MA101 Mathematics I

July-November, 2013

Tutorial & Additional Problem Set - 3

Notation

LI Linearly Independent.

$\setminus$ Asymmetric set difference. $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

SECTION - A (for Tutorial - 3)

1. True or False? Give justifications.

   (a) Given an $m \times n$ matrix A, there can exist $\mathbf{b}$ and $\mathbf{b}'$ such that $A\mathbf{x} = \mathbf{b}$ has a unique solution but $A\mathbf{x} = \mathbf{b}'$ has infinitely many solutions.

   (b) If $\mathbf{x}, \mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$ with $\mathbf{x}^T\mathbf{y} = 0$, then $\mathbf{x}$ and $\mathbf{y}$ are LI.

   (c) If for three distinct subsets S_1, S_2 and S_3 of $\mathbb{R}^n$, $span(S_1 \cup S_2) = span(S_1 \cup S_3)$, then $span(S_2) = span(S_3)$.

   (d) If $\tilde{A}$ is the RREF of A then the column spaces of A and $\tilde{A}$ are equal.

> **Solution:**
>
> (a) False. Suppose $S_H = \{x | A\mathbf{x} = \mathbf{0}\}$. If $A\mathbf{x} = \mathbf{b}$ is consistent, then $S = \{x | A\mathbf{x} = \mathbf{b}\} = S_H + x$, for some $x \in S$. So, if $A\mathbf{x} = \mathbf{b}$ has a unique solution, then $A\mathbf{x} = \mathbf{b}'$ has a unique solution.
>
> (b) True. Suppose $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ are LD. Then, $\mathbf{x} = c\mathbf{y}$ for some $c \neq 0$. Then, $\mathbf{x}^T\mathbf{y} = c\mathbf{y}^T\mathbf{y} = 0$ implies $\mathbf{y} = \mathbf{0}$.
>
> (c) False. For example, take $n = 2$, $S_1 = \{[1, 0]^T\}$, $S_2 = \{[1, 1]^T\}$ and $S_3 = \{[1, 2]^T\}$.
>
> (d) False. For example, take $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, the RREF of A is $\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

2. Consider the linear system with the augmented matrix $[A|\mathbf{b}]$ as given below:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 2 \\ 5 & 6 & 7 & 8 & 5 \\ 9 & 10 & 11 & 12 & 8 \end{array}\right].$$

   (a) Find all the solutions of the system.

   (b) Find $\mathbf{b}'$ such that $A\mathbf{x} = \mathbf{b}'$ does not have a solution.

   (c) By changing exactly one entry of A, find an A' such that $A'\mathbf{x} = \mathbf{b}'$ will be consistent for all $\mathbf{b}' \in \mathbb{R}^3$.

Solution:

(a) Solution set is $\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{4} \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} | \alpha, \beta \in \mathbb{R} \right\}$.

(b) Note that in A, $R_3 = 2R_2 - R_1$, where R_i is the ith row of A. For any $\mathbf{b}'$ such that $b_3' \neq 2b_2' - b_1'$, $A\mathbf{x} = \mathbf{b}'$ will have no solution.

(c) Since $R_3 = 2R_2 - R_1$, no two rows of A are LD. If you take A' obtained by changing any one entry of A, then the rows of the new A' will be LI, i.e., $rank(A') = 3$. Thus, $A'\mathbf{x} = \mathbf{b}'$ will be consistent for every $\mathbf{b}'$.

3. Check whether the set $S = \left\{ \begin{bmatrix} 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is LI.

Solution: Yes, it is LI.

4. Consider $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 = 2x_3 + x_2 \right\}$.

(a) Show that S is a subspace of $\mathbb{R}^3$.

(b) Find $\{\mathbf{u}, \mathbf{v}\}$ such that $span\{\mathbf{u}, \mathbf{v}\} = S$.

(c) Find a $\mathbf{v}'$ such that $span\{\mathbf{u}, \mathbf{v}'\} = span\{\mathbf{v}, \mathbf{v}'\} = S$.

(d) Find an $\mathbf{u}'$ such that $span\{\mathbf{u}', \mathbf{v}'\}$ is not a subspace of S. Geometrically what will be the picture of S and $span\{\mathbf{u}', \mathbf{v}'\}$?

Solution:

(a) Being the solution set of a homogeneous system, S is a subspace.

(b) Solving the system we have $S = \left\{ \alpha \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} | \alpha, \beta \in \mathbb{R} \right\}$. One choice of $\{\mathbf{u}, \mathbf{v}\}$ can be $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(c) Take any $\mathbf{v}' \in S$ but not in $span\{\mathbf{u}\}$ or $span\{\mathbf{v}\}$. For example take $\mathbf{v}' = \mathbf{u} + \mathbf{v}$.

(d) Take any $\mathbf{u}'$ not in S. Then $span\{\mathbf{u}', \mathbf{v}'\}$ will correspond to a plane in $\mathbb{R}^3$ and will intersect the plane associated with S in a line given by $span\{\mathbf{v}'\}$.

5. Let S be a subspace of $\mathbb{R}^4$ and $\mathbf{x}, \mathbf{y} \in S$ are LI.
   (a) Show that if $\mathbf{u} \in \mathbb{R}^4 \setminus S$ then $\{\mathbf{x}, \mathbf{y}, \mathbf{u}\}$ is LI.
   (b) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4 \setminus S$ are LI then does it imply that $\{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$ is LI?

Solution:

(a) Suppose, if possible, $\{\mathbf{x}, \mathbf{y}, \mathbf{u}\}$ is LI. Then there exists c_1, c_2, c_3, not all zeros, such that $c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{u} = \mathbf{0}$. Now, $c_3 \neq 0$, otherwise it will contradict that $\mathbf{x}, \mathbf{y} \in S$ are LI. But then $\mathbf{u} = (-\frac{c_1}{c_3})\mathbf{x} + (-\frac{c_2}{c_3})\mathbf{y}$. This contradicts that $\mathbf{u} \in \mathbb{R}^4 \setminus S$.

(b) No. For example take $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $S = span\{\mathbf{x}, \mathbf{y}\}$.

6. Show that for any matrix $\text{row}(A^TA) = \text{row}(A)$. Moreover, if A is square, then show that A^TA and A are row equivalent.

Solution: Note that $A\mathbf{x} = \mathbf{0}$ and $A^TA\mathbf{x} = \mathbf{0}$ are equivalent: [$A\mathbf{x} = \mathbf{0} \Rightarrow A^TA\mathbf{x} = \mathbf{0}$. $A^TA\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x}^TA^TA\mathbf{x} = \mathbf{y}^T\mathbf{y} = 0$, which implies $\mathbf{y} = A\mathbf{x} = \mathbf{0}$.] Thus,

$$\dim(\text{row}(A^TA)) = \text{rank}(A^TA) = \text{rank}(A) = \dim(\text{row}(A)). \quad (1)$$

Further, any row of A^TA is a linear combination of rows of A. Thus, $\text{row}(A^TA) \subseteq \text{row}(A)$. Take any basis for $\text{row}(A^TA)$, which must span $\text{row}(A)$, otherwise (1) will be contradicted.

The second part is a corollary of the fact that $A\mathbf{x} = \mathbf{0}$ and $A^TA\mathbf{x} = \mathbf{0}$ are equivalent and that A^TA and A are of same order.

7. Show that the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ are LI if and only if $P\mathbf{x}, P\mathbf{y}, P\mathbf{z}$ are LI for any $n \times n$ invertible matrix P.

Solution: For any c_1, c_2, c_3, $c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z} = \mathbf{0}$ implies $c_1P\mathbf{x} + c_2P\mathbf{y} + c_3P\mathbf{z} = P(c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z}) = \mathbf{0}$. Conversely $c_1P\mathbf{x} + c_2P\mathbf{y} + c_3P\mathbf{z}) = P(c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z}) = \mathbf{0}$ implies $c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z} = \mathbf{0}$, (since for an invertible P, $P\mathbf{x} = \mathbf{0}$ implies has only the trivial solution). Hence the result follows.

8. If the RREF $\tilde{A}$, of a 5×5 matrix A has the 1^{st}, 3^{rd} and the 5^{th} columns as the only leading columns, then

(a) Find two LI solutions of $A\mathbf{x} = \mathbf{0}$.

(b) If $\mathbf{a}_i$ is the ith column of A, show that $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_5$ are LI and span the column space of A.

(c) Can the sets $\{\mathbf{a}_1, \mathbf{a}_2\}$, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ and $\{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ be LI?

Solution:

(a) $\mathbf{u} = \begin{bmatrix} -\tilde{a}_{12} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -\tilde{a}_{14} \\ 0 \\ -\tilde{a}_{24} \\ 1 \\ 0 \end{bmatrix}$, where $\tilde{a}_{ij}$ is the (i,j)th entry of $\tilde{A}$.

(b) That the columns $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_5$ are LI follows from problem 7.
By inspection one can check that $\tilde{\mathbf{a}}_2 = \tilde{a}_{12}\tilde{\mathbf{a}}_1$ and $\tilde{\mathbf{a}}_4 = \tilde{a}_{14}\tilde{\mathbf{a}}_1 + \tilde{a}_{24}\tilde{\mathbf{a}}_2$. Hence again by the result of problem 7, $\mathbf{a}_2 = \tilde{a}_{12}\mathbf{a}_1$ and $\mathbf{a}_4 = \tilde{a}_{14}\mathbf{a}_1 + \tilde{a}_{24}\mathbf{a}_2$ and the result follows.

(c) The sets $\{\mathbf{a}_1, \mathbf{a}_2\}$, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ will not be LI follows from part (b). $\{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ will be LI if $\tilde{a}_{14} \neq \mathbf{0}$.

SECTION - B: ADDITIONAL PROBLEMS

(No solutions provided. These are take home exercises.)

9. True or False? Give justifications.

(a) If $\{\mathbf{x}, \mathbf{y}\}$ and $\{\mathbf{u}, \mathbf{v}\}$ are two different LI subsets of $\mathbb{R}^2$, then $\{\mathbf{x}, \mathbf{u}\}$ and $\{\mathbf{y}, \mathbf{v}\}$ are also LI sets.

(b) If $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\}$ is LI in $\mathbb{R}^3$ then $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\}$ is LI in $\mathbb{R}^2$.

(c) If S is a subspace of $\mathbb{R}^n$ then $\mathbf{x} + S$ is a subspace if and only if $\mathbf{x} \in S$.

(d) If the diagonal entries of a 4×4 upper triangular matrix A are $1, 2, 3$ and 4 then $S_1 = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = 2\mathbf{x}\}$ is a subspace of $\mathbb{R}^4$ but $S_2 = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = 5\mathbf{x}\}$ is not.

Solution:

(a) False.

(b) False.

(c) True.

(d) False. Both will be subspaces, S_2 will be a $\{\mathbf{0}\}$ subspace.

10. By using Gauss Jordan elimination find the inverse of the matrix
$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 12 \end{bmatrix}.$$

11. Using LU factorization of the matrix A solve the system of linear equations with the augmented matrix $[A|\mathbf{b}]$ as given below:
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 1 & 2 & 3 & 4 & 30 \\ 1 & 4 & 8 & 15 & 93 \\ 1 & 3 & 6 & 10 & 65 \end{array}\right].$$

12. Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 = 2x_3 - x_2,\ 2x_2 = x_3 \right\}$ is a subspace of $\mathbb{R}^3$.
Find an $\mathbf{u}$ such that $span\{\mathbf{u}\} = S$. Find an $\mathbf{u}'$ such that $span\{\mathbf{u}, \mathbf{u}'\}$ gives a plane in $\mathbb{R}^3$. Find a $\mathbf{v}$ such that $span\{\mathbf{v}\}$ is not a subspace of $span\{\mathbf{u}, \mathbf{u}'\}$. What will be the $span\{\mathbf{u}, \mathbf{u}', \mathbf{v}\}$?

13. Give four LI sets of three vectors each, in $\mathbb{R}^3$.

14. Let $S = \left\{ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} a \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \right\}$. Find the values of a for which $span(S) \neq \mathbb{R}^3$.

15. If any of the diagonal entries of a 3×3 upper triangular matrix is zero, then show that the columns are linearly dependent. Hint: Look at the similar problem in tutorial 2.