End Semester Examination (PH101)

Time - 180 Minutes, Marks: 50, Date: 23rd November, 2014

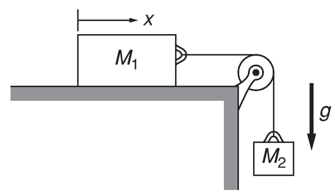
Part A: Short questions

1. [Marks: 2] Consider a non-uniform rod of length l and total mass M. The mass per unit length of is given by $\lambda = \frac{\pi M}{2l}\cos\left(\frac{\pi x}{2l}\right)$, where x is the position measured from one end of the rod. Find the $x-$coordinate of the center of mass.

Solution:

$$\begin{aligned} X_{cm} &= \frac{1}{M}\int_0^l x\lambda dx \\ &= \frac{1}{M}\frac{\pi M}{2l}\int_0^l x\cos\left(\frac{\pi x}{2l}\right)dx \\ &= \frac{\pi}{2l}\left[\frac{2l^2}{\pi} - \frac{4l^2}{\pi^2}\right] = l\left(1-\frac{2}{\pi}\right) \end{aligned}$$

2. [Marks: 2] The two blocks M_1 and M_2 shown in the sketch are connected by a string of negligible mass. If the system is released from rest, find how far block M_1 slides in time t. Neglect friction.



Solution: We have

$$M_1 a = T \tag{1}$$

$$M_2 a = M_2 g - T \tag{2}$$

From Eq. (1) and (2) we get, $a = \frac{M_2 g}{M_1+M_2}$ and $x = \frac{1}{2}at^2 = \frac{M_2 g t^2}{2(M_1+M_2)}$

3. [Marks: 2] A thin cylindrical shell with open ends of mass M and radius R rolls without slipping on a plank which is accelerated at a rate A. Find the acceleration of the cylinder.

Solution: In accelerated frame, let the acceleration of the shell be a'. The equations are

$$ma' = F - mA$$

$$I\alpha' = -FR$$

and

$$\alpha' = a'/R$$

Thus

$$a' = -\frac{1}{2}A$$

Then in the inertial frame the acceleration would be

$$a = a' + A = \frac{1}{2}A$$

4. [Marks: 2] Consider a rotating frame and an inertial frame with common origin and common z axis. The angular velocity of the rotating frame about the z axis is $\vec{\omega}$. Derive the relation between the accelerations of a particle measured in the rotating frame and in the inertial frame.

Solution: The velocity of the particle in fixed frame given by

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r} \\ \vec{v} &= \vec{v}' + (\omega \times \vec{r}) \end{aligned}$$

Then acceleration is given by

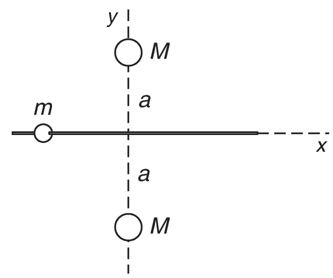
$$\begin{aligned} \frac{d\vec{v}}{dt} &= \left(\frac{d\vec{v}}{dt}\right)_{rot} + \vec{\omega} \times \vec{v} \\ \frac{d\vec{v}}{dt} &= \left(\frac{d\vec{v}'}{dt}\right)_{rot} + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\ \vec{a} &= \vec{a}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \end{aligned}$$

5. [Marks: 2] A particle of mass m moves under conservative force with potential energy $U(x) = \frac{ax}{x^2+b^2}$, where a and b are constants. Find positions of all equilibrium points.

Solution: Condition of equilibrium is given by

$$\begin{aligned} \frac{dU}{dx} &= 0 \\ \frac{d}{dx}\left(\frac{ax}{x^2+b^2}\right) &= 0 \\ \frac{a\left(b^2-x^2\right)}{x^2+b^2} &= 0 \\ x &= \pm b \end{aligned}$$

6. [Marks: 2] A bead of mass m slides without friction on a smooth rod along the $x-$axis. The rod is equidistant between two spheres of mass M. The spheres are located at $x = 0$, $y = \pm a$ as shown, and attract the bead gravitationally. Starting from the expression of potential energy obtain the force on the bead along $x-$direction.



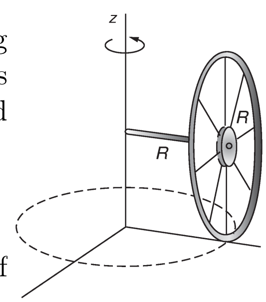
Solution: Total potential energy due to two masses M is given by

$$\begin{aligned} U(x) &= -2 \times \frac{GmM}{r}, \text{ where } r^2 = a^2 + x^2 \\ &= -2 \times \frac{GmM}{r} \end{aligned}$$

Force along $x-$direction is

$$F_x = -\frac{\partial U}{\partial x} = -2GmM\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -2GmM\left(\frac{x}{r^3}\right)$$

7. [Marks: 2] A thin hoop of mass M and radius R rolls without slipping about the $z-$axis. It is supported by an axle of length R through its center, as shown. The hoop circles around the $z-$axis with angular speed Ω.



(a) What is the instantaneous angular velocity $\vec{\omega}$ of the hoop?

(b) What is the angular momentum $\vec{L}$ of the hoop? (The moment of inertia of a hoop for an axis along its diameter is $\frac{1}{2}MR^2$.)

Solution:

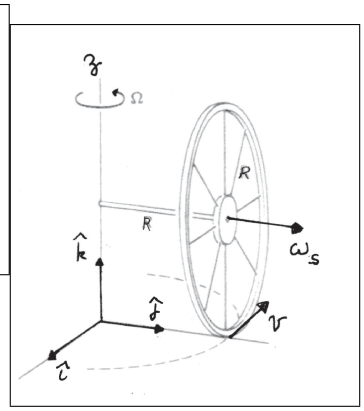
(a) $\omega_s = \frac{v}{R} = \frac{\Omega R}{R} = R$

Then, $\vec{\omega} = \vec{\omega}_s + \vec{\Omega} = \Omega\left(\hat{j} + \hat{k}\right)$

(b) $\vec{L} = I_s\vec{\omega}_s + I_z\vec{\Omega}$

We have $I_s = MR^2$ and $I_z = \frac{3}{2}MR^2$

Thus, $\vec{L} = MR^2\left(\vec{\omega_s} + \frac{3}{2}\vec{\Omega}\right) = MR^2\Omega\left(\hat{j} + \frac{3}{2}\hat{k}\right)$



8. [Marks: 2] A particle of proper mean life of 1 μsec travels at speed $0.9c$. What is the particles life measured by an observer in the ground? What is the distance the particle travels before it disintegrates?

Solution:

$$\gamma = \left(1 - 0.9^2\right)^{-1/2} = 2.294$$

$$\Delta t' = \gamma \Delta t = 2.294 \times 10^{-6} \simeq 2.3 \ \mu\text{sec}$$

Distance traveled by the particle is

$$d = 0.9c \times 2.3 \times 10^{-6} = 621 \text{ meters}$$

9. [Marks: 2] The length of a spaceship is measured to be exactly half its proper length. What is the speed of the spaceship relative to the observer's frame?

Solution: Given, $L = L_0/2$.
Using length contraction relation, we have

$$\begin{aligned} L &= L_0/\gamma \\ \gamma &= 2 \\ \sqrt{1-\left(\frac{v^2}{c^2}\right)} &= \frac{1}{2} \\ \Rightarrow v &= \frac{\sqrt{3}}{2}c \end{aligned}$$

10. [Marks: 2] A particle of rest mass m_0 travels with relativistic speed v. Find the phase velocity of the matter wave in terms of v.

Solution:
Energy

$$E = mc^2 = h\nu \Rightarrow \nu = \frac{mc^2}{h}$$

Phase velocity

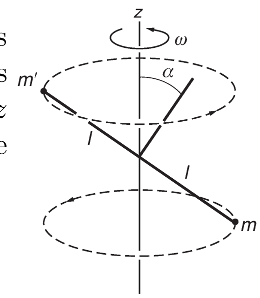
$$\begin{aligned} v_p &= \frac{w}{k} \\ &= \nu\lambda \\ &= \left(\frac{mc^2}{h}\right)\left(\frac{h}{mv}\right) \\ &= \frac{c^2}{v} \end{aligned}$$

11. [Marks: 2] The wave function of a particle is given by $\psi(x) = \sqrt{\frac{2}{l}}\ \sin\frac{\pi x}{l}$ when $0 < x < l$ and $\psi(x) = 0$ everywhere else. What is the probability of finding the particle between $l/4$ and $3l/4$?

Solution:
Probability is given by

$$\begin{aligned} P &= \frac{2}{l}\int_{l/4}^{3l/4} \sin^2\frac{\pi x}{l} dx \\ &= \frac{2}{l}\frac{1}{2}\int_{l/4}^{3l/4}\left(1-\cos\frac{2\pi x}{l}\right)dx \\ &= \frac{1}{l}\left[\frac{3l}{4}-\frac{l}{4}\right] - \frac{1}{l}\int_{l/4}^{3l/4}\cos\frac{2\pi x}{l}dx \\ &= \frac{1}{2} - \frac{1}{l}\frac{l}{2\pi}\int_{\pi/2}^{3\pi/2}\cos t\, dt \\ &= \frac{1}{2} - \frac{1}{2\pi}\left[\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right] \\ &= \frac{1}{2} + \frac{1}{\pi} \end{aligned}$$

12. [Marks: 2] Consider a simple rigid body consisting of two particles of mass m separated by a massless rod of length $2l$. The midpoint of the rod is attached to a vertical axis that rotates at angular speed ω around the z axis. The rod is skewed at angle α, as shown in the sketch. Find the torque of the system?



Solution:
$\omega_{\|}$ is along the rod and $\omega_{\perp}$ is perpendicular to the rod. For point masses, $\omega_{\|}$ does not produce angular momentum. The angular momentum is $L = I\omega_{\perp} = 2ml^2\omega\cos\alpha$ and is along the direction of $\omega_{\perp}$.
$\vec{L}$ can be decomposed along $z-$direction as L_z (= constnat) and along horizontal direction as $L_h = L\sin\alpha$. If L_h lies in the xy plane and at $t = 0$, L_h coincides with the $x-$axis, then

$$L_x = L_h \cos\omega t = L\sin\alpha\cos\omega t$$

and

$$L_y = L_h \sin\omega t = L\sin\alpha\sin\omega t$$

Thus

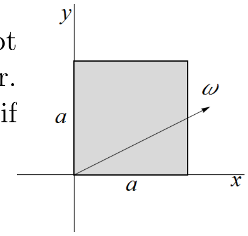
$$\vec{L} = L\sin\alpha\left(\hat{\imath}\cos\omega t + \hat{\jmath}\sin\omega t\right) + L\cos\alpha\hat{k}$$

So, the torque is given by

$$\begin{aligned}\vec{\tau} &= \frac{d\vec{L}}{dt}\\ &= L\omega\sin\alpha\left(-\hat{\imath}\sin\omega t + \hat{\jmath}\cos\omega t\right)\end{aligned}$$

Part B: Long questions

13. [Marks: 8] Consider a uniform square plate of mass M and side a. It is kept in the xy plane as shown in the figure. Find the moment of inertia tensor. Find principal axes and principal momenta. Find angular momentum $\vec{L}$ if the instantaneous angular speed is $\vec{\omega} = \left(2\hat{\imath} + \hat{\jmath}\right)\omega$.



Solution:
The density of the plate is $\sigma = M/a^2$. Then

$$I_{xx} = \frac{M}{a^2}\int_0^a dx \int_0^a y^2 dy = \frac{1}{3}Ma^2.$$

Similarly, $I_{yy} = \frac{1}{3}Ma^2$. And

$$I_{xy} = -\frac{M}{a^2}\int_0^a x dx \int_0^a y dy = -\frac{1}{4}Ma^2.$$

It is easy to see that $I_{xz} = I_{yz} = 0$, since the z coordinate of the plate is 0. And finally $I_{zz} = \frac{2}{3}Ma^2$.
Thus the MI tensor is

$$\overleftrightarrow{I} = \frac{1}{12}Ma^2 \begin{bmatrix} 4 & -3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

The principal momenta are $\frac{1}{12}Ma^2$, $\frac{7}{12}Ma^2$ and $\frac{2}{3}Ma^2$ with principal axes $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ respectively.
The angular momentum will be

$$\vec{L} = \frac{1}{12}Ma^2\omega \begin{bmatrix} 4 & -3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{12}Ma^2\omega\left(5\hat{x} - 2\hat{y}\right).$$

14. (a) [Marks: 4] A rocket is traveling at speed $0.9c$ along the x axis of frame S. It shoots a bullet whose velocity (measured in the rocket's rest frame S') v' is $0.9c$ along the y' axis of S'. What is the bullet's velocity (magnitude and direction) as measured in S?

Solution
Let $\gamma = \left(1 - 0.9^2\right)^{-1/2} = 2.29$. The velocity in S' is given to be $\left(v'_x, v'_y\right) = (0, 0.9c)$. By velocity transformation formula (relative speed of rocket is $v = 0.9c$)

$$v_x = \frac{v + u'_x}{1 + vu'_x/c^2} = v = 0.9c$$

And

$$v_y = \frac{v'_y}{\gamma\left(1 + vu'_x/c^2\right)} = \frac{v'_y}{\gamma} = 0.9c/\gamma$$

The magnitude of the velocity is

$$0.9c\left(1 + \frac{1}{\gamma^2}\right)^{1/2} = 1.09 \times 0.9c = 0.98c$$

and the velocity is

$$\vec{v} = 0.9c\left(\hat{x} + \frac{1}{\gamma}\hat{y}\right)$$

(b) [Marks: 4] An excited atom of mass m_0, initially at rest in frame S, emits a photon and recoils. The internal energy of the atom decreases by ΔE and the energy of the photon is $h\nu$. Express ν in terms of ΔE.

Solution
The rest mass of the atom after emitting the photon be m'_0. Then

$$\Delta E = \left(m_0 - m'_0\right) c^2$$

Momentum conservation:

$$p_{atom} = p_{photon} = h\nu/c$$

Energy conservation:

$$\begin{aligned} m_0c^2 &= m'c^2 + h\nu \\ \implies \left(m_0c^2 - h\nu\right)^2 &= p_{atom}^2c^2 + m_0'^2c^4 \\ \implies m_0^2c^4 - 2m_0c^2h\nu + h^2\nu^2 &= h^2\nu^2 + \left(m_0c^2 - \Delta E\right)^2 \\ \implies 2m_0c^2h\nu &= 2m_0c^2\Delta E - \Delta E^2 \\ \implies h\nu &= \Delta E\left(1 - \Delta E/2m_0c^2\right) \end{aligned}$$

15. [Marks: 4] The wave function of a particle in an infinite potential well of width l is given by $\psi(x) = A$ when $0 < x < l/2$ and $\psi(x) = 0$ everywhere else. Here, A is a constant. Normalize the wave function. Calculate the probability that the system will be found in ground state upon energy measurement.

Solution

Now

$$\int_0^l |\psi(x)|^2 \, dx = \int_0^{l/2} A^2 dx = A^2 l/2$$

For normalized wave function this must be equal to 1, hence

$$A = \sqrt{2/l}$$

Let $u_n(x)$ be n^{th} energy eigenstate. Let $\psi(x) = \sum_n c_n u_n(x)$. Then

$$\begin{aligned} c_1 &= \int_0^l u_n^*(x)\psi(x)dx \\ &= \frac{2}{l}\int_0^{l/2} \sin\left(\frac{\pi x}{l}\right) dx = \frac{2}{\pi}. \end{aligned}$$

Then by measurement postulate, the probability that the state will be found in ground state is

$$|c_1|^2 = 4/\pi^2$$

16. [Marks: 6] Consider the wave function $\psi(x) = A \exp\left(-\frac{\alpha x^2}{2}\right)$. (a) Normalize the wave function. (b) Sketch the probability as function of x and find its maximum value. (c) Obtain $< x >$, $< x^2 >$, $< p >$ and $< p^2 >$. (d) Find $\Delta x \, \Delta p$.

Solution:

(a) For normalization

$$\begin{aligned}
\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx &= 1 \\
A^2 \int_{-\infty}^{\infty} \exp\left(-\alpha x^2\right) dx &= 1 \\
A^2 \sqrt{\frac{\pi}{\alpha}} &= 1 \implies A = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}}
\end{aligned}$$

(b) Sketch

(c) Clearly

$$\langle x \rangle = \int_{-\infty}^{\infty} x \, |\psi(x)|^2 \, dx = 0$$

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} x^2 \, |\psi(x)|^2 \, dx = \frac{1}{2\alpha}$$

and

$$\langle p \rangle = 0$$

$$\begin{aligned}
\left\langle p^2 \right\rangle &= \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha x^2/2} \left(-\hbar^2 \frac{d^2}{dx^2} e^{-\alpha x^2/2}\right) dx \\
&= \hbar^2 \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \left(\alpha - \alpha^2 x^2\right) e^{-\alpha x^2} dx \\
&= \hbar^2 \frac{1}{2} \alpha
\end{aligned}$$

(d) Now $\Delta x = \sqrt{\frac{1}{2\alpha}}$ and $\Delta p = \hbar \sqrt{\frac{\alpha}{2}}$ and hence $\Delta x \Delta p = \frac{1}{2}\hbar$

Useful integral:

1. $\int_{-\infty}^{\infty} e^{-ax^2} dx = \left(\frac{\pi}{a}\right)^{1/2}$
2. $\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n a^n} \left(\frac{\pi}{a}\right)^{1/2}$
3. $\int_{-\infty}^{\infty} x^{2n+1} e^{-ax^2} dx = 0$