**PH 101: Physics I**

Mid-Semester Examination (30 Marks)

Duration: 2 Hrs | Date: 23 Sept 2014

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*Answer all questions. Write sub-parts of a question together.*

## Section 1: Short Questions [6×2 Marks]

1. The polar coordinates of a traveling particle at time $t$ are given by $(r(t), \theta(t))$. Show that the velocity vector is $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$.

   *Since* $\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, $\frac{d}{dt}(\hat{\mathbf{r}}) = \dot{\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \dot{\theta}\hat{\theta}$. *Let the position vector of the particle be* $\mathbf{r} = r\hat{\mathbf{r}}$. *Then*

$$\frac{d}{dt}\mathbf{r} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d}{dt}\hat{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$$

2. Show that the force field, given by, $\vec{F} = (4x^3y^2 + y)\hat{\mathbf{i}} + (2x^4y + x)\hat{\mathbf{j}}$ is conservative and obtain the potential energy function.

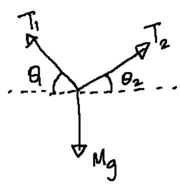
   *Given* $F_x = 4x^3y^2 + y$ *and* $F_y = 2x^4y + x$. *Then*

$$\frac{\partial F_x}{\partial y} = 8x^3y + 1 \quad \textit{and} \quad \frac{\partial F_y}{\partial x} = 8x^3y + 1$$

   *Since these expressions are equal, The force is conservative. The potential energy is given by*

$$\begin{aligned} U(x,y) &= -\int F_x(x,y)dx - \int F_y(0,y)dy \\ &= -x^4y^2 - xy \end{aligned}$$

3. A mass $M$ is hung using two strings as shown in figure. What is the tension in each string?




   *The free body diagram is shown in the figure. Clearly*

$$\begin{aligned} T_1\cos\theta_1 - T_2\cos\theta_2 &= 0 \\ T_1\sin\theta_1 + T_2\sin\theta_2 &= Mg \end{aligned}$$

   *Solving for* $T_1$ *and* $T_2$, *we get*

$$T_1 = \frac{Mg\cos\theta_2}{\sin(\theta_1+\theta_2)} \quad \textit{and} \quad T_2 = \frac{Mg\cos\theta_1}{\sin(\theta_1+\theta_2)}$$

4. Sand drops vertically at a constant rate $\sigma$ Kg/s (ignore the height) onto a moving conveyor belt. What force must be applied to the belt in order to keep it moving at a constant speed $v$? How much kinetic energy does the sand gain per unit time?

   *At time* $t$, *there are* $m$ *kgs of sand on the belt. Momentum of the sand will be* $p(t) = mv$. *In next time interval* $\Delta t$, *another* $\sigma\Delta t$ *kg of sand is added to the system such that the momentum* $p(t+\Delta t) = (m+\sigma\Delta t)v$. *The force needed to keep the system moving with same speed would be*

$$F = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(p(t+\Delta t) - p(t)\right) = \sigma v.$$

   *And gain in KE per unit time will be*

$$K = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\frac{1}{2}(m+\sigma\Delta t)v^2 - \frac{1}{2}mv^2\right) = \frac{1}{2}\sigma v^2.$$

5. Find the moment of inertia of a non-uniform semi-circular disk of radius $R$ about its straight edge (X axis). The mass density of the disk is $\sigma(r,\theta) = \sigma_0 r$.

*The moment of inertia about X axis is*

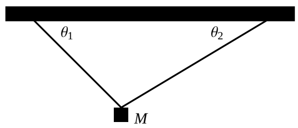
$$\begin{aligned} I_{xx} &= \int y^2 dm = \int_0^R \int_0^\pi r^2 \sin^2\theta \,(\sigma_0 r)\,(r dr d\theta) \\ &= \sigma_0 \frac{R^5}{5}\frac{\pi}{2} \end{aligned}$$

6. A 500-ton train is traveling to the east (X axis) with speed 300 km/h at latitude $60^\circ$ north. Find the magnitude and direction of the coriolis force. (Z axis is pointing upwards).

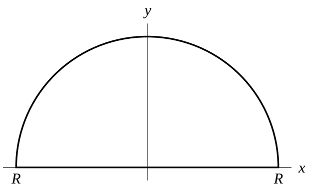
*The earth angular velocity vector is given by* $\vec{\Omega} = \Omega(\cos 60\mathbf{j} + \sin 60\mathbf{k})$ *where* $\Omega = 7.27\times 10^{-5}$ *rad/s and the velocity vector is given by* $\vec{v} = v\mathbf{i}$ *where* $v = 83.3$ *m/s. The coriolis force is*

$$\vec{F}_c = -2m\vec{\Omega}\times\vec{v} = -m\Omega v\left(\mathbf{j} + \sqrt{3}\mathbf{k}\right)\times\mathbf{i} = m\Omega v\left(\mathbf{k} - \sqrt{3}\mathbf{j}\right) = 303\left(\mathbf{k} - \sqrt{3}\mathbf{j}\right)$$

*The magnitude is 606 N. and direction is 30 deg to up towards south.*



(a) Question 3



(b) Question 5

## Section 2: Long Questions [3×6 Marks]

1. A particle is traveling on a trajectory given by $\vec{\mathbf{r}}(t) = a\cos(\omega t)\,\hat{\mathbf{i}} + b\sin(\omega t)\,\hat{\mathbf{j}}$ where $a$, $b$, and $\omega$ are positive constants.

   (a) Find the equation of the orbit of the particle.

   (b) Find the net force on the particle.

   (c) What is the net work done on the particle during $t = 0$ to $t = \pi/2\omega$?

   *Solution:*

   (a) *Since* $x(t) = a\cos\omega t$ *and* $y(t) = b\cos\omega t$,

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   *This is an elliptical orbit.*

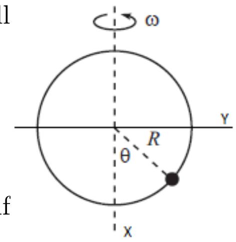
   (b) *Net force is*

   $$\vec{F} = m\ddot{\mathbf{r}} = -m\omega^2\left(a\cos(\omega t)\,\mathbf{i} + b\sin(\omega t)\,\mathbf{j}\right) = -m\omega^2\mathbf{r}$$

   (c) *Here* $d\mathbf{r} = \left(-a\omega\sin(\omega t)\,\mathbf{i} + b\omega\cos(\omega t)\,\mathbf{j}\right)dt$. *Then*

   $$\begin{aligned} W &= \int_0^{2\pi/\omega} \vec{F}\cdot d\mathbf{r} = -m\omega^3 \int_0^{2\pi/\omega} \left(a\cos(\omega t)\,\mathbf{i} + b\sin(\omega t)\,\mathbf{j}\right)\cdot\left(-a\sin(\omega t)\,\mathbf{i} + b\cos(\omega t)\,\mathbf{j}\right)dt \\ &= -m\omega^3\left(b^2 - a^2\right)\int_0^{2\pi/\omega} \cos(\omega t)\sin(\omega t)\,dt \\ &= -m\omega^3\left(b^2 - a^2\right)\frac{1}{2\omega} \end{aligned}$$

2. A bead of mass $m$ is free to slide along a frictionless hoop of radius $R$. The hoop rotates with constant angular speed $\omega$ around a vertical diameter (see Fig). Do all calculations in frame of reference of hoop.



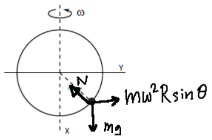

   (a) Draw free body diagram.

   (b) Write equations of motion in polar coordinates.

   (c) What are the equilibrium points if $\omega^2 R > g$? What are the equilibrium points if $\omega^2 R < g$?

   (d) For $\omega^2 R > g$, find the frequency of small oscillations about equilibrium point which is not on the axis of rotation of hoop.

   *Solution*

(a) *Free Body Diagram* 


(b) *In hoop's frame, bead is constrained to the circle of radius R. Equations are motion for the bead are*

$$-mR\dot{\theta}^2 = -N + m\omega^2 R\sin^2\theta + mg\cos\theta \tag{1}$$

*and*

$$\begin{aligned} mR\ddot{\theta} &= m\omega^2 R\sin\theta\cos\theta - mg\sin\theta \\ &= m\sin\theta\left(\omega^2 R\cos\theta - g\right) \end{aligned} \tag{2}$$

(c) *Let $\omega^2 R > g$. For equilibrium:*

$$\begin{aligned} mR\ddot{\theta} &= 0 \\ \implies m\sin\theta\left(\omega^2 R\cos\theta - g\right) &= 0 \end{aligned}$$

*That is, when $\theta = 0, \pi$ or $\theta_0$, where*

$$\cos\theta_0 = \frac{g}{\omega^2 R}.$$

*When $\omega^2 R < g$, then there are only two equilibrium points $0$ or $\pi$.*

(d) *For $\omega^2 R > g$, about $\theta_0$: Let $\theta = \theta_0 + \phi$ where $\phi$ is a small angle.*

$$\begin{aligned} mR\ddot{\theta} &= m\left(\sin\theta_0 + \phi\cos\theta_0\right)\left(\omega^2 R\left(\cos\theta_0 - \phi\sin\theta_0\right) - g\right) \\ &= m\left(\sin\theta_0 + \phi\cos\theta_0\right)\left(-\omega^2 R\phi\sin\theta_0\right) \\ &= -m\omega^2 R\sin^2\theta_0\phi \\ \implies \ddot{\phi} &= -(\omega^2\sin^2\theta_0)\phi \end{aligned}$$

*Frequency of oscillation is*

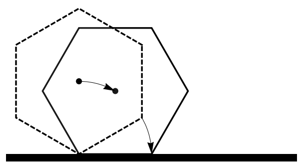
$$\omega\sin\theta_0 = \sqrt{\omega^2 - \frac{g^2}{\omega^2 R^2}}$$

3. A hexagonal pencil with side length $a$, is kept on a plane. Now it is disturbed from rest and begins uneven rolling over the edges. Assume that it rolls only over the edges, does not loose contact with plane and that there is no sliding. Also assume that the normal reaction and friction due to the table act only at the edges of the pencil. The moment of inertia about the center of the pencil $I_0 = \frac{5}{12}Ma^2$ where $M$ is the mass of the pencil.

   (a) Let the angular speed just before the given edge hits the plane is $\omega_i$ and just after is $\omega_f$. Write $\omega_f$ in terms of $\omega_i$.

   (b) Let $K_i$ and $K_f$ be kinetic energies just before and just after the impact, respectively. Write $K_f$ in terms of $K_i$.

   (c) What is the minimum value of $K_i$, such that the pencil will roll over to the next edge.



*Solution*

(a) *Let O be the center of the pencil. Assume that first the pencil is rolling on edge A and is about to hit the ground at edge B. Just before impact the velocity of center of mass (O point) makes an angle of $\pi/2$ with AO and $\pi/6$ with BO. The angular momentum about B is*

$$\begin{aligned} L_{before} &= M\left|\vec{v}_{cm} \times \vec{BO}\right| + I_0\omega_i \\ &= Ma^2\omega_i\sin\frac{\pi}{6} + \frac{5}{12}Ma^2\omega_i = \frac{11}{12}Ma^2\omega_i \end{aligned}$$

*After the impact there is pure rotation about B edge, thus $L_{after} = I_B\omega_f$, and $I_B = \frac{17}{12}Ma^2$ by parallel axis theorem. Angular momentum is conserved during the impact. Thus*

$$\omega_f = \frac{11}{17}\omega_i.$$

(b) *In the same way*

$$K_{before} = \frac{1}{2}Ma^2\omega_i^2 + \frac{1}{2}I_0\omega_i^2 = \frac{1}{2}I_B\omega_i^2$$

*And*

$$K_{after} = \frac{1}{2}I_B\omega_f^2$$

*Thus,*

$$K_{after} = K_{before}\left(\frac{\omega_f}{\omega_i}\right)^2 = \frac{121}{289}K_{before}$$

(c) *To continue rolling, the KE after the bounce must be at least* $Mga\left(1-\sqrt{3}/2\right)$, *that is*

$$\begin{aligned} K_{after} &= Mga\left(1-\sqrt{3}/2\right) \\ K_{before} &= \frac{289}{121}\left(1-\sqrt{3}/2\right)Mga \end{aligned}$$