WELCOME

to

MA 101: Mathematics I

IIT Guwahati

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

July – November, 2014

MA101: MATHEMATICS I [3-1-0-8]

Syllabus and Course Plan

Linear Algebra: Systems of linear equations and their solutions; vector space R^n and its subspaces; spanning set and linear independence; matrices, inverse and determinant; range space and rank, null space and nullity, eigenvalues and eigenvectors; diagonalization of matrices; similarity; inner product, Gram-Schmidt process; vector spaces (over the field of real and complex numbers), linear transformations.

Calculus: Convergence of sequences and series of real numbers; continuity of functions; differentiability, Rolle's theorem, mean value theorem, Taylor's theorem; power series; Riemann integration, fundamental theorem of calculus, improper integrals; application to length, area, volume and surface area of revolution.

Texts:

1. D. Poole, **Linear Algebra: A Modern Introduction**, 2nd Edition, Brooks/Cole, 2005.
2. G. B. Thomas and R. L. Finney, **Calculus and Analytic Geometry**, 9th Edition, Pearson Education India, 1996.

References:

1. G. Strang, **Linear Algebra and Its Applications**, 4th Edition, Brooks/Cole India, 2006.
2. W. Cheney and D. Kincaid, **Linear Algebra: Theory and Applications,** 1st Edition, Jones & Bartlett, 2010.
3. R. G. Bartle and D. R. Sherbert, **Introduction to Real Analysis**, 3rd Edition, Wiley India, 2005.

Instructors:

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anjankc@iitg.ernet.in

Gautam K. Das (Calculus Part), Office: E1- 109 (Maths Dept.), Ph. 2626, Email: gkd@iitg.ernet.in

Course Web sites: http://www.iitg.ernet.in/b.bikash/ and http://www.iitg.ac.in/anjankc/

Attendance: It is expected that you attend all the classes.

Grading: Grading will be done based on the total marks obtained in the quizzes, mid semester exam and end semester exam. Exams / quizzes will have the following weight-ages:

Quiz I: 15 marks (on 19th August 2014, Tuesday).

Mid Semester Exam: 35 marks (on 26th September 2014, Friday).

Quiz II: 10 marks (on 11th November 2014, Tuesday).

End Semester Exam: 40 marks (on 26th November 2014, Wednesday).

Availability of Course Materials: All course related materials like detailed syllabus, Tutorial Problem Set and Solutions, Practice Problem Set and Hints, lecture slides, summary of lectures etc., will be available for photocopying in the stationary shop at Core I Academic Building, and will also be uploaded in the course websites.

The following sections of the textbook "**Linear Algebra: A Modern Introduction**" by David Poole are of particular relevance for the topics to be covered in MA 101 (Linear Algebra part). The references are as per the 2nd edition (International Student Edition) published by Brooks/Cole.

Chapters	**Sections**	**Page Numbers**
Chapter 2	Section 2.2, Section 2.3	68 – 85, 90 -- 101
Chapter 3	Section 3.1, Section 3.2, Section 3.3, Section 3.5	136 – 178, 189 -- 209
Chapter 4	Section 4.1, Section 4.2, Section 4.3, Section 4.4	253 – 282, 289 -- 308
Chapter 5	Section 5.1, Section 5.2, Section 5.3	365 -- 389
Chapter 6	Section 6.1, Section 6.2, Section 6.3, Section 6.4, Section 6.5, Section 6.6	433 – 463, 467 -- 510

Plan

- System of Linear Equations
- Row Echelon Form
- Elementary Row Operations
- Gaussian Elimination Method
- Reduced Row Echelon Form
- Gauss-Jordan Elimination Method
- Rank

System of Linear Equations and Their Solutions

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An Example for Motivation

Solve the simultaneous linear equations:

$$x - y - z = 2, \qquad 3x - 3y + 2z = 16, \qquad 2x - y + z = 9.$$

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Solve the simultaneous linear equations:

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Step 1: Represent the given equations in a rectangular array form as follows.

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 2 \\ 3x & - & 3y & + & 2z & = & 16 \\ 2x & - & y & + & z & = & 9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

Step 2: Subtract 3 times the 1st equation from the 2nd equation; **Subtract** 3 **times the 1st row from the 2nd row.**

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 2 \\ & & & & 5z & = & 10 \\ 2x & - & y & + & z & = & 9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

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Step 3: Subtract 2 times the 1st equation from the 3rd equation; **Subtract** 2 **times the 1st row from the 3rd row.**

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 2 \\ & & & & 5z & = & 10 \\ & & y & + & 3z & = & 5 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

Step 4: Interchange the 2nd and 3rd equation; **Interchange the 2nd and 3rd row.**

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 2 \\ & & y & + & 3z & = & 5 \\ & & & & 5z & = & 10 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

Step 4: Interchange the 2nd and 3rd equation; **Interchange the 2nd and 3rd row.**

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 2 \\ & & y & + & 3z & = & 5 \\ & & & & 5z & = & 10 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

By backward substitution, we find $z = 2, y = -1, x = 3$ is a solution of the given system of equations.

Definition

- An $m \times n$ matrix $A = [a_{ij}]$ is an array of m rows and n columns as shown below:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

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- The number a_{ij} is called the (i, j)-th entry of A.
- A $1 \times n$ matrix is called a **row matrix** (row vector).
- An $n \times 1$ matrix is called a **column matrix** (column vector).

- Matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if they are of same size and $a_{ij} = b_{ij}$ for each i, j.
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- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the sum $A + B$ is defined to be the matrix $C = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$. Similarly, the difference $A - B$ is defined.

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- For a matrix $A = [a_{ij}]$ and $c \in \mathbb{C}$ (**set of complex numbers**), we define cA to be the matrix $[ca_{ij}]$.

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- For a matrix $A = [a_{ij}]$ and $c \in \mathbb{C}$ (**set of complex numbers**), we define cA to be the matrix $[ca_{ij}]$.
- Let $A = [a_{ij}]$ and $B = [b_{jk}]$ be two $m \times n$ and $n \times r$ matrices, respectively. Then the product AB is defined to be the $m \times r$ matrix $AB = [c_{ik}]$, where

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}.$$

Row Echelon Form: A matrix A is said to be in row echelon form if it satisfies the following properties:

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- The following matrices are in row echelon form:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 3 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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- If a matrix A is in row echelon form, then in each column of A containing a leading entry, the entries below that leading entry are zero.

- The following matrices are **not** in row echelon form:

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -1 & 2 \\ 1 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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2. Multiply a row R_i by a non-zero constant c ($R_i \to cR_i$).
3. Add a multiple of a row R_j to another row R_i ($R_i \to R_i + cR_j$).

Example

Transform the following matrix to row echelon form

$$\begin{bmatrix} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{bmatrix}.$$

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- A given matrix can be reduced to several row echelon form.

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Row Equivalent Matrices: Matrices A and B are said to be row equivalent if there is a sequence of elementary row operations that converts A into B.

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Row Equivalent Matrices: Matrices A and B are said to be row equivalent if there is a sequence of elementary row operations that converts A into B.

Result

*Matrices A and B are row equivalent if and only if (**iff**) they can be reduced to the same row echelon form.*

Linear System of Equations:

A linear system of m equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \quad &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned} \tag{1}$$

where $a_{ij}, b_i \in \mathbb{R}$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$.

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where $a_{ij}, b_i \in \mathbb{R}$ for each $1 \le i \le m$ and $1 \le j \le n$. Letting

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

we can represent the above system of equations as $A\mathbf{x} = \mathbf{b}$.

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- The matrix $[A|\ \mathbf{b}]$, as given below, is called the augmented matrix of the system of equations $A\mathbf{x} = \mathbf{b}$.

$$[A|\ \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

- The vertical bar is used in the augmented matrix $[A|\ \mathbf{b}]$ only to distinguish the column vector $\mathbf{b}$ from the coefficient matrix A.

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- The vertical bar is used in the augmented matrix $[A|\ \mathbf{b}]$ only to distinguish the column vector $\mathbf{b}$ from the coefficient matrix A.

- If $\mathbf{b} \neq \mathbf{0}$ then $A\mathbf{x} = \mathbf{b}$ is called an non-homogeneous system of equations.

- If $\mathbf{b} = \mathbf{0} = [0, 0, \ldots, 0]^t$, i.e., if $b_1 = b_2 = \ldots = b_m = 0$, the system $A\mathbf{x} = \mathbf{0}$ is called a homogeneous system of equations.

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- A solution of the linear system $A\mathbf{x} = \mathbf{b}$ is a column vector $\mathbf{y} = [y_1, y_2, \ldots, y_n]^t$ such that the linear system (1) is satisfied by substituting y_i in place of x_i. That is, $A\mathbf{y} = \mathbf{b}$ holds true.

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- The solution $\mathbf{0}$ of $A\mathbf{x} = \mathbf{0}$ is called the **trivial** solution and any other solutions of $A\mathbf{x} = \mathbf{0}$ are called **non-trivial** solutions.

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Result

Let $Cx = d$ be the linear system obtained from the linear system $Ax = b$ by a single elementary operation. Then the linear systems $Ax = b$ and $Cx = d$ have the same set of solutions.

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Two equivalent system of linear equations have the same set of solutions.

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Leading and Free Variable:

- Consider the linear system $A\mathbf{x} = \mathbf{b}$ in n variables and m equations.

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Leading and Free Variable:

- Consider the linear system $A\mathbf{x} = \mathbf{b}$ in n variables and m equations.

- Let $[R \mid \mathbf{r}]$ be a row echelon form of the augmented matrix $[A \mid \mathbf{b}]$.

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Leading and Free Variable:

- Consider the linear system $A\mathbf{x} = \mathbf{b}$ in n variables and m equations.
- Let $[R \mid \mathbf{r}]$ be a row echelon form of the augmented matrix $[A \mid \mathbf{b}]$.
- The variables corresponding to the leading columns in the first n columns of $[R \mid \mathbf{r}]$ are called the leading variables or basic variables.

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Leading and Free Variable:

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- Let $[R \mid \mathbf{r}]$ be a row echelon form of the augmented matrix $[A \mid \mathbf{b}]$.
- The variables corresponding to the leading columns in the first n columns of $[R \mid \mathbf{r}]$ are called the leading variables or basic variables.
- The variables which are not leading are called free variables.

Gaussian Elimination Method: Use the following steps to solve a system of equations $A\mathbf{x} = \mathbf{b}$.

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Example

Use Gaussian Elimination method to solve the system:

$(a)\quad y + z = 1, \quad x + y + z = 2, \quad x + 2y + 2z = 3$

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Use Gaussian Elimination method to solve the system:

$(a)\quad y + z = 1, \quad x + y + z = 2, \quad x + 2y + 2z = 3$

$(b)\quad y + z = 1, \quad x + y + z = 2, \quad x + 2y + 3z = 4$

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Use Gaussian Elimination method to solve the system:

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- The following matrices are in reduced row echelon form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Result

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Proof: RREF is unique (Using Induction)

We will apply induction on number of columns.

Base Case: Consider a matrix with one column i.e. $A_{m \times 1}$. Observe that its RREF is unique. What are the possibilities? $\mathbf{O}_{m \times 1}$ and e_1. Can you transform $\mathbf{O}_{m \times 1}$ into e_1?

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$\Rightarrow$ there must be leading 1 in the k^{th} column of B and C
$\Rightarrow$ the location of 1 is different in the k^{th} column
$\Rightarrow$ numbers of nonzero rows in B_1 and C_1 are different,
a contradiction.

Method of transforming a given matrix to reduced row echelon form: Let A be an $m \times n$ matrix. The following step by step method is used to obtain the reduced row echelon form of A.

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(2) Forget the first row and first i columns. Start with the lower $(m-1) \times (n-i)$ sub matrix of the matrix obtained in the first step and proceed as in **Step 1**.

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(4) Now use the leading term in each of the leading column to make (by elementary row operations) all other entries in that column equal to zero. Use this step starting form the rightmost leading column.

(5) Scale all non-zero entries (leading entries) of the matrix obtained in the previous step, by multiplying the rows by suitable constants, to make all the leading entries equal to 1, ending with the unique reduced row echelon form of A.

Gauss-Jordan Elimination Method: Use the following steps to solve a system of equations $A\mathbf{x} = \mathbf{b}$.

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Example

Solve the system

$w - x - y + 2z = 1,\ 2w - 2x - y + 3z = 3,\ -w + x - y = -3$

using Gauss-Jordan elimination method.

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Solve the following systems using Gauss-Jordan elimination method:

$(a)\quad 2y + 3z = 8, \quad 2x + 3y + z = 5, \quad x - y - 2z = -5$

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Result

Let $A\mathbf{x}=\mathbf{b}$ be a consistent system of equations with n variables. Then number of free variables is equal to $n-rank(A)$.

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Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of equations with n variables. If $\mathrm{rank}(A) < n$ then the system has infinitely many solutions.

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