Problem : 1

At time $t = 0$, a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} A\dfrac{x}{a} & \text{if } 0 \le x \le a \\ \dfrac{A(b-x)}{(b-a)} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

where A, a, and b are constants.

a) Normalize $\Psi(x,0)$

b) Sketch $\Psi(x,0)$, as a function of x.

c) Where is the particle most likely to be found, at $t=0$?

d) What is the probability of finding the particle to the left of 'a'?

e) What is the expectation value of x?

Ans: a)

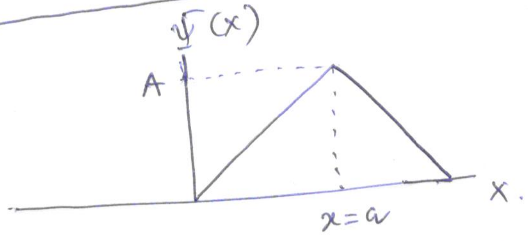
$$\int_{-\infty}^{\infty} |\Psi|^2 dx = \int_0^a \frac{A^2x^2}{a^2} dx + \int_a^b \frac{A^2(b-x)^2}{(b-a)^2} dx$$

$$= \frac{A^2x^3}{3a^2}\Bigg|_0^a + \frac{A^2}{(b-a)^2}\frac{(b-x)^3}{3}(-1)\Bigg|_a^b$$

$$= \frac{A^2a}{3} + \frac{A^2}{(b-a)^2}\frac{(b-a)^3}{3} = \frac{A^2}{3}\left[a+b-a\right] = \frac{A^2b}{3} = 1$$

Therefore : $\boxed{A = \sqrt{\frac{3}{b}}}$

b)



c) Near $x=a$, the particle is most likely to be found.

d) $$P_{(0,a)} = \int_0^a |\Psi(x)|^2 dx = \int_0^a \frac{A^2x^2}{a^2} = \frac{A^2a}{3} = \frac{3}{b}\cdot\frac{a}{3} = \frac{a}{b}$$

e) $$\langle x \rangle = \int |\psi(x)|^2 x \, dx$$

$$= \int_0^a \frac{A^2 x^2}{a^2} x \, dx + \int_a^b \frac{A^2 (b-x)^2}{(b-a)^2} x \, dx$$

$$= \frac{A^2 x^4}{4a^2} \Bigg|_0^a + \frac{A^2}{(b-a)^2} \left[\frac{x(b-x)^3(-1)}{3} - \frac{(b-x)^4}{4.3}(-1)^2 \right]_a^b$$

$$= \frac{A^2 a^2}{4} + \frac{A^2}{(b-a)^2} \left[\frac{a(b-a)^3}{3} + \frac{(b-a)^4}{4.3} \right]$$

$$= \frac{A^2 a^2}{4} + A^2 \left[\frac{a(b-a)}{3} + \frac{(b-a)^2}{4.3} \right]$$

$$= \frac{A^2 a^2}{4} + \frac{A^2}{12}(b-a)(3a+b) = \frac{A^2}{12}\left[3a^2 + (b-a)(3a+b)\right]$$

(2) A particle of mass m is in the state

$$\Psi = A e^{-a\left(\frac{mx^2}{\hbar} + it\right)}$$

a) Find A

b) For what potential energy function $V(x)$ does Ψ satisfy the Schrödinger equation ?

c) Compute Δx and Δp. Show that the product is consistent with uncertainity relation

d) Calculate the corresponding probability current.

Ans: Wave function of the particle is given as

$$\Psi = A e^{-a\left[\frac{mx^2}{\hbar} + it\right]}$$

Normalization: $$I = \int_{-\infty}^{\infty} A^2 e^{-2a\left[\frac{mx^2}{\hbar}\right]} dx = 1$$

using Gaussian integral formula:

$$I = A^2 \sqrt{\frac{\pi\hbar}{2ma}} = 1 \Rightarrow A = \left(\frac{2ma}{\pi\hbar}\right)^{\frac{1}{4}}.$$

Hence $$\Psi = \left(\frac{2ma}{\pi\hbar}\right)^{\frac{1}{4}} e^{-a\left(\frac{mx^2}{\hbar} + it\right)}.$$

Now the potential corresponding to 'Ψ' is V

Hence $$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[-\frac{2am}{\hbar} + \frac{4a^2m^2}{\hbar^2}x^2\right] + V = a\hbar$$

Hence $$\boxed{V = 2ma^2x^2}$$

Uncertainity in momentum $\Delta p = \sqrt{\langle (p - \langle p \rangle)^2 \rangle} = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

& position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

Now $\langle x \rangle = \int_{-\infty}^{\infty} A^2 e^{-\frac{2amx^2}{\hbar}} x \, dx = 0$ odd function

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} A^2 x^2 e^{-\frac{2amx^2}{\hbar}} dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} \qquad \alpha = \frac{2am}{\hbar}$$

$$= -A^2 \frac{\partial}{\partial \alpha}\left[\int_{-\infty}^{\infty} e^{-\alpha x^2} dx\right] = -A^2 \frac{\partial}{\partial \alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{A^2}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\langle x^2 \rangle = \frac{A^2}{2}\sqrt{\frac{\pi}{\alpha^3}} = \frac{1}{2}\left(\frac{2ma}{\pi\hbar}\right)^{\frac{1}{2}}\left(\frac{\pi\hbar^3}{(2ma)^3}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\sqrt{\frac{\pi}{\alpha^3}} =$$

$$\langle p \rangle = A^2 \int_{-\infty}^{\infty} e^{-\frac{amx^2}{\hbar}}\left(-i\hbar\frac{\partial}{\partial x}\right)\left(e^{-\frac{amx^2}{\hbar}}\right) dx = 0 \leftarrow \text{odd function}$$

$$\langle p^2 \rangle = A^2 \int_{-\infty}^{\infty} e^{-\frac{amx^2}{\hbar}}\left[\left(+\hbar^2 \frac{2am}{\hbar}\right) e^{-\frac{amx^2}{\hbar}} - \hbar^2 \frac{4a^2m^2x^2}{\hbar^2} e^{-\frac{amx^2}{\hbar}}\right] dx$$

$$= i\hbar A^2 \int_{-\infty}^{\infty}\left[e^{-\frac{2amx^2}{\hbar}}\left(\frac{2am}{\hbar}\right) - \frac{4a^2m^2}{\hbar^2} x^2 e^{-\frac{2amx^2}{\hbar}}\right] dx$$

$$= \hbar^2 A^2\left[\alpha \cdot \sqrt{\frac{\pi}{\alpha}} - \frac{\alpha^2}{2}\sqrt{\frac{\pi}{\alpha^3}}\right] = \hbar^2\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}}\left[\sqrt{\pi\alpha} - \frac{1}{2}\sqrt{\pi\alpha}\right]$$

$$\boxed{\langle p^2 \rangle = \frac{\hbar^2}{2}\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\sqrt{\pi\alpha}}$$

Hence $\Delta x \, \Delta p = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \sqrt{\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\sqrt{\frac{\pi}{\alpha^3}}\,\frac{\hbar^2}{2}\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\sqrt{\pi\alpha}} = \frac{\hbar}{2}$

[Minimum uncertainity state]

Probability current

Probability current is defined as.

$$J = \frac{\hbar}{2mi}\left[\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right]$$

$$= \frac{\hbar}{2mi}\left[e^{-\frac{2ma}{\hbar}x^2}\,\frac{2max}{\hbar} - e^{-\frac{2ma}{\hbar}x^2}\,\frac{2max}{\hbar}\right]$$

$$= 0$$

Since the probability density is conserved in time for the given state, probability current turned out to be zero.

The given state is a ~~stati~~ state with particular eigen energy.

Now if we ~~take~~ consider ~~two~~ gensal state of a form

$$\Psi = \bar{a}\, e^{-\alpha x^2 + iE_0 t} + \bar{b}\, x\, e^{-\alpha x^2 + iE_1 t}$$ so that $|\bar{a}|^2 + |\bar{b}|^2 = 1$

For this state

$$J = \frac{\hbar}{2mi}\left[\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x}\psi\right] \neq 0.$$

One can check $\frac{\partial P}{\partial t} = \frac{\partial |\psi|^2}{\partial t} = 2\bar{a}\bar{b}\, x\, e^{-2\alpha x^2} \sin(E_1 - E_0)t \times (E_1 - E_0)$

For discussion only.

Problem: 3

A free particle has the initial wave function

$$\Psi(x) = A e^{-a|x|}$$

where A and a are positive real constants.

a) Normalize $\Psi(x)$

b) Find the momentum wave by using the inverse Fourier transformation. $\Phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x)\, e^{-\frac{ip.x}{\hbar}}\, dp$.

c) Show that $\int_{-\infty}^{\infty} |\phi(k)|^2\, dk = 1$

Ans: a)

$$\int_{-\infty}^{\infty} A^2 e^{-2a|x|}\, dx = 2A^2 \int_0^{\infty} e^{-2ax}\, dx = \frac{2A^2 e^{-2ax}}{-2a}\Bigg|_0^{\infty}$$

$$= \frac{A^2}{a} = 1 \qquad \boxed{A = \sqrt{a}}$$

b)

$$\Phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} A e^{-a|x|}\, e^{-\frac{ip.x}{\hbar}}\, dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{0} A e^{+ax}\, e^{-\frac{ipx}{\hbar}}\, dx + \frac{1}{\sqrt{2\pi\hbar}} \int_0^{\infty} A e^{-ax - \frac{ipx}{\hbar}}\, dx$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \left[\frac{e^{(a - \frac{ip}{\hbar})x}}{(a - \frac{ip}{\hbar})}\Bigg|_{-\infty}^{0} + \frac{e^{-(a + \frac{ip}{\hbar})x}}{-(a + \frac{ip}{\hbar})}\Bigg|_0^{\infty} \right]$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \left[\frac{1}{a - \frac{ip}{\hbar}} + \frac{1}{a + \frac{ip}{\hbar}} \right] = \frac{A}{\sqrt{2\pi\hbar}} \frac{2a}{a^2 + \frac{p^2}{\hbar^2}}$$

$$\phi(p) = \frac{2a\sqrt{a}}{\sqrt{2\pi\hbar}\,(a^2 + p^2/\hbar^2)}$$

c) $I = \int_{-\infty}^{\infty} |\phi(p)|^2 dp = \frac{4a^3}{2\pi\hbar} \int_{-\infty}^{\infty} \frac{dp}{(a^2 + p^2/\hbar^2)^2}$

assume: $p = \hbar a \tan\theta$

$$I = \frac{4a^3}{2\pi\hbar} \int_{-\pi/2}^{\pi/2} \frac{\hbar a \sec^2\theta \, d\theta}{a^4 (1+\tan^2\theta)^2} = \frac{4}{2\pi} \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{4}{2\pi} \int_{-\frac{\pi}{2}}^{\pi/2} \frac{1}{2}(1+\cos 2\theta) \, d\theta = \frac{4}{2\pi} \cdot \frac{\pi}{2} = 1.$$