1. Let A and B be two $m \times n$ matrices that are in reduced row echelon form. If the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set then show that $A = B$.

Solution: We will apply induction on the number of columns to prove that "if the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set and A, B are RREF of the same size then $A = B$".

Base Case: Let both A and B have 1 column. Since there are only two RREF of an $m \times 1$ matrix, *viz.,* $\mathbf{O}_{m\times 1}$ and e_1, we find easily that if the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set then $A = B$.

Induction Hypothesis: Assume that for any two RREF P, Q of size $m \times k$, if the systems $P\mathbf{x} = \mathbf{0}$ and $Q\mathbf{x} = \mathbf{0}$ have the same solution set then $P = Q$.

Inductive Case: Let A and B be two RREF of size $m \times (k + 1)$. Let the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set. We want to show that $A = B$.

Let $\hat{A}$ and $\hat{B}$ be the matrices obtained by deleting the last columns of A and B, respectively. It is clear that $\hat{A}$ and $\hat{B}$ are also RREF, but having k columns. Let $\mathbf{u} = [u_1, \ldots, u_k]^t$. We have

$$\hat{A}\mathbf{u} = \mathbf{0} \Leftrightarrow A \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix} = \mathbf{0} \Leftrightarrow B \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix} = \mathbf{0} \Leftrightarrow \hat{B}\mathbf{u} = \mathbf{0}.$$

Thus the systems $\hat{A}\mathbf{x} = \mathbf{0}$ and $\hat{B}\mathbf{x} = \mathbf{0}$ have the same solution set. By induction hypothesis, we find that $\hat{A} = \hat{B}$.

Let, if possible, $A \neq B$. Let $\mathbf{v} = [v_1, \ldots, v_k, v_{k+1}]^t$ be a solution of $A\mathbf{x} = \mathbf{0}$. Then we have $(A - B)\mathbf{v} = \mathbf{0}$ which gives that $v_{k+1} = 0$. Hence the $(k + 1)$-th column of A is pivotal. Assume that $\hat{A}$ has r pivots. Then $(k + 1)$-th column of A must be e_{r+1}. Similarly, $(k + 1)$-th column of B will also be e_{r+1}. Hence $A = B$, which is a contradiction.

Hence by mathematical induction, we conclude that if the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set and A, B are RREF of the same size then $A = B$.

Aliter

Suppose, if possible, $A \neq B$. Let k be the smallest positive integer such that the k-th columns $\mathbf{a}_k$ and $\mathbf{b}_k$ of A and B, respectively, differ. Let A_{k-1} (resp. B_{k-1}) be the matrix formed by the first $k - 1$ columns of A (resp. B). Then $A_{k-1} = B_{k-1}$.

Now, it cannot happen that **both** $\mathbf{a}_k$ and $\mathbf{b}_k$ are leading columns, otherwise the leading 1's on them are at the same positions, and consequently we will get $\mathbf{a}_k = \mathbf{b}_k$.

Next neither of $\mathbf{a}_k$ and $\mathbf{b}_k$ is a zero column. If $\mathbf{a}_k$ is a zero column, then $\mathbf{b}_k$ is not (since they differ). So, $\mathbf{e}_k = [0, \ldots, 1, \ldots, 0]^t$ is a solution of $A\mathbf{x} = \mathbf{0}$ but not of $B\mathbf{x} = \mathbf{0}$.

Suppose $\mathbf{a}_k$ is a non-zero, non-leading column and suppose $a_{rk} \neq b_{rk}$. Consider the vector $\mathbf{s} = [s_1, s_2, \ldots, s_k, 0 \ldots, 0]^t$, where

$$s_i = \begin{cases} -a_{pk}, & \text{if the } i\text{-th column of } A \text{ is } \mathbf{e}_p \text{ (leading)} \\ 0, & \text{if the } i\text{-th column of } A \text{ is non-leading} \\ 1, & \text{if } i = k. \end{cases}$$

It is easy to see that $\mathbf{s}$ is a solution of $A\mathbf{x} = \mathbf{0}$ but not of $B\mathbf{x} = \mathbf{0}$. In fact the r-th equation of $B\mathbf{x} = \mathbf{0}$ is not satisfied by $\mathbf{s}$. Thus, there is a contradiction. □