# Plan

- Vector Space
- Subspace
- Linear Dependence and Linear Independence
- Basis and Dimension
- Linear Transformation
- Kernel and Range
- The Rank-Nullity Theorem
- Isomorphism
- The Matrix of a Linear Transformation

# Vector Space

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The above properties are sufficient to do vector algebra in $\mathbb{R}^n$.

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- If $p(x), q(x), r(x), \mathbf{0} \in \mathbb{R}_2[x]$ (set of all polynomials of degree at most two with real coefficients) and $c, d \in \mathbb{R}$, we get all the previous ten properties.

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*Let $\mathbb{R}_2[x] = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}$. For $p(x) = a_0 + b_0x + c_0x^2, \quad q(x) = a_1 + b_1x + c_1x^2 \in \mathbb{R}_2[x]$ and $k \in \mathbb{R}$, define*

$$p(x) + q(x) = (a_0 + a_1) + (b_0 + b_1)x + (c_0 + c_1)x^2$$

$$k.p(x) = (ka_0) + (kb_0)x + (kc_0)x^2.$$

*Then $\mathbb{R}_2[x]$ is a vector space over $\mathbb{R}$.*

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- If there is no confusion, $c.\mathbf{u}$ is simply written as $c\mathbf{u}$.

- We write $V(\mathbb{F})$ to denote that $V$ is a vector space over $\mathbb{F}$.
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**Result**

*Let $V$ be a vector space over $\mathbb{F}$. Let $\mathbf{u} \in V$ and $c \in \mathbb{F}$. Then*

1. $0.\mathbf{u} = \mathbf{0}$;
2. $c.\mathbf{0} = \mathbf{0}$;
3. $(-1).\mathbf{u} = -\mathbf{u}$; *and*
4. *If $c.\mathbf{u} = \mathbf{0}$ then either $c = 0$ or $\mathbf{u} = \mathbf{0}$.*

Subspace: Let $V$ be a vector space and $(\emptyset \neq) W \subseteq V$. Then $W$ is called a subspace of $V$ if and only if $a\mathbf{u} + b\mathbf{v} \in W$ for every $\mathbf{u}, \mathbf{v} \in W$ and for every $a, b \in \mathbb{F}$.

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*Let $W = \{[x, y, z]^t \in \mathbb{R}^3 : x + y - z = 0\}$. Then $S$ is a subspace of $\mathbb{R}^3$.*

**Spanning Set:** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a vector space $V(\mathbb{F})$. Then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, and is denoted by $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $\text{span}(S)$. That is,

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## Result

*Let $S$ be a subset of a vector space $V(\mathbb{F})$. Then span$(S)$ is a subspace of $V$.*

**Linear Dependence:** A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space $V(\mathbb{F})$ is said to be linearly dependent (LD) if there are scalars $c_1, c_2, \ldots, c_k$, at least one of them non-zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$.

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An infinite set $S \subseteq V$ is linearly dependent if there is some finite linearly dependent subset of $S$.

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- If $\mathbf{0} \in S$, then $S$ is always linearly dependent as $S$ contains a LD set $\{\mathbf{0}\}$.

## Example

*The set* $\{A, B, C\}$ *is linearly dependent in* $M_2(\mathbb{R})$*, where*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

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## Result

*The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space $V$ are linearly dependent iff either $\mathbf{v}_1 = \mathbf{0}$ or there is an integer $r$ such that $\mathbf{v}_r$ can be expressed as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}$.*

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$\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ *is a basis for* $M_2(\mathbb{R})$, *where*

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$B = \{1 + x, x + x^2, 1 + x^2\}$ *is a basis for* $\mathbb{R}_2[x]$.

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- $\{1+x, x+x^2, 1+x^2\}$ is LI iff $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is LI.

Coordinate: Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered bssis for a vector space $V(\mathbb{F})$ and let $\mathbf{v} \in V$. Let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$. Then the scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ with respect to $B$, and the column vector

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is called the coordinate vector of **v** with respect to $B$.

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## Example

*The coordinate vector $[p(x)]_B$ of $p(x) = 1 - 3x + 4x^2$ with respect to basis $B = \{1, x, x^2\}$ of $\mathbb{R}_2[x]$ is $[p(x)]_B = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$.*

## Result

*Let* $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *be an ordered bssis for a vector space* $V$*, let* $\mathbf{u}, \mathbf{v} \in V$ *and let* $c \in \mathbb{F}$*. Then*

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1. *Any set of more than $n$ vectors in $V$ must be linearly dependent.*
2. *Any set of fewer than $n$ vectors in $V$ cannot span $V$.*

**Result (The Basis Theorem)**

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**Example**

*$\dim(\mathbb{R}^n) = n$, $\dim \mathbb{C}(\mathbb{C}) = 1$, $\dim \mathbb{C}(\mathbb{R}) = 2$, $\dim M_2(\mathbb{R}) = 4$ and $\dim \mathbb{R}_n[x] = n + 1$.*

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Change of Basis: Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $C = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be bases for a vector space $V$. The $n \times n$ matrix whose columns are the coordinate vectors $[\mathbf{u}_1]_C, [\mathbf{u}_2]_C, \ldots, [\mathbf{u}_n]_C$ is denoted by $P_{C \leftarrow B}$, and is called the change of basis matrix from $B$ to $C$. That is,

$$P_{C \leftarrow B} = [[\mathbf{u}_1]_C, [\mathbf{u}_2]_C, \ldots, [\mathbf{u}_n]_C].$$

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## Example

*Find the change of basis matrix* $P_{C \leftarrow B}$ *and* $P_{B \leftarrow C}$ *for the bases* $B = \{1, x, x^2\}$ *and* $C = \{1 + x, x + x^2, 1 + x^2\}$ *of* $\mathbb{R}_2[x]$*. Then find the coordinate vector of* $p(x) = 1 + 2x - x^2$ *with respect to the basis* $C$*.*

# Linear Transformations

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or, equivalently, $F(\alpha\mathbf{u}+\mathbf{v}) = \alpha F(\mathbf{u}) + F(\mathbf{v})$. Such functions are called linear transformations (LT).

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**Definition**

A linear transformation from a vector space $V$ into a vector space $W$ is a mapping $T : V \to W$ such that for all $\mathbf{u}, \mathbf{v} \in V$ and for all $a \in \mathbb{F}$

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## Example

*Let $A$ be an $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then $T$ is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$.*

## Example

*The map* $T : \mathbb{R}^2 \to \mathbb{R}^2$*, defined by* $T([x, y]^t) = [2x, x + y]^t$ *for all* $[x, y]^t \in \mathbb{R}^2$*, is a linear transformation.*

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*Let* $V$ *be a vector space. The map* $I : V \to V$*, defined by* $I(\mathbf{v}) = \mathbf{v}$ *for all* $\mathbf{v} \in V$*, is a linear transformation. The map* $I$ *is called the identity transformation.*

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*The map* $T : \mathbb{R} \to \mathbb{R}$*, defined by* $T(x) = x + 1$ *for all* $x \in \mathbb{R}$*, is* *not* *a linear transformation.*

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## Result

*Let $T : V \to W$ be a linear transformation. Then*

1. $T(\mathbf{0}) = \mathbf{0}$;
2. $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v} \in V$; *and*
3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ *for all* $\mathbf{u}, \mathbf{v} \in V$.

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## Example

*Suppose* $T : \mathbb{R}^2 \to \mathbb{R}_2[x]$ *is a linear transformation such that* $T[1, 0]^t = 2 - 3x + x^2$ *and* $T[0, 1]^t = 1 - x^2$*. Find* $T[2, 3]^t$ *and* $T[a, b]^t$.

# Composition of Linear Transformation

Let $T : U \to V$ and $S : V \to W$ be two linear transformations. The composition of $S$ with $T$ is the mapping $S \circ T : U \to W$ defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u})) \quad \text{for all } \mathbf{u} \in U.$$

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**Result**

*Let $T : U \to V$ and $S : V \to W$ be two linear transformations. Then the composition $S \circ T$ is also a linear transformation.*

Inverse of a function: A function $f : X \to Y$ is said to be invertible if there is another function $g : Y \to X$ such that

$$g \circ f = I_X \quad \text{and} \quad f \circ g = I_Y.$$

- If $f$ is invertible, the the function $g$ satisfying $g \circ f = I_X, f \circ g = I_Y$ is called inverse of $f$.

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- Inverse of a linear transformation is linear.

### Example

*Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$T[x, y]^t = [x - y, -3x + 4y]^t \;\; \textit{and} \;\; S[x, y]^t = [4x + y, 3x + y]^t$$

*for all $[x, y]^t \in \mathbb{R}^2$. Then $S$ is the inverse of $T$.*

**Kernel and Range:** Let $T : V \to W$ be a linear transformation. Then the kernel of $T$ (null space of $T$), denoted $\ker(T)$, and the range of $T$, denoted $\mathrm{range}(T)$, are defined as

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}, \text{ and}$$

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## Result

*Let $T : V \to W$ be a linear transformation and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a spanning set for $V$. Then $T(B) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ spans the range of $T$.*

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## Example

*Let $A$ be an $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then $\ker(T) = \text{null}(A)$ and $\text{range}(T) = \text{col}(A)$.*

**Result**

*Let* $T : V \to W$ *be a linear transformation. Then* $ker(T)$ *is a subspace of* $V$ *and* $range(T)$ *is a subspace of* $W$*.*

## Result

*Let $T : V \longrightarrow W$ be a linear transformation. Then ker$(T)$ is a subspace of $V$ and range$(T)$ is a subspace of $W$.*

## Definition

Let $T : V \longrightarrow W$ be a linear transformation. We define

- rank$(T) =$ dimension of range$(T)$; and
- nullity$(T) =$ dimension of ker$(T)$.

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### Definition

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### Example

*Let $D : \mathbb{R}_3[x] \to \mathbb{R}_2[x]$ be defined by $D(p(x)) = \frac{d}{dx} p(x)$. Then rank$(D) = 3$ and nullity$(D) = 1$.*

Result (**The Rank-Nullity Theorem**)

*Let $T : V \to W$ be a linear transformation from a finite dimensional vector space $V$ into a vector space $W$. Then*

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**Definition**

Let $T : V \to W$ be a linear transformation. Then

1. $T$ is called one-one if $T$ maps distinct vectors in $V$ into distinct vectors in $W$.
2. $T$ is called onto if $\text{range}(T) = W$.

Let $T : V \to W$ be a linear transformation.

- For all $\mathbf{u}, \mathbf{v} \in V$, if $\mathbf{u} \neq \mathbf{v}$ implies that $T(\mathbf{u}) \neq T(\mathbf{v})$, then $T$ is one-one.

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- For all $\mathbf{w} \in W$, if there is at least one $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$, then $T$ is onto.

## Example

- $T : \mathbb{R} \to \mathbb{R}^2$ *defined by* $T(x) = [x, 0]^t$ *for* $x \in \mathbb{R}$ *is one-one but not onto.*

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*A linear transformation* $T:V\to W$ *is one-one iff* $ker(T)=\{\mathbf{0}\}$.

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## Isomorphism:

- A linear transformation $T : V \to W$ is called an isomorphism if it is one-one and onto.
- If $T : V \to W$ is an isomorphism then we say that $V$ and $W$ are isomorphic, and we write $V \cong W$.

**Example**

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*The vector spaces* $\mathbb{R}^n$ *and* $\mathbb{R}_n[x]$ *are not isomorphic.*

# The Matrix of a Linear Transformation

**Result**

*Let $V$ and $W$ be two vector spaces with bases $B$ and $C$ respectively, where $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $dim(W) = m$. If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix $A$ defined by*

$$A = [[T(\mathbf{v}_1)]_C, [T(\mathbf{v}_2)]_C, \ldots, [T(\mathbf{v}_n)]_C]$$

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- The above matrix $A$ is called the matrix of $T$ with respect to the bases $B$ and $C$.
- The matrix $A$ is also written as $[T]_{C \leftarrow B}$.
- If $B = C$, then $[T]_{C \leftarrow B}$ is written as $[T]_B$.

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## Example

*Let* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *be defined by*

$$T([x, y, z]^t) = [x - 2y, x + y - 3z]^t \quad \text{for } [x, y, z]^t \in \mathbb{R}^3.$$

*Let* $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *and* $C = \{\mathbf{e}_2, \mathbf{e}_1\}$ *be bases for* $\mathbb{R}^3$ *and* $\mathbb{R}^2$, *respectively. Find* $[T]_{C \leftarrow B}$ *and verify the previous result for* $\mathbf{v} = [1, 3, -2]^t$.

## Result

*Let $U, V$ and $W$ be three vector spaces with bases $B, C$ and $D$, respectively. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then*

$$[S \circ T]_{D \leftarrow B} = [S]_{D \leftarrow C}[T]_{C \leftarrow B}.$$

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*Let $T : V \to W$ be a linear transformation between two $n$-dimensional vector spaces $V$ and $W$ with bases $B$ and $C$, respectively. Then $T$ is invertible if and only if the matrix $[T]_{C\leftarrow B}$ is invertible. In this case,*

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## Example

*Let $T : \mathbb{R}^2 \to \mathbb{R}_1[x]$ be defined by $T([a, b]^t) = a + (a + b)x$ for $[a, b]^t \in \mathbb{R}^2$. Show that $T$ is invertible, and hence find $T^{-1}$.*