Indian Institute of Technology Guwahati
Department of Mathematics

MA 101 – Mathematics-I **Date:** 31-Aug-2015
Tutorial Sheet-4 **Time:** 08:00 – 09:00

Linear Algebra

Topics Covered:
Linear transformations, determinants, Cramer's rule, eigenvalues-eigenvectors, similarity of matrices, diagonalization.

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1. If a matrix A is *idempotent*, i.e. if $A^2 = A$, then find all possible values of $\det(A)$.
2. If a matrix A is *nilpotent*, i.e. if $A^n = 0$ for some $n \in \mathbb{N}$, then find all possible values of $\det(A)$.
3. For an $n \times n$ matrix A, show that
$$\det(adj(A)) = \det(A)^{n-1}.$$
4. Let A be a square matrix such that A can be partitioned as $A = \begin{bmatrix} P & \mid & Q \\ R & \mid & S \end{bmatrix}$, where P, Q, R and S are square matrices. Then is the following statement true:
$$\det(A) = \det(P)\det(S) - \det(Q)\det(R).$$
Justify your argument.
5. Prove that the range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is equal the column space of its standard matrix $[T]$.
6. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then show that the eigenvalues of A are the solutions of the equation $\lambda^2 - tr(A)\lambda + \det A = 0$, where $tr(A)$ is the sum of the entries on the main diagonal of A.
Express the trace and determinant of A in terms of eigenvalues of A. Can you generalize this for an $n \times n$ matrix?
7. For each of the following matrix, compute the characteristc polynomial, eigenvalues, basis for the eigenspaces corresponding to each eigenvalue, algebraic and geometric multiplicity.
(a) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 2 \\ 3 & 0 & 1 \end{bmatrix}$
8. Let A, B be square matrices. Then prove or disprove (using counter example) the following statements:
   (a) If λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda + \mu$ is an eigenvalue of $A + B$.
   (b) If λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB.
   (c) If $v \in \mathbb{R}^n$ is such that $Av = \lambda v$ and $Bv = \mu v$, then $\lambda + \mu$ is an eigenvalue of $A + B$ and $\lambda\mu$ is an eigenvalue of AB.
9. If $A \sim B$ then show that $A^T \sim B^T$.
10. In the following, check whether the matrices A and B are similar. If yes, find the matrix P such that $B = P^{-1}AP$.
    (a) $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
    (b) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
    (c) $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$ and B is a diagonal matrix.
11. If A, B are similar matrices, then show that the geometric multiplicities of the eigenvalues of A and B are same.
12. If A is an $n \times n$ diagonalizable matrix whose eigenvalues are 0 & 1, then for each $k \in \mathbb{N}$, compute A^k.