MA 101 | END SEM

MATHEMATICS I | 01:00–04:00PM

IIT GUWAHATI | 22^{nd} NOV. 2011

INSTRUCTIONS

1. To 'print' means to write legibly in capital letters.
2. Print your tutorial group:→ | T | | |
3. Put your signature in the space provided:→
4. Print your roll number:→
5. Print your name as registered:
6. Do not write anything else on this page.
7. Write your answers in this booklet and only in the space marked for answers. Additional space is to be found towards the end of the booklet. Use the provided blank sheets *only* for rough work.
8. Your writing should be legible and neat. When in doubt, print.
9. Check that you have 12 printed pages. This exam has 13 questions, for a total of 50 points.
10. The first question has 28 parts. Each part is worth half a point. They require only the correct answer for full credit. Remaining questions need all necessary steps for full credit.
11. Although not necessary, you may use a non–programmable calculator.
12. Time management: Spend 4 minutes over this page, 56 minutes over question 1 (@ 2 minutes/part) and 120 minutes over questions 2–13 (@ 10 minutes/question).

SPACE FOR RECHECK CRIBS

FOR OFFICE USE ONLY

Jobs	Invigilator	Grader	Scrutinizer	Rechecker	Rechecker
Initials					

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
Points:	14	2	4	3	3	3	3	3	3	3	3	3	3	50
Score:														

[14pnts.] 1. Write your answers NEATLY in the spaces earmarked.

	QUESTION	ANSWER
(i)	Let S be a spanning set for the vector space $M_3(\mathbb{R})$ over $\mathbb{R}$. Then, the minimum number of vectors that S must have is	
(ii)	Consider the basis $B = \{1+x, x+x^2, 1+x^2\}$ for $\mathbb{R}_2[x]$ over $\mathbb{R}$ and let $p(x) = 3x + x^2$. Then, the coordinate vector $[p(x)]_B$ equals	
(iii)	Consider the subspace $V = \text{span}\{1, x, x^2, 1+2x^2\}$ of $\mathbb{R}_5[x]$. Then, $\dim(V)$ equals	
(iv)	Consider the vector space $V = \{p(x) \in \mathbb{R}_2[x] : p(0) = 0\}$ over $\mathbb{R}$. Then, $\dim(V)$ equals	
(v)	Let B and C be two bases for a vector space V such that $P_{B\leftarrow C} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then, $P_{C\leftarrow B}$ equals	
(vi)	Let $T : \mathbb{R}^2 \to \mathbb{R}_2[x]$ be a linear transformation such that $T([1,0]^t) = 2 - 3x + x^2$ and $T([1,1]^t) = x$. Then, $T([4,3]^t)$ equals	
(vii)	Consider the linear transformation $T : \mathbb{R}_2[x] \to \mathbb{R}^2$ defined by $T(p(x)) = [p(0), p(1)]^t$ for all $p(x) \in \mathbb{R}_2[x]$. Then, nullity(T) equals	
(viii)	Let $T : \mathbb{R}^{15} \to \mathbb{R}^8$ be a linear transformation such that nullity$(T) = 9$. Then, rank(T) equals	
(ix)	$\lim\limits_{n\to\infty} \frac{2n^2+4}{6n^2+9}$ evaluates to	
(x)	An example for reals α and β for which the sequence $\left((-1)^n\alpha + (-1)^{n+1}\beta + \frac{1}{n^2}\right)$ converges is	$\alpha =$ $\beta =$
(xi)	$\lim\limits_{n\to\infty} \left((3 \cdot (14)^n + 2 \cdot (21)^n)^{\frac{1}{n}}\right)$ evaluates to	
(xii)	Suppose $a_n = \frac{(-1)^{n+1}}{2n-1}$ for each natural n. Consider the following three statements about the series $\sum a_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ (A) $\sum a_n$ is divergent (B) $\sum a_n$ is conditionally convergent (C) $\sum a_n$ is absolutely convergent. Which one of (A), (B) or (C) is TRUE?	
(xiii)	Given any conditionally convergent series, it can be rearranged to get a sum equal to	
(xiv)	Let k be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = kx^2$ if $x \leq 2$ and $f(x) = 5$ if $x > 2$. If f is continuous at $x = 2$, then k equals	
(xv)	Which of the following statement(s) is (are) TRUE for increasing functions from reals to reals? (A) Sum of any two such functions is increasing (B) Difference of any two such functions is increasing (C) Product of any two such functions is increasing (D) Composition of any two such functions is increasing	

	QUESTION	ANSWER
(xvi)	Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \|x-1\| + \|x-2\| + \|x-3\|$ if $x \in \mathbb{Q}$ and $f(x) = 6 - \|x-1\| - \|x-2\| - \|x-3\|$ otherwise. All points of continuity of f are	
(xvii)	Let J be an interval such that every function $f : J \to \mathbb{R}$ is bounded. Then J can be of only one type which is	
(xviii)	Let I be an interval such that every continuous $f : I \to \mathbb{R}$ is bounded. Then I can be of only two types which are	
(xix)	Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f(g(x)) = x^3 + 6x$ for all real x. It is known that $g(2) = 5$ and $g'(2) = 6$. Then, $f'(5)$ equals	
(xx)	Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 7(x-11)^3 + 13$. The point(s) at which f^{-1} is not differentiable (is) are	
(xxi)	Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax^3 + 3bx^2 + 3cx + d$ for reals a, b and c with $a \neq 0$. If f is bijective, then $b^2 - ac$ has to be	
(xxii)	Given $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^3 + 2x + 1$ for $x \in \mathbb{R}$, the value of $(h^{-1})'(4)$ is	
(xxiii)	The number of points of relative extrema of the function $f(x) = \|x^2 - 1\|$ on the domain $-4 \leq x \leq 4$ is	
(xxiv)	For each $x \in \mathbb{R}$, let $f(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4$. This defines a function $f : \mathbb{R} \to \mathbb{R}$. There is a unique natural number k such that every anti–derivative of f has either k roots or $k+2$ roots. The natural k equals	
(xxv)	$\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{n} \frac{i}{\sqrt{i^2+n^2}}$ evaluates to	
(xxvi)	Consider the function $f : [0,3] \to \mathbb{R}$ given by $f(x) = 1$ if $x < 1$, $f(x) = 2$ if $1 \leq x < 2$ and $f(x) = 3$ otherwise. The number of points of discontinuity of the indefinite integral $\mathcal{A}_f(x) = \int_0^x f(t)dt$ on $[0,3]$ is	
(xxvii)	Consider the function $g : [0,3] \to \mathbb{R}$ given by $g(x) = \|x - \frac{1}{2}\| + \|x-1\| + \|x - \frac{3}{2}\| + \|x-2\| + \|x - \frac{5}{2}\|$. The number of points of non–differentiability of the indefinite integral $\mathcal{A}_g(x) = \int_0^x g(t)dt$ on $(0,3)$ is	
(xxviii)	Given three distinct reals a, b and c, verify that we can perform 12 types of addition, 12 types of multiplication and 24 operations involving one addition and one multiplication, for a total of 48 different operations. Under this interpretation, the number of different ways of performing operations with addition or multiplication on four distinct real numbers is	

[2pnts.] 2. Let W_1 and W_2 be subspaces of a vector space V such that $W_1 \cup W_2$ is also a subspace of V. Show that either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

[4pnts.] 3. Let the mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T([x, y, z]^t) = [x - y, y - z]^t$ for all $[x, y, z]^t \in \mathbb{R}^3$. Show that T is a linear transformation. Also, find the matrix $[T]_{C \leftarrow B}$, where B is the standard basis for $\mathbb{R}^3$ and $C = \{[1, 1]^t, [1, -1]^t\}$ is a basis for $\mathbb{R}^2$.

[3pnts.] 4. If $a, b \in \mathbb{R}$ and $a \neq 0$ and $b \neq 0$, show that $ab \neq 0$.

[3pnts.] 5. Let a and b be reals satisfying $0 < a < b$. For any natural number n, show that $0 < a^n < b^n$.

[3pnts.] 6. Let A and B be non–empty subsets of $\mathbb{R}$ which are both bounded above. Define $A+B=\{a+b \mid a \in A, b \in B\}$. Show the following:
(a) $A+B$ is bounded above (b) $\sup(A+B)=\sup(A)+\sup(B)$.

[3pnts.] 7. Let $a<b$ be reals. If (α_n) is a cauchy sequence of reals in $[a,b]$, show that (α_n) is convergent. Further, if (α_n) converges to a real α show that $\alpha \in [a,b]$.

[3 pnts.] 8. Define a function $\sigma : \mathbb{N} \to \{-1, 1\}$ as follows. For any natural n, $\sigma(n) = 1, -1$ or 1 accordingly as whether n leaves a remainder 1, 2 or 0 upon division by 3.
Explicitly, $\sigma(1) = 1, \sigma(2) = -1, \sigma(3) = 1, \sigma(4) = 1, \sigma(5) = -1, \sigma(6) = 1, \ldots$.
Investigate whether the series $\sum_{n=1}^{\infty} \frac{\sigma(n)}{n}$ converges or diverges.

[3pnts.] 9. Find a root of the polynomial $x^4 + 4x^3 + 20$ to within two decimal places using bisection method.

ITERATE	LEFT POINT	RIGHT POINT	MID POINT	MID VALUE	ERROR
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					

A root of the given polynomial to within two decimal places is ____________.

[3pnts.] 10. Let $a < b$ be reals and $f : [a, b] \to \mathbb{R}$ be continuous. If f has an absolute maximum at a point $c \in (a, b)$, show that f is not injective.

[3pnts.] 11. Consider $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $g(x) = 1/x^2$. Let $c \in \mathbb{R} \setminus \{0\}$ and consider the difference quotient $q_c(h) = \frac{g(c+h)-g(c)}{h}$. Note that q_c is defined for $0 < |h| < |c|$.
Given any real $\epsilon > 0$, find a real $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|q_c(h) + \frac{2}{c^3}| < \epsilon$.
{Write your proof respecting the following sketch.}

Given a real $\epsilon > 0$, MY $\delta = \boxed{}$

CHECK: $\delta > 0$? YES/NO and δ does not depend on x? YES/NO
START: Assume x is a real such that $0 < |x - c| < \delta$.
THEN: ...

[3pnts.] 12. Let $f : [-1, 1] \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{\sqrt[3]{(x^2+1)(x^2+2)(x^2+3)}}.$$

Prove that f has an anti–derivative.

[3pnts.] 13. For reals $a < b$, let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that there exists a $c \in (a, b)$ such that $(b - a) \int_a^c f(t)dt = (c - a) \int_a^b f(t)dt$. Prove that there is a $\xi \in (a, b)$ such that $f'(\xi) = 0$.

EXTRA SPACE FOR ANSWERS/NOT FOR ROUGH WORK