DEPARTMENT OF MATHEMATICS, IIT Guwahati
MA101: Mathematics I, July - November 2014
Summary of Lectures (Set - IV)

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1 Eigenvalue, Eigenvector and Diagonalizability

Just like the space $\mathbb{R}^n$, we also define the space $\mathbb{C}^n$. Indeed,

$$\mathbb{C}^n = \{[x_1, x_2, \ldots, x_n]^t : x_1, x_2, \ldots, x_n \in \mathbb{C}\}.$$

The definitions of vector addition and scalar multiplication *etc.*, and most of the results that we have studied so far in case of $\mathbb{R}^n$, can also be accomplished for the space $\mathbb{C}^n$, in a similar manner.

Definition 1.1. *Let A be an $n \times n$ matrix. A complex number λ is called an **eigenvalue** of A if there is a vector $\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Such a vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.*

Example 1.1. *The numbers $4, -2$ are eigenvalues of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ with corresponding eigenvectors $[1, 1]^t$ and $[1, -1]^t$, respectively.*

Definition 1.2. *Let λ be an eigenvalue of a matrix A. Then the collection of all eigenvectors of A corresponding to λ, together with the zero vector, is called the eigenspace of λ, and is denoted by E_λ.*

Result 1.1. *Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Then*

- *λ is an eigenvalue of A iff $det(A - \lambda I) = 0$.*
- *0 is an eigenvalue of A iff A is not invertible.*
- *$E_\lambda = null(A - \lambda I)$, that is, E_λ is a subspace of $\mathbb{C}^n$.*
- *Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be eigenvectors of A corresponding to λ and $\mathbf{v} = \alpha_1\mathbf{v}_1 + \ldots + \alpha_k\mathbf{v}_k \neq \mathbf{0}$. Then $\mathbf{v}$ is also an eigenvector of A corresponding to λ.*
- *Eigenvalues of a **triangular** matrix are its **diagonal** entries.*
- *Eigenvaules of $\left[\begin{array}{c|c} A_p & C \\ \hline O & B_q \end{array}\right]$ are the eigenvalues of A and B.*

Definition 1.3. *Let A be an $n \times n$ matrix. Then*

- *$P_A(x) = det(A - xI)$ is called **characteristic polynomial** of A.*
- *$P_A(x) = 0$ is called **characteristic equation** of A.*

Example 1.2. *Find the eigenvalues and the corresponding eigenspaces of the following matrices:*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Result 1.2 (The Fundamental Theorem of Invertible Matrices: Version II). *Let A be an $n \times n$ matrix. Then the following statements are equivalent.*

1. *A is invertible.*
2. *A^t is invertible.*
3. *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
4. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
6. *The reduced row echelon form of A is I_n.*

7. *The rows of A are linearly independent.*

8. *The columns of A are linearly independent.*

9. *rank*$(A) = n$.

10. *A is a product of elementary matrices.*

11. *nullity*$(A) = 0$.

12. *The column vectors of A span $\mathbb{R}^n$.*

13. *The column vectors of A form a basis for $\mathbb{R}^n$.*

14. *The row vectors of A span $\mathbb{R}^n$.*

15. *The row vectors of A form a basis for $\mathbb{R}^n$.*

16. ***det*** $A \neq 0$**.**

17. 0 ***is not an eigenvalue of*** A**.**

Result 1.3. *Let A be a matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.*

1. *For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.*

2. *If A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$.*

3. *If A is invertible then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.*

Result 1.4. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be eigenvectors of a matrix A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, respectively. Let $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$. Then for any positive integer k,*

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \ldots + c_m\lambda_m^k\mathbf{v}_m.$$

Result 1.5. *Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of a matrix A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, respectively. Then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly independent.*

Similar Matrices:

Let A and B be two $n \times n$ matrices. Then A is said to be **similar** to B if there is an $n \times n$ invertible matrix T such that $T^{-1}AT = B$.

- If A is similar to B, we write $A \approx B$.
- If $A \approx B$, we can equivalently write that $A = TBT^{-1}$ or $AT = TB$.

Example 1.3. *Let* $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ *and* $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. *Then $A \approx B$ since $AT = TB$.*

Result 1.6. *Let A, B and C be $n \times n$ matrices. Then*

1. $A \approx A$.

2. *If $A \approx B$ then $B \approx A$.*

3. *If $A \approx B$ and $B \approx C$ then $A \approx C$.*

Result 1.7. *Let A and B be two matrices such that $A \approx B$. Then*

1. *det* $A =$ *det* B.

2. *A is invertible iff B is invertible.*

3. *A and B have the same rank.*

4. *A and B have the same characteristic polynomial.*

5. *A and B have the same set of eigenvalues.*

6. λ *is an eigenvalue of* B *with corresponding eigenvector* $\mathbf{v}$ ***iff*** λ *is an eigenvalue of* A *with corresponding eigenvector* $T\mathbf{v}$.

7. *The* $dim(E_\lambda)$ *for* A *is same as* $dim(E_\lambda)$ *for* B.

Diagonalizable Matrix: A matrix A is said to be **diagonalizable** if there is a diagonal matrix D such that $A \approx D$, that is, if there is an invertible matrix T and a diagonal matrix D such that $AT = TD$.

Example 1.4. *The matrix* $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ *is diagonalizable, since if* $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ *and* $T = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ *then* $AT = TD$.

Result 1.8. *Let* A *be an* $n \times n$ *matrix. Then* A *is diagonalizable iff* A *has* n *linearly independent eigenvectors.*

- Let A be an $n \times n$ matrix. Then there exists an invertible matrix T and a diagonal matrix D satisfying $T^{-1}AT = D$ **iff** the columns of T are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the columns (eigenvectors of A) of T in the same order.

Example 1.5. *Check for the diagonalizablity of the following matrices. If they are diagonalizable, find invertible matrices* T *that diagonalizes them:*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Result 1.9. *If* A *is an* $n \times n$ *matrix with* n *distinct eigenvalues then* A *is diagonalizable.*

Result 1.10. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be distinct eigenvalues of a matrix* A. *If* $\mathcal{B}_i$ *is a basis for the eigenspace* E_{λ_i}, *then* $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_k$ *is a linearly independent set.*

Definition 1.4. *Let* λ *be an eigenvalue of a matrix* A.

- *The* ***algebraic multiplicity*** *of* λ *is the multiplicity of* λ *as a root of the characteristic polynomial of* A.
- *The* ***geometric multiplicity*** *of* λ *is the dimension of* E_λ.

Result 1.11. *The geometric multiplicity of each eigenvalue of a matrix is less than or equal to its algebraic multiplicity.*

Result 1.12 (The Diagonalization Theorem). *Let* A *be an* $n \times n$ *matrix whose distinct eigenvalues are* $\lambda_1, \lambda_2, \ldots, \lambda_k$. *Then the following statements are equivalent:*

1. A *is diagonalizable.*
2. *The union* $\mathcal{B}$ *of the bases of the eigenspaces of* A *contains* n *vectors.*
3. *The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.*