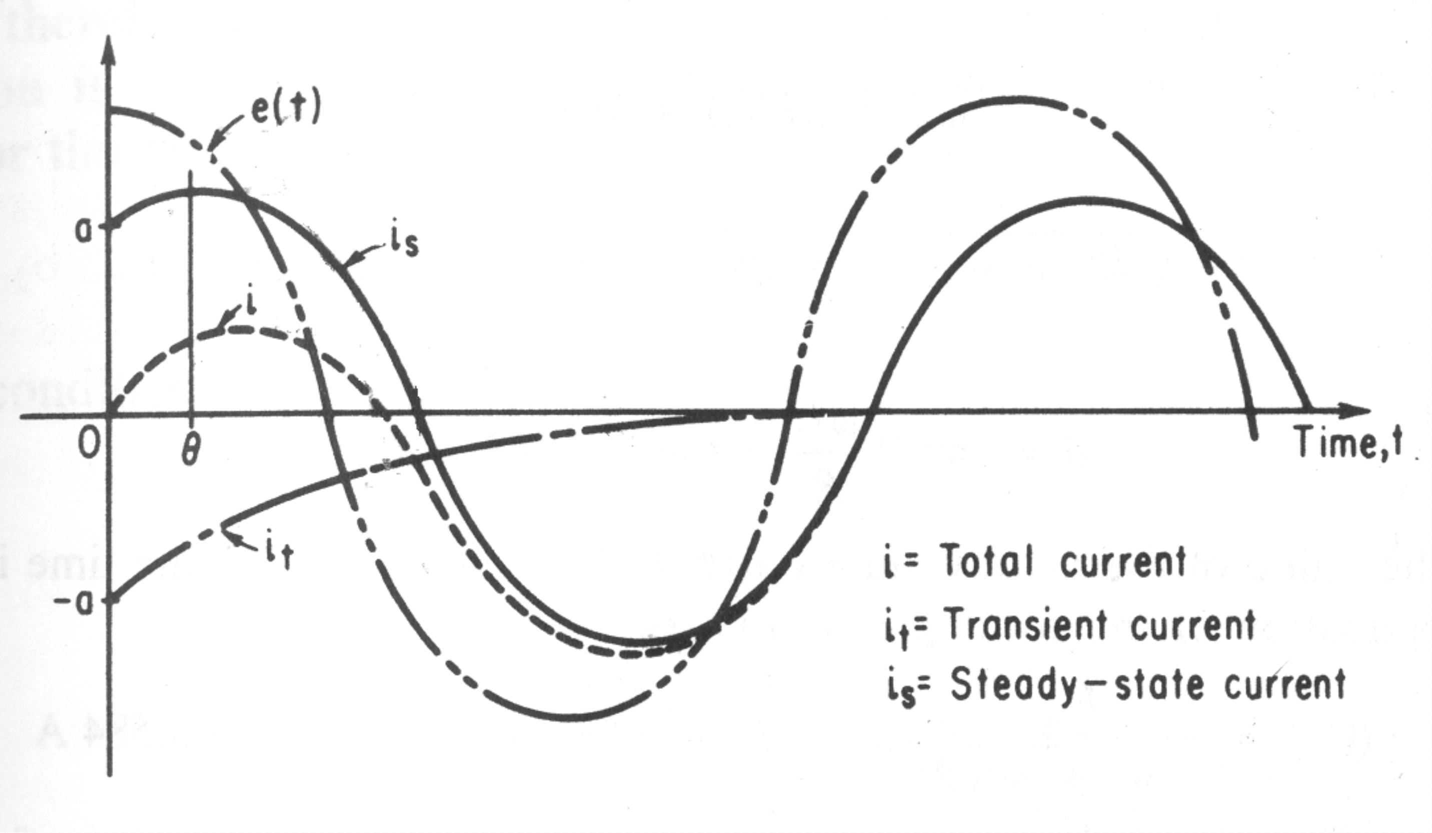
$$K = \frac{E_m}{Z}\cos(-\theta) = -\frac{E_m}{Z}\cos(\theta) \qquad (15)$$

- Thus, the complete response is:

$$i(t) = \frac{E_m}{Z}\left(\cos(\omega t - \theta) - e^{-(R/L)t}\cos(\theta)\right) \qquad (16)$$

- A graphical representation of eq. 15 is shown in Fig. 3.
- In the Fig. 3, it is assumed that the switch is closed at the instant when the voltage is at its positive maximum value.
- It is seen from Fig.3 that the value of the steady state current is not zero at t=0+. Since the initial condition demands that the current at this instant be zero, it is necessary for the transient term to have a value equal an opposite.

Complete Response of RL Circuit to Sinusoidal Input



At the instant of switching, the magnitude of the transient current is equal and opposite to the instantaneous value of the steady state current.

Thank you!