DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
MA101 MATHEMATICS-I
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Solutions to Tutorial Sheet - 4
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Linear transformations,determinants, Cramer's rule, eigenvalues, similarity of matrices, diagonalization.

Recall:

- Let A be an $m \times n$ matrix. The null space of A is the subspace $null(A)$ of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $A\mathbf{X} = \mathbf{0}$.
- A transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if
  1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$.
  2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^n$ and all scalars c.
- The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as
  1. $\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij}$ **(cofactor expansion along the i^{th} row)**
  2. $\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij}$ **(cofactor expansion along the j^{th} column)**

  where
  $C_{ij} = (-1)^{i+j}\det(A_{ij})$,
  A_{ij} is the sub-matrix of a matrix A obtained by deleting row i and column j.
- The determinant of a triangular matrix is the product of the entries on its main diagonal.
- If A is an $n \times n$ matrix, then $\det(kA) = k^n \det(A)$
- If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$
- For any square matrix $adj(A)\, A = A\, adj(A) = \det(A).I$
- Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a non-zero vector x such that $Ax = \lambda x$. Such a vector x is called an eigenvector of A corresponding to λ.
- Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the eigenspace of λ and is denoted by E_λ.
- Algebraic multiplicity of an eigenvalue is the multiplicity of root of the characteristic equation.
- Geometric multiplicity of an eigenvalue λ is the dimension of its corresponding eigenspace.
- Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$.

- An $n \times n$ matrix A is diagonalizable if there is a diagonal matrix D such that A is similar to D.

Theoretical

1. If a matrix A is idempotent, i.e. if $A^2 = A$, then find all possible value of $\det(A)$.
   Soln.
   $$\det(A^2) = \det(A) \Rightarrow (\det(A))^2 = \det(A) \Rightarrow \det(A) = 0 \text{ or } 1$$

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2. If a matrix A is nilpotent, i.e. if $A^n = \mathbf{0}$ for some $n \in \mathbb{N}$, then find all possible eigenvalue of A.
   Soln. Let λ is the eigenvalue of A and x is the eigenvector corresponding to the eigenvalue λ then
   $$Ax = \lambda x \Rightarrow A^n x = \lambda^n x$$
   Since A is nilpotent hence
   $$\lambda^n x = 0 \Rightarrow \lambda^n = 0 \Rightarrow \lambda = 0$$

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3. For an $n \times n$ matrix A, show that
   $$\det(adj(A)) = \det(A)^{n-1}$$
   Solution:
   Case I $\det(A) \neq 0$
   We know that
   $$A\, adj(A) = adj(A)\ A = \det(A)\ I$$
   which implies
   $$\begin{aligned} & \det(A\, adj(A)) = \det(\det(A)\ I) \\ \Rightarrow\ & \det(A)\ \det(adj(A)) = (\det(A))^n \det(I) \\ \Rightarrow\ & \det(adj(A)) = \det(A)^{n-1}. \end{aligned}$$
   Case II $\det(A) = 0$
   When $\det(A) = 0$ we need to show that $\det(adj(A)) = 0$
   If possible let $\det(adj(A)) \neq 0$ which implies that $adj(A)$ is invertible.
   $$\begin{aligned} & A\, adj(A) = \det(A)\ I \\ \Rightarrow\ & A\, adj(A)(adj(A))^{-1} = \det(A)\ adj(A)^{-1} \\ \Rightarrow\ & A = \det(A)\ adj(A)^{-1} \\ \Rightarrow\ & A = 0 \\ \Rightarrow\ & adj(A) = 0 \\ \Rightarrow\ & \det((adj(A)) = 0. \end{aligned}$$
   Contradiction.

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4. Let A be a square matrix such that A can be partitioned as $A = \begin{bmatrix} P & | & Q \\ R & | & S \end{bmatrix}$, where P, Q, R and S are square matrices. Then is the following statement true:

$$\det(A) = \det(P)\det(S) - \det(Q)\det(R)$$

Justify your argument
Soln: **FALSE**

Let $P = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ 0 & 5 \end{bmatrix}$, then $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 5 \end{bmatrix}$

$$\det(P)\det(S) - \det(Q)\det(R) = 2 * 5 - 3 * 4 = -2 \neq 0 = \det(A)$$

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5. Prove that the range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is equal the column space of its standard matrix $[T]$.
Soln: Since Range of T consists of all vectors $y \in \mathbb{R}^m$ which are expressible in the form $[T]x$ where $x = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$.

$$x_1c_1 + x_2c_2 + \cdots + x_nc_n = y$$

where $c_1, c_2, \ldots, c_n$ are column vectors of matrix $[T]$. Thus vector y in $\mathbb{R}^m$ belongs to range(T) iff it is a linear combination of the column vector of $[T]$. Therefore range(T) is spanned by these column vectors.

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Alter: The standard matrix of $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$[T] = [Te_1 \quad Te_2 \quad \cdots \quad Te_n],$$

where $\{e_1, \cdots, e_n\}$ represents the standard basis for $\mathbb{R}^n$.
Observe that $y \in \mathit{Range}(T)$ if and only if $\exists x = [x_1 \quad \cdots \quad x_n]^T \in \mathbb{R}^n$ such that

$$y = Tx = T(x_1e_1 + \cdots + x_ne_n) = x_1T(e_1) + \cdots + x_nT(e_n) \in \mathit{Col}([T])$$

This completes the proof. $\square$

Eigenvalue

6. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. then show that the eigenvalue of A are the solution of the equation $\lambda^2 - tr(A)\lambda + \det(A) = 0$ where $tr(A)$ is the sum the entries on the main diagonal of A.
Express the trace and determinant of A in terms of eigenvalues of A.

Can you generalize it for an $n \times n$ matrix ?

Soln: The characteristic polynomial corresponding to A is given by

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} &= 0 \\ (a-\lambda)(d-\lambda) - cb &= 0 \\ \lambda^2 - (a+d)\lambda + (ad - bc) &= 0 \\ \lambda^2 - tr(A)\lambda + \det(A) &= 0. \end{aligned} \tag{1}$$

Let λ_1 and λ_2 are the eigenvalue of matrix A. then characteristic polynomial will be

$$\begin{aligned} & (\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \\ \Rightarrow \quad & \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0 \end{aligned} \tag{2}$$

from (1) and (2) $tr(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$.

In general, suppose $\lambda_1, \cdots, \lambda_n$ are eigenvalues of $A_{n\times n}$, then

$$tr(A) = \sum_{i=1}^{n} \lambda_i, \qquad \det(A) = \prod_{i=1}^{n} \lambda_i.$$

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7. For each of the following matrix, compute the characteristic polynomial, eigenvalue, basis for the eigenspace corresponding to each eigenvalue , algebraic and geometric multiplicity.

(a) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 2 \\ 3 & 0 & 1 \end{bmatrix}$

Soln: (a) Characteristic polynomial $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$(\lambda - 2)((1-\lambda)^2 - 1) = 0$$

$$(\lambda)(2-\lambda)^2 = 0 \Rightarrow \lambda = 0, 2$$

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 0$. Thus, the eigenvalue 2 has algebraic multiplicity 2 and the eigenvalue 0 has algebraic multiplicity 1.

For $\lambda_1 = \lambda_2 = 2$, we compute

$$[A - 2I|0] = \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

from which it follows that

$$E_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis of eigenspace corresponding to eigenvalue 2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Geometric multiplicity of eigenvalue 2 is 2.
For $\lambda_3 = 0$, we compute

$$[A - 0I|0] = [A|0] = \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 2 & 0 & | & 0 \\ -1 & 1 & 1 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

from which it follows that

$$E_0 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} \right\} = \left\{ x3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis of eigenspace corresponding to eigenvalue 0 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Geometric multiplicity of eigenvalue 0 is 1.

(b) Characteristic polynomial $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 3-\lambda & 1 & 0 & 0 \\ -1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 4 \\ 0 & 0 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$((3-\lambda)(1-\lambda)+1)((1-\lambda)^2 - 4) = 0$$

$$(\lambda - 3)(\lambda + 1)(\lambda - 2)^2 = 0 \Rightarrow \lambda = 3, -1, 2$$

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = -1$ and $\lambda_4 = 3$. Thus, the eigenvalue 2 has algebraic multiplicity 2, eigenvalue -1 has algebraic multiplicity 1 and the eigenvalue 3 has algebraic multiplicity 1.

For $\lambda_1 = \lambda_2 = 2$, we compute

$$[A - 2I|0] = \begin{bmatrix} 1 & 1 & 0 & 0 & | & 0 \\ -1 & -1 & 0 & 0 & | & 0 \\ 0 & 0 & -1 & 4 & | & 0 \\ 0 & 0 & 1 & -1 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & -4 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

from which it follows that

$$E_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} = span \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Basis of eigenspace corresponding to eigenvalue 2 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Geometric multiplicity of eigenvalue 2 is 1.

For $\lambda_3 = -1$, we compute

$$[A + I|0] = \begin{bmatrix} 4 & 1 & 0 & 0 & | & 0 \\ -1 & 2 & 0 & 0 & | & 0 \\ 0 & 0 & 2 & 4 & | & 0 \\ 0 & 0 & 1 & 2 & | & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 4 & 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

from which it follows that

$$E_{-1} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ -2x_4 \\ x_4 \end{bmatrix} \right\} = \left\{ x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} = span \left\{ \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \end{bmatrix} \right\}$$

Basis of eigenspace corresponding to eigenvalue -1 is $\left\{ \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

Geometric multiplicity of eigenvalue -1 is 1.

For $\lambda_4 = 3$, we compute

$$[A - 3I|0] = \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 \\ -1 & -2 & 0 & 0 & | & 0 \\ 0 & 0 & -2 & 4 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

from which it follows that

$$E_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 2x_4 \\ x_4 \end{bmatrix} \right\} = \left\{ x_4 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} = span \left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Basis of eigenspace corresponding to eigenvalue 3 is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

Geometric multiplicity of eigenvalue 3 is 1.

(c) Characteristic polynomial $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 1-\lambda & 0 & 3 \\ 2 & -2-\lambda & 2 \\ 3 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$(\lambda + 2)((1-\lambda)^2 - 9) = 0$$

$$(\lambda - 4)(\lambda + 2)^2 = 0 \Rightarrow \lambda = 4, -2$$

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = -2$ and $\lambda_3 = 4$. Thus, the eigenvalue -2 has algebraic multiplicity 2 and the eigenvalue 4 has algebraic multiplicity 1.
For $\lambda_1 = \lambda_2 = -2$, we compute

$$[A + 2I|0] = \begin{bmatrix} 3 & 0 & 3 & | & 0 \\ 2 & 0 & 2 & | & 0 \\ 3 & 0 & 3 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

from which it follows that

$$E_{-2} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = span \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis of eigenspace corresponding to eigenvalue -2 is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Geometric multiplicity of eigenvalue -2 is 2.
For $\lambda_3 = 4$, we compute

$$[A - 4I|0] = \begin{bmatrix} -3 & 0 & 3 & | & 0 \\ 2 & -6 & 2 & | & 0 \\ 3 & 0 & -3 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & -3 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

from which it follows that

$$E_4 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x_3 \\ \frac{2}{3}x_3 \\ x_3 \end{bmatrix} \right\} = \left\{ \frac{x_3}{3} \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\} = span \left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Basis of eigenspace corresponding to eigenvalue 4 is $\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$

Geometric multiplicity of eigenvalue 4 is 1.

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8. Let A, B be square matrices. Then prove or disprove(using counter example) the following statements:

   (a) If λ is an eigenvalue of A and μ is the eigenvalue of B, then $\lambda + \mu$ is an eigenvalue of $A + B$.

(b) If λ is an eigenvalue of A and μ is the eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB.

(c) If $v \in \mathbb{R}^n$ is such that $Av = \lambda v$ and $Bv = \mu v$, then $\lambda + \mu$ is an eigenvalue of $A + B$ and $\lambda\mu$ is an eigenvalue of AB.

Soln: (a) **FALSE** $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ then $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2 is the eigenvalue of A and -1 is the eigenvalue of B but $2 + (-1) = 1$ is not eigenvalue of $A + B$.

(b) **FALSE** $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

1 is the eigenvalue of A and 2 is the eigenvalue of B but $1 * 2 = 2$ is not eigenvalue of AB.

(c) **TRUE**

$$(A + B)v = Av + Bv = \lambda v + \mu v = (\lambda + \mu)v$$

hence $\lambda + \mu$ is the eigenvalue of $A + B$.

$$(AB)v = A(Bv) = A(\mu v) = \mu(Av) = \mu(\lambda v) = (\lambda\mu)v$$

$\lambda\mu$ is an eigenvalue of AB.

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9. If $A \sim B$ then show that $A^T \sim B^T$.
Soln: Since $A \sim B$ so there exists an invertible matrix P such that $P^{-1}AP = B$.

$$(P^{-1}AP)^T = B^T \Rightarrow P^T A^T (P^{-1})^T = B^T \Rightarrow P^T A^T (P^T)^{-1} = B^T \Rightarrow A^T \sim B^T$$

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10. In the following, check whether the matrices A aand B are similar. If yes, find the matrix P such that $B = P^{-1}AP$.

(a) $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$ and B is a diagonal matrix.

Soln: (a) If possible A and B are similar then there exits invertible 2×2 matrix P such that $B = P^{-1}AP$ since $B = I_2$ then $A = PBP^{-1} = PI_2P^{-1} = I_2 \neq A$
hence A and B are not similar.
(b)Eigenvalue of A are 1 and 3 on the other hand eigenvalue of B are -1 and 3. Since eigenvalue of A and B are not same hence A and B are not similar.
(c) characteristic polynomial $det(A - \lambda I) = 0$

$$\lambda^2(2 + \lambda) = 0 \Rightarrow \lambda = 0, 0, -2$$

Eigenvector corresponding to eigenvalue $\lambda = 0$
$(A - 0I)x = Ax = 0.$

Consider the augmented matrix $\begin{bmatrix} -1 & 0 & 1 & | & 0 \\ 3 & 0 & -3 & | & 0 \\ 1 & 0 & -1 & | & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$

$$x = \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

eigenvector corresponding to eigenvalue $\lambda = 0$ are $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Eigenvector corresponding to eigenvalue $\lambda = -2$
$(A + 2I)x = 0.$

Consider the augmented matrix $\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 3 & 2 & -3 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

eigenvector corresponding to eigenvalue $\lambda = -2$ is $v_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

Let

$$P = [v_1 \; v_2 \; v_3] = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = B$$

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11. If A, B are similar matrices, then show that the geometric multiplicities of eigenvalues of A and B are same.
Soln : Since $A \sim B$ so there exists an invertible matrix P such that $P^{-1}AP = B$.

$$P^{-1}(A - \lambda I)P = P^{-1}AP - \lambda I = B - \lambda I$$

$$Rank(A - \lambda I) = Rank(B - \lambda I)$$

$$Nullity(A - \lambda I) = Nullity(B - \lambda I)$$

hence geometric multiplicities of of eigenvalues of A and B are same.

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12. If A is an $n \times n$ diagonalizable matrix whose eigenvalue are 0 & 1, then for each $k \in \mathbb{N}$, compute A^k.
Soln: Since A is diagonalizable so there is a invertible matrix P such that

$$P^{-1}AP = D$$

where D is the diagonal matrix whose diagonal entries are 0 or 1.

$$A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1} = PDP^{-1} = A$$