DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Guwahati

MA101: Mathematics I, July - November, 2014

Tutorial Sheet: LA - 5

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1. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Prove that

   (a) $|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$ (**Cauchy-Schwarz** inequality);

   (b) $|\mathbf{x}.\mathbf{y}| = \|\mathbf{x}\| \, \|\mathbf{y}\|$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent;

   (c) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

2. Let $V = \{[x, y, z, w]^t \in \mathbb{C}^4 : x = z + iw, y = z - w\}$. Show that V is a subspace of $\mathbb{C}^4$. Find a basis for each of V and $V^{\perp}$.

3. Let A be a 2×2 orthogonal matrix. Show that there exists a real number θ such that $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. In the first case, A rotates the vectors of $\mathbb{R}^2$ by the angle θ counterclockwise, and in the second case, A reflects the vectors of $\mathbb{R}^2$ about a line; in this case find the line.

4. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Show that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{x}.\mathbf{y} = \sum_{i=1}^{n} (\mathbf{x}.\mathbf{v}_i)(\mathbf{v}_i.\mathbf{y}) \quad \text{and} \quad \mathbf{x}.\mathbf{x} \geq \sum_{i=1}^{k} |(\mathbf{x}.\mathbf{v}_i)|^2 \quad \text{for} \ \ 1 \leq k \leq n.$$

   Further, show that $\mathbf{x}.\mathbf{x} = \sum\limits_{i=1}^{k} |(\mathbf{x}.\mathbf{v}_i)|^2$ iff $\mathbf{x} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$, where $1 \leq k \leq n$.

5. Let W be a subspace of $\mathbb{C}^n$ and let $\mathbf{u} \in \mathbb{C}^n$. Show that $\mathbf{v} \in W$ is the projection of $\mathbf{u}$ onto W if and only if $\|\mathbf{u} - \mathbf{v}\| \leq \|\mathbf{u} - \mathbf{w}\|$ for every $\mathbf{w} \in W$.

6. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthogonal set of non-zero vectors in $\mathbb{C}^n$. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ be some vectors in $\mathbb{C}^n$ that are orthogonal to S. If $p + q > n$ then show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ are linearly dependent.

7. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, n \geq 2$ such that $\|\mathbf{v}\| = \|\mathbf{w}\| = 1$. Prove that there exists an orthogonal matrix A such that $A(\mathbf{v}) = \mathbf{w}$ and $\det(A) = 1$.

Tutorial Sheet: LA - 6

1. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a linearly independent set of a vector space V. Write $\mathbf{w}_1 = \mathbf{v}_1$ and $\mathbf{w}_k = \sum\limits_{i=1}^{k} c_{ik}\mathbf{v}_i$ for $2 \leq k \leq n$ such that $c_{ik} > 0$ for all $1 \leq i \leq n, 2 \leq k \leq n$. Show that the set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is also linearly independent.

2. Consider the vector space $\mathbb{R}_4[x]$ of all polynomials with real coefficients of degree at most 4. Show that $\{p(x) \in \mathbb{R}_4[x] : p(1) = p(-1) = 0\}$ is a subspace of $\mathbb{R}_4[x]$. Determine a basis and the dimension of this subspace.

3. Let $M_n(\mathbb{R})$ denote the vector space of all $n \times n$ real matrices. Show that the set $S_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A = A^t\}$ is a subspace of $M_n(\mathbb{R})$. Also, find a basis and the dimension of $S_n(\mathbb{R})$.

4. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for a vector space V. Show that the set $\{\mathbf{x}_1+\mathbf{x}_2, \mathbf{x}_2+\mathbf{x}_3, \ldots, \mathbf{x}_n+\mathbf{x}_1\}$ is also a basis for V if and only if n is an odd integer.

5. Let W_1 and W_2 be two subspaces of a finite dimensional vector space V. Show that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

6. Let V be a finite dimensional vector space and let V_1 and V_2 be two subspaces of V. If $\dim(V_1 + V_2) = \dim(V_1 \cap V_2) + 1$, show that either $V_1 + V_2 = V_1, V_1 \cap V_2 = V_2$ or $V_1 + V_2 = V_2, V_1 \cap V_2 = V_1$. (Equivalently, for subspaces V_1 and V_2 of V, if neither contains the other, then
$$\dim(V_1 + V_2) \geq \dim(V_1 \cap V_2) + 2.)$$

7. Find the coordinates of the vector $[1, 2, 3]^t$ with respect to the bases B and C for $\mathbb{R}^3$, where
$$B = \{[1, 1, 0]^t, [0, 1, 1]^t, [1, 0, 1]^t\} \text{ and } C = \{[1, 1, 1]^t, [1, 1, -1]^t, [1, -1, 1]^t\}.$$
Also, find the matrix P such that $[\mathbf{x}]_B = P[\mathbf{x}]_C$ for all $\mathbf{x} \in \mathbb{R}^3$.
