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MA101 MATHEMATICS-I

Solutions to Tutorial - 5

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1. Show that any two similar matrices have the same trace.

Solution: Let A and B are similar matrices. Therefore, $\exists$ an invertible matrix P such that

$$P^{-1}AP = B.$$

We know that for any two $n \times n$ matrix X, Y

$$tr(XY) = tr(YX).$$

Therefore,

$$tr(B) = tr(P^{-1}AP) = tr(AP^{-1}P) = tr(A).$$

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Alter: Since similar matrices have the same characteristic polynomial, so all the eigenvalues are equal for both A and B. Suppose $\{\lambda_1, \cdots, \lambda_n\}$(not necessarily distinct) be the set of eigenvalues for both A and B. Then we have

$$tr(A) = \sum_i \lambda_i = tr(B).$$

2. Let A be an invertible matrix. Prove that if A is diagonalizable, then so is A^{-1}.

Solution: Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ (need not be all distinct). Since A is invertible, so $\lambda_i \neq 0 \ \forall \ i$ and hence λ_i^{-1} exists for all i. A is diagonalizable $\Rightarrow \exists$ a non-singular matrix P such that

$$\begin{aligned} & P^{-1}AP = D = \text{diag}\{\lambda_1, \cdots, \lambda_n\} \\ \Rightarrow \ & (P^{-1}AP)^{-1} = D^{-1} = \text{diag}\{\lambda_1{}^{-1}, \cdots, \lambda_n{}^{-1}\} \\ \Rightarrow \ & P^{-1}A^{-1}P = D^{-1}. \end{aligned}$$

So A^{-1} is similar to a diagonal matrix D^{-1} i.e. A^{-1} is diagonalizable.

3. Let A be a diagonalizable matrix such that characteristic polynomial of A has only one root. Then find out the diagonal matrix D such that $A \sim D$. Is such a matrix D unique ?

Solution: Since A is diagonalizable so there exists an invertible matrix P such that

$$P^{-1}AP = D = \text{diag}\{\lambda_1, \cdots, \lambda_n\}.$$

Also since the characteristic polynomial has only one root (λ^* say), so

$$P^{-1}AP = D = \text{diag}\{\lambda^*, \cdots, \lambda^*\} = \lambda^* I_n,$$

which is unique. (assuming A to be an $n \times n$ matrix)
(Note that, $A = \lambda^* I_n$. Such matrix is called **Scalar Matrix**.)

4. With the help of diagonalization, calculate A^{2015} where

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

Solution: On solving the characteristic equation, we get the eigenvalues of A to be: $0, 0$ and -2. Also the eigenvectors corresponding to these eigenvalues are respectively

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

So A is diagonalizable and for $P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ we have

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \implies A = PDP^{-1} \Rightarrow A^{2015} = PD^{2015}P^{-1}.$$

Since $P^{-1} = \begin{bmatrix} \frac{3}{2} & 1 & -\frac{3}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$, so we have

$$A^{2015} = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} P^{-1} = \begin{bmatrix} -2^{2014} & 0 & 2^{2014} \\ 3 \cdot 2^{2014} & 0 & -3 \cdot 2^{2014} \\ 2^{2014} & 0 & -2^{2014} \end{bmatrix}.$$

5. Let A be an $n \times n$ matrix and let $P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ be the characteristic polynomial of A. Then show that $P(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_0 I_n$ is the zero matrix if

   (a) A is a diagonal matrix.

   (b) A is a diagonalizable matrix.

   [*Cayley Hamilton theorem states, this statement holds for any square matrix.*]

Solution:

(a) Suppose $A = diag\{a_1, \cdots, a_n\}$ is a diagonal matrix with characteristic polynomial $P(\lambda)$. Then a_i (for $i = 1, \cdots, n$) being roots of the characteristic polynomial, $P(a_i) = 0$ for all $1 \leq i \leq n$. Therefore we have

$$P(A) = diag\{P(a_1), \cdots, P(a_n)\} = diag\{0, \cdots, 0\} = \mathbf{0}.$$

(b) Suppose A is diagonalizable. Then $A \sim D$, where $D = diag\{\lambda_1, \cdots, \lambda_n\}$ is a diagonal matrix. Therefore $\exists$ a non-singular matrix X such that $X^{-1}AX = D \Rightarrow A = XDX^{-1}$.
But since similar matrices have the same characteristic polynomial, and the same set of eigenvalues therefore

$$P(A) = P(XDX^{-1}) = XP(D)X^{-1} = \mathbf{0}.$$

(Note that, $A^k = (XDX^{-1})^k = XD^kX^{-1}$ for all $0 \leq k \leq n$)

6. (a) For any $u, v \in \mathbb{R}^n$, show that $|u \cdot v| \leq \|u\|\|v\|$ (Cauchy Schwartz inequality).
   (b) For any $u, v \in \mathbb{R}^n$, show that $\|u + v\| \leq \|u\| + \|v\|$ (Triangle inequality).

Solution:

(a) Note that, if $\mathbf{u}, \mathbf{v}$ are scalar multiple of each other, i.e. $\mathbf{v} = \lambda \mathbf{u}$ (or vice versa), then

$$LHS = |\mathbf{u} \cdot \lambda \mathbf{u}| = |\lambda| \|\mathbf{u}\|^2 = \|\mathbf{u}\|\|\mathbf{v}\| = RHS, \tag{1}$$

and we are done.
Suppose $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Let $t \in \mathbb{R}$. Consider

$$\begin{aligned} 0 < \langle t\mathbf{u} + \mathbf{v}, t\mathbf{u} + \mathbf{v} \rangle &= t^2 \langle \mathbf{u}, \mathbf{u} \rangle + 2t \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= t^2 \|\mathbf{u}\|^2 + 2t \langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \end{aligned}$$

Since $\mathbf{u}, \mathbf{v}$ are linearly independent, the quadratic equation in t given by

$$t^2 \|\mathbf{u}\|^2 + 2t \langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 = 0$$

cannot have a real root, which implies

$$4 \langle \mathbf{u}, \mathbf{v} \rangle^2 - 4 \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 < 0 \implies |\mathbf{u} \cdot \mathbf{v}| < \|\mathbf{u}\|\|\mathbf{v}\|. \tag{2}$$

From (1) and (2) the result follows.

(b) We will use Cauchy Schwartz inequality to prove this part.

$$\begin{aligned} (\|u\| + \|v\|)^2 &= \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \\ &\geq \|u\|^2 + \|v\|^2 + 2|u \cdot v| \quad \text{(Cauchy Schwartz inequality)} \\ &\geq \|u\|^2 + \|v\|^2 + 2(u \cdot v) \\ &= \|u + v\|^2 \\ \Rightarrow \|u\| + \|v\| &\geq \|u + v\|. \qquad (\because \text{ both quantities are non-negative}) \end{aligned}$$

7. Let A be a real symmetric matrix.

   (a) Show that all the eigenvalues of A are real.

   (b) Show that any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Solution: A is real & symmetric $\Rightarrow A^* = \overline{A}^T = A$.

(a) Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{R}^{n \times n}$. A is real and symmetric i.e. $A^* = A$. Therefore,

$$\begin{aligned} Au = \lambda u &\Rightarrow (Au)^* = \lambda^* u^* \\ &\Rightarrow u^* A^* = \lambda^* u^* \Rightarrow u^* A = \lambda^* u^* \\ &\Rightarrow (u^* A)u = (\lambda^* u^*)u \Rightarrow u^*(\lambda u) = \lambda^* \|u\| \qquad \text{(since } Au = \lambda u\text{)} \\ &\Rightarrow \lambda \|u\| = \lambda^* \|u\| \\ &\Rightarrow \lambda = \lambda^* \qquad \text{(since } u \text{ is non-zero)} \\ &\Rightarrow \lambda \in \mathbb{R}. \end{aligned}$$

(b) Let λ_1, λ_2 $(\lambda_1 \neq \lambda_2)$ are two eigenvalues of A and the corresponding eigenvectors are u_1, u_2 respectively. Then

$$Au_1 = \lambda_1 u_1 \qquad Au_2 = \lambda_2 u_2.$$

Now,

$$\begin{aligned}
\lambda_1 u_2^T u_1 &= u_2^T(Au_1) = (u_2^T A)u_1 \\
&= (A^T u_2)^T u_1 = (Au_2)^T u_1 \quad (\text{since } A^T = A) \\
&= \lambda_2 u_2^T u_1 \\
\Rightarrow \lambda_1 u_2^T u_1 &= \lambda_2 u_2^T u_1 \\
\Rightarrow (\lambda_1 - \lambda_2) u_2^T u_1 &= 0 \Rightarrow u_2^T u_1 = 0.
\end{aligned}$$

This completes the proof. □

8. If A and B are $n \times n$ matrices with n distinct eigenvalues. Then show that $AB = BA$ if and only if A and B have the same eigenvectors.

Solution: Suppose, A has distinct eigenvalues $\lambda_1, \cdots, \lambda_n$ and the corresponding eigenvectors $v_1, \cdots, v_n$ respectively. Also let B has distinct eigenvalues $\mu_1, \cdots, \mu_n$ and the corresponding eigenvectors $u_1, \cdots, u_n$ respectively. Also suppose $AB = BA$. Then we have

$$\begin{aligned}
ABu_i &= A(\mu_i u_i) \\
\Rightarrow BAu_i &= \mu_i(Au_i) \qquad (\because AB = BA)
\end{aligned}$$

This implies that, if $Au_i \neq 0$ then Au_i is an eigenvector of B corresponding to the eigenvalue μ_i.
Otherwise if $Au_i = 0$ then, u_i is an eigenvector of A corresponding to the eigenvalue 0.
But since all the μ_i's are distinct, so dimension of each eigenspace is 1 and therefore

$$Au_i = cu_i \quad \text{for some } c \in \{\lambda_1, \cdots, \lambda_n\}.$$

Hence u_i is an eigenvector of A.
Conversely, suppose A and B both have same set of eigenvectors $v_1, \cdots, v_n$ i.e.

$$Av_i = \lambda_i v_i \qquad Bv_i = \mu_i v_i \qquad \text{for } i = 1, \cdots, n$$

where all λ_i's are distinct and also all μ_i's are distinct. Then we have

$$(AB)v_i = A(\mu_i v_i) = \mu_i(Av_i) = \mu_i \lambda_i v_i = \lambda_i(\mu_i v_i) = \lambda_i B v_i = B(\lambda_i v_i) = (BA)v_i,$$

for all $i = 1, \cdots, n$.
This implies $AB = BA$.

9. Find an orthogonal basis for $\mathbb{R}^4$ containing the vectors: $\mathbf{v_1} = [1 \quad -1 \quad 1 \quad -1]^T$ and $\mathbf{v_2} = [1 \quad 1 \quad 1 \quad 1]^T$.
Remark 1. A set of orthogonal vectors can be extended to a basis.

Solution: Let us first extend this set to a basis of $\mathbb{R}^4$. Then by Gram-Schmidt process, we can find a orthogonal basis.
Clearly, $\mathbf{v_3} = [1 \quad 0 \quad 0 \quad 0]^T$ and $\mathbf{v_4} = [0 \quad 1 \quad 0 \quad 0]^T$ extend the given set of vectors to a basis of

$\mathbb{R}^4$.
By Gram-Schmidt process, we get

$$\begin{aligned}
\mathbf{u_1} &= \mathbf{v_1} \\
\mathbf{u_2} &= \mathbf{v_2} - \frac{\langle \mathbf{u_1}, \mathbf{v_2} \rangle}{\|\mathbf{u_1}\|^2}\mathbf{u_1} = \mathbf{v_2} \qquad (\because \langle \mathbf{u_1}, \mathbf{v_2} \rangle = 0) \\
\mathbf{u_3} &= \mathbf{v_3} - \frac{\langle \mathbf{u_1}, \mathbf{v_3} \rangle}{\|\mathbf{u_1}\|^2}\mathbf{u_1} - \frac{\langle \mathbf{u_2}, \mathbf{v_3} \rangle}{\|\mathbf{u_2}\|^2}\mathbf{u_2} \\
&= \mathbf{v_3} - \frac{1}{4}\mathbf{u_1} - \frac{1}{4}\mathbf{u_2} \quad = \quad \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{bmatrix}^T \\
\mathbf{u_4} &= \mathbf{v_4} - \frac{\langle \mathbf{u_1}, \mathbf{v_4} \rangle}{\|\mathbf{u_1}\|^2}\mathbf{u_1} - \frac{\langle \mathbf{u_2}, \mathbf{v_4} \rangle}{\|\mathbf{u_2}\|^2}\mathbf{u_2} - \frac{\langle \mathbf{u_3}, \mathbf{v_4} \rangle}{\|\mathbf{u_3}\|^2}\mathbf{u_3} \\
&= \mathbf{v_4} + \frac{1}{4}\mathbf{u_1} - \frac{1}{4}\mathbf{u_2} - 0\ \mathbf{u_3} \quad = \quad \begin{bmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}^T
\end{aligned}$$

Then $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ is a required basis.

10. Let W be the row space of the matrix

$$\begin{bmatrix} 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 3 & 1 & 3 & 1 & 3 \\ 1 & 3 & 1 & 3 & 1 \\ 1 & 4 & 1 & 4 & 1 \end{bmatrix}.$$

Compute $W^\perp$ and the orthogonal decomposition of the vector $\mathbf{v} = [1 \quad 2 \quad 3 \quad 4 \quad 5]$ with respect to W.

Solution: Since $W = row(A)$, so $W^\perp = null(A)$. Therefore

$$\begin{bmatrix} 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 3 & 1 & 3 & 1 & 3 \\ 1 & 3 & 1 & 3 & 1 \\ 1 & 4 & 1 & 4 & 1 \end{bmatrix} \begin{matrix} R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 - R_2 \\ R_5 \leftarrow R_5 - R_2 \end{matrix} \begin{bmatrix} 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix} \begin{matrix} R_1 \leftarrow R_1 - 2R_3 \\ R_2 \leftarrow R_2 - R_3 \\ R_5 \leftarrow R_5 - 2R_4 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_4 \leftarrow R_4 - R_1 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \leftrightarrow R_1 \\ R_1 \leftrightarrow R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore x_3, x_4, x_5 are free variables. Let

$$x_3 = s, \quad x_4 = t, \quad x_5 = r.$$

Therefore $x_1 = -x_3 - x_5 = -s - r$, and $x_2 = -x_4 = -t$.
Hence we have

$$W^\perp = \left\{ \begin{bmatrix} -s-r \\ -t \\ s \\ t \\ r \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = su_1 + tu_2 + ru_3 \colon s, t, r \in \mathbb{R} \right\}$$

Here $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ forms a basis of $W^\perp$. We need to have an orthogonal basis. First two are already orthogonal. By Gram-Schmidt process, we get

$$\mathbf{u_3}' = \mathbf{u_3} - \frac{\langle \mathbf{u_1}, \mathbf{u_3} \rangle}{\|\mathbf{u_1}\|^2}\mathbf{u_1} - \frac{\langle \mathbf{u_2}, \mathbf{u_3} \rangle}{\|\mathbf{u_2}\|^2}\mathbf{u_2} = \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}^T$$

By the orthogonal decomposition theorem, we can write

$$\mathbf{v} = \mathrm{proj}_{W^\perp}(\mathbf{v}) + \mathrm{perp}_{W^\perp}(\mathbf{v}).$$

Since an orthogonal basis of $W^\perp$ is given by $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3'}$, hence

$$\begin{aligned} \mathrm{proj}_{W^\perp}(\mathbf{v}) &= \mathrm{proj}_{\mathbf{u_1}}(\mathbf{v}) + \mathrm{proj}_{\mathbf{u_2}}(\mathbf{v}) + \mathrm{proj}_{\mathbf{u_3'}}(\mathbf{v}) \\ &= \frac{\langle \mathbf{u_1}, \mathbf{v} \rangle}{\|\mathbf{u_1}\|^2}\mathbf{u_1} + \frac{\langle \mathbf{u_2}, \mathbf{v} \rangle}{\|\mathbf{u_2}\|^2}\mathbf{u_2} + \frac{\langle \mathbf{u_3'}, \mathbf{v} \rangle}{\|\mathbf{u_3'}\|^2}\mathbf{u_3'} \\ &= (1)\mathbf{u_1} + (1)\mathbf{u_2} + (2)\mathbf{u_3'} \\ &= [-2 \quad -1 \quad 0 \quad 1 \quad 2]^T \\ \Rightarrow \mathrm{perp}_W(\mathbf{v}) &= \mathbf{v} - \mathrm{proj}_{W^\perp}(\mathbf{v}) \\ &= [3 \quad 3 \quad 3 \quad 3 \quad 3]^T. \end{aligned}$$

11. Let $S = \{\mathbf{v_1}, \cdots, \mathbf{v_k}\}$ be an orthonormal set in $\mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$ be a vector. Then show that

$$\|\mathbf{x}\|^2 \geq |\mathbf{x} \cdot \mathbf{v_1}|^2 + |\mathbf{x} \cdot \mathbf{v_2}|^2 + \cdots + |\mathbf{x} \cdot \mathbf{v_k}|^2.$$

Also show that the above becomes an equality if and only if $x \in Span(S)$.

Solution: Since every orthonormal set in $\mathbb{R}^n$ can be extended to a basis of $\mathbb{R}^n$, therefore let $\{\mathbf{v_1}, \cdots, \mathbf{v_k}, \mathbf{u_{k+1}}, \cdots, \mathbf{u_n}\}$ be a orthogonal basis of $\mathbb{R}^n$. (This is possible by Gram-Schmidt process)
Then for any $\mathbf{x} \in \mathbb{R}^n$ we can write

$$\begin{aligned} \mathbf{x} &= \mathrm{proj}_{\mathbf{v_1}}(\mathbf{x}) + \cdots + \mathrm{proj}_{\mathbf{v_k}}(\mathbf{x}) + \mathrm{proj}_{\mathbf{u_{k+1}}}(\mathbf{x}) + \cdots + \mathrm{proj}_{\mathbf{u_n}}(\mathbf{x}) \\ &= \frac{\langle \mathbf{v_1}, \mathbf{x} \rangle}{\|\mathbf{v_1}\|^2}\mathbf{v_1} + \cdots + \frac{\langle \mathbf{v_k}, \mathbf{x} \rangle}{\|\mathbf{v_k}\|^2}\mathbf{v_k} + \frac{\langle \mathbf{u_{k+1}}, \mathbf{x} \rangle}{\|\mathbf{u_{k+1}}\|^2}\mathbf{u_{k+1}} + \cdots + \frac{\langle \mathbf{u_n}, \mathbf{x} \rangle}{\|\mathbf{u_n}\|^2}\mathbf{u_n} \\ &= \langle \mathbf{v_1}, \mathbf{x} \rangle \mathbf{v_1} + \cdots + \langle \mathbf{v_k}, \mathbf{x} \rangle \mathbf{v_k} + \frac{\langle \mathbf{u_{k+1}}, \mathbf{x} \rangle}{\|\mathbf{u_{k+1}}\|^2}\mathbf{u_{k+1}} + \cdots + \frac{\langle \mathbf{u_n}, \mathbf{x} \rangle}{\|\mathbf{u_n}\|^2}\mathbf{u_n} \\ &\qquad \text{(since } S \text{ is an orthonormal set, so } \|\mathbf{v_i}\| = 1 \text{ for all } 1 \leq i \leq n) \\ \Rightarrow \|\mathbf{x}\|^2 &= |\mathbf{x} \cdot \mathbf{v_1}|^2 + |\mathbf{x} \cdot \mathbf{v_2}|^2 + \cdots + |\mathbf{x} \cdot \mathbf{v_k}|^2 + \underbrace{|\mathbf{x} \cdot \mathbf{u_{k+1}}|^2 + \cdots + |\mathbf{x} \cdot \mathbf{u_n}|^2}_{(\geq 0)} \\ &\geq |\mathbf{x} \cdot \mathbf{v_1}|^2 + |\mathbf{x} \cdot \mathbf{v_2}|^2 + \cdots + |\mathbf{x} \cdot \mathbf{v_k}|^2. \end{aligned}$$

This completes the proof.

12. Let A be a 2×2 orthogonal matrix. Show that there exists a real number θ such that

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

In the first case, A rotates the vectors of $\mathbb{R}^2$ by the angle θ counter-clockwise, and in the second case, A reflects the vectors of $\mathbb{R}^2$ about a line; in this case find the line.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since A is orthogonal, therefore

$$AA^T = I_2 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow a^2 + b^2 = 1 = c^2 + d^2, \; ac + bd = 0.$$

Take $a = \cos\theta, b = \sin\theta$ and $c = \cos\phi, d = \sin\phi$.
From this $ac + bd = 0 \Rightarrow \cos\theta\cos\phi + \sin\theta\sin\phi = 0 \Rightarrow \cos(\theta - \phi) = 0 \Rightarrow \theta - \phi = \frac{\pi}{2}, -\frac{\pi}{2}$.
Case 1: $\theta - \phi = -\frac{\pi}{2} \Rightarrow \phi = (\frac{\pi}{2} + \theta)$. In this case,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \cos\phi & \sin\phi \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

which rotates the vectors of $\mathbb{R}^2$ by the angle θ counter-clockwise.
Case 2: $\theta - \phi = \frac{\pi}{2} \Rightarrow \phi = -(\frac{\pi}{2} - \theta)$. In this case,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \cos\phi & \sin\phi \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$

which reflects the vectors of $\mathbb{R}^2$ about a line L.
Let us determine equation the line L. Note that any point $(x, y) \in L$ if and only if $A[x \quad y]^T = [x \quad y]^T$. Therefore L is precisely the null space of

$$A - I = \begin{bmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{bmatrix}.$$

$$[A - I|0] \quad \xrightarrow{REF} \quad \begin{bmatrix} \cos\theta - 1 & \sin\theta & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

Therefore $null(A - I) = \text{span}([\cot\frac{\theta}{2}, 1]^T)$, which gives the equation of the line

$$L : y = \tan\frac{\theta}{2}x.$$