DEPARTMENT OF MATHEMATICS, IIT Guwahati

MA101: Mathematics I, July - November 2014

Solutions of Tutorial Sheet: LA - 1

1. Supply two examples each and explain their geometrical meaning.

   (a) Two linear equations in two variables with exactly one solution.

   (b) Two linear equations in two variables with infinitely many solutions.

   (c) Two linear equations in two variables with no solutions.

   (d) Three linear equations in two variables with exactly one solution.

   (e) Three linear equations in two variables with no solutions.

   Solution:

   (a) Take $x + y = 2, \quad x - y = 0$. They represent two lines in $\mathbb{R}^2$ intersecting at a point.

   (b) Take $x + y = 2, \quad 2x + 2y = 4$. They represent the same line in $\mathbb{R}^2$.

   (c) Take $x + y = 2, \quad 2x + 2y = 1$. They represent two parallel lines in $\mathbb{R}^2$, no intersection.

   (d) Take $x + y = 2, \quad x - y = 0, \quad 2x - y = 1$. They represent three lines in $\mathbb{R}^2$ with a single point in common.

   (e) Take $x + y = 2, \quad x - y = 0, \quad 2x + 2y = 1$. They represent three lines in $\mathbb{R}^2$ with no point in common. $\square$

2. For what values of $\lambda \in \mathbb{R}$, the following system of equations has (i) no solution, (ii) a unique solution, and (iii) infinitely many solutions?

$$(5-\lambda)x + 4y + 2z = 4, \quad 4x + (5-\lambda)y + 2z = 4, \quad 2x + 2y + (2-\lambda)z = 2.$$

   Also, find the solutions whenever they exist.

   Solution: The augmented matrix of the given system of equations is

$$\left[\begin{array}{ccc|c} 5-\lambda & 4 & 2 & 4 \\ 4 & 5-\lambda & 2 & 4 \\ 2 & 2 & 2-\lambda & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 2 & 2 & 2-\lambda & 2 \\ 4 & 5-\lambda & 2 & 4 \\ 5-\lambda & 4 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 2 & 2 & 2-\lambda & 2 \\ 0 & 1-\lambda & -2+2\lambda & 0 \\ 5-\lambda & 4 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2}(5-\lambda)R_1} \left[\begin{array}{ccc|c} 2 & 2 & 2-\lambda & 2 \\ 0 & 1-\lambda & -2+2\lambda & 0 \\ 0 & \lambda-1 & 2-\frac{1}{2}(2-\lambda)(5-\lambda) & \lambda-1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 2 & 2-\lambda & 2 \\ 0 & 1-\lambda & -2+2\lambda & 0 \\ 0 & 0 & -\frac{1}{2}(\lambda^2 - 11\lambda + 10) & \lambda-1 \end{array}\right].$$

   (i) **No Solution:** In this case, the rank of the coefficient matrix $\neq$ the rank of the augmented matrix. Thus $\lambda^2 - 11\lambda + 10 = 0$ and $\lambda - 1 \neq 0$. That is, if $\lambda = 10$ then the given system has no solution.

   (ii) **Unique Solution:** In this case, the rank of the coefficient matrix = the rank of the augmented matrix = 3. Thus $\lambda^2 - 11\lambda + 10 \neq 0$. That is, if $\lambda \neq 1, 10$ then the given system has a solution. The unique solution is given by $\left[\frac{4}{10-\lambda}, \frac{4}{10-\lambda}, \frac{2}{10-\lambda}\right]$.

   (iii) **Infinitely Many Solutions:** In this case, the rank of the coefficient matrix = the rank of the augmented matrix < 3. In this case, $\lambda^2 - 11\lambda + 10 = 0$ and $\lambda - 1 = 0$, that is, $\lambda = 1$. Thus, if $\lambda = 1$ then the given system has infinitely many solutions. The solution set is given by

$$\left\{ \left[1 - s - \frac{t}{2}, \ s, \ t\right]^t : s, t \in \mathbb{R} \right\}.$$

$\square$

3. Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

Solution: Let A be an $m \times n$ matrix, where $m \geq 2$. Let the i-th and j-th rows of A be $\mathbf{u}$ and $\mathbf{v}$, respectively.

Let application of $R_j \to R_j + R_i$ on A transform A into A_1, application of $R_i \to R_i + (-1)R_j$ on A_1 transform A_1 into A_2, application of $R_j \to R_j + R_i$ on A_2 transform A_2 into A_3, and application of $R_i \to (-1)R_i$ on A_3 transform A_3 into A_4. Then the i-th and j-th rows of A_1 are $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$, respectively. The i-th and j-th rows of A_2 are $-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$, respectively. The i-th and j-th rows of A_3 are $-\mathbf{v}$ and $\mathbf{u}$, respectively. Finally, the i-th and j-th rows of A_4 are $\mathbf{v}$ and $\mathbf{u}$, respectively.

Thus, we see that the i-th and the j-th row of a matrix A can be interchanged by applying the sequence $R_j \to R_j + R_i, \;\; R_i \to R_i + (-1)R_j, \;\; R_j \to R_j + R_i$ and $R_i \to (-1)R_i$ of elementary row operations on A. □

4. Let A be an $n \times n$ matrix. If the system $A^2\mathbf{x} = \mathbf{0}$ has a non-trivial solution then show that the system $A\mathbf{x} = \mathbf{0}$ also has a non-trivial solution.

Solution: Assume that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then for $\mathbf{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} A^2\mathbf{y} = \mathbf{0} &\Rightarrow A(A\mathbf{y}) = \mathbf{0} \\ &\Rightarrow A\mathbf{y} = \mathbf{0}, \;\; \text{since the system } A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution} \\ &\Rightarrow \mathbf{y} = \mathbf{0}, \;\; \text{since the system } A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution} \\ &\Rightarrow \text{the system } A^2\mathbf{x} = \mathbf{0} \text{ has no non-trivial solution.} \end{aligned}$$

Aliter: Let $\mathbf{y}$ be a non-trivial solution of $A^2\mathbf{x} = \mathbf{0}$. Then we have $A^2\mathbf{y} = \mathbf{0}$. Now if $A\mathbf{y} = \mathbf{0}$ then $\mathbf{y}$ is a non-trivial solution of $A\mathbf{x} = \mathbf{0}$. If $A\mathbf{y} \neq \mathbf{0}$ then $A\mathbf{y}$ is a non-trivial solution of $A\mathbf{x} = \mathbf{0}$. □

5. Let A and B be two 2×3 matrices that are in reduced row echelon form. If the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set then show that $A = B$.

Solution: There are only 7 possible 2×3 matrices that are in reduced row echelon form. These are

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \;\; A_2 = \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}, \;\; A_3 = \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}, \;\; A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}, \;\; A_6 = \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_7 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } a, b \in \mathbb{R}.$$

Let $\mathbf{x} = [x, y, z]^t \in \mathbb{R}^3$. Then we have

$$\begin{aligned} &A_1\mathbf{x} = \mathbf{0} \Rightarrow x, y \text{ and } z \text{ are arbitrary;} \\ &A_2\mathbf{x} = \mathbf{0} \Rightarrow x = -ay - bz, \text{ where } y \text{ and } z \text{ are arbitrary;} \\ &A_3\mathbf{x} = \mathbf{0} \Rightarrow y = -az, \text{ where } x \text{ and } z \text{ are arbitrary;} \\ &A_4\mathbf{x} = \mathbf{0} \Rightarrow z = 0, \text{ where } x \text{ and } y \text{ are arbitrary;} \\ &A_5\mathbf{x} = \mathbf{0} \Rightarrow x = -az, y = -bz, \text{ where } z \text{ is arbitrary;} \\ &A_6\mathbf{x} = \mathbf{0} \Rightarrow x = -ay, z = 0, \text{ where } y \text{ is arbitrary; and} \\ &A_7\mathbf{x} = \mathbf{0} \Rightarrow y = z = 0, \text{ where } x \text{ is arbitrary.} \end{aligned}$$

We see that the above seven system of equations have seven distinct solution sets. Hence none of the above seven system of equations are equivalent. So, if A and B are two 2×3 matrices that are in reduced row echelon form and if the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are equivalent, then A must be equal to B. □

6. Prove or disprove: There exist two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of some consistent non-homogeneous system $A\mathbf{x} = \mathbf{b}$ such that $\mathbf{x}_1 + \mathbf{x}_2$ is also a solution of $A\mathbf{x} = \mathbf{b}$.

Solution: Consider a consistent non-homogeneous system $A\mathbf{x} = \mathbf{b}$. Here we have $\mathbf{b} \neq \mathbf{0}$. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two solutions of $A\mathbf{x} = \mathbf{b}$ so that $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$. Then we have $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b} + \mathbf{b} \neq \mathbf{b}$, since $\mathbf{b} \neq \mathbf{0}$. Hence $\mathbf{x}_1 + \mathbf{x}_2$ can never be a solution of $A\mathbf{x} = \mathbf{b}$. Thus the given statement is disproved. □

7. Prove or disprove: If two matrices of the same order have the same rank then they must be row equivalent.

Solution: Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Here A and B have the same rank but they are not row equivalent (why?). Thus the given statement is disproved by this counterexample. □