MA101 Mathematics I

Department of Mathematics
Indian Institute of Technology Guwahati

Jul – Nov 2013

Slides originally created by: Dr. Bikash Bhattacharjya

Instructors:
Rafikul Alam, Bhaba K. Sarma, Sriparna Bandyopadhyay, Kalpesh Kapoor

ver. 21 Sept 2013, 10:18 (bks)

Slides-3

Plan

- Linear Span
- Subspaces
- Linear Independence
- Basis, Dimension & Rank

http://www.iitg.ernet.in/maths/ext/ma101/

An example revisited:

Consider our earlier homogeneous system $A\mathbf{x} = \mathbf{0}$, where
$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}.$$
The solutions set for $A\mathbf{x} = \mathbf{0}$ is

$$S_h = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \; s, t \in \mathbb{R} \right\}.$$

Can we describe S_h with a few of the solutions? How? Can we derive some special properties of solution sets like S_h?

Linear Combinations:

A vector $\mathbf{v}$ in $\mathbb{R}^n$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ if there exist real numbers $c_1, c_2, \ldots, c_k$ such that
$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k.$$

- The numbers $c_1, c_2, \ldots, c_k$ are called the coefficients of the linear combination.

Example

Is the vector $[1, 2, 3]^T$ *a linear combination of* $[1, 0, 3]^T$ *and* $[-1, 1, -3]^T$*?*

Result

A system of linear equations with augmented matrix $[A \mid \mathbf{b}]$ *is consistent if and only if* $\mathbf{b}$ *is a linear combination of the columns of* A*.*

Span of Vectors: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Then the collection of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called the span of S (or span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$), and is denoted by $\text{span}(S)$ (or $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$).

Thus

$$\text{span}(S) = \{\mathbf{v} \in \mathbb{R}^n | \ \mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k \text{ for some } c_1, \ldots, c_k \in \mathbb{R}\}.$$

- Convention: $\text{span}(\emptyset) = \{\mathbf{0}\}$.
- If $\text{span}(S) = \mathbb{R}^n$, then S is called a spanning set for $\mathbb{R}^n$.
- $\mathbb{R}^2 = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$, where $\mathbf{e}_1 = [1, 0]^T$ and $\mathbf{e}_2 = [0, 1]^T$.

Example

Let $\mathbf{u} = [1, 2, 3]^T$ and $\mathbf{v} = [-1, 1, -3]^T$. Describe $\text{span}(\mathbf{u}, \mathbf{v})$ geometrically.

Subspaces of $\mathbb{R}^n$

A set $U\ (\neq \emptyset) \subseteq \mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if $a\mathbf{u} + b\mathbf{v} \in U$ for every $\mathbf{u}, \mathbf{v} \in U$ and for every $a, b \in \mathbb{R}$.

- $U = \{\mathbf{0}\}$ and $U = \mathbb{R}^n$ are subspaces of $\mathbb{R}^n$, called the trivial subspaces of $\mathbb{R}^n$.
- Any subspace contains $\mathbf{0}$.
- U is a subspace iff U is closed under addition and scalar multiplication.
- For any finite subset S of $\mathbb{R}^n$, $\text{span}(S)$ is a subspace of $\mathbb{R}^n$.

Example

Examine whether the sets
$S = \{[x, y, z]^T \in \mathbb{R}^3 : x = y + 1\}$, $\ T = \{[x, y, z]^T \in \mathbb{R}^3 : x = 5y\}$ *and* $U = \{[x, y, z]^T \in \mathbb{R}^3 : x = z^2\}$ *are subspaces of* $\mathbb{R}^3$.

Result

Let A be an $m \times n$ matrix. Then $U = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the nullspace of A.

Result

Let U and V be two subspaces of $\mathbb{R}^n$. Then $U + V = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$ is also a subspace of $\mathbb{R}^n$.

If U and V are subspaces of $\mathbb{R}^n$ such that $U \cap V = \{\mathbf{0}\}$, then $U + V$ is called an internal direct sum. Notation: $U \oplus V$.

Linear Dependence

A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly dependent if one of the vectors $\mathbf{v}_i$ is a linear combination of the rest, i.e., if there are real numbers $c_1, c_2, \ldots, c_k$, at least one of them is non-zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}.$$

- We say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent, to mean that the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent.
- Any set of vectors containing the $\mathbf{0}$ is linearly dependent.

Example

Examine whether the sets $T = \{[1, 2, 0]^T, [1, 1, -1]^T, [1, 4, 2]^T\}$ and $S = \{[1, 4]^T, [-1, 2]^T\}$ are linearly dependent.

Linear Independence

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is said to be linearly independent if S is not linearly dependent.

- S is linearly independent iff
  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = c_2 = \ldots = c_k = 0.$
- We say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent, to mean that the set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is linearly independent.

Example

Let $\mathbf{e}_i \in \mathbb{R}^n$ be the i-th column of the identity matrix I_n. Is $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ linearly independent?

Linear combinations of rows

Suppose $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ is an $m \times n$ matrix. Then

- For $c_i \in \mathbb{R}$, $\mathbf{a} = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T$ is a linear combination of the rows of A. Note that $\mathbf{a}$ is an $1 \times n$ matrix and $\mathbf{a}^T \in \mathbb{R}^n$.
- Note: $c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = [c_1, \ldots, c_m]A$. Thus, for any $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{c}^T A$ is a linear combination of rows of A.
- The rows of A are linearly dependent iff
  $\mathbf{c}^T A = c_1\mathbf{a}_1^T + \ldots c_m\mathbf{a}_m^T = \mathbf{0}^T$ (zero row) for some nonzero $\mathbf{c} \in \mathbb{R}^m$.
- The rows of A are linearly dependent iff $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are linearly dependent, i.e., the columns of A^T are linearly dependent.

Suppose $R = \text{rref}(A)$ has a zero row. Are the rows of $A_{m\times n}$ linearly dependent?

Yes. $R = PA$ for some invertible P. If $[p_{m1}, p_{m2}, \ldots, p_{mm}]$ is the m-th row of P, then $[p_{m1}, p_{m2}, \ldots, p_{mm}]A$ is the m-th row of R.

Example

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 4 & 2 \end{bmatrix} \xrightarrow[E_{31}(-1)]{E_{21}(-1)} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow[E_{12}(-2)]{E_{32}(2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{E_{2}(-1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

So, $PA = R$ *where* $P = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$. *Verify that* $-3\mathbf{a}_1^T + 2\mathbf{a}_2^T + \mathbf{a}_3^T = \mathbf{0}^T$.

Result

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$ *and* $A = [\mathbf{v}_1 \cdots \mathbf{v}_m]$. *Then the following are equivalent.*

1. *S is linearly dependent.*
2. *Columns of A are linearly dependent.*
3. *$A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.*
4. *Rows of A^T are linearly dependent.*
5. *$\text{rank}(A^T) < m$.*
6. *$\text{rref}(A^T)$ has a zero row.*

Result

If $m > n$ then any set of m vectors in $\mathbb{R}^n$ is linearly dependent.

Basis:

Let U be a subspace of $\mathbb{R}^n$ and $B \subseteq U$. Then B is said to be a basis for U if B is linearly independent and $\text{span}(B) = U$.

- The set $\{1\}$ is a basis for $\mathbb{R}^1$ $(= \mathbb{R})$.
- Let $\mathbf{e}_i \in \mathbb{R}^n$ be the i-th column of the identity matrix I_n. The set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$. The vectors $\mathbf{e}_i$ (for $i = 1, 2, \ldots, n$) are called the standard unit vectors.

Result

For a subspace U, a subset $B = \{v_1, \ldots, \mathbf{v}_r\} \subseteq U$ is a basis of U iff every element of U is a unique linear combination of $v_1, \ldots, \mathbf{v}_r$.

Example

Find a basis for the subspace $U = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}.$$

Result

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \in \mathbb{R}^n$ and $T \subseteq span(S)$ such that $m = |T| > r$. Then T is linearly dependent.

Proof. Let $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$. Write

$$\mathbf{u}_i = a_{i1}\mathbf{v}_1 + a_{i2}\mathbf{v}_2 + \cdots + a_{ir}\mathbf{v}_r, \; 1 \leq i \leq m.$$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mr} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$. So $u_i = \mathbf{a}_i^T \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_r \end{bmatrix}$.

Since $m > r$, the rows of A are linearly dependent. Suppose $\alpha_1 \mathbf{a}_1^T + \cdots + \alpha_m \mathbf{a}_m^T = \mathbf{0}^T$. Then

$$\sum_{i=1}^{m} \alpha_i \mathbf{u}_i = \sum_{i=1}^{m} \alpha_i \mathbf{a}_i^T \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_r \end{bmatrix} = \mathbf{0}^T \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_r \end{bmatrix} = \mathbf{0}^T.$$

Result

Let U be a subspace of $\mathbb{R}^n$. Then U has a basis and any two bases for S have the same number of elements.

Dimension: The number of elements in a basis for U (a subspace of $\mathbb{R}^n$) is called the dimension, denoted $\dim(U)$, of U.

- $\dim(\mathbb{R}^n) = n$.
- $\dim(\{\mathbf{0}\}) = 0$, since $\text{span}(\{\}) = \{\mathbf{0}\}$.
- If $\mathbf{v}_1, \dots, \mathbf{v}_m$ are linearly dependent, then $\dim(\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m)) = m$.
- A set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq \mathbb{R}^n$ is a basis of $\mathbb{R}^n$ iff S is linearly independent iff S is a spanning set of $\mathbb{R}^n$, i.e., $\text{span}(S) = \mathbb{R}^n$.

Fundamental subspaces associated to a matrix

Definition

Let A be an $m \times n$ matrix.

1. The column space / range space of A, denoted $\text{col}(A)$, is the subspace of $\mathbb{R}^m$ spanned by the columns of A.
   In other words, $\text{col}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$.
2. The row space of A, denoted $\text{row}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the rows of A.
   In other words, $\text{row}(A) = \{\mathbf{x}^T A \mid \mathbf{x} \in \mathbb{R}^m\}$

   [Here, elements of $\text{row}(A)$ are row vectors. How can they be elements of $\mathbb{R}^n$. In strict sense, $\text{row}(A) := \text{col}(A^T)$.]
3. The null space of A, denoted $\text{null}(A)$, is the subspace of $\mathbb{R}^n$ consisting of the solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. In other words, $\text{null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$
4. The null space of A^T: $\text{null}(A^T) = \{\mathbf{x} \in \mathbb{R}^m \mid A^T\mathbf{x} = \mathbf{0}\}$

Result

If two matrices A and B are row equivalent, then row(B) = row(A).

Proof. A and B are row equivalent $\Rightarrow$ $B = PA$, for some invertible P. Thus,

$$\text{row}(B) = \{\mathbf{x}^T B \mid \mathbf{x} \in \mathbb{R}^n\} = \{(\mathbf{x}^T P)A \mid \mathbf{x} \in \mathbb{R}^n\} \subseteq \text{row}(A).$$

Similarly, row$(A) \subseteq$ row(B), since $A = P^{-1}B$.

Corollary

For any A, row(A) = row(rref(A)).

Corollary

For any A, the non-zero rows of rref(A) forms a basis of row(A).

Suppose A and B are row-equivalent. Are col(A) and col(B) equal? No. Take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Suppose A and B are row-equivalent. Do col(A) and col(B) have same dimension? Yes. We will see soon.

Result

Let P be an invertible matrix. Then a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ in $\mathbb{R}^n$ is linearly independent iff the set $\{P\mathbf{v}_1, P\mathbf{v}_2, \ldots, P\mathbf{v}_m\}$ is linearly independent.

Corollary

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n]$ and $R = [\mathbf{b}_1\ \mathbf{b}_2\ \ldots\ \mathbf{b}_n]$ = rref(A). If the leading columns of R are $\mathbf{b}_{j_1}, \mathbf{b}_{j_2}, \ldots, \mathbf{b}_{j_r}$, then $\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_r}\}$ is a basis for col(A).

A way of Computing a basis of null(R):

INPUT: An $m \times n$ matrix A.
OUTPUT: A full column rank matrix X whose columns span the null space of A.

1. Compute $R = \mathrm{rref}(A)$.

2. Suppose that R has p-nonzero rows. So it has p-pivot columns. Interchange columns of R (this means choose a permutation matrix P) so that

$$RP = \begin{bmatrix} I_p & F \\ 0 & 0 \end{bmatrix} = \text{column interchanged form of } R,$$

   where I_p is the identity matrix of size p.

Computing a basis of null space (cont.)

3. Set $Y := \begin{bmatrix} -F \\ I_{n-p} \end{bmatrix}$, where I_{n-p} is the identity matrix of size $n - p$.

4. Now interchange rows of Y according to the permutation P. This means compute

$$X := PY.$$

   Then $\mathrm{rank}(X) = n - p$ and $RX = RPY = 0$. Thus columns of X span the null space of R and hence the null space of A.

Example

Find bases for the column, row and null spaces of

$$A = \begin{bmatrix} 1 & 3 & 5 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}.$$

We have $R = \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore

- $\{[1, 3, 0, -1], [0, 0, 1, 1]\}$ is a basis for the row space of A.
- $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \right\}$ is a basis for the column space of A.
- Solve $R\mathbf{x} = \mathbf{0}$ and find a basis for null(R), or use the previous algorithm.

Example (contd.)

Note that $RE_{23} = \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & F \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Put $Y = \begin{bmatrix} -F \\ I_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Note that $R(E_{23}Y) = (RE_{23})Y = \mathbf{0}$. Therefore, the columns of

$E_{23}Y = \begin{bmatrix} -3 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$ give a basis for null(A).

Example

Find bases for the row space, column space and null space of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 4 & 6 & 2 \end{bmatrix}.$$

Result

The row space and the column space of a matrix A have the same dimension, and $dim(row(A)) = dim(col(A)) = rank(A)$.

So, we have several definitions for rank(A).

Result

For any matrix A, we have $rank(A^T) = rank(A)$.

Nullity: The nullity of a matrix A is the dimension of its null space, and is denoted by nullity(A).

Result (**Rank Nullity Theorem**)

Let A be an $m \times n$ matrix. Then

$$rank(A) + nullity(A) = n.$$

Result (**The Fundamental Theorem of Invertible Matrices: Version II**)

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. *A is invertible.*
2. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
3. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
4. *The reduced row echelen form of A is I_n.*
5. *A is a product of elementary matrices.*
6. *$rank(A) = n$.*

7. $\text{nullity}(A) = 0$.
8. The column vectors of A are linearly independent.
9. The column vectors of A span $\mathbb{R}^n$.
10. The column vectors of A form a basis for $\mathbb{R}^n$.
11. The row vectors of A are linearly independent.
12. The row vectors of A span $\mathbb{R}^n$.
13. The row vectors of A form a basis for $\mathbb{R}^n$.

Example

Show that the vectors $[1, 2, 3]^T, [-1, 0, 1]^T$ *and* $[4, 9, 7]^T$ *form a basis for* $\mathbb{R}^3$.

Result

Let A be an $m \times n$ matrix. Then

1. $rank(A^T A) = rank(A)$.
2. *The $n \times n$ matrix $A^T A$ is invertible if and only if $rank(A) = n$.*

Proof.

1. $A\mathbf{x} = \mathbf{0}$ and $A^T A\mathbf{x} = \mathbf{0}$ are equivalent:

   Note that $A\mathbf{x}_0 = \mathbf{0} \Rightarrow A^T A\mathbf{x}_0 = \mathbf{0}$. On the other hand

   $A^T A y_0 = \mathbf{0} \Rightarrow \mathbf{y}_0^T A^T A\mathbf{y}_0 = \mathbf{0} \Rightarrow (A\mathbf{y}_0)^T (A\mathbf{y}_0) = \mathbf{0} \Rightarrow$

   $A\mathbf{y}_0 = \mathbf{0}$, because $\mathbf{x}^T\mathbf{x} = 0 \Rightarrow \mathbf{x} = 0$.
2. Follows from the first part.