MA101 Mathematics I

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Series

Plan

- Series and their convergence
- Convergence criteria
- Absolute convergence
- Test for absolute convergence
- Conditional convergence
- Test for conditional convergence

Convergence of series

- An infinite series in $\mathbb{R}$ is an expression $\sum_{n=1}^{\infty} x_n$, where (x_n) is a sequence in $\mathbb{R}$.

  More formally, it is an ordered pair $((x_n), (s_n))$, where (x_n) is a sequence in $\mathbb{R}$ and $s_n = x_1 + \cdots + x_n$ for all $n \in \mathbb{N}$.

- x_n : nth term of the series

  s_n : nth partial sum of the series

- Convergence of series: $\sum_{n=1}^{\infty} x_n$ is convergent if (s_n) is convergent.

  Otherwise $\sum_{n=1}^{\infty} x_n$ is divergent (not convergent).

Examples

- Sum of a convergent series: $\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n$
- Examples:
  1. The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ (where $a \neq 0$) converges iff $|r| < 1$.
  2. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.
  3. The series $1 - 1 + 1 - 1 + \cdots$ is not convergent.

  Ex. If $a, b \in \mathbb{R}$, show that the series $a + (a+b) + (a+2b) + \cdots$ is not convergent unless $a = b = 0$.

Convergence criteria

- Algebraic operations on series: Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent with sums x and y respectively.
  Then
  1. $\sum_{n=1}^{\infty} (x_n + y_n)$ is convergent with sum $x + y$
  2. $\sum_{n=1}^{\infty} \alpha x_n$ is convergent with sum αx, where $\alpha \in \mathbb{R}$
- Cauchy criterion and Monotone sequence criterion for series
- Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Necessary condition for convergence

Result: If $\sum_{n=1}^{\infty} x_n$ is convergent, then $x_n \to 0$.

Hence if $x_n \not\to 0$, then $\sum_{n=1}^{\infty} x_n$ cannot be convergent.

Examples: The following series are not convergent.

(i) $\sum_{n=1}^{\infty} \frac{n^2+1}{(n+3)(n+4)}$ (ii) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

Test for convergence

Comparison test: Let (x_n) and (y_n) be sequences in $\mathbb{R}$ such that for some $n_0 \in \mathbb{N}$, $0 \leq x_n \leq y_n$ for all $n \geq n_0$.
Then

(i) $\sum_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ is convergent.

(ii) $\sum_{n=1}^{\infty} x_n$ is divergent $\Rightarrow \sum_{n=1}^{\infty} y_n$ is divergent.

Limit comparison test: Let (x_n) and (y_n) be sequences of positive real numbers such that $\frac{x_n}{y_n} \to \ell \in \mathbb{R}$.

(i) If $\ell \neq 0$, then $\sum_{n=1}^{\infty} x_n$ is convergent iff $\sum_{n=1}^{\infty} y_n$ is convergent.

(ii) If $\ell = 0$, then $\sum_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ is convergent.

Condensation and integral tests

Cauchy's condensation test: Let (x_n) be a decreasing sequence of nonnegative real numbers. Then $\sum\limits_{n=1}^{\infty} x_n$ is convergent iff $\sum\limits_{n=1}^{\infty} 2^n x_{2^n}$ is convergent.

Examples:

1. p-series: $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is convergent iff $p > 1$.
2. $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent iff $p > 1$.

Integral Test: Let $f : [1, \infty) \to \mathbb{R}$ be monotone decreasing and $f(t) \geq 0$ for all $t \in [1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\lim_{n\to\infty} \int_1^n f(t)dt$ exists.

Example: p-series: $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is convergent iff $p > 1$.

Absolute convergence

Ex. Examine whether the following series are convergent.

(i) $\sum\limits_{n=1}^{\infty} \frac{1+\sin n}{1+n^2}$ (ii) $\sum\limits_{n=1}^{\infty} \frac{1}{2^n+n}$ (iii) $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$

(iv) $\sum\limits_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ (v) $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ (vi) $\sum\limits_{n=1}^{\infty} \frac{n}{4n^3-2}$

Definitions: $\sum\limits_{n=1}^{\infty} x_n$ is called absolutely convergent if $\sum\limits_{n=1}^{\infty} |x_n|$ is convergent.

$\sum\limits_{n=1}^{\infty} x_n$ is called conditionally convergent if $\sum\limits_{n=1}^{\infty} x_n$ is convergent but $\sum\limits_{n=1}^{\infty} |x_n|$ is divergent.

Ratio test

Result: Every absolutely convergent series is convergent.

Ratio test: Let (x_n) be a sequence of nonzero real numbers such that $\left|\frac{x_{n+1}}{x_n}\right| \to \ell$.

(i) If $\ell < 1$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

(ii) If $\ell > 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.

Examples: (i) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ (ii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (iii) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

Ex. Find all real values of x for which $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges.

Root test

Root test: Let (x_n) be a sequence in $\mathbb{R}$ such that $|x_n|^{\frac{1}{n}} \to \ell$.

(i) If $\ell < 1$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

(ii) If $\ell > 1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.

Examples: (i) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$ (ii) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ (iii) $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$

Leibniz test for alternating series

Ex. Test the convergence of the series
$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + 2x^7 + \cdots$, where $x \in \mathbb{R}$.

Leibniz's test: Let (x_n) be a decreasing sequence of positive real numbers such that $x_n \to 0$.

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} x_n$ is convergent.

Examples: (i) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p}$, $p \in \mathbb{R}$ (ii) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$

(iii) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

Conditional convergence

Conditional convergence: If a series is convergent but is NOT absolutely convergent then the series is called conditionally convergent.

Example: The series $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Consider a series $\sum x_n$. Define $a_n := (|x_n| + x_n)/2$ and $b_n := (|x_n| - x_n)/2$. Then $a_n \geq 0$, $b_n \geq 0$, $|x_n| = a_n + b_n$ and $x_n = a_n - b_n$ for all $n \in \mathbb{N}$.

Result: (i) The series $\sum x_n$ converges absolutely if and only if both $\sum a_n$ and $\sum b_n$ are convergent.

(ii) The series $\sum x_n$ converges conditionally if and only if $\sum x_n$ is convergent but both $\sum a_n$ and $\sum b_n$ are divergent.

Grouping and rearrangement

Result: Grouping of terms (putting bracket inside the infinite sum) of a convergent series does not change the convergence and the sum.

However, a divergent series can become convergent after grouping of terms.

Rearrangement of terms: Consider a series $\sum x_n$. Let ϕ be a permutation of $\mathbb{N}$, that is, $\phi : \mathbb{N} \to \mathbb{N}$ is a bijective map. Then the new series $\sum_{n=1}^{\infty} x_{\phi(n)}$ is called a rearrangement of $\sum x_n$.

Result: If $\sum x_n$ is absolutely convergent then any rearrangement of $\sum x_n$ is convergent and converges to the same limit.

Rearrangement of conditionally convergent series

Riemann's rearrangement theorem: Let $\sum\limits_{n=1}^{\infty} x_n$ be a conditionally convergent series.

(i) If $s \in \mathbb{R}$, then there exists a rearrangement of terms of $\sum\limits_{n=1}^{\infty} x_n$ such that the rearranged series has the sum s.

(ii) There exists a rearrangement of terms of $\sum\limits_{n=1}^{\infty} x_n$ such that the the rearranged series diverges.

Example: Consider $\sum \frac{(-1)^{n+1}}{n}$.

*** End ***