Indian Institute of Technology Guwahati
Department of Mathematics

MA 101 – Mathematics-I **Date:** 17-Aug-2015
Tutorial Sheet-3 **Time:** 08:00 – 09:00

Linear Algebra

Topics Covered:
Linear independence, subspace, row/column space, null space, basis, dimension, linear transformations.

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1. State TRUE or FALSE. Give a brief justification.

   (a) For a matrix A in its row echelon form, the non-zero rows are linearly independent.

   (b) If v_1 and v_2 are linearly independent vectors, and if $\{v_1, v_2, v_3\}$ is a linearly dependent set, then $v_3 \in Span\{v_1, v_2\}$.

   (c) The vectors u, v and w are in $Span(u, u+v, u+v+w)$.

   (d) If all the columns of an $m \times n$ nonzero matrix (it has at least one nonzero entry) A are equal then $rank(A) = 1$.

   (e) If A and B are square matrices such that AB is invertible then both A and B are invertible.

   (f) If the equation $AX = b$ has at least one solution for each $b \in \mathbb{R}^n$, then the solution is unique for each b.

   (g) Let A be an invertible matrix. If the vectors $\{x_1, x_2, \ldots, x_r\}$ are linearly independent then the vectors $\{Ax_1, Ax_2, \ldots, Ax_r\}$ are linearly independent.

   (h) Let $\{v_1, \ldots, v_n\}$ be a linearly independent set. Suppose there exists scalars α_i and β_i such that
$$\sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i.$$
   Then for each i, $\alpha_i = \beta_i$.

2. Examine whether the following sets are subspaces of $\mathbb{R}^n$.

   (a) For $n \geq 3$, $S_1 = \{\ [x_1, \ldots, x_n]^T \in \mathbb{R}^n \ :\ x_1 + x_2 = 4x_3\}$

   (b) For $n \geq 3$, $S_1 = \{\ [x_1, \ldots, x_n]^T \in \mathbb{R}^n \ :\ x_1 + x_2 \leq 4x_3\}$

   (c) A line given by equation $y = mx + c$ in $\mathbb{R}^2$.

   (d) For a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, the range of T.

3. Show that the matrix
$$A = \begin{bmatrix} 2 & 5 & 2 & 2 & 7 \\ 0 & 3 & 5 & 0 & 8 \\ 6 & 2 & 7 & 9 & 4 \\ 0 & 2 & 5 & 2 & 2 \\ 4 & 7 & 5 & 7 & 1 \end{bmatrix}$$
   is equivalent to another matrix B whose last row is $\begin{bmatrix} 20604 & 53227 & 25755 & 20927 & 78421 \end{bmatrix}$.

4. Compute the null space of the matrix
$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$
   What is $\dim(null(A))$?

5. Under what conditions on the scalars α such that the vectors $[1+\alpha \quad 1-\alpha]^T$ and $[1-\alpha \quad 1+\alpha]^T$ in $\mathbb{R}^2$ are linearly independent.

6. Let A be a 3×3 matrix such that $rank(A) = 2$ and the columns of A satisfy the relation $C_3 = C_1 + C_2$. Then show that there exists a matrix X, not equal to identity matrix, such that $AX = A$.

7. Let A be an $n \times m$ matrix and let B be an $m \times n$ matrix. Prove that the matrix $I_m - BA$ is invertible if and only if the matrix $I_n - AB$ is invertible.

8. Show that if A is a $m \times n$ and B is an $n \times p$ matrix then:

   (a) $col(AB) \subseteq col(A)$.

   (b) $row(AB) \subseteq row(B)$. If $m = n$ and A is invertible, what can you say in addition?

9. Let A be an $m \times n$ matrix with entries in $\mathbb{R}$. Let T be the corresponding linear transformation. Then find out the domain and codomain of T.

10. Show that in $\mathbb{R}^2$, the rotation by 90° is a linear transformation.

11. Examine whether the following maps $T : V \to W$ are linear transformations.

    (a) $V = W = \mathbb{R}^3$. $T\left(\begin{bmatrix} x & y & z \end{bmatrix}^T\right) = \begin{bmatrix} 3x + y & z & |x| \end{bmatrix}^T$

    (b) $V = W = \mathbb{R}^2$. T is the reflection in the line $y = -x$.

    (c) $V = \mathbb{R}^2, W = \mathbb{R}^3$. $T\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) = \begin{bmatrix} x - y + 5 & z^2 & xyz \end{bmatrix}^T$.

    (d) $V = \mathbb{R}^3, W = \mathbb{R}^3$. $T\left(\begin{bmatrix} x & y & z \end{bmatrix}^T\right) = \begin{bmatrix} x - y + z & 2z - 3y + x \end{bmatrix}^T$.

    (e) $V = W = \mathbb{R}^2$. T is the projection onto Y-axis.

12. In the previous exercise, if the map T is a linear transformation, then compute its *standard matrix*.