# Indian Institute of Technology Guwahati

Mid-Semester Examination, July-November 2012

*MA 101 Mathematics-I*

Time: 2 Hrs | Marks: 25

Model Solution

1. With proper justifications, prove or disprove the following statements.

(a) The set $\{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + 5z = 3\}$ is a subspace of $\mathbb{R}^3$. [1]

Ans: If the zero vector $(0, 0, 0)$ is in the given set, then $0 = 3$. This is an absurd. Thus, the given set is not a subspace of $\mathbb{R}^3$.

(b) If $A$ is any $m \times n$ matrix, then $\text{rank}(A^T A) = \text{rank}(A)$. [1]

Ans: Note that the number of columns of $A$ and $A^T A$ are equal and it is $n$. By rank theorem, $\text{rank}(A) + \text{nullity}(A) = n = \text{rank}(A^T A) + \text{nullity}(A^T A)$. Observe that $\mathbf{x} \in \text{null}(A)$ if and only if $\mathbf{x} \in \text{null}(A^T A)$, so that $\text{nullity}(A) = \text{nullity}(A^T A)$. Hence, $\text{rank}(A) = \text{rank}(A^T A)$.

(c) If $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for eigenspaces corresponding to two distinct eigenvalues of a matrix, then $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$. [1]

Ans: Let $\lambda_1$ and $\lambda_2$ be the corresponding distinct eigenvalues of a matrix $A$. If $v \in \mathcal{B}_1 \cap \mathcal{B}_2$, then clearly $v \neq \mathbf{0}$. Further, $Av = \lambda_1 v$ and $Av = \lambda_2 v$, so that $\lambda_1 v = \lambda_1 v$. Thus, $(\lambda_1 - \lambda_2)v = \mathbf{0}$. Since $v \neq \mathbf{0}$, we get $\lambda_1 = \lambda_2$; a contradiction. Hence, $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$.

OR

Refer to the result "the eigenvectors $v_1$ and $v_2$ corresponding to distinct eigenvalues $\lambda_1$ and $\lambda_2$ are linearly independent", and conclude the given statement.

(d) If $A$ and $B$ are $2 \times 2$ matrices such that $\det A = \det B$, then $A$ must be similar to $B$. [1]

Ans: Consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Note that $\det A = \det B$. However, observe that their characteristic polynomials are different. Hence, $A$ is not similar to $B$.

(e) The basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for $\mathbb{R}^3$ is orthonormal. [1]

Ans: Write $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Observe that $e_1 \cdot e_2 = e_3 \cdot e_2 = e_1 \cdot e_3 = 0$. Further, $\|e_1\| = \|e_2\| = \|e_3\| = 1$.

2. (a) Define all the three types of elementary matrices. [1]

Ans: A matrix that is obtained by performing any one of the three elementary row operations (row exchange, multiply a nonzero constant to a row, or add a multiple of a row to another row) on an identity matrix is called as an elementary matrix.

Write $E_{i,j}$ to denote the elementary matrix obtained by exchanging the $i$th row with $j$th row in an identity matrix.

Write $E_{ki}$ to denote the elementary matrix obtained by multiplying a nonzero constant $k$ to the $i$th row in an identity matrix.

Write $E_{ki+j}$ to denote the elementary matrix obtained by multiplying a constant $k$ to the $i$th row and adding to $j$th row in an identity matrix.

(b) Write the inverse of each of the three types of elementary matrices. [1]

Ans: $E_{i,j}^{-1} = E_{i,j}$, $E_{ki}^{-1} = E_{\frac{1}{k}i}$, and $E_{ki+j}^{-1} = E_{-ki+j}$. This can be observed from the fact that the elementary row operations are reversible.

(c) Write the determinant of each of the three types of elementary matrices. [1]

Ans: $\det E_{i,j} = -1$, $\det E_{ki} = k$ and $\det E_{ki+j} = 1$.

(d) Show that every invertible matrix is a product of elementary matrices. [2]

Ans: Let $A_{n\times n}$ be an invertible matrix. Then, observe that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. That is, the Gauss-Jordan elimination applied to the augmented matrix $[A \mid \mathbf{0}]$ of the system gives $[I_n \mid \mathbf{0}]$ so that the reduced row echelon form of $A$ is $I_n$. Hence, there is a finite sequence of elementary row operations which reduces $A$ to $I_n$. Let $E_1, \ldots, E_k$ be the corresponding sequence of elementary matrices (in the order) which are left multiplied to $A$ to get $I_n$. That is,

$$E_k \cdots E_1 A = I_n.$$

Hence, $A = E_1^{-1} \cdots E_k^{-1}$. Since inverse of an elementary matrix is also an elementary matrix, we have $A$ as a product of elementary matrices.

3. (a) Prove that the range of a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the column space of its standard matrix $[T]$. [1]

Ans:

$$\begin{aligned}
\text{Range of } T &= \{T(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n\} = \{[T]\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} \\
&= \left\{ [C_1 \quad \cdots \quad C_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_i \in \mathbb{R} \right\}, \\
&\quad \text{where } C_1, \ldots, C_n \text{ are the columns of } [T] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \\
&= \{C_1 x_1 + \cdots + C_n x_n \mid x_i \in \mathbb{R}\} \\
&= \text{col}([T]), \text{ being the set of all linear combinations of columns of } [T].
\end{aligned}$$

(b) Prove that the linear transformation defined by an orthogonal matrix is angle-preserving in $\mathbb{R}^n$. [1]

Ans: Let $A$ be an $n \times n$ orthogonal matrix and $T_A$ be the linear transformation defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. For any nonzero vectors $u, v \in \mathbb{R}^n$, the angle $\theta$ between $u$ and $v$ is given by $\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$. Since $\|A\mathbf{x}\| = \|\mathbf{x}\|$ and $A\mathbf{x} \cdot A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have the angle between $T_A(u)$ and $T_A(v)$ is the same as $\theta$. (Here, $T_A(u) \neq \mathbf{0} \neq T_A(v)$.)

(c) If the null space of a matrix $A$ is $\{\mathbf{0}\}$, then prove that the matrix transformation $T_A$ is injective (one-one). [1]

Ans: For two vectors $u, v$, assume $T_A(u) = T_A(v)$. Then, $Au = Av \Longrightarrow A(u - v) = \mathbf{0}$. Consequently, $u - v \in \text{null}(A)$. Hence, $u - v = \mathbf{0}$, so that $u = v$.

(d) Let $A$ be a matrix. Prove that each vector in row$(A)$ is orthogonal to every vector in null$(A)$. [1]

Ans: Let $A = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}$ be an $m \times n$ matrix with rows $R_1, \ldots, R_m$. Let $\mathbf{x} \in \text{null}(A)$ be arbitrary. Then, $\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = A\mathbf{x} = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} \mathbf{x} = \begin{bmatrix} R_1 \cdot \mathbf{x} \\ \vdots \\ R_m \cdot \mathbf{x} \end{bmatrix}$, so that $R_i \cdot \mathbf{x} = 0$ for all $i$. Thus, each row of $A$ is orthogonal to $\mathbf{x}$. Hence, every vector in row$(A)$ is orthogonal to $\mathbf{x}$.

(e) If the algebraic multiplicity of an eigenvalue $\lambda$ of a matrix is 1, then prove that any two eigenvectors corresponding to $\lambda$ are parallel (a scalar multiple of one another). [1]

Ans: Since the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity, the geometric multiplicity of $\lambda$ equals 1. So, any nonzero vector in the eigen space of $\lambda$ will form a basis and hence, any two eigenvectors corresponding to $\lambda$ are scalar multiple of each other.

4. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(a) Write all the eigenvalues of $A$. [1]

Ans: Eigenvalues are 1, 2, 3.

(b) Write an eigenvector corresponding to each of the eigenvalues of $A$. [1]

Ans: An eigenvector corresponding to the eigenvalue 1 is $[1 \quad 0 \quad 0 \quad 0]^T$. An eigenvector corresponding to the eigenvalue 2 is $[1 \quad 1 \quad 0 \quad 0]^T$. An eigenvector corresponding to the eigenvalue 3 is $[3 \quad 2 \quad 2 \quad 2]^T$.

(c) Write an invertible matrix $P$ such that $PAP^{-1} = D$, where $D$ is a diagonal matrix. [2]

Ans: Note that the algebraic and geometric multiplicities of the eigenvalue 2 are equal and another eigenvector of 2 which is independent of the one given above is $[1 \quad 0 \quad 1 \quad 0]^T$. By arranging these linearly independent eigenvectors as columns a matrix we get $P^{-1}$, as indicated here. Thus, $P^{-1} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Hence, the corresponding $P = \begin{bmatrix} 1 & -1 & -1 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(d) For $k \geq 0$, compute $A^k$. [1]

Ans: We have $PAP^{-1} = D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, so that $A = P^{-1}DP$ and $A^k = P^{-1}D^kP$.

Thus,

$$\begin{aligned} A^k &= \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2^k & 0 & 0 \\ 0 & 0 & 2^k & 0 \\ 0 & 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & \frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2^k - 1 & 2^k - 1 & \frac{3^{k+1} - 2^{k+2} + 1}{2} \\ 0 & 2^k & 0 & 3^k - 2^k \\ 0 & 0 & 2^k & 3^k - 2^k \\ 0 & 0 & 0 & 3^k \end{bmatrix} \end{aligned}$$

5. (a) Use Cayley-Hamilton theorem to compute the inverse of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 0 \end{bmatrix}$. [2]

Ans: Characteristic equation $\det(A - \lambda I) = 0$ will be $\lambda^3 - 2\lambda^2 - 22\lambda = 10$. By Cayley-Hamilton theorem, we have $A^3 - 2A^2 - 22A = 10I$ so that $10A^{-1} = A^2 - 2A - 22I$. Thus, $A^{-1} = \frac{1}{10}\begin{bmatrix} -12 & 4 & 14 \\ 3 & -1 & -1 \\ 7 & 1 & -9 \end{bmatrix}$.

(b) If $A$ is an $n \times n$ matrix and $B$ is the matrix obtained by interchanging any two rows of $A$, then prove that $\det B = -\det A$. [3]

Ans: Refer to the textbook (3rd edition), page no. 289, Lemma 4.14.

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