DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
MA101 MATHEMATICS-I
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Solutions to Tutorial Sheet - 3
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Linear independence, Subspace, Row/Column space, Null space, Basis, Dimension, Linear transformations.

Recall:

- If $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}$ is called the *span* of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}$ and is denoted by

$$span(\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}) \text{ or } span(S).$$

  If $span(S) = \mathbb{R}^n$, then S is called a spanning set for $\mathbb{R}^n$.

- A set of vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}$ is linearly dependent if there are scalars $c_1, c_2, \cdots, c_k$, atleast one of which is not zero, such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}.$$

  A set of vectors that is not linearly dependent is called linearly independent.

  Theorem 1. $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_m}$ *be (column) vectors in* $\mathbb{R}^n$ *and let* A *be the* $n \times m$ *matrix* $[\mathbf{v_1} \quad \mathbf{v_2} \quad \cdots \quad \mathbf{v_m}]$ *with these vectors as its columns. Then* $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_m}$ *are linearly dependent if and only if the homogeneous linear system with augmented matrix* $[A|\mathbf{0}]$ *has a nontrivial solution.*

- A subspace of $\mathbb{R}^n$ is any collection $\mathcal{S}$ of vectors in $\mathbb{R}^n$ such that

  1. The zero vector $\mathbf{0}$ is in $\mathcal{S}$.
  2. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathcal{S}$, then $\mathbf{u} + \mathbf{v}$ is in $\mathcal{S}$. ($\mathcal{S}$ is closed under addition.)
  3. If $\mathbf{u}$ is in $\mathcal{S}$ and c is a scalar, then $c\mathbf{u}$ is in $\mathcal{S}$. ($\mathcal{S}$ is closed under scalar multiplication.)

- Let A be an $m \times n$ matrix.

  1. The row space of A is a subspace $row(A)$ of $\mathbb{R}^n$ spanned by the rows of A.
  2. The column space of A is a subspace $col(A)$ of $\mathbb{R}^n$ spanned by the columns of A.

- Let A be an $m \times n$ matrix. The null space of A is the subspace $null(A)$ of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $A\mathbf{X} = \mathbf{0}$.
  The null space of A is a subspace of $\mathbb{R}^n$.

- Two matrices A, B are row equivalent if there is a sequence of elementary row operations that converts A into B.

- A basis for a subspace $\mathcal{S}$ of $\mathbb{R}^n$ is a set of vectors in $\mathcal{S}$ that

1. spans $\mathcal{S}$ and
2. is linearly independent.

- If $\mathcal{S}$ is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for $\mathcal{S}$ is called the dimension of $\mathcal{S}$, denoted by dim $\mathcal{S}$.
- A transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if
  1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$.
  2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^n$ and all scalars c.

Theoretical

1. State TRUE or FALSE. Give a brief justification.

(a) For a matrix A in its row echelon form, the non-zero rows are linearly independent.
TRUE.
Justification. If not, then there is a non-zero row R_i in $REF(A)$ such that R_i can be written as a linear combination of the remaining non-zero rows, say

$$R_i = c_1R_1 + \cdots + c_{i-1}R_{i-1}.$$

(WLOG we can assume that R_i is the last non-zero row)
Then the following sequence of elementary row operations increase the number of zero rows in $REF(A)$, $\to$ a contradiction to the definition of $rank(A)$.

(b) If $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent vectors, and if $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly dependent set, then $v_3 \in span\{\mathbf{v_1}, \mathbf{v_2}\}$.
TRUE.

Proof. Since $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly dependent set, therefore there exist scalars c_j's, atleast one of them not zero such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}.$$

But then $c_3 = 0 \Rightarrow c_1 = c_2 = 0$. So $c_3 \neq 0$. Then

$$\mathbf{v_3} = \frac{c_1}{c_3}\mathbf{v_1} + \frac{c_2}{c_3}\mathbf{v_2} \in span\{\mathbf{v_1}, \mathbf{v_2}\}.$$

□

(c) The vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are in $Span\{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$.
TRUE. *Justification.* Consider the following relations:

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{u}+\mathbf{v} \\ \mathbf{u}+\mathbf{v}+\mathbf{w} \end{bmatrix}$$

(d) If all the columns of an $m \times n$ non-zero matrix (it has atleast one non-zero entry) A are equal then $rank(A) = 1$.
TRUE.
Justification. Note that

$$rank(A) = rank(A^T) = \text{No. of non-zero rows in } RREF(A^T).$$

But A^T has all the rows same and so the first row must contain atleast one non-zero entry. So $RREF(A)$ will have exactly one non-zero row and rest are all zero. Thus

$$rank(A^T) = 1 = rank(A).$$

(e) If A and B are square matrices such that AB is invertible then both A and B are invertible.
TRUE.
Justification. Let A be an $n \times n$ matrix. Then so are B (otherwise product AB not defined) and AB. Recall that,

$$rank(AB) \leq \min\{rank(A), rank(B)\}.$$

Since AB is invertible, so $rank(AB) = n$. Therefore,

$$n \leq rank(A), rank(B) \leq n \Rightarrow rank(A) = rank(B) = n.$$

Hence A and B are invertible.

(f) If the equation $AX = b$ has atleast one solution for each $b \in \mathbb{R}^n$, then the solution is unique for each b.
FALSE.
The above statement is true if A is a square matrix. Otherwise it may not. Consider the following example where $n = 2$:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(g) Let A be an invertible matrix and the vectors $\{x_1, x_2, \cdots, x_r\}$ are linearly independent. Then the vectors $Ax_1, Ax_2, \cdots, Ax_r$ are linearly independent.
TRUE.

Proof. Consider the equation

$$c_1 Ax_1 + c_2 Ax_2 + \cdots + c_r Ax_r = 0.$$

Since A is invertible, so by left multiplying A^{-1} to both sides of the above equation we get

$$c_1 A^{-1}Ax_1 + c_2 A^{-1}Ax_2 + \cdots + c_r A^{-1}Ax_r = 0 \Rightarrow c_1x_1 + c_2x_2 + \cdots + c_rx_r = 0 \Rightarrow c_j = 0 \;\forall\; j.$$

So $Ax_1, Ax_2, \cdots, Ax_r$ are linearly independent. $\square$

(h) Let $\{v_1, \cdots, v_n\}$ be a linearly independent set. Suppose there exists scalars α_i and β_i such that $\sum_{i=1}^n \alpha_i v_i = \sum_{i=1}^n \beta_i v_i$. Then for each i, $\alpha_i = \beta_i$.
TRUE.

Proof. Suppose, $u = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i$. Then

$$0 = u - u = \sum_{i=1}^{n} \alpha_i v_i - \sum_{i=1}^{n} \beta_i v_i = \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i.$$

But since $\{v_1, \cdots, v_n\}$ is linearly independent, therefore, for each i, $\alpha_i - \beta_i = 0$. $\square$

Subspaces

2. Examine whether the following sets are subspaces of $\mathbb{R}^n$.

   (a) For $n \geq 3$, $S_1 = \{[x_1, \cdots, x_n]^T \in \mathbb{R}^n \colon x_1 + x_2 = 4x_3\}$

   (b) For $n \geq 3$, $S_1 = \{[x_1, \cdots, x_n]^T \in \mathbb{R}^n \colon x_1 + x_2 \leq 4x_3\}$

   (c) A line given by equation $y = mx + c$ in $\mathbb{R}^2$.

   (d) For a linear transformation $T \colon \mathbb{R}^m \to \mathbb{R}^n$, the range of T.

Soln. Suppose $x = [x_1, \cdots, x_n]^T, y = [y_1, \cdots, y_n]^T \in S_1$ and $c \in \mathbb{R}$. Then

(a) $x + y = [x_1 + y_1, \cdots, x_n + y_n]^T$ and so

$$(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = 4x_3 + 4y_3 = 4(x_3 + y_3).$$

Also, $cx = [cx_1, \cdots, cx_n]^T$ and so

$$cx_1 + cx_2 = c(x_1 + x_2) = 4(cx_3).$$

So, $x + y, cx \in S_1$. Hence S_1 is a subspace.

(b) Here consider x with $x_1 + x_2 - 4x_3 \leq 0$. Take $-1 \in \mathbb{R}$. Then, $(-1)x = [-x_1, \cdots, -x_n]^T$ but

$$-x_1 - x_2 + 4x_3 \geq 0 \Rightarrow -x \notin S_2.$$

Hence S_2 is not a subspace.

(c) This is not a subspace if $c \neq 0$. Since, $\mathbf{0} \notin y = mx + c$.
If $c = 0$, then it is a line passing through the origin and hence a subspace.

(d) T being a linear transformation, $T(0) = 0 \in range(T)$. Let $u, v \in range(T)$ and $c \in \mathbb{R}$. Then $u = T(x), v = T(y)$ for some $x, y \in \mathbb{R}^m$. Then

$$u + v = T(x) + T(y) = T(x + y) \in range(T) \quad cu = cT(x) = T(cx) \in range(T).$$

Hence range(T) is a subspace.

Equivalent Matrices

3. Show that the following matrix $A = \begin{bmatrix} 2 & 5 & 2 & 2 & 7 \\ 0 & 3 & 5 & 0 & 8 \\ 6 & 2 & 7 & 9 & 4 \\ 0 & 2 & 5 & 2 & 2 \\ 4 & 7 & 5 & 7 & 1 \end{bmatrix}$ is equivalent to another matrix B whose last row is $[20604 \quad 53227 \quad 25755 \quad 20927 \quad 78421]$.

Soln. Applying the elementary row operations

$$\begin{aligned} R_5 &\leftarrow R_5 + 10R_4 \\ R_5 &\leftarrow R_5 + 10^2 R_3 \\ R_5 &\leftarrow R_5 + 10^3 R_2 \\ R_5 &\leftarrow R_5 + 10^4 R_1 \end{aligned}$$

on the given matrix, the entries in the last row will become $[20604 \quad 53227 \quad 25755 \quad 20927 \quad 78421]$..

Null Space

4. Compute the null space of the following matrix $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$. What is $\dim(null(A))$?

Soln. Note that $null(A)$ is a subspace of $\mathbb{R}^5$ and is a solution space of the homogeneous equation system $AX = 0$. Consider the augmented matrix $\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.
Note that x_1 and x_3 are leading variables and x_2, x_4, x_5 are the free variables. Let r, s, t be the parameters for x_2, x_4, x_5.
Hence the corresponding system of equations is :

$$\begin{aligned} x_1 &= 2r + s - 3t \\ x_2 &= r \\ x_3 &= -2s + 2t \\ x_4 &= s \\ x_5 &= t \end{aligned}$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $null(A)$ is spanned by $\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\2\\0\\1\end{bmatrix}$. It is easy to check that they are linearly independent. Hence these three vectors form a basis for $null(A)$.
Hence $\dim(null(A))=3$.

Rank & Linear Independence

5. Under what conditions on the scalar $\alpha \in \mathbb{Q}$ the vectors $[1+\alpha \quad 1-\alpha]^T$ and $[1-\alpha \quad 1+\alpha]^T$ in $\mathbb{R}^2$ are linearly independent ?

   Soln. Consider the solution of two equations in two unknowns :

$$\begin{aligned} a(1+\alpha)+b(1-\alpha) &= 0 \\ a(1-\alpha)+b(1+\alpha) &= 0 \end{aligned}$$

   in the unknowns a and b. If $\alpha \neq 0$, then a and b must be 0; the only case of linear dependence is the trivial one $(1,-1)$, i.e. $\alpha = 0$.

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6. Let A be a 3×3 matrix such that $rank(A) = 2$ and the columns of A satisfy $C_3 = C_1 + C_2$. Then show that there exists a matrix X such that $AX = A$, $X \neq I_3$.

   Proof. We have $A = [C_1 \quad C_2 \quad C_3] = [C_1 \quad C_2 \quad C_1 + C_2] = A[\mathbf{e_1} \quad \mathbf{e_2} \quad \mathbf{e_1} + \mathbf{e_2}]$, where $\mathbf{e_j}$ denotes the j^{th} column in I_3.
   Clearly, $X = [\mathbf{e_1} \quad \mathbf{e_2} \quad \mathbf{e_1} + \mathbf{e_2}]$ gives a required matrix. $\square$

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7. Let A be an $n \times m$ matrix and let B be an $m \times n$ matrix. Prove that the matrix $I_m - BA$ is invertible if and only if the matrix $I_n - AB$ is invertible.

   Soln. We shall prove that the matrix $I_m - BA$ is not invertible if and only if the matrix $I_n - AB$ is not invertible.
   Let the matrix $I_m - BA$ be not invertible, so that $(I_m - BA)x = 0$ has a non trivial solution. Let $u \neq 0$ be such that $(I_m - BA)u = 0$, that is, $u = BAu$. Now

$$(I_n - AB)Au = Au - ABAu = Au - Au = 0.$$

   Moreover, $Au \neq 0$ since $u = BAu$ and $u \neq 0$. Thus $(I_n - AB)x = 0$ has a non trivial solution, and hence the matrix $I_n - AB$ is not invertible.
   Similarly, if the matrix $I_n - AB$ is not invertible then the matrix $I_m - BA$ is not invertible.

   Alter. Suppose that the matrix $I_n - AB$ is invertible.
   Let $C = B(I_n - AB)^{-1}A + I_m$. Then $(I_m - BA)C = I_m$, and hence the matrix $I_m - BA$ is invertible. Similarly, if the matrix $I_m - BA$ is invertible then the matrix $I_n - AB$ is invertible.

Row & Column Spaces

8. Show that if A is a $m \times n$ and B is an $n \times p$ matrix then:

   (a) $col(AB) \subseteq col(A)$.

   (b) $row(AB) \subseteq row(B)$. If $m = n$ and A is invertible, what can you say in addition?

   Soln.

   (a) To show $col(AB) = \{ABx : x \in \mathbb{R}^p\} \subseteq col(A) = \{Ay : y \in \mathbb{R}^n\}$. Let $p \in col(AB)$ be arbitrary. Then $p = ABx$, for $x \in \mathbb{R}^p \implies p = Ay$ for $y = Bx \in \mathbb{R}^n \implies p \in col(A)$.

   (b) Note that $row(AB) = col((AB)^T) = col(B^T A^T) \subseteq col(B^T) = row(B)$.
   If $m = n$ and A is invertible then $row(AB) = row(B)$.

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9. Let A be an $m \times n$ matrix with entries in $\mathbb{R}$. Let T be the corresponding linear transformation. Then find out the domain and codomain of T.

   Soln. Here the linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is given by $T(x) = Ax$, for $x \in \mathbb{R}^n$. So the domain of T is $\mathbb{R}^n$ and codomain is $\mathbb{R}^m$.

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10. Show that in $\mathbb{R}^2$, the rotation by 90° is a linear transformation.

    Soln. The rotation (by 90°) map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by

    $$T([x \quad y]^T) = [-y \quad x]^T.$$

    Consider $\mathbf{u} = [x_1 \quad y_1]^T$ and $\mathbf{v} = [x_2 \quad y_2]^T$. Then

    $$T(\mathbf{u}+\mathbf{v}) = T([x_1 + x_2 \quad y_1 + y_2]^T) = [-(y_1 + y_2) \quad (x_1 + x_2)]^T = [-y_1 \quad x_1]^T + [-y_2 \quad x_2]^T = T(\mathbf{u}) + T(\mathbf{v}).$$

    Now let us consider $\mathbf{v} = [x \quad y]^T$ and c be a scalar. Then

    $$T(c\mathbf{v}) = T([cx \quad cy]^T) = [-cy \quad cx]^T = c[-y \quad x]^T = cT(\mathbf{v}).$$

    (**DONE IN BOOK.**)

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11. Examine whether the following maps $T : V \longrightarrow W$ are linear transformations.

    (a) $V = W = \mathbb{R}^3, \quad T([x \quad y \quad z]^T) = [3x + y \quad z \quad |x|]^T$.

    (b) $V = W = \mathbb{R}^2, \quad T$ is the reflection in the line $y = -x$.

    (c) $V = W = \mathbb{R}^3, \quad T([x \quad y \quad z]^T) = [x - y + 5 \quad z^2 \quad xyz]^T$.

    (d) $V = \mathbb{R}^3, \quad W = \mathbb{R}^2, \quad T([x \quad y \quad z]^T) = [x - y + z \quad 2z - 3y + x]^T$.

    (e) $V = W = \mathbb{R}^2, \quad T$ is the projection onto Y-axis.

Soln.

(a) T is not a linear transformation. Consider $\mathbf{u} = [1 \quad 0 \quad 0]^T$ and $\mathbf{v} = [-1 \quad 0 \quad 0]^T$. Then

$$T(\mathbf{u}+\mathbf{v}) = [0 \quad 0 \quad 0]^T$$

but

$$T(\mathbf{u}) + T(\mathbf{v}) = [0 \quad 0 \quad 2]^T \neq T(\mathbf{u}+\mathbf{v}).$$

(b) Note that T is the reflection in the line $y = -x$, i.e, $T([x \quad y]^T) = [-y \quad -x]^T$. Here T is a linear transformation. Consider $\mathbf{u} = [x_1 \quad y_1]^T$ and $\mathbf{v} = [x_2 \quad y_2]^T$. Then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) = T([x_1+x_2 \quad y_1+y_2]^T) &= [-(y_1+y_2) \quad -(x_1+x_2)]^T \\ &= [-y_1 \quad -x_1]^T + [-y_2 \quad -x_2]^T \\ &= T(\mathbf{u}) + T(\mathbf{v}). \end{aligned}$$

Now let us consider $\mathbf{v} = [x \quad y]^T$ and c be a scalar. Then

$$T(c\mathbf{v}) = T([cx \quad cy]^T) = [-cy \quad -cx]^T = c[-y \quad -x]^T = cT(\mathbf{v}).$$

(c) This is not a linear transformation. Consider $\mathbf{v} = [0 \quad 0 \quad 1]^T$ and c be a scalar. Then

$$T(c\mathbf{v}) = T([0 \quad 0 \quad c]^T) = [5 \quad c^2 \quad 0]^T \neq cT(\mathbf{v}).$$

(d) T is a linear transformation. Consider $\mathbf{u} = [x_1 \quad y_1 \quad z_1]^T$ and $\mathbf{v} = [x_2 \quad y_2 \quad z_2]^T$. Then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T([x_1+x_2 \quad y_1+y_2 \quad z_1+z_2]^T) \\ &= [(x_1+x_2)-(y_1+y_2)+(z_1+z_2) \quad 2(z_1+z_2)-3(y_1+y_2)+(x_1+x_2)]^T \\ &= [x_1-y_1+z_1 \quad 2z_1-3y_1+x_1]^T + [x_2-y_2+z_2 \quad 2z_2-3y_2+x_2]^T \\ &= T(\mathbf{u}) + T(\mathbf{v}). \end{aligned}$$

Now let us consider $\mathbf{v} = [x \quad y \quad z]^T$ and c be a scalar. Then

$$T(c\mathbf{v}) = T([cx \quad cy \quad cz]^T) = [cx-cy+cz \quad 2cz-3cy+cx]^T = c[x-y+z \quad 2z-3y+x]^T = cT(\mathbf{v}).$$

(e) Note that T is the projection onto Y-axis, i.e,

$$T([x \quad y]^T) = [0 \quad y]^T.$$

Here T is a linear transformation. Consider $\mathbf{u} = [x_1 \quad y_1]^T$ and $\mathbf{v} = [x_2 \quad y_2]^T$. Then

$$T(\mathbf{u}+\mathbf{v}) = T([x_1+x_2 \quad y_1+y_2]^T) = [0 \quad (y_1+y_2)]^T = [0 \quad y_1]^T + [0 \quad y_2]^T = T(\mathbf{u}) + T(\mathbf{v}).$$

Now let us consider $\mathbf{v} = [x \quad y]^T$ and c be a scalar. Then

$$T(c\mathbf{v}) = T([cx \quad cy]^T) = [0 \quad cy]^T = c[0 \quad y]^T = cT(\mathbf{v}).$$

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12. In the previous exercise, if the map T is a linear transformation, then compute its *standard matrix.*

Soln. The matrices are as follows :

(a) NOT a Linear Transformation.

(b) $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(c) NOT a Linear Transformation.

(d) $[T] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$.

(e) $[T] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
