DEPARTMENT OF MATHEMATICS
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MA101 MATHEMATICS-I

First Semester of Academic Year 2015 - 2016

Solutions to Tutorial Sheet - 2

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Reduced Row-Echelon form (RREF), Gauss-Jordan elimination, Homogeneous systems, Rank of a matrix, Inverse of a matrix, Vector space $\mathbb{R}^n$, Spanning set, Linear independence

Recall:

- A matrix is in row echelon form if it satisfies the following properties:
  1. Any rows consisting entirely of zeros are at the bottom,
  2. In each non-zero row, the first non-zero entry (called the leading entry) is in a column to the left of any leading entries below it.
- A matrix is in reduced row echelon form if it satisfies the following properties:
  1. It is in row echelon form.
  2. The leading entry in each non-zero row is a 1 (called a leading 1),
  3. Each column containing a leading 1 has zeros everywhere else.
- Two system of linear equations are equivalent if and only if they have the same set of solutions.
- Rank of a matrix A can be thought of as the number of non-zero rows in the matrix $rref(A)$.
- Two matrices A, B are row equivalent if there is a sequence of elementary row operations that converts A into B.
- A system of linear equations $AX = \mathbf{b}$ is said to be homogeneous if $\mathbf{b} = \mathbf{0}$.
- If $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}$ is called the *span* of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}$ and is denoted by

  $$span(\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}) \text{ or } span(S).$$

  If $span(S) = \mathbb{R}^n$, then S is called a spanning set for $\mathbb{R}^n$.

  Theorem 1. *A system of linear equations with augmented matrix $[A|\mathbf{b}]$ is consistent iff $\mathbf{b}$ is a linear combination of the columns of A.*

- A set of vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}$ is linearly dependent if there are scalars $c_1, c_2, \cdots, c_k$, atleast one of which is not zero, such that

  $$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}.$$

  A set of vectors that is not linearly dependent is called linearly independent.

Rank of a Matrix

1. Compute the rank of the following matrices.

$$(a)\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \qquad (b)\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \qquad (c)\begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\\ 13 & 14 & 15 & 16\end{bmatrix}.$$

Soln. Consider the general $n \times n$ matrix

$$A_n = \begin{bmatrix}1 & 2 & \cdots & n\\ n+1 & n+2 & \cdots & 2n\\ & \vdots & \vdots & \\ n(n-1)+1 & n(n-2)+2 & \cdots & n^2\end{bmatrix} \xrightarrow[\text{for all } 2 \le i \le n]{R_i \leftarrow R_i - R_{i-1}} \begin{bmatrix}1 & 2 & \cdots & n\\ n & n & \cdots & n\\ & \vdots & \vdots & \\ n & n & \cdots & n\end{bmatrix}$$

$$\xrightarrow{R_i \leftarrow R_i - R_2} \begin{bmatrix}1 & 2 & 3 & \cdots & n\\ n & n & n & \cdots & n\\ 0 & 0 & 0 & \cdots & 0\\ & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0\end{bmatrix} \xrightarrow{R_2 \leftarrow \frac{1}{n}R_2} \begin{bmatrix}1 & 2 & 3 & \cdots & n\\ 1 & 1 & 1 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\\ & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & -1 & \cdots & 2-n\\ 0 & 1 & 2 & \cdots & n-1\\ 0 & 0 & 0 & \cdots & 0\\ & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}$$

Similarly we can compute the rref of the augmented matrix corresponding to all three systems and we can see that $rref(A_n)$ will always have exactly two non-zero rows (for $n \geq 2$).
Hence $rank(A_n) = 2$ for all three matrices given in part $(a), (b)$ and (c).

Gauss-Jordan Method

2. Using Gauss-Jordan method, check whether the following matrix is invertible or not! If yes, compute the inverse. Can you write down A as a product of elementary matrices ?

$$A = \begin{bmatrix}1 & 0 & -2\\ 3 & 1 & -2\\ -5 & -2 & 3\end{bmatrix}.$$

Soln. Consider the following augmented matrix :

$$\left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 3 & 1 & -2 & 0 & 1 & 0\\ -5 & -2 & 3 & 0 & 0 & 1\end{array}\right].$$

Now let us perform elementary row operations to compute the $rref(A)$. Then we have,

$$\left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 3 & 1 & -2 & 0 & 1 & 0\\ -5 & -2 & 3 & 0 & 0 & 1\end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + 5R_1]{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 0 & 1 & 4 & -3 & 1 & 0\\ 0 & -2 & -7 & 5 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_2 \leftarrow R_2 - 4R_3]{R_1 \leftarrow R_1 + 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 1 & -7 & -4 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right].$$

So $rref(A) = I_3$ and therefore A is invertible with inverse given by

$$\begin{bmatrix} -1 & 4 & 2 \\ 1 & -7 & -4 \\ -1 & 2 & 1 \end{bmatrix}.$$

Furthermore, A can be expressed as product of elementary matrices. For instance we can write

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Gaussian Elimination

3. In the following cases find out the conditions on b_i's so that the system is consistent / inconsistent.

(a) $A = \begin{pmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Soln. By applying elementary row operation on the augmented matrix $(A|b)$ we get

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 5R_1]{R_2 \leftarrow R_2 + 3R_1} \left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 14 & 12 & b_3 - 5b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 + 2b_2 + b_1 \end{array}\right]$$

Thus the system is consistent iff $b_3 + 2b_2 + b_1 = 0$.

(b) $A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Soln. By applying elementary row operation on the augmented matrix $(A|b)$ we get

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right].$$

Thus the system is consistent iff $b_3 + b_2 - 5b_1 = 0$.

4. Determine if the vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a_1}, \mathbf{a_2}$ and $\mathbf{a_3}$ where

$$\mathbf{a_1} = [1, -2, 0]^T, \quad \mathbf{a_2} = [0, 1, 2]^T, \quad \mathbf{a_3} = [5, -6, 8]^T, \quad \mathbf{b} = [2, -1, 6]^T.$$

Soln. Consider the augmented matrix

$$A = \begin{bmatrix} 1 & 0 & 5 & | & 2 \\ -2 & 1 & -6 & | & -1 \\ 0 & 2 & 8 & | & 6 \end{bmatrix}$$

Then it is enough to show this represents a consistent system. By using elementary row operations on A we get

$$\begin{bmatrix} 1 & 0 & 5 & | & 2 \\ -2 & 1 & -6 & | & -1 \\ 0 & 2 & 8 & | & 6 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 5 & | & 2 \\ 0 & 1 & 4 & | & 3 \\ 0 & 2 & 8 & | & 6 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 5 & | & 2 \\ 0 & 1 & 4 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Therefore the system is consistent. Hence $\mathbf{b}$ can be expressed as a linear combination of the vectors $\mathbf{a_1}, \mathbf{a_2}$ and $\mathbf{a_3}$. In particular we can take,

$$\mathbf{b} = 2\mathbf{a_1} + 3\mathbf{a_2} + 0\mathbf{a_3}.$$

Theoretical

5. State TRUE or FALSE. Give a brief justification.

(a) If the columns of an $m \times n$ matrix A spans $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b} \in \mathbb{R}^m$. **TRUE**.

Proof. Let $\mathbf{b} \in \mathbb{R}^m$ be arbitrary. Since columns of $A = [a_1 \cdots a_n]$ spans $\mathbb{R}^m$, so $\exists\, c_1, \cdots, c_n \in \mathbb{R}$ such that

$$\mathbf{b} = c_1 a_1 + \cdots + c_n a_n \Rightarrow \mathbf{b} = A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Then $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is a solution to the system $A\mathbf{x} = \mathbf{b}$. Since $\mathbf{b}$ is arbitrary, this completes the proof. $\square$

(b) Every homogeneous system has infinitely many solutions. **FALSE**.

Justification. The statement is true only if (no of equations)<(no of unknowns). For example, consider the system

$$I_3\mathbf{x} = \mathbf{0}.$$

(c) If the RREF of a 5×5 matrix A has the third column as $[1, 2, 0, 0, 0]^T$ then $[-1, -2, 1, 0, 0]^T$ is a solution of the homogeneous system $AX = 0$.
TRUE.

Justification. Notice that the first two columns in $RREF(A)$ are leading columns and the third is not (it contains two non-zero entries), hence the $RREF(A)$ looks like

$$\begin{bmatrix} 1 & 0 & 1 & * & * \\ 0 & 1 & 2 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

Then it can be easily verified that

$$RREF(A)[-1, -2, 1, 0, 0]^T = 0.$$

(d) For an $n \times n$ matrix A, the system $AX = 0$ and $A^T X = 0$ are equivalent.
FALSE.

Justification. Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(e) Let A be a 4×3 matrix with $rank(A) = 3$, then there exists another 3×4 matrix B such that $BA = I_3$.
TRUE.

Justification. Since rank of A is 3, therefore $RREF(A) = [I_3, 0]^T$. So there exist a invertible 4×4 matrix P such that $PA = [I_3, 0]^T$. Now take $B = [I_3, 0]P$. Then

$$BA = [I_3, 0]PA = [I_3, 0]_{3\times4}[I_3, 0]^T_{4\times3} = I_3$$

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(f) Let A and B be two matrices of the same order having the same rank, then they are row equivalent.
FALSE.

Justification. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

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6. Does there exists a 2×2 matrix such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} \frac{1}{a} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{d} \end{pmatrix}.$$

Justify your argument.
Ans. Consider the augmented matrix $(A|I_2)$ given by,

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

Then by applying elementary row operations we get, (assuming $a, b, c, d \neq 0$)

$$\begin{bmatrix} a & b & \mid & 1 & 0 \\ c & d & \mid & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - (c/a)R_1} \begin{bmatrix} a & b & \mid & 1 & 0 \\ 0 & d - \frac{bc}{a} & \mid & -\frac{c}{a} & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow (\frac{a}{ad-bc})R_2} \begin{bmatrix} a & b & \mid & 1 & 0 \\ 0 & 1 & \mid & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - bR_2} \begin{bmatrix} a & 0 & \mid & \frac{ad}{ad-bc} & -b\frac{a}{ad-bc} \\ 0 & 1 & \mid & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \xrightarrow{R_1 \leftarrow (1/a)R_1} \begin{bmatrix} 1 & 0 & \mid & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & \mid & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

But then, comparing 1st element in A^{-1} we must have

$$ad = ad - bc \Rightarrow bc = 0,$$

- a contradiction.

Hence such A cannot exist.

Alter. Consider the equation, $AA^{-1} = I_2$. Then we have,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{1}{a} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{d} \end{bmatrix} = \begin{bmatrix} 1 + \frac{b}{c} & \frac{a}{b} + \frac{b}{d} \\ \frac{c}{a} + \frac{d}{c} & 1 + \frac{c}{b} \end{bmatrix} \Rightarrow \frac{b}{c} = 0 = \frac{c}{b}$$

– a contradiction.

Hence such A cannot exist.

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7. Give an example of a subset $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\} \subset \mathbb{R}^3$ which is linearly dependent but any two of these are linearly independent.

   Proof. Consider the set $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_1} + \mathbf{e_2}\}$, where $\mathbf{e_i}$ denotes the i-th column of I_3. □
