DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Guwahati

Tutorial and practice problems on Single Variable Calculus

MA-101 : Mathematics-I | Tutorial Problem Set - 12 | November 13, 2013

PART-A (Tutorial)

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f''(c)$ exists, where $c \in \mathbb{R}$. Prove that

$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

Give an example of an $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$ for which $f''(c)$ does not exist but the above limit exists.

> **Solution:** Define $F(h) := f(c+h)+f(c-h)-2f(c)$. Then $F(h) \to 0$ and $\lim F'(h)/2h \to f''(c)$ as $h \to 0$. Hence by L'Hospital rule, $\lim_{h\to 0} F(h)/h^2 = f''(c)$.
>
> For the converse, consider $f(x) := x|x|$ for $x \in \mathbb{R}$. Then for $c := 0$, the given limit exists although $f''(0)$ does not exists (here $f'(x) = 2|x|$ for $x \in \mathbb{R}$).

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := x^3 + 2x + 1$. Show that $f^{-1} : \mathbb{R} \to \mathbb{R}$ exists. Assume that f^{-1} is differentiable and determine $(f^{-1})'(1), (f^{-1})'(4)$ and $(f^{-1})'(-2)$.

> **Solution:** Since $f'(x) = 3x^2 + 2 > 0$ for $x \in \mathbb{R}$, f is strictly increasing. Consequently, f^{-1} exists. By chain rule, we have $(f^{-1})'(f(x)) = 1/f'(x)$. Now $f(0) = 1, f(1) = 4, f(-1) = -1$ give the desired results.

3. Find the points of local extrema of the following functions on the specified domain:

   (i) $f(x) := x|x^2 - 12|$ for $-2 \le x \le 3$; (ii) $f(x) := 1 - (1-x)^{2/3}$ for $0 \le x \le 2$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := 2x^4 + x^4 \sin(1/x)$ for $x \neq 0$ and $f(0) := 0$. Show that f has a global minimum at $c := 0$, but $f'(x)$ takes both negative and positive values in every neighbourhood of 0.

> **Solution:** Note that $f(x) = x^4(2 + \sin(1/x)) \ge 0$ for all $x \in \mathbb{R}$. Hence f has global minimum at 0. Now for $x \neq 0$, we have $f'(x) = 8x^3 + 4x^3 \sin(1/x) - x^2 \cos(1/x)$. For $n \ge 2$, we have $f'(1/2n\pi) = 1/n^3\pi^3 + 0 - 1/4n^2\pi^2 < 0$ and $f'(2/(4n+1)\pi) > 0$. Hence the result follows.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) := 0$. Show that $f'(0) = 1$, but f is not monotone on any neighbourhood of 0.

> **Solution:** That $f'(0) = 1$ is easy to see. Now for $x \neq 0$, we have $f'(x) = 1 + 4x \sin(1/x) - 2\cos(1/x)$. Thus for $n \in \mathbb{N}$, we have $f'(1/2n\pi) = 1+0-2 < 0$ and $f'(1/(4n+1)\pi) = 1-0+2 > 0$. Hence the assertion follows.

6. Show that $|\sin(x) - \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)| < \frac{1}{5040}$ for $|x| \le 1$.

7. Derive the Taylor series of $f(x) := \log(1+x)$ at $x_0 = 0$ and determine the radius of convergence.

Solution: We have $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$ for $n \in \mathbb{N}$. The Taylor's series of f at 0 is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$. The radius of convergence of the Taylor series is 1, that is, the Taylor series converges for $|x| < 1$ and diverges for $|x| > 1$.

8. Suppose that $f \in R([a,b])$ and $P_n \in \mathcal{P}([a,b])$ is such that $\|P_n\| \to 0$ as $n \to \infty$. Show that $\lim_{n\to\infty} S(P_n, f) = \int_a^b f(t)dt$.

Solution: Choose $\epsilon > 0$. Then there exists $\delta > 0$ such that for any $P \in \mathcal{P}([a,b])$ with $\|P\| < \delta \Rightarrow |S(P,f) - \int_a^b f(t)dt| < \epsilon$. Since $\|P_n\| \to 0$ as $n \to \infty$, there exists $m \in \mathbb{N}$ such that $n \geq m \Rightarrow \|P_n\| < \delta \Rightarrow |S(P_n, f) - \int_a^b f(t)dt| < \epsilon$. Hence the result follows.

9. Define $f : [0,1] \to \mathbb{R}$ by $f(x) := 0$ if x is rational and $f(x) = 1/x$ if x is irrational. Show that $f \notin R([a,b])$. Show that there exists partitions $P_n \in \mathcal{P}([a,b])$ with $\|P_n\| \to 0$ as $n \to \infty$ such that $\lim_{n\to\infty} S(P_n, f)$ exists.

Solution: Since f is unbounded, $f \notin R([a,b])$. In the Riemann sum $S(P,f) := \sum f(c_i)(x_i - x_{i-1})$, choose c_i to be rational numbers. Then $S(P,f) = 0$. Hence the assertion follows.

10. Let $f : [a,b] \to \mathbb{R}$ be continuous. Show that there exists $c \in (a,b)$ such that $\int_a^b f(t)dt = (b-a)f(c)$. If $g : [a,b] \to \mathbb{R}$ is continuous and $\int_a^b f(t)dt = \int_a^b g(t)dt$ then show that $f(\lambda) = g(\lambda)$ for some $\lambda \in (a,b)$.

Solution: Let m and M be the global minimum and global maximum of f on $[a,b]$. Then $m(b-a) \leq \int_a^b f(t)dt \leq M(b-a) \Rightarrow m \leq (\int_a^b f(t)dt)/(b-a) \leq M$. Hence by the IVT, there exists $c \in (a,b)$ such that $f(c)(b-a) = \int_a^b f(t)dt$.

Next, set $h := f - g$. Then there exists $c \in (a,b)$ such that $h(c)(b-a) = \int_a^b h(t)dt = 0$. Consequently, $f(c) = g(c)$.

PART-B (Homework/Practice problems)

1. Let $f, g : [0,1] \to \mathbb{R}$ be given by $f(x) := x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, and $g(x) := x^2$. Then f and g are differentiable on $[0,1]$ and $g'(x) \neq 0$ on $(0,1)$. Show that $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$ and that $\lim_{x\to 0} f(x)/g(x)$ does not exist. Does this contradict L'Hospital rule?

   Next, consider $g(x) := \sin(x)$ and show that $\lim_{x\to 0} f(x)/g(x) = 0$ but $\lim_{x\to 0} f'(x)/g'(x)$ does not exist. Does this contradict L'Hospital rule?

2. Determine the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$, where a_n is given by:

   (i) $1/\log(n), n \geq 2$; (ii) $n^n/n!$; (iii) $(n!)^2/(2n)!$; (iv) $n^p/n!$.

Solution: The radius of convergence R is given by $1/R = \lim_{n\to\infty} |a_{n+1}|/|a_n| = \lim_{\infty} |a_n|^{1/n}$.

(i) By ratio test, we have $R = 1$.

(ii) Again by ratio test, we have $R = 1/e$.

(iii) By ratio test, we have $R = 4$.

(iv) The ratio test shows that $R = \infty$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := e^{-1/x^2}$ for $x \neq 0$ and $f(0) := 0$. Show (by induction) that $f^{(n)}(0) = 0$ for $n \in \mathbb{N}$. Is f represented by its Taylor series in a neighbourhood of 0? What is the moral of this example?

Solution: It can be shown by induction that $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$, where P_n is a polynomial of degree $3n$. It is easy to see that $f^{(n)}(x) \to 0$ and $\frac{f^{(n)}(x)}{x} \to 0$ as $x \to 0$. Hence $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. This shows that f cannot be represented by a power series in a neighbourhood of 0 even though f is infinitely differentiable.

Moral: The requirement that a function be represented by a power series (so called *real analytic* function) is more demanding than the smoothness requirement.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)\,dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Solution: Assume that $f(c) \neq 0$ for some $c \in (a, b)$, so that $f(c) > 0$. Since f is continuous at c, there exists $\delta > 0$ such that$|f(x) - f(c)| < \frac{1}{2}f(c)$ for all $x \in (c - \delta, c + \delta)$. This implies that $f(x) > \frac{1}{2}f(c)$ for all $x \in (c-\delta, c+\delta)$. So $\int\limits_a^b f(x)\,dx = \int\limits_a^{c-\delta/2} f(x)\,dx + \int\limits_{c-\delta/2}^{c+\delta/2} f(x)\,dx + \int\limits_{c+\delta/2}^{b} f(x)\,dx \geq \frac{1}{2}\delta f(c) > 0$, a contradiction. Almost similar arguments work if $c = a$ or $c = b$.

5. For each of the function f given below, determine the intervals on which f is increasing/decreasing. Also, determine the intervals of convexity/concavity, points of local extrma, and points of inflection.
(i) $f(x) := 2x^3 + 2x^2 - 2x - 1$; (ii) $f(x) := x^2/(x^2 + 1)$; (iii) $f(x) := 1 + 12|x| - 3x^2, x \in [-2, 5]$.
Give an example of a nonconstant function $f : (-1, 1) \to \mathbb{R}$ such that f has a local extremum (i.e. a maximum or a minimum) at 0 as well as a point of inflection at 0.

Solution: (i) Note that $f'(x) = 2(x + 1)(3x - 1)$. Thus $f'(x) > 0$ in $(-\infty, -1) \cup (1/3, \infty)$ so that f is strictly increasing in those intervals, and $f'(x) < 0$ in $(-1, 1/3)$ so that f is strictly decreasing in that interval. This shows that f has a local maximum at $x = -1$ and a local minimum at $x = 1/3$.

Since $f''(x) = 12x + 4$, we conclude that f is convex in $(-1/3, \infty)$ and concave in $(-\infty, -1/3)$ with a point of inflection at $x = -1/3$.

(ii) Since $f'(x) = 2x/(1 + x^2)^2$, we conclude that f is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$. Further, $f''(x) = -\frac{2(3x^2-1)}{(x^2+1)^3}$ implies that $f''(x) > 0$ if $|x| < 1/\sqrt{3}$, and $f''(x) < 0$ if $|x| > 1/\sqrt{3}$. Therefore, f is convex in $(-1/\sqrt{3},\, 1/\sqrt{3})$ and concave in $\mathbb{R} \setminus (-1/\sqrt{3},\, 1/\sqrt{3})$ with the points $x = \pm 1/\sqrt{3}$ being the points of inflection.

(iii) f is not differentiable at $x = 0$; $f(0) = 1$. Further $f'(x) = 0$ at $x = \pm 2$, $f'(x) < 0$ in $(-2, 0) \cup (2, 5]$, $f'(x) > 0$ in $(0, 2)$, and $f''(x) = -6$ in $(-2, 0) \cup (0, 5)$. Thus f is concave in $(-2, 0) \cup (0, 5)$, decreasing in $(-2, 0) \cup (2, 5)$, and increasing in $(0, 2)$. Further f has a global maximum at $x = \pm 2$.

Define $f : [-1, 1] \to \mathbb{R}$ be $f(x) := \begin{cases} -\sin(\pi x), & \text{if } x \in [-1, 0], \\ x^2, & \text{if } x \in [0, 1]. \end{cases}$ Then f is concave on $[-1, 0]$ and convex on $[0, 1]$. Note that $f(0) = 0$ is the global minimum of f.

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