Indian Institute of Technology Guwahati
Department of Mathematics

MA 101 – Mathematics-I (2017-18) Tutorial Sheet – Extra

Topics Covered:
Vector spaces and subspaces, Linear independence, basis, dimension, Linear transformation, kernel, range, rank-nullity theorem, matrix of a linear transformation.
Corresponding sections in the textbook [Poole]: 6.1, 6.2, 6.3, 6.4, 6.5

Note: If the field of scalars is not mentioned for a vector space, you may assume the same to be $\mathbb{R}$.
Recall:

- $\mathscr{F}$ is the set of all real valued functions defined on the real line.
- $M_{r,c}(\mathbb{F})$ denotes the set of all $r \times c$ matrices with entries from $\mathbb{F}$.
- $\mathscr{P}_n(\mathbb{F})$ denotes the set of all polynomials of degree atmost n, with coefficients in $\mathbb{F}$.

1. In each of the following, determine whether the given set is a vector space, as specified.
   (a) The set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} x \\ x \end{bmatrix}$, with the usual vector addition and scalar multiplication, where scalars are the real numbers.
   (b) The set of all vectors in $\mathbb{C}^2$ of the form $\begin{bmatrix} z \\ \bar{z} \end{bmatrix}$, with the usual vector addition and scalar multiplication, where scalars are the complex numbers.
   (c) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x \geq y \right\}$, with the usual vector addition and scalar multiplication, where scalars are the real numbers.
   (d) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R}, ad = 0 \right\}$, with the usual matrix addition and scalar multiplication, where scalars are the real numbers.
   (e) The set of all vectors in $\mathbb{Z}_2^n$ with even number of 1s, with the usual vector addition and scalar multiplication, where scalars are from $\mathbb{Z}_2$.

2. $\mathbf{H} = \left\{ a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R} \text{ such that } \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \right\}$. Is it possible to define operations on $\mathbf{H}$ so that $\mathbf{H}$ becomes a vector space over $\mathbb{R}$.

3. In each of the following, determine whether W is a subspace of V.
   (a) $V = M_{2,2}$ and $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; ad \geq bc \right\}$.
   (b) $V = M_{r,r}$ and W be the set of invertible $r \times r$ matrices.
   (c) $V = \mathscr{F}$ and $W = \{f \in \mathscr{F} \mid f(0) = 0\}$.
   (d) $V = \mathscr{F}$ and $W = \{f \in \mathscr{F} \mid f(1) = 0\}$.
   (e) $V = \mathscr{F}$ and $W = \{f \in \mathscr{F} \mid f(0) = 1\}$.
   (f) $V = \mathscr{F}$ and W be the set of all continuous real valued functions on $\mathbb{R}$.
   (g) $V = \mathscr{F}$ and W be the set of all differentiable real valued functions on $\mathbb{R}$.
   (h) $V = \mathscr{F}$ and W be the set of all integrable real valued functions on $\mathbb{R}$.
   (i) $\mathbf{H} = \left\{ a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R} \text{ such that } \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \right\}$. Is it possible to define the operations on $\mathbf{H}$ so that $\mathbf{H}$ becomes a vector space over $\mathbb{R}$.
   (j) $V = \mathbb{C}$ and $W = \mathbb{R}$ (both trated as a vector spaces over $\mathbb{R}$.)

4. (a) Prove that zero element of a vector space is unique.
   (b) Prove that for every vector v in a vector space V, there is a unique $v' \in V$ such that $v + v' = \mathbf{0}$. For such v, v', is it possible that $v + v' = \mathbf{0}$ but $v' + v \neq \mathbf{0}$?

5. Check whether the following sets are linear independent in the given vector space.
   (a) $\{x, 1 + x\}$ in $\mathscr{P}_1$.
   (b) $\{x, 2x - x^2, 3x + 2x^2\}$ in $\mathscr{P}_2$.

(c) $\{x, 2x - x^2, 3x + 2x^2\}$ in $\mathscr{P}_3$, in $\mathscr{P}_n$ for $n \geq 2$.

(d) $\{1, \sin x, \cos x\}$ in $\mathscr{F}$.

(e) $\{\sin x, \sin 2x, \sin 3x\}$ in $\mathscr{F}$.

6. Let $\mathcal{B}$ be a set of vectors in a vector space V with the property that every vector in V is a unique linear combination of vectors in $\mathcal{B}$.

   Then prove or disprove that $\mathcal{B}$ is a basis for V.

7. For each of the following vector spaces, find the dimension and give a basis.

   (a) $\{p(x) \in \mathscr{P}_2 \mid p(1) = 0\}$

   (b) $\{A \in M_{2,2} \mid A \text{ is upper triangular }\}$

   (c) $\{A \in M_{3,3} \mid A \text{ is diagonal }\}$

   (d) $\mathbb{C}^2$ as a vector space over $\mathbb{R}$.

8. Determine whether each of the following functions is a linear transformation.

   (a) $T : M_{2,2} \to M_{2,2}$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix}$.

   (b) $T : M_{2,2} \to \mathbb{R}$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = abcd$.

   (c) $T : M_{2,2} \to \mathbb{R}$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

   (d) Let $B \in M_{n,n}$ be a matrix. $T : M_{n,n} \longrightarrow M_{n,n}$ defined by $T(A) = AB$.

   (e) Let $B \in M_{n,n}$ be a matrix. $T : M_{n,n} \longrightarrow M_{n,n}$ defined by $T(A) = A + B$.

   (f) $T : \mathscr{P}_2 \longrightarrow \mathscr{P}_2$ defined by $T(a + bx + cx^2) = (a+1) + (b+1)x + (c+1)x^2$.

9. Let $T : \mathscr{P}_2 \longrightarrow \mathscr{P}_2$ be a linear transformation for which $T(1+x) = 1 + x^2$, $T(x + x^2) = x - x^2$ and $T(1 + x^2) = 1 + x + x^2$. Find $T(4 - x + 3x^2)$ and $T(a + bx + cx^2)$.

10. Let V and W be vector spaces over the same set of scalars. Prove that the set $\mathscr{L}(V, W)$ of all linear transformations is a vector space, with the vector addition and scalar multiplication as defined over $\mathscr{F}$.

11. Describe $\ker(T)$ and $\text{range}(T)$ of the following linear transformations and compute a basis for each. Also determine the rank and the nullity of T and hence verify the *rank-nullity theorem*.

    (a) Let $T : M_{2,2} \longrightarrow M_{2,2}$ be the linear transformation defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

    (b) $T : M_{2,2} \longrightarrow \mathbb{R}$ defined by $T(A) = \text{tr}(A)$.

    (c) $T : M_{2,2} \longrightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - b \\ c - d \end{bmatrix}$.

    (d) $T : \mathscr{P}_2 \longrightarrow \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

    (e) $T : \mathbb{C} \to \mathbb{R}$ given by $T(2) = 2$ and $T(1 + i) = 1$. (Considering $\mathbb{C}$ as a vector space over $\mathbb{R}$.)

    (f) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(A) = AB - BA$, where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

12. Let T be a linear transformation on a vector space V. Let $\dim V = n \geq 1$. Suppose there exists $x \in V$ such that $T^{n-1}(x) \neq 0$ but $T^n(x) = 0$. Then show that the set $\{x, T(x), \ldots, T^{n-1}(x)\}$ is a basis for V. Also, find the matrix representation of T with respect to this basis. (What is the codomain of T?)

References

[Poole] David Poole, *Linear algebra : a modern introduction.* 3rd ed., Brooks/Cole, 2011.