Plan

- Inner Product
- Gram-Schmidt Process

Inner Product and Gram-Schmidt Process

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- The dot product or the standard inner product $\mathbf{u}.\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u}.\mathbf{v} = \overline{u_1}v_1 + \overline{u_2}v_2 + \ldots + \overline{u_n}v_n = \mathbf{u}^*\mathbf{v}.$$

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- Observe that $\|\alpha\mathbf{u}\| = |\alpha| \, \|\mathbf{u}\|$.

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3. $(c\mathbf{u}).\mathbf{v}=\overline{c}(\mathbf{u}.\mathbf{v})$;
4. $\mathbf{u}.\mathbf{u}\geq 0$ and $\mathbf{u}.\mathbf{u}=0$ if and only if $\mathbf{u}=\mathbf{0}$.

Orthogonal Set: $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ $(\subseteq \mathbb{C}^n)$ is said to be an orthogonal set if

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Orthogonal Basis: An orthogonal basis of a subspace W is a basis for W that is an orthogonal set.

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Notice that $\{[2, 1, -1]^t, [0, 1, 1]^t, [1, -1, 1]^t\}$ *is an orthogonal basis for* $\mathbb{R}^3$*. Take* $[1, 1, 1]^t \in \mathbb{R}^3$*. Find* a, b, c *such that* $[1, 1, 1]^t = a[2, 1, -1]^t + b[0, 1, 1]^t + c[1, -1, 1]^t$*.*

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Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *be an orthogonal basis for a subspace* W *and let* $\mathbf{w} \in W$. *Then the unique scalars* $c_1, c_2, \ldots, c_k$ *such that* $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$ *are given by*

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$$\mathbf{w} = (\mathbf{u}_1.\mathbf{w})\mathbf{u}_1 + (\mathbf{u}_2.\mathbf{w})\mathbf{u}_2 + \ldots + (\mathbf{u}_k.\mathbf{w})\mathbf{u}_k,$$

and this representation is unique.

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Orthogonal Complement: Let W be a subspace.

- A vector $\mathbf{v}$ is said to be orthogonal to W if $\mathbf{v}$ is orthogonal to every vector in W.

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- The orthogonal complement of W, denoted $W^\perp$, is defined as

$$W^\perp = \{\mathbf{v} \in \mathbb{C}^n : \mathbf{v}.\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

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4. *Let* $A = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k]$. *Let* T *be an invertible matrix such that* $TA = \begin{bmatrix} I_k \\ \hline \mathbf{O} \end{bmatrix}$. *Then the last* $n-k$ *columns of* T^* *will form a basis for* $W^\perp$.

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5. $W \oplus W^\perp = \mathbb{C}^n$.

Example

Consider $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}$. *A basis for* W *is* $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. *A basis for* $W^\perp$ *is* $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

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- *Notice that if* $\mathbf{v} \in \mathbb{R}^3$ *satisfies*

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  then $\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{w}_2' - \mathbf{w}_1' \in W \cap W^\perp$. *Hence* $\mathbf{w}_1 = \mathbf{w}_2$ *and* $\mathbf{w}_1' = \mathbf{w}_2'$.

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  then $\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{w}'_2 - \mathbf{w}'_1 \in W \cap W^\perp$. *Hence* $\mathbf{w}_1 = \mathbf{w}_2$ *and* $\mathbf{w}'_1 = \mathbf{w}'_2$.
- *Thus* $\mathbf{v}$ *can be written as a sum of a vector in* W *and a vector in* $W^\perp$ *uniquely.*

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- *Thus* $\mathbf{v}$ *can be written as a sum of a vector in* W *and a vector in* $W^\perp$ *uniquely.*
- *Consider* $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$*. Find (somehow)* $\mathbf{w} \in W$ *and* $\mathbf{w}' \in W^\perp$ *such that* $\mathbf{v} = \mathbf{w} + \mathbf{w}'$*. This* $\mathbf{w}$ *is called the orthogonal projection of* $\mathbf{v}$ *on* W*.*

Result

Let A be an $m \times n$ matrix. Then $(row(\overline{A}))^{\perp} = null(A)$, $row(\overline{A}) = null(A)^{\perp}$ *and* $(col(A))^{\perp} = null(A^*)$.

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Exercise

Let A and B be two $m \times n$ matrices and let the consistent linear systems $A\mathbf{x} = \mathbf{c}$ and $B\mathbf{x} = \mathbf{d}$ have the same solution set. Show that the matrix A is row equivalent to B.

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Find a basis for $W^{\perp}$, where $W = span\left(\begin{bmatrix} 1 \\ -3 \\ 5 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 4 \\ -1 \\ 5 \end{bmatrix}\right)$.

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- Let W be a subspace of $\mathbb{C}^n$. Assume $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is an orthonormal basis for $\mathbb{C}^n$ such that $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is a basis for W and $\{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$ is a basis for $W^{\perp}$.

- Let $\mathbf{v} \in \mathbb{C}^n$. Then

  $$\mathbf{v} = (\mathbf{w}_1.\mathbf{v})\mathbf{w}_1 + \ldots + (\mathbf{w}_n.\mathbf{v})\mathbf{w}_n$$

  $$= [(\mathbf{w}_1.\mathbf{v})\mathbf{w}_1 + \ldots + (\mathbf{w}_k.\mathbf{v})\mathbf{w}_k] + [(\mathbf{w}_{k+1}.\mathbf{v})\mathbf{w}_{k+1} + \ldots + (\mathbf{w}_n.\mathbf{v})\mathbf{w}_n] = \mathbf{w} + \mathbf{w}'.$$

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Orthogonal Projection: Let W be a subspace of $\mathbb{C}^n$ and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be an orthogonal basis for W. If $\mathbf{v} \in \mathbb{C}^n$, then the orthogonal projection of $\mathbf{v}$ onto W is defined as

$$\mathrm{proj}_W(\mathbf{v}) = \frac{\mathbf{u}_1.\mathbf{v}}{\mathbf{u}_1.\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2.\mathbf{v}}{\mathbf{u}_2.\mathbf{u}_2}\mathbf{u}_2 + \ldots + \frac{\mathbf{u}_k.\mathbf{v}}{\mathbf{u}_k.\mathbf{u}_k}\mathbf{u}_k.$$

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- Notice that $\mathbf{w}$ is the projection of $\mathbf{v}$ onto W.

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$$\mathrm{proj}_W(\mathbf{v}) = \frac{\mathbf{u}_1.\mathbf{v}}{\mathbf{u}_1.\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2.\mathbf{v}}{\mathbf{u}_2.\mathbf{u}_2}\mathbf{u}_2 + \ldots + \frac{\mathbf{u}_k.\mathbf{v}}{\mathbf{u}_k.\mathbf{u}_k}\mathbf{u}_k.$$

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Let W be a subspace of $\mathbb{C}^n$ and let $\mathbf{v} \in \mathbb{C}^n$. Then there are unique vectors $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$. That is, $W \oplus W^\perp = \mathbb{C}^n$.

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Let S be a subspace of $\mathbb{C}^n$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ form a basis for $S^\perp$. Consider the $k \times n$ matrix A whose i-th row is $\mathbf{v}_i^*$. Show that $S = null(A)$.

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Let A be an $m \times n$ matrix. Then $rank(A) + nullity(A) = n$.

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In particular, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is an orthogonal basis for* W*.*

Example

Apply the Gram-Schmidt process to find an orthonormal basis of the subspace spanned by $\mathbf{x}_1 = [1, -1, 1]^t$, $\mathbf{x}_2 = [0, 3, -3]^t$ *and* $\mathbf{x}_3 = [3, 2, 2]^t$.

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- $\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{v}_1.\mathbf{x}_3}{\mathbf{v}_1.\mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2.\mathbf{x}_3}{\mathbf{v}_2.\mathbf{v}_2}\right)\mathbf{v}_2$

  $= \mathbf{x}_3 - \left(\frac{3}{3}\right)\mathbf{v}_1 - \left(\frac{6}{6}\right)\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$.
- $\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$, $\frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.
- Orthonormal basis is $\left\{ \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$.

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- The vectors in S corresponding to the elements of B are linearly independent.