DEPARTMENT OF MATHEMATICS, IIT Guwahati

MA101: Mathematics I, July - November 2014

Solutions of Tutorial Sheet: LA - 3

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1. Check, without calculating the actual value, whether the determinant of the following matrix is even or odd:

$$\begin{bmatrix} 2 & 2 & 5 & 4 \\ 5 & 3 & 2 & 2 \\ 2 & 5 & 7 & 5 \\ 1 & 3 & 9 & 2 \end{bmatrix}.$$

Solution: Note that an integer is even or odd according as the remainder, after division by 2, is 0 or 1. Replacing all the even numbers by 0 and the odd numbers by 1, the given matrix reduces to

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Adding the first two rows to the fourth row, and then replacing 2 by 0, we get

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

whose determinant is zero. Hence the determinant of the given matrix is **even**. $\square$

2. Find the determinant of the following matrices:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 2 & 3 & 4 & \dots & n \\ 3 & 3 & 3 & 4 & \dots & n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n & n & n & n & \dots & n \end{bmatrix}.$$

Solution: For $n \geq 2$, let A_n denote the first given matrix (this matrix is known as **Vandermonde** Matrix). We shall use method of induction on n to prove that $\det(A_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$. We can assume that all x_i's are distinct, as otherwise the result is obvious.

For $n = 2$, clearly $\det(A_2) = x_2 - x_1 = \prod_{1 \leq i < j \leq 2} (x_j - x_i)$. Assume that $\det(A_{k-1}) = \prod_{1 \leq i < j \leq (k-1)} (x_j - x_i)$, where $k \geq 3$. Now we have

$$\begin{aligned} \det(A_k) &= \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{k-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_k & x_k^2 & \dots & x_k^{k-1} \end{vmatrix} \\ &= \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{k-1} \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 & \dots & x_2^{k-1} - x_1^{k-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & x_k - x_1 & x_k^2 - x_1^2 & \dots & x_k^{k-1} - x_1^{k-1} \end{vmatrix} \quad (\text{applying } R_i \to R_i - R_1 \text{ for each } i \geq 2) \\ &= \begin{vmatrix} x_2 - x_1 & x_2^2 - x_1^2 & \dots & x_2^{k-1} - x_1^{k-1} \\ \dots & \dots & \dots & \dots \\ x_k - x_1 & x_k^2 - x_1^2 & \dots & x_k^{k-1} - x_1^{k-1} \end{vmatrix} \quad (\text{expanding through the first column}) \end{aligned}$$

$$= (x_2 - x_1)\ldots(x_k - x_1)\begin{vmatrix} 1 & x_2 + x_1 & \ldots & x_2^{k-2} + x_2^{k-3}x_1 + \ldots + x_1^{k-2} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & x_k + x_1 & \ldots & x_k^{k-2} + x_k^{k-3}x_1 + \ldots + x_1^{k-2} \end{vmatrix}$$

$$= (x_2 - x_1)\ldots(x_k - x_1)\begin{vmatrix} 1 & x_2 & \ldots & x_2^{k-2} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & x_k & \ldots & x_k^{k-2} \end{vmatrix} \quad \text{(applying } C_i \to C_i - x_1C_{i-1}, \quad \text{for each } i \geq 2)$$

$$= (x_2 - x_1)\ldots(x_k - x_1)\prod_{2\leq i<j\leq k}(x_j - x_i) \quad \text{(by induction hypothesis)}$$

$$= \prod_{1\leq i<j\leq k}(x_j - x_i).$$

Hence, by induction we conlude that $\det(A_n) = \prod\limits_{1\leq i<j\leq n}(x_j - x_i)$.

Let the second given matrix be B. Applying the sequence $R_n \to R_n - R_{n-1}, R_{n-1} \to R_{n-1} - R_{n-2}, \ldots, R_2 \to R_2 - R_1$ of elementary row operations on B, the matrix B is transformed to

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & 1 & \ldots & 0 \end{bmatrix}.$$

Let B_{1n} be the matrix obtained by deleting the first row and the last column of the above matrix. Then B_{1n} will be a lower triangular matrix with all diagonal entries equal to 1.
Hence $\det(B) = (-1)^{1+n}n.\det(B_{1n}) = (-1)^{n+1}n$. □

3. Let A be an $n \times m$ matrix and let B be an $m \times n$ matrix. Prove that the matrix $I_m - BA$ is invertible if and only if the matrix $I_n - AB$ is invertible.

Solution: We shall prove that the matrix $I_m - BA$ is not invertible if and only if the matrix $I_n - AB$ is not invertible.

Let the matrix $I_m - BA$ be not invertible, so that $(I_m - BA)\mathbf{x} = \mathbf{0}$ has a non-trivial solution. Let $\mathbf{u}(\neq \mathbf{0})$ be such that $(I_m - BA)\mathbf{u} = \mathbf{0}$, that is, $\mathbf{u} = BA\mathbf{u}$. Now

$$\mathbf{u} = BA\mathbf{u} \Rightarrow A\mathbf{u} = ABA\mathbf{u} \Rightarrow (I_n - AB)A\mathbf{u} = \mathbf{0}.$$

Moreover, $A\mathbf{u} \neq \mathbf{0}$ since $\mathbf{u} = BA\mathbf{u}$ and $\mathbf{u} \neq \mathbf{0}$. Thus $(I_n - AB)\mathbf{x} = \mathbf{0}$ has a non-trivial solution, and hence the matrix $I_n - AB$ is not invertible.

Similarly, if the matrix $I_n - AB$ is not invertible then the matrix $I_m - BA$ is also not invertible.

Aliter

Suppose that $I_n - AB$ is invertible. Let $C = B(I_n - AB)^{-1}A + I_m$. Then $(I_m - BA)C = I_m$, and hence $I_m - BA$ is invertible. Similarly, if the matrix $I_m - BA$ is invertible then the matrix $I_n - AB$ is also invertible. □

4. Let A be a nilpotent matrix (*i.e.*, $A^m = \mathbf{O}$ for some $m \geq 1$). Show that the matrix $I + A$ is invertible.

Solution: Take $B = I - A + A^2 - A^3 + \ldots + (-1)^{m-1}A^{m-1}$. Then we see that

$$\begin{aligned}(I + A)B &= (I + A)[I - A + A^2 - A^3 + \ldots + (-1)^{m-1}A^{m-1}] \\ &= [I - A + A^2 - A^3 + \ldots + (-1)^{m-1}A^{m-1}] + \\ &\qquad [A - A^2 + A^3 - A^4 + \ldots + (-1)^{m-2}A^{m-1}], \text{ since } A^m = \mathbf{O} \\ &= I.\end{aligned}$$

Hence $I + A$ is invertible, and $B = I - A + A^2 - A^3 + \ldots + (-1)^{m-1}A^{m-1}$ is the inverse of $I + A$.

Aliter

Consider the system $(I + A)\mathbf{x} = \mathbf{0}$, *i.e.,* $\mathbf{x} + A\mathbf{x} = \mathbf{0}$. Then we get $\mathbf{x} = -A\mathbf{x}$, and so

$$\mathbf{x} = -A\mathbf{x}, \quad A\mathbf{x} = -A^2\mathbf{x}, \quad A^2\mathbf{x} = -A^3\mathbf{x}, \quad \ldots, \quad A^{m-2}\mathbf{x} = -A^{m-1}\mathbf{x}, \quad A^{m-1}\mathbf{x} = -A^m\mathbf{x} = \mathbf{O}, \quad \text{as } A^m = \mathbf{O}.$$

This gives that $\mathbf{x} = \mathbf{0}$. Thus the system $(I + A)\mathbf{x} = \mathbf{0}$ has only the trivial solution, and hence the matrix $I + A$ is invertible. □

5. Let A and B be two $n \times n$ matrices. If AB is invertible then show that each of the matrices A and B must be invertible, in at least three different ways.

   Solution 1: Use $\det(AB) = \det(A)\det(B)$.

   Solution 2: Use $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

   Solution 3: Let AB be invertible and let C be the inverse of AB. Then we have $(AB)C = I \Rightarrow A(BC) = I \Rightarrow A$ is invertible. Again, $C(AB) = I \Rightarrow (CA)B = I \Rightarrow B$ is invertible.

   Solution 4: Let $\mathbf{x} \in \mathbb{R}^n$. Then we have,

$$\begin{aligned} B\mathbf{x} = \mathbf{0} &\Rightarrow AB\mathbf{x} = A\mathbf{0} = \mathbf{0} \\ &\Rightarrow \mathbf{x} = \mathbf{0}, \quad \text{since } AB \text{ is invertible} \end{aligned}$$

   Thus the system $B\mathbf{x} = \mathbf{0}$ has only the trivial solution, and hence the matrix B is invertible. Finally, since product of two invertible matrices is invertible, the matrix $A = (AB)B^{-1}$ is also invertible. □

6. Let A be an $n \times n$ skew-symmetric matrix. Prove that $\mathbf{x}^t A\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Further, prove that the matrix $I_n + A$ is invertible.

   Solution: Let $A = [a_{ij}]$ and $\mathbf{x} = [x_1, x_2, \ldots, x_n]^t \in \mathbb{R}^n$. Then we have

$$\mathbf{x}^t A\mathbf{x} = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{1 \leq i < j \leq n} (a_{ij} + a_{ji})x_i x_j.$$

   If A is an skew-symmetric matrix then we have $a_{ii} = 0$ and $a_{ij} = -a_{ji}$. Hence $\mathbf{x}^t A\mathbf{x} = 0$.

   Let $\mathbf{x} \in \mathbb{R}^n$. Then we have

$$(I_n + A)\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x}^t(I_n + A)\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x}^t\mathbf{x} + \mathbf{x}^t A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x}^t\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}.$$

   Thus the system $(I_n + A)\mathbf{x} = \mathbf{0}$ has only the trivial solution, and hence the matrix $I_n + A$ is invertible. □

7. Let $A = [a_{ij}]$ be an invertible matrix and let $B = [k^{i-j}a_{ij}]$, where $k(\neq 0) \in \mathbb{R}$. Find B^{-1} and $\det(B)$.

   Solution: We notice that

$$B = \begin{bmatrix} k & 0 & \ldots & 0 \\ 0 & k^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & k^n \end{bmatrix} A \begin{bmatrix} k^{-1} & 0 & \ldots & 0 \\ 0 & k^{-2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & k^{-n} \end{bmatrix}.$$

   Hence

$$\begin{aligned} B^{-1} &= \begin{bmatrix} k^{-1} & 0 & \ldots & 0 \\ 0 & k^{-2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & k^{-n} \end{bmatrix}^{-1} A^{-1} \begin{bmatrix} k & 0 & \ldots & 0 \\ 0 & k^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & k^n \end{bmatrix}^{-1} \\ &= \begin{bmatrix} k & 0 & \ldots & 0 \\ 0 & k^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & k^n \end{bmatrix} A^{-1} \begin{bmatrix} k^{-1} & 0 & \ldots & 0 \\ 0 & k^{-2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & k^{-n} \end{bmatrix}. \end{aligned}$$

   Thus, if $A^{-1} = [x_{ij}]$ then $B^{-1} = [k^{i-j}x_{ij}]$.

   We have

$$\det(B) = \begin{vmatrix} k & 0 & \ldots & 0 \\ 0 & k^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & k^n \end{vmatrix} \det(A) \begin{vmatrix} k^{-1} & 0 & \ldots & 0 \\ 0 & k^{-2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & k^{-n} \end{vmatrix} = k^{\frac{n(n+1)}{2}}.\det(A).k^{-\frac{n(n+1)}{2}} = \det(A).$$

□