MA101 Mathematics I

Department of Mathematics
Indian Institute of Technology Guwahati

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Slides originally created by: Dr. Anjan Kumar Chakrabarty

Instructors: RA, BKS, SB, KK

Outline of Syllabus

- Sequence
- Series
- Continuity
- Derivative
- Integral

Books

- Calculus and Analytic Geometry - Thomas & Finney
- Introduction to Real Analysis - Bartle & Sherbert
- A Course in Calculus and Real Analysis - Ghorpade & Limaye

Tests

- Quiz - 2 (10 marks / October 30, 2013)
- End-semester Exam. (Total 50 marks)
  40 marks from Single Variable Calculus & 10 marks from Linear Algebra.

Problem solving

Three types of problems

- Examples in lectures
- Tutorial problems
- Additional practice problems

$$\lim_{n\to\infty}\left[\frac{1}{1.n}+\frac{1}{2.(n-1)}+\cdots+\frac{1}{n.1}\right]$$
$$=\lim_{n\to\infty}\frac{1}{1.n}+\lim_{n\to\infty}\frac{1}{2.(n-1)}+\cdots=0+0+\cdots=0$$

Method is wrong but answer is correct.

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right]$$
$$=\lim_{n\to\infty}\frac{1}{n+1}+\lim_{n\to\infty}\frac{1}{n+2}+\cdots=0+0+\cdots=0$$

Method is wrong and answer is also wrong.

$\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\, dx = [\frac{3}{2}x^{\frac{2}{3}}]_{-1}^{1} = 0$

Method is wrong but answer is correct.

$\int_{-1}^{1} \frac{1}{x^2}\, dx = [-\frac{1}{x}]_{-1}^{1} = -2$

Method is wrong and answer is also wrong.

Let $f(x) = \begin{cases} x^2 + x & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$

So $f''(x) = \begin{cases} 2 & \text{if } x \geq 0, \\ 2 & \text{if } x < 0. \end{cases}$

i.e. $f''(x) = 2$ for all $x \in \mathbb{R}$

Method ? Answer ? Think!

Order properties of real numbers

$$\begin{aligned}
\mathbb{N} &= \{1, 2, \ldots, n, \ldots\} = \text{ set of natural numbers} \\
\mathbb{Z} &= \{0, \pm 1, \pm 2, \ldots, \} = \text{ set of integers} \\
\mathbb{Q} &= \{p/q : p \in \mathbb{Z} \text{ and } q \in \mathbb{N}\} = \text{ set of rationals} \\
\mathbb{R} &= \text{ set of real numbers } = \text{ the real line}
\end{aligned}$$

Fact: $\mathbb{R}$ is an ordered field.

1. If $a, b \in \mathbb{R}$ then exactly one of the following is true:
   $a < b; \qquad a = b; \qquad b < a.$
2. $a < b$ and $b < c \Longrightarrow a < c$.
3. $a < b$ and $c \in \mathbb{R} \Longrightarrow a + c < b + c$.
4. $a < b$ and $c > 0 \Longrightarrow ac < bc$; $a < b$ and $c < 0 \Longrightarrow bc < ac$.

Absolute value

For $a, b \in \mathbb{R}$, define $a \le b$ by $a < b$ or $a = b$.

Then for $a, b \in \mathbb{R}$, either $a \le b$ or $b \le a$ (also written as $a \ge b$).

Absolute value: $|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$|x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x < 0. \end{cases}$$

Then the absolute value function satisfies the following:

1. $|x| \ge 0$ and $|x| = 0 \iff x = 0$.
2. $|xy| = |x|\,|y|$ for $x, y \in \mathbb{R}$.
3. $|x + y| \le |x| + |y|$ for $x, y \in \mathbb{R}$.

Bounded sets

Let $S \subset \mathbb{R}$ be finite. Then there exists $x_{\min}, x_{\max} \in S$ such that

$$x_{\min} \leq x \leq x_{\max} \text{ for all } x \in S.$$

What happens if $S \subset \mathbb{R}$ is infinite?

Examples:

1. Let $S_1 := \{1/n : n \in \mathbb{N}\}$. Then $x_{\max} = 1$ and $x_{\min} =?$.
2. Let $S_2 = \{1 - 1/n : n \in \mathbb{N}\}$. Then $x_{\min} = 0$ and $x_{\max} =?$
3. Let $S_3 = \{x \in \mathbb{R} : 0 < x < 1)$. Then $x_{\min} =?$ and $x_{\max} =?$.

Definition: Let $S(\neq \emptyset) \subset \mathbb{R}$ and $u, \ell \in \mathbb{R}$.

u is an upper bound of S in $\mathbb{R}$ if $x \leq u$ for all $x \in S$.

S is called bounded above if there is an upper bound of S in $\mathbb{R}$.

ℓ is a lower bounded of S in $\mathbb{R}$ if $\ell \leq x$ for all $x \in S$.

S is called bounded below if there a lower bound of S in $\mathbb{R}$.
S is called bounded if it is bounded above and bounded below.

Supremum and infimum

Definition: Let $S(\neq \emptyset) \subset \mathbb{R}$ and $u \in \mathbb{R}$.Then u is called the supremum (least upper bound = lub) of S in $\mathbb{R}$ if

1. u is an upper bound of S in $\mathbb{R}$, and
2. u is the least among all the upper bounds of S in $\mathbb{R}$, *i.e.* if u' is any upper bound of S in $\mathbb{R}$, then $u \leq u'$.

Notation: $\sup S$, $\mathrm{lub} S$.

Definition: Let $S(\neq \emptyset) \subset \mathbb{R}$ and $\ell \in \mathbb{R}$.Then ℓ is called the infimum (greatest lower bound = glb) of S in $\mathbb{R}$ if

1. ℓ is a lower bound of S in $\mathbb{R}$, and
2. ℓ is the greatest among all the lower bounds of S in $\mathbb{R}$, *i.e.* if ℓ' is any lower bound of S in $\mathbb{R}$, then $\ell' \leq \ell$.

Notation: $\inf S$, $\mathrm{glb} S$.

Examples: $\sup S_1 = 1 \in S_1$ and $\inf S_1 = 0 \notin S_1$.
$\inf S_2 = 0 \in S_2$ and $\sup S_2 = 1 \notin S_2$.
$\inf S_3 = 0 \notin S_3$ and $\sup S_3 = 1 \notin S_3$.

Completeness property of $\mathbb{R}$

If $S \subset \mathbb{R}$ is nonempty and bounded above then does S have a supremum?

Completeness property/lub property: Let $S \subset \mathbb{R}$ be nonempty. If S is bounded above then S has a supremum ($\sup S$ exits).

Ex. If $S \subset \mathbb{R}$ is nonempty and is bounded below then S has an infimum ($\inf S$ exits).

Archimedean property: Let $a \in \mathbb{R}$. Then there exits $n \in \mathbb{N}$ such that $n > a$.

Density of rationals: Let $a, b \in \mathbb{R}$ with $a < b$. Then exists $r \in \mathbb{Q}$ such that $a < r < b$.

Ex. Let $a, b \in \mathbb{R}$ with $a < b$. Then exists an irrational number s such that $a < s < b$.

*** end ***