Mathematics I (MA101)

MID SEM

IIT GUWAHATI

26^{th} Sep 2013, Thu.

Questions: 10, Marks: 30, Pages: 9

1400–1600 (02 hrs.)

To be filled by the student

Roll No.:

Tut. Group:

Name:

Sign:

Instructions

1. Write all required information in the appropriate places above.
2. Verify the number of printed pages.
3. Write your answers in this booklet and only in the space marked for answers.
4. Additional space for writing/continuing your answers has been provided towards the end of the booklet. You must clearly indicate the question number(s) while using the additional space.
5. Supplementary sheets will be provided for **rough work only**. **These sheets will NOT be evaluated.**
6. No clarifications about the questions will be provided during examination.

For office use

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	5	3	2	2	2	3	3	4	2	4	30
Score:											

1. Fill in the blanks.

(a) If $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & a & b & 2 \\ 0 & 0 & c & 0 \end{bmatrix}$ is in RREF, then (2 pts.)

$a = \boxed{0}$ $b = \boxed{1}$ $c = \boxed{0}$

(b) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 2 & \lambda & 6 \end{bmatrix}$, then the system $A\mathbf{x} = \mathbf{0}$ will have infinitely many solutions, if (1 pt.)

$\lambda = \boxed{4}$

(c) If $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 & 3 \\ x & -1 & -1 \\ y & z & -1 \end{bmatrix}$, then (2 pts.)

$x = \boxed{-1}$ $y = \boxed{-3}$ $z = \boxed{0}$

(Partial mark for (a) and (c): 1 mark for only two correct answers.)

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2. Let A be an $m \times n$ matrix with linearly independent columns.

(a) Prove that $A^T A$ is invertible. (2 pts.)

(b) Must AA^T also be invertible? Explain. (1 pt.)

Solution:

(a) Step 1. $x \in \text{null}(A^T A) \Rightarrow A^T Ax = \mathbf{0} \Rightarrow x^T A^T Ax = \mathbf{0}$
$\Rightarrow (Ax)^T Ax = \mathbf{0} \Rightarrow Ax = \mathbf{0}$. (1 mark)

Step 2. $\Rightarrow x = \mathbf{0}$, since columns of A are LI. (1/2 mark)

Step 3. Hence $A^T A$, being a square matrix, is invertible. (1/2 mark)

Aliter to Step 1:

(1) A linear combination $A^T Ay$ of columns of $A^T A$ is zero implies Ay is zero.

(2) The systems $A^T Ax = \mathbf{0}$ and $Ax = \mathbf{0}$ are equivalent.

(3) $\text{rank}(A^T A) = \text{rank}(A)$.

Marking: For any of the above: with justification 1 mark, without 1/2 mark.

(b) No.

If $m > n$, then $rank(AA^T) \leq n < m$. (1 mark)

Hence AA^T being an $m \times m$ matrix is not invertible.

Aliter: Take for example $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

3. Consider the following subspaces of $\mathbb{R}^3$:

$$\mathbb{U} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = 2x_3 - x_2,\ 2x_2 = x_3 \right\} \text{ and } \mathbb{W} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = 2x_3 - x_2 \right\}.$$

(a) Find a vector $\mathbf{u}$ such that $\operatorname{span}\{\mathbf{u}\} = \mathbb{U}$. (1 pt.)

(b) Find a vector $\mathbf{v}$ such that $\operatorname{span}\{\mathbf{u}, \mathbf{v}\} = \mathbb{W}$. (1 pt.)

Solution:

(a) Solving the system $x_1 = 2x_3 - x_2,\ 2x_2 = x_3$, we get $\mathbb{U} = \left\{ t \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$.

So, $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ (or any scalar multiple of this). (1 mark)

(For giving a correct $\mathbf{u}$ without any justification - 1/2 mark)

(b) $\mathbb{W} = \left\{ t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$. (1/2 mark)

You can take $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ or any nonzero linear combination of these two vectors which is not a scalar multiple of $\mathbf{u}$. (1/2 mark)

(For giving a correct $\mathbf{v}$ without any justification - 1/2 mark)

4. Prove or disprove: There exists a linear transformation T from $\mathbb{R}^4 \to \mathbb{R}^3$ such that (2 pts.)

$$\text{range}(T) = \{[x, y, z]^T : x + y + z = 0\} \text{ and } \ker(T) = \{[x, y, z, w]^T : x + y - 2z = 0\}.$$

Solution:

Step 1. rank$(T) = 2$ (with or without justification). (1/2 mark)

Step 2. nullity$(T) = 3$ (with or without justification). (1/2 mark)

Step 3. By rank-nullity theorem,

$$4 = \text{rank}(T) + \text{nullity}(T) = 5, \quad \text{(1 mark)}$$

which is not possible.

(No mark for just stating that the statement is FALSE. Wrong justification in any of the steps invites deduction of mark.)

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5. Prove or disprove: If A and B are non-square matrices such that both AB and BA are defined, then either AB or BA has zero as an eigenvalue. (2 pts.)

Solution: Suppose A is an $m \times n$ matrix, where $m \neq n$. Then B is an $n \times m$ matrix. (1/2 mark)

Suppose $m > n$, then rank$(AB) \leq \min\{m, n\} = n < m$. (1 mark)

Hence, 0 is an eigenvalue of AB (because AB is $m \times m$ with rank $< m$.) (1/2 mark)

Similarly, if $n > m$, then 0 is an eigenvalue of BA.

6. Let $\mathbb{R}_4[x]$ be the vector space of all real polynomials of degree less than or equal to 4.

   (a) Show that $\mathbb{W} = \{p(x) \in \mathbb{R}_4[x] \ : \ p''(0) = 2p(0)\}$ is a subspace of $\mathbb{R}_4[x]$. (1 pt.)

   (b) Find the dimension of $\mathbb{W}$. (2 pts.)

Solution:

(a) $\mathbb{W} \neq \emptyset$, because the zero polynomial is in $\mathbb{W}$.

For $p_1, p_2 \in \mathbb{W}$, $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}(\alpha p_1 + \beta p_2)''(0) &= \alpha p_1''(0) + \beta p_2''(0) \\ &= 2(\alpha p_1(0) + \beta p_2(0)) \\ &= 2(\alpha p_1 + \beta p_2)(0).\end{aligned}$$

(1 mark)

(Alternatively, you may show $p_1 + p_2 \in \mathbb{W}$ and $\alpha p \in \mathbb{W}$ for any $p_1, p_2 \in \mathbb{W}$ and $\alpha \in \mathbb{R}$. Then, 1/2 mark for each of these two steps is awarded.)

Thus, $\alpha p_1 + \beta p_2 \in \mathbb{W}$ and so $\mathbb{W}$ is a subspace.

Aliter:

If you show $\mathbb{W} = \{p(x) = a_0(1 + x^2) + a_1 x + a_3 x^3 + a_4 x^4 : a_0, a_1, a_3, a_4 \in \mathbb{R}\}$, as in part (b) below, and then justify that $\mathbb{W}$ is a subspace using this.

(b) If $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$, then $p''(0) = 2p(0) \Rightarrow a_0 = a_2$. (1 mark)
Then $\mathbb{W} = \{p(x) = a_0(1 + x^2) + a_1 x + a_3 x^3 + a_4 x^4 : a_0, a_1, a_3, a_4 \in \mathbb{R}\}$
Therefore $\dim(\mathbb{W}) = 4$. (1 mark)

Aliter:

(a) $T : \mathbb{R}_4[x] \to \mathbb{R}$ defined as $T(p) = p''(0) - 2p(0)$ is a LT. (1/2 mark)
$\mathbb{W} = \ker(T)$, and therefore is a subspace of $\mathbb{R}_4[x]$. (1/2 mark)

(b) $\text{rank}(T) = 1$, since $\text{range}(T) = \mathbb{R}$. (1 mark)

Since $\dim(\mathbb{R}_4[x]) = 5$, by rank-nullity theorem,

$\dim(\mathbb{W}) = \text{nullity}(T) = 4$. (1 mark)

7. Let $\mathcal{M}_2(\mathbb{R})$ denote the vector space of all 2×2 real matrices. Let $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ be defined by $T(A) = A + A^T$.

   (a) Show that T is a linear transformation. (1 pt.)

   (b) Find a basis for range(T) and a basis for $\ker(T)$. (2 pts.)

Solution:

(a) For $A, B \in \mathcal{M}_2(\mathbb{R})$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} T(A+B) &= (A+B)+(A+B)^T = (A+A^T)+(B+B^T) \\ &= T(A)+T(B). \end{aligned}$$ (1/2 mark)

$T(\alpha A) = \alpha A + (\alpha A)^T = \alpha(A + A^T) = \alpha T(A)$. (1/2 mark)

(Alternatively, showing $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$. (1 mark))
Hence T is a LT.

(b) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $T(A) = A + A^T = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$. Therefore,

$$\text{range}(T) = \left\{ \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Thus, $A \in \text{range}(T) \Leftrightarrow A$ is symmetric. (1/2 mark)

A basis for range(T) $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. (1/2 mark)

(For writing the basis without any justification also gets full 1 mark.)

$\dim(\ker(T)) = 1$ (by rank nullity theorem). (1/2 mark)

A basis for $\ker(T)$ $= \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$. (1/2 mark)

(For writing the basis without any justification also gets full 1 mark.)

8. Consider the 10×10 matrix $A = \mathbf{x}\mathbf{y}^T$, where $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 9 \\ 10 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -1 \end{bmatrix}$.

   (a) Show that A is diagonalizable. (2 pts.)

   (b) Find two eigenvectors corresponding to two distinct eigenvalues of A. (2 pts.)

Solution:

(a) rank$(A) = 1$, and hence nullity$(A) = 9$. (1/2 mark)

So 0 is an eigenvalue of A with geometric multiplicity 9 (i.e., A has 9 LI eigenvectors corresponding to the eigenvalue 0). (1/2 mark)

$\mathbf{y}^T\mathbf{x} \neq 0$ is another eigenvalue of A. (1/2 mark)

Thus, for each eigenvalue of A, geometric multiplicity equals its algebraic multiplicity. (1/2 mark)
(i.e., A has ten LI eigenvectors.)

Hence, A is diagonalizable.

(Alternatively, one can actually find a matrix P (e.g., finding ten LI eigenvectors) and a diagonal matrix D such that $AP = PD$. One can take P as

$$P = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & 0 & \dots & 0 & 2 \\ 0 & -1 & \dots & 0 & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 10 \end{bmatrix}.$$

(Appropriate marks are awarded.)

(b) Note that $\mathbf{y}^T\mathbf{x}$ is a scalar, and therefore, $(\mathbf{x}\mathbf{y}^T)\mathbf{x} = \mathbf{x}(\mathbf{y}^T\mathbf{x}) = (\mathbf{y}^T\mathbf{x})\mathbf{x}$, i.e., $\mathbf{x}$ is an eigenvector of A corresponding to eigenvalue $\mathbf{y}^T\mathbf{x} \neq 0$. (1 mark)

For an eigenvector corresponding to 0, any $\mathbf{z}$ such that $\sum_{i=1}^{10} z_i = 0$ will be an eigenvector.

For example $\mathbf{z} = [1, -1, 0, \dots, 0]^T$. (1 mark)

9. Applying Gram-Schmidt process to $\left\{\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ find an orthonormal basis of $\mathbb{R}^3$. (2 pts.)

Solution:

$\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}.$ (1/2 mark)

$\mathbf{u}_2 = \mathbf{v}_2 - (\mathbf{w}_1 \cdot \mathbf{v}_2)\mathbf{w}_1.$

$= \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ (1/2 mark)

Therefore $w_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}.$

$\mathbf{u}_3 = \mathbf{v}_3 - (\mathbf{w}_1 \cdot \mathbf{v}_3)\mathbf{w}_1 - (\mathbf{w}_2 \cdot \mathbf{v}_3)\mathbf{w}_2.$ (1/2 mark)

$= \frac{1}{2}\begin{bmatrix}1\\0\\-1\end{bmatrix}.$

$w_3 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}.$ (1/2 mark)

(For finding an **orthogonal** basis in stead of **orthonormal** only partial mark is awarded.)

10. Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ 0 \end{bmatrix}$. Consider the ordered bases

$\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ and $\mathcal{C} = \{\mathbf{e}_5, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1\}$ of $\mathbb{R}^5$.

(a) Find the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ of T with respect to the bases $\mathcal{B}$ and $\mathcal{C}$. (2 pts.)

(b) Find the change of basis matrix (also known as transition matrix) $P_{\mathcal{C}\leftarrow\mathcal{B}}$. (1 pt.)

(c) Find rank(T) and nullity(T). (1 pt.)

Solution:

(a) $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = [[T(\mathbf{e}_1)]_{\mathcal{C}}, \ldots, [T(\mathbf{e}_5)]_{\mathcal{C}}\ ]$. (1 mark)

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$ (1 mark)

(If only the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ is written correctly without doing the first step (that is without writing the form), then also gets full 2 marks.)

(b) $$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ (1 mark)

(c) rank$(T) = 4$. (1/2 mark)
nullity$(T) = 1$. (1/2 mark)