DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati
Tutorial and practice problems on Single Variable Calculus

MA-101 : Mathematics-I   Tutorial Problem Set - 10   October 30, 2013

PART-A (Tutorial)

Question 1:

(i) Give an example of a function $f : [0, 1] \to \mathbb{R}$ such that f is discontinuous at each $x \in [0, 1]$ but $|f|$ is continuous on $[0, 1]$.

(ii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at $c \in \mathbb{R}$. Define $H, K : \mathbb{R} \to \mathbb{R}$ by $H(x) := \max(f(x), g(x))$ and $K(x) := \min(f(x), g(x))$ for $x \in \mathbb{R}$. Discuss the continuity of H and K at c.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) := \begin{cases} 2x, & \text{if } x \text{ is rational,} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$
Show that f is continuous only at $c := 1/3$.

(iv) Let $f : [a, b] \to \mathbb{R}$ be such that whenever $a \le x_1 < x_2 \le b$ and λ lies between $f(x_1)$ and $f(x_2)$, then there is some $c \in [x_1, x_2]$ such that $f(c) = \lambda$. Must f be continuous?

Solution: (i) Consider $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ -1, & \text{if } x \text{ is irrational.} \end{cases}$
Then $|f|$ is a constant function and hence continuous.

Now we show that f is discontinuous at each x in $[0, 1]$. Let $a \in [0, 1]$. Suppose that a is rational. Then $f(a) = 1$. Choose a sequence (x_n) in $[0, 1]$ of irrational numbers such that $x_n \to a$. Since $f(x_n) = -1$ for all n, it follows that the sequence $(f(x_n))$ does not converge to $f(a) = 1$.

Next suppose that a is irrational. Then $f(a) = -1$. Choose a sequence (y_n) in $[0, 1]$ of rational numbers such that $y_n \to a$. Since $f(y_n) = 1$, it follows that the sequence $(f(y_n))$ does not converge to $f(a) = -1$. Hence f is not continuous at a. Since a is arbitrary, the result follows. $\blacksquare$

(ii) It is easy to see that $H(x) = (|f(x)-g(x)|+f(x)+g(x))/2$ and $K(x) = (f(x)+g(x)-|f(x)-g(x)|)/2$. Hence the result follows. $\blacksquare$

(iii) Let $c \in \mathbb{R}$. First, we show that if f is continuous at c then c must be equal to $1/3$, that is, $c = 1/3$. Suppose that f is continuous at c. Choose sequences (x_n) and (y_n) of rational and irrational numbers, respectively, both converging to c. Now $f(x_n) = 2x_n \to 2c$ and $f(y_n) = 1 - y_n \to 1 - c$. Since f is continuous at c, we must have $2c = 1 - c$. This shows that $c = 1/3$.

Now, we show that f is continuous at $1/3$. Choose any $\epsilon > 0$ and set $\delta := \epsilon/2$. We have

$$f(x) - f(1/3) = \begin{cases} 2(x - 1/3), & \text{if } x \text{ is rational,} \\ 1/3 - x, & \text{if } x \text{ is irrational.} \end{cases}$$

This shows that $|x - 1/3| < \delta \Longrightarrow |f(x) - f(1/3)| < \epsilon$. $\blacksquare$

(iv) Consider $f : [-1, 1] \to \mathbb{R}$ given by $f(x) := \sin(1/x)$ and $f(0) := 0$. Then it is easy to see that the given condition is satisfied by f even though f is discontinuous at $c := 0$. $\blacksquare$

Question 2:

(i) Show that the equation $17x^7 - 19x^5 - 1 = 0$ has a solution p which satisfies $-1 < p < 0$.

(ii) Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that for every $x \in [a, b]$ there exists a $y \in [a, b]$ such that $|f(y)| \le |f(x)|/2$. Show that there exists $p \in [a, b]$ such that $f(p) = 0$.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $f(x) \to 0$ as $x \to \pm\infty$. Prove that f attains either a maximum or a minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained.

Solution:

(i) Define $f : [-1, 0] \to \mathbb{R}$ by $f(x) := 17x^7 - 19x^5 - 1$. Then f is continuous on $[-1, 0]$ and $f(-1)f(0) < 0$. Hence by IVT there exists $p \in (-1, 0)$ such that $f(p) = 0$. ■

(ii) Let $x_0 \in [a, b]$. Then there exists a sequence (x_n) in $[a, b]$ such that $|f(x_n)| \leq \frac{1}{2}|f(x_{n-1})| \leq \frac{1}{2^n}|f(x_0)|$. This shows that $f(x_n) \to 0$ as $n \to \infty$. Since (x_n) is a bounded sequence, by Bolzano-Weierstrass theorem, there is a subsequence (y_n) of (x_n) such that $y_n \to y$ for some $y \in [a, b]$. Since f is continuous, $f(y_n) \to f(y)$. Now, by the uniqueness of limit, we have $f(y) = 0$. ■

(iii) Assume that f is not identically equal to zero. It is evident that the set $f(\mathbb{R})$ is bounded. So, let m and M, respectively, be the infimum and supremum of $f(\mathbb{R})$. Then there exist sequences (x_n) and (y_n) in $\mathbb{R}$ such that $f(x_n) \to m$ and $f(y_n) \to M$ as $n \to \infty$. Note that if (x_n) (resp., (y_n)) is bounded then by passing to a subsequence (courtesy Bolzano-Weierstrass), we see that the value m (resp., M) is attained by f on $\mathbb{R}$.

Hence to complete the proof, we have to show that both (x_n) and (y_n) cannot be unbounded. Indeed, if (x_n) and (y_n) are both unbounded, then there exist subsequences (α_n) of (x_n) and (β_n) of (y_n) diverging to either ∞ or $-\infty$. Consequently, $f(\alpha_n) \to 0$ and $f(\beta_n) \to 0$. Since $f(\alpha_n) \to m$ and $f(\beta_n) \to M$, we have $m = 0 = M$. Showing that f is identically equal to zero - a contradiction. ■

Alternative soln: If there exists $c \in \mathbb{R}$ such that $f(c) > 0$, then we show that f has a maximum. So, suppose that $f(c) > 0$ for some $c \in \mathbb{R}$. Then taking $\epsilon := f(c)$, there exists M_1 and M_2 with $M_1 < M_2$ such that $f(x) < \epsilon = f(c)$ for all x outside the interval $[M_1, M_2]$. It is evident that $c \in [M_1, M_2]$. Since f is continuous on $[M_1, M_2]$, f attains its maximum at p for some $p \in [M_1, M_2]$. This shows that $f(x) \leq \max(f(c), f(p)) = f(p)$ for all $x \in \mathbb{R}$.

Similarly, if $f(c) < 0$ for some $c \in \mathbb{R}$, then taking $\epsilon := -f(c)$, it is easy to see that f attains its minimum on $\mathbb{R}$. ■

Example: Consider $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) := e^{-|x|}$ and $g(x) := -e^{-|x|}$. Then f attains its maximum on $\mathbb{R}$ but not the minimum. On the other hand, g attains its minimum on $\mathbb{R}$ but not the maximum. ■

Question 3: Find the limits of the following functions whenever they exist. $[x]$ denotes the largest integer $\leq x$.

(i) $\lim_{x\to3}([x] - [2x - 1])$; (ii) $\lim_{x\to2}([x] - x^2)$; (iii) $\displaystyle\lim_{x\to1} \frac{|x-1| + 1}{x + |x+1|}$.

Solution:

(i) $\lim_{x\to3^+} f(x) = -2 = \lim_{x\to3^-} f(x)$. Hence $\lim_{x\to3} f(x) = -2$. ■

(ii) $\lim_{x\to2^+} f(x) = -2$ and $\lim_{x\to2^-} f(x) = -3$. Hence $\lim_{x\to2} f(x)$ does not exist. ■

(iii) $\lim_{x\to1^+} f(x) = 1/3 = \lim_{x\to1^-} f(x)$. Hence $\lim_{x\to1} f(x) = 1/3$. ■

Question 4:

(i) Let $f : (1, 2) \to \mathbb{R}$ be such that $-16 - \sin^2(x - 2) < f(x) < \frac{x^2|4x-8|}{x-2}$, for $x \in (1, 2)$. Show that $\lim_{x\to2} f(x)$ exists and find the limit.

(ii) Let $f : [1, 3] \to \mathbb{R}$ be such that $x/[x] \leq f(x) \leq \sqrt{6 - x}$ for $x \in [1, 3]$, $f(2) = 1$ and f is continuous on $[1, 2) \cup (2, 3]$. Show that $\lim_{x\to2^-} f(x)$ exists and find the limit. Is f continuous at 2?

(iii) Let $f : (a, b) \to \mathbb{R}$. Define $|f| : (a, b) \to \mathbb{R}$ by $|f|(x) := |f(x)|$ for $x \in (a, b)$. Let $c \in (a, b)$. If $\lim_{x\to c} f(x)$ exists and is equal to L then show that $\lim_{x\to c} |f|(x)$ exists and is equal to $|L|$. Is the converse true?

Solution: (i) Set $h(x) := -16 - \sin^2(x - 2)$ and $k(x) := \frac{x^2|4x-8|}{x-2}$. Then $\lim_{x\to 2} h(x) = \lim_{x\to 2} k(x) = -16$. Hence by sandwich theorem $\lim_{x\to 2} f(x)$ exists and is equal to -16. ■

(ii) Set $h(x) := x/[x]$ and $k(x) := \sqrt{6 - x}$. Let (x_n) be a sequence in $[1, 3]$ such that $x_n < 2$ and $x_n \to 2$ as $n \to \infty$. Then $h(x_n) = x_n \to 2$ and $k(x_n) = \sqrt{6 - x_n} \to 2$ as $n \to \infty$. Hence by sandwich theorem $f(x_n) \to 2$ as $n \to \infty$, if $x_n < 2$ and $x_n \to 2$ as $n \to \infty$.
Since $f(2) = 1$, we see that $f(x_n)$ does not converge to $f(2)$. Therefore, f is not continuous at 2. ■

(iii) Since $|(|f(x)| - |L|)| \leq |f(x) - L|$, the result follows.

For the converse, consider $f : (-1, 1) \to \mathbb{R}$ given by $f(x) := x/|x|$ for $x \neq 0$ and $f(0) = 1$. Then $\lim_{x\to 0} |f|(x) = 1$ but $\lim_{x\to 0} f(x)$ does not exist. ■

PART-B (Homework/Practice problems)

Question 5:

(a) Let $f : [0, 1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Show that

1. there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{2}$.
2. there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{3}$.

(b) Let p be an odd degree polynomial with real coefficients in one real variable. If $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

In particular, this shows that

1. every odd degree polynomial with real coefficients in one real variable has at least one real zero.
2. the equation $x^9 - 4x^6 + x^5 + \dfrac{1}{1 + x^2} = \sin 3x + 17$ has at least one real solution.
3. the range of every odd degree polynomial with real coefficients in one real variable is $\mathbb{R}$.)

(c) Let $f : (0, \infty) \to \mathbb{R}$ be such that $f(x) := \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ \frac{1}{q}, & \text{if } x = p/q, \text{ and } \gcd(p, q) = 1. \end{cases}$
Show that f is continuous at each irrational in $(0, \infty)$ but discontinuous at each rational in $(0, \infty)$.

Solution: (iv) Let $a \in (0, \infty)$ be rational. Let (x_n) be a sequence of irrational numbers in $(0, \infty)$ such that $x_n \to a$. Since $f(x_n) = 0$ for all n, it follows that the sequence $(f(x_n))$ does not converge to $f(a) > 0$. Hence f is not continuous at a. Since a is arbitrary, f is discontinuous at each rational in $(0, \infty)$.

Next, let a be any irrational number in $(0, \infty)$. Then $f(a) = 0$. Choose $\epsilon > 0$. Then there exists $m \in \mathbb{N}$ such that $1/m < \epsilon$. Note that $(a - 1, a + 1)$ contains only a finite numbers of rational numbers p/q (with $\gcd(p, q) = 1$) with $q < m$. Now choose $\delta > 0$ such that $(a - \delta,\ a + \delta)$ does not contain any of those rational numbers (for which $q < m$). Then for $x \in (a - \delta,\ a + \delta)$, we have

$$f(x) - f(a) = f(x) = \begin{cases} 1/q \leq 1/m < \epsilon, & \text{if } x = p/q \text{ with } \gcd(p, q) = 1, \\ 0 < \epsilon, & \text{if } x \text{ is irrational.} \end{cases}$$

This shows that f is continuous at a. Since a is arbitrary, f is continuous at each irrational in $(0, \infty)$. ■

Question 6: Let $f : [a, b] \to \mathbb{R}$ be continuous and one-to-one.

(i) If $f(a) < f(b)$ then show that f is strictly increasing, that is, $x < y \Longrightarrow f(x) < f(y)$.

(ii) If $f(a) > f(b)$ then show that f is strictly decreasing, that is, $x < y \Longrightarrow f(x) > f(y)$.

Solution: (i) First, we show that $f(b)$ is the maximum of f on $[a, b]$. Suppose that f attains its maximum at $c \in [a, b]$. Obviously $c \neq a$. If $c \neq b$ then $f(a) < f(b) < f(c)$. Hence by IVT, there exists $x_0 \in (a, c)$ such that $f(x_0) = f(b)$. This contradicts that f is injective. Hence $c = b$.

Next, we show that f is strictly increasing. Let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$. If possible suppose that $f(x_1) > f(x_2)$. Let $f(x_2) < \lambda < f(x_1)$. Then there exists $\alpha \in (x_1, x_2)$ such that $f(\alpha) = \lambda$. Since $f(x_2) < \lambda < f(b)$ there exists $\beta \in (x_2, b)$ such that $f(\beta) = \lambda$. Consequently, $f(\alpha) = f(\beta)$ and $\alpha < \beta$. This contradicts that f is injective. Hence $f(x_1) < f(x_2)$. ■

(ii) When $f(a) > f(b)$, a similar proof shows that f is strictly decreasing. ■

Question 7: Suppose that $f : [a, b] \to \mathbb{R}$ is increasing, that is, $x \leq y \Longrightarrow f(x) \leq f(y)$. Let $c \in (a, b)$. Show that

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x < c\} \text{ and } \lim_{x \to c^+} f(x) = \inf\{f(x) : x > c\}.$$

Solution: Since f is increasing, the set $S := \{f(x) : x < c\}$ is nonempty and bounded above by $f(c)$. Let $L := \sup S$. Choose $\epsilon > 0$. Then there exists x_0 such that $f(x_0) \in S$ and $L - \epsilon < f(x_0) \leq L$. Set $\delta := c - x_0$. Then $\delta > 0$. Now $c - \delta < x < c \Longrightarrow x_0 < x < c \Longrightarrow L - \epsilon < f(x_0) \leq f(x) \leq L < L + \epsilon$. This shows that $\lim_{x \to c^-} f(x) = L$.

Similarly, we have $\lim_{x \to c^+} f(x) = \inf\{f(x) : x > c\}$ ■

*** End ***