MA101 Mathematics I

Tutorial & Additional Problem Set - 5

SECTION - A (for Tutorial -5)

1. True or False? Give justifications.

   (a) There exists distinct linear transformations $S, T : \mathbb{V} \to \mathbb{W}$ such that $ker(S) = ker(T)$ and $range(S) = range(T)$.

   (b) There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that none of T, T^2, T^3 is the identity transformation, but $T^4 = I$ (identity transformation).

   (c) If $T : \mathbb{V} \to \mathbb{W}$ is a linear transformation then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is LI in $\mathbb{V}$ if and only if $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is LI in $\mathbb{W}$.

   (d) There exists a linear transformation T from $\mathbb{R}^2 \to \mathbb{R}^4$ such that $range(T) = \{[x, y, z, w]^T : x + y + z = 0\}$.

Solution:

(a) True. Take $S, T : \mathbb{R}^2 \to \mathbb{R}^2$, such that $S([x, y]^T) = [x, y]^T$ and $T([x, y]^T) = [y, x]^T$.

(b) True. Rotate every element of $\mathbb{R}^2$ by 90 degrees, that is $T([x, y]^T) = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} [x, y]^T$.

(c) False. If $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is LI in $\mathbb{W}$ then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is LI in $\mathbb{V}$, but the converse is not true, for example the $\mathbf{0}$ transformation.

(d) False. From the rank nullity theorem, $rank(T) \leq 2$, but if $S = \{[x, y, z, w]^T : x+y+z = 0\}$, then $dim(S) = 3$.

2. Determine a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ such that $range(T) = \{[x, y, z]^T : x + 2y + z = 0\}$. If possible give two more such linear transformations with the same range.

Solution: It is enough to define T in a basis.
Take any basis, say $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$, and consider the LT T such that $T(\mathbf{e}_1) = [2, -1, 0]^T$, $T(\mathbf{e}_2) = [0, -1, 2]^T$ and $T(\mathbf{e}_3) = \mathbf{0}$. Then $range(T) = span\{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$.
Note that $\{[2, -1, 0]^T, [0, -1, 2]^T\}$ is a basis of $S = \{[x, y, z]^T : x + 2y + z = 0\}$.
Hence $range(T) = span\{T(\mathbf{e}_1), T(\mathbf{e}_2)\} = S$ and $Ker(T)$ is the $z-$axis.
If $T(\mathbf{e}_3) = \alpha(2, -1, 0)^T + \beta(0, -1, 2)^T$ then again $range(T)$ is same but $Ker(T)$ is different. For the same basis by making suitable choices one can also get $Ker(T)$ as the x-axis or the y-axis.
By considering different basis of $\mathbb{R}^3$, one can get many more T's.

3. If $dim(\mathbb{V}) = dim(\mathbb{W})$, then show that a linear transformation $T : \mathbb{V} \to \mathbb{W}$ is one-one if and only if it is onto.

Solution: IF T is onto, then $range(T) = \mathbb{W}$, hence $rank(T) = dim(\mathbb{W}) = dim(\mathbb{V})$. Hence from the rank nullity theorem $ker(T) = \{\mathbf{0}\}$, or T is one-one.
If T is one-one, then $ker(T) = \{\mathbf{0}\}$, hence from the rank nullity theorem, $rank(T) = dim(\mathbb{V}) = dim(\mathbb{W})$. Since $range(T)$ is a subspace of $\mathbb{W}$, $rank(T) = dim(range(T)) = dim(\mathbb{W})$ implies $range(T) = \mathbb{W}$, or T is onto.

4. If possible find linear transformations $S : \mathbb{R}^2 \to \mathbb{R}_2[x]$ and $T : \mathbb{R}_2[x] \to \mathbb{R}^2$, such that,
   (a) $S \circ T = I$.
   (b) $T \circ S = I$.
   (c) $range(T \circ S)$ is a line.
   (d) Neither S not T is the zero transformation, but $S \circ T = 0$ (zero transformation).

Solution:

(a) $S \circ T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$.
Note that $range(S \circ T) \preccurlyeq range(S)$ and $rank(S) \leq 2$ (from the rank nullity theorem), hence $rank(S \circ T) \leq 2$. An identity map from $\mathbb{R}_2[x] \to \mathbb{R}_2[x]$ will have rank 3, hence not possible.

(b) $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$.
Define $S([a, b]^T) = a + bx$ and $T : \mathbb{R}_2[x] \to \mathbb{R}^2$ as $T(a_0 + a_1x + a_2x^2) = [a_0, a_1]^T$, then check that $T \circ S = I$.

(c) Define $S([a, b]^T) = a + bx$ and $T : \mathbb{R}_2[x] \to \mathbb{R}^2$ as $T(a_0 + a_1x + a_2x^2) = [a_0, 0]^T$, then check that $range(T \circ S)$ is a line.

(d) Take $T(a_0 + a_1x + a_2x^2) = [a_0, 0]^T$ and $S([a, b]^T) = bx$.

5. Give three linear transformations from $\mathbb{R}^3$ to $\mathbb{W} = \{\mathbf{w} \in \mathbb{R}^5 | w_1 - w_2 + w_3 - w_4 + w_5 = 0\}$. Give their coordinate matrices w.r.t the ordered basis $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ on $\mathbb{R}^3$ and some ordered basis of $\mathbb{W}$.

Solution: A basis for $\mathbb{W}$ is $B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$.

Take $T_1 \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then $[T] = \Big[[T(v_1)]_{B'}, [T(v_2)]_{B'}, [T(v_3)]_{B'} \Big] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Take $T_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then $[T] = \Big[[T(v_1)]_{B'}, [T(v_2)]_{B'}, [T(v_3)]_{B'} \Big] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Take $T_3 \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ y \\ 0 \\ -x \end{bmatrix}$. Then

$$[T] = \Big[[T(v_1)]_{B'}, [T(v_2)]_{B'}, [T(v_3)]_{B'} \Big] = \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \\ -1 \end{bmatrix}_{B'} & \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ -1 \end{bmatrix}_{B'} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}_{B'} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix},$$

where

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \end{bmatrix}$$

In general this can be solved using GJE. Form the matrix

$$\left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{array}\right]$$

RREF is

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

the right-top matrix.

6. Let $\mathbb{V}, \mathbb{W}$ be finite dimensional vector spaces with bases B and B' respectively. Take a linear transformation $T : \mathbb{V} \to \mathbb{W}$. Is it true that $\operatorname{rank}(T) = \operatorname{rank}([T]_{B' \leftarrow B})$? Is it true that $\operatorname{nullity}(T) = \operatorname{nullity}([T]_{B' \leftarrow B})$?

Solution: Yes. Let $B = \{v_1, \dots, v_n\}$ and $B' = \{w_1, \dots, w_m\}$. Note that $A = [T]_{B' \leftarrow B} \in \mathcal{M}_{m \times n}$.

Recall: Let $x, y \in \mathbb{V}$ and $a = [x]_B$, $b = [y]_B$. This means $x = \sum a_i v_i$ and $y = \sum b_i v_i$. So $\alpha x + \beta y = \sum(\alpha a_i + \beta b_i)v_i$. So $[\alpha x + \beta y]_B = \alpha a + \beta b = \alpha[x]_B + \beta[y]_B$.

Let $A = [A_1, A_2, \ldots, A_n]$. Then a set of columns $A_{i_1}, \ldots, A_{i_k}$ of A are linearly independent if and only if $Tv_{i_1}, \ldots, Tv_{i_k}$ are linearly independent. This is because

$w = \sum \alpha_j T v_{i_j} = 0$ iff $[w]_{B'} = 0$ iff $[\sum \alpha_j T v_{i_j}]_{B'} = 0$ iff $\sum \alpha_j [T v_{i_j}]_{B'} = 0$ iff $\sum \alpha_j A_{i_j} = 0$.

So, $\text{rank}(T) = \text{rank}(A)$. Also, this implies $\text{nullity}(T) = \text{nullity}(A)$.

Alternatively since $[Tv_1 \ldots Tv_n] = [w_1 \ldots w_m][T]_{B' \leftarrow B}$,

$[Tv_1 \ldots Tv_n] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = [w_1 \ldots w_m][T]_{B' \leftarrow B} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ for any scalars $\alpha_1, \ldots, \alpha_n$.

Since $\{w_1, \ldots, w_m\}$ is LI, $\sum_{i=1}^n \alpha_i T v_i = 0$ iff $\sum_{i=1}^n \alpha_i A_i = 0$, hence whatever is the linear independence and dependence relationship between the Tv_i's, the same holds for the A_i's.

SECTION - B: ADDITIONAL PROBLEMS

1. True or False? Give justifications.

   (a) A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T([x, y]^T) = [x, y]^T$ for $x \neq 0$ and $T([0, y]^T) = [0, 0]^T$ satisfies $T(c[x, y]^T) = cT([x, y]^T)$ but is not a linear transformation.

   (b) Given vector spaces $\mathbb{V}$ and $\mathbb{W}$, for any $\mathbf{v}_1, \mathbf{v}_2$ in V and $\mathbf{w}_1, \mathbf{w}_2$ in W, there exists a linear transformation $T : \mathbb{V} \to \mathbb{W}$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$.

   (c) There exists a 2×2 matrix which transforms $[2, 6]^T$ to $[1, 0]^T$ and $[1, 0]^T$ to $[1, 5]^T$.

   (d) Given an invertible linear transformation $T : \mathbb{V} \to \mathbb{W}$ there exists basis B and B' in $\mathbb{V}$ and $\mathbb{W}$ respectively such that I (identity matrix) is the matrix of T.

Solution:

(a) True. $T(c[x, y]^T) = cT([x, y]^T)$ is easy to check. But for $y \neq 0$, $T([-1, y]^T) + T([1, y]^T) = [-1, y]^T + [1, y]^T = [0, 2y]^T \neq [0, 0]^T = T([-1, y]^T + [1, y]^T)$.

(b) False. For example if $\mathbf{v}_1, \mathbf{v}_2$ is LD, say $\mathbf{v}_1 = [1, 0]^T$ and $\mathbf{v}_2 = [2, 0]^T$ and $\mathbf{w}_1 = [1, 0]^T$ and $\mathbf{w}_2 = [0, 1]^T$, then there exists no LT $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$.

(c) True. This is just the change of basis matrix.

(d) True. Since T is invertible it takes every basis of $\mathbb{U}$ to a basis of $\mathbb{W}$. Consider any basis B of $\mathbb{U}$, then T takes the basis B to a basis say B' of $\mathbb{W}$. Then $[T]_{B \leftarrow B'} = I$.

2. Determine a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ such that $Ker(T) = \{[x, y]^T : 2x + y = 0\}$.

Solution: Take an $[x, y]^T$ satisfying $2x + y = 0$, for example $[-1, 2]^T$ and take T such that $T([-1, 2]^T) = \mathbf{0}$. Since T has to be defined on a basis, take any other $[a, b]^T$ such that $\{[-1, 2]^T, [a, b]^T\}$ forms a basis of $\mathbb{R}^2$ and define $T([a, b]^T)$, such that $T([a, b]^T) \neq \mathbf{0}$, then this will give a required T. For example take $[a, b]^T = [1, 0]^T$ and $T([1, 0]^T) = [1, 0]^T$.

3. Show that there exists a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that none of T, T^2, T^3 is a zero transformation, but $T^4 = \mathbf{0}$. Is it possible to get such a transformation from $\mathbb{R}^3 \to \mathbb{R}^3$?

Solution:

(a) Note that for any LT T in $\mathbb{R}^4$, $nullity(T) \le nullity(T^2) \le nullity(T^3) \le nullity(T^4)$.
If $nullity(T) = 0$ then check that $nullity(T^2) = nullity(T^3) = nullity(T^4) = 0$. Then T^4 becomes invertible.
If $nullity(T) = nullity(T^2)$, then check that $nullity(T) = nullity(T^2) = nullity(T^3) = nullity(T^4)$. Hence if $T^4 = \mathbf{0}$, then $T = T^2 = T^3 = T^4 = \mathbf{0}$.
Similarly if $nullity(T^2) = nullity(T^3)$, then check that $nullity(T^2) = nullity(T^3) = nullity(T^4)$. Hence if $T^4 = \mathbf{0}$, then $T^2 = T^3 = T^4 = \mathbf{0}$.
By similar argument we get $nullity(T^3) \ne nullity(T^4)$.
Hence in order to get the required LT we have to take a T such that $1 \le nullity(T) < nullity(T^2) < nullity(T^3) < nullity(T^4)$, which will imply $nullity(T^4) = 4$, or $T^4 = \mathbf{0}$.
For example take $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$ or $T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}) = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ 0 \end{bmatrix}$.
This is not possible in $\mathbb{R}^3$ since for the above to happen $nullity(T) \ge 1$ and hence $nullity(T^4) \ge 4$, but since T^4 is a LT from $\mathbb{R}^3$ to $\mathbb{R}^3$, $nullity(T^4) \le 3$.

4. Let $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ be defined as:
$T(A) = A - A^T$ for all $A \in \mathcal{M}_2(\mathbb{R})$. Find a basis for $range(T)$ and for $null(T)$.

Solution: Since T is taking every 2×2 real matrix to a skew symmetric matrix and $T(\begin{bmatrix} x & y \\ z & w \end{bmatrix}) = \begin{bmatrix} 0 & y-z \\ z-y & 0 \end{bmatrix}$, $range(T) = span\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$, and $null(T) = span\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by, $T(x, y, z)^T = (x+y, x+y-z)^T$. Fix ordered basis $\{v_1, v_2, v_3\}$ and $\{w_1, w_2\}$, in $\mathbb{V} = \mathbb{R}^3$ and $\mathbb{W} = \mathbb{R}^2$ respectively, where $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) Find the matrix of the linear transformation with respect to these ordered bases. Find the coordinates of $(1, 2, 3)^T$ and $T(1, 2, 3)^T$ with respect to the ordered bases.

(b) Find the matrix of the linear transformation if the ordered basis in $\mathbb{R}^3$ is changed to $\{v_2, v_1, v_3\}$, the basis of $\mathbb{R}^2$ is as given.

(c) Find the matrix of the linear transformation if the ordered basis in $\mathbb{R}^3$ is changed to $\{3v_1, v_2, v_3\}$, the basis of $\mathbb{R}^2$ is as given.

(d) Find the matrix of the linear transformation if the ordered basis in $\mathbb{R}^2$ is changed to $\{w_2, w_1\}$, the basis of $\mathbb{R}^3$ is as given.

6. (*) Suppose all vectors in the the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ are transformed to $A(x, y)^T$, where A is a 2×2 matrix.

   (a) For which A is the region square again and has area 1?

   (b) For which A is the region square again?

   (c) For which A is the region a line?

   (*) means optional.