DEPARTMENT OF MATHEMATICS, IIT Guwahati

MA101: Mathematics I, July - November 2014

Hints to Problems in Practice Problem Set

1. There are four and eight reduced row echelon forms of a 2×2 and a 3×3 matrices, respectively. These matrices can be obtained by considering the cases based on the ranks and the position of the leading entries.

2. The reduced row echelon form of the given matrices are the following (in order):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. (a) $\{[0,0,0]^t\}$, (b) $\{[r,r,-s,s]^t : r,s \in \mathbb{R}\}$, (c) $\{[1,1,1]^t\}$, (d) Inconsistent, (e) $\{[s,s,-s]^t : s \in \mathbb{R}\}$, (f) $\left\{\left[\frac{5}{22} - s,\ \frac{15}{22} - 2s,\ \frac{5}{2} - 4s,\ s\right]^t : \ s \in \mathbb{R}\right\}$.

4. (b) If $a = 0$ or $ac \neq 0$ or $c = 0$ then the reduced row echelon form of A is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

   (c) If $a = c = 0$ then the reduced row echelon form of A is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If $(ac \neq 0)$ or $(a = 0, c \neq 0)$ or $(a \neq 0, c = 0)$, then the reduced row echelon form of A is $\begin{bmatrix} 1 & \alpha \\ 0 & 0 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$.

5. (a) If $c = 1, k \neq 7$ then no solution. If $c \neq 1$ then unique solution: $\left[k - 5 - \frac{k-7}{c-1},\ 8 - k,\ \frac{k-7}{c-1}\right]^t$.
   If $c = 1, k = 7$ then infinitely many solutions: $\{[2-s, 1, s]^t : s \in \mathbb{R}\}$.

   (b) If $c = 4, k \neq 5$ then no solution. If $c \neq 4$ then unique solution: $\left[6 - k,\ 2 + \frac{(c-2)(k-5)}{c-4},\ \frac{k-5}{4-c}\right]^t$.
   If $c = 4, k = 5$ then infinitely many solutions: $\{[1, 2-2s, s]^t : s \in \mathbb{R}\}$.

   (c) If $k = 0$ then no solution. If $k \neq 0$ then unique solution: $\left[\frac{k^2-k+3}{k^2},\ \frac{k-2}{k^2},\ \frac{k-1}{k^2}\right]^t$.

6. The augmented matrix can be reduced to

$$\left[\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 0 & b-1 & 1 & 0 \\ 0 & 0 & b+1 & 2(b-1) \end{array}\right].$$

   If $a = 0, b = -1$, then no solution.
   If $a = 0, b = 1$, then infinitely many solutions: $\{[s, 1, 0]^t : s \in \mathbb{R}\}$.
   If $a = 0, b = 5$, then infinitely many solutions: $\{[s, -1/3, 4/3]^t : s \in \mathbb{R}\}$.
   If $a = 0, b \neq 5, \pm 1$, then no solution.
   If $a \neq 0, b = -1$, then no solution.
   If $a \neq 0, b = 1$, then infinitely many solutions: $\{[\frac{1-s}{a}, s, 0]^t : s \in \mathbb{R}\}$.
   If $a \neq 0, b \neq \pm 1$, then unique solution: $\left[\frac{5-b}{a(b+1)}, -\frac{2}{b+1}, \frac{2(b-1)}{b+1}\right]^t$.

7. The coefficient matrix can be transformed to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1-n \\ 0 & 1 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 1 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

   Hence $x_n = x_{n-1} = \ldots\ldots = x_2 = x_1$.

8. Apply the following sequence of elementary rwo operations:

$$R_s \leftrightarrow R_{s-1}, \quad R_{s-1} \leftrightarrow R_{s-2}, \quad \ldots\ldots, R_{r+1} \leftrightarrow R_r, \quad R_{r+1} \leftrightarrow R_{r+2}, \quad R_{r+2} \leftrightarrow R_{r+3}, \quad \ldots\ldots, R_{s-1} \leftrightarrow R_s.$$

9. Take $\mathbf{y}_0 = \mathbf{x}_0 - \mathbf{x}_1$ for the second part.

10. For the 'only if' part ($\Rightarrow$), take $\mathbf{b} = \mathbf{e}_i$ for $i = 1, 2, \ldots, n$. For the 'if' part ($\Leftarrow$), $m = \text{rank}(A) \leq \text{rank}([A \mid \mathbf{b}]) \leq m$.

11. Prove the contrapositive for the 'only if' part.

12. The reduced row echelon form is $\begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & -\frac{1}{2}(1-i) \\ 0 & 0 & 0 \end{bmatrix}$. A basis for the solution space is $\{[i, \frac{1}{2}(1-i), 1]^t\}$.

13. For the 'only if' part ($\Rightarrow$), prove the contrapositive, so that there will be at least one free variable. For the 'if' part ($\Leftarrow$), there will be no free variable.

14. Similar to Problem 13.

15. Similar to Problem 4 of the first tutorial sheet (LA-1). The statement is also true in general. Indeed, if $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ are two systems with equal solution sets (**non-empty**) then $[A \mid \mathbf{b}]$ is row equivalent to $[C \mid \mathbf{d}]$. This has been partialy covered in the additional question for **LA-1**.

16. (a) $\{[0, 2]^t\}$, (b) $\{[0, 1, 1]^t, [1, 0, 0]^t\}$, (c) Inconsistent, (d) $\{[2, 0]^t, [2, 2]^t, [2, 4]^t\}$.

17. There are p choices to assign values to each of the $n - r$ free variables.

18. Show that $\text{rank}(A^*A) \leq \text{rank}([A^*A \mid A^*\mathbf{b}]) \leq \text{rank}(A^*[A \mid \mathbf{b}]) \leq \text{rank}(A^*)$ and $\text{rank}(A^*A) = \text{rank}(A^*)$. Hence $\text{rank}(A^*A) = \text{rank}([A^*A \mid A^*\mathbf{b}])$.

19. Easy.

20. $x_{n+2}, x_{n+3}, \ldots, x_{2n}$ are the free variables. A basis for the solutions space is $\{f_1, f_2, \ldots, f_{n-1}\}$, where

$$f_i = [-1, 0, 0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0], \ 1 \leq i \leq n-1.$$

Note that, the first and the $(n+1)$-th entry of each f_i is -1. The two positive entries of f_i is at the $(i+1)$-th position and at the $(n+i+1)$-th position, respectively.

21. Only the first two sets are subspaces.

22. Only the second set is a subspace.

23. Only the S_3 is a subspace.

24. The span of the first three sets is the plane $x - 2y + z = 0$. The last two sets span $\mathbb{R}^3$.

25. Easy.

26. For any $(\emptyset \neq) S \subseteq \mathbb{R}^3$, we have $\dim \text{span}(S) \leq \dim \mathbb{R}^3 = 3$.

27. Only the first and the third sets span $\mathbb{R}^3$.

28. The set is linearly independent iff $t \neq 1$.

29. Only part (c) is true.

30. A basis is $\{f_1, f_2, \ldots, f_{n-1}\}$, where

$$f_i = [-1, 0, 0, \ldots, 0, 1, 0, \ldots, 0], \ 1 \leq i \leq n-1.$$

Note that, the first entry of each f_i is -1 and the other positive entry of f_i is at the $(i+1)$-th position.

31. Similar to **Result** 2.4 of **Lecture** 4.

32. Take hint from **Result** 2.3 of **Lecture** 4.

33. Same as Problem 32.

34. For example, $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

35. The reason is $AB \neq BA$.

36. Use induction on m. Find counterexamples for the last part.

37. $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$. For the last part, take $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

38. Solve $CA = B$, for $C = \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$. One solution is $C = \begin{bmatrix} 1 & 1 & 0 \\ -4 & 0 & 0 \end{bmatrix}$.

39. For the first part, use $\operatorname{tr}(C) = \operatorname{tr}(AB - BA)$. For the second part, if $a \neq 0$ then take $A = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & \frac{b+c}{a} \end{bmatrix}$. If $a = 0$, then solve $C = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} - \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ to obtain $A = \begin{bmatrix} 1 & 0 \\ 0 & 1+c \end{bmatrix}, B = \begin{bmatrix} 0 & -b/c \\ 1 & 0 \end{bmatrix}$ for $b \neq 0$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 1-b \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -c/b & 0 \end{bmatrix}$ for $c \neq 0$.

40. $a = d, \ b = c = 0$

41. Easy.

42. $\operatorname{tr}(AA^t) = \sum_{i,j=1}^{n} a_{ij}^2$.

43. Use the binomial expansion of $(A+B)^k$ and the facts $AB = \mathbf{O}, \ \operatorname{tr}(XY) = \operatorname{tr}(YX)$.

44. Use induction on k.

45. Show that A^2 has $n-2$ non-zero rows, and so on.

46. If $[a_{ij}]$ and $[b_{ij}]$ are lower triangular matrices then show that $\sum_{k=1}^{n} a_{ik}b_{kj} = 0$ for $i < j$.

47. AB is symmetric.

48. Take $\mathbf{x} = \mathbf{e}_i$ to show that the i-th column of A is zero.

49. Take $X = \begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix}$ solve the system $XA = I_2$ and $AX = I_3$.

50. Compare the (i,i)-entries of AA^* and A^*A.

51. Use induction on n.

52. Take two invertible matrices which do not commute.

53. Take $A = B = I_2$.

54. The given condition gives $I = (2I - A)A$.

55. The solutions are (in order): $X = A^{-3}, \ X = AB^{-2}$ and $X = B^{-1}A^{-1}BA + A$.

56. Compare the (i,i)-entries of AA^t and $\mathbf{O}$.

57. Show that $AB = I$ is never possible.

58. For each matrix A if you get the answer as B, then to be sure verify that $AB = I$.

59. For each matrix A if you get the answer as B, then to be sure verify that $AB = I$.

60. For each matrix A if you get the answer as B, then to be sure verify that $AB = I$.

61. $A[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = [\mathbf{u}_1, 2\mathbf{u}_2, 3\mathbf{u}_3]$.

62. $A^2 = A \Rightarrow (\det A)^2 = \det A$.

63. Apply the row operations $R_n \to R_n - R_{n-1}, R_{n-1} \to R_{n-1} - R_{n-2}, \ldots, R_2 \to R_2 - R_1$ and then expand through the last column. We get $\det A = (-1)^{n+1} n$.

64. Consider the Vandermonde matrix, where $x_i = i$ for $i = 1, 2, \ldots, n$.

65. Use **Rule** 5 of **Result** 5.3 repeatedly. We find $\det(A + B) = 40$ an $\det C = 5$.

66. $A^2 = -I \Rightarrow (\det A)^2 = (-1)^n$. For the 2nd part, using the fact $i^2 = -1$ try to find a counterexample for $n = 3$.

67. $\det A = \pm 1$ and $\det(A + I) = \det(A + AA^t)$. We find $\det(A + I) = 0$.

68. For square matrices, $XY = I \Leftrightarrow YX = I$.

69. $B = A(A+2I)(A-I)$. Also $A^3 = 2I \Rightarrow I = (A-I)(A^2+A+I)$ and $A^3+8I = 10I \Rightarrow (A+2I)(A^2-4A+4I) = 10I$. Thus $\det(A - I) \neq 0$ and $\det(A + 2I) \neq 0$, and hence $\det B \neq 0$.

70. Use induction on n. [Note that we need to prove that $\det(A) = \frac{a^{n+1}-b^{n+1}}{a-b}$.] Take limit for the second part.

71. (a). The numbers $i+j$ and $i-j$ are either both even or both odd. Thus, multiplying the entries at odd positions of A by -1 is same as multiplying each entry of A by $(-1)^{i-j}$. Now use **Problem** 7 of **LA-3**.

    (b). Multiplying the entries at even positions of A by -1 is same as multiplying each entry of B by -1.

72. Expand through the last row. The answer is $\lambda^{10} - 10^{10}$.

73. What is the determinant of a triangular matrix?

74. $AB = A - B \Rightarrow (A + I)(I - B) = I \Rightarrow (A + I)(I - B) = I = (I - B)(A + I) \Rightarrow AB = BA$

75. $A + B$ and $A - B$ need not be non-singular (find counterexapmles.) Also, $\det(-A) = (-1)^n \det(A)$.

76. Apply suitable elementary row operations.

77. To show that $\det(A) \neq 0$. Subtract the last row of $\det(A)$ from the $n-1$ preceding rows. Now take the factors $\frac{1}{n+1}, \frac{1}{n+2}, \ldots, \frac{1}{2n-1}, \frac{1}{2n}$ common from the respective columns and also take the factors $n-1, n-2, \ldots, 2$ common from the respective rows. Now in the remaining determinant, subtract the last column from each of the preceding columns and take suitable factors common from the rows as well as from the columns. Finally, expand the remaining determinant through a suitable row (column) to get a similar determinant of size $n-1$. Now use mathematical induction on n.

78. Here $A = J - I$, where each entry of J is 1. We have $(J - I)\left(\frac{1}{n-1}J - I\right) = I$.

79. Apply appropriate elementary column operations on $A - I$ to make the first column of $A - I$ zero.

80. $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution iff $\det(A) = 0$.

81. For each i, there is a column vector $\mathbf{c}_i$, each of whose entries are integers, such that $A\mathbf{c}_i = \mathbf{e}_i$. Set $C = [\mathbf{c}_1, \ldots, \mathbf{c}_n]$ so that $AC = I$.

82. Easy.

83. $\det(AA^t) = 1$.

84. $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$ and $A^{-1} = E_2 E_1, \quad A = E_1^{-1} E_2^{-1}$.

85. Use the fact that B is a product of elementary matrices, **OR** diectly show that both the systems have the same solution sets.

86. If $B = [b_{ij}]$ is the inverse of A, then $\sum\limits_{j=1}^{5}\sum\limits_{i=1}^{5}\left(\sum\limits_{k=1}^{5} b_{ik}a_{kj}\right) = 5 \Rightarrow \sum\limits_{k=1}^{5}\left[\sum\limits_{i=1}^{5} b_{ik}\left(\sum\limits_{j=1}^{5} a_{kj}\right)\right] = 5.$

87. Each elementary row operation corresponds to an elementary matrix. (a) $PA = B = QA \Rightarrow P = Q$.

    (b) Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

88. Find P and Q from the multiplication of elementary matrices corresponding to the respective elementary row and column operations. We get $P = \begin{bmatrix} 3/2 & -2 \\ -1/2 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

89. (a) $\det(\overline{A}^t) = \det(-A) \Rightarrow \overline{\det(A)} = (-1)^n \det(A)$. (b) Similar to first part. (c) $AA^t = I$. (d) $AA^* = I$.

90. Easy.

91. $\det(AB) = 0$.

92. Apply sufficient numbers of elementary row and column operations on A.

93. The given statement is correct. The proof is similar to **Problem** 77.

94. $\mathbf{x}_i$ is the solution of $A\mathbf{x} = \mathbf{e}_i$, where $\mathbf{x}_1 = [2, -1, 0, \ldots, 0]^t, \mathbf{x}_n = [0, \ldots, 0, -1, 1]^t$ and $\mathbf{x}_i = [0, \ldots, 0, -1, 2, -1, 0, \ldots, 0]^t$ for $1 < i < n$. Then $A^{-1} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$.

95. Apply the row operations $R_1 \leftrightarrow R_n,\ R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1, \ldots, R_n \to R_n - R_{n-1}$ and then use induction to find $\det(A) = (-1)^{n-1}(n-1)$. Find A^{-1} as in Problem 78 **OR** solve $A\mathbf{x} = \mathbf{e}_i$ for each $i = 1, 2, \ldots, n$.

96. Check that $(A - B)\left[A^{-1} + A^{-1}(B^{-1} - A^{-1})^{-1}A^{-1}\right] = I$.

97. Direct computation gives $S^2 = I$. Use induction to show that $\det(S) = (-1)^{\frac{n^2+3n}{2}}$. The (i, j)-th entry of SAS is $a_{n-i+1, n-j+1}$. (Notice that if $S = [u_{ij}]$ then $u_{ij} = 1$ iff $i + j = n + 1$.)

98. The reduced row echelon forms are given in **Problem** 2. Bases for the row space, column space and the null space are:

    First matrix: $\{[1,0,0,0]^t, [0,1,0,0]^t, [0,0,1,0]^t, [0,0,0,1]^t\}$, $\{[1,0,-1,1]^t, [-1,5,2,2]^t, [2,6,4,-1]^t, [3,2,3,2]^t\}$ and $\emptyset$, respectively.

    Second matrix: $\{[1,0,-1,-2]^t, [0,1,2,3]^t\}$, $\{[1,5,9,13]^t, [2,6,10,14]^t\}$ and $\{[1,-2,1,0]^t, [2,-3,0,1]^t\}$, respectively.

    Third matrix: $\{[1,0,0,0]^t, [0,1,0,0]^t, [0,0,1,0]^t, [0,0,0,1]^t\}$, $\{[3,2,0,5]^t, [4,3,2,-5]^t, [5,1,0,5]^t, [-6,1,0,5]^t\}$ and $\emptyset$, respectively.

    Fourth matrix: $\{[1,0,0,1,0]^t, [0,1,0,1,0]^t, [0,0,1,0,0]^t, [0,0,0,0,1]^t\}$, $\{[1,0,2,4]^t, [2,2,-2,2]^t, [1,2,4,5]^t, [2,4,8,10]^t\}$ and $\{[-1,-1,0,1,0]^t\}$, respectively.

99. Show that the non-zero rows of R are linearly independent.

100. Two examples were given in class. Find other examples of your own.

101. Not possible since $[1,2,1][1,-2,1]^t \neq 0$.

102. There are invertible matrices P and Q such that $PAQ = \begin{bmatrix} I_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$.

103. $t = 3$.

104. $t = -3$.

105. If $AB = \mathbf{O}$ then $\mathrm{col}(B) \subseteq \mathrm{null}(A)$.

106. $I = (I - A) + A \Rightarrow n \le \mathrm{rank}(I - A) + \mathrm{rank}(A)$. Also $A^3 = A^2 \Rightarrow \mathrm{rank}(I - A) \le \mathrm{nullity}(A^2) = n - \mathrm{rank}(A^2)$. Now use **Problem** 108. Take $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ as a counterexample.

107. If A_s is the matrix obtained by deleting s rows of A, then $\mathrm{rank}(A) - s \le \mathrm{rank}(A_s)$.

108. $n \le \mathrm{rank}(A - I) + \mathrm{rank}(A)$. Also $A^2 = A \Rightarrow \mathrm{rank}(A - I) \le \mathrm{nullity}(A) = n - \mathrm{rank}(A)$. Conversely, $n = \mathrm{rank}(A - I) + \mathrm{rank}(A) \Rightarrow \mathrm{rank}(A - I) = \mathrm{nullity}(A)$. Then $\mathbf{z} \in \mathrm{null}(A) \Rightarrow \mathbf{z} = -(A - I)\mathbf{z} \in \mathrm{col}(A - I)$. Thus $\mathrm{null}(A) \subseteq \mathrm{col}(A - I) \Rightarrow \mathrm{null}(A) = \mathrm{col}(A - I)$, and so $A(A - I)\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$. Hence $A^2 = A$.

109. $\lambda \neq -3$ (transform the matrix to row echelon form).

110. $\text{null}(A) \subseteq \text{null}(A^2)$. Also $\text{rank}(A) = \text{rank}(A^2) \Rightarrow \text{nullity}(A) = \text{nullity}(A^2)$.

111. The formula $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$ will be discussed in vector space.

$$\begin{aligned}\dim[\text{null}(A) \cap \text{null}(B)] &= \dim[\text{null}(A)] + \dim[\text{null}(B)] - \dim[\text{null}(A) + \text{null}(B)] \\ &= n - \text{rank}(A) + n - \text{rank}(B) - \dim[\text{null}(A) + \text{null}(B)] > n - \dim[\text{null}(A) + \text{null}(B)] \geq 0.\end{aligned}$$

112. $\text{nullity}(AB) = n - \text{rank}(AB) \geq n - \max\{n-l, n-m\} = \max\{l, m\}$. For the second part, $\mathbb{R}^n = \text{null}(A) \cup \text{null}(B) \Rightarrow \mathbb{R}^n = \text{null}(A)$ or $\mathbb{R}^n = \text{null}(B) \Rightarrow A = \mathbf{O}$ or $B = \mathbf{O}$.

113. Let $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_r\}$ a basis for $\text{col}(A)$. If $\mathbf{a}_i = \sum_{j=1}^{r} \alpha_{ij}\mathbf{b}_j$ then take $B = [\mathbf{b}_1, \ldots, \mathbf{b}_r]$ and $C = [\alpha_{ji}]$.

114. $\text{col}(AB) \subseteq \text{col}(A), \text{rank}(A) = \text{rank}(AB) \Rightarrow \text{col}(AB) = \text{col}(A)$. Now if $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n], AB = [\mathbf{b}_1, \ldots, \mathbf{b}_k]$ and $\mathbf{a}_i = \sum_{j=1}^{k} \alpha_{ij}\mathbf{b}_j$ then take $X = [\alpha_{ji}]$.

115. $\det(S^{-1}AS - \lambda I) = \det[S^{-1}(A - \lambda I)S]$. Take the counterexample $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ for the second part.

116. For each of the matrices, the ordered pairs given below consist of an eigenvalue and a basis for the corresponding eigenspace:
1st matrix: $(0, \{[0,1]^t\})$, $(1, \{[1,0]^t\})$, **2nd matrix:** $(1+\sqrt{2}, \{[\sqrt{2}, 1-i]^t\})$, $(1-\sqrt{2}, \{[-\sqrt{2}, 1-i]^t\})$,
3rd matrix: $(2+\sqrt{3}, \{[2, 1+\sqrt{3}]^t\})$, $(2-\sqrt{3}, \{[2, 1-\sqrt{3}]^t\})$, **5th matrix:** $(1, \{[-3,5]^t\})$, $(4, \{[0,1]^t\})$,
4th matrix: $(1+\sqrt{2i}, \{[1, -i\sqrt{2i}]^t\})$, $(1-\sqrt{2i}, \{[1, i\sqrt{2i}]^t\})$,
6th matrix: $(\frac{5}{2} + \frac{\sqrt{33}}{2}, \{[3-\sqrt{33}, -6]^t\})$, $(\frac{5}{2} - \frac{\sqrt{33}}{2}, \{[3+\sqrt{33}, -6]^t\})$,
7th matrix: $(0, \{[0,1,-1]^t\})$, $(2, \{[1,-2,3]^t\})$,
8th matrix: $(1, \{[0,0,1]^t, [1,1,0]^t\})$, $(-1, \{[1,-1,0]^t\})$,
9th matrix: $(3, \{[1,1,1]^t\})$, $(i\sqrt{2}, \{[\frac{2(2-i\sqrt{2})}{9}, \frac{i\sqrt{2}}{3}, 1]^t\})$, $(-i\sqrt{2}, \{[\frac{2(2+i\sqrt{2})}{9}, -\frac{i\sqrt{2}}{3}, 1]^t\})$
10th matrix: $(1, \{[1,-2,1]^t\})$, $(\frac{3+\sqrt{5}}{2}, \{[-1-\sqrt{5}, -1-\sqrt{5}, 1-\sqrt{5}]^t\})$, $(\frac{3-\sqrt{5}}{2}, \{[-1+\sqrt{5}, -1+\sqrt{5}, 1+\sqrt{5}]^t\})$,
11th matrix: $(0, \{[-1,1,0]^t, [-1,0,1]^t\})$, $(3, \{[1,1,1]^t\})$.

117. A has 3 distinct eigenvalues and $S_1 = \begin{bmatrix} 2 & \frac{-8-2i\sqrt{11}}{9} & \frac{8-2i\sqrt{11}}{9} \\ 2 & \frac{1}{3}(5-i\sqrt{11}) & -\frac{1}{3}(5+i\sqrt{11}) \\ 3 & -1-i\sqrt{11} & 1-i\sqrt{11} \end{bmatrix}$.

B has 3 distinct eigenvalues and $S_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

C has eigenvalues 0 and 3, but E_0 has dimension 2, and $S_3 = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

118. (a) Both the matrices are similar to $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$. If $U^{-1}AU = D$ and $V^{-1}AV = D$ then take $P = UV^{-1}$.
(b) Both the matrices are similar to $\text{diag}[2, -2, 1]$.

119. $A^{-1}(AB)A = BA$.

120. $B = P^{-1}AP \Rightarrow A$ and B have the same characteristic polynomial. Also any eigenvector of B must be of the form $P^{-1}\mathbf{v}$ for some eigenvector $\mathbf{v}$ of A.

121. Find the algebraic and geometric multiplicities of the eigenvalues.

122. $A\mathbf{u} = \lambda\mathbf{u} \Rightarrow \overline{\mathbf{u}}^t A = \overline{\lambda}\overline{\mathbf{u}}^t \Rightarrow \overline{\mathbf{u}}^t A\mathbf{u} = \overline{\lambda}\overline{\mathbf{u}}^t\mathbf{u}$.

123. $A^t = -A \Rightarrow \det(A) = (-1)^n \det(A)$.

124. $\det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$. Compare the constant term and the coefficient of λ^{n-1} on both the sides.

125. $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\det(AB) = \det(A)\det(B)$.

126. -1 is an eigenvalue of A. So, $A = B^{52} \Rightarrow \lambda^{52} = -1$ for some eigenvalue λ of B.

127. $(a)\ \ A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ \ B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad (b)\ \ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \ B = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \quad (c)$ Easy.

128. Easy.

129. $\det(A - \lambda I) = \det(A^t - \lambda I)$. Take the counterexample $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ for the second part.

130. Complex roots of a polynomial equation occur in pairs.

131. Similar to Problem 122.

132. $(a)\ \ A\mathbf{x} = \lambda\mathbf{x}, \mathbf{x} \neq \mathbf{0} \Rightarrow \lambda^2\mathbf{x} = \lambda\mathbf{x}. \quad (b)\ \ A\mathbf{x} = \lambda\mathbf{x} \Rightarrow \mathbf{0} = \lambda^m\mathbf{x}.$

133. $(a)\ \ A\mathbf{v} = \lambda\mathbf{v}$ and $\mathrm{adj}(A)A = \det(A)I$.
(b) If $A\mathbf{v} = \lambda\mathbf{v}$ and $\lambda \neq 0$, then use part (a). Now let $A\mathbf{v} = \mathbf{0}$ *i.e.*, $\mathbf{v} \in \mathrm{null}(A)$. If $\mathrm{rank}(A) \leq n-2$ then $\mathrm{adj}(A) = \mathbf{O}$. If $\mathrm{rank}(A) = n-1$ then $\mathrm{null}(A) = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$ and also $A.\mathrm{adj}(A)\mathbf{v} = \det(A)\mathbf{v} = \mathbf{0} \Rightarrow \mathrm{adj}(A)\mathbf{v} \in \mathrm{null}(A)$.

134. $A^{-1}(BA^{-1})A = A^{-1}B$.

135. $\mathrm{tr}(A)$ and $\mathrm{tr}(A^2)$ are real numbers. Also $\mathrm{tr}(A^2) = \sum\limits_{k=1}^{n} \lambda_k^2$.

136. **1st matrix:** $E_0 = \{[0, \alpha]^t : \alpha \in \mathbb{C}\},\ E_1 = \{[\alpha, 0]^t : \alpha \in \mathbb{C}\}$,
2nd matrix: $E_{1+\sqrt{2}} = \{\alpha[\sqrt{2},\ 1-i]^t : \alpha \in \mathbb{C}\},\ E_{1-\sqrt{2}} = \{\alpha[-\sqrt{2},\ 1-i]^t : \alpha \in \mathbb{C}\}$,
3rd matrix: $E_{i+i\sqrt{2}} = \{\alpha[i\sqrt{2},\ i-1]^t : \alpha \in \mathbb{C}\},\ E_{i-i\sqrt{2}} = \{\alpha[-i\sqrt{2},\ i-1]^t : \alpha \in \mathbb{C}\}$,
4th matrix: If $\theta \neq n\pi$ then $E_{e^{i\theta}} = \{\alpha[i, 1]^t : \alpha \in \mathbb{C}\}$ and $E_{e^{-i\theta}} = \{\alpha[-i, 1]^t : \alpha \in \mathbb{C}\}$. If $\theta = 2n\pi$ then $E_1 = \mathbb{R}^2$. If $\theta = (2n+1)\pi$ then $E_{-1} = \mathbb{R}^2$.
5th matrix: Consider each of the cases $\theta \neq n\pi, n\pi + \frac{\pi}{2},\ \theta = 2n\pi, \theta = (2n+1)\pi, \theta = 2n\pi + \frac{\pi}{2}$ and $\theta = (2n+1)\pi + \frac{\pi}{2}$ one by one.

137. If the given matrices are A, B, C, D (in order), then $A^{-1} = \frac{1}{3}(3I - A),\ B^{-1} = \frac{1}{3}(B - 2I),\ C^{-1} = \frac{1}{8}(-C^2 + 5C - 2I)$ and $D^{-1} = -D^2 + D + I$.

138. **1st matrix:** $k \neq 1$, **2nd matrix:** $k = 0$, **3rd matrix:** $k \in \mathbb{R}$, **4th matrix:** $k = 0$, **5th matrix:** $k \neq -2$.

139. A and B have the same set of eigenvalues.

140. Take $P = [\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2]$ and $Q = [\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1]$. Then $AP = PB$ and $AQ = QC$. Also $BC = CB \Rightarrow a^3 + b^3 + c^3 = 3abc$. Use this equation in computing $\det(A - \lambda I)$.

141. Take transpose of both sides of $P^{-1}AP = B$.

142. Use Problem 141.

143. Take inverse of both sides of $P^{-1}AP = B$.

144. $P^{-1}AP = D \Rightarrow P^{-1}A^mP = D^m$.

145. (a) When does the equation $\det(A - \lambda I) = 0$ have distinct real roots? $\quad (b)\ \ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \ B = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

146. $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2,\quad A^k\mathbf{x} = [2^{1-k} + 3.2^k,\ 3.2^k - 2^{1-k}]^t$.

147. Easy.

148. Easy.

149. Expand $\det(C(p) - \lambda I)$ through the last column and use induction.

150. $a_0 = \det(A) \neq 0$. Use Cayley-Hamilton Theorem.

151. (a) $xy = ab, x + y = a + b \Rightarrow \{x, y\} = \{a, b\}$. (b) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

152. Easy.

153. For the first four parts, applly distributive and commutative laws in $\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y}).(\mathbf{x} + \mathbf{y})$.
(e) $\|s\mathbf{x} + t\mathbf{y}\| \leq s\|\mathbf{x}\| + t\|\mathbf{y}\|$. Also for $t \geq s$, $\|s\mathbf{x} + t\mathbf{y}\| = \|t(\mathbf{x} + \mathbf{y}) - (t - s)\mathbf{x}\| \geq t\|\mathbf{x} + \mathbf{y}\| - (t - s)\|\mathbf{x}\|$.
Similarly, for $t < s$, $\|s\mathbf{x} + t\mathbf{y}\| \geq s\|\mathbf{x}\| + t\|\mathbf{y}\|$.

154. Use $\mathbf{v}_i.\mathbf{v}_j = 0$ in the expansion of the RHS of $\left\|\sum\limits_{i=1}^{n} \mathbf{v}_i\right\|^2 = (\sum\limits_{i=1}^{n} \mathbf{v}_i).(\sum\limits_{i=1}^{n} \mathbf{v}_i)$.

155. $S^\perp = \{[u, v, w]^t \in \mathbb{R}^3 : u + 2v = 0, 3u - 2w = 0\}$ and a basis for $S^\perp$ is $\{[2, -1, 3]^t\}$.
$M^\perp = \{[u, v, w]^t \in \mathbb{R}^3 : 2u + 2v - w = 0\}$ and a basis for $M^\perp$ is $\{[-1, 1, 0]^t, [1, 0, 2]^t\}$.
$W^\perp = \{[u, v, w]^t \in \mathbb{R}^3 : u - v + 3w = 0\}$ and a basis for $W^\perp$ is $\{[1, 1, 0]^t, [3, 0, -1]^t\}$.

156. $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, and $A^{-1} = A^t$.

157. Show that $(M + N)^\perp \subseteq M^\perp \cap N^\perp$ and $M^\perp \cap N^\perp \subseteq (M + N)^\perp$. For the 2nd part, replace M by $M^\perp$ and N by $N^\perp$ in the 1st part.

158. $A\mathbf{x} = \lambda\mathbf{x}, A\mathbf{y} = \mu\mathbf{y} \Rightarrow \lambda\mathbf{x}^t\mathbf{y} = \mathbf{x}^t A\mathbf{y} = \mu\mathbf{x}^t\mathbf{y}$.

159. Rearranging the columns of Q^t will not change the orthogonality of its columns.

160. See **Theorem 5.4** of the text book.

161. Use **Problem 160**.

162. If $A = [a_{ij}]$ is orthogonal and upper triangular, then we have $a_{11}^2 = a_{11}^2 + a_{12}^2 + \ldots + a_{1n}^2$.

163. See **Theorem 5.6** of the text book.

164. Use **Problem 163**.

165. (a) $\{[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]^t, [-\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{2\sqrt{2}}{3}]^t\}$, (b) $\{[-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]^t, [\frac{1}{\sqrt{6}}, -\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{6}}]^t, [\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}]^t\}$,
(c) $\{[\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}]^t, [0, \frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{\sqrt{2}}{\sqrt{7}}]^t\}$,
(d) $\{[\frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}]^t, [-\frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, -\frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}]^t, [-\frac{2}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}]^t\}$.

166. $\{[0, 1, 1]^t, [1, -\frac{1}{2}, \frac{1}{2}]^t, [\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}]^t\}$ and $\{[1, 1, -1, 1]^t, [0, -2, 2, 4]^t, [0, 1, 1, 0]^t\}$.

167. Apply Gram-Schmidt Process to $\{[3, 1, 5]^t\} \cup B$, where B is a basis for $S^\perp$ and $S = \text{span}([3, 1, 5]^t)$.

168. Similar to **Problem 167**.

169. Apply Gram-Schmidt Process to a basis for the subspace.

170. $\text{proj}_W(\mathbf{v}) = [-\frac{1}{2}, \frac{1}{2}, 3]^t$.

171. $\text{proj}_W(\mathbf{v}) = [0, 1, 1, 0]^t$.

172. $m - k$.

173. $AA^t = I = A^t A$.

174. Similar to **Problem 168**.

175. The given statement is true.

176. (a) Direct use of the definition of $\text{proj}_W(\mathbf{v})$. (c) Use part (a).

177. See **Q.2** of **Quiz I**.

178. Routine Check.

179. Routine Check.

180. $(a),(b)$ $1[2,2]^t \neq [2,2]^t$. (c) Addition is not commutative. (d) $(1+1)[2,2]^t \neq 1[2,2]^t + 1[2,2]^t$.
(e) Addition is not commutative. $(f),(i)$ $\mathbf{0} \notin V$. (g) V is a vector space iff $c=0$. (h) Routine Check.

181. Routine Check.

182. Routine Check.

183. Routine Check.

184. Routine Check.

185. Routine Check.

186. If $\mathbf{0}$ and $\mathbf{0}'$ are two zeros then $\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'$.

187. $\mathbf{v} + \mathbf{v}' = \mathbf{0} = \mathbf{v} + \mathbf{v}'' \Rightarrow -\mathbf{v} + (\mathbf{v} + \mathbf{v}') = -\mathbf{v} + (\mathbf{v} + \mathbf{v}'')$.

188. No. Compare coefficients of $s(x) = ap(x) + bq(x) + cr(x)$ and solve for a, b, c.

189. Consider $a\mathbf{v} + \sum\limits_{i=1}^{k} a_i \mathbf{v}_i = \mathbf{0}$. Then $a \neq 0 \Rightarrow \mathbf{v} \in \text{span}(S)$.

190. $\text{span}(S) = \text{span}(S \setminus \{\mathbf{u}_k\})$.

191. $\text{span}(S) \in \mathcal{A} \Rightarrow \bigcap\limits_{W \in \mathcal{A}} W \subseteq \text{span}(S)$. Also $S \subseteq W \Rightarrow \text{span}(S) \subseteq W \Rightarrow \text{span}(S) \subseteq \bigcap\limits_{W \in \mathcal{A}} W$.

192. (a) Consider $\mathbf{u}_1 = [1,0]^t, \mathbf{u}_2 = [2,0]^t, \mathbf{u}_3 = [0,1]^t$. (a) Consider $\mathbf{u}_1 = [1,0,0]^t, \mathbf{u}_2 = [0,1,0]^t, \mathbf{u}_3 = [1,1,0]^t$.
(c) Consider $\mathbf{u}_1 = [1,0]^t, \mathbf{u}_2 = [2,0]^t, \mathbf{u}_3 = [0,1]^t, \mathbf{u}_4 = [0,2]^t$.

193. $a\mathbf{v} + \sum\limits_{i=1}^{k} a_i \mathbf{u}_i = \mathbf{0} \Rightarrow a \neq 0$. Also $\sum\limits_{i=1}^{k} b_i \mathbf{u}_i = \mathbf{v} = \sum\limits_{i=1}^{k} c_i \mathbf{u}_i \Rightarrow b_i = c_i$.

194. $a_1\mathbf{u}_1 + a_2(\mathbf{u}_1 + \mathbf{u}_2) + \ldots + a_n(\mathbf{u}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_n) = \mathbf{0} \Rightarrow a_1 + \ldots + a_n = 0, \ldots, a_{n-1} + a_n = 0, a_n = 0$. The converse is also true.

195. $(a),(d),(f),(g)$ Linearly independent, (b) Linearly dependent, $[1,2,0,0]^t = 2[1,1,0,0]^t - [1,0,0,0]^t + 0[1,1,1,1]^t$,
(c) Linearly dependent, $[i+2,-1,2]^t = i[1,i,0]^t + 2[1,0,1]^t$,
(e) $\{1, i\}$ is linearly independent in $\mathbb{C}(\mathbb{R})$, but linearly dependent in $\mathbb{C}(\mathbb{C})$, $1 = -i.i$.

196. (a) Yes. (b) $(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w}) = \mathbf{u} - \mathbf{w}$.

197. Similar to **Problem** 194. Or, the reduced ro echelon form of A is I_n.

198. $\mathbf{v}_1 = \frac{1}{3}[2(\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{v}_1 - 2\mathbf{v}_2)]$, $\mathbf{v}_2 = \frac{1}{3}[(\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{v}_1 - 2\mathbf{v}_2)]$.

199. None of them are bases. Notice that $x - x^2 = (1 - x^2) - (1 - x)$, and the second set is linearly independent but not spanning.

200. $a \neq 0, 1, -1$.

201. $\mathbf{z} = -\frac{1}{7}(\mathbf{x} + 2\mathbf{y}), \mathbf{y} \in \text{span}(\mathbf{x}, \mathbf{y}) \Rightarrow \text{span}(\mathbf{y}, \mathbf{z}) \subseteq \text{span}(\mathbf{x}, \mathbf{y})$.

202. $[1,0]^t + i[i,0]^t = [0,0]^t$.

203. The set is linearly independent for all values of α.

204. $\sum a_k \mathbf{x}_k + \sum b_k (i\mathbf{x}_k) = \mathbf{0} \Rightarrow \sum (a_k + ib_k)\mathbf{x}_k = \mathbf{0}$.

205. The 1st and the 4th sets are subspaces of $\mathbb{C}^3(\mathbb{R})$ but not of $\mathbb{C}^3(\mathbb{C})$. The 2nd set is a subspace of each of $\mathbb{C}^3(\mathbb{R})$ and $\mathbb{C}^3(\mathbb{C})$. The 3rd set is neither a subspace of $\mathbb{C}^3(\mathbb{R})$ nor a subspace of $\mathbb{C}^3(\mathbb{C})$.

206. Take $U = \{[x,0]^t : x \in \mathbb{R}\}$ and $W = \{[0,x]^t : x \in \mathbb{R}\}$.

207. Routine check. $U + W$ will represent the xy-plane in $\mathbb{R}^3$.

208. If $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is a basis for W then $\{(\mathbf{w}_i, \mathbf{w}_i) : i = 1, 2, \ldots, k\}$ is a basis for Δ.

209. $\dim(U \times V) = \dim(U).\dim(V)$.

210. $\sum\limits_{k\neq i} a_k\mathbf{x}_k + a_i(c\mathbf{x}_i) = \mathbf{0} \Rightarrow a_k = \mathbf{0}$ for $k \neq i$ and $ca_i = \mathbf{0}$.

211. $\{1-x, x-x^2\}$.

212. $\{1, 1+x, 1+x+x^2\}$. (Some other answer is also possible).

213. $\left\{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$. (Some other answer is also possible).

214. $(a), (d)$ True statements. (b) Take $\mathbf{x}_1 = [1,0]^t, \mathbf{x}_2 = [2,0]^t, \mathbf{x}_3 = [0,1]^t$. (c) If $W_1 = \text{span}(1, x, x+x^2, x^3),\ W_2 = \text{span}(1, x, x^2, x^4, x^5)$ then $W_1 \cap W_2 = \text{span}(1, x, x^2) = \text{span}(1, x, x+x^2)$.

215. (a) $\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$, (b) $\left\{\begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 0 & 6 \end{bmatrix}\right\}$, (c) $\{2+x, 1-x^3\}$,
(d) Let $A_{lk} = [a_{ij}]$, where $a_{l1} = 1, a_{lk} = -1$ and $a_{ij} = 0$ otherwise. Then $\{A_{lk} : 1 \leq l \leq m, 2 \leq k \leq n\}$ is a basis.

216. (a) $\{A_{lk} : 1 \leq l \leq n, 1 \leq k \leq n, l \leq k\}$, where $A_{lk} = [a_{ij}]$ and $a_{ij} = \begin{cases} 1 & \text{if } i = l, j = k, \\ 0 & \text{otherwise.} \end{cases}$

(b) $\{B_{lk} : 1 \leq l \leq n, 1 \leq k \leq n, l \geq k\}$, where $B_{lk} = [b_{ij}]$ and $b_{ij} = \begin{cases} 1 & \text{if } i = l, j = k, \\ 0 & \text{otherwise.} \end{cases}$

(c) $\{A_l : 1 \leq l \leq n\}$, where $A_l = [a_{ij}]$ and $a_{ij} = \begin{cases} 1 & \text{if } i = j = l, \\ 0 & \text{otherwise.} \end{cases}$

(d) $\{A_{lk} : 1 \leq l \leq n, 1 \leq k \leq n, l \neq k\} \cup \{B_l : 2 \leq l \leq n\}$, where $A_{lk} = [a_{ij}],\quad a_{ij} = \begin{cases} 1 & \text{if } i = l, j = k, \\ 0 & \text{otherwise,} \end{cases}$ and

$B_l = [b_{ij}],\quad b_{ij} = \begin{cases} 1 & \text{if } i = j = 1, \\ -1 & \text{if } i = j = l, \\ 0 & \text{otherwise.} \end{cases}$

(e) $\{A_{lk} : 1 \leq l \leq n, 1 \leq k \leq n, l < k\}$, where $A_{lk} = [a_{ij}]$ and $a_{ij} = \begin{cases} 1 & \text{if } i = l, j = k, \\ -1 & \text{if } i = k, j = l, \\ 0 & \text{otherwise.} \end{cases}$

217. The required coordinate is $[1, -2, 3]^t$.

218. $(a), (c), (d), (e)$ n^2, (b) $2n^2$.

219. A basis for V is $\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right\}$.

A basis for W is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\right\}$.

220. $B = \{[1, -1, 0, 1]^t, [1, 0, -1, 0]\}$ is a basis for W. An extension is $B \cup \{[1, 0, 0, 0]^t, [0, 1, 0, 0]^t\}$.

221. $B = \{[0, 0, 0, 0, 1, -1]^t, [0, 1, -1, 0, 0, 0]^{\cdot}[1, 0, -1, 1, 0, 0]^t\}$ is a basis for W. An extension is $B \cup \{[1, 0, 0, 0, 0, 0]^t, [0, 1, 0, 0, 0, 0]^t, [0, 0, 0, 0, 1, 0]^t\}$.

222. $a_1\mathbf{v}_1 + \sum\limits_{i=2}^{n} a_i(\alpha\mathbf{v}_1 + \mathbf{v}_i) = \mathbf{0} \Rightarrow a_1 + \alpha a_2 + \ldots + \alpha a_n = 0, a_2 = 0, \ldots, a_n = 0$.

223. Examine that the given set is linearly independent.

224. The given set is linearly independent over $\mathbb{C}$. An extension is $\{[1, 0, 0]^t, [1, 1, 1]^t, [1, 1, -1]^t, [i, 0, 0]^t, [i, i, i]^t, [i, i, -i]^t\}$.

225. A basis for W is $\{1-x, 1-x^2\}$.

226. Only part (c) is not a subspace.

227. If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ then because of the uniqueness, $\sum a_i\mathbf{v}_i = \mathbf{0} \Rightarrow a_i = 0$.

228. $\dim(W_1) = 3, \dim(W_2) = 3, \dim(W_1 + W_2) = 4$ and $\dim(W_1 \cap W_2) = 2$.

229. $\dim(W_1 \cap W_2) \leq \dim(W_2) = 5$. Also $\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) \geq 6 + 5 - 8 = 3$.

230. $\dim(M \cap N) = \dim(M) + \dim(N) - \dim(M + N) \geq 4 + 4 - 7 = 1$

231. If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a basis for M and $B \cup \{\mathbf{u}_1, \ldots, \mathbf{u}_{n-m}\}$ is an extension of B, then consider $N = \text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_{n-m})$.

232. $\{[-1,2,0]^t, [-2,0,1]^t\}$, $\{[1,1,0]^t, [1,0,-1]^t\}$, $\{[-5,-2,3]^t\}$ and $\{[1,1,0]^t, [1,0,-1]^t, [-1,2,0]^t\}$ are bases for W_1, $W_2, W_1 \cap W_2$ and $W_1 + W_2$, respectively.

233. $\{[1,1,0]^t, [-1,1,0]^t, [1,0,2]^t\}$ is linearly independent. Also $[1,1,1]^t = [0,1,0]^t + [1,0,1]^t = [1,1,0]^t + [0,0,1]^t$.

234. Both the spaces have the equal dimension.

235. The set $\{[1,0,0]^t, [1,1,0]^t, [1,1,1]^t\}$ is linearly independent. Also $[x,y,z]^t = [x-z, y-z, 0]^t + [z,z,z]^t$.

236. $f \in V_e \cap V_o \Rightarrow f(x) = f(-x), f(-x) = -f(x)$ for all $x \in \mathbb{R} \Rightarrow f(x) = -f(x)$ for all $x \in \mathbb{R} \Rightarrow f = \mathbf{0}$. Also $f = f_e + f_o$, where $f_e(x) = \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$.

237. $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_1' + \mathbf{v}_2' \Rightarrow \mathbf{v}_1 - \mathbf{v}_1' = \mathbf{v}_2 - \mathbf{v}_2' \in V_1 \cap V_2 = \{\mathbf{0}\}$.

238. $[\frac{1}{2}, \frac{3}{2}, 3]^t$.

239. $[-1,-1,-1,4]^t$.

240. $[p(x)]_B = [1,-1,1]^t$, $[p(x)]_C = [1,0,1]^t$, $P_{C \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $P_{B \leftarrow C} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

241. $(1+x^n) + (-x^n) = 1$. If B is the given basis then $[2x-1]_B = [\frac{1}{2}, \frac{3}{4}, -\frac{3}{4}]^t$, $[x^2+1]_B = [\frac{1}{2}, \frac{1}{4}, -\frac{3}{4}]^t$ and $[x^2+5x-1]_B = [2,2,-1]^t$.

242. Let $\mathbf{u}_1 = [1,1,\ldots,1]^t, \mathbf{u}_2 = [1,2,3,\ldots,n]^t$ and $\mathbf{u}_3 = [0,\ldots,0,1,0,\ldots,0]^t$. Then $[\mathbf{u}_1]_B = \mathbf{e}_1, [\mathbf{u}_2]_B = \mathbf{u}_1$ and $[\mathbf{u}_3]_B = [0,\ldots,0,1,-1,0,\ldots,0]^t$, where 1 is at the i-th place.

243. $P_{C \leftarrow B} = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$, $P_{B \leftarrow C} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

244. $[\mathbf{e}_1]_B = [0,0,1,-2]^t, [\mathbf{e}_2]_B = [1,0,-1,2]^t, [\mathbf{e}_3]_B = [0,1,0,-\frac{1}{2}]^t$ and $[\mathbf{e}_4]_B = [0,0,0,\frac{1}{2}]^t$.

245. $\mathbf{v}_1 = -i\mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{v}_2 = (2-i)\mathbf{u}_1 + i\mathbf{u}_2$, $[\mathbf{u}_1]_B = [\frac{1-i}{2}, \frac{1+i}{2}]^t$, $[\mathbf{u}_2]_B = [\frac{i+1}{2}, \frac{i-1}{2}]^t$.

246. $[\mathbf{v}]_C = [\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}]^t$, $[\mathbf{w}]_B = [1,1,\ldots,1]^t$ and $[\mathbf{w}]_C = [\frac{1}{a_1}, \ldots, \frac{1}{a_n}]^t$.

247. Let $\text{span}(A) = V$ and $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $\mathbf{x} = [x_1, \ldots, x_n]^t$. Take $\mathbf{v} = \sum\limits_{i=1}^{n} x_i \mathbf{v}_i$.
Then $\mathbf{v} = \sum\limits_{i=1}^{m} y_i \mathbf{u}_i \Rightarrow \mathbf{x} = [\mathbf{v}]_B = y_1[\mathbf{u}_1]_B + \ldots + y_m[\mathbf{u}_m]_B$.

248. $P_{B \leftarrow C} = (P_{C \leftarrow B})^{-1}$ and $C = \{1 - 2x + 2x^2, 2x - x^2, 1 - 3x + 2x^2\}$.

249. $a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n = \mathbf{0} \Rightarrow P\mathbf{a} = \mathbf{0}$, where $\mathbf{a} = [a_1, \ldots, a_n]^t \Rightarrow \mathbf{a} = \mathbf{0}$.

250. Yes: $(b)i, (b)v, (b)vi, (c)i, (d)$. No: $(a), (b)ii, (b)iii, (b)iv, (c)ii, (c)iii$.

251. $T[a,b]^t = \frac{1}{4}[(a+3b) - (a+7b)x + 2(a-b)x^2]$.

252. $T(a + bx + cx^2) = a + cx + \frac{(3a-b-c)}{2}x^2$.

253. $(T \circ S)[x,y]^t = [0,y]^t$, $(S \circ T)[x,y]^t = [x,0]^t$.

254. $T(i.1) \neq iT(1)$.

255. $T(\mathbf{v}) = T(\sum \mathbf{a}_i\mathbf{u}_i) = \sum \mathbf{a}_i T(\mathbf{u}_i) = \sum \mathbf{a}_i \mathbf{u}_i = \mathbf{v}$ for all $\mathbf{v} \in V$.

256. For $\mathbf{v} = \sum \mathbf{a}_i\mathbf{v}_i$, define $T(\mathbf{v}) = \sum \mathbf{a}_i\mathbf{w}_i$.

257. (a) $\ker(T) = \left\{ \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$, $\text{range}(T) = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{R} \right\}$.
(b) $\ker(T) = \text{span}(1 + x - x^2)$, $\text{range}(T) \cong \mathbb{R}^2$, (c) $\ker(T) = \{\mathbf{0}\}$, $\text{range}(T) = \{[x, y, z]^t \in \mathbb{R}^3 : 2x - y - z = 0\}$,
(d) $\ker(T) = \{\mathbf{0}\}$, $\text{range}(T) = \text{span}(x, x^2)$, (e) $\ker(T) = \left\{ \begin{bmatrix} a & a \\ c & c \end{bmatrix} : a, c \in \mathbb{R} \right\}$, $\text{range}(T) \cong \mathbb{R}^2$.

258. (a) $\ker(T) = \{\mathbf{0}\}$, (b) $\ker(T) \neq \{\mathbf{0}\}$.

259. (a) Define $T(\mathbf{e}_1) = [1, 1, 1]^t, T(\mathbf{e}_2) = T(\mathbf{e}_3) = \mathbf{0}$. (b) Define $T(\mathbf{e}_1) = [1, 2, 3]^t, T(\mathbf{e}_2) = [1, 3, 2]^t, T(\mathbf{e}_3) = \mathbf{0}$.

260. (a) $T(\mathbf{x}) = \mathbf{0} \Rightarrow (S \circ T)(\mathbf{x}) = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$. (b) For $\mathbf{w} \in W$, $S(T\mathbf{u}) = \mathbf{w}$ where $\mathbf{u} \in U$.

261. $T[x, y, z]^t = [x, y, y - z]^t$, $\text{nullity}(T) = 0$, $\text{rank}(T) = 3$, $T(T[x, y, z]^t) = [x, y, z]^t$.

262. If $k \geq n + 1$ then $f \in \ker(T) \Rightarrow f$ will have at least $n + 1$ roots $\Rightarrow \ker(T) = \{\mathbf{0}\}$.
If $k = n$ then $\ker(T) = \text{span}(f(z))$, where $f(z) = (z - z_1)(z - z_2) \ldots (z - z_n)$.
If $k < n$ then $\ker(T) = \{(z - z_1)(z - z_2) \ldots (z - z_k)q(z) :\ q(z)$ is a polynomial of degree at most $n - k\}$. Now use $\text{rank}(T) + \text{nullity}(T) = n + 1$.

263. All are isomorphisms. One method may be by showing that $\ker(T) = \{\mathbf{0}\}$.

264. (a) $T(a + ib) = [a, b]^t$, (b) $\dim(V) \neq \dim(\mathbb{R}^2)$, (c) $T(\text{diag}[x, y, z]) = [x, y, z]^t$, (d) $\dim(V) \neq \dim(W)$.

265. Consider $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$. Now $m < n \Rightarrow T$ is not one-one, and $m > n \Rightarrow T$ is not onto.

266. $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\text{range}(T)$, $\{\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_3\}$ is a basis for $\ker(T)$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\ker(T) + \text{range}(T)$ and $\{\mathbf{u}_1 - \mathbf{u}_2\}$ is a basis for $\ker(T) \cap \text{range}(T)$.

267. $\text{rank}(T) = \text{rank}(T^2) \Rightarrow \text{nullity}(T) = \text{nullity}(T^2)$. Also $\ker(T) \subseteq \ker(T^2)$. Hence $\ker(T) = \ker(T^2)$. Now $\mathbf{x} \in \ker(T) \cap \text{range}(T) \Rightarrow T\mathbf{x} = \mathbf{0}, \mathbf{x} = T\mathbf{y}$ for some $\mathbf{y}$ $\Rightarrow T^2\mathbf{y} = \mathbf{0} \Rightarrow T\mathbf{y} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$.

268. (c) Define $S : \ker(T) \to U \cap W$ by $S(\mathbf{w}, \mathbf{w}) = \mathbf{w}$. Then S is an isomorphism. (d) Use Rank-Nullity Theorem.

269. (a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, (e) $\begin{bmatrix} 0 & i \\ 1 & 1 - i \end{bmatrix}$, (c) $[\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}]$, (d) $[\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5]$,
(b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, (f) $\begin{bmatrix} 1 + i & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 1 + i \end{bmatrix}$.

270. If $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ then $T(\mathbf{e}_i) = \mathbf{a}_i$.

271. $T[x, y, z]^t = A[x, y, z]^t = [x + 2y - 4z,\ 2x - 3y,\ -4x + 5y + z]^t$. Also A is invertible $\Rightarrow T$ is invertible, and $T^{-1}[x, y, z]^t = A^{-1}[x, y, z]^t$.

272. Use $A[\mathbf{x}]_B = [T(\mathbf{x})]_B$. We have $T[x, y, z]^t = \left[3x + 5y - 4z,\ \frac{-5x - 4y + 9z}{2},\ x + 3y - 2z\right]$.

273. Use $A[\mathbf{x}]_B = [T(\mathbf{x})]_C$. We have $T[x, y, z]^t = (-2x + 2y) + (-4y + 2z)t$.

274. (a) A basis for $\text{null}(A)$ is $\{\mathbf{x}, \mathbf{y}\}$, where $\mathbf{x} = [-2, -\frac{3}{2}, 1, 0]^t, \mathbf{y} = [-1, -2, 0, 1]^t$. We have $\ker(T) = \text{span}(\mathbf{v}, \mathbf{w})$, where $\mathbf{v} = -2\mathbf{u}_1 - \frac{3}{2}\mathbf{u}_2 + \mathbf{u}_3$ and $\mathbf{w} = -\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_4$.
Now $\text{col}(A) = \text{span}(\mathbf{a}_1, \mathbf{a}_2)$, where $\mathbf{a}_1, \mathbf{a}_2$ are the 1st and the 2nd column of A. Therefore $\text{range}(T) = \text{span}(T\mathbf{u}_1, T\mathbf{u}_2)$, since $\mathbf{a}_1 = [T\mathbf{u}_1]_B,\ \mathbf{a}_2 = [T\mathbf{u}_2]_B$.
(b) $C = \{\mathbf{v}, \mathbf{w}, \mathbf{u}_1, \mathbf{u}_2\}$ will be a basis for V. Also $[T]_C = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 9/2 & 1 \end{bmatrix}$.

275. (a) $\begin{bmatrix} 8 & 9 \\ 4/3 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 7 & 8 \\ 13 & 14 \end{bmatrix}$, (c) $[3, 5]^t$, (d) $[9, 6]^t$.