DEPARTMENT OF MATHEMATICS, IIT Guwahati

MA101: Mathematics I, July - November 2014

Solutions of Tutorial Sheet: LA - 2

1. Examine whether the following sets are subspaces of $\mathbb{R}^n$ $(n \geq 3)$.

$$S_1 = \{[x_1, \ldots, x_n]^t \in \mathbb{R}^n : x_1 + x_2 = 4x_3\}, \quad S_2 = \{[x_1, \ldots, x_n]^t \in \mathbb{R}^n : x_1 + x_2 \leq 4x_3\},$$

$$S_3 = \{[x_1, \ldots, x_n]^t \in \mathbb{R}^n : x_1 = 1 + x_2\} \text{ and } S_4 = \{[x_1, \ldots, x_n]^t \in \mathbb{R}^n : x_1x_2 = 0\}.$$

Remark: If a set S is a subspace then show that the rules

$$\textbf{Rule 1}: \quad \mathbf{0} \in S \quad (\text{to claim that } S \neq \emptyset);$$
$$\textbf{Rule 2}: \quad \alpha\mathbf{u} + \beta\mathbf{v} \in S \quad \text{for every } \mathbf{u}, \mathbf{v} \in S \text{ and } \alpha, \beta \in \mathbb{R}.$$

are satisfied. If S is **not** a subspace then show that either $\mathbf{0} \notin S$ **or** show, *by a counter-example*, that **Rule 2** is not satisfied.

Solution: Clearly $\mathbf{0} \in S_1$ and hence $S_1 \neq \emptyset$. (**This is an important step.**) Let $\mathbf{u} = [x_1, \ldots, x_n]^t$, $\mathbf{v} = [y_1, \ldots, y_n]^t \in S_1$ and $a, b \in \mathbb{R}$. Then we have $x_1 + x_2 = 4x_3$ and $y_1 + y_2 = 4y_3$. Now

$$a\mathbf{u} + b\mathbf{v} = [ax_1 + by_1, ax_2 + by_2, ax_3 + by_3, \ldots, ax_n + by_n]^t.$$

We have $(ax_1 + by_1) + (ax_2 + by_2) = a(x_1 + x_2) + b(y_1 + y_2) = 4(ax_3 + by_3)$. Hence $a\mathbf{u} + b\mathbf{v} \in S_1$ for any $\mathbf{u}, \mathbf{v} \in S_1$ and $a, b \in \mathbb{R}$, and therefore S_1 is a subspace of $\mathbb{R}^n$.

We have $\mathbf{u} = [1, 1, 1, \ldots, 1]^t \in S_2$ but $-\mathbf{u} \notin S_2$ since $-1 - 1 > -4$. Hence S_2 is not a subspace of $\mathbb{R}^n$.

We see that $\mathbf{0} \notin S_3$. Hence S_3 is not a subspace of $\mathbb{R}^n$.

We see that $\mathbf{e}_1 = [1, 0, 0, \ldots, 0]^t, \mathbf{e}_2 = [0, 1, 0, \ldots, 0]^t \in S_4$. However, $1\mathbf{e}_1 + 1\mathbf{e}_2 = \mathbf{e}_1 + \mathbf{e}_2 \notin S_4$. Hence S_4 is not a subspace of $\mathbb{R}^n$. □

2. Find all the subspaces of $\mathbb{R}^2$.

Solution: Clearly, $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^2$. Let S $(\neq \{\mathbf{0}\})$ be another subspace of $\mathbb{R}^2$. Then there exists $\mathbf{u} = [u_1, u_2]^t \in S$ such that $\mathbf{u} \neq \mathbf{0}$. Notice that $\{\alpha\mathbf{u} : \alpha \in \mathbb{R}\} = \text{span}(\mathbf{u}) \subseteq S$. Now either $\text{span}(\mathbf{u}) = S$ or $\text{span}(\mathbf{u}) \subsetneq S$. In the first case, S is the straight line in $\mathbb{R}^2$ passing through the origin and $\mathbf{u}$.

If $\text{span}(\mathbf{u}) \subsetneq S$, then there exists $\mathbf{v} = [v_1, v_2]^t \in S \setminus \text{span}(\mathbf{u})$, and therefore $\mathbf{v} \neq s\mathbf{u}$ for any $s \in \mathbb{R}$. Thus the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, that is, the rank of the matrix $A = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$ is 2. Therefore for any $\mathbf{b} = [b_1, b_2]^t \in \mathbb{R}^2$, we have $\text{rank}(A) = \text{rank}([A \mid \mathbf{b}])$, and hence the system $A\mathbf{x} = \mathbf{b}$ will have a solution $\mathbf{w} = [\alpha, \beta]^t$. Thus $A\mathbf{w} = \mathbf{b}$, which implies that $\mathbf{b} = \alpha\mathbf{u} + \beta\mathbf{v}$. Thus $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v}) \subseteq S$, and hence $\mathbb{R}^2 = S$.

Thus the subspaces of $\mathbb{R}^2$ are $\{\mathbf{0}\}, \mathbb{R}^2$ and the straight lines passing through the origin. □

3. Show that $W = \{[x, y, z, w]^t \in \mathbb{R}^4 : w - z = y - x\}$ is a subspace of $\mathbb{R}^4$, spanned by the vectors $[1, 0, 0, -1]^t, [0, 1, 0, 1]^t$ and $[0, 0, 1, 1]^t$. Also, determine dim W.

Solution: We have

$$\begin{aligned} W &= \{[x, y, z, w]^t \in \mathbb{R}^4 : w - z = y - x\} \\ &= \{[x, y, z, y - x + z]^t : x, y, z \in \mathbb{R}\} \\ &= \{x[1, 0, 0, -1]^t + y[0, 1, 0, 1]^t + z[0, 0, 1, 1]^t : x, y, z \in \mathbb{R}\} \\ &= \text{span}([1, 0, 0, -1]^t, [0, 1, 0, 1]^t, [0, 0, 1, 1]^t). \end{aligned}$$

Thus W is a subspace of $\mathbb{R}^4$, spanned by $[1, 0, 0, -1]^t, [0, 1, 0, 1]^t$ and $[0, 0, 1, 1]^t$.

Now $a[1, 0, 0, -1]^t + b[0, 1, 0, 1]^t + c[0, 0, 1, 1]^t = [0, 0, 0, 0]^t \Rightarrow a = b = c = 0$.
Thus the set $\{[1, 0, 0, -1]^t, [0, 1, 0, 1]^t, [0, 0, 1, 1]^t\}$ is linearly independent. Hence $\{[1, 0, 0, -1]^t, [0, 1, 0, 1]^t, [0, 0, 1, 1]^t\}$ is a basis for W and consequently dim $W = 3$. □

4. Find three vectors in $\mathbb{R}^3$ which are linearly dependent but any two of them are linearly independent.

Solution: The idea is to take three vectors corresponding to three distinct straight lines through origin where all of them lie on the same plane. For example, take $\mathbf{u} = [1, 0, 0]^t, \mathbf{v} = [0, 1, 0]^t$ and $\mathbf{w} = [1, 1, 0]^t$. □

5. Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, where $\mathbf{x}_1 = [1, 0, 0, 0]^t, \mathbf{x}_2 = [1, 1, 0, 0]^t, \mathbf{x}_3 = [1, 2, 0, 0]^t$ and $\mathbf{x}_4 = [1, 1, 1, 0]^t$. Determine all $\mathbf{x}_i \in S$ such that $\operatorname{span}(S) = \operatorname{span}(S \setminus \{\mathbf{x}_i\})$.

   Solution: It is easy to observe that $\mathbf{x}_1 = 2\mathbf{x}_2 - \mathbf{x}_3$. Thus $\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_3$ and $\mathbf{x}_3 = 2\mathbf{x}_2 - \mathbf{x}_1$. Hence $\operatorname{span}(S) = \operatorname{span}(S \setminus \{\mathbf{x}_i\})$ for $i = 1, 2, 3$.

   However, $\mathbf{x}_4 = a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3 \Rightarrow a + b + c = 1, b + 2c = 1, 0 = 1$, which is absurd.
   Hence $\operatorname{span}(S) \neq \operatorname{span}(S \setminus \{\mathbf{x}_4\})$. □

6. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be linearly dependent vectors in $\mathbb{R}^n$ and let $\mathbf{v}, \mathbf{w}, \mathbf{x}$ be linearly independent vectors in $\mathbb{R}^n$, where $n \geq 3$. Show that **(i)** $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$, and **(ii)** $\mathbf{x}$ is not a linear combination of $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.

   Solution: (i) Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent, there exist $a, b, c \in \mathbb{R}$, not all zero, such that $a\mathbf{u}+b\mathbf{v}+c\mathbf{w} = \mathbf{0}$. Now if $a = 0$, then we have $b\mathbf{v} + c\mathbf{w} = \mathbf{0}$, where $(b, c) \neq (0, 0)$. Therefore $b\mathbf{v} + c\mathbf{w} + 0\mathbf{x} = \mathbf{0}$, where $(b, c) \neq (0, 0)$. This implies that $\mathbf{v}, \mathbf{w}, \mathbf{x}$ are linearly dependent, a contradiction. Thus $a \neq 0$, and therefore $\mathbf{u} = \left(-\frac{b}{a}\right)\mathbf{v}+\left(-\frac{c}{a}\right)\mathbf{w}$.

   (ii) Let, if possible, $\mathbf{x} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. In **Part (i)**, we have got that $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Therefore $\mathbf{x} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ gives that $\mathbf{x}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$. This implies that the vectors $\mathbf{v}, \mathbf{w}, \mathbf{x}$ are linearly dependent, a contradiction. Hence $\mathbf{x}$ is not a linear combination of $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. □

7. Let $M = \{[x, y, z]^t \in \mathbb{R}^3 : x + y + 4z = 0\}$ and $N = \{[x, y, z]^t \in \mathbb{R}^3 : x + y + z = 0\}$. Show that M and N are subspaces of $\mathbb{R}^3$. Determine a basis for each of $M, N, M + N$ and $M \cap N$. Also, interpret your result geometrically.

   Solution: It is left to the students to show that M and N are subspaces of $\mathbb{R}^3$.

   For any $[x, y, z]^t \in M$, we have $x = -y - 4z$. Hence $[x, y, z]^t = [-y - 4z, y, z]^t = y[-1, 1, 0]^t + z[-4, 0, 1]^t$. Also, $\{[-1, 1, 0]^t, [-4, 0, 1]^t\}$ is linearly independent (**Check!**). Hence $\{[-1, 1, 0]^t, [-4, 0, 1]^t\}$ is a basis for M. Similarly, $\{[-1, 1, 0]^t, [-1, 0, 1]^t\}$ is a basis for N.

   Since $[-1, 0, 1]^t \notin M$, we find that $\{[-1, 0, 1]^t, [-1, 1, 0]^t, [-4, 0, 1]^t\}$ is a linearly independent subset of $M + N$. Moreover, $\operatorname{span}([-1, 0, 1]^t, [-1, 1, 0]^t, [-4, 0, 1]^t) = M + N$. Hence $\{[-1, 0, 1]^t, [-1, 1, 0]^t, [-4, 0, 1]^t\}$ is a basis for $M + N$.

   For any $[x, y, z]^t \in M \cap N$, we have $x + y + 4z = 0, x + y + z = 0$, that is, $x + y = 0, z = 0$. Hence $\{[-1, 1, 0]^t\}$ is a basis for $M \cap N$.

   Geometric Interpretation: The subspaces M and N represent two planes in $\mathbb{R}^3$ (which are not parallel), each passing through the origin. The subspace $M \cap N$ is the straight line in $\mathbb{R}^3$ passing through the origin and $[-1, 1, 0]^t$.

   The basis $\{[-1, 0, 1]^t, [-1, 1, 0]^t, [-4, 0, 1]^t\}$ is also a basis for $\mathbb{R}^3$. Hence $M + N = \mathbb{R}^3$. Thus any point in $\mathbb{R}^3$ can be expressed as a sum of two points, one lying on the plane M and the other lying on the plane N. □