EE 101
Electrical Sciences

Department of Electronics & Electrical Engineering

Lectures 7-9

AC Circuits
Sinusoidal Analysis

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Sinusoidal Functions

- A sinusoidal wave form is generated by the vertical component of a vector rotating counterclockwise with uniform angular velocity ω as shown in Fig. 1

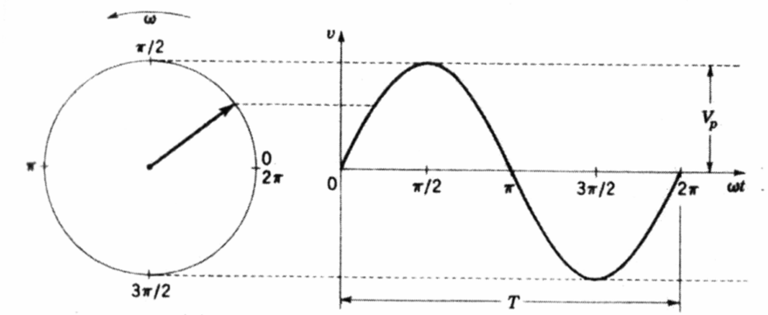


Fig. 4: Generation of sine wave by vertical component of rotating vector.

$$v = V_p \sin(\omega t) = V_p \sin(2\pi ft) \qquad (1)$$

where,	v	is instantaneous voltage, and
	V_p	is the peak or the maximum value (amplitude)

- One cycle: is a complete revolution
- Period (T): time required for one revolution (s)
- Frequency (f): Number of cycles per second ($f = 1/T$) Hz
- Angular speed (ω): $2\pi f$ rad/s

- Such rotating vectors are termed *Phasors*. It may be noted that the horizontal component of the rotating vector will give the cosine wave.

Sinusoidal Functions

- In Fig. 2, there are two sinusoids, denoting two voltages and the corresponding phasors.
- It is possible that the sinusoids pass through zero at different times, but it is necessary for the two sinusoids to have the same frequency to be included in the same diagram.

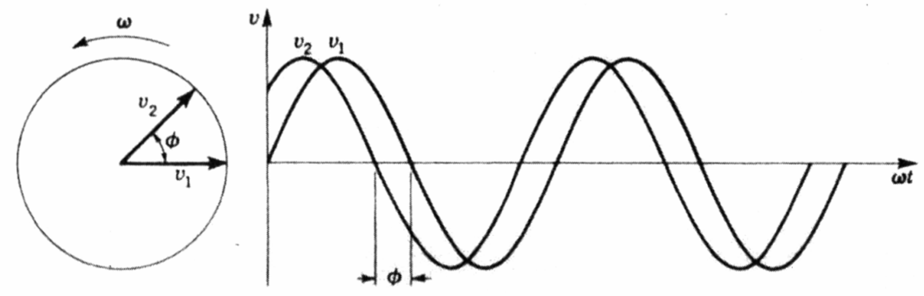


Fig. 5: Illustrating phase angle between two sinusoidal voltages.

- If, $v_1 = V_{p1}\sin(\omega t) = V_{p1}\sin(2\pi ft)$ (2)

$$v_2 = V_{p2}\sin(\omega t + \varphi) = V_{p2}\sin(2\pi ft + \varphi) \quad (3)$$

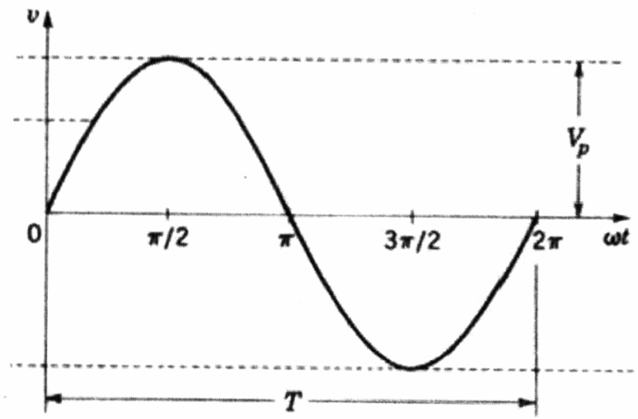
Then, ϕ is said to be the phase difference between v_1 and v_2.and the phasor that passes through zero at t=0 is called the ***reference***.

In Figure 2, v_1 is the reference and v_2 is said to lead v_1 by ϕ.

Average Value of Sinusoids

- A general definition of the average value of any function *f(t)* over the specified interval between t_1 and t_2 is expressed as

$$F_{av} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t)dt \qquad (4)$$



- The average value of a cyclic function is

$$F_{av} = \frac{1}{2\pi} \int_0^{2\pi} f(\omega t)d(\omega t) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha)d\alpha \qquad (5)$$

- Hence the average value of a sinusoidal voltage is

$$v_{av} = \int_0^{2\pi} V_p \sin(\omega t) d(\omega t) = 0 \qquad (6)$$

The average value of a sinusoid over *one complete cycle* is equal to zero.

- A finite and more meaningful average value can be found for the sinusoid for the *positive* or *negative* half cycle. The half cycle average value for the waveform shown in Fig.1 is given by

$$v_{av} = \frac{1}{\pi} \int_0^{\pi} V_p \sin(\omega t) d(\omega t) = \frac{2}{\pi} V_p = 0.636 V_p \qquad (7)$$

The average value of either the positive or negative half of a sine function can be found by multiplying the amplitude of the wave by *0.636*.

Effective (RMS) Value of Sinusoids

- Although the criterion of the *average value* of current works well in describing the energy transferring capacity for direct sources, it is less meaningful for symmetrical periodic functions.
- A more suitable definition of the *average value* for a symmetric periodic functions is *effective current*. It is expressed as

$$I_{eff} = I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} \qquad (8)$$

- If the function *I(t)* in eq.19 is sinusoidal then the *effective value* is obtained as

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \sin^2 \omega t} \qquad (9)$$

- Using the trigonometric identity $\sin^2 \omega t = \frac{1}{2}(1-\cos(2\omega t))$

  The *effective value* , called the rms value, is obtained as

$$I_{eff} = \sqrt{\frac{I_m^2}{2T}\int_0^T (1-\cos(2\omega t))} = \frac{I_m}{\sqrt{2}} \qquad (10)$$

Instantaneous Power

- Let *v(t)* and *i(t)* be the instantaneous voltage and instantaneous current across a network given by

$$v(t) = V_m \sin(\omega t) \qquad (11)$$

$$i(t) = I_m \sin(\omega t - \theta) \qquad (12)$$

- The expression for instantaneous power is given by

$$p(t) = v(t)i(t) = V_m I_m \sin(\omega t)\sin(\omega t - \theta) \qquad (13)$$

- Using, $\sin(\omega t - \theta) = \sin(\omega t)\cos\theta - \cos\omega t \sin\theta,$

$$\sin^2 \omega t = \frac{1}{2}(1 - \cos 2\omega t), \quad \text{and} \quad \sin(\omega t)\cos\omega t = \frac{1}{2}\sin(2\omega t)$$

$$\begin{aligned} p(t) &= V_m I_m \sin^2(\omega t)\cos\theta - \sin(\omega t)\cos(\omega t)\sin\theta \\ &= \frac{V_m I_m}{2}\cos\theta - \frac{V_m I_m}{2}\cos(2\omega t - \theta) \qquad (14) \end{aligned}$$

- It can be seen that for a given value of angle θ the instantaneous power consists of two components; a constant part and a time varying part.
- The time varying part has a frequency which is *twice* that of the voltage and current sinusoids.

Instantaneous Power

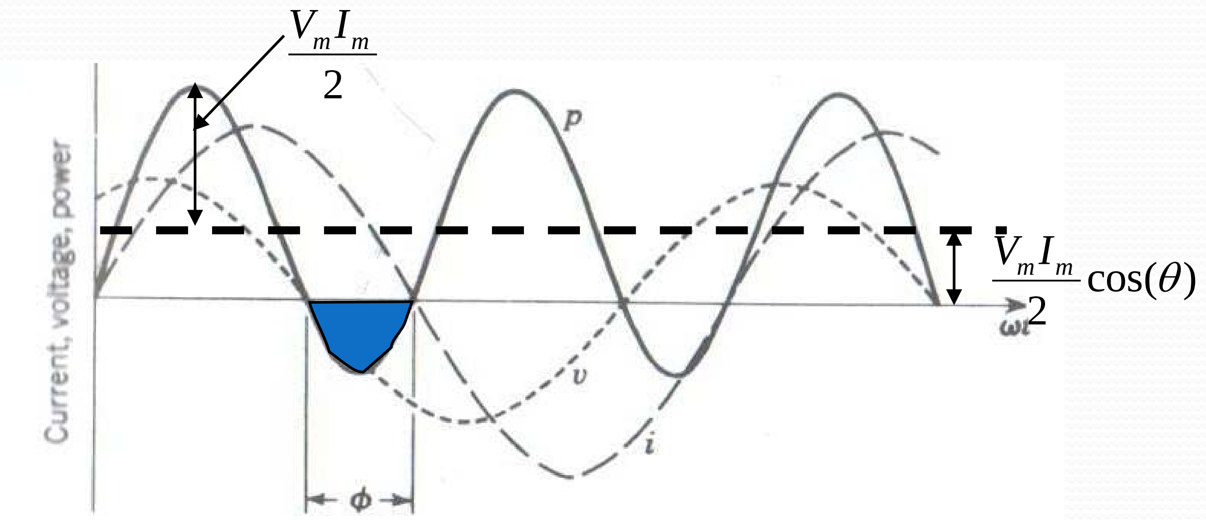


Fig. 6: Instantaneous power in ac circuit.

- The shaded part of the power in Fig.2 refers to those time intervals when the power is negative. The negative power in effect means that the circuit is returning power to the source during these intervals.
- It should be noted from Form Fig.2 that the time varying component oscillates about the constant power axis, which gives the average power.

Instantaneous Power

- As the angle θ is made smaller and smaller, i.e. as the current I is brought nearly in phase to the voltage v, the negative area gets smaller and smaller.
- As $\theta=0$, the current and voltage are in phase, there is no negative area associated with *p(t)* curve and all the power in consumed between the circuit branch terminals. This circuit is purely resistive.
- When θ is increased, the negative area increases and less power is consumed by the circuit and more returned to the source.
- At the extreme value of θ i.e. $\theta=\pi/2$, the *p(t)* curve such that the negative are is equal to the positive ar In this case no power is consumed between the ci terminals.

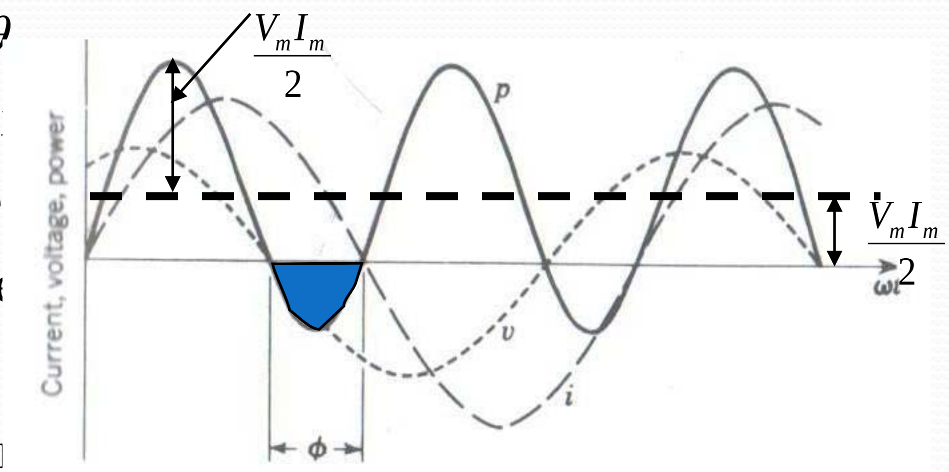


Fig. 6: Instantaneous power in ac circuit.

Average Power

- The useful quantity in terms of the capability of the circuit to do work is the average power.
- The average power is given by

$$P_{av} = \frac{1}{T}\left[\int_0^T \frac{V_m I_m}{2}\cos\theta dt - \int_0^T \frac{V_m I_m}{2}\cos(2\omega t - \theta)dt\right] \qquad (15)$$

- The second term in eq.22 involves the integration of a sine function over a time interval of two period, hence its value is equal to zero.
- The first term is independent of time t, the average power is obtained as

$$P_{av} = \frac{V_m I_m}{2}\cos\theta \qquad (16)$$

- More commonly, the average power is written as:

$$P_{av} = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos\theta = V_{eff} I_{eff} \cos\theta \qquad (17)$$

Power Factor

- While the product of V and I (VI) gives power in dc circuits It should be noted that this product (VI) does not give the average power when sinusoidal voltages and currents are involved. In ac circuits, the product (VI) is called the *apparent power* or *Volt Amperes*.
- In ac circuits, $P_{av} = \dfrac{V_m I_m}{2}\cos\theta = \dfrac{V_m I_m}{\sqrt{2}\sqrt{2}}\cos\theta = VI\cos\theta$ (18)
- The product of V and I, or the apparent power must be multiplied by the factor (cosθ) to obtain the average power.
- This important factor

$$\cos\theta = \frac{P_{av}}{VI} = \frac{Average\ Power}{Volt\ Amps\ \text{or}\ Apparent\ Power} \qquad (19)$$

is called the power factor.

so that, Average Power = Apparent Power × power factor. (20)

Example 1

- Find the average value of the periodic function shown in Fig. 4

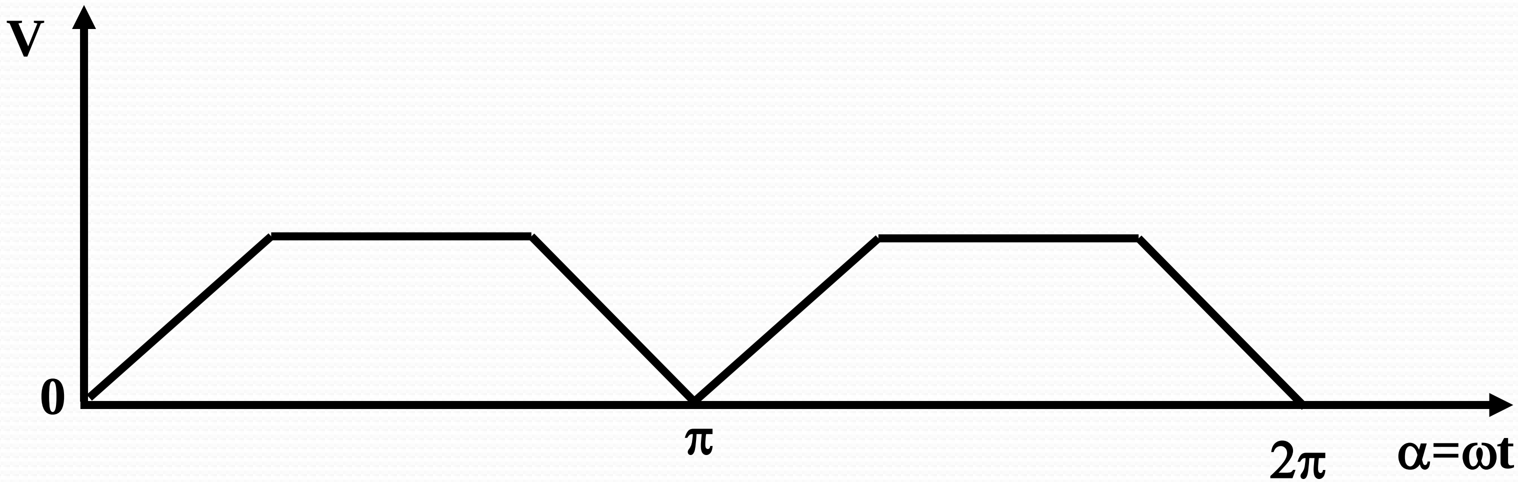


Fig 4: Plot of a periodic function

Example 1 - Solution

- The entire information about the waveform is contained in the period 0 to π/2.

$$v(t)=\begin{cases} \dfrac{V_m}{\pi/3}(\omega t)=\dfrac{V_m}{\pi/3}\alpha \ \text{ for } 0\le\alpha\le\dfrac{\pi}{3} \\ V_m \qquad \text{for } \dfrac{\pi}{3}\le\alpha\le\dfrac{\pi}{2} \end{cases}$$

- Hence the average value is obtained as

$$V_{av}=\frac{1}{\pi/2}\left\{\int_0^{\pi/3}\frac{V_m}{\pi/3}\alpha\,d\alpha+\int_{\pi/3}^{\pi/2}V_m\,d\alpha\right\}$$

$$=\frac{1}{\pi/2}\left\{\frac{V_m}{\pi/3}\left[\frac{\alpha^2}{2}\right]_0^{\pi/3}+V_m\left[\alpha\right]_{\pi/3}^{\pi/2}\right\}$$

$$=\frac{V_m}{\pi/2}\left\{\frac{\pi}{6}+\frac{\pi}{6}\right\}=\frac{2}{3}V_m$$

Example 2

A voltage given by

$$v(\mathrm{t})=170\sin(377t+10°)$$

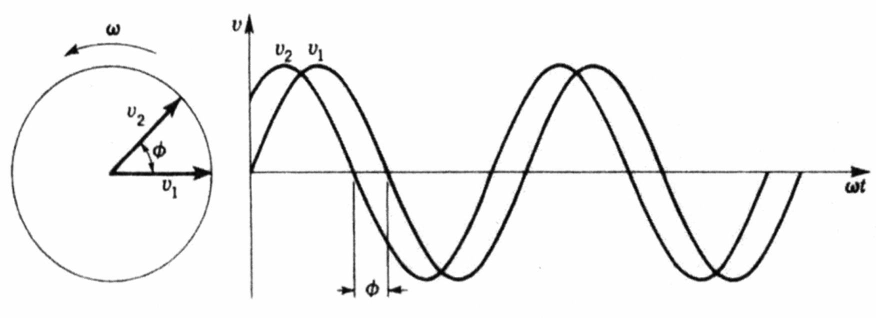
is applied to a circuit. It causes a steady state current to flow which is given by

$$i(\mathrm{t})=14.14\sin(377t-20°).$$

Determine the power factor and the average power delivered to the circuit.

Phasor Representation

- Consider two sinusoids, denoting two voltages corresponding to two phasors as shown.



Illustrating phase angle between two sinusoidal voltages.

$$v_1 = V_{p1}\sin(\omega t)$$

$$\Rightarrow \quad \mathrm{V}_{p1} = \mathrm{Im}\,(V_{p1}e^{j\omega t})$$

$$= \mathrm{Im}(V_{p1}\angle 0^\circ)$$

$$v_2 = V_{p2}\sin(\omega t + \varphi)$$

$$\Rightarrow \quad \mathrm{V}_{p2} = \mathrm{Im}\,(V_{p2}e^{j(\omega t+\varphi)})$$

$$= \mathrm{Im}(V_{p2}\angle \varphi^\circ)$$

- Algebraic treatment of sinusoids are made simpler by phasor representation, eg,

$$\begin{aligned} v_1 + v_2 \quad &= \mathrm{Im}\,(V_{p1}\angle 0^\circ + V_{p2}\angle \varphi^\circ) \\ &= \mathrm{Im}\,(V_{p1}\cos 0^\circ + V_{p2}\cos\varphi^\circ + j(V_{p1}\sin 0^\circ + V_{p2}\sin\varphi^\circ)) \\ &= \mathrm{Im}\,(V_t\angle \varphi_t{}^\circ) \quad = V_t\sin(\omega t + \varphi_t{}^\circ) \qquad (21) \end{aligned}$$

where $\quad V_t = \sqrt{(V_{p1}\cos 0^\circ + V_{p2}\cos\varphi^\circ)^2 + (V_{p1}\sin 0^\circ + V_{p2}\sin\varphi^\circ)^2}$, and,

$$\varphi_t = \tan^{-1}\frac{(V_{p1}\sin 0^\circ + V_{p2}\sin\varphi^\circ)}{(V_{p1}\cos 0^\circ + V_{p2}\cos\varphi^\circ)} \qquad (22)$$

Phasor Addition - Example

$$v_1 = V_m \sin(\omega t) \quad \Rightarrow \quad \mathrm{V}_1 = (V_m \angle 0)$$

$$v_2 = V_m \sin(\omega t + 90°) \quad \Rightarrow \quad \mathrm{V}_2 = V_m \angle 90°$$

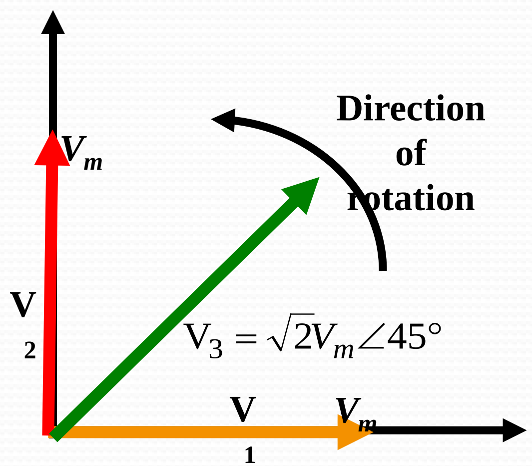


Figure 8a

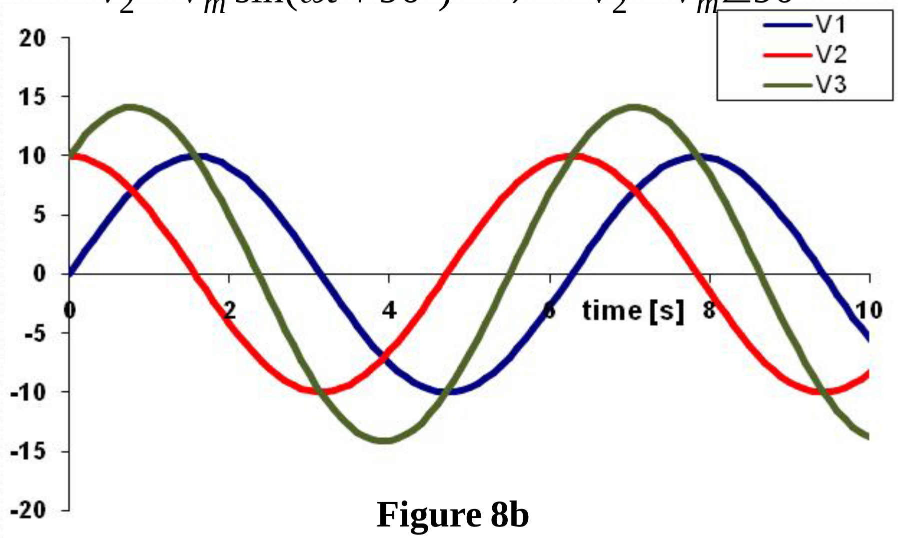


Figure 8b

$$\mathrm{V}_3 = \mathrm{V}_1 + \mathrm{V}_2 = V_m \angle 0° + V_m \angle 90° = \sqrt{2} V_m \angle 45° \quad \Rightarrow \quad v_3 = \sqrt{2} V_m \sin(\omega t + 45°)$$

- Commonly, the phasors use rms values for magnitude in frequency domain (rather than the peak values) which are more convenient in circuit analysis.
- Lower case letters like *v* or *i* are used to indicate instantaneous values
- Normal letters like V or $\overline{\mathrm{V}}$ or $\overline{V}$ are used to indicate vectors or phasors
- Italic letters like *V* or *I* are used to indicate magnitudes of vectors and phasors.

Representation of Circuit Parameters in Frequency Domain

- If a current $i = I_m \sin(\omega t - \theta)$ ie, $\mathrm{I} = I\angle -\theta$
  is made to flow through a resistor R, the voltage across R is:

$$v_R = i \times R = RI_m \sin(\omega t - \theta) \Rightarrow \quad V_R = RI\angle -\theta$$

Therefore, $\mathrm{Z_R} = R$ (23)

- If a current $i = I_m \sin(\omega t - \theta)$ ie, $\mathrm{I} = I\angle -\theta$ is made to flow through an inductor L,
- the voltage across L is: $v_L = L\dfrac{di}{dt} = \omega LI_m \cos(\omega t - \theta) = \omega LI_m \sin(\omega t - \theta + 90°)$

$$\Rightarrow \quad \mathrm{V_L} = \omega LI\angle(90° - \theta) = j\omega L\mathrm{I}$$

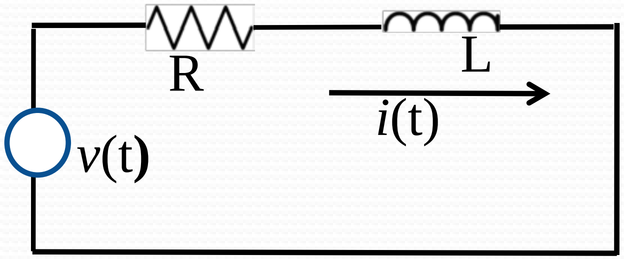
Therefore, $\mathrm{Z_L} = j\omega L$ (24)

- If a voltage $v = V_m \sin(\omega t + \theta)$ ie, $\mathrm{V} = V\angle\theta$ is applied across a capacitor C,
  the current through C is: $i_C = C\dfrac{dv}{dt} = \omega CV_m \cos(\omega t - \theta) = \omega CV_m \sin(\omega t - \theta + 90°)$

$$\Rightarrow \quad \mathrm{I_C} = \omega CV\angle(90° - \theta) = j\omega C\mathrm{V}$$

Therefore, $\mathrm{Z_C} = \dfrac{1}{j\omega C} = -j\dfrac{1}{\omega C}$ (25)

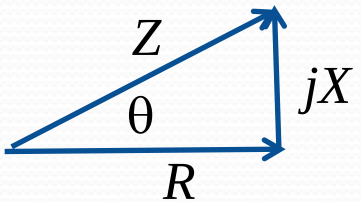
RL Circuit – Steady State Analysis (Revisit)



- Consider the RL circuit
  Let the input voltage be:

$$v = V_m \sin(\omega t) \quad \Rightarrow \quad \mathrm{V} = V\angle 0^\circ \;\; ref.$$
$$V = V_m / \sqrt{2}$$

- Impedance of the circuit is formed as:



Impedance Diagram

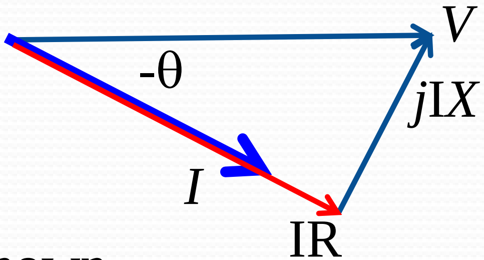
$$\mathrm{Z} = R + \omega L = Z\angle\theta$$

where, $Z = \sqrt{R^2 + (\omega L)^2}$ and, $\theta = \tan^{-1}\dfrac{\omega L}{R}$

This is commonly depicted in the impedance triangle as shown.

Then, $\mathrm{I} \;= \dfrac{\mathrm{V}}{\mathrm{Z}} = \dfrac{V\angle 0^\circ}{Z\angle\theta} = \dfrac{V}{Z}\angle -\theta = I\angle -\theta,$ and

$$\mathrm{V_R} = \mathrm{IR} = IR\angle -\theta,$$
$$\mathrm{V_L} = \mathrm{IX} = I\angle -\theta \times j\omega L = I\omega L\angle 90^\circ - \theta$$



Vector Diagram

The relationships between various quantities are shown in a phasor/vector diagram.

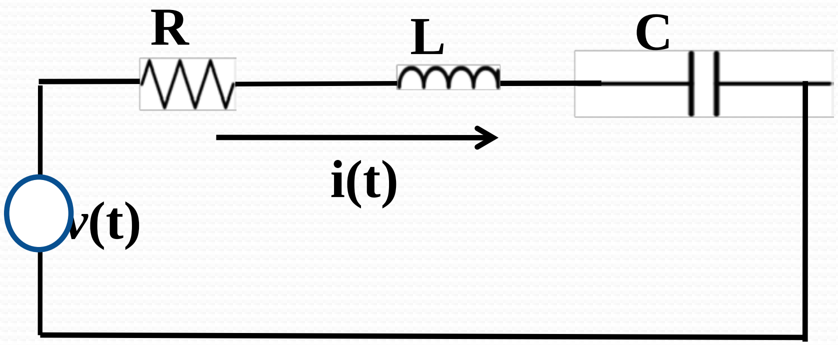
The current can be written as: $i = I_m \sin(\omega t - \theta)$, $I_m = \sqrt{2}(V/Z)$

Example 1

Question:

A voltage v(t)=220 sin(ωt) Volts is applied to a series combination of a resistance of 15 Ω, an inductive reactance 10 Ω and a capacitive reactance of 5 Ω.

i. Find the current in the circuit
ii. Sketch the phasor diagram



Solution

- The applied voltage is : $V(t) = 220 \sin(\omega t)$
- The phasor representation of the voltage is:

$$V = \frac{220}{\sqrt{2}}\angle 0^\circ = 155.56\angle 0^\circ$$

Solution

- The impedance of the circuit is

$$\bar{Z} = 15 + j10 - j5 = 15 + j5 = 15.81\angle 18.43^\circ$$

- The current in the circuit is

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{155.56\angle 0^\circ}{15.81\angle 18.43^\circ} = 9.84\angle -18.43^\circ$$

- The instantaneous current in the circuit is

$$i(t) = \sqrt{2} \times 9.84 \sin(\omega t - 18.43 \times pi/180) = 13.92 \sin(\omega t - 0.32)$$

- The voltage across the resistor is

$$V_R = 15 \times \bar{I} = 15 \times 9.84\angle -18.43 = 147.6\angle -18.43$$

Solution - The Phasor Diagram

- The voltage across the inductor is

$$V_L = 10\angle 90^\circ \times \bar{I} = 10\angle 90^\circ \times 9.84\angle -18.43^\circ = 98.4\angle 71.57$$

- The voltage across the capacitor

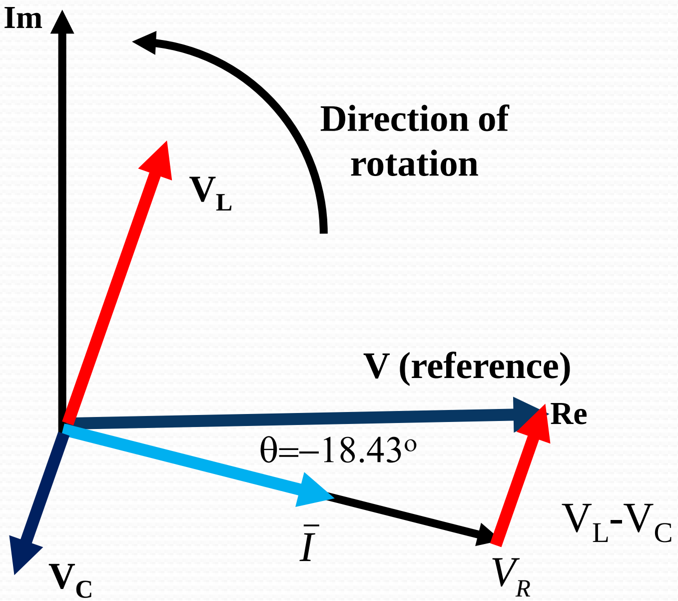
$$V_C = 5\angle -90^\circ \times \bar{I} = 5\angle -90^\circ \times 9.84\angle -18.43^\circ = 49.2\angle -108.43^\circ$$

- The resultant voltage is

$$\mathrm{V} = \mathrm{V_R} + \mathrm{V_L} + \mathrm{V_C} = 155.56\angle 0^\circ$$

- The current is

$$I = 9.84\angle -18.43^\circ$$



Multiplication and Division of Complex Quantities

- In dealing with the sinusoidal steady state response of electric circuits the need frequently arises to multiply and divide complex numbers.
- As an illustration consider the phasors $\bar{I} = Ie^{j\theta}$ and $\bar{Z} = Ze^{j\phi}$. The product of these two phasors is

$$\overline{IZ} = Ie^{j\theta} Ze^{j\phi} = IZe^{j(\theta+\phi)} = IZ\angle\theta+\phi \qquad (26)$$

- Hence, *the product of two complex numbers is found by taking the product of their magnitudes and the sum of their angles.*
- To illustrate the division of the complex numbers consider the phasors $\bar{V} = Ve^{j\theta}$ and $\bar{Z} = Ze^{j\phi}$

  The division of these two phasors is given by

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{Ve^{j\theta}}{Ze^{j\phi}} = \frac{V}{Z} e^{j(\theta-\phi)} = \frac{V}{Z}\angle\theta-\phi \qquad (27)$$

- *The division of one complex number by another involves the division of their magnitudes and difference in their phase angles.*

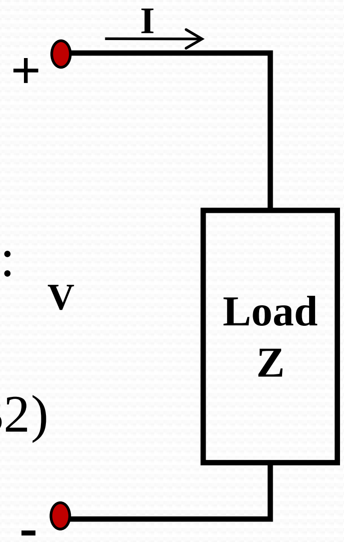
Power and Roots of Complex Quantities

- The n^{th} power of the complex quantity $\bar{Z} = Ze^{j\phi}$ is obtained as
  $$\bar{Z}^n = \left(Ze^{j\phi}\right)^n = Z^n e^{jn\phi} = Z^n \angle n\phi \qquad (28)$$
- *The n^{th} power of a complex number is a complex number whose magnitude is the n^{th} power ot the magnitude of the original complex number and whose angle is n times as large as that of the original complex number.*
- To find the root of a complex number the exponent n is made a proper fraction in eq. 40. The angle of the original complex number is increased by $2k\pi$ in order to determine all the roots that satisfy eq. 28.
- Accordingly, the fourth power of $\bar{Z} = Ze^{j\phi}$ is
  $$\bar{Z}^{\frac{1}{4}} = \left(Ze^{j(\phi+2k\pi)}\right)^{\frac{1}{4}} = Z^{\frac{1}{4}} \angle \frac{\phi}{4} + \frac{k\pi}{2} \qquad (29)$$
  Then the four distinct roots are,
  $$Z^{\frac{1}{4}} = Z^{\frac{1}{4}} \angle \frac{\phi}{4}, \qquad Z^{\frac{1}{4}} \angle \frac{\phi}{4} + \frac{\pi}{2}, \qquad Z^{\frac{1}{4}} \angle \frac{\phi}{4} + \pi, \qquad Z^{\frac{1}{4}} \angle \frac{\phi}{4} + \frac{3\pi}{4}$$
  for k=1, 2, 3, and 4 respectively.

Complex Power

- Consider the ac load shown in Fig. The voltage and current in the network are: $v = V_m \sin(\omega t)$ (30)

  $i = I_m \sin(\omega t - \theta)$ (31)



- The phasors in terms of rms values can be written as:

$$\mathrm{V} = \frac{V_m}{\sqrt{2}}\angle 0^\circ = V\angle 0^\circ, \quad \text{and} \qquad \mathrm{I} = \frac{I_m}{\sqrt{2}}\angle -\theta^\circ = I\angle -\theta^\circ$$

- The complex power S absorbed by the ac load is defined as:

$$\mathrm{S} = \mathrm{V} \times \mathrm{I}^* = V\angle 0^\circ \times (I\angle -\theta^\circ)^* = VI\angle \theta^\circ$$

$$= VI\cos\theta + jVI\sin\theta = S\cos\theta + jS\sin\theta = P + jQ \qquad (32)$$

- The complex power may also be written as:
  - In terms of load impedance and V as:

$$\mathrm{S} = \mathrm{V} \times \mathrm{I}^* = \mathrm{V} \times \left[\frac{\mathrm{V}}{\mathrm{Z}}\right]^* = \frac{\mathrm{VV}^*}{\mathrm{Z}^*}2 = \frac{V^2}{\mathrm{Z}^*} \qquad (33)$$

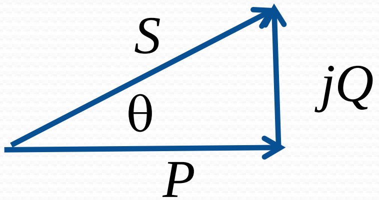
  - In terms of circuit parameters and I as:

$$\mathrm{S} = \mathrm{V} \times \mathrm{I}^* = \mathrm{IZ} \times \mathrm{I}^* = I^2\mathrm{Z} = I^2(R + jX) = I^2R + jI^2X = P + jQ \qquad (34)$$

Power Triangle

- The complex power

$$S = V \times I^* = V\angle 0^\circ \times (I\angle -\theta^\circ)^* = VI\angle\theta^\circ$$
$$= VI\cos\theta + jVI\sin\theta = S\cos\theta + jS\sin\theta = P + jQ$$



Power Triangle

is commonly represented in the power triangle as shown.

where, S = VI, is the apparent power,

P = VI cos θ is the real power

Q = VI sin θ is the reactive power , and

cos θ = *P*/*S* is the power factor

- It may be noted that the angle θ in the power triangle, in the current, and the power triangle is the same, except for the sign, when the voltage is the reference.

Example 1

- The voltage across a load is $v(t) = 60\cos(\omega t - 10^o)$ V and the current through the element in direction of voltage drop is $i(t) = 1.5\cos(\omega t + 50^o)$ A.
- Find

  (a) The complex power and the apparent power

  (b) The real and reactive power

  (c) The power factor and the load impedance

Solution

a. $V_{rms} = \frac{60}{\sqrt{2}} \angle -10^\circ, \; I_{rms} = \frac{1.5}{\sqrt{2}} \angle 50^\circ$

The complex power is

$$\mathbf{S} = \mathbf{V}_{rms} \mathbf{I}^*_{rms} = \left(\frac{60}{\sqrt{2}} \angle -10^o \right) \left(\frac{1.5}{\sqrt{2}} \angle -50^o \right) = 45 \angle -60^o$$

The apparent power is $S = |\mathbf{S}| = 45$ Volt-Ampere [VA]

b. $\mathbf{S} = P + jQ = 45\cos(-60^o) + j\sin(-60^o) = 22.5 - j38.97$

The active power is P=22.5 watts [w]

c. The power factor is $pf = \cos(\text{-}60^o) = 0.5(leading)$

and the load impedance is $\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle -10^o}{1.5\angle 50^o} = 40\angle -60^o$

Network Theorems for Sinusoidal Steady State Analysis

Procedure to Analyze AC Circuits

- The basic steps involved in applying network theorems to AC circuits are

  Step1: Transform the circuit to the phasor or frequency domain, when required

  Step 2: Solve the problem using the circuit techniques such as nodal, analysis, mesh analysis, superposition theorem, etc.

  Step 3: Transform the resulting phasor to the time domain, when required.

- The Step 1 is not necessary if the problem is specified in the frequency domain.
- In Step 2, the analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.

Network Theorems for AC Circuits

- The network theorems discussed in DC circuits are also applicable to AC circuits.
- Kirchoff's laws are obviously applicable to AC circuits. Series and parallel combinations are applicable to impedances.
- Star=Delta transformations are applicable by replacing the resistances with proper impedances.

$$Z_{23} = Z_2 + Z_3 + \frac{Z_2 Z_3}{Z_1}, \qquad Z_{31} = Z_3 + Z_1 + \frac{Z_3 Z_1}{Z_2}, \qquad Z_{12} = Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3}$$

$$Z_1 = \frac{Z_{12} Z_{13}}{Z_{12} + Z_{13} + Z_{23}}, \qquad Z_2 = \frac{Z_{21} Z_{23}}{Z_{12} + Z_{13} + Z_{23}}, \qquad Z_3 = \frac{Z_{31} Z_{32}}{Z_{12} + Z_{13} + Z_{23}}$$

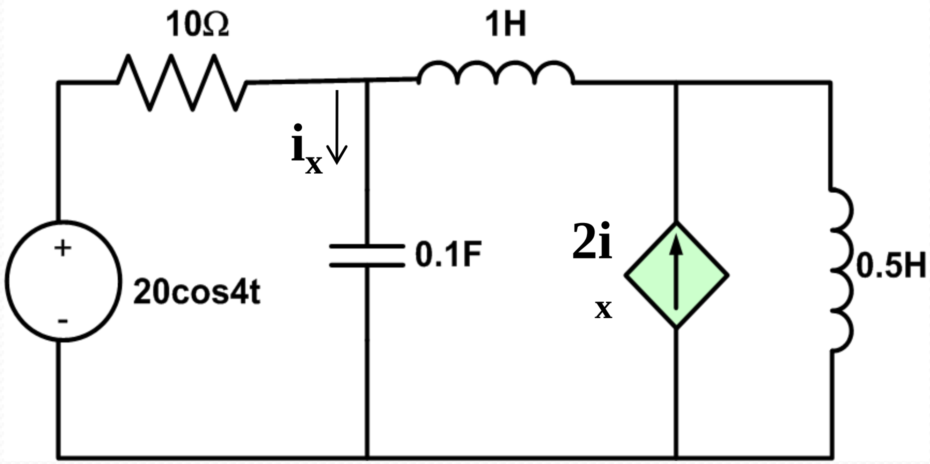
- The following sections, highlight more important aspects and features of some procedures and theorems when applied to AC circuits.

Nodal Analysis

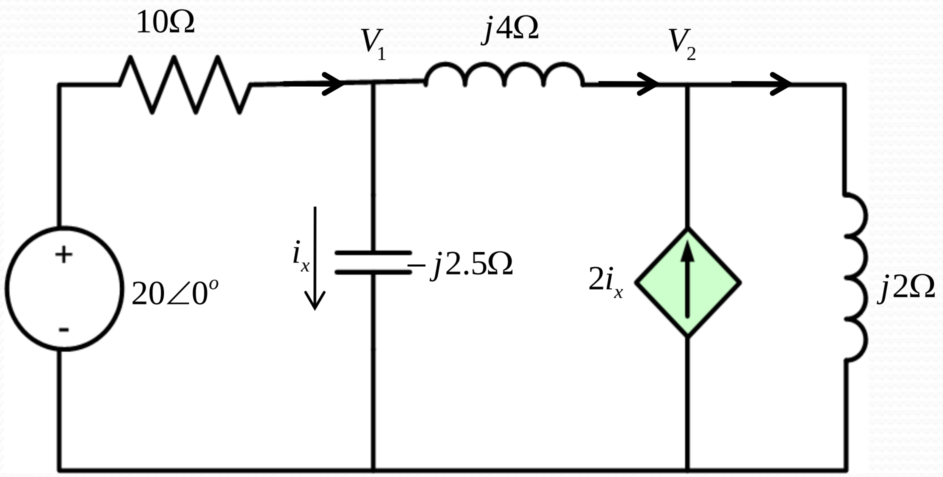
- Consider the network given for nodal analysis.
- Convert the entire circuit to the frequency domain

$$20\cos 4t \;\Rightarrow\; 20\angle 0^o,$$
$$1\text{ H} \;\Rightarrow\; j\omega L = j4\ \Omega$$
$$0.5H \;\Rightarrow\; j\omega L = j2\ \Omega$$
$$0.1F \;\Rightarrow\; \frac{1}{j\omega C} = -j2.5\ \Omega$$
$$\text{since,}\quad \omega = 4\ rad/s$$

- The frequency domain equivalent circuit is as shown.



Network for Nodal Analysis



Frequency domain equivalent of the circuit

Nodal Analysis

- Applying KCL at node 1

$$\frac{20-V_1}{10}=\frac{V_1}{-j2.5}+\frac{V_1-V_2}{j4} \qquad (1)$$

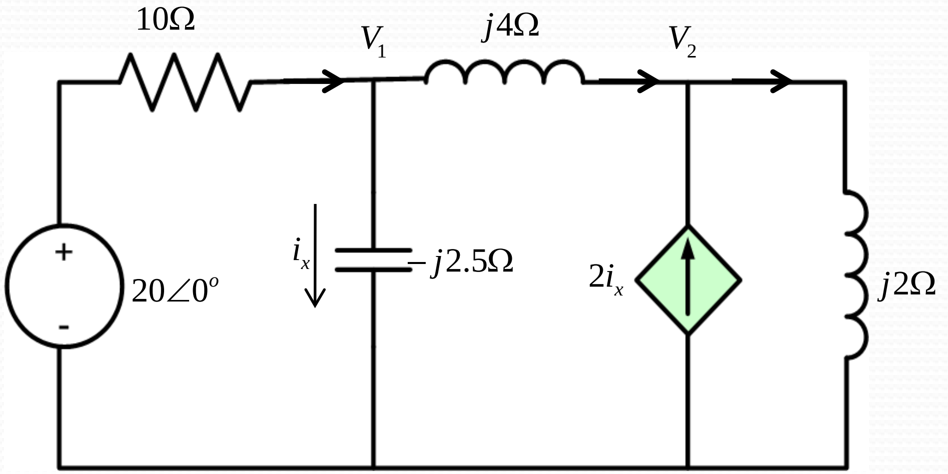
- The KCL at node 2

$$2I_x+\frac{V_1-V_2}{j4}=\frac{V_2}{j2} \qquad (2)$$

- Solution of eq. 1 and eq. 2 gives

$$V_1=18.97\angle 18.43^o\ V$$

$$V_2=13.91\angle 198.3^o\ V$$



Frequency domain equivalent of the circuit

- The current

$$I_x=\frac{V_1}{-j2.5}=\frac{18.97\angle 18.43^o}{2.5\angle -90^o}=7.59\angle 108.4^o\,A$$

- Transforming i_x to the time domain gives

$$i_x=7.59\cos(4t+108.4^o)\,A$$

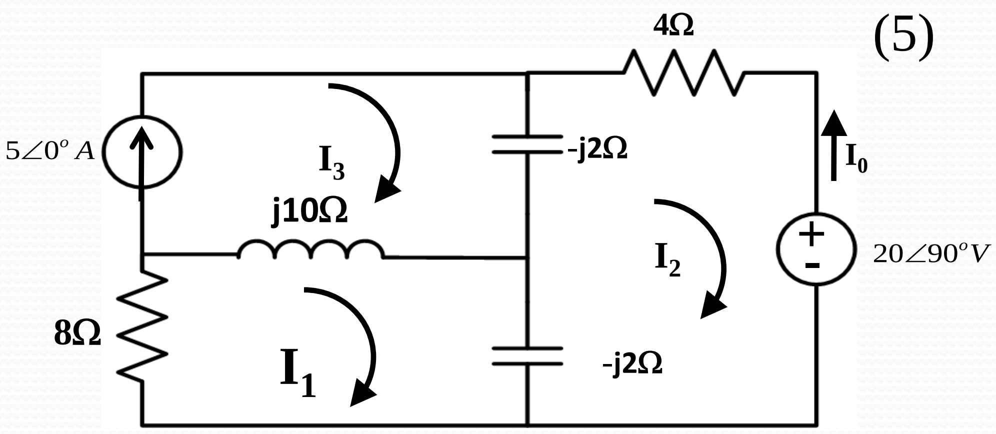
Mesh Analysis

- Consider the circuit shown. It is desired to find current I_0.
- Applying KVL to mesh 1: $(8+j10-j2)I_1-(-j2)I_2-j10I_3=0$ (1)
- The KVL for mesh 2: $(4-j2-j2)I_2-(-j2)I_1-(-j2)I_3+20\angle 90^o=0$ (2)
- For mesh 3, $I_3=5$ (3)
- Substituting eq. 3 in eq. 1 and eq. 2 gives

$$(8+j8)I_1+j2I_2=j50 \quad (4)$$
$$j2I_1+(4-j4)I_2=-j20-j10 \quad (5)$$

- Solving eq 4. and eq.5 gives

$$I_2=6.12\angle -35.22^o\,A$$
$$\Rightarrow I_0=-I_2=6.12\angle 144.78^o\,A$$



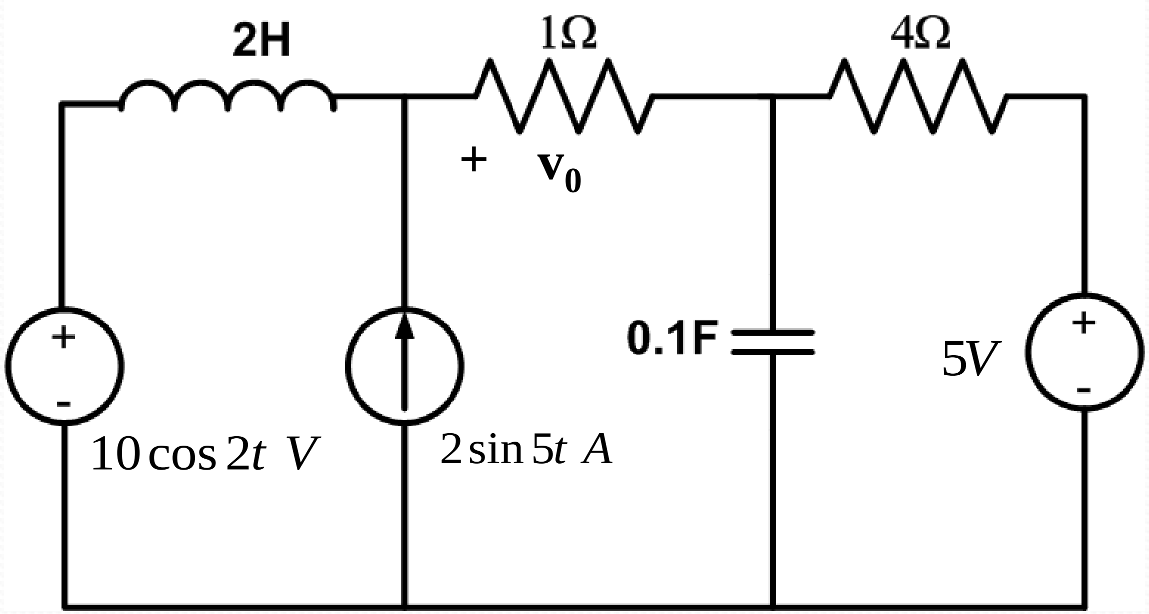
The network for nodal analysis

Superposition Theorem

- Since AC circuits are linear, the superposition theorem applies to AC circuits the same way it applies to dc circuits.
- The theorem becomes important if the circuit has sources operating at different frequencies. In this case, since the impedances depend in frequency, **it is required to have a different frequency domain circuit for each frequency.**
- In case of sources with **different frequencies, the total response must be obtained by adding the individual responses in time domain.** It is incorrect to add the responses in the phasor or frequency domain.

Superposition Theorem

- Consider the circuit shown. It is required to find the voltage across the 1 Ω resistor.



The Network for Superposition Theorem

Superposition Theorem

- Since the circuit operates at three different frequencies, the problem is divided into single frequency problems. Then,

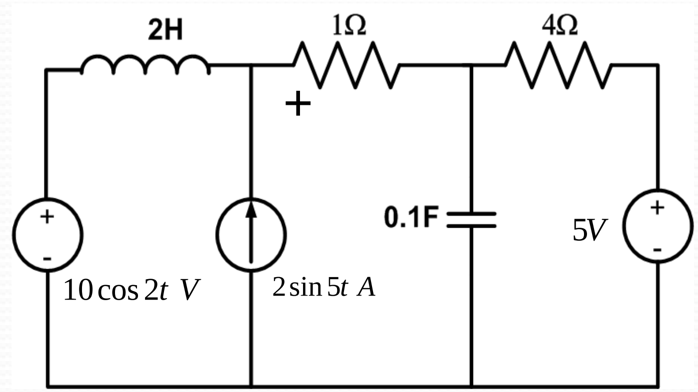
$$v_0 = v_1 + v_2 + v_3$$

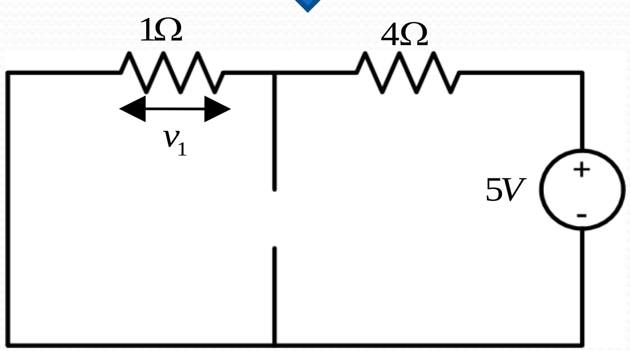
where

v_1 *is due to* $5\ V$ *dc voltage source*

v_2 *is due to the* $10\cos 2t\ V$ *voltage source*

v_3 *is due to the* $2\sin 5t\ A$ *current source*



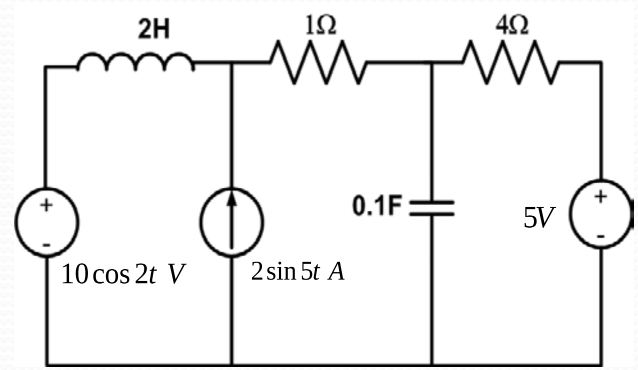


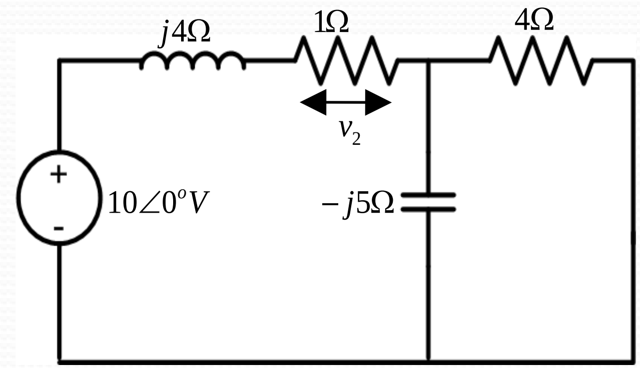
- To find v_1, remove all sources except the 5V dc source. In steady state, a capacitor is an open circuit to dc while an inductor is a short circuit to dc. The equivalent circuit is as shown. By voltage division

$$-v_1 = \frac{1}{1+4} \times 5 = 1\ \text{V}$$

Superposition Theorem

- To find v_2, the 5 v voltage source is short circuited and the current source is open circuited. The equivalent circuit is as shown in the figure.





$$10\cos 2t \Rightarrow 10\angle 0^o, \;\; \omega = 2\text{rad/s}$$

$$2H \Rightarrow j\omega L = j4\Omega$$

$$0.1F \Rightarrow \frac{1}{j\omega C} = -j5\Omega$$

The parallel combination of -j5Ω and 4Ω is

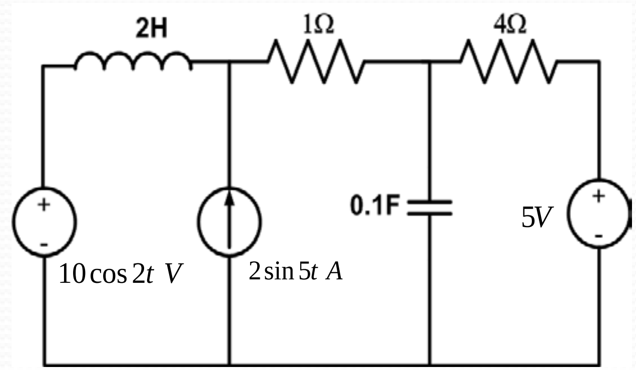
$$Z = \frac{-j5\times 4}{4-j5} = 2.439 - j1.951$$

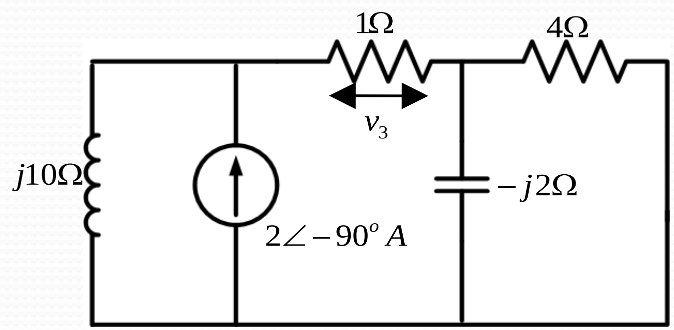
- By voltage division

$$V_2 = \frac{1}{1+j4+Z}\times\left(10\angle 0^o\right) \quad = \frac{10}{3.439+j2.049} = 2.498\angle -30.79^o \text{ V}$$

In time domain, $\quad v_2 \quad = 2.498\cos(2t - 30.79^o)$ V

Superposition Theorem





- To obtain v_3, set the voltage sources to zero and transform what is left to the frequency domain

$$2\cos 5t \Rightarrow 2\angle -90^o,\ \ \omega\text{=5rad/s}$$

$$2H \Rightarrow j\omega L = j10\Omega$$

$$0.1F \Rightarrow \frac{1}{j\omega C} = -j2\Omega$$

The parallel combination of -j2Ω and 4Ω is

$$Z = \frac{-j2\times 4}{4-j2} = 0.8 - j1.6$$

- By current division

$$I_3 = \frac{j10}{j10+1+Z}\left(2\angle -90^o\right)\text{A},\ \ v_3 = I_1\times 1 = \frac{j10}{1.8+j8.4}(-j2) = 2.328\angle -80^o\text{V}$$

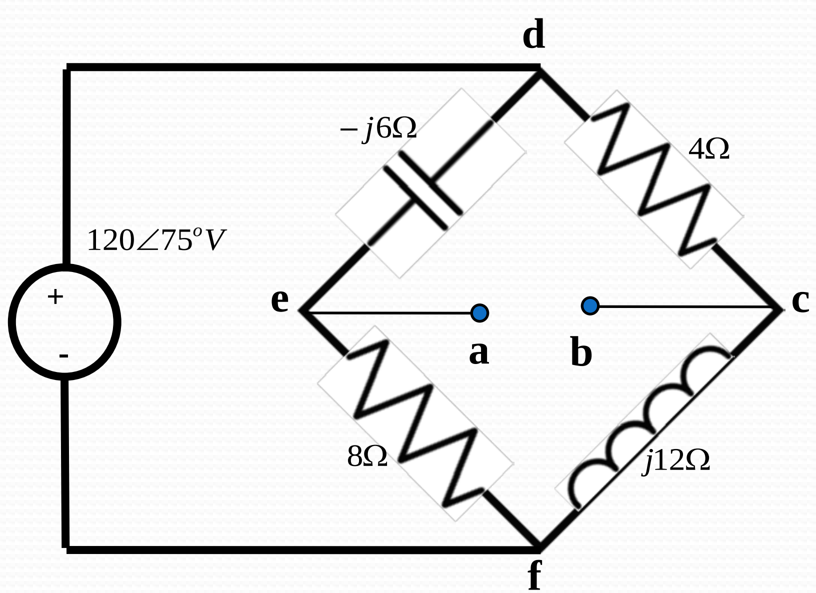
- The final output is summation of all three:

$$v_0(t) = -1 + +2.498\cos(2t - 30.79^o) + 2.33\sin(5t + 10^o)\ \text{V}$$

Thevenin and Norton Equivalent Circuits

- Thevnin’s and Norton’s Equivalents are equally applicable to AC circuits.

- The procedure is illustrated by finding the Thevnin’s equivalent between nodes a and b.



The Network for Thevenin and Norton’s Theorem

Thevenin's Equivalent

- The value of Thevenin's impedance Z_{th} is obtained by setting the voltage source to zero. It is seen that 8 Ω and –j6 Ω are in parallel:

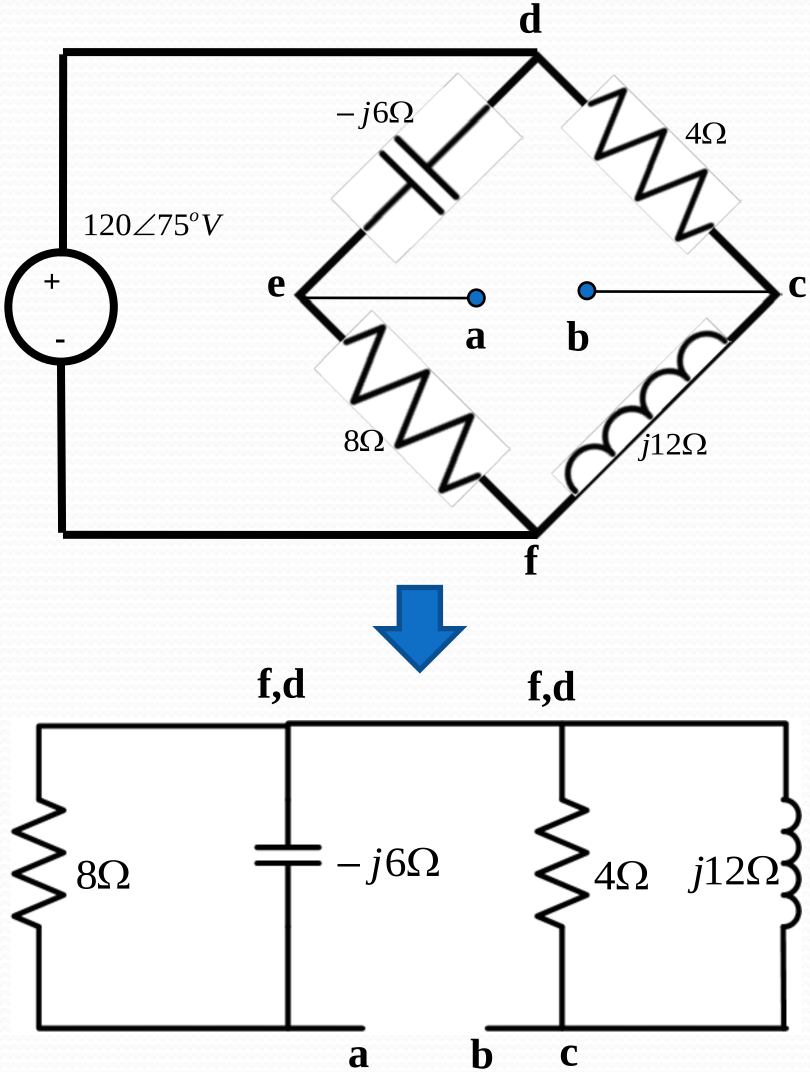
$$Z_1 = \frac{-j6 \times 8}{8 - j6} = 2.88 - j3.84\Omega$$

- The 4 Ω resistance is in parallel with the j12 Ω reactance:

$$Z_2 = \frac{j12 \times 4}{4 + j12} = 3.6 + j1.2\ \Omega$$

- The Thevenin impedance is the series combination of Z_1 and Z_2, i.e.

$$Z_{th} = Z_1 + Z_2 = 6.48 - j2.64\ \Omega$$



Thevenin's Equivalent

- To find V_{th}, consider the circuit and currents I_1 and I_2 are obtained as

$$I_1 = \frac{120\angle 75^o}{8 - j6} \text{ A}, \qquad I_2 = \frac{120\angle 75^o}{4 + j12} \text{ A}$$

- Applying KVL around loop *bcdeab*

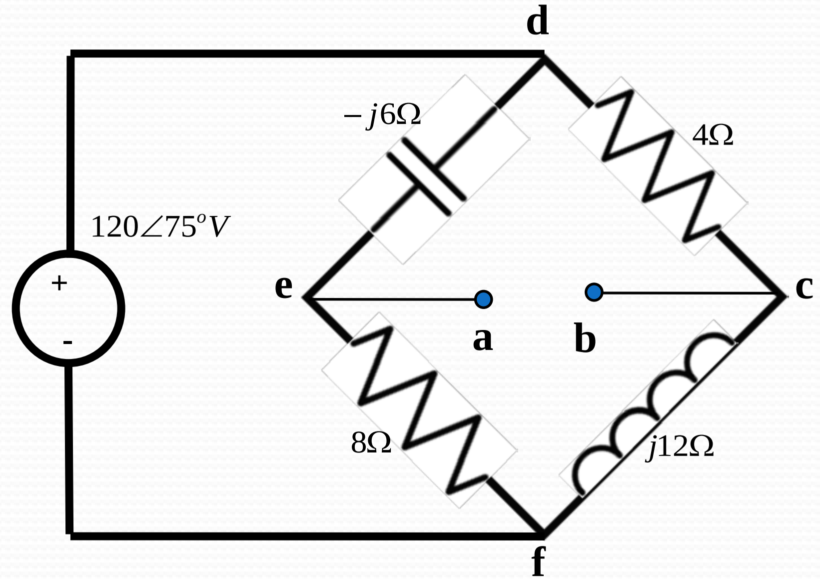
$$V_{th} - 4I_2 + (-j6)I_1 = 0$$

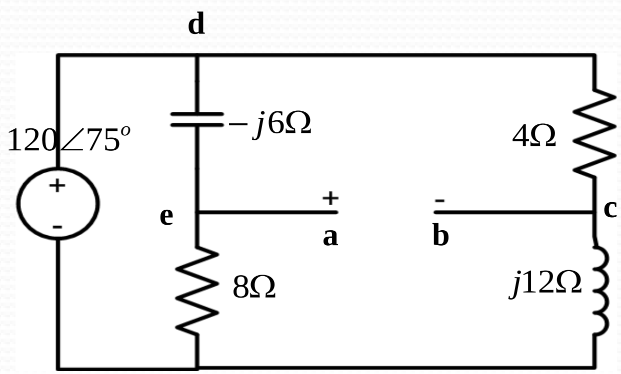
$$V_{th} = 4I_2 + j6I_1$$

$$= \frac{480\angle 75^o}{4 + j12} + \frac{720\angle 75^o + 90^o}{8 - j6}$$

$$= 37.95\angle 3.43^o + 72\angle 201.87^o$$

$$= 37.95\angle 220.31^o \text{ V}$$





Example (Supernode)

- Compute V_1 and V_2 in the network shown in Fig.8

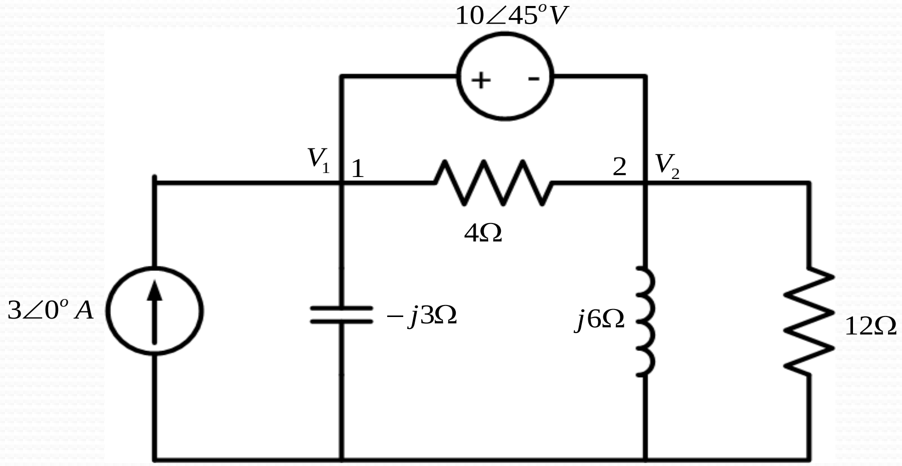
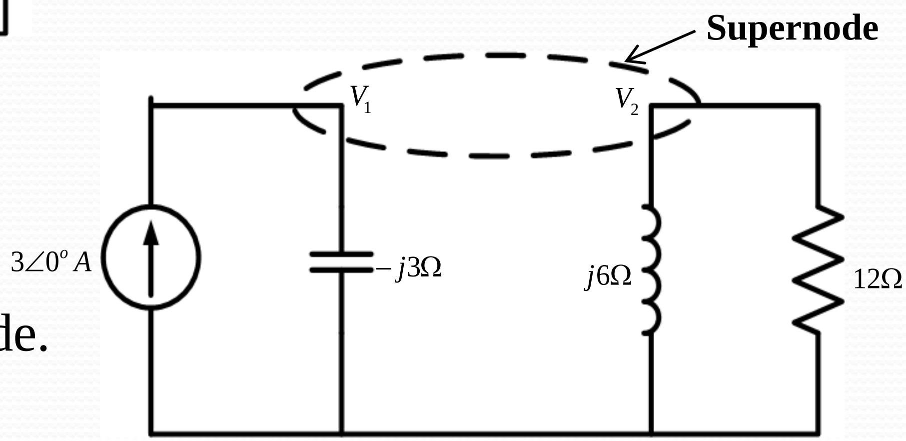


Fig.8: The Network for Example



- Node 1 and Node 2 form a supernode.

Example (Supernode)

- Applying KCL at the supernode gives

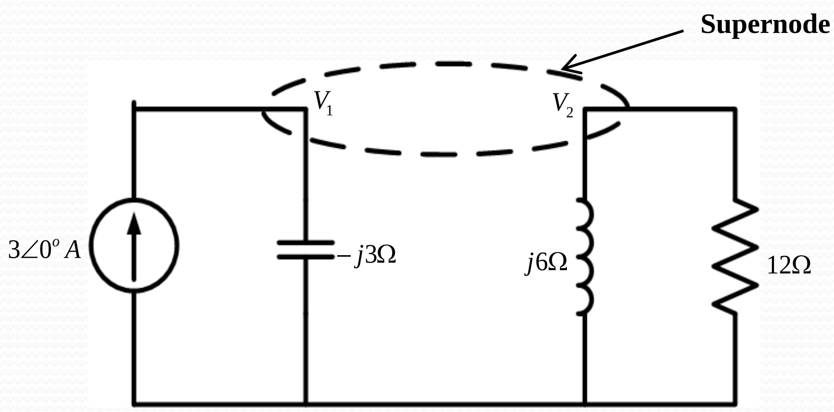
$$3 = \frac{V_1}{-j3} + \frac{V_2}{j6} + \frac{V_2}{12}$$

$$36 = j4V_1 + (1 - j2)V_2$$

- A voltage source is connected between nodes 1 and 2, hence

$$V_1 = V_2 + 10\angle 45^o$$

- Substituting for V_1 gives



$$36 - 40\angle 135^o = (1 + j2)V_2$$

$$V_2 = 31.41\angle -87.18^o \text{ V}$$

$$V_1 = V_2 + 10\angle 45^o = 25.78\angle -70.48^o \text{ V}$$

Thank you!