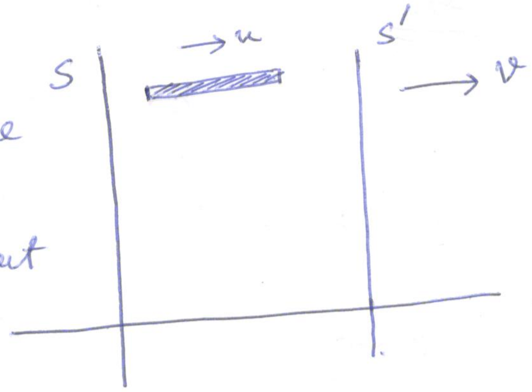
(1) A rod of proper length 'l_0' oriented parallel to the x-axis moves with speed 'u' along x-axis in S. What is the length measured by an observer in S', moving with velocity 'v' with respect to S.



Ans:

For this problem we have to use velocity addition theorem, Assume 'u'' is the velocity of the rod with respect to 'S'',

$$u' = \frac{u-v}{1-\frac{uv}{c^2}} \quad \text{--- (i)}$$

Again from formula of length contraction

$$l' = l_0\sqrt{1-\frac{u'^2}{c^2}}$$

Therefore,

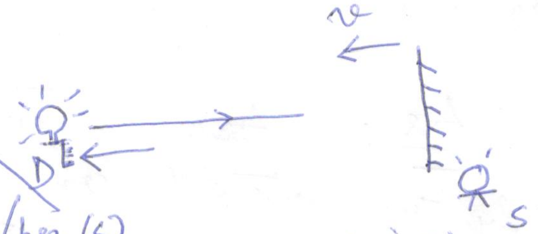
$$l' = l_0\sqrt{1-\frac{(u-v)^2}{\left(1-\frac{uv}{c^2}\right)^2}\frac{1}{c^2}}$$

$$= \frac{l_0}{c\left(1-\frac{uv}{c^2}\right)}\sqrt{\left(1-\frac{uv}{c^2}\right)^2c^2-(u-v)^2}$$

$$= \frac{l_0}{(c^2-uv)}\sqrt{c^4-c^2u^2+u^2v^2-c^2v^2}$$

$$\boxed{l' = \frac{l_0}{(c^2-uv)}\sqrt{(c^2-u^2)(c^2-v^2)}}.$$

(2) The frequency of light reflected from a moving mirror (2) undergoes a Doppler shift. Find the Doppler shift of light reflected directly back from a mirror which is approaching the observer with speed v.



Ans: Method: I

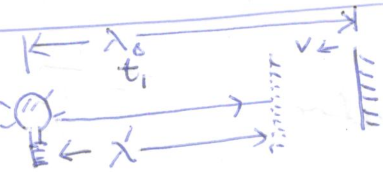
1st assume an observer is attached (S) with the mirror. With respect to him/her (S) the light source is approaching towards him with 'v'. The frequency shift (For this we have already calculated in the class)

$$\boxed{\nu_s = \nu_0 \sqrt{\frac{c+v}{c-v}}}$$

Now we can assume the original observer 'D' receives the light from source 'S' whose frequency is ν_s. Therefore, we can use same formula as above

$$\nu_D = \nu_s \sqrt{\frac{c+v}{c-v}} = \nu_0 \left(\sqrt{\frac{c+v}{c-v}}\right)^2 = \nu_0 \frac{c+v}{c-v}.$$

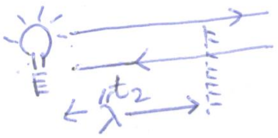
Therefore Dopper shift : $$\boxed{\frac{\nu_D - \nu_0}{\nu_0} = \frac{\cancel{c}+v-\cancel{c}+v}{c-v} = \frac{2v}{c-v}}$$



stage-I

Method-II

Assume λ_0 is the proper wave length of the light. And final wavelength observed by the observer is λ''.



. Stage-II.

In stage-I : $\lambda' = \lambda_0 - vt_1 = \lambda_0 - v\frac{\lambda'}{c} \Rightarrow \lambda' = \frac{\lambda_0}{1+v/c}$

In stage-II : $\lambda'' = \lambda' - vt_2 = \lambda' - v\cancel{t_1} = \lambda' - v\frac{\lambda'}{c} = \lambda'(1 - v/c)$

Therefore : $\lambda'' = \lambda_0 \frac{1-v/c}{1+v/c} \Rightarrow \boxed{\nu'' = \nu_D = \nu_0 \frac{c+v}{c-v}}$

(3) (3) A particle of mass m is moving in the direction $+x$ with speed u_x, and has momentum p_x and energy E in the frame S. If S' is moving at speed V_x along x-direction, determine the momentum p_x', and energy E' observed in S', and show that $E'^2 - p_x'^2 c^2 = E^2 - p_x^2 c^2$.

Solution: We know the following transformation laws:

$$x' = \gamma(x - v_x t) \quad ; \quad t' = \gamma\left(t - \frac{v_x x}{c^2}\right)$$

$$u_x' = \frac{u_x - v_x}{1 - \frac{u_x v_x}{c^2}} \quad ; \quad \gamma = \sqrt{1 - \frac{v_x^2}{c^2}}$$

Now in : S frame we know

$$p_x = m u_x = \frac{m_0 u_x}{\sqrt{1 - \frac{u_x^2}{c^2}}}$$

$$E = m c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{u_x^2}{c^2}}}$$

In S' frame

$$p_x' = m' u_x' = \frac{m_0 u_x'}{\sqrt{1 - \frac{u_x'^2}{c^2}}}$$

$$E' = \frac{m_0 c^2}{\sqrt{1 - \frac{u_x'^2}{c^2}}}$$

Our goal is to find the relation between (p_x', p_x) and (E', E)

Let us consider

$$1 - \frac{u_x'^2}{c^2} = 1 - \frac{(u_x - v_x)^2}{\left(1 - \frac{u_x v_x}{c^2}\right)^2 c^2} = \frac{c^2 + \frac{u_x^2 v_x^2}{c^2} - 2u_x v_x - u_x^2 - v_x^2 - 2u_x v_x}{\left(1 - \frac{u_x v_x}{c^2}\right)^2 c^2}$$

$$= \frac{c^2\left(1 - \frac{u_x^2}{c^2}\right) - v_x^2\left(1 - \frac{u_x^2}{c^2}\right)}{\left(1 - \frac{u_x v_x}{c^2}\right)^2 c^2} = \frac{\left(1 - \frac{u_x^2}{c^2}\right)\left(1 - \frac{v_x^2}{c^2}\right)}{\left(1 - \frac{u_x v_x}{c^2}\right)}$$

Hence : $$E' = \frac{m_0 c^2}{\sqrt{1 - \frac{u_x'^2}{c^2}}} = \frac{m_0 c^2}{\sqrt{\left(1 - \frac{u_x^2}{c^2}\right)\left(1 - \frac{v_x^2}{c^2}\right)}}\left(1 - \frac{u_x v_x}{c^2}\right) = \gamma\left(E - v_x p_x\right)$$

Similarly : $$p_x' = \frac{m_0}{\sqrt{\left(1 - \frac{u_x^2}{c^2}\right)\left(1 - \frac{v_x^2}{c^2}\right)}}(u_x - v_x) = \gamma\left(p_x - \frac{v_x E}{c^2}\right)$$

Now we have to show

$$\underbrace{E'^2 - c^2 p_x'^2 - c^2 p_y'^2 - c^2 p_z'^2}_{\Downarrow} = E^2 - c^2 p_x^2 - c^2 p_y^2 - c^2 p_z^2$$

$$\gamma^2 (E - v_x p_x)^2 - \gamma^2 \left(p_x - \frac{v_x E}{c^2}\right)^2 c^2 - c^2 p_y^2 - c^2 p_z^2$$

$$= \gamma^2 \left(\frac{E^2}{\gamma^2} - \frac{c^2 p_x^2}{\gamma^2}\right) - p_y^2 c^2 - p_z^2 c^2$$

$$= E^2 - c^2 p_x^2 - c^2 p_y^2 - c^2 p_z^2$$

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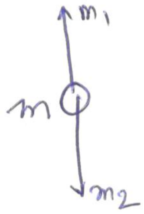
4 a) A body of mass 'm' at rest breaks up spontaneously into two parts, having rest masses m_1 and m_2 and respective speed v_1 and v_2. Show that $m > m_1 + m_2$.



Ans:

energy conservation

$$E_0 = E_1 + E_2$$

$$mc^2 = T_1 + m_1c^2 + T_2 + m_2c^2$$

Hence $$mc^2 = (T_1 + T_2) + (m_1 + m_2)c^2$$

Hence $$mc^2 > (m_1 + m_2)c^2$$

$$\boxed{m > m_1 + m_2}$$

where we know in general

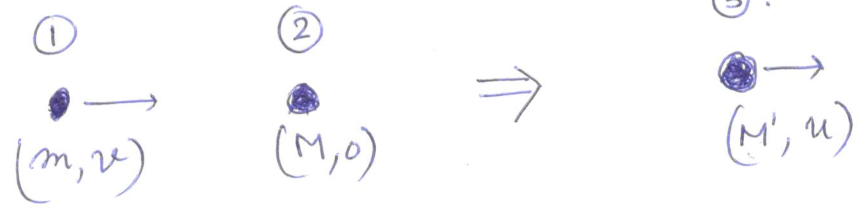
$$E = T_0 + m_0c^2$$

Hence $E \geq m_0c^2$

$$\boxed{T > 0}$$

4 6 A particle of rest mass m and speed v collides and sticks to a stationary particle of mass M. What is the final speed of the composit particle?

Ans:



Four momentum of particle 1 $\Rightarrow p_\mu^{(1)} \equiv \left(\frac{iE^{(1)}}{c}, \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right)$,

Four momentum of particle 2 $\Rightarrow p_\mu^{(2)} \equiv \left(\frac{iE^{(2)}}{c}, 0\right)$

Four momentum of particle 3 $\Rightarrow p_\mu^{(3)} \equiv \left(\frac{iE^{(3)}}{c}, \frac{M'u}{\sqrt{1-u^2/c^2}}\right)$

According to conservation of four-momentum

$$p_\mu^{(1)} + p_\mu^{(2)} = p_\mu^{(3)}$$

Energy conservation

$$\Rightarrow p_1^{(1)} + p_1^{(2)} = p_1^{(3)} \Rightarrow \frac{E^{(1)}}{c} + \frac{E^{(2)}}{c} = \frac{E^{(3)}}{c}$$

$$\Rightarrow \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + Mc^2 = \frac{M'c^2}{\sqrt{1-u^2/c^2}}$$

$$\text{or} \quad \boxed{(\gamma m + M) = \frac{M'}{\sqrt{1-u^2/c^2}}} \quad \ldots (1)$$

Momentum conservation: $p_2^{(1)} + p_2^{(2)} = p_3^{(3)}$

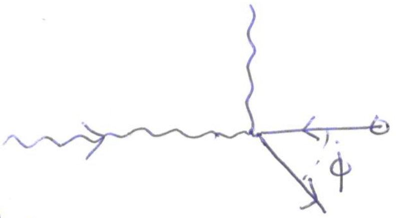
$$\frac{mv}{\sqrt{1-v^2/c^2}} = \frac{M'u}{\sqrt{1-u^2/c^2}} \Rightarrow \boxed{mv\gamma = \frac{M'u}{\sqrt{1-u^2/c^2}}} \ldots (2)$$

Now combining equations (1) & (2)

Speed of the combined particle: $\boxed{u = \frac{mv\gamma}{\gamma m + M}}$ where $\boxed{\gamma = \frac{1}{\sqrt{1-v^2/c^2}}}$

(6) (7)

(5) A photon of energy E_0 ~~and wavelength λ_0~~ collides head on with a free electron of rest mass m_e^0 and speed v as shown. The photon is scattered at 90°. Find the E of the scattered photon.



Ans: In terms of four vector notation

~~Init;~~ Initial values.

$$p_\mu^{\gamma} = \left(\frac{iE_0}{c}, \frac{E_0}{c}\right);$$

$$p_\mu^{e} = \left(i\frac{m_e c^2}{c}, m_e v\right)$$

where $m_e = \dfrac{m_e^0}{\sqrt{1-v^2/c^2}}$

Final values

$$p_\mu^{\gamma'} = \left(\frac{iE_0'}{c}, \frac{E_0'}{c}\right)$$

$$p_\mu^{e'} = \left(i\frac{m_e' c^2}{c}, m_e' v'\right).$$

$$m_e' = \frac{m_e^0}{\sqrt{1-\frac{v'^2}{c^2}}}$$

Now from conservation laws:

Energy: $$\frac{E_0}{c} + m_e c = \frac{E_0'}{c} + m_e' c \quad \ldots (1)$$

Momentum: $$\frac{E_0}{c} - m_e v = \frac{E_0'}{c}\cos 90^\circ + m_e' v' \cos\phi \quad \ldots (2) \text{ — x: component}$$

$$m_e' v' \sin\phi = \frac{E_0'}{c} \ldots (3) \text{ — y: component}$$

$$\sin\phi = \frac{E_0'}{c}\frac{1}{m_e' v'}$$

from (2) $\Rightarrow$ $$\left(\frac{E_0}{c} - m_e v\right)^2 = m_e'^2 v'^2 (1-\sin^2\phi) = m_e'^2 v'^2\left(1 - \frac{E_0'^2}{c^2 m_e'^2 v'^2}\right)$$

$$\boxed{\left(\frac{E_0}{c} - m_e v\right)^2 = \left(m_e'^2 v'^2 - \frac{E_0'^2}{c^2}\right)} \ldots (4)$$

Now for electron: $$m_e'^2 c^2 = m_e'^2 v'^2 + m_e^{02} c^2 \quad \text{[Relativistic energy]}.$$

$$\boxed{m_e'^2 v'^2 = m_e'^2 c^2 - m_e^{02} c^2} \ldots (5)$$

using (5) into (4) $\Rightarrow$ $$\left(\frac{E_0}{c} - m_e v\right)^2 = m_e'^2 c^2 - m_e^{02} c^2 - \frac{E_0'^2}{c^2}$$

Now using eqn (1) $\Rightarrow$ $$\left(\frac{E_0}{c} - m_e v\right)^2 = \left(\frac{E_0}{c} + m_e c\right)^2 - 2\left(\frac{E_0}{c} + m_e c\right)\frac{E_0'}{c} - m_e^{02} c^2$$

$$\Rightarrow \boxed{\frac{E_0'}{c} = \frac{\left(\frac{E_0}{c} + m_e c\right)^2 - m_0^2 c^2 - \left(\frac{E_0}{c} - m_e v\right)^2}{2\left(\frac{E_0}{c} + m_e c\right)}}$$