DEPARTMENT OF MATHEMATICS, IIT Guwahati
MA101: Mathematics I, July - November 2014
Summary of Lectures (Set - VI)

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1 Vector Space

For $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{0} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$, we have

1. $\mathbf{u} + \mathbf{v} \in \mathbb{R}^n$;
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
6. $c.\mathbf{u} \in \mathbb{R}^n$;
7. $c.(\mathbf{u} + \mathbf{v}) = c.\mathbf{u} + c.\mathbf{v}$;
8. $(c + d).\mathbf{u} = c.\mathbf{u} + d.\mathbf{u}$;
9. $c.(d.\mathbf{u}) = (cd).\mathbf{u}$; and
10. $1.\mathbf{u} = \mathbf{u}$.

The above properties are sufficient to do vector algebra in $\mathbb{R}^n$.

- If $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{0} \in \mathbb{C}^n$ and $c, d \in \mathbb{R}$, we get all the previous ten properties.
- If $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{0} \in \mathbb{C}^n$ and $c, d \in \mathbb{C}$, we get all the previous ten properties.
- If $A, B, C, \mathbf{O} \in M_2(\mathbb{R})$ (set of all 2×2 real matrices) and $c, d \in \mathbb{R}$, we get all the previous ten properties.
- If $p(x), q(x), r(x), \mathbf{0} \in \mathbb{R}_2[x]$ (set of all polynomials of degree at most two with real coefficients) and $c, d \in \mathbb{R}$, we get all the previous ten properties.
- In our discussion, the symbol $\mathbb{F}$ will be used as a representative of $\mathbb{R}$ or $\mathbb{C}$. That is, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.
- The elements of $\mathbb{F}$ will be termed as **scalars**.

Definition: Let V be a non-empty set. For every $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{F}$, let the addition $\mathbf{u} + \mathbf{v}$ (called the vector addition) and the multiplication $c.\mathbf{u}$ (called the scalar multiplication) be defined. Then V is called a **vector space** over $\mathbb{F}$ if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c, d \in \mathbb{F}$, the following properties are satisfied:

1. $\mathbf{u} + \mathbf{v} \in V$;
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;
4. There is an element $\mathbf{0}$, called a zero, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
5. For each $\mathbf{u} \in V$, there is an element $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
6. $c.\mathbf{u} \in V$;
7. $c.(\mathbf{u} + \mathbf{v}) = c.\mathbf{u} + c.\mathbf{v}$;
8. $(c + d).\mathbf{u} = c.\mathbf{u} + d.\mathbf{u}$;
9. $c.(d.\mathbf{u}) = (cd).\mathbf{u}$; and
10. $1.\mathbf{u} = \mathbf{u}$.

Example 1.1. *For any $n \geq 1$, the set $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ with respect to usual operations of addition and scalar multiplication.*

Example 1.2. *For any $n \geq 1$, the set $\mathbb{C}^n$ is a vector space over $\mathbb{R}$ with respect to usual operations of addition and scalar multiplication.*

Example 1.3. *For any $n \geq 1$, the set $\mathbb{C}^n$ is a vector space over $\mathbb{C}$ with respect to usual operations of addition and scalar multiplication.*

Example 1.4. *The set $\mathbb{R}^n$ is **not** a vector space over $\mathbb{C}$ with respect to usual operations of addition and scalar multiplication.*

Example 1.5. *The set $\mathbb{Z}$ is **not** a vector space over $\mathbb{R}$ with respect to usual operations of addition and scalar multiplication.*

Example 1.6. *The set $M_2(\mathbb{R})$ of all 2×2 real matrices is a vector space over $\mathbb{R}$ with respect to usual operations of matrix addition and matrix scalar multiplication.*

Example 1.7. *The set $\mathbb{R}^2$ is **not** a vector space over $\mathbb{R}$ with respect to usual operations of addition and the following definition of scalar multiplication:*

$$c.[x, y]^t = [cx, 0]^t \quad \text{for } [x, y]^t \in \mathbb{R}^2, c \in \mathbb{R}.$$

Example 1.8. *Let $\mathbb{R}_2[x]$ denote the set of all polynomials of degree at most two with real coefficients. That is,*

$$\mathbb{R}_2[x] = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}.$$

For $p(x) = a_0 + b_0x + c_0x^2, q(x) = a_1 + b_1x + c_1x^2 \in \mathbb{R}_2[x]$ and $k \in \mathbb{R}$, define

$$p(x) + q(x) = (a_0 + a_1) + (b_0 + b_1)x + (c_0 + c_1)x^2 \text{ and } k.p(x) = (ka_0) + (kb_0)x + (kc_0)x^2.$$

Then $\mathbb{R}_2[x]$ is a vector space over $\mathbb{R}$.

- If V is a vector space, then the elements of V are called **vectors**.
- If there is no confusion, $c.\mathbf{u}$ is simply written as $c\mathbf{u}$.
- We write $V(\mathbb{F})$ to denote that V is a vector space over $\mathbb{F}$.
- We call V a **real vector space** or **complex vector space** according as $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Result 1.1. *Let V be a vector space over $\mathbb{F}$. Let $\mathbf{u} \in V$ and $c \in \mathbb{F}$. Then*

1. $0.\mathbf{u} = \mathbf{0}$;
2. $c.\mathbf{0} = \mathbf{0}$;
3. $(-1).\mathbf{u} = -\mathbf{u}$; *and*
4. *If $c.\mathbf{u} = \mathbf{0}$ then either $c = 0$ or $\mathbf{u} = \mathbf{0}$.*

Subspace: Let V be a vector space and $(\emptyset \neq)W \subseteq V$. Then W is called a **subspace** of V if and only if $a\mathbf{u} + b\mathbf{v} \in W$ for every $\mathbf{u}, \mathbf{v} \in W$ and for every $a, b \in \mathbb{F}$.

- If W is a subspace of a vector space $V(\mathbb{F})$, then $W(\mathbb{F})$ is also a vector space.
- If W is a subspace of a vector space $V(\mathbb{F})$ then $\mathbf{0} \in W$.
- The sets $\{\mathbf{0}\}$ and V are always subspaces of any vectors space V. These are called the **trivial** subspaces.

Example 1.9. *Let W be the set of all 2×2 real symmetric matrices. Then W is a subspace of $M_2(\mathbb{R})$.*

Example 1.10. *Let $W = \{[x, y, z]^t \in \mathbb{R}^3 : x + y - z = 0\}$. Then S is a subspace of $\mathbb{R}^3$.*

Spanning Set: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a vector space $V(\mathbb{F})$. Then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, and is denoted by $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $\text{span}(S)$. That is,

$$\text{span}(S) = \{\mathbf{v} \mid \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k \text{ for some scalars } c_1, c_2, \ldots, c_k\}.$$

Let $S \subseteq V$ (may be infinite!) The span of S is defined by

$$\text{span}(S) := \left\{ \sum_{i=1}^{m} \alpha_i \mathbf{v}_i \mid \mathbf{v}_i \in S, \alpha_i \in \mathbb{F}, m \text{ a nonnegative integer} \right\}.$$

- Convention: $\text{span}(\emptyset) = \{\mathbf{0}\}$.
- If $\text{span}(S) = V$, then S is called a **spanning set** for V.
- For example, $\mathbb{R}_2[x] = \text{span}(1, x, x^2)$.
- $\mathbb{R}[x] = \text{span}(1, x, x^2, \ldots)$ [$\mathbb{R}[x]$ = set of all polynomials in x].

Result 1.2. *Let S be a subset of a vector space $V(\mathbb{F})$. Then span(S) is a subspace of V.*

Linear Dependence: A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space $V(\mathbb{F})$ is said to be **linearly dependent** if there are scalars $c_1, c_2, \ldots, c_k$, at least one of them non-zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$.

An **infinite** set $S \subseteq V$ is linearly dependent if there is some **finite** linearly dependent subset of S.

Linear Independence: The set S of vectors in a vector space $V(\mathbb{F})$ is said to be **linearly independent** (LI) if it is **not** linearly dependent. Thus
if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ then S is **linearly independent** (LI) if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = c_2 = \ldots = c_k = 0$;
if S is **infinite** then S is **linearly independent** (LI) if every **finite subset** of S is *linearly independent.*

- Set $\{\mathbf{0}\}$ is linearly dependent as $1.\mathbf{0} = \mathbf{0}$. [A non-trivial linear combination of $\mathbf{0}$ is $\mathbf{0}$.]
- If $\mathbf{0} \in S$, then S is always linearly dependent as S contains a linearly dependent set $\{\mathbf{0}\}$.

Example 1.11. *The set $\{A, B, C\}$ is linearly dependent in $M_2(\mathbb{R})$, where*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

Example 1.12. *The set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathbb{R}_n[x]$.*

Example 1.13. *The set $\{1, x, x^2, \ldots\}$ is linearly independent in $\mathbb{R}[x]$.*

Result 1.3. *The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are linearly dependent **iff** either $\mathbf{v}_1 = \mathbf{0}$ or there is an integer r such that $\mathbf{v}_r$ can be expressed as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}$.*

Basis: A subset B of a vector space V is said to be a **basis** for V if $\text{span}(B) = V$ and if B is linearly independent.

Example 1.14. *$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{F}^n$. This basis is called the **standard basis** for $\mathbb{F}^n$.*

Example 1.15. *$\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathbb{R}_n[x]$, known as the **standard basis** for $\mathbb{R}_n[x]$.*

Example 1.16. *$\{1, x, x^2, \ldots\}$ is a basis for $\mathbb{R}[x]$, known as the **standard basis** for $\mathbb{R}[x]$.*

Example 1.17. *$\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis for $M_2(\mathbb{R})$, where*

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 1.18. *$B = \{1 + x, x + x^2, 1 + x^2\}$ is a basis for $\mathbb{R}_2[x]$.*

Let $a, b, c \in \mathbb{R}$. Then

$$\begin{aligned}
a(1+x)+b(x+x^2)+c(1+x^2)=0 &\Rightarrow \begin{bmatrix}1+x & x+x^2 & 1+x^2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=0\\
&\Rightarrow \begin{bmatrix}1 & x & x^2\end{bmatrix}\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=0\\
&\Rightarrow \begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\mathbf{0},\ \text{as } \{1,x,x^2\} \text{ is LI}\\
&\Rightarrow \begin{bmatrix}a\\b\\c\end{bmatrix}=\mathbf{0},\quad \text{as } \begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix} \text{ is invertible}\\
&\Rightarrow \{1+x, x+x^2, 1+x^2\} \quad \text{is LI.}
\end{aligned}$$

- Note the correspondence $1+x \longleftrightarrow \begin{bmatrix}1\\1\\0\end{bmatrix}$, $x+x^2 \longleftrightarrow \begin{bmatrix}0\\1\\1\end{bmatrix}$, $1+x^2 \longleftrightarrow \begin{bmatrix}1\\0\\1\end{bmatrix}$.
- $\{1+x, x+x^2, 1+x^2\}$ is LI **iff** $\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\}$ is LI.

Coordinate: Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space $V(\mathbb{F})$ and let $\mathbf{v} \in V$. Let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$. Then the scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates of v with respect to** B, and the column vector

$$[\mathbf{v}]_B = \begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix}$$

is called the **coordinate vector of v with respect to** B.
★ Coordinate of a vector is always associated with an **ordered** basis.

Example 1.19. *The coordinate vector $[p(x)]_B$ of $p(x) = 1 - 3x + 4x^2$ with respect to basis $B = \{1, x, x^2\}$ of $\mathbb{R}_2[x]$ is* $[p(x)]_B = \begin{bmatrix}1\\-3\\4\end{bmatrix}$.

Result 1.4. *Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, let $\mathbf{u}, \mathbf{v} \in V$ and let $c \in \mathbb{F}$. Then*

$$[\mathbf{u}+\mathbf{v}]_B = [\mathbf{u}]_B + [\mathbf{v}]_B \qquad \textit{and} \qquad [c\mathbf{u}]_B = c[\mathbf{u}]_B.$$

Result 1.5. *Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $V(\mathbb{F})$, and let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be vectors in V. Then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent in V if and only if $\{[\mathbf{u}_1]_B, [\mathbf{u}_2]_B, \ldots, [\mathbf{u}_k]_B\}$ is linearly independent in $\mathbb{F}^n$.*

Result 1.6. *Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V.*

1. *Any set of more than n vectors in V must be linearly dependent.*
2. *Any set of fewer than n vectors in V cannot span V.*

Result 1.7 (The Basis Theorem). *If a vector space V has a basis with n vectors, then every basis for V has exactly n vectors.*

Dimension: Let V be a vector space.

- The **dimension** of V, denoted dim V, is the number of vectors in a basis for V. We write dim $V = \infty$ if V does not have a finite basis.
- The dimension of the zero space $\{\mathbf{0}\}$ is defined to be 0.

Example 1.20. *dim*$(\mathbb{R}^n) = n$, *dim* $\mathbb{C}(\mathbb{C}) = 1$, *dim* $\mathbb{C}(\mathbb{R}) = 2$, *dim* $M_2(\mathbb{R}) = 4$ *and dim* $\mathbb{R}_n[x] = n + 1$.

Result 1.8. *Let V be a vector space with dim $V = n$.*

1. *Any linearly independent set in V contains at most n vectors.*

2. *Any spanning set for V contains at least n vectors.*

3. *Any linearly independent set of exactly n vectors in V is a basis for V.*

4. *Any spanning set for V of exactly n vectors is a basis for V.*

5. *Any linearly independent set in V can be extended to a basis for V.*

6. *Any spanning set for V can be reduced to a basis for V.*

Change of Basis: Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $C = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be bases for a vector space V. The $n \times n$ matrix whose columns are the coordinate vectors $[\mathbf{u}_1]_C, [\mathbf{u}_2]_C, \ldots, [\mathbf{u}_n]_C$ is denoted by $P_{C \leftarrow B}$, and is called the **change of basis matrix** from B to C. That is,

$$P_{C \leftarrow B} = [[\mathbf{u}_1]_C, [\mathbf{u}_2]_C, \ldots, [\mathbf{u}_n]_C].$$

Result 1.9. *Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $C = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be bases for a vector space V and let $P_{C \leftarrow B}$ be the change of basis matrix from B to C. Then*

1. $P_{C \leftarrow B}[\mathbf{x}]_B = [\mathbf{x}]_C$ *for all* $\mathbf{x} \in V$*;*

2. $P_{C \leftarrow B}$ *is the unique matrix P with the property that $P[\mathbf{x}]_B = [\mathbf{x}]_C$ for all $\mathbf{x} \in V$;*

3. $P_{C \leftarrow B}$ *is invertible and* $(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C}$.

Example 1.21. *Find the change of basis matrix $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ for the bases $B = \{1, x, x^2\}$ and $C = \{1 + x, x + x^2, 1 + x^2\}$ of $\mathbb{R}_2[x]$. Then find the coordinate vector of $p(x) = 1 + 2x - x^2$ with respect to C.*

2 Linear Transformation

- Suppose $A \in \mathcal{M}_{m \times n}$. Take $\mathbf{v} \in \mathbb{R}^n$. Then $A\mathbf{v} \in \mathbb{R}^m$. Thus, we have a map (function) $F : \mathbb{R}^n \to \mathbb{R}^m$ given by $F(\mathbf{v}) = A\mathbf{v}$.
- Take $F : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $F(p(x)) = p'(x)$.
- Take $F : \mathbb{R}[x] \to \mathbb{R}$ given by $F(p(x)) = p(3)$.

What is common in all of these? Well, they are maps (functions) with domains and codomains as vector space's. We have

$$F(\mathbf{u} + \mathbf{v}) = F(\mathbf{u}) + F(v), F(\alpha\mathbf{v}) = \alpha F(\mathbf{v}),$$

or, equivalently, $F(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha F(\mathbf{u}) + \beta F(\mathbf{v})$. Such functions are called linear transformations (LT).

Definition 2.1. *A* ***linear transformation*** *from a vector space V into a vector space W is a mapping $T : V \to W$ such that for all $\mathbf{u}, \mathbf{v} \in V$ and for all $a \in \mathbb{F}$*

$$T(a\mathbf{u} + \mathbf{v}) = aT(\mathbf{u}) + T(\mathbf{v}).$$

Example 2.1. *Let A be an $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then T is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$.*

Example 2.2. *The map $T : \mathbb{R} \to \mathbb{R}$, defined by $T(x) = x + 1$ for all $x \in \mathbb{R}$, is* ***not*** *a linear transformation.*

Example 2.3. *The map $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T([x, y]^t) = [2x, x + y]^t$ for all $[x, y]^t \in \mathbb{R}^2$, is a linear transformation.*

Example 2.4. *Let V and W be two vector spaces. The map $T_0 : V \to W$, defined by $T_0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$, is a linear transformation. The map T_0 is called the* ***zero transformation***.

Example 2.5. *Let V be a vector space. The map $I : V \to V$, defined by $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$, is a linear transformation. The map I is called the* ***identity transformation***.

Result 2.1. *Let $T : V \to W$ be a linear transformation. Then*

1. $T(\mathbf{0}) = \mathbf{0}$*;*

2. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v} \in V$; and

3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$.

Example 2.6. *Suppose $T : \mathbb{R}^2 \to \mathbb{R}_2[x]$ is a linear transformation such that $T[1,0]^t = 2 - 3x + x^2$ and $T[0,1]^t = 1 - x^2$. Find $T[2,3]^t$ and $T[a,b]^t$.*

Composition of Linear Transformation: Let $T : U \to V$ and $S : V \to W$ be two linear transformations. Then the composition of S with T is the mapping $S \circ T : U \to W$ defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u})) \quad \text{for all } \mathbf{u} \in U.$$

Result 2.2. *Let $T : U \to V$ and $S : V \to W$ be two linear transformations. Then the composition $S \circ T$ is also a linear transformation.*

Inverse of a Function: A function $f : X \to Y$ is said to be **invertible** if there is another function $g : Y \to X$ such that

$$g \circ f = I_X \quad \text{and} \quad f \circ g = I_Y.$$

- If f is invertible, the the function g satisfying $g \circ f = I_X, \quad f \circ g = I_Y$ is called inverse of f.
- Inverse of a function, if exists, is **unique**.
- The symbol f^{-1} is used to denote the inverse of f.
- Inverse of a linear transformation is linear.

Example 2.7. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$T[x,y]^t = [x - y, -3x + 4y]^t \text{ and } S[x,y]^t = [4x + y, 3x + y]^t \text{ for all } [x,y]^t \in \mathbb{R}^2.$$

Then S is the inverse of T.

Kernel and Range: Let $T : V \to W$ be a linear transformation. Then the **kernel** of T (null space of T), denoted $\ker(T)$, and the range of T, denoted range(T), are defined as

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}, \text{ and}$$

$$\text{range}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}.$$

Result 2.3. *Let $T : V \to W$ be a linear transformation and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a spanning set for V. Then $T(B) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ spans the range of T.*

Example 2.8. *Let A be an $m \times n$ matrix. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then ker$(T)$ = null(A) and range(T) = col(A).*

Result 2.4. *Let $T : V \to W$ be a linear transformation. Then ker(T) is a subspace of V and range(T) is a subspace of W.*

Definition 2.2. *Let $T : V \to W$ be a linear transformation. Then we define*

- ***rank**(T) = dimension of range(T); and*
- ***nullity**(T) = dimension of ker(T).*

Example 2.9. *Let $D : \mathbb{R}_3[x] \to \mathbb{R}_2[x]$ be defined by $D(p(x)) = \frac{d}{dx}p(x)$. Then rank$(D) = 3$ and nullity$(D) = 1$.*

Result 2.5 (The Rank-Nullity Theorem). *Let $T : V \to W$ be a linear transformation from a finite dimensional vector space V into a vector space W. Then*

$$\textit{rank}(T) + \textit{nullity}(T) = \textit{dim}(V).$$

Definition 2.3. *Let $T : V \to W$ be a linear transformation. Then*

*1. T is called **one-one** if T maps distinct vectors in V into distinct vectors in W.*

*2. T is called **onto** if range$(T) = W$.*

- For all $\mathbf{u}, \mathbf{v} \in V$, if $\mathbf{u} \neq \mathbf{v}$ implies that $T(\mathbf{u}) \neq T(\mathbf{v})$, then T is one-one.
- For all $\mathbf{u}, \mathbf{v} \in V$, if $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$, then T is one-one.
- For all $\mathbf{w} \in W$, if there is at least one $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$, then T is onto.

Example 2.10. *Some examples of one-one and onto linear transformation.*

- $T : \mathbb{R} \to \mathbb{R}^2$ *defined by* $T(x) = [x, 0]^t, x \in \mathbb{R}$ *is one-one but not onto.*
- $T : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $T[x, y]^t = x$*, for* $[x, y]^t \in \mathbb{R}^2$ *is onto but not one-one.*
- $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $T[x, y]^t = [-x, -y]^t$*, for* $[x, y]^t \in \mathbb{R}^2$ *is one-one and onto.*

Result 2.6. *A linear transformation* $T : V \to W$ *is one-one iff* $ker(T) = \{\mathbf{0}\}$.

Result 2.7. *Let* $dim(V) = dim(W)$*. Then a linear transformation* $T : V \to W$ *is one-one iff* T *is onto.*

Result 2.8. *Let* $T : V \to W$ *be a one-one linear transformation. If* $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is a linearly independent set in* V *then* $T(S) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ *is a linearly independent set in* W.

Result 2.9. *Let* $dim(V) = dim(W)$*. Then a one-one linear transformation* $T : V \to W$ *maps a basis for* V *onto a basis for* W.

Isomorphism:

- A linear transformation $T : V \to W$ is called an **isomorphism** if it is one-one and onto.
- If $T : V \to W$ is an isomorphism then we say that V and W are isomorphic, and we write $V \cong W$.

Example 2.11. *The vector spaces* $\mathbb{R}^3$ *and* $\mathbb{R}_2[x]$ *are isomorphic.*

Result 2.10. *Let* $V(\mathbb{F})$ *and* $W(\mathbb{F})$ *be two finite dimensional vector spaces. Then* $V \cong W$ *iff* $dim(V) = dim(W)$.

Example 2.12. *The vector spaces* $\mathbb{R}^n$ *and* $\mathbb{R}_n[x]$ *are not isomorphic.*

The Matrix of a Linear Transformation:

Result 2.11. *Let* V *and* W *be two finite-dimensional vector spaces with bases* B *and* C *respectively, where* $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *and dim* $W = m$*. If* $T : V \to W$ *is a linear transformation, then the* $m \times n$ *matrix* A *defined by*

$$A = [[T(\mathbf{v}_1)]_C, [T(\mathbf{v}_2)]_C, \ldots, [T(\mathbf{v}_n)]_C]$$

satisfies

$$A[\mathbf{v}]_B = [T(\mathbf{v})]_C \qquad \text{for all } \mathbf{v} \in V.$$

- The above matrix A is called the **matrix of T with respect to the bases B and C**.
- The matrix A is also written as $[T]_{C \leftarrow B}$.
- If $B = C$, then $[T]_{C \leftarrow B}$ is written as $[T]_B$.

Remark 2.1. *The above result means:*
Suppose we know $[T]_{C \leftarrow B}$ *with respect to given bases* B *and* C*. Then we know* T *in the following sense:*

If $\mathbf{v} = \sum\limits_{i=1}^{n} \mathbf{a}_i \mathbf{v}_i$ *and* $[T]_{C \leftarrow B} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$*, then* $T(\mathbf{v}) = \sum\limits_{j=1}^{m} b_j \mathbf{u}_j$.

Example 2.13. *Let* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *be the linear transformation defined by*

$$T([x, y, z]^t) = [x - 2y, x + y - 3z]^t \quad \text{for } [x, y, z]^t \in \mathbb{R}^3.$$

Let $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *and* $C = \{\mathbf{e}_2, \mathbf{e}_1\}$ *be bases for* $\mathbb{R}^3$ *and* $\mathbb{R}^2$*, respectively. Find* $[T]_{C \leftarrow B}$ *and verify the previous result for* $\mathbf{v} = [1, 3, -2]^t$.

Result 2.12. *Let U, V and W be three finite-dimensional vector spaces with bases B, C and D, respectively. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then*

$$[S \circ T]_{D \leftarrow B} = [S]_{D \leftarrow C}[T]_{C \leftarrow B}.$$

Result 2.13. *Let $T : V \to W$ be a linear transformation between two n-dimensional vector spaces V and W with bases B and C, respectively. Then T is invertible if and only if the matrix $[T]_{C \leftarrow B}$ is invertible. In this case,*

$$([T]_{C \leftarrow B})^{-1} = [T^{-1}]_{B \leftarrow C}.$$

Example 2.14. *Let $T : \mathbb{R}^2 \to \mathbb{R}_1[x]$ be defined by $T([a, b]^t) = a + (a + b)x$ for $[a, b]^t \in \mathbb{R}^2$. Show that T is invertible, and hence find T^{-1}.*