Friday Time Table will be followed

on

6th August (Wednesday)

Plan

- Algebra of Vectors (in $\mathbb{R}^n$)
- Subspace of $\mathbb{R}^n$
- Linear Dependence and Linear Independence
- Basis and Dimension
- Matrices
- The Inverse of a Matrix
- Elementary Matrix
- Gauss-Jordan Method for Computing Inverse

The Euclidean Space $\mathbb{R}^n$ and its Subspaces

The Euclidean Space $\mathbb{R}^n$ and its Subspaces

Definition: Let $n \in \mathbb{N}$. The space $\mathbb{R}^n$, as defined below, is called the n-dimensional Euclidean space.

$$\mathbb{R}^n = \left\{ [x_1, x_2, \ldots, x_n]^t : x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

The Euclidean Space $\mathbb{R}^n$ and its Subspaces

Definition: Let $n \in \mathbb{N}$. The space $\mathbb{R}^n$, as defined below, is called the n-dimensional Euclidean space.

$$\mathbb{R}^n = \left\{ [x_1, x_2, \ldots, x_n]^t : x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

- Elements of $\mathbb{R}^n$ are called n-vectors or simply vectors.

The Euclidean Space $\mathbb{R}^n$ and its Subspaces

Definition: Let $n \in \mathbb{N}$. The space $\mathbb{R}^n$, as defined below, is called the n-dimensional Euclidean space.

$$\mathbb{R}^n = \left\{ [x_1, x_2, \ldots, x_n]^t : x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}.$$

- Elements of $\mathbb{R}^n$ are called n-vectors or simply vectors.
- Note that $[x_1, x_2, \ldots, x_n]^t = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector.

- Sometimes, an element $[x_1, x_2, \ldots, x_n]^t$ of $\mathbb{R}^n$ is also written as a row vector $[x_1, x_2, \ldots, x_n]$ or $(x_1, x_2, \ldots, x_n)$.

- Sometimes, an element $[x_1, x_2, \ldots, x_n]^t$ of $\mathbb{R}^n$ is also written as a row vector $[x_1, x_2, \ldots, x_n]$ or $(x_1, x_2, \ldots, x_n)$.
- The element $(x_1, x_2, \ldots, x_n)$ is also termed as an *n*-tuple.

- Sometimes, an element $[x_1, x_2, \ldots, x_n]^t$ of $\mathbb{R}^n$ is also written as a row vector $[x_1, x_2, \ldots, x_n]$ or $(x_1, x_2, \ldots, x_n)$.
- The element $(x_1, x_2, \ldots, x_n)$ is also termed as an *n*-tuple.
- The vector $[0, 0, \ldots, 0]^t$ of $\mathbb{R}^n$, called the zero vector, is denoted by the symbol $\mathbf{0}$.

- Sometimes, an element $[x_1, x_2, \ldots, x_n]^t$ of $\mathbb{R}^n$ is also written as a row vector $[x_1, x_2, \ldots, x_n]$ or $(x_1, x_2, \ldots, x_n)$.
- The element $(x_1, x_2, \ldots, x_n)$ is also termed as an *n*-tuple.
- The vector $[0, 0, \ldots, 0]^t$ of $\mathbb{R}^n$, called the zero vector, is denoted by the symbol $\mathbf{0}$.
- If A is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$, then a solution of $A\mathbf{x} = \mathbf{b}$, if any, is an element of $\mathbb{R}^n$.

- Sometimes, an element $[x_1, x_2, \ldots, x_n]^t$ of $\mathbb{R}^n$ is also written as a row vector $[x_1, x_2, \ldots, x_n]$ or $(x_1, x_2, \ldots, x_n)$.
- The element $(x_1, x_2, \ldots, x_n)$ is also termed as an n-tuple.
- The vector $[0, 0, \ldots, 0]^t$ of $\mathbb{R}^n$, called the zero vector, is denoted by the symbol $\mathbf{0}$.
- If A is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$, then a solution of $A\mathbf{x} = \mathbf{b}$, if any, is an element of $\mathbb{R}^n$.
- Normally, while discussing a system of linear equations, elements of $\mathbb{R}^n$ are regarded a column vectors.
- Otherwise, elements of $\mathbb{R}^n$ may be regarded as row vectors.

Definition

Let $\mathbf{u} = [u_1,\ u_2,\ \ldots,\ u_n]^t, \mathbf{v} = [v_1,\ v_2, \ldots,\ v_n]^t \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We define

Definition

Let $\mathbf{u} = [u_1, \ u_2, \ \ldots, \ u_n]^t, \mathbf{v} = [v_1, \ v_2, \ldots, \ v_n]^t \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We define

1. $\mathbf{u} + \mathbf{v} = [u_1 + v_1, \ u_2 + v_2, \ \ldots, \ u_n + v_n]^t$;

Definition

Let $\mathbf{u} = [u_1,\ u_2,\ \dots,\ u_n]^t, \mathbf{v} = [v_1,\ v_2, \dots,\ v_n]^t \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We define

1. $\mathbf{u} + \mathbf{v} = [u_1 + v_1,\ u_2 + v_2,\ \dots,\ u_n + v_n]^t$;
2. $c\mathbf{u} = [cu_1,\ cu_2,\ \dots,\ cu_n]^t$;

Definition

Let $\mathbf{u} = [u_1,\ u_2,\ \ldots,\ u_n]^t, \mathbf{v} = [v_1,\ v_2, \ldots,\ v_n]^t \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We define

1. $\mathbf{u} + \mathbf{v} = [u_1 + v_1,\ u_2 + v_2,\ \ldots,\ u_n + v_n]^t$;
2. $c\mathbf{u} = [cu_1,\ cu_2,\ \ldots,\ cu_n]^t$;
3. $-\mathbf{u} = (-1)\mathbf{u} = [-u_1,\ -u_2,\ \ldots,\ -u_n]^t$;

Definition

Let $\mathbf{u} = [u_1,\ u_2,\ \ldots,\ u_n]^t, \mathbf{v} = [v_1,\ v_2, \ldots,\ v_n]^t \in \mathbb{R}^n$ and $c \in \mathbb{R}$.
We define

1. $\mathbf{u} + \mathbf{v} = [u_1 + v_1,\ u_2 + v_2,\ \ldots,\ u_n + v_n]^t$;
2. $c\mathbf{u} = [cu_1,\ cu_2,\ \ldots,\ cu_n]^t$;
3. $-\mathbf{u} = (-1)\mathbf{u} = [-u_1,\ -u_2,\ \ldots,\ -u_n]^t$;
4. $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v} = [u_1 - v_1,\ u_2 - v_2,\ \ldots\ ,\ u_n - v_n]^t$.

Definition

Let $\mathbf{u} = [u_1,\ u_2,\ \ldots,\ u_n]^t, \mathbf{v} = [v_1,\ v_2, \ldots,\ v_n]^t \in \mathbb{R}^n$ and $c \in \mathbb{R}$.
We define

1. $\mathbf{u} + \mathbf{v} = [u_1 + v_1,\ u_2 + v_2,\ \ldots,\ u_n + v_n]^t$;
2. $c\mathbf{u} = [cu_1,\ cu_2,\ \ldots,\ cu_n]^t$;
3. $-\mathbf{u} = (-1)\mathbf{u} = [-u_1,\ -u_2,\ \ldots,\ -u_n]^t$;
4. $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v} = [u_1 - v_1,\ u_2 - v_2,\ \ldots\ ,\ u_n - v_n]^t$.

- The vector $\mathbf{u} + \mathbf{v}$ is called the vector addition of $\mathbf{u}$ and $\mathbf{v}$.

Definition

Let $\mathbf{u} = [u_1,\ u_2,\ \ldots,\ u_n]^t, \mathbf{v} = [v_1,\ v_2, \ldots,\ v_n]^t \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We define

1. $\mathbf{u} + \mathbf{v} = [u_1 + v_1,\ u_2 + v_2,\ \ldots,\ u_n + v_n]^t$;
2. $c\mathbf{u} = [cu_1,\ cu_2,\ \ldots,\ cu_n]^t$;
3. $-\mathbf{u} = (-1)\mathbf{u} = [-u_1,\ -u_2,\ \ldots,\ -u_n]^t$;
4. $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v} = [u_1 - v_1,\ u_2 - v_2,\ \ldots\ ,\ u_n - v_n]^t$.

- The vector $\mathbf{u} + \mathbf{v}$ is called the vector addition of $\mathbf{u}$ and $\mathbf{v}$.
- The vector $c\mathbf{u}$ us called the scalar multiplication of c and $\mathbf{u}$.

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition);

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition);
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity over scalar addition);

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition);
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity over scalar addition);
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$;

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition);
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity over scalar addition);
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
- $0\mathbf{u} = \mathbf{0}$;

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition);
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity over scalar addition);
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
- $0\mathbf{u} = \mathbf{0}$;
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition);
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity over scalar addition);
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
- $0\mathbf{u} = \mathbf{0}$;
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
- $1\mathbf{u} = \mathbf{u}$;

Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition);
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity over scalar addition);
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
- $0\mathbf{u} = \mathbf{0}$;
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
- $1\mathbf{u} = \mathbf{u}$;
- $c(d\mathbf{u}) = (cd)\mathbf{u}$.

An example revisited:

Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix},$$

An example revisited:

Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

An example revisited:

Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions set for $A\mathbf{x} = \mathbf{0}$ is

$$S_h = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

An example revisited:

Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions set for $A\mathbf{x} = \mathbf{0}$ is

$$S_h = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} : \; s, t \in \mathbb{R} \right\}.$$

Can we describe S_h with a few of the solutions?

An example revisited:

Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions set for $A\mathbf{x} = \mathbf{0}$ is

$$S_h = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} : \ s, t \in \mathbb{R} \right\}.$$

Can we describe S_h with a few of the solutions? How?

An example revisited:

Consider the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions set for $A\mathbf{x} = \mathbf{0}$ is

$$S_h = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

Can we describe S_h with a few of the solutions? How? Can we derive some special properties of solution sets like S_h?

Linear Combination:

A vector $\mathbf{v}$ in $\mathbb{R}^n$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ if there exist real numbers $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k.$$

Linear Combination:

A vector $\mathbf{v}$ in $\mathbb{R}^n$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ if there exist real numbers $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k.$$

- The numbers $c_1, c_2, \ldots, c_k$ are called the coefficients of the linear combination.

Linear Combination:

A vector $\mathbf{v}$ in $\mathbb{R}^n$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ if there exist real numbers $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k.$$

- The numbers $c_1, c_2, \ldots, c_k$ are called the coefficients of the linear combination.

Example

Is the vector $[1, 2, 3]^t$ *a linear combination of* $[1, 0, 3]^t$ *and* $[-1, 1, -3]^t$*?*

Linear Combination:

A vector $\mathbf{v}$ in $\mathbb{R}^n$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ if there exist real numbers $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k.$$

- The numbers $c_1, c_2, \ldots, c_k$ are called the coefficients of the linear combination.

Example

Is the vector $[1, 2, 3]^t$ *a linear combination of* $[1, 0, 3]^t$ *and* $[-1, 1, -3]^t$*?*

Result

A system of linear equations with augmented matrix $[A \mid \mathbf{b}]$ *is consistent if and only if* $\mathbf{b}$ *is a linear combination of the columns of A.*

Span of Vectors: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Then the collection of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called the span of S (or span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$), and is denoted by $\mathrm{span}(S)$ (or $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$).

Span of Vectors: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Then the collection of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called the span of S (or span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$), and is denoted by $\text{span}(S)$ (or $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$).

Thus

$$\text{span}(S) = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k \text{ for some } c_1, \ldots, c_k \in \mathbb{R}\}.$$

Span of Vectors: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Then the collection of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called the span of S (or span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$), and is denoted by $\text{span}(S)$ (or $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$).

Thus

$$\text{span}(S) = \{\mathbf{v} \in \mathbb{R}^n \,|\, \mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k \text{ for some } c_1, \ldots, c_k \in \mathbb{R}\}.$$

- Convention: $\text{span}(\emptyset) = \{\mathbf{0}\}$.

Span of Vectors: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Then the collection of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called the span of S (or span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$), and is denoted by $\mathrm{span}(S)$ (or $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$).

Thus

$$\mathrm{span}(S) = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k \text{ for some } c_1, \ldots, c_k \in \mathbb{R}\}.$$

- Convention: $\mathrm{span}(\emptyset) = \{\mathbf{0}\}$.
- If $\mathrm{span}(S) = \mathbb{R}^n$, then S is called a spanning set for $\mathbb{R}^n$.

Span of Vectors: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Then the collection of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called the span of S (or span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$), and is denoted by $\text{span}(S)$ (or $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$).

Thus

$$\text{span}(S) = \{\mathbf{v} \in \mathbb{R}^n | \, \mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k \text{ for some } c_1, \ldots, c_k \in \mathbb{R}\}.$$

- Convention: $\text{span}(\emptyset) = \{\mathbf{0}\}$.
- If $\text{span}(S) = \mathbb{R}^n$, then S is called a spanning set for $\mathbb{R}^n$.
- $\mathbb{R}^2 = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$, where $\mathbf{e}_1 = [1, 0]^t$ and $\mathbf{e}_2 = [0, 1]^t$.

Span of Vectors: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Then the collection of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called the span of S (or span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$), and is denoted by $\mathrm{span}(S)$ (or $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$).

Thus

$$\mathrm{span}(S) = \{\mathbf{v} \in \mathbb{R}^n \,|\, \mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k \text{ for some } c_1, \ldots, c_k \in \mathbb{R}\}.$$

- Convention: $\mathrm{span}(\emptyset) = \{\mathbf{0}\}$.
- If $\mathrm{span}(S) = \mathbb{R}^n$, then S is called a spanning set for $\mathbb{R}^n$.
- $\mathbb{R}^2 = \mathrm{span}(\mathbf{e}_1, \mathbf{e}_2)$, where $\mathbf{e}_1 = [1, 0]^t$ and $\mathbf{e}_2 = [0, 1]^t$.

Example

Let $\mathbf{u} = [1, 2, 3]^t$ *and* $\mathbf{v} = [-1, 1, -3]^t$. *Describe span*$(\mathbf{u}, \mathbf{v})$ *geometrically.*

Subspace of $\mathbb{R}^n$

Let S $(\neq \emptyset) \subseteq \mathbb{R}^n$. Then S is called a subspace of $\mathbb{R}^n$ iff $a\mathbf{u} + b\mathbf{v} \in S$ for every $\mathbf{u}, \mathbf{v} \in S$ and for every $a, b \in \mathbb{R}$.

Subspace of $\mathbb{R}^n$

Let S $(\neq \emptyset) \subseteq \mathbb{R}^n$. Then S is called a subspace of $\mathbb{R}^n$ iff $a\mathbf{u} + b\mathbf{v} \in S$ for every $\mathbf{u}, \mathbf{v} \in S$ and for every $a, b \in \mathbb{R}$.

- $S = \{\mathbf{0}\}$ and $S = \mathbb{R}^n$ are (trivial) subspaces of $\mathbb{R}^n$.

Subspace of $\mathbb{R}^n$

Let $S\ (\neq \emptyset) \subseteq \mathbb{R}^n$. Then S is called a subspace of $\mathbb{R}^n$ iff $a\mathbf{u} + b\mathbf{v} \in S$ for every $\mathbf{u}, \mathbf{v} \in S$ and for every $a, b \in \mathbb{R}$.

- $S = \{\mathbf{0}\}$ and $S = \mathbb{R}^n$ are (trivial) subspaces of $\mathbb{R}^n$.
- If S is a subspace of $\mathbb{R}^n$ and if $\mathbf{u}, \mathbf{v} \in S$ then $0\mathbf{u} + 0\mathbf{v} = \mathbf{0} \in S$, $\quad a\mathbf{u} + 0\mathbf{u} = a\mathbf{u} \in S$ and $1\mathbf{u} + 1\mathbf{v} = \mathbf{u} + \mathbf{v} \in S$.

Subspace of $\mathbb{R}^n$

Let $S\ (\neq \emptyset) \subseteq \mathbb{R}^n$. Then S is called a subspace of $\mathbb{R}^n$ iff $a\mathbf{u} + b\mathbf{v} \in S$ for every $\mathbf{u}, \mathbf{v} \in S$ and for every $a, b \in \mathbb{R}$.

- $S = \{\mathbf{0}\}$ and $S = \mathbb{R}^n$ are (trivial) subspaces of $\mathbb{R}^n$.
- If S is a subspace of $\mathbb{R}^n$ and if $\mathbf{u}, \mathbf{v} \in S$ then $0\mathbf{u} + 0\mathbf{v} = \mathbf{0} \in S$, $\quad a\mathbf{u} + 0\mathbf{u} = a\mathbf{u} \in S$ and $1\mathbf{u} + 1\mathbf{v} = \mathbf{u} + \mathbf{v} \in S$.

Example

Examine whether the sets
$S = \{[x, y, z]^t \in \mathbb{R}^3 : x = y + 1\}$, $T = \{[x, y, z]^t \in \mathbb{R}^3 : x = 5y\}$ *and* $U = \{[x, y, z]^t \in \mathbb{R}^3 : x = z^2\}$ *are subspaces of* $\mathbb{R}^3$.

Subspace of $\mathbb{R}^n$

Let $S\ (\neq \emptyset) \subseteq \mathbb{R}^n$. Then S is called a subspace of $\mathbb{R}^n$ iff $a\mathbf{u} + b\mathbf{v} \in S$ for every $\mathbf{u}, \mathbf{v} \in S$ and for every $a, b \in \mathbb{R}$.

- $S = \{\mathbf{0}\}$ and $S = \mathbb{R}^n$ are (trivial) subspaces of $\mathbb{R}^n$.
- If S is a subspace of $\mathbb{R}^n$ and if $\mathbf{u}, \mathbf{v} \in S$ then $0\mathbf{u} + 0\mathbf{v} = \mathbf{0} \in S$, $\quad a\mathbf{u} + 0\mathbf{u} = a\mathbf{u} \in S$ and $1\mathbf{u} + 1\mathbf{v} = \mathbf{u} + \mathbf{v} \in S$.

Example

Examine whether the sets $S = \{[x, y, z]^t \in \mathbb{R}^3 : x = y + 1\}$, $T = \{[x, y, z]^t \in \mathbb{R}^3 : x = 5y\}$ *and* $U = \{[x, y, z]^t \in \mathbb{R}^3 : x = z^2\}$ *are subspaces of* $\mathbb{R}^3$.

Example

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$ *then* $span(S)$ *is a subspace of* $\mathbb{R}^n$.

Example

Let A be an $m \times n$ matrix. Then $U = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the nullspace of A..

Example

Let A be an $m \times n$ matrix. Then $U = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the nullspace of A..

Example

Let U and V be two subspaces of $\mathbb{R}^n$. Then $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a subspace of $\mathbb{R}^n$.

Example

Let A be an $m \times n$ matrix. Then $U = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the nullspace of A..

Example

Let U and V be two subspaces of $\mathbb{R}^n$. Then $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a subspace of $\mathbb{R}^n$.

If U and V are subspaces of $\mathbb{R}^n$ such that $U \cap V = \{\mathbf{0}\}$, then $U + V$ is called an internal direct sum. Notation: $U \oplus V$.

Example

Let A be an $m \times n$ matrix. Then $U = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the nullspace of A..

Example

Let U and V be two subspaces of $\mathbb{R}^n$. Then $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a subspace of $\mathbb{R}^n$.

If U and V are subspaces of $\mathbb{R}^n$ such that $U \cap V = \{\mathbf{0}\}$, then $U + V$ is called an internal direct sum. Notation: $U \oplus V$.

Result

Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with n variables. Then exactly one of the following is true:

Example

Let A be an $m \times n$ matrix. Then $U = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the nullspace of A..

Example

Let U and V be two subspaces of $\mathbb{R}^n$. Then $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a subspace of $\mathbb{R}^n$.

If U and V are subspaces of $\mathbb{R}^n$ such that $U \cap V = \{\mathbf{0}\}$, then $U + V$ is called an internal direct sum. Notation: $U \oplus V$.

Result

Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with n variables. Then exactly one of the following is true:

1. *there is no solution of the system $A\mathbf{x} = \mathbf{b}$;*

Example

Let A be an $m \times n$ matrix. Then $U = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the nullspace of A..

Example

Let U and V be two subspaces of $\mathbb{R}^n$. Then $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a subspace of $\mathbb{R}^n$.

If U and V are subspaces of $\mathbb{R}^n$ such that $U \cap V = \{\mathbf{0}\}$, then $U + V$ is called an internal direct sum. Notation: $U \oplus V$.

Result

Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with n variables. Then exactly one of the following is true:

1. *there is no solution of the system $A\mathbf{x} = \mathbf{b}$;*
2. *there is a unique solution of the system $A\mathbf{x} = \mathbf{b}$;*

Example

Let A be an $m \times n$ matrix. Then $U = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the nullspace of A..

Example

Let U and V be two subspaces of $\mathbb{R}^n$. Then $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a subspace of $\mathbb{R}^n$.

If U and V are subspaces of $\mathbb{R}^n$ such that $U \cap V = \{\mathbf{0}\}$, then $U + V$ is called an internal direct sum. Notation: $U \oplus V$.

Result

Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with n variables. Then exactly one of the following is true:

1. *there is no solution of the system $A\mathbf{x} = \mathbf{b}$;*
2. *there is a unique solution of the system $A\mathbf{x} = \mathbf{b}$;*
3. *there are infinitely many solutions of the system $A\mathbf{x} = \mathbf{b}$.*

Linear Dependence

A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$, at least one of them is non-zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}.$$

Linear Dependence

A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$, at least one of them is non-zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}.$$

- If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent, then we also say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent.

Linear Dependence

A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$, at least one of them is non-zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}.$$

- If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent, then we also say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent.
- A set of vectors containing the $\mathbf{0}$ is always linearly dependent.

Linear Dependence

A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$, at least one of them is non-zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}.$$

- If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent, then we also say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent.
- A set of vectors containing the $\mathbf{0}$ is always linearly dependent.

Example

Examine whether the sets $T = \{[1, 2, 0]^t, [1, 1, -1]^t, [1, 4, 2]^t\}$ *and* $S = \{[1, 4]^t, [-1, 2]^t\}$ *are linearly dependent.*

Linear Independence

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly independent if S is not linearly dependent.

Linear Independence

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly independent if S is not linearly dependent.

- S is linearly independent iff
  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow$

Linear Independence

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly independent if S is not linearly dependent.

- S is linearly independent iff
  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = c_2 = \ldots = c_k = 0.$

Linear Independence

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly independent if S is not linearly dependent.

- S is linearly independent iff $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = c_2 = \ldots = c_k = 0.$
- If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, then we also say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Linear Independence

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly independent if S is not linearly dependent.

- S is linearly independent iff
  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = c_2 = \ldots = c_k = 0.$
- If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, then we also say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Example

Let $\mathbf{e}_i \in \mathbb{R}^n$ be the i-th column of the identity matrix I_n.

Linear Independence

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly independent if S is not linearly dependent.

- S is linearly independent iff
  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = c_2 = \ldots = c_k = 0.$

- If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, then we also say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Example

Let $\mathbf{e}_i \in \mathbb{R}^n$ be the i-th column of the identity matrix I_n. Is $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ linearly independent?

Linear Independence

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly independent if S is not linearly dependent.

- S is linearly independent iff $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = c_2 = \ldots = c_k = 0$.
- If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, then we also say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Example

Let $\mathbf{e}_i \in \mathbb{R}^n$ be the i-th column of the identity matrix I_n. Is $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ linearly independent?

Result

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are linearly dependent iff at least one of these vectors can be expressed as a linear combination of the others.

Linear combinations of rows

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A.

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A. Note that $\mathbf{a}$ is an $1 \times n$ matrix and $\mathbf{a}^T \in \mathbb{R}^n$.

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A. Note that $\mathbf{a}$ is an $1 \times n$ matrix and $\mathbf{a}^T \in \mathbb{R}^n$.
- Note: $c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = [c_1, \ldots, c_m]A$.

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A. Note that $\mathbf{a}$ is an $1 \times n$ matrix and $\mathbf{a}^T \in \mathbb{R}^n$.
- Note: $c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = [c_1, \ldots, c_m]A$. Thus, for any $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c}^T A$ is a linear combination of rows of A.

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A. Note that $\mathbf{a}$ is an $1 \times n$ matrix and $\mathbf{a}^T \in \mathbb{R}^n$.
- Note: $c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = [c_1, \ldots, c_m]A$. Thus, for any $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c}^T A$ is a linear combination of rows of A.
- The rows of A are linearly dependent iff $\mathbf{c}^T A = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = \mathbf{0}^T$ (zero row) for some nonzero $\mathbf{c} \in \mathbb{R}^m$.

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A. Note that $\mathbf{a}$ is an $1 \times n$ matrix and $\mathbf{a}^T \in \mathbb{R}^n$.
- Note: $c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = [c_1, \ldots, c_m]A$. Thus, for any $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c}^T A$ is a linear combination of rows of A.
- The rows of A are linearly dependent iff $\mathbf{c}^T A = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = \mathbf{0}^T$ (zero row) for some nonzero $\mathbf{c} \in \mathbb{R}^m$.

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A. Note that $\mathbf{a}$ is an $1 \times n$ matrix and $\mathbf{a}^T \in \mathbb{R}^n$.
- Note: $c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = [c_1, \ldots, c_m]A$. Thus, for any $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c}^T A$ is a linear combination of rows of A.
- The rows of A are linearly dependent iff $\mathbf{c}^T A = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = \mathbf{0}^T$ (zero row) for some nonzero $\mathbf{c} \in \mathbb{R}^m$.
- The rows of A are linearly dependent iff $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are linearly dependent,

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A. Note that $\mathbf{a}$ is an $1 \times n$ matrix and $\mathbf{a}^T \in \mathbb{R}^n$.
- Note: $c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = [c_1, \ldots, c_m]A$. Thus, for any $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c}^T A$ is a linear combination of rows of A.
- The rows of A are linearly dependent iff $\mathbf{c}^T A = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = \mathbf{0}^T$ (zero row) for some nonzero $\mathbf{c} \in \mathbb{R}^m$.
- The rows of A are linearly dependent iff $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are linearly dependent, i.e., the columns of A^T are linearly dependent.

Result

Suppose $RREF(A)$ *has a zero row. Then the rows of* A *are linearly dependent.*

Result

Suppose RREF(A) has a zero row. Then the rows of A are linearly dependent.

Result

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$ *and consider the* $n \times m$ *matrix* $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_m]$. *Then S is linearly dependent iff the system* $A\mathbf{x} = \mathbf{0}$ *has a non-trivial solution.*

Result

Suppose $RREF(A)$ has a zero row. Then the rows of A are linearly dependent.

Result

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$ and consider the $n \times m$ matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_m]$. Then S is linearly dependent iff the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

Result

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$ and consider the $m \times n$ matrix

$$A = \begin{bmatrix} \mathbf{v}_1^t \\ \mathbf{v}_2^t \\ \vdots \\ \mathbf{v}_m^t \end{bmatrix}.$$

Then S is linearly dependent iff $rank(A) < m$.

Result

Suppose $RREF(A)$ has a zero row. Then the rows of A are linearly dependent.

Result

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$ and consider the $n \times m$ matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \ \ldots\ \mathbf{v}_m]$. Then S is linearly dependent iff the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

Result

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$ and consider the $m \times n$ matrix

$$A = \begin{bmatrix} \mathbf{v}_1^t \\ \mathbf{v}_2^t \\ \vdots \\ \mathbf{v}_m^t \end{bmatrix}.$$

Then S is linearly dependent iff $rank(A) < m$.

Result

If $m > n$ then any set of m vectors in $\mathbb{R}^n$ is linearly dependent.

Basis: Let S be a subspace of $\mathbb{R}^n$ and $B \subseteq S$. Then B is said to be a basis for S iff B is linearly independent and $\text{span}(B) = S$.

Basis: Let S be a subspace of $\mathbb{R}^n$ and $B \subseteq S$. Then B is said to be a basis for S iff B is linearly independent and $\text{span}(B) = S$.

- The set $\{1\}$ is a basis for $\mathbb{R}^1$ $(= \mathbb{R})$.

Basis: Let S be a subspace of $\mathbb{R}^n$ and $B \subseteq S$. Then B is said to be a basis for S iff B is linearly independent and span$(B) = S$.

- The set $\{1\}$ is a basis for $\mathbb{R}^1$ $(= \mathbb{R})$.
- If $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots, 0]^t$, where 1 is at the i-th entry and the other entries are zero, then $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$.

Basis: Let S be a subspace of $\mathbb{R}^n$ and $B \subseteq S$. Then B is said to be a basis for S iff B is linearly independent and $\text{span}(B) = S$.

- The set $\{1\}$ is a basis for $\mathbb{R}^1$ $(= \mathbb{R})$.
- If $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots, 0]^t$, where 1 is at the i-th entry and the other entries are zero, then $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$.
- The vectors $\mathbf{e}_i$ (for $i = 1, 2, \ldots, n$) are called the standard unit vector.

Basis: Let S be a subspace of $\mathbb{R}^n$ and $B \subseteq S$. Then B is said to be a basis for S iff B is linearly independent and $\text{span}(B) = S$.

- The set $\{1\}$ is a basis for $\mathbb{R}^1$ $(= \mathbb{R})$.
- If $\mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots, 0]^t$, where 1 is at the i-th entry and the other entries are zero, then $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$.
- The vectors $\mathbf{e}_i$ (for $i = 1, 2, \ldots, n$) are called the standard unit vector.

Result

For a subspace U, a subset $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq U$ is a basis of U iff every element of U is a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_r$.

Example

Find a basis for the subspace $S = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$, *where*

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example

Find a basis for the subspace $S = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$, *where*

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Result

Let S be a subspace of $\mathbb{R}^n$. Then S has a basis and any two bases for S have the same number of elements.

Example

Find a basis for the subspace $S = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$, *where*

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Result

Let S be a subspace of $\mathbb{R}^n$. Then S has a basis and any two bases for S have the same number of elements.

Dimension: The number of elements in a basis for S (a subspace of $\mathbb{R}^n$) is called the dimension, denoted $\dim(S)$, of S.

Example

Find a basis for the subspace $S = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$*, where*

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad RREF(A) = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Result

Let S be a subspace of $\mathbb{R}^n$. Then S has a basis and any two bases for S have the same number of elements.

Dimension: The number of elements in a basis for S (a subspace of $\mathbb{R}^n$) is called the dimension, denoted $\dim(S)$, of S.

- $\dim(\mathbb{R}^n) = n$.

Matrices

Matrices

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix.

Matrices

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix.

- If $m = n$, then A is called a square matrix.

Matrices

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix.

- If $m = n$, then A is called a square matrix.
- If A is a square matrix, then the entries a_{ii} are called the diagonal entries of A.

Matrices

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix.

- If $m = n$, then A is called a square matrix.
- If A is a square matrix, then the entries a_{ii} are called the diagonal entries of A.
- If A is a square matrix and if $a_{ij} = 0$ for all $i \neq j$, then A is called a diagonal matrix.

Matrices

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix.

- If $m = n$, then A is called a square matrix.
- If A is a square matrix, then the entries a_{ii} are called the diagonal entries of A.
- If A is a square matrix and if $a_{ij} = 0$ for all $i \neq j$, then A is called a diagonal matrix.
- If an $n \times n$ diagonal matrix has all diagonal entries equal to 1, then it is called the identity matrix of size n, and is denoted by I_n (or simply by I).

Matrices

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix.

- If $m = n$, then A is called a square matrix.
- If A is a square matrix, then the entries a_{ii} are called the diagonal entries of A.
- If A is a square matrix and if $a_{ij} = 0$ for all $i \neq j$, then A is called a diagonal matrix.
- If an $n \times n$ diagonal matrix has all diagonal entries equal to 1, then it is called the identity matrix of size n, and is denoted by I_n (or simply by I).
- If all the entries of A are equal to 0 then A is called a zero matrix, denoted $\mathbf{O}_{m \times n}$ (or simply by $\mathbf{O}$)

- A matrix B is said to be a sub matrix of A if B is obtained by deleting some rows and/or columns of A.

- A matrix B is said to be a sub matrix of A if B is obtained by deleting some rows and/or columns of A.
- The transpose A^t of $A = [a_{ij}]$ is defined as $A^t = [b_{ji}]$, where $b_{ji} = a_{ij}$ for all i, j.

- A matrix B is said to be a sub matrix of A if B is obtained by deleting some rows and/or columns of A.
- The transpose A^t of $A = [a_{ij}]$ is defined as $A^t = [b_{ji}]$, where $b_{ji} = a_{ij}$ for all i, j.
- The matrix A is said to be symmetric if $A^t = A$.

- A matrix B is said to be a sub matrix of A if B is obtained by deleting some rows and/or columns of A.
- The transpose A^t of $A = [a_{ij}]$ is defined as $A^t = [b_{ji}]$, where $b_{ji} = a_{ij}$ for all i, j.
- The matrix A is said to be symmetric if $A^t = A$.
- The matrix A is said to be skew-symmetric if $A^t = -A$.

- A matrix B is said to be a sub matrix of A if B is obtained by deleting some rows and/or columns of A.
- The transpose A^t of $A = [a_{ij}]$ is defined as $A^t = [b_{ji}]$, where $b_{ji} = a_{ij}$ for all i, j.
- The matrix A is said to be symmetric if $A^t = A$.
- The matrix A is said to be skew-symmetric if $A^t = -A$.
- If A is a complex matrix, then $\overline{A} = [\overline{a}_{ij}]$ and $A^* = \overline{A}^t$.

- A matrix B is said to be a sub matrix of A if B is obtained by deleting some rows and/or columns of A.
- The transpose A^t of $A = [a_{ij}]$ is defined as $A^t = [b_{ji}]$, where $b_{ji} = a_{ij}$ for all i, j.
- The matrix A is said to be symmetric if $A^t = A$.
- The matrix A is said to be skew-symmetric if $A^t = -A$.
- If A is a complex matrix, then $\overline{A} = [\overline{a}_{ij}]$ and $A^* = \overline{A}^t$.
- The matrix A^* is called the conjugate transpose of A.

- A matrix B is said to be a sub matrix of A if B is obtained by deleting some rows and/or columns of A.
- The transpose A^t of $A = [a_{ij}]$ is defined as $A^t = [b_{ji}]$, where $b_{ji} = a_{ij}$ for all i, j.
- The matrix A is said to be symmetric if $A^t = A$.
- The matrix A is said to be skew-symmetric if $A^t = -A$.
- If A is a complex matrix, then $\overline{A} = [\overline{a}_{ij}]$ and $A^* = \overline{A}^t$.
- The matrix A^* is called the conjugate transpose of A.
- The (complex) matrix A is said to be Hermitian if $A^* = A$, and skew-Hermitian if $A^* = -A$.

- A square matrix A is said to be upper triangular if $a_{ij} = 0$ for all $i > j$.

- A square matrix A is said to be upper triangular if $a_{ij} = 0$ for all $i > j$.
- A square matrix A is said to be lower triangular if $a_{ij} = 0$ for all $i < j$.

- A square matrix A is said to be upper triangular if $a_{ij} = 0$ for all $i > j$.
- A square matrix A is said to be lower triangular if $a_{ij} = 0$ for all $i < j$.
- Let A be an $n \times n$ square matrix. Then we define $A^0 = I_n, \quad A^1 = A \quad \text{and} \quad A^2 = AA.$

- A square matrix A is said to be upper triangular if $a_{ij} = 0$ for all $i > j$.
- A square matrix A is said to be lower triangular if $a_{ij} = 0$ for all $i < j$.
- Let A be an $n \times n$ square matrix. Then we define $A^0 = I_n, \quad A^1 = A \quad \text{and} \quad A^2 = AA$.
- In general, if k is a positive integer, we define the power A^k as follows

$$A^k = \underbrace{AA \ldots A}_{k \text{ times}}.$$

- A square matrix A is said to be upper triangular if $a_{ij} = 0$ for all $i > j$.
- A square matrix A is said to be lower triangular if $a_{ij} = 0$ for all $i < j$.
- Let A be an $n \times n$ square matrix. Then we define $A^0 = I_n, \quad A^1 = A \quad \text{and} \quad A^2 = AA.$
- In general, if k is a positive integer, we define the power A^k as follows
$$A^k = \underbrace{AA \dots A}_{k \text{ times}}.$$

If A and $\mathbf{O}$ are matrices of the same size, then
$A + \mathbf{O} = A = \mathbf{O} + A, \; A - \mathbf{O} = A, \; \mathbf{O} - A = -A, \; A - A = \mathbf{O}.$

Matrices

Matrices

Result

Let A be an $m \times n$ matrix, $\mathbf{e}_i$ an $1 \times m$ standard unit vector, and $\mathbf{e}_j$ an $n \times 1$ standard unit vector. Then $\mathbf{e}_i A$ is the i-th row of A and $A\mathbf{e}_j$ is the j-th column of A.

Matrices

Result

Let A be an $m \times n$ matrix, $\mathbf{e}_i$ an $1 \times m$ standard unit vector, and $\mathbf{e}_j$ an $n \times 1$ standard unit vector. Then $\mathbf{e}_i A$ is the i-th row of A and $A\mathbf{e}_j$ is the j-th column of A.

Result

Let A be a square matrix and let r and s be non-negative integers. Then $A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.

Matrices

Result

Let A be an $m \times n$ matrix, $\mathbf{e}_i$ an $1 \times m$ standard unit vector, and $\mathbf{e}_j$ an $n \times 1$ standard unit vector. Then $\mathbf{e}_i A$ is the i-th row of A and $A\mathbf{e}_j$ is the j-th column of A.

Result

Let A be a square matrix and let r and s be non-negative integers. Then $A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.

Result

Let A, B and C be matrices of size $m \times n$, and let $s, r \in \mathbb{R}$. Then

Matrices

Result

Let A be an $m \times n$ matrix, $\mathbf{e}_i$ an $1 \times m$ standard unit vector, and $\mathbf{e}_j$ an $n \times 1$ standard unit vector. Then $\mathbf{e}_i A$ is the i-th row of A and $A\mathbf{e}_j$ is the j-th column of A.

Result

Let A be a square matrix and let r and s be non-negative integers. Then $A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.

Result

Let A, B and C be matrices of size $m \times n$, and let $s, r \in \mathbb{R}$. Then

1. *Commutative Law: $A + B = B + A$.*

Matrices

Result

Let A be an $m \times n$ matrix, $\mathbf{e}_i$ an $1 \times m$ standard unit vector, and $\mathbf{e}_j$ an $n \times 1$ standard unit vector. Then $\mathbf{e}_i A$ is the i-th row of A and $A\mathbf{e}_j$ is the j-th column of A.

Result

Let A be a square matrix and let r and s be non-negative integers. Then $A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.

Result

Let A, B and C be matrices of size $m \times n$, and let $s, r \in \mathbb{R}$. Then

1. *Commutative Law: $A + B = B + A$.*
2. *Associative Law: $(A + B) + C = A + (B + C)$.*

Matrices

Result

Let A be an $m \times n$ matrix, $\mathbf{e}_i$ an $1 \times m$ standard unit vector, and $\mathbf{e}_j$ an $n \times 1$ standard unit vector. Then $\mathbf{e}_i A$ is the i-th row of A and $A\mathbf{e}_j$ is the j-th column of A.

Result

Let A be a square matrix and let r and s be non-negative integers. Then $A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.

Result

Let A, B and C be matrices of size $m \times n$, and let $s, r \in \mathbb{R}$. Then

1. *Commutative Law: $A + B = B + A$.*
2. *Associative Law: $(A + B) + C = A + (B + C)$.*
3. *$1A = A$ and $s(rA) = (sr)A$.*

Matrices

Result

Let A be an $m \times n$ matrix, $\mathbf{e}_i$ an $1 \times m$ standard unit vector, and $\mathbf{e}_j$ an $n \times 1$ standard unit vector. Then $\mathbf{e}_i A$ is the i-th row of A and $A\mathbf{e}_j$ is the j-th column of A.

Result

Let A be a square matrix and let r and s be non-negative integers. Then $A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.

Result

Let A, B and C be matrices of size $m \times n$, and let $s, r \in \mathbb{R}$. Then

1. *Commutative Law*: $A + B = B + A$.
2. *Associative Law*: $(A + B) + C = A + (B + C)$.
3. $1A = A \quad \text{and} \quad s(rA) = (sr)A$.
4. $s(A + B) = sA + sB \quad \text{and} \quad (s + r)A = sA + rA$.

Result

Let A, B *and* C *be matrices, and let* $s \in \mathbb{R}$. *Then*

Result

Let A, B and C be matrices, and let $s \in \mathbb{R}$. Then

1. *Associative Law*: $(AB)C = A(BC)$, *if the respective matrix products are defined.*

Result

Let A, B and C be matrices, and let $s \in \mathbb{R}$. Then

1. *Associative Law: $(AB)C = A(BC)$, if the respective matrix products are defined.*

2. *Distributive Law:*
   $A(B + C) = AB + AC, \quad (A + B)C = AC + BC,$
   if the respective matrix sum and matrix products are defined.

Result

Let A, B and C be matrices, and let $s \in \mathbb{R}$. Then

1. *Associative Law: $(AB)C = A(BC)$, if the respective matrix products are defined.*
2. *Distributive Law:*
   $A(B+C) = AB + AC, \quad (A+B)C = AC + BC,$
   if the respective matrix sum and matrix products are defined.
3. *$s(AB) = (sA)B = A(sB)$, if the respective matrix products are defined.*

Result

Let A, B and C be matrices, and let $s \in \mathbb{R}$. Then

1. *Associative Law*: $(AB)C = A(BC)$, *if the respective matrix products are defined.*

2. *Distributive Law*:
   $A(B + C) = AB + AC, \quad (A + B)C = AC + BC,$
   if the respective matrix sum and matrix products are defined.

3. $s(AB) = (sA)B = A(sB)$, *if the respective matrix products are defined.*

4. $I_m A = A = A I_n$, *if A is of size $m \times n$.*

Result

Let A and B be two matrices and $k \in \mathbb{R}$. Then

Result

Let A and B be two matrices and $k \in \mathbb{R}$. Then

1. $(A^t)^t = A, \quad (kA)^t = kA^t.$

Result

Let A and B be two matrices and $k \in \mathbb{R}$. Then

1. $(A^t)^t = A, \quad (kA)^t = kA^t.$
2. $(A+B)^t = A^t + B^t$ *if A and B are of the same size.*

Result

Let A and B be two matrices and $k \in \mathbb{R}$. Then

1. $(A^t)^t = A, \quad (kA)^t = kA^t$.
2. $(A+B)^t = A^t + B^t$ *if A and B are of the same size.*
3. $(AB)^t = B^t A^t$ *if the matrix product AB is defined.*

Result

Let A and B be two matrices and $k \in \mathbb{R}$. Then

1. $(A^t)^t = A, \quad (kA)^t = kA^t$.
2. $(A+B)^t = A^t + B^t$ *if A and B are of the same size.*
3. $(AB)^t = B^t A^t$ *if the matrix product AB is defined.*
4. $(A^r)^t = (A^t)^r$ *for any non-negative integer r.*

Partitioned Matrix

- By introducing vertical and horizontal lines into a given matrix, we can partition it into some blocks of smaller sub-matrices.

Partitioned Matrix

- By introducing vertical and horizontal lines into a given matrix, we can partition it into some blocks of smaller sub-matrices.
- For example, three partitions of the matrix A are given below:

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad A = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right],$$

$$A = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right], \quad A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ \hline 0 & 0 & 1 & 5 \end{array}\right].$$

- A given matrix can have several partitions possible.

- A given matrix can have several partitions possible.
- A block matrix $A = [A_{ij}]$ is a matrix, where each entry A_{ij} is itself a matrix.

- A given matrix can have several partitions possible.
- A block matrix $A = [A_{ij}]$ is a matrix, where each entry A_{ij} is itself a matrix.
- Thus a partition of a given matrix give us a block matrix.

- A given matrix can have several partitions possible.
- A block matrix $A = [A_{ij}]$ is a matrix, where each entry A_{ij} is itself a matrix.
- Thus a partition of a given matrix give us a block matrix.
- The first partition of the previous matrix A is the block matrix $\begin{bmatrix} I & B \\ \mathbf{O} & C \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right]$, where

- A given matrix can have several partitions possible.
- A block matrix $A = [A_{ij}]$ is a matrix, where each entry A_{ij} is itself a matrix.
- Thus a partition of a given matrix give us a block matrix.
- The first partition of the previous matrix A is the block matrix $\begin{bmatrix} I & B \\ \mathbf{O} & C \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right]$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$.

Result

Let A and B be two matrices of sizes $m \times n$ and $n \times r$, resp.

Result

Let A and B be two matrices of sizes $m \times n$ and $n \times r$, resp.

1. *If* $B = [\mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_r]$ *then* $AB = [A\mathbf{b}_1 \mid A\mathbf{b}_2 \mid \ldots \mid A\mathbf{b}_r]$.

Result

Let A and B be two matrices of sizes $m \times n$ and $n \times r$, resp.

1. *If* $B = [\mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_r]$ *then* $AB = [A\mathbf{b}_1 \mid A\mathbf{b}_2 \mid \ldots \mid A\mathbf{b}_r]$.

2. *If* $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$ *then* $AB = \begin{bmatrix} \mathbf{a}_1 B \\ \hline \mathbf{a}_2 B \\ \hline \vdots \\ \hline \mathbf{a}_m B \end{bmatrix}$.

Result

Let A and B be two matrices of sizes $m \times n$ and $n \times r$, resp.

1. *If* $B = [\mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_r]$ *then* $AB = [A\mathbf{b}_1 \mid A\mathbf{b}_2 \mid \ldots \mid A\mathbf{b}_r]$.

2. *If* $A = \left[\begin{array}{c} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right]$ *then* $AB = \left[\begin{array}{c} \mathbf{a}_1 B \\ \hline \mathbf{a}_2 B \\ \hline \vdots \\ \hline \mathbf{a}_m B \end{array}\right]$.

3. *If* $m = n = r$ *and if* $A = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \ldots \mid \mathbf{a}_n]$ *and* $B = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \vdots \\ \hline \mathbf{b}_n \end{array}\right]$

   then
   $$AB = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \ldots + \mathbf{a}_n\mathbf{b}_n.$$

The Inverse of a Matrix

Definition

An $n \times n$ matrix A is said to be invertible if there exists a matrix B satisfying $AB = I_n = BA$, and B is called an inverse of A.

The Inverse of a Matrix

Definition

An $n \times n$ matrix A is said to be invertible if there exists a matrix B satisfying $AB = I_n = BA$, and B is called an inverse of A.

- We can talk of invertibility only for square matrices.

The Inverse of a Matrix

Definition

An $n \times n$ matrix A is said to be invertible if there exists a matrix B satisfying $AB = I_n = BA$, and B is called an inverse of A.

- We can talk of invertibility only for square matrices.
- For example, the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is invertible since

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

The Inverse of a Matrix

Definition

An $n \times n$ matrix A is said to be invertible if there exists a matrix B satisfying $AB = I_n = BA$, and B is called an inverse of A.

- We can talk of invertibility only for square matrices.
- For example, the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is invertible since

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

- It is easy to see that the zero matrix $\mathbf{O}$ is never invertible.

The Inverse of a Matrix

Definition

An $n \times n$ matrix A is said to be invertible if there exists a matrix B satisfying $AB = I_n = BA$, and B is called an inverse of A.

- We can talk of invertibility only for square matrices.
- For example, the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is invertible since

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

- It is easy to see that the zero matrix $\mathbf{O}$ is never invertible.
- The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible.

Matrices

Matrices

Result

If A is an invertible matrix, then its inverse is unique.

Matrices

Result

If A is an invertible matrix, then its inverse is unique.

- A^{-1} denotes the inverse of an invertible matrix A.

Matrices

Result

If A is an invertible matrix, then its inverse is unique.

- A^{-1} denotes the inverse of an invertible matrix A.
- If A is invertible then $AA^{-1} = I_n = A^{-1}A$.

Matrices

Result

If A is an invertible matrix, then its inverse is unique.

- A^{-1} denotes the inverse of an invertible matrix A.
- If A is invertible then $AA^{-1} = I_n = A^{-1}A$.

Result

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- *If $ad - bc \neq 0$ then A is invertible, and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Matrices

Result

If A is an invertible matrix, then its inverse is unique.

- A^{-1} denotes the inverse of an invertible matrix A.
- If A is invertible then $AA^{-1} = I_n = A^{-1}A$.

Result

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- *If $ad - bc \neq 0$ then A is invertible, and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- *If $ad - bc = 0$ then A is not invertible.*

Result

Let A and B be two invertible matrices of the same size.

Result

Let A and B be two invertible matrices of the same size.

1. *The matrix A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.*

Result

Let A and B be two invertible matrices of the same size.

1. *The matrix A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.*
2. *If $c \neq 0$ then cA is also invertible, and $(cA)^{-1} = \frac{1}{c}A^{-1}$.*

Result

Let A and B be two invertible matrices of the same size.

1. *The matrix A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.*
2. *If $c \neq 0$ then cA is also invertible, and $(cA)^{-1} = \frac{1}{c}A^{-1}$.*
3. *The matrix AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.*

Result

Let A and B be two invertible matrices of the same size.

1. *The matrix A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.*
2. *If $c \neq 0$ then cA is also invertible, and $(cA)^{-1} = \frac{1}{c}A^{-1}$.*
3. *The matrix AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.*
4. *The matrix A^t is invertible, and $(A^t)^{-1} = (A^{-1})^t$.*

Result

Let A and B be two invertible matrices of the same size.

1. *The matrix A^{-1} is also invertible, and* $(A^{-1})^{-1} = A$.
2. *If $c \neq 0$ then cA is also invertible, and* $(cA)^{-1} = \frac{1}{c}A^{-1}$.
3. *The matrix AB is invertible, and* $(AB)^{-1} = B^{-1}A^{-1}$.
4. *The matrix A^t is invertible, and* $(A^t)^{-1} = (A^{-1})^t$.
5. *For any non-negative integer k, the matrix A^k is invertible, and* $(A^k)^{-1} = (A^{-1})^k$.

Result

Let A and B be two invertible matrices of the same size.

1. *The matrix A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.*
2. *If $c \neq 0$ then cA is also invertible, and $(cA)^{-1} = \frac{1}{c}A^{-1}$.*
3. *The matrix AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.*
4. *The matrix A^t is invertible, and $(A^t)^{-1} = (A^{-1})^t$.*
5. *For any non-negative integer k, the matrix A^k is invertible, and $(A^k)^{-1} = (A^{-1})^k$.*

Elementary Matrix: An elementary matrix is a matrix that can be obtained by performing an elementary row operation on the identity matrix.

- There are three types of elementary matrices.

- There are three types of elementary matrices.
- For example, the following are the three types of elementary matrices of size 3.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

- There are three types of elementary matrices.
- For example, the following are the three types of elementary matrices of size 3.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

- There are three types of elementary matrices.
- For example, the following are the three types of elementary matrices of size 3.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

- There are three types of elementary matrices.
- For example, the following are the three types of elementary matrices of size 3.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

- The matrix E_1 is obtained by performing $R_2 \leftrightarrow R_3$ on I_3.

- There are three types of elementary matrices.
- For example, the following are the three types of elementary matrices of size 3.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

- The matrix E_1 is obtained by performing $R_2 \leftrightarrow R_3$ on I_3.
- E_2 is obtained by performing $R_2 \to 5R_2$ on I_3.

- There are three types of elementary matrices.
- For example, the following are the three types of elementary matrices of size 3.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

- The matrix E_1 is obtained by performing $R_2 \leftrightarrow R_3$ on I_3.
- E_2 is obtained by performing $R_2 \to 5R_2$ on I_3.
- E_3 is obtained by performing $R_3 \to R_3 - 2R_1$ on I_3.

- Let A be the 3×3 matrix as given below:

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}.$$

- Let A be the 3×3 matrix as given below:

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}.$$

Then

$$E_1 A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix},$$

- Let A be the 3×3 matrix as given below:

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}.$$

Then

$$E_1A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}, \quad E_2A = \begin{bmatrix} a & b & c \\ 5x & 5y & 5z \\ p & q & r \end{bmatrix} \text{ and}$$

- Let A be the 3×3 matrix as given below:

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}.$$

Then

$$E_1A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}, \quad E_2A = \begin{bmatrix} a & b & c \\ 5x & 5y & 5z \\ p & q & r \end{bmatrix} \text{ and}$$

$$E_3A = \begin{bmatrix} a & b & c \\ x & y & z \\ p-2a & q-2b & r-2c \end{bmatrix}.$$

- The matrix E_1A is the matrix obtained from A by performing the elementary row operation $R_2 \leftrightarrow R_3$.

- The matrix E_1A is the matrix obtained from A by performing the elementary row operation $R_2 \leftrightarrow R_3$.
- The matrix E_2A is the matrix obtained from A by performing the elementary row operation $R_2 \longrightarrow 5R_2$.

- The matrix $E_1 A$ is the matrix obtained from A by performing the elementary row operation $R_2 \leftrightarrow R_3$.
- The matrix $E_2 A$ is the matrix obtained from A by performing the elementary row operation $R_2 \rightarrow 5R_2$.
- The matrix $E_3 A$ is the matrix obtained from A by performing the elementary row operation $R_3 \rightarrow R_3 - 2R_1$.

- The matrix E_1A is the matrix obtained from A by performing the elementary row operation $R_2 \leftrightarrow R_3$.
- The matrix E_2A is the matrix obtained from A by performing the elementary row operation $R_2 \to 5R_2$.
- The matrix E_3A is the matrix obtained from A by performing the elementary row operation $R_3 \to R_3 - 2R_1$.

Result

- *Let E be an elementary matrix obtained by an elementary row operation on I_n. If the same elementary row operation is performed on an $n \times r$ matrix A, then the resulting matrix is equal to EA.*
- *The matrix B is row equivalent to A if there are elementary matrices $E_1, E_2, \cdots, E_k$ such that $B = E_k \cdots E_2 E_1 A$.*

- Elementary matrices are always invertible.

- Elementary matrices are always invertible.
- Applying $R_2 \leftrightarrow R_3$ on I_3 we find E_1^{-1}.

- Elementary matrices are always invertible.
- Applying $R_2 \leftrightarrow R_3$ on I_3 we find E_1^{-1}.
- Applying $R_2 \to \frac{1}{5}R_2$ on I_3 we find E_2^{-1}.

- Elementary matrices are always invertible.
- Applying $R_2 \leftrightarrow R_3$ on I_3 we find E_1^{-1}.
- Applying $R_2 \rightarrow \frac{1}{5}R_2$ on I_3 we find E_2^{-1}.
- Applying $R_3 \rightarrow R_3 + 2R_1$ on I_3 we find E_3^{-1}.

- Elementary matrices are always invertible.
- Applying $R_2 \leftrightarrow R_3$ on I_3 we find E_1^{-1}.
- Applying $R_2 \to \frac{1}{5}R_2$ on I_3 we find E_2^{-1}.
- Applying $R_3 \to R_3 + 2R_1$ on I_3 we find E_3^{-1}.
- We have

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1,$$

- Elementary matrices are always invertible.
- Applying $R_2 \leftrightarrow R_3$ on I_3 we find E_1^{-1}.
- Applying $R_2 \to \frac{1}{5}R_2$ on I_3 we find E_2^{-1}.
- Applying $R_3 \to R_3 + 2R_1$ on I_3 we find E_3^{-1}.
- We have

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

- Elementary matrices are always invertible.
- Applying $R_2 \leftrightarrow R_3$ on I_3 we find E_1^{-1}.
- Applying $R_2 \to \frac{1}{5}R_2$ on I_3 we find E_2^{-1}.
- Applying $R_3 \to R_3 + 2R_1$ on I_3 we find E_3^{-1}.
- We have

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

- Elementary matrices are always invertible.
- Applying $R_2 \leftrightarrow R_3$ on I_3 we find E_1^{-1}.
- Applying $R_2 \to \frac{1}{5}R_2$ on I_3 we find E_2^{-1}.
- Applying $R_3 \to R_3 + 2R_1$ on I_3 we find E_3^{-1}.
- We have

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Result

Every elementary matrix is invertible, and its inverse is an elementary matrix of the same type.

Result (Fundamental Theorem of Invertible Matrices: I)

Result (Fundamental Theorem of Invertible Matrices: I)

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

Result (Fundamental Theorem of Invertible Matrices: I)

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. A is *invertible*.
2. A^t is *invertible*.
3. $A\mathbf{x} = \mathbf{b}$ has a *solution* for every $\mathbf{b}$ in $\mathbb{R}^n$.
4. $A\mathbf{x} = \mathbf{b}$ has a *unique solution* for every $\mathbf{b}$ in $\mathbb{R}^n$.
5. $A\mathbf{x} = \mathbf{0}$ has only the *trivial solution*.
6. The reduced row echelon form of A is I_n.
7. The rows of A are linearly independent.
8. The columns of A are linearly independent.
9. $rank(A) = n$.
10. A is a *product of* elementary matrices.

Result

Let A be a square matrix. If B is a square matrix such that either $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$.

Result

Let A be a square matrix. If B is a square matrix such that either $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$.

Result

Let A be a square matrix. If a sequence of elementary row operations transforms A to the identity matrix I, then the same sequence of elementary row operations transforms I to A^{-1}.

Gauss-Jordan Method for Computing Inverse

Gauss-Jordan Method for Computing Inverse

Let A be an $n \times n$ matrix.

Gauss-Jordan Method for Computing Inverse

Let A be an $n \times n$ matrix.

- Apply elementary row operations on the matrix $[A \mid I_n]$.

Gauss-Jordan Method for Computing Inverse

Let A be an $n \times n$ matrix.

- Apply elementary row operations on the matrix $[A \mid I_n]$.
- If A is invertible, then $[A \mid I_n]$ will be transformed to $[I_n \mid A^{-1}]$.

Gauss-Jordan Method for Computing Inverse

Let A be an $n \times n$ matrix.

- Apply elementary row operations on the matrix $[A \mid I_n]$.
- If A is invertible, then $[A \mid I_n]$ will be transformed to $[I_n \mid A^{-1}]$.
- If A is not invertible, then $[A \mid I_n]$ can never be transformed to a matrix of the type $[I_n \mid B]$.

Gauss-Jordan Method for Computing Inverse

Let A be an $n \times n$ matrix.

- Apply elementary row operations on the matrix $[A \mid I_n]$.
- If A is invertible, then $[A \mid I_n]$ will be transformed to $[I_n \mid A^{-1}]$.
- If A is not invertible, then $[A \mid I_n]$ can never be transformed to a matrix of the type $[I_n \mid B]$.

Example

Find the inverse of the following matrix A, if it exists:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$