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1 Determinant

Definition 1.1. *Let $A = [a_{ij}]$ be an $n \times n$ matrix.*

- *For an 1×1 matrix $A = [a]$, we define the **determinant** of A as $det(A) = a$.*
- *If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we define $det(A) = ad - bc$.*
- *In general, if A_{ij} is the submatrix of A obtained by deleting the i-th row and the j-th column of A, then $det(A)$ is defined recursively as follows:*

$$det(A) = a_{11}\,det(A_{11}) - a_{12}\,det(A_{12}) + \ldots + (-1)^{1+n} a_{1n}\,det(A_{1n}) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\,det(A_{1j}).$$

- *Sometimes $det(A)$ is also denoted by $|A|$.*
- *We define $det(A_{ij})$ to be the (i,j)-**minor** of A.*
- *The number $C_{ij} = (-1)^{i+j} det(A_{ij})$ is called the (i,j)-**cofactor** of A.*
- *Thus we can write*

$$det(A) = |A| = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

★ Take a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$. Put $\sigma_{ij} = (-1)^{i+j}$.

Then

$$|A| = \sigma_{11} a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + \sigma_{12} a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + \sigma_{13} a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + \sigma_{14} a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.$$

⋆ Expand each of them. Do you get 12 terms? Do you get a term (...) $\begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}$? Do you get this matrix twice?

⋆ So, the coefficient of $\begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix}$ is $\sigma_{12} a_{12} \quad \sigma_{12} a_{23} + \sigma_{13} a_{13}\, \sigma_{12} a_{22} = (-1)^{1+2+2+3} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$.

⋆ The coefficient of $|A_{1,2|1,3}| = \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}$ is $(-1)^{1+2+1+3} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$.

⋆ In general, the coefficient of $|A_{1,2|i,j}|$ in the double expansion of $|A|$ is $(-1)^{1+2+i+j} \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}$.

⋆ Hence $|A| = \sum_{i<j} (-1)^{1+2+i+j} \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} |A_{1,2|i,j}|$.

Result 1.1 (**Properties of Determinants**).

- *If B is obtained by interchanging the first two rows of A, then $det(B) = -det(A)$.*
- *(*By Induction*) If B is obtained by interchanging any two consecutive rows of A, then $det(B) = -det(A)$.*
- *If B is obtained by interchanging any two rows of A, then $det(B) = -det(A)$.*
- *If A has a zero row then $det(A) = 0$.*
- *If A has two identical rows then $det(A) = 0$.*
- *If B is obtained by multiplying a row of A by k, then $det(B) = k.det(A)$.*
- *If B is obtained by adding a multiple of one row of A to another row, then $det(B) = det(A)$.*

Result 1.2 (**Determinants of Elementary Matrices**). *Let E be an $n \times n$ elementary matrix and A be any $n \times n$ matrix. Then*

1. $det(E) = -1, k$ *or* 1.
2. $det(EA) = det(E)det(A)$.
3. E^t *is also an elementary matrix and* $det(E) = det(E^t)$.

Result 1.3.

- *A square matrix A is invertible if and only if* $det(A) \neq 0$.
- *Let A be an $n \times n$ matrix. Then* $det(kA) = k^n det(A)$.
- *Let A and B be two $n \times n$ matrices. Then* $det(AB) = det(A)det(B)$.
- *If the matrix A is invertible then* $det(A^{-1}) = \frac{1}{det(A)}$.

★ A matrix A is said to be **singular** or **non-singular** according as $\det(A) = 0$ or $\det(A) \neq 0$.

Result 1.4.

- *The determinant of a triangular matrix is the product of the diagonal entries. That is, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix then* $det(A) = a_{11}a_{22} \ldots a_{nn}$.
- *If R is in row echelon form having a **zero** row, then* $det(R) = 0 = det(R^t)$.
- *For any square matrix A,* $det(A^t) = det(A)$.

★ Thus a determinant can be expanded column-wise.

★ All the previous results based on row-wise expansion of determinant are also valid for column-wise expansion.

Result 1.5 (**Laplace Expansion Theorem**). *The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as*

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij},$$

*(this is the **cofactor expansion along the i-th row**), and also as*

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij},$$

*(this is the **cofactor expansion along the j-th column**).*

Result 1.6.

- *If B is obtained by interchanging any two columns of A, then* $det(B) = -det(A)$.
- *If A has a zero column then* $det(A) = 0$.
- *If A has two identical columns then* $det(A) = 0$.
- *If B is obtained by multiplying a column of A by k, then* $det(B) = k det(A)$.
- *If B is obtained by adding a multiple of one column of A to another column, then* $det(B) = det(A)$.

Definition 1.2. *Let A be an $n \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^n$. Then $A_i(\mathbf{b})$ denotes the matrix obtained by replacing the i-th column of A by $\mathbf{b}$. That is, if $A = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n]$, then $A_i(\mathbf{b}) = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1}\ \ldots\ \mathbf{a}_n]$.*

Result 1.7 (**Cramer's Rule**). *Let A be an $n \times n$ invertible matrix and let $\mathbf{b} \in \mathbb{R}^n$. Then the unique solution $\mathbf{x} = [x_1, x_2, \ldots, x_n]^t$ of the system $A\mathbf{x} = \mathbf{b}$ is given by*

$$x_i = \frac{det\ (A_i(\mathbf{b}))}{det(A)} \quad for\ \ i = 1, 2, \ldots, n.$$

The Adjoint of a Matrix: Let $A = [a_{ij}]$ be an $n \times n$ matrix and let C_{ij} be the (i,j)-cofactor of A. Then the **adjoint** of A, denoted $\text{adj}(A)$, is defined as

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} = [C_{ij}]^t.$$

Result 1.8. *Let A be an $n \times n$ invertible matrix. Then $A^{-1} = \frac{1}{det(A)}.adj(A)$.*

Example 1.1. *Use the adjoint method to find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}.$$

2 Subspaces Associated with Matrices

Definition 2.1. *Let A be an $m \times n$ matrix.*

1. *The **null space** of A, denoted **null**(A), is the subspace of $\mathbb{R}^n$ consisting of the solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. In other words, null$(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$.*

2. *The **column space** of A, denoted **col**(A), is the subspace of $\mathbb{R}^m$ spanned by the columns of A. In other words, col$(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$.*

3. *The **row space** of A, denoted **row**(A), is the subspace of $\mathbb{R}^n$ spanned by the rows of A. In other words, row$(A) = \{\mathbf{x}^T A \mid \mathbf{x} \in \mathbb{R}^m\}$.*

[Here, elements of row(A) are row vectors. How can they be elements of $\mathbb{R}^n$? In strict sense, row$(A) :=$ col(A^T).]

Result 2.1. *Let B be a matrix that is row equivalent to the matrix A. Then row$(B) =$ row(A).*

Corollary 2.1. *For any A, row$(A) =$ row(RREF(A)).*

Corollary 2.2. *For any A, the non-zero rows of RREF(A) forms a basis of row(A).*

Suppose A and B are row-equivalent. Are col(A) and col(B) equal? No. Take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
Suppose A and B are row-equivalent. Do col(A) and col(B) have same dimension? Yes. We will see soon.

Method of Finding Bases of the Row Space, the Null Space and the Column Space of a Matrix:

Let A be a given matrix and let R be the reduced row echelon form of A.

1. Use the non-zero rows of R to form a basis for row(A).

2. Solve the leading variables of $R\mathbf{x} = \mathbf{0}$ in terms of the free variables, set the free variables equal to parameters, substitute back into $\mathbf{x}$, write the result as a linear combination of k vectors (where k is the number of free variables). These k vectors form a basis for null(A).

3. A basis for row(A^t) will also be a basis for col(A). **Or**, Use the columns of A that correspond to the columns of R containing the leading 1's to form a basis for col(A).

Example 2.1. *Find bases for the row space, column space and the null space of the following matrix:*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 4 & 6 & 2 \end{bmatrix}.$$

Result 2.2. *Let $R = [\mathbf{b}_1\ \mathbf{b}_2\ \dots\ \mathbf{b}_n]$ be the reduced row echelen form of a matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \dots\ \mathbf{a}_n]$ of rank r. Let $\mathbf{b}_{j_1}, \mathbf{b}_{j_2}, \dots, \mathbf{b}_{j_r}$ be the columns of R such that $\mathbf{b}_{j_k} = \mathbf{e}_k$ for $k = 1, \dots, r$. Then $\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \dots, \mathbf{a}_{j_r}\}$ is a basis for col(A).*

Result 2.3. *The row space and the column space of a matrix A have the same dimension, and $dim(row(A)) = dim(col(A)) = rank(A)$.*

Result 2.4. *For any matrix A, we have $rank(A^t) = rank(A)$.*

Nullity: The **nullity** of a matrix A is the dimension of its null space, and is denoted by **nullity**(A).

Result 2.5 (Rank Nullity Theorem). *Let A be an $m \times n$ matrix. Then*

$$rank(A) + nullity(A) = n.$$

Result 2.6 (The Fundamental Theorem of Invertible Matrices: Version II). *Let A be an $n \times n$ matrix. Then the following statements are equivalent.*

1. *A is invertible.*
2. *A^t is invertible.*
3. *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
4. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
6. *The reduced row echelon form of A is I_n.*
7. *The rows of A are linearly independent.*
8. *The columns of A are linearly independent.*
9. *$rank(A) = n$.*
10. *A is a product of elementary matrices.*
11. *$nullity(A) = 0$.*
12. *The column vectors of A span $\mathbb{R}^n$.*
13. *The column vectors of A form a basis for $\mathbb{R}^n$.*
14. *The row vectors of A span $\mathbb{R}^n$.*
15. *The row vectors of A form a basis for $\mathbb{R}^n$.*

Example 2.2. *Show that the vectors $[1, 2, 3]^t, [-1, 0, 1]^t$ and $[4, 9, 7]^t$ form a basis for $\mathbb{R}^3$.*

Result 2.7. *Let A be an $m \times n$ matrix. Then*

1. *$rank(A^tA) = rank(A)$.*
2. *The $n \times n$ matrix A^tA is invertible if and only if $rank(A) = n$.*