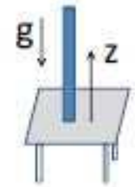
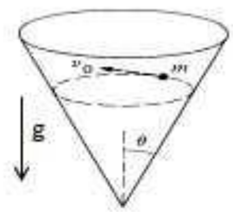
`Marks will be deducted for illegible presentation.`

1. (a) A rod of uniform cross-sectional area is maintained vertically on a table. The linear density of the rod varies as $\rho(z) = \rho_0\, z/l$, where l is the length of the rod and ρ_0 is a constant. Gravity acts along negative z-direction. Find the potential energy of the rod with respect to the table, in terms of $\rho_0,\ l,\ g$. [3]



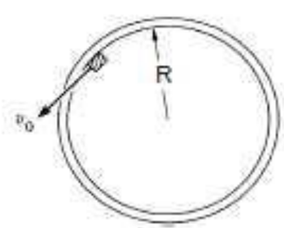
   (b) A particle of mass m slides without friction on the inside of a cone (see figure). The axis of the cone is vertical, and gravity acts downward. The apex half-angle of the cone is θ. The path of the particle happens to be a circle in a horizontal plane. The speed of the particle is v_0. Find the radius of the circular path in terms of v_0, g and θ. [3]



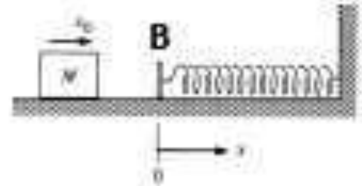
2. A block of mass m slides on a friction less table. It is constrained to move along the inner surface of a ring of radius R, which is fixed on the table. At $t = 0$ the block has a velocity v_0 as shown in the figure. The coefficient of friction between the block and the ring is μ.

   i. Find the angular frequency, $\omega(t)$, of the block as a function of time. [4]

   ii. Find the time taken by the block to make one full revolution of the ring (starting from $t = 0$). [2]



3. A block of mass M slides along a horizontal table. From $x = 0$ the mass starts experiencing forces due to a spring and the frictional force of the table (see figure). The spring has a spring-constant k. The coefficient of friction in this case is a function of x and is given by $\mu = bx$ (for $x > 0$), where b is a constant. The spring and board B are massless. At $x = 0$ the block has a speed v_0. Find the loss of mechanical energy of the block during its motion from $x = 0$ to the point where it comes momentarily to rest, for the first time. [5]



4. (a) A block moves along the x-axis on a horizontal friction less table. The table is covered with dust of linear density ρ. As the block moves it gathers all the dust in the path and gets heavier. At $t = 0$, the block's velocity is v_0 and its mass is M_0.

    i. Find the velocity of the block as a function of distance, x. [3]

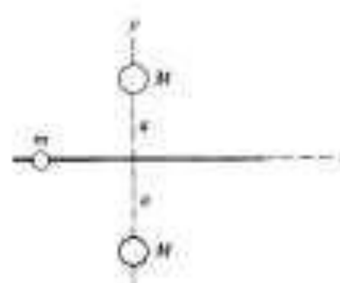
    ii. What horizontal force, $F(x)$, must be applied on the block, if the block has to move with constant velocity v_0. [2]

   (b) A rubber ball of mass m dropped from a height H under gravity hits the ground and rebounds to a height h. Find the impulse it received from the ground. [2]

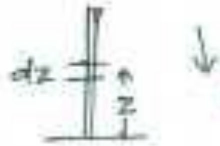
5. A bead of mass m slides without friction on a smooth rod along the X-axis. Two particles each of mass M are located at $(0, +a)$ and $(0, -a)$ as shown in the figure. The bead is under the gravitational attraction of the masses, M.

   i. Starting from the expression of the total potential energy obtain the force on the bead along the x-direction. [2]

   ii. Find the frequency of small oscillation of the bead about the equilibrium. [4]



a) Given $\rho(z) = \frac{\rho_0 z}{\ell}$ (linear density)



mass of the small element

$$dm = \rho(z)\,dz = \frac{\rho_0}{\ell} z\, dz$$

The Potential Energy of the element,

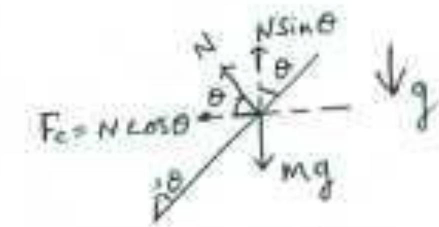
$$dU = dm\, g\, z = \frac{\rho_0 g}{\ell} z^2 dz$$

$$\text{Total P.E} = \int_{z=0}^{z=\ell} dU = \frac{\rho_0 g}{\ell} \int_0^{\ell} z^2 dz$$

i.e. $$U = \frac{\rho_0 g \ell^2}{3}$$

b)

From the Force diagram with components resolved along horizontal & vertical directions,



Force Diagram (Not required)

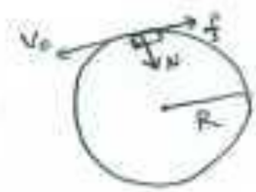
$$N\sin\theta = mg \quad -①$$

$$N\cos\theta = \frac{m v_0^2}{r} \quad -②$$

$$\frac{①}{②} \Rightarrow \tan\theta = \frac{\cancel{m} g}{\cancel{m} v_0^2/r} = \frac{rg}{v_0^2}$$

So $$r = \frac{v_0^2 \tan\theta}{g}$$

Q.2
i) Given at $t=0$; $v = v_0$
or $\omega = v_0/R$



Total force on the particle

$$\bar{F} = N(-\hat{r}) + f(-\hat{\theta})$$ (f - friction, N - normal reaction)

The general form of acceleration

$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

Since $r = R$; $\dot{r} = \ddot{r} = 0$.

$$\text{i.e. } \bar{a} = -R\dot{\theta}^2\hat{r} + R\ddot{\theta}\hat{\theta} \quad -(1)$$

Comparing forces and accelerations along $\hat{r}$ & $\hat{\theta}$ directions

$$\left.\begin{aligned} N &= mR\dot{\theta}^2 \quad (\text{along } \hat{r}) \quad -(2) \\ f &= -mR\ddot{\theta} \quad (\text{along } \hat{\theta}) \quad -(3) \end{aligned}\right\}$$

Since $f = \mu N$, $\mu m R\dot{\theta}^2 = -mR\ddot{\theta}$

$$\dot{\theta} = \omega, \quad \mu\omega^2 = -\frac{d\omega}{dt}$$

$$\text{or} \quad -\frac{d\omega}{\omega^2} = \mu\, dt$$

Integrating. $\frac{1}{\omega} = \mu t + C$

Using Initial condition, $t=0$, $\omega = v_0/R$

$$\frac{1}{\omega} = \mu t + R/v_0$$

$$\text{Or} \quad \omega(t) = \frac{v_0}{(R + \mu v_0 t)}$$

Q.2

ii) $$\omega(t) = \frac{d\theta}{dt} = \frac{v_0}{(R+\mu v_0 t)}$$

or $$d\theta = \frac{v_0\, dt}{(R+\mu v_0 t)}$$

Integrating between $t=0$ to $t=T$ where T is the time taken for the 1st full revolution.

$$\int_0^T d\theta = v_0 \int_0^T \frac{dt}{(R+\mu v_0 t)}$$

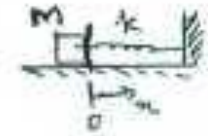
$\therefore$ $$2\pi = v_0 \frac{1}{\mu v_0} \ln(R+\mu v_0 t)\Big|_0^T$$

Or $$2\pi\mu = \ln\left(\frac{R+\mu v_0 T}{R}\right)$$

Or $$1+\frac{\mu v_0 T}{R} = e^{2\pi\mu}$$

$\therefore$ $$\underline{\underline{T = \frac{R(e^{2\pi\mu}-1)}{\mu v_0}}}$$

Q.3. Using Work energy theorem



$$E_i - E_f = -W^{nc} \quad \text{—} (1)$$

where E_i - initial mechanical energy (total)

E_f - final mechanical energy (total)

W^{nc} — Work done by non-conservative forces (here friction, $f = \mu Mg$)

$$E_i = \frac{1}{2}Mv_0^2 \;;\; E_f = \frac{1}{2}kx_0^2 \quad \text{—} (2)$$

x_0 - unknown, the distance at which the block comes to rest.

$$W^{nc} = -\int_0^{x_0} f\,dx = -\int_0^{x_0} \mu Mg\,dx$$

Since $\mu = bx$

$$W^{nc} = -Mgb\int_0^{x_0} x\,dx = -\frac{Mgb\,x_0^2}{2} \quad \text{—} (3)$$

Using (1) & (2), $\dfrac{Mgb\,x_0^2}{2} = \frac{1}{2}Mv_0^2 - \frac{1}{2}kx_0^2$

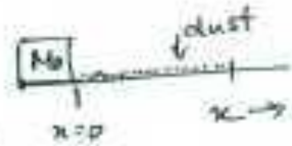
ie $\quad x_0^2 = \dfrac{Mv_0^2}{(k+Mgb)}$

Mechanical energy lost is $-W^{nc}$, say ΔE,

Using (3), $\quad \Delta E = \dfrac{Mgb}{2}\dfrac{Mv_0^2}{(k+Mgb)}$

Q.4.

a) i) Given at $t=0$, $M = M_0$ and $V = V_0$



Since there is no external forces along the x-direction the total linear momentum is conserved.

That is, $M_0 V_0 = M(x)\,V(x)$

$$V(x) = \frac{M_0 V_0}{M(x)}$$

The mass of the block at $\underline{M(x) = M_0 + \rho . x}$ (1)

$\rho . x$ – being the mass accumulated over the distance x.

Thus $V(x) = \underline{\underline{M_0 V_0 / (M_0 + \rho . x)}}$

a) i) 2nd solution:

$$F_x = \frac{dP_x}{dt} = V\frac{dM}{dt} + M\frac{dV}{dt} = 0$$

or $\dfrac{dM}{M} + \dfrac{dV}{V} = 0$

Integrating $\ln M = -\ln V + \underline{\ln C}$ some constant

or $M = C/V$

at $x=0$; $M = M_0$ and $V = V_0$

Hence $C = M_0 V_0$

that $M(x) = \dfrac{M_0 V_0}{V(x)}$

or $$\underline{\underline{V(x) = \frac{M_0 V_0}{(M_0 + \rho . x)}}}$$

Q4 a) ii)

$$F(x) = \frac{d}{dt}\left(M(x)V(x)\right)$$

$$= M(x)\frac{dV(x)}{dt} + V(x)\frac{dM(x)}{dt}$$

but $V(x) = V_0$ in this case

So $$F(x) = V_0 \frac{dM(x)}{dt}$$

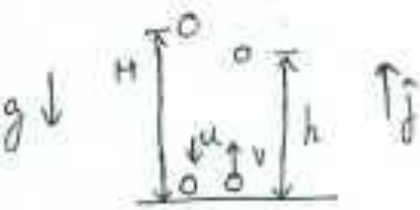
$$= V_0 \frac{dM(x)}{dx}\frac{dx}{dt} \rightarrow V_0$$

$$F(x) = V_0^2 \rho$$

ie $$\underline{\underline{F(x) = \rho V_0^2}} \text{ (a const.)}$$

b) The velocity with the ball hits the ground

$u = \sqrt{2gH}$ — (1)



The velocity with which it rebounds $v = \sqrt{2gh}$ — (2)

Impulse it received from the ground

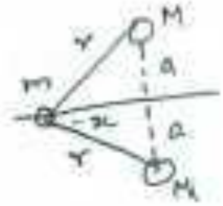
$$= \int_0^{\tau} F\,dt = p_f - p_i$$ (p_i & p_f are the initial & final momenta)

$$= mv\hat{j} - mu(-\hat{j}) = m(v+u)\hat{j}$$

Impulse $= \underline{\underline{m\left(\sqrt{2gH} + \sqrt{2gh}\right)\hat{j}}}$

Q.5

i) The total potential energy of the mass "m" due to the two masses "M" is



$$U = -\frac{2GMm}{r} \quad \text{where } r = \sqrt{a^2+x^2}$$

$$F_x = -\frac{\partial U}{\partial x} = 2GMm\frac{d(1/r)}{dx}$$

$$= -\frac{2GMm}{r^2}\frac{dr}{dx}$$

$$r^2 = x^2+a^2$$

$$2r\frac{dr}{dx} = 2x$$

$$\frac{dr}{dx} = x/r$$

i.e., $$F_x = -\frac{2GMm\,x}{r^3} \quad \text{—①}$$

ii) At equilibrium $F_x = 0$

From ① equilibrium point is, $x_0 = 0$.

The effective force constant for small oscilla-tion of "m" about $x = 0$ is given by

$$k = \left.\frac{d^2U}{dx^2}\right|_{x=0}$$

i.e. $$k = 2GMm\frac{d}{dx}(x/r^3) = 2GMm\left(\frac{1}{r^3} + x\cdot\frac{-3}{r^4}\cdot\frac{dr}{dx}\right)\Bigg|_{x=0}$$

$$= 2GMm\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)\Bigg|_{x=0}$$

i.e., $$k = \frac{2GMm}{a^3} \quad (\text{Since at, } x=0;\ r=a)$$

Frequency $$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2GM}{a^3}}$$