Indian Institute of Technology Guwahati
Department of Mathematics

MA 101 – Mathematics-I **Date:** 07-Sep-2015
Tutorial Sheet-5 **Time:** 08:00 – 09:00

Linear Algebra

Topics Covered:
Diagonalization, Orthogonality, Orthonormal Basis, Orthogonal Complement and Orthogonal Projections, Gram-Schmidt Process.

---

1. Show that any two similar matrices have the same trace.

2. Let A be an invertible matrix. Prove that if A is diagonalizable, then so is A^{-1}.

3. Let A be a diagonalizable matrix such that characteristic polynomial of A has only one root. Then find out the diagonal matrix D such that $A \sim D$. Is such a matrix D unique?

4. With the help of diagonlization, calculate A^{2015} where
$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

5. (a) Let A be a diagonal matrix and let $P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0$ be the characteristic polynomial of A. Then show that $P(A) = A^n + a_{n-1}A^{n-1} + \ldots + a_0 I_n$ is the zero matrix.

   (b) Show that the statement holds for diagonalizable matrices as well.

   [*Cayley-Hamilton theorem states that this statement holds for any square matrix, i.e. a matrix A satisfies its characteristic polynomial.*]

6. (a) For any $u, v \in \mathbb{R}^n$, show that $|u \cdot v| \leq \|u\|\|v\|$ (Cauchy-Schwartz inequality).

   (b) For any $u, v \in \mathbb{R}^n$, show that $\|u + v\| \leq \|u\| + \|v\|$ (Triangle inequality).

7. Let A be a real symmetric matrix.

   (a) Show that all the eigenvalues of A are real.

   (b) Show that any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

8. If A and B are $n \times n$ matrices with n distinct eigenvalues. Then show that $AB = BA$ if and only if A and B have the same eigenvectors.

9. Find an orthogonal basis for $\mathbb{R}^4$ containing the vectors: $v_1 = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

10. Let W be the row space of the matrix
$$A = \begin{bmatrix} 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 3 & 1 & 3 & 1 & 3 \\ 1 & 3 & 1 & 3 & 1 \\ 1 & 4 & 1 & 4 & 1 \end{bmatrix}.$$
Compute $W^{\perp}$ and the orthogonal decomposition of the vector $v = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$ with respect to W.

11. Let $S = \{v_1, \ldots, v_k\}$ be an orthonormal set in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be a vector. Then show that
$$\|x\|^2 \geq |x \cdot v_1|^2 + |x \cdot v_2|^2 + \ldots + |x \cdot v_k|^2.$$
Also show that the above becomes an equality if and only if $x \in Span(S)$.

12. Let A be a 2×2 orthogonal matrix. Show that there exists a real number θ such that
$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \text{or} \qquad A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$
In the first case, A rotates the vectors of $\mathbb{R}^2$ by the angle θ counterclockwise, and in the second case, A reflects the vectors of $\mathbb{R}^2$ about a line; in this case find the line.