DEPARTMENT OF MATHEMATICS, IIT GUWAHATI

MA101: Mathematics I — **Mid Semester Exam** (Maximum Marks: 30)

Date: September 20, 2011 — **Time:** 2 pm - 4 pm

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1. (a) Prove or disprove: If A and B are two matrices of the same size such that the linear system of equations $A\mathbf{x} = \mathbf{a}$ and $B\mathbf{x} = \mathbf{b}$ have the same set of solutions then the matrices $[A \mid \mathbf{a}]$ and $[B \mid \mathbf{b}]$ must be row-equivalent. $\boxed{\mathbf{2}}$

   (b) Find all real values of k for which the following system of equations has (i) no solution, (ii) unique solution, and (iii) infinitely many solutions: $\boxed{\mathbf{3}}$

$$kx + y + z = 1, \quad x + ky + z = 1, \quad x + y + kz = 1.$$

2. (a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three linearly independent vectors in $\mathbb{R}^n$, where $n \geq 3$. For what real values of k, are the vectors $\mathbf{v} - \mathbf{u}, k\mathbf{w} - \mathbf{v}$ and $\mathbf{u} - \mathbf{w}$ linearly independent? $\boxed{\mathbf{2}}$

   (b) Find a basis for the subspace V, where $V = \{[x_1, x_2, \ldots, x_6]^t \in \mathbb{R}^6 : x_i = 0 \text{ if } i \text{ is even}\}$. $\boxed{\mathbf{3}}$

3. (a) Prove or disprove: There exist 2×2 matrices A and B such that $AB - BA = I_2$. $\boxed{\mathbf{2}}$

   (b) Let A be an invertible matrix with integer entries. Show that A^{-1} has all entries integer if and only if $\det(A) = \pm 1$. $\boxed{\mathbf{3}}$

4. Let A be an $n \times n$ real matrix and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for $\mathbb{R}^n$. Show that $\text{rank}(A) = n$ if and only if $\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$. $\boxed{\mathbf{5}}$

5. (a) Let A be a diagonalizable matrix such that every eigenvalue of A is either 0 or 1. Show that $A^2 = A$. $\boxed{\mathbf{2}}$

   (b) Let λ_1 and λ_2 be two distinct eigenvalues of a matrix A and let $\mathbf{u}_1$ and $\mathbf{u}_2$ be eigenvectors of A corresponding to λ_1 and λ_2, respectively. Show that $\mathbf{u}_1 + \mathbf{u}_2$ is not an eigenvector of A. $\boxed{\mathbf{3}}$

6. (a) Let W be a subspace of $\mathbb{R}^5$ and $\mathbf{v} \in \mathbb{R}^5$. Suppose that $\mathbf{w}$ and $\mathbf{w}'$ are orthogonal vectors with $\mathbf{w} \in W$ and that $\mathbf{v} = \mathbf{w} + \mathbf{w}'$. Is it necessarily true that $\mathbf{w}' \in W^\perp$? Either prove that it is true or find a counterexample. $\boxed{\mathbf{2}}$

   (b) Find a basis for $M^\perp$, where $M = \{[x, y, z]^t : x = s, y = -s, z = 3s, s \in \mathbb{R}\}$. $\boxed{\mathbf{3}}$

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