Mathematics I (MA101)

Quiz 1 (5 Questions, 10 Marks)

28 Aug 2013 (Wed) Time: 0800–0855 (55 min.) Pages: 4

Roll Number:		**Tutorial Group:**	**T**
Name:		**Invigilator:**	

1. True or False. (No justification required.)

   (a) There exists a linear system with exactly two distinct solutions. $\boxed{1/2}$

   (b) If x and y are two non-zero vectors in $\mathbb{R}^n$ then $\text{rank}(xy^T) = 1$. $\boxed{1/2}$

   (c) Every 2×2 real matrix can be expressed as a product of a lower triangular matrix and an upper triangular matrix. $\boxed{1/2}$

   (d) Let A be an $m \times n$ matrix. If $\text{rank}(A) = 1$ and exactly one entry of A is changed then $\text{rank}(A)$ is either 0, 1 or 2. $\boxed{1/2}$

   Solution: False, True, False, True.

2. Give 5 vectors in $\mathbb{R}^2$ such that any two vectors among these 5 vectors are linearly independent. $\boxed{2}$

   Solution:

   - The five points given should be such that none is multiple of another. (Full Marks: **2 Marks**)
   - Only three or four pair wise linearly independent vectors. (Partial Marks: **1 Mark**)
   - Three or more pair wise linearly independent vectors and vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (Partial Marks: **1 Mark**)

3. Let $u_1, \ldots, u_k$ be vectors in $\mathbb{R}^n$ and P be an invertible matrix. Prove that $\boxed{2}$

$$\dim(\text{span}(\{u_1, \ldots, u_k\})) = \dim(\text{span}(\{Pu_1, \ldots, Pu_k\}))$$

Solution: For any set $v_1, \ldots, v_p$ in $\mathbb{R}^n$,

$$\sum_{i=1}^{p} \alpha_i v_i = 0 \Leftrightarrow \sum_{i=1}^{p} \alpha_i P v_i = 0$$

Thus, if $\{u_{i_1}, \ldots, u_{i_p}\}$ is a basis for $\text{span}(v_1, \ldots, v_k)$ then $\{Pu_{i_1}, \ldots, Pu_{i_p}\}$ is a basis for $\text{span}(Pv_1, \ldots, Pv_k)$ noting that P is invertible. **2 Marks**

Alternative

- Choose the basis as $\{u_{i_1}, \ldots, u_{i_p}\}$. **1 mark**
- Linearly independence of two sets of vectors does not change as P is invertible. **1 Mark**

If only second point is written. **1 Mark**

Arguments for which marks are **NOT** awarded.

- Choosing $\{u_1, \ldots, u_k\}$ as basis or linearly independent.
- Showing $\text{span}(u_1, \ldots, u_k) = \text{span}(Pu_1, \ldots, Pu_k)$.
- Showing $\text{span}(u_{i1}, \ldots, u_{il}) = \text{span}(Pu_{i1}, \ldots, Pu_{il})$.
- Stating given conditions and reaching the conclusion without saying about preservation of dependency relation or linear indenpendence.

4. Consider [2]

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 6 \\ 19 \\ \alpha \end{bmatrix}, \text{ where } \alpha \in \mathbb{R}.$$

Find the RREF of $[A \mid b]$ and determine the value of α for which the system $Ax = b$ is consistent.

Solution:

$$\text{RREF}([A \mid b]) = \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{44}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & 0 & 0 & \alpha - 7 \end{array}\right]$$

Condition for consistency: $\alpha - 7 = 0 \Rightarrow \alpha = 7$. Marking scheme:

- Computation of correct RREF. **1 Mark**
  - In case there is only one incorrect entry in RREF. $\frac{1}{2}$ **Mark**

- If more than one entry is incorrect in RREF. **0 Mark**

- Correct computation of α **1 Mark**
  - Wrong value of α but correct reasoning. **1 Mark**
  - Value of α without any reasoning. **0 Mark**

5. Consider the matrix B and its RREF given by

$$B = \begin{bmatrix} 1 & 3 & 3 & -1 & 2 & 17 \\ 2 & 6 & -2 & 14 & -3 & -19 \\ 4 & 12 & 2 & 16 & 1 & 7 \\ 3 & 9 & 1 & 13 & -1 & -2 \end{bmatrix} \qquad \mathrm{RREF}(B) = \begin{bmatrix} 1 & 3 & 0 & 5 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis of row(B). $\boxed{1/2}$

Solution: First three rows of RREF(B), i.e.,

$$\begin{bmatrix} 1 \\ 3 \\ 0 \\ 5 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

For correct answer $\frac{1}{2}$ mark, otherwise 0.

(b) Find a basis of col(B). $\boxed{1/2}$

Solution: 1^{st}, 3^{rd} and 5^{th} columns of B, i.e.,

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}$$

For correct answer $\frac{1}{2}$ mark, otherwise 0.

(c) Find a basis of null(B). $\boxed{1}$

Solution:

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

Marking scheme:

- Proper explanation with correct calculation and correct answer **1 Mark**
- Proper explanation but without correct calculation/correct answer $\frac{1}{2}$ **Mark**
- Proper explanation with correct calculation but only two vectors $\frac{1}{2}$ **Mark**
- Vectors written as rows are also accepted as answers.