DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Guwahati

MA101: Mathematics I, July - November, 2014

Tutorial Sheet: LA - 7

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1. Examine whether the following maps $T : V \to W$ are linear transformations.

   (a) $V = W = \mathbb{C}^2(\mathbb{C})$ and $T[z_1, z_2]^t = [\overline{z}_1, \overline{z}_2]^t$ for all $[z_1, z_2]^t \in \mathbb{C}^2$.

   (b) $V = W = M_n(\mathbb{R})$ and fix $B \in M_n(\mathbb{R})$. Consider $T(A) = AB - BA$ for all $A \in V$.

2. Let $T : V \to V$ be a linear transformation such that $T \circ T = I$ and let $\mathbf{v} \in V$.

   (a) Show that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent if and only if $T(\mathbf{v}) = \pm\mathbf{v}$.

   (b) Give an example of such a linear transformation with $V = \mathbb{R}^2$.

3. Find bases for the range space and the null space of the linear transformation $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(A) = AB - BA$ for all $A \in M_2(\mathbb{R})$, where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Hence find the rank and the nullity of T.

4. Let V and W be two finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Show that

   (a) if $\dim(V) < \dim(W)$ then T is not onto; and

   (b) if $\dim(V) > \dim(W)$ then T is not one-one.

5. Let T be a linear transformation on a vector space V and let $\dim V = n$. If

$$T^{n-1}(\mathbf{x}) \neq \mathbf{0} \text{ but } T^n(\mathbf{x}) = \mathbf{0} \text{ for some } \mathbf{x} \in V,$$

   then show that the set $\{\mathbf{x}, T(\mathbf{x}), \ldots, T^{n-1}(\mathbf{x})\}$ is a basis for V. Also, find the matrix representation of T with respect to this basis.

6. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$, defined by

$$T[x, y, z]^t = [2x + 3y - z, 4x - y + 2z]^t \text{ for all } [x, y, z]^t \in \mathbb{R}^3.$$

   Find $[T]_{C \leftarrow B}$, where $B = \{[1, 1, 0]^t, [1, 2, 3]^t, [1, 3, 5]^t\}$ and $C = \{[1, 2]^t, [2, 3]^t\}$.

7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $A = [a_{ij}]$ be the matrix of T with respect to an orthonormal basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$. Show that $a_{ij} = \mathbf{v}_i.(T\mathbf{v}_j)$ for all i, j.

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1. Examine whether the following maps $T : V \to W$ are linear transformations.

   (a) $V = W = \mathbb{C}^2(\mathbb{C})$ and $T[z_1, z_2]^t = [\overline{z}_1, \overline{z}_2]^t$ for all $[z_1, z_2]^t \in \mathbb{C}^2$.

   (b) $V = W = M_n(\mathbb{R})$ and fix $B \in M_n(\mathbb{R})$. Consider $T(A) = AB - BA$ for all $A \in V$.

   Solution:

   (a) T is not a linear transformation, as

   $$\begin{bmatrix} -i \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = T\left(i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \neq iT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} i \\ 0 \end{bmatrix}.$$

   (b) Let $X, Y \in M_n(\mathbb{R})$ and $a, b \in \mathbb{R}$. We have

   $$\begin{aligned} T(aX + bY) &= (aX + bY)B - B(aX + bY) \\ &= a(XB - BX) + b(YB - BY) \\ &= aT(X) + bT(Y). \end{aligned}$$

   Hence T is a linear transformation.

$\square$

2. Let $T : V \to V$ be a linear transformation such that $T \circ T = I$ and let $\mathbf{v} \in V$.

   (a) Show that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent if and only if $T(\mathbf{v}) = \pm\mathbf{v}$.

   (b) Give an example of such a linear transformation with $V = \mathbb{R}^2$.

   Solution:

   (a) If $T(\mathbf{v}) = \pm\mathbf{v}$, then it is clear that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent.

   Conversely, suppose that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent. If $\mathbf{v} = \mathbf{0}$ then clearly $T(\mathbf{v}) = \pm\mathbf{v}$. If $\mathbf{v} \neq \mathbf{0}$, then $T(\mathbf{v}) = a\mathbf{v}$ for some $a \in \mathbb{F}$. We have

   $$\begin{aligned} T(\mathbf{v}) = a\mathbf{v} \Rightarrow T(T(\mathbf{v})) = T(a\mathbf{v}) &\Rightarrow T^2(\mathbf{v}) = aT(\mathbf{v}) \\ &\Rightarrow \mathbf{v} = a^2\mathbf{v} \\ &\Rightarrow (1 - a^2)\mathbf{v} = \mathbf{0} \\ &\Rightarrow 1 - a^2 = 0, \quad \text{since } \mathbf{v} \neq \mathbf{0} \\ &\Rightarrow a = \pm 1 \\ &\Rightarrow T(\mathbf{v}) = \pm\mathbf{v}. \end{aligned}$$

   (b) Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T([x, y]^t) = [-x, -y]^t$ for all $[x, y]^t \in \mathbb{R}^2$.

$\square$

3. Find bases for the range space and the null space of the linear transformation $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(A) = AB - BA$ for all $A \in M_2(\mathbb{R})$, where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Hence find the rank and the nullity of T.

Solution: Let $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \ker(T)$ (the null space of T). Then we have

$$
\begin{aligned}
T(A) = \mathbf{O} &\Rightarrow AB - BA = \mathbf{O} \\
&\Rightarrow \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} z-y & w-x \\ x-w & y-z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&\Rightarrow z-y=0, w-x=0, x-w=0, y-z=0 \\
&\Rightarrow y=z, x=w.
\end{aligned}
$$

Thus $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ y & x \end{bmatrix} = x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also, the set $\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ is linearly independent. Hence $\mathcal{B}$ is a basis for $\ker(T)$.

Now let $X \in \text{range}(T)$. Then $X = T(A) = AB - BA$, for some $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_2(\mathbb{R})$. We have

$$
X = AB - BA = \begin{bmatrix} z-y & w-x \\ x-w & y-z \end{bmatrix} = (z-y)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (w-x)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
$$

Also, the set $\mathcal{C} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$ is linearly independent. Hence $\mathcal{C}$ is a basis for range(T).

We see that nullity$(T) = 2$ and rank$(T) = 2$. □

4. Let V and W be two finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Show that

(a) if $\dim(V) < \dim(W)$ then T is not onto; and

(b) if $\dim(V) > \dim(W)$ then T is not one-one.

Solution: Let $\dim(V) = n$ and $\dim(W) = m$.

(a) Let T be onto. Then we have range$(T) = W$, and so rank$(T) = \dim(W) = m$. We have

$$\text{rank}(T) + \text{nullity}(T) = n \Rightarrow \text{nullity}(T) = n - m \geq 0 \Rightarrow n \geq m.$$

(b) Let T be one-one. Then we have $\ker(T) = \{\mathbf{0}\}$, and so nullity$(T) = 0$. We have

$$\text{rank}(T) + \text{nullity}(T) = n \Rightarrow n = \text{rank}(T) \leq \dim(W) = m.$$

□

5. Let T be a linear transformation on a vector space V and let dim $V = n$. If

$$T^{n-1}(\mathbf{x}) \neq \mathbf{0} \text{ but } T^n(\mathbf{x}) = \mathbf{0} \text{ for some } \mathbf{x} \in V,$$

then show that the set $\{\mathbf{x}, T(\mathbf{x}), \ldots, T^{n-1}(\mathbf{x})\}$ is a basis for V. Also, find the matrix representation of T with respect to this basis.

Solution: Let $a_1\mathbf{x}+a_2T(\mathbf{x})+\ldots+a_nT^{n-1}(\mathbf{x}) = \mathbf{0}$. Applying T^k on both sides of this equation, successively for $k = n-1, \ldots, 2, 1$, we find that $a_1 = a_2 = \ldots = a_{n-1} = a_n = 0$. Hence the vectors $\mathbf{x}, T(\mathbf{x}), \ldots, T^{n-1}(\mathbf{x})$ are linearly independent.

We have

$$\begin{aligned}
T(\mathbf{x}) &= 0\mathbf{x} + T(\mathbf{x}) + 0T^2(\mathbf{x}) + \ldots + 0T^{n-1}(\mathbf{x}) \\
T(T(\mathbf{x})) &= 0\mathbf{x} + 0T(\mathbf{x}) + T^2(\mathbf{x}) + \ldots + 0T^{n-1}(\mathbf{x}) \\
\vdots \quad & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
T(T^{n-2}(\mathbf{x})) &= 0\mathbf{x} + 0T(\mathbf{x}) + 0T^2(\mathbf{x}) + \ldots + T^{n-1}(\mathbf{x}) \\
T(T^{n-1}(\mathbf{x})) &= 0\mathbf{x} + 0T(\mathbf{x}) + 0T^2(\mathbf{x}) + \ldots + 0T^{n-1}(\mathbf{x}).
\end{aligned}$$

Hence the required matrix is $[a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

□

6. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$, defined by

$$T[x, y, z]^t = [2x + 3y - z, 4x - y + 2z]^t \text{ for all } [x, y, z]^t \in \mathbb{R}^3.$$

Find $[T]_{C \leftarrow B}$, where $B = \{[1, 1, 0]^t, [1, 2, 3]^t, [1, 3, 5]^t\}$ and $C = \{[1, 2]^t, [2, 3]^t\}$.

Solution: We have

$$\begin{aligned}
T([1, 1, 0]^t) &= [5, 3]^t = -9[1, 2]^t + 7[2, 3]^t \\
T([1, 2, 3]^t) &= [5, 8]^t = [1, 2]^t + 2[2, 3]^t \\
T([1, 3, 5]^t) &= [6, 11]^t = 4[1, 2]^t + [2, 3]^t.
\end{aligned}$$

Hence

$$[T]_{C \leftarrow B} = [[T([1, 1, 0]^t)]_C, [T([1, 2, 3]^t)]_C, [T([1, 3, 5]^t)]_C] = \begin{bmatrix} -9 & 1 & 4 \\ 7 & 2 & 1 \end{bmatrix}.$$

□

7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $A = [a_{ij}]$ be the matrix of T with respect to an orthonormal basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$. Show that $a_{ij} = \mathbf{v}_i.(T\mathbf{v}_j)$ for all i, j.

Solution: For $1 \leq j \leq n$, we have

$$\begin{aligned}
& T\mathbf{v}_j = a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + \ldots + a_{ij}\mathbf{v}_i + \ldots + a_{nj}\mathbf{v}_n \\
\Rightarrow & \mathbf{v}_i.(T\mathbf{v}_j) = \mathbf{v}_i.(a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + \ldots + a_{ij}\mathbf{v}_i + \ldots + a_{nj}\mathbf{v}_n, \quad (1 \leq i \leq n) \\
\Rightarrow & \mathbf{v}_i.(T\mathbf{v}_j) = a_{1j}(\mathbf{v}_i.\mathbf{v}_1) + a_{2j}(\mathbf{v}_i.\mathbf{v}_2) + \ldots + a_{ij}(\mathbf{v}_i.\mathbf{v}_i) + \ldots + a_{nj}(\mathbf{v}_i.\mathbf{v}_n) \\
\Rightarrow & \mathbf{v}_i.(T\mathbf{v}_j) = a_{ij}.
\end{aligned}$$

□