DEPARTMENT OF MATHEMATICS, IIT Guwahati
MA101: Mathematics I, July - November 2014
Solutions of Tutorial Sheet: LA - 4

---

1. For all invertible matrices A and B of the same size, show that $\text{adj}(AB) = \text{adj}(B)\text{adj}(A)$. (The result is also true for non-invertible matrices, but the proof is beyond the present scope of this course.)

   Solution: We have

$$\begin{aligned}(AB)\text{adj}(AB) = \det(AB)I \Rightarrow \text{adj}(AB) &= \det(AB)(AB)^{-1}\\ &= \det(A)\det(B)B^{-1}A^{-1}\\ &= (\det(B)B^{-1})(\det(A)A^{-1})\\ &= \text{adj}(B)\text{adj}(A).\end{aligned}$$

   □

2. If A is an $n \times n$ matrix then prove that $\det(\text{adj}(A)) = (\det\ A)^{n-1}$.

   Solution: We have

$$A\ \text{adj}(A) = (\det\ A)I_n \Rightarrow (\det\ A)\det(\text{adj}(A)) = (\det\ A)^n.$$

   If $\det\ A \neq 0$, then $(\det\ A)\det(\text{adj}(A)) = (\det\ A)^n \Rightarrow \det(\text{adj}(A)) = (\det\ A)^{n-1}$.

   If $\det\ A = 0$ then we must have $\det(\text{adj}(A)) = 0$. Otherwise, if $\text{adj}(A)$ is invertible, then $A\ \text{adj}(A) = (\det\ A)I_n = \mathbf{O} \Rightarrow A\ \text{adj}(A)\text{adj}(A)^{-1} = \mathbf{O} \Rightarrow A = \mathbf{O} \Rightarrow \text{adj}(A) = \mathbf{O}$, a contradiction to the assumption that $\text{adj}(A)$ is invertible. Hence, if $\det\ A = 0$ then $\det(\text{adj}(A)) = 0 = (\det\ A)^{n-1}$.

   Thus $\det(\text{adj}(A)) = (\det\ A)^{n-1}$. □

3. Let A and B be two matrices, where $\text{rank}(A) = r$. Show that

   (a) if AB is defined then $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$;

   (b) there exist invertible matrices T, S such that

$$TAS = \begin{bmatrix} I_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix};$$

   (c) there exist matrices $P_{m\times r}, Q_{r\times n}$ such that $A = PQ$; (this is known as Rank-Factorization Theorem)

   (d) A can be expressed as a sum of r rank one matrices;

   (e) $r = 1$ if and only if $A = \mathbf{u}\mathbf{v}^t$ for some $\mathbf{u}(\neq \mathbf{0}) \in \mathbb{R}^m$ and $\mathbf{v}\ (\neq \mathbf{0}) \in \mathbb{R}^n$;

   (f) if $A + B$ is defined then $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

   Solution:

   (a) Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix.

   Let P be an invertible matrix such that the last $m - r$ rows of PA are all zero. Then in PAB also, the last $m - r$ rows are all zero. Hence $\text{rank}(AB) = \text{rank}(PAB) \leq r = \text{rank}(A)$. Also $\text{rank}(AB) = \text{rank}((AB)^t) = \text{rank}(B^tA^t) \leq \text{rank}(B^t) = \text{rank}(B)$. Hence $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

   (b) Let $R = RREF(A)$. So, there exists an invertible matrix T such that $TA = R = \begin{bmatrix} R_1 \\ \vdots \\ R_r \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$, where $R_1, \ldots, R_r$ are the non-zero rows of R. Notice that $R^t = [R_1^t \ \ \ldots \ \ R_r^t \mid \mathbf{0} \ \ \ldots \ \ \mathbf{0}]$ also has rank r. So, there exists an invertible matrix P such that PR^t is the RREF of R^t, that is, $PR^t = \begin{bmatrix} I_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$. Taking transpose, we find that $RP^t = \begin{bmatrix} I_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}^t = \begin{bmatrix} I_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$. Finally, putting $S = P^t$, we have

$$TAS = RP^t = \begin{bmatrix} I_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}.$$

(c) Note that

$$\left[\begin{array}{c|c} I_r & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right] = \left[\begin{array}{c} I_r \\ \hline \mathbf{O} \end{array}\right][I_r \mid \mathbf{O}].$$

Therefore we have

$$TAS = \left[\begin{array}{c|c} I_r & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right] \Rightarrow A = T^{-1}\left[\begin{array}{c} I_r \\ \hline \mathbf{O} \end{array}\right][I_r \mid \mathbf{O}]S^{-1} = PQ,$$

where $P = T^{-1}\left[\begin{array}{c} I_r \\ \hline \mathbf{O} \end{array}\right]$ has r columns and $Q = [I_r \mid \mathbf{O}]S^{-1}$ has r rows.

(d) Let $T^{-1} = [\mathbf{t}_1 \quad \dots \quad \mathbf{t}_m]$ and $S^{-1} = \begin{bmatrix} \mathbf{s}_1^t \\ \vdots \\ \mathbf{s}_n^t \end{bmatrix}$, where $\mathbf{t}_1, \dots, \mathbf{t}_m$ are the columns of T^{-1} and $\mathbf{s}_1^t, \dots, \mathbf{s}_m^t$ are the rows of S^{-1}. Then we see that

$$P = T^{-1}\left[\begin{array}{c} I_r \\ \hline \mathbf{O} \end{array}\right] = [\mathbf{t}_1 \quad \dots \quad \mathbf{t}_r] \text{ and } Q = [I_r \mid \mathbf{O}]S^{-1} = \begin{bmatrix} \mathbf{s}_1^t \\ \vdots \\ \mathbf{s}_r^t \end{bmatrix}.$$

Therefore

$$A = PQ = [\mathbf{t}_1 \quad \dots \quad \mathbf{t}_r]\begin{bmatrix} \mathbf{s}_1^t \\ \vdots \\ \mathbf{s}_r^t \end{bmatrix} = \mathbf{t}_1\mathbf{s}_1^t + \dots + \mathbf{t}_r\mathbf{s}_r^t.$$

Note that, being columns/rows of invertible matrices, $\mathbf{t}_1, \dots, \mathbf{t}_r, \mathbf{s}_1, \dots, \mathbf{s}_r$ are all non-zero.
Further, $\text{row}(\mathbf{t}_i\mathbf{s}_i^t) = \text{span}(\mathbf{s}_i)$ and so $\text{rank}(\mathbf{t}_i\mathbf{s}_i^t) = 1$.
Thus $A = \mathbf{t}_1\mathbf{s}_1^t + \dots + \mathbf{t}_r\mathbf{s}_r^t$ is a sum of r rank one matrices.

(e) Follows from **part (d)**.

(f) Let $\text{rank}(B) = k$. Then we have $A = \mathbf{t}_1\mathbf{s}_1^t + \dots + \mathbf{t}_r\mathbf{s}_r^t$ and $B = \mathbf{a}_1\mathbf{b}_1^t + \dots + \mathbf{a}_k\mathbf{b}_k^t$ for some $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{R}^m, \mathbf{b}_1, \dots, \mathbf{b}_k \in \mathbb{R}^n$. We have

$$A + B = \mathbf{t}_1\mathbf{s}_1^t + \dots + \mathbf{t}_r\mathbf{s}_r^t + \mathbf{a}_1\mathbf{b}_1^t + \dots + \mathbf{a}_k\mathbf{b}_k^t = [\mathbf{t}_1 \quad \dots \quad \mathbf{t}_r \quad \mathbf{a}_1 \quad \dots \quad \mathbf{a}_k]\begin{bmatrix} \mathbf{s}_1^t \\ \vdots \\ \mathbf{s}_r^t \\ \mathbf{b}_1^t \\ \vdots \\ \mathbf{b}_k^t \end{bmatrix}.$$

Now

$$\begin{aligned} \text{rank}(A + B) = \text{rank}([\mathbf{t}_1 \quad \dots \quad \mathbf{t}_r \quad \mathbf{a}_1 \quad \dots \quad \mathbf{a}_k]\begin{bmatrix} \mathbf{s}_1^t \\ \vdots \\ \mathbf{s}_r^t \\ \mathbf{b}_1^t \\ \vdots \\ \mathbf{b}_k^t \end{bmatrix}) &\leq \text{rank}([\mathbf{t}_1 \quad \dots \quad \mathbf{t}_r \quad \mathbf{a}_1 \quad \dots \quad \mathbf{a}_k]) \\ &\leq r + k \\ &= \text{rank}(A) + \text{rank}(B). \end{aligned}$$

□

4. Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 6 & 4 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. Determine all $\mathbf{b} \in \mathbb{R}^3$ for which the system $A\mathbf{x} = \mathbf{b}$ is consistent.

Solution: We have, $\{\mathbf{b} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{b} \text{ is consistent}\} = \text{col}(A)$. Thus for every $\mathbf{b} \in \text{col}(A)$, the system $A\mathbf{x} = \mathbf{b}$ is consistent. □

5. Let A be an $n \times n$ matrix. Find the eigenvalues of $A - 3I$ in terms of the eigenvalues of A. Also, show that their corresponding eigenspaces are equal.

Solution: From $|A - \lambda I| = |(A - 3I) - (\lambda - 3)I|$, we find that λ is an eigenvalue of A iff $\lambda - 3$ is an eigenvalue of $A - 3I$.

Let $\mathbf{x}$ be an eigenvector of A with corresponding eigenvalue λ. Then we have

$$(A - 3I)\mathbf{x} = A\mathbf{x} - 3\mathbf{x} = (\lambda - 3)\mathbf{x}.$$

Again if $\mathbf{y}$ is an eigenvector of $A - 3I$ with corresponding eigenvalue $\lambda - 3$, then we have

$$A\mathbf{y} = (A - 3I)\mathbf{y} + 3\mathbf{y} = (\lambda - 3)\mathbf{y} + 3\mathbf{y} = \lambda\mathbf{y}.$$

From the above, we have seen that, if E_λ is the eigenspace of A corresponding to λ and if $E_{\lambda-3}$ is the eigenspace of $A - 3I$ corresponding to $\lambda - 3$ then $E_\lambda = E_{\lambda-3}$. $\square$

6. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let $k \in \mathbb{R}$. Suppose that $\sum\limits_{j=1}^{n} a_{ij} = k$ for $i = 1, 2, \ldots, n$. Prove that k is an eigenvalue of A. Also, find an eigenvector of A corresponding to the eigenvalue k.

   Solution: Let $\mathbf{u} = [1, 1, 1, \ldots, 1]^t \in \mathbb{R}^n$. Then it is easy to see that $A\mathbf{u} = k\mathbf{u}$. Hence k is an eigenvalue of A and $\mathbf{u}$ is an eigenvector corresponding to k. $\square$

7. Find all real values of a, b, c, d, e, f for which the matrix $\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.

   Solution: Let the given matrix be A. Since all the eigenvalues of A are 1, if A is diagonalizable then there exists an invertible matrix T such that $T^{-1}AT = I$, which gives that $A = I$. Hence we must have $a = b = c = d = e = f = 0$ for the diagonalizability of A. $\square$

8. Let A be a 6×6 matrix with characteristic polynomial $p(\lambda) = (1 + \lambda)(1 - \lambda)^2(2 - \lambda)^3$.

   (a) Prove that it is not possible to find three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^6$ such that $A\mathbf{v}_1 = \mathbf{v}_1, A\mathbf{v}_2 = \mathbf{v}_2$ and $A\mathbf{v}_3 = \mathbf{v}_3$.

   (b) If A is diagonalizable, find the dimensions of the eigenspaces E_{-1}, E_1 and E_2?

   Solution:

   (a) Since the characteristic polynomial of A is $p(\lambda) = (1 + \lambda)(1 - \lambda)^2(2 - \lambda)^3$, 1 is an eigenvalue of A and the algebraic multiplicity of 1 is 2. Thus the geometric multiplicity of 1 (dimension of E_1) can be at most 2. Therefore there cannot be three linearly independent vectors in E_1. Hence there do not exist three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^6$ such that $A\mathbf{v}_1 = \mathbf{v}_1, A\mathbf{v}_2 = \mathbf{v}_2$ and $A\mathbf{v}_3 = \mathbf{v}_3$.

   (b) If A is diagonalizable, then the geometric multiplicities of each eigenvalues of A will be equal to the corresponding algebraic multiplicities. Hence $\dim E_{-1} = 1, \dim E_1 = 2$ and $\dim E_2 = 3$. $\square$

9. Let A and B be two $n \times n$ matrices satisfying $AB = BA$ and let B have n distinct eigenvalues. Show that the matrix A is diagonalizable.

   Solution: Since B is an $n \times n$ matrix with n distinct eigenvalues, B must be diagonalizable. Let Q be an invertible matrix such that $QBQ^{-1} = D$, where D is a diagonal matrix. Then using $AB = BA$, we have

$$(QAQ^{-1})(QBQ^{-1}) = (QBQ^{-1})(QAQ^{-1}) \Rightarrow YD = DY, \text{ where } Y = QAQ^{-1}.$$

Let $Y = [y_{ij}]$ and let the diagonal entries of D be $d_1, d_2, \ldots, d_n$. Since the diagonal entries of D are the n distinct eigenvalues of B, we find that $d_i \neq d_j$ for $i \neq j$. Now for $i \neq j$, we have

$$DY - YD = \mathbf{O} \Rightarrow (DY - YD)_{ij} = 0 \Rightarrow (d_i - d_j)y_{ij} = 0, 1 \leq i, j \leq n \Rightarrow y_{ij} = 0, \text{ since } i \neq j.$$

Thus Y is a diagonal matrix. Hence $Y = QAQ^{-1}$ implies that the matrix A is also diagonalizable. $\square$
