DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati
Tutorial and practice problems on Single Variable Calculus

MA-101 : Mathematics-I   Tutorial Problem Set - 8   October 09, 2013

PART-A (Tutorial)

1. Find the supremum and the infimum (if they exist) of the sets defined below.

   (i) $S_1 = \{1/n : n \in \mathbb{N}\}$; (ii) $S_2 = \{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\}$; (iii) $S_3 = \{\frac{2n^2+1}{3n+2} : n \in \mathbb{N}\}$.

2. Let S be a nonempty subset of $\mathbb{R}$ and $m, M \in \mathbb{R}$.

   (i) Show that $M = \sup S$ if and only if $x \leq M$ for all $x \in S$ and for any $\epsilon > 0$ there exists $x \in S$ such that $M - \epsilon < x \leq M$.

   (ii) Show that $m = \inf S$ if and only if $x \geq m$ for all $x \in S$ and for any $\epsilon > 0$ there exists $x \in S$ such that $m \leq x < m + \epsilon$.

   Solution: (i) If M is the supremum then it is evident that for any $\epsilon > 0$ there exists $x \in S$ such that $M \geq x > M - \epsilon$. Conversely, suppose that M is an upper bound of S and that for any $\epsilon > 0$ there exists $x \in S$ such that $x > M - \epsilon$. If possible suppose that $M > \sup S$. Then for $\epsilon := M - \sup S$, there exists $x \in S$ such that $x > M - \epsilon = \sup S$ which is a contradiction. Hence we must have $M = \sup S$.

   (ii) Proof is similar. Leave as an exercise. ∎

3. Use the definition of convergence of a sequence to examine whether the sequences (x_n) defined below are convergent or not.

   $$\text{(i) } x_n = \frac{n^2}{n^2+n}; \quad \text{(ii) } x_n = \frac{2}{\sqrt{n}} + \frac{1}{n} + 3; \quad \text{(iii) } x_n = \frac{3n+2}{n+1}; \quad \text{(iv) } x_n = \frac{5}{n^{3/2}}.$$

4. Examine whether the sequences (x_n) defined below are convergent or not. Also, find the limits when they exist.

   (i) $x_n := \sin(\frac{n\pi}{2})$; (ii) $x_n := (-1)^n$; (iii) $x_n := n^k x^n$, where $k \in \mathbb{N}$ and $|x| < 1$;

   (iv) $x_n := \frac{n}{x^n}$, where $x > 1$; (v) $x_n := n^{3/2}(\sqrt{n+1} - \sqrt{n})$.

   Solution: (i) Note that $x_{2n} = 0$ and $x_{2n-1} = \pm 1$. Hence (x_n) does not converge.

   (ii) The subsequences (x_{2n}) and (x_{2n-1}), respectively, converge to 1 and -1. Hence (x_n) does not converge.

   (iii) We have $|x_{n+1}/x_n| = |x|(1 + 1/n)^k \to |x| < 1$. Hence $x_n \to 0$.

   (iv) We have $|x_{n+1}/x_n| = (1 + 1/n)/|x| \to 1/|x| < 1$. Hence $x_n \to 0$.

   (v) It follows that $x_n \to \infty$. ∎

5. Let (x_n) be a sequence of real numbers.

   (i) If $x_n := x^{1/n}$, where $x > 0$, then show that $x_n \to 1$ as $n \to \infty$.

(ii) If $x_n := n^{1/n}$ then show that $x_n \to 1$ as $n \to \infty$.

(iii) If $x_n := x^n$, where $|x| < 1$, then show that $x_n \to 0$ as $n \to \infty$.

Solution: (i) If $x > 1$ then $x^{1/n} = 1 + d_n$ for some $d_n > 0$. Hence $x = (1 + d_n)^n > nd_n$ [binomial theorem]. Consequently $|x^{1/n} - 1| = d_n < x/n$. This shows that $x_n \to 1$ as $n \to \infty$. Indeed, for any $\epsilon > 0$, by Archimedean property, there exists $m \in \mathbb{N}$ such that $1/m < \epsilon/x$. Hence $|x_n - 1| < x/n < \epsilon$ for all $n \geq m$.

If $x < 1$ then the result follows by considering $(1/x)^{1/n}$.

The case $x = 1$ is trivial.

(ii) For $n > 1$, we have $n^{1/n} = 1 + d_n$ for some $d_n > 0$. Hence $n = (1 + d_n)^n > 1 + n(n-1)d_n^2/2$ for $n > 1$ [binomial theorem]. This shows that $d_n^2 < 2/n$ for $n > 1$. Choose $\epsilon > 0$. Then there exists [Archimedean property] $m \in \mathbb{N}$ such that $1/m < \epsilon^2/2$. Therefore $d_n^2 < 2/n < \epsilon^2$ for all $n \geq \max(2, m)$. Hence the result follows.

(iii) Note that $|x_n| = 1/(1 + \delta_x)^n$ for some $\delta_x > 0$. Hence the result follows. ■

6. Let (x_n) be a sequence of real numbers.

   (i) Suppose that $x_1 := 2$ and $x_{n+1} := 2 + 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

   (ii) Suppose that $x_1 := 1$ and $x_{n+1} := x_n/(1 + 2x_n)$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

   (iii) If $x_n \to L$ as $n \to \infty$ then show that $(x_1 + \cdots + x_n)/n \to L$ as $n \to \infty$.

Solution: (i) Note that $x_n \geq 2$ for $n \in \mathbb{N}$. Now $|x_{n+1} - x_n| = |1/x_n - 1/x_{n-1}| = \frac{|x_n - x_{n-1}|}{x_n x_{n-1}} \leq \frac{1}{4}|x_n - x_{n-1}|$ shows that (x_n) is a Cauchy sequence. Suppose that $x_n \to x$. Then we have $x = 2 + 1/x$ which gives $x = 1 \pm \sqrt{2}$. Since $x \geq 2$, we have $x = 1 + \sqrt{2}$.

(ii) It follows that $x_n \geq 0$ and $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Hence by monotone convergence theorem (x_n) converges. Let x be the limit. Then by the limit theorem $x = x/(1 + 2x)$ which gives $x = 0$.

(iii) Choose $\epsilon > 0$. Then there exists $m \in \mathbb{N}$ such that $|x_n - L| < \epsilon/2$ for $n \geq m$. Let $y_n := (x_1 + \cdots + x_n)/n$. Then $|y_n - L| \leq (|x_1 - L| + \cdots + |x_m - L|)/n + (n-m)\epsilon/2n$ for $n \geq m$. By Archimedean Property there exists $k \in \mathbb{N}$ such that $(|x_1 - L| + \cdots + |x_m - L|)/k < \epsilon/2$. Hence for $n \geq \max(k, m)$ we have $|y_n - L| < \epsilon/2 + (1 - m/n)\epsilon/2 < \epsilon$. Consequently $y_n \to L$. ■

7. Let (x_n) be a sequence of nonzero real numbers. Prove or disprove the following:

   (i) If (x_n) is not bounded, then $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = 0$.

   (ii) If (x_n) does not have any convergent subsequence, then $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = 0$.

Solution: (i) Need not be true. For example, the sequence $(x_n) = (1, 2, 1, 3, 1, 4, ...)$ is not bounded, but $\frac{1}{x_n} \not\to 0$, because $(\frac{1}{x_n})$ has a subsequence $(1, 1, ...)$ converging to 1.

(ii) True. If $\lim\limits_{n\to\infty} \dfrac{1}{x_n} \neq 0$, then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exists a positive integer $m > n$ satisfying $|\frac{1}{x_m}| \geq \varepsilon$, *i.e.* $|x_m| \leq \frac{1}{\varepsilon}$. Thus we get positive integers $n_1 < n_2 < \cdots$ such that $|x_{n_k}| \leq \frac{1}{\varepsilon}$ for each $k \in \mathbb{N}$. So (x_{n_k}) is a bounded subsequence of (x_n) and hence by Bolzano-Weierstrass theorem, (x_{n_k}) has a convergent subsequence, which is also a convergent subsequence of (x_n), which contradicts the hypothesis. ■

PART-B (Homework/Practice problems)

1. Let $a, b \in \mathbb{R}$. If $|a-b| < \dfrac{1}{n}$ for all $n \in \mathbb{N}$ then show that $a = b$.

   If $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$ then show that $a \leq b$.

   Let $a \in \mathbb{R}$. Show that for any $n \in \mathbb{N}$ there is a rational number $r_n \in \mathbb{Q}$ such that $|a - r_n| < \dfrac{1}{n}$. (This shows the denseness of $\mathbb{Q}$ in $\mathbb{R}$.)

   Solution: If $a \neq b$ then by Archimedean property there exists $n \in \mathbb{N}$ such that $n > 1/|b-a|$, that is, $|b-a| > 1/n$ which is a contradiction. Hence $a = b$.

   The proof is immediate. Indeed, if $a > b$ then there exists $n \in \mathbb{N}$ such that $n > 1/(a-b)$, that is, $a > b + 1/n$ which is a contradiction.

   Finally, for each $n \in \mathbb{N}$, by the density of rational, there is a rational number between $a - 1/n$ and $a + 1/n$, that is, there exists $r_n \in \mathbb{Q}$ such that $a - 1/n < r_n < a + 1/n$. Hence $|a - r_n| < 1/n$. ■

2. Given any $a, b \in \mathbb{R}$ with $a \neq b$, show that there exists $\delta > 0$ such that the intervals $(a-\delta,\, a+\delta)$ and $(b-\delta,\, b+\delta)$ have no point in common. (This is sometimes called the Housdorff property.)

   Solution: WLOG, suppose that $a < b$. Then by the density of rational, there exists $\delta \in \mathbb{Q}$ such that $0 < \delta < (b-a)/2$. Since $a+\delta < a+(b-a)/2 = (a+b)/2 = b-(b-a)/2 < b-\delta$, we conclude that the intervals $(a-\delta, a+\delta)$ and $(b-\delta, b+\delta)$ are disjoint. ■

3. Let $a, b \in \mathbb{R}$ with $a > 0$. Show that there exists $n \in \mathbb{N}$ such that $na > b$. (This is equivalent to the Archimedean property.)

   Solution: Archimedean property says that if $x \in \mathbb{R}$ then there is some $n \in \mathbb{N}$ such that $n > x$. So taking $x = b/a$, we have $na > b$. ■

4. Let (x_n) be a sequence in $\mathbb{R}$.

   (i) Suppose that $x_n \geq a$ for all $n \in \mathbb{N}$, where $a \in \mathbb{R}$. If $x_n \to x$ as $n \to \infty$ then show that $x \geq a$. Give an example where $x_n > a$ but $x = a$.

   (ii) Let (y_n) be a sequence satisfying $a - x_n \leq x \leq a - y_n$ for all $n \in \mathbb{N}$, where $a, x \in \mathbb{R}$. If $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$ then show that $x = a$.

   (iii) If $x_n \to x$ as $n \to \infty$ then show that $|x_n| \to |x|$ as $n \to \infty$. Is the converse true?

   (iv) Suppose that $x_n \geq 0$ for $n \in \mathbb{N}$. If $x_n \to x$ as $n \to \infty$ then show that $\sqrt{x_n} \to \sqrt{x}$ as $n \to \infty$.

   Solution: (i) If possible, suppose that $x < a$. Then taking $\epsilon := a - x$, for all large n, we have $x_n < x + \epsilon = a$ which is a contradiction. Considering $x_n = 1/n$ and $a = 0$ we have $x_n > a$ but $x = 0 = a$.

   (ii) Since $a - x \leq x_n$ and $x_n \to 0$, by (i) we have $a - x \leq 0$. Again, since $y_n \leq a - x$ and $y_n \to 0$, we have $0 \leq a - x$. Hence we conclude that $x = a$.

   (iii) Since $||x_n| - |x|| \leq |x - x_n|$, the result follows. The converse need not be true. Consider $x_n := (-1)^n$.

   (iv) If $x = 0$ then the result follows. Suppose that $x > 0$. Then $|\sqrt{x_n} - \sqrt{x}| = |x_n - x|/(\sqrt{x_n} + \sqrt{x}) \leq |x_n - x|/\sqrt{x}$. Hence the result follows. ■

5. Suppose that $|x_n - x_{n+1}| \le r^n$ for $n \in \mathbb{N}$, where $0 < r < 1$. Show that (x_n) is a Cauchy sequence. Give an example of a sequence (x_n) such that $|x_n - x_{n+1}| \to 0$ as $n \to \infty$ but (x_n) is not a Cauchy sequence.

   Solution: Note that $|x_n - x_{n+p}| \le |x_n - x_{n+1}| + \cdots + |x_{n+p-1} - x_{n+p}| \le r^n(1 + \cdots + r^{p-1})$. Since $1 + \cdots + r^{p-1} = \frac{1-r^p}{1-r} < 1/(1-r)$, we have $|x_n - x_{n+p}| < r^n/(1-r)$ for all $n, p \in \mathbb{N}$. Since $r^n \to 0$, for any $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that $r^n < (1-r)\epsilon$ for $n \ge m$. This shows that $|x_n - x_{n+p}| < r^n/(1-r) < \epsilon$ for all $n > m$ and $p \in \mathbb{N}$. Hence (x_n) is a Cauchy sequence.

   Consider $x_n := \sqrt{n}$. Then $|x_n - x_{n+1}| \to 0$ but (x_n) does not converge. ■

6. Let (x_n) be a sequence defined by $x_1 > 0$ and $x_{n+1} := (2 + x_n)^{-1}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

   Solution: Show that $|x_{n+1} - x_n| \le r|x_n - x_{n-1}|$ for some $0 < r < 1$. ■

7. Examine whether the sequences (x_n) defined below are convergent or not. Also, find the limits when they exist.

   (i) $x_n := n^2 a^n$, where $0 < a < 1$; (ii) $x_n := \dfrac{x^n}{n^2}$, where $x > 1$.

   (iii) $x_n := \dfrac{x^n}{n!}$; (iv) $x_n := \dfrac{n!}{n^n}$.

   Solution: Use the test for null sequence: Let $|\dfrac{x_{n+1}}{x_n}| \to L$ as $n \to \infty$.

   (i) If $L < 1$ then $x_n \to 0$ as $n \to \infty$.

   (ii) If $L > 1$ then $|x_n| \to \infty$ as $n \to \infty$. ■

8. Consider the sequences (x_n) and (y_n). Prove or disprove the following.

   (i) The sequence $(x_n y_n)$ converges if (x_n) converges.

   (ii) The sequence $(x_n y_n)$ converges if (x_n) converges and (y_n) is bounded.

   Solution: Easy. Left as an exercise. ■

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