Indian Institute of Technology Guwahati
Department of Mathematics

MA 101 – Mathematics-I **Date:** 14-Sep-2015
Tutorial Sheet-6 **Time:** 08:00 – 09:00

Linear Algebra

Topics Covered:
Vector spaces, subspace, basis, change-of-basis, linear transformations, matrix of linear transformation, kernel and range of a linear transformation

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1. In the following examples,

   (i) find the co-ordinate vectors $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{C}}$ with respect to the given bases $\mathcal{B}$ and $\mathcal{C}$

   (ii) Find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$, and then find $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

   (iii) Use the answer in (ii) to compute $[v]_{\mathcal{C}}$ from $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{B}}$ from $[v]_{\mathcal{C}}$. Compare with answer in (i):

   (a) $v = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix} \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix} \right\}$ in $\mathbb{R}^3$.

   (b) $v = 1 + x^2, \quad \mathcal{B} = \{1, x, x^2\}, \quad \mathcal{C} = \{1 + x + x^2, x + x^2, x^2\}$ in $\mathcal{P}_2(R)$.

2. In each of the following, find the matrix of T with respect to the standard bases. Is the matrix invertible? If so, find the inverse matrix and hence the inverse linear map of T.

   (i) $T : \mathbb{R}^2 \longrightarrow \mathcal{P}_1, \quad \begin{bmatrix}a\\b\end{bmatrix} \mapsto a + (a - b)x.$

   (ii) $D : \mathrm{Span}(e^x, xe^x, x^2e^x) \longrightarrow \mathcal{F}$ where D is the differential operator and $\mathcal{F}$ is the space of all real valued functions from $\mathbb{R}$ to $\mathbb{R}$.

3. Let V be a vector space and $T : V \to V$ be a linear transformation such that $T \circ T = I$ and let $v \in V$ be a non-zero vector. Then show that the set $\{v, T(v)\}$ is linearly dependent if and only if $T(v) = \pm v$.

4. Let V, W be two vector spaces over $\mathbb{R}$, and $T : V \longrightarrow W$ be a linear map. Let $w_1, \ldots, w_n$ be linearly independent vectors in W, and let $v_1, \ldots, v_n$ be vectors in V such that $T(v_i) = w_i$. Show that $v_1, \ldots, v_n$ are linearly independent.

5. Let V, W be finite dimensional vector spaces over the field $\mathbb{F}$ and $T : V \to W$ be a linear map.

   (a) Show that
   $$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V).$$

   (b) Assume that $\dim V = \dim W$, then show that T is injective if and only if T is surjective.

   (c) If $\dim(V) < \dim(W)$ then show that T is not surjective.

   (d) If $\dim(V) > \dim(W)$ then show that T is not injective.

6. Let V be a vector space over a field $\mathbb{F}$ and $T : V \longrightarrow V$ be a linear map such that $T^n = 0$ for some $n \in \mathbb{N}$. Show that the linear map $(ID_V - T)$ is one-one and onto, where ID_V is the identity map on V.

7. Show that there is no linear transformation $T : \mathbb{R}^3 \longrightarrow \mathcal{P}_2$ for which
   $$T\left(\begin{bmatrix}2\\-1\\0\end{bmatrix} \right) = 1 + 2x + x^2, \qquad T\left(\begin{bmatrix}3\\0\\-2\end{bmatrix} \right) = 1 - x^2, \qquad T\left(\begin{bmatrix}0\\3\\-4\end{bmatrix} \right) = 2x + 3x^2.$$

8. For each of the maps given below, show that T is a linear transformation. Describe $\ker(T)$ and $\mathrm{range}(T)$ of the following. Hence find the rank or the nullity of the following

   (a) $T : M_{2\times 2} \longrightarrow \mathbb{R}$ defined by $T(A) = \mathrm{tr}(A)$.

   (b) $T : \mathcal{P}_3 \longrightarrow \mathbb{R}^2, \quad T(a + bx + cx^2 + dx^3) = \begin{bmatrix}a\\b-a\\c-a+b\end{bmatrix}$

(c) $T : M_{2\times 2} \longrightarrow M_{2\times 2}$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

(d) $T : M_{2\times 2} \longrightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b \\ c-d \end{bmatrix}$.

(e) $T : \mathcal{P}_2 \longrightarrow \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

9. Let V be vector space and $T : V \to V$ be a linear transformation. Suppose $\dim V = n$. If there exists a vector $x \in V$ such that $T^n(x) = 0$ but $T^{n-1}(x) \neq 0$, then show that the set $\{x, T(x), \ldots, T^{n-1}(x)\}$ is a basis for V. Also, find the matrix representation of T with respect to this basis.

Practice Problems

1. If $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{R}^3$ such that the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is given by $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

   If $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, find $\mathcal{B}$.

2. Let $T : \mathbb{R}^2 \longrightarrow \mathcal{P}_2(\mathbb{R})$ be a linear map for which $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 1 + 2x + x^2$ and $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 3 + 4x^2$. Find $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$.

3. Find the matrix $[S]$, $[T]$ and $[S \circ T]$ with respect to the standard bases and then verify that $[S \circ T] = [S][T]$, where

$$\begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathcal{P}_2, & \begin{bmatrix} a \\ b \end{bmatrix} &\mapsto a + (a-b)x + bx^2, \\ S : \mathcal{P}_2 &\longrightarrow \mathcal{P}_1, & a + bx + cx^2 &\mapsto (3a + 2b + c) + (a+b)x. \end{aligned}$$

4. Show that a linear map $T : V \longrightarrow W$ is uniquely determined by its values on the elements of a basis of V.

5. Let W be a subspace of a finite dimensional vector space V. Show that W is a finite dimensional vector space and $\dim W \leq \dim V$.

6. Let $\mathcal{B}$ be a set of vectors in a vector space V with the property that every vector in V is a unique linear combination of vectors in $\mathcal{B}$. Prove that $\mathcal{B}$ is a basis for V.

7. Let V and W be vector spaces over $\mathbb{R}$. Prove that the set $\mathscr{L}(V, W)$ of all linear transformations forms a vector space over $\mathbb{R}$, with the vector addition and scalar multiplication as defined over $\mathscr{F}$, the space of functions from $\mathbb{R}$ to $\mathbb{R}$.

8. Let W_1 and W_2 be two subspaces of a finite dimensional vector space V. Show that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$
