Plan

- Eigenvalues and Eigenvectors
- Similar Matrix
- Diagonalizable Matrix

Eigenvalues and Eigenvectors

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The definitions of vector addition and scalar multiplication *etc.*, and most of the results that we have studied so far in case of $\mathbb{R}^n$, can also be accomplished for the space $\mathbb{C}^n$, in a similar manner.

Definition

Let A be an $n \times n$ matrix.

- A complex number λ is called an eigenvalue of A if there is $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

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Example

For $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$*, eigenvalues:* $4, -2$*; resp. eigenvectors* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

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- *λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$.*

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- λ *is an eigenvalue of A iff* $\det(A - \lambda I) = 0$.
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★ The eigenvalues can be directly computed for 2×2 matrices.

★ Use Gauss Jordan Elimination on $[A - \lambda I | \mathbf{0}]$ to find all possible eigenvectors for λ of A.

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- $E_\lambda = null(A - \lambda I)$, *that is,* E_λ *is a subspace of* $\mathbb{C}^n$.
- *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *be eigenvectors of* A *corresponding to* λ *and* $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_k \mathbf{v}_k \neq \mathbf{0}$. *Then* $\mathbf{v}$ *is an eigenvector of* A.

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- *Eigenvalues of a triangular matrix are its diagonal entries.*
- *Eigenvaules of* $\left[\begin{array}{c|c} A_p & C \\ \hline O & B_q \end{array}\right]$ *are the eigenvalues of A and B.*

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$$Take\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \quad and \ \ B = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

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- $P_A(x) = -(x-1)^2(x-2)$; $E_1 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}), E_2 = \text{span}(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix})$.

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- $P_B(x) = -x^2(x+2)$; $E_0 = \text{span}(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}), E_{-2} = \text{span}(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix})$.

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Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. *A is invertible.*
2. *A^t is invertible.*
3. *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
4. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
6. *The reduced row echelon form of A is I_n.*
7. *The rows of A are linearly independent.*
8. *The columns of A are linearly independent.*

9. $\text{rank}(A) = n$.
10. A is a product of elementary matrices.
11. $\text{nullity}(A) = 0$.
12. The column vectors of A span $\mathbb{R}^n$.
13. The column vectors of A form a basis for $\mathbb{R}^n$.
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16. $\det A \neq 0$.
17. 0 is not an eigenvalue of A.

Result

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3. *If A is invertible then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.*

Result

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ *be eigenvectors of a matrix A with corresponding eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_m$*, respectively. Let* $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$*. Then for any positive integer* k*,*

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \ldots + c_m\lambda_m^k\mathbf{v}_m.$$

Result

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be eigenvectors of a matrix A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, respectively. Let $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$. Then for any positive integer k,

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Result

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of a matrix A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, respectively. Then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly independent.

Similar Matrix

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- If A is similar to B, we write $A \approx B$.
- If $A \approx B$, we can equivalently write that $A = TBT^{-1}$ or $AT = TB$.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ *and* $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Then $A \approx B$ *since* $AT = TB$.

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Let A, B *and* C *be* $n \times n$ *matrices. Then*

1. $A \approx A$.

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★ *λ is an eigenvalue of B with an eigenvector $\mathbf{v}$ iff λ is an eigenvalue of A with an eigenvector $T\mathbf{v}$.*

6. *The $\dim(E_\lambda)$ for A is same as $\dim(E_\lambda)$ for B.*

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- Then the columns of T are the linearly independent eigenvectors of A.
- The diagonal entries of D are the eigenvalues of A corresponding to the columns (eigenvectors of A) of T in the same order.

Example

Check for the diagonalizablity. If diagonalizable, find a T that diagonalizes it. [Use GJE]

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

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- For A, $E_1 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}), E_2 = \text{span}(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix})$.
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- A is not diagonalizable. B is diagonalizable, $T = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

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Let λ be an eigenvalue of a matrix A.

- The algebraic multiplicity of λ is the multiplicity of λ as a root of the characteristic polynomial of A.
- The geometric multiplicity of λ is the dimension of E_λ.

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Result (**The Diagonalization Theorem**)

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the following statements are equivalent:

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Result (The Diagonalization Theorem)

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the following statements are equivalent:

1. *A is diagonalizable.*
2. *The union $\mathcal{B}$ of the bases of the eigenspaces of A contains n vectors.*
3. *The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.*