MA101 Mathematics I

July-November, 2013

Tutorial & Additional Problem set - 1

SECTION - A (for Tutorial - 1)

1. True or false? Give justifications.

   (a) Product of two symmetric matrices is symmetric.

   (b) Product of two upper triangular matrices is upper triangular.

   (c) Every matrix in $\mathcal{M}_2(\mathbb{R})$ with nonzero entries can be expressed as a product of a lower triangular and an upper triangular matrix.

   (d) Every matrix is row equivalent to a unique matrix of row echelon form.

   (e) If $\mathbf{x}_0$ is a solution of the system $A\mathbf{x} = \mathbf{b}$, then any solution of the system is given by $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1$ for some solution $\mathbf{x}_1$ of the system $A\mathbf{x} = \mathbf{0}$.

2. The *trace* of a matrix $A = [a_{ij}] \in \mathcal{M}_n$ is the sum of its diagonal entries and is denoted by $\operatorname{tr}(A)$, i.e. $\operatorname{tr}(A) = \sum_i a_{ii}$.
   Prove the following: if $A, B \in \mathcal{M}_n$ and α is scalar, then

   (a) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$;

   (b) $\operatorname{tr}(\alpha A) = \alpha \operatorname{tr}(A)$;

   (c) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

   Also, for any matrix $C = [c_{ij}] \in \mathcal{M}_{m\times n}(\mathbb{R})$ can you tell what $\operatorname{tr}(CC^T)$ is?

3. Express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, where

   (a) $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$;

   (b) $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

4. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two distinct solutions of the system $A\mathbf{x} = \mathbf{b}$. Prove that there are infinitely many solutions to this system. Interpret your findings geometrically.

5. Decide whether the following pairs are row-equivalent:

   (a) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 4 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \end{bmatrix}$

SECTION - B: ADDITIONAL PROBLEMS

1. Consider vectors $\mathbf{u} = [a_1, b_1]^T$ and $\mathbf{v} = [a_2, b_2]^T$ in $\mathbb{R}^2$. Use parallelogram law to justify the vector addition $\mathbf{u} + \mathbf{v} = [a_1 + a_2, b_1 + b_2]^T$. Do the same for vectors $\mathbf{u} = [a_1, b_1, c_1]^T$ and $\mathbf{v} = [a_2, b_2, c_2]^T$ in $\mathbb{R}^3$.

2. Give two examples of matrices in $\mathcal{M}_3(\mathbb{C})$, one of which is and another which is not a

   (a) diagonal matrix,

   (b) symmetric matrix.

   (c) skew-symmetric matrix,

   (d) Hermitian matrix,

   (e) skew-Hermitian matrix,

   (f) upper triangular,

   (g) lower triangular.

3. True or false? Give justifications.

   (a) A real Hermitian matrix is symmetric.

   (b) The diagonal entries of a skew-Hermitian matrix are imaginary.

   (c) Any real square matrix can be expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

   (d) For any vector $\mathbf{b} \in \mathbb{R}^n$ there is a system $A\mathbf{x} = \mathbf{b}$ with two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\mathbf{x}_1 + \mathbf{x}_2$ is also a solution.

4. Find the coefficients a, b, c, d so that the graph of $y = ax^3 + bx^2 + cx + d$ passes through $(1, 2), (-1, 6)$, $(2, 3)$ and $(0, 1)$.

5. Supply two examples each and explain their geometrical meaning.
   a) Two linear equations in two variables with exactly one solution.
   b) Two linear equations in two variables with infinitely many solutions.
   c) Two linear equations in two variables with no solutions.
   d) Three linear equations in two variables with exactly one solution.
   e) Three linear equations in two variables with no solutions.

6. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}$. For which vectors $\mathbf{y} = [y_1, y_2, y_3]^T$ does the system $A\mathbf{x} = \mathbf{y}$ has a solution?