1. **Fill up the blanks.**

(1 pt.) (a) The dimension of the space of all $n \times n$ real symmetric matrices with trace zero is $\boxed{\frac{1}{2}n(n+1) - 1 = \frac{1}{2}[n^2 + n - 2]}$

(1 pt.) (b) Let A be an $n \times n$ upper triangular matrix with $a_{ij} = 1$ for all $i \leq j$. Then the geometric multiplicity of 1 as an eigenvalue of A is $\boxed{1}$

(1 pt.) (c) If the vector space $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0, x_1 + x_3 = 0, x_2 + x_4 = 0 \right\}$ is isomorphic to $\mathbb{R}^n$, then n equals $\boxed{2}$

(1 pt.) (d) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that for each $x \in \mathbb{R}$, $f(x) \in \mathbb{Q}$. If $f(1) = 1$ then $f(0) =$ $\boxed{1}$

(1 pt.) (e) If $0 < a < b$, then the sequence $\left(\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n} \right)$ converges to $\boxed{b}$

(1 pt.) (f) The radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{2^n}$ is $\boxed{2}$

(1 pt.) (g) Let $f : [1, 3] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 2 & \text{if } x \text{ is irrational.} \end{cases}$

Then, for any partition P of $[1, 3]$, $U(P, f) - L(P, f)$ equals $\boxed{2}$

(1 pt.) (h) A limit point of $\left\{ \left(-\dfrac{n+1}{n} \right)^n + \left(\dfrac{n+1}{n} \right)^n : n \in \mathbb{N} \right\}$ is $\boxed{2e}$

(1 pt.) (i) $\displaystyle\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$ equals $\boxed{0}$

(1 pt.) (j) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^5 + 2x + 3$. Then $(f^{-1})'(6)$ equals $\boxed{\frac{1}{7}}$

(2 pts.) 2. Prove or disprove: There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that none of T and T^2 is the identity transformation, but T^3 is the identity transformation.

True.

Example:

Rotation of $\mathbb{R}^2$ about the origin by an angle $\frac{2\pi}{3}$ $\boxed{\textbf{.5 Mark}}$.

Formally, take $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} -\dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{bmatrix}. \qquad \boxed{\textbf{1.5 Marks}}$$

(2 pts.) 3. If A is a 3×3 matrix with eigenvalues $0, 1$ and 10 and D is a 2×2 matrix with eigenvalues 0 and 10, then show that the matrix $\begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix}$ has eigenvalues $0, 1, 10$ with algebraic multiplicities $2, 1$ and 2, respectively.

Given, the characteristic polynomials of the matrix A and D as $\det(A - \lambda I_3) = (-1)^3(\lambda - 0)(\lambda - 1)(\lambda - 10)$ and $\det(D - \lambda I_2) = (\lambda - 0)(\lambda - 10)$ respectively. The characteristic polynomial of the given matrix say X is

$$\det(X - \lambda I_5) = \det \begin{bmatrix} A - \lambda I_3 & B \\ \mathbf{0} & D - \lambda I_2 \end{bmatrix}$$

$$= \det(A - \lambda I_3) \cdot \det(D - \lambda I_2)$$

$$= (-1)^3(\lambda - 0)(\lambda - 1)(\lambda - 10)(\lambda - 0)(\lambda - 10)$$

$$= (-1)^3\lambda^2(\lambda - 10)^2(\lambda - 1)$$

$\boxed{\textbf{1.5 Mark}}$

Since $0, 1$ and 10 are the zeroes of the characteristic polynomial of the given matrix X with multiplicities $2, 1$ and 2, the eigenvalues of the given matrix are $0, 1, 10$ with algebraic multiplicities $2, 1$ and 2 respectively. $\boxed{\textbf{.5 Mark}}$

(3 pts.) 4. Let A be an $n \times n$ upper triangular matrix with all diagonal entries equal to a. If A is diagonalizable then show that A is a diagonal matrix.

Since A is upper triangular with all diagonal entries equal to a, a is the only eigenvalue of A with algebraic multiplicity n. **1 Mark**

Since A is diagonalizable, the geometric multiplicity of a is same as the algebraic multiplicity is equal to n (or A has n linearly independent eigenvectors corresponding to the eigenvalue a) which implies $nullity(A - aI) = n$. **1 Mark**

Hence $rank(A - aI) = 0$ or $A - aI$ is the zero matrix, i.e., $A = aI$ is a diagonal matrix. **1 Mark**

Alternative

Since A is upper triangular with all diagonal entries equal to a, a is the only eigenvalue of A with algebraic multiplicity n. **1 Mark**

Since A is diagonalizable A has n linearly independent eigenvectors corresponding to the eigenvalue a, hence there exists an invertible matrix P, such that $P^{-1}AP = D = aI$. **1 Mark**

Hence $A = (P(aI))P^{-1} = (aP)P^{-1} = aI$. **1 Mark**

Note: For claiming that $P = I$, without any justification and then carrying out further calculation, was not given any credit. Since for any diagonalizable matrix A, P need not be I, although the columns of P forms a basis of R^n.

(2 pts.) 5. Show that $x^5 + 4x - \sin x = 0$ has exactly one real solution.

Let $f(x) = x^5 + 4x - \sin x$.

Observe that $f(x)$ is differentiable (hence continuous) in $(-\infty, \infty)$.

Also, $f(\frac{\pi}{2}) = (\frac{\pi}{2})^5 + 2\pi - 1 > 0$, and $f(-\frac{\pi}{2}) = -(\frac{\pi}{2})^5 - 2\pi + 1 < 0$

Alternatively Since $|\sin x| \leq 1$ and $x^5 + 4x \to \infty$ as $x \to \infty$ and $x^5 + 4x \to -\infty$ as $x \to -\infty$, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

Therefore, by Intermediate Value Theorem (IVT) for continuous functions, $f(x)$ must be 0 for some $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. $\boxed{\textbf{1 Mark}}$

Alternatively By verification, $f(0) = 0$. $\boxed{\textbf{1 Mark}}$

Since $f'(x) = 5x^4 + 4 - \cos x > 0$ for all $x \in \mathbb{R}$, f is strictly increasing, $f(x) = 0$ can have at most one real solution, hence the equation $f(x) = 0$ has exactly one real solution. $\boxed{\textbf{1 Mark}}$

Alternatively since $f'(x) > 0$ for all $x \in \mathbb{R}$, by Rolle's theorem $f(x) = 0$ can have at most one real solution, hence the equation $f(x) = 0$ has exactly one real solution. $\boxed{\textbf{1 Mark}}$

Note: 1/2 Marks have been deducted for writing expressions such as $f(\infty) = \infty, f(-\infty) = -\infty$, etc.

(3 pts.) 6. Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$ converges absolutely, converges conditionally or diverges.

Let $x_n = \sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

For absolute convergence, consider the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}x_n)\right| = \sum_{n=1}^{\infty} x_n$

Note that $0 \leq \dfrac{1}{2\sqrt{2}\sqrt{n}} = \dfrac{1}{\sqrt{n+n}+\sqrt{n+n}} \leq \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$ for all $n \in \mathbb{N}$.

Using comparison test and result about p series, we conclude that
$\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})\right|$ is divergent. **1.5 Marks**

Observe that $x_n > 0$, $x_{n+1} < x_n$ and $x_n \to 0$. Therefore Leibniz test is applicable and the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ is convergent.

Hence, the given series is conditionally convergent. **1.5 Marks**

Alternative

Telescopic summation can be used conclude divergence of $\sum |(-1)^{n+1}x_n|$.

Let $y_n = \dfrac{1}{\sqrt{n}}$. Then, $\dfrac{x_n}{y_n} \to \dfrac{1}{2}$. Since $\sum y_n$ is divergent, by limit comparison test, $\sum x_n$ is divergent.

Marks Deduction

For missing one of the condition for Leibniz test, 0.5 is deducted.

Application of Leibniz test without checking conditions is not awarded any mark.

(3 pts.) 7. Suppose that $f : (0,1) \to \mathbb{R}$ is differentiable and that f' is bounded. Show that $(f(1/n))$ is a cauchy sequence and that $\lim_{n\to\infty} f(1/n)$ exists.

By Mean Value Theorem (MVT)

$$|f(1/n) - f(1/m)| = |f'(x_{mn})| \;\; |1/n - 1/m|$$

for all $m \geq 2, n \geq 2$, where x_{mn} lies between $1/n$ and $1/m$. **1 Mark**

We are given that f' is bounded. Let $\alpha > 0$ be such that $|f'(x)| \leq \alpha$ for $x \in (0,1)$. Therefore,

$$|f(1/n) - f(1/m)| \leq \alpha\,|1/n - 1/m|$$

for all $m \geq 2, n \geq 2$.

Hence it follows that $(f(1/n))$ is a Cauchy sequence. **1 Mark**

Since a Cauchy sequence is convergent, $\lim_{n\to\infty} f(1/n)$ exists. **1 Mark**

Incorrect Solution

Use of integration such as $\displaystyle\int_0^{1/n} f'$ is not correct. This is because f' may not be integrable. Moreover, the domain of f does not include 0.

(3 pts.) 8. Let $f : [0,1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Show that there exists $x_1, x_2 \in [0,1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{3}$.

Consider the function $g(x) = f(x + \frac{1}{3}) - f(x)$ for all $x \in [0, \frac{2}{3}]$. **1 Mark**

Since f is continuous, $g : [0, \frac{2}{3}] \to \mathbb{R}$ is also continuous.

Observe that $g(\frac{1}{3}) = f(\frac{2}{3}) - f(\frac{1}{3})$ and $g(\frac{2}{3}) = f(1) - f(\frac{2}{3})$

Also $g(0) + g(\frac{1}{3}) + g(\frac{2}{3}) = f(1) - f(0) = 0$. If at least one of $g(0)$, $g(\frac{1}{3})$ and $g(\frac{2}{3})$ is 0, then the result follows immediately. **1 Mark**

Otherwise, at least two of $g(0)$, $g(\frac{1}{3})$ and $g(\frac{2}{3})$ are of opposite signs.

Hence by the intermediate value theorem, there exists $c \in (0, \frac{2}{3})$ such that $g(c) = 0$, *i.e.* $f(c + \frac{1}{3}) = f(c)$. We take $x_1 = c + \frac{1}{3}$ and $x_2 = c$. **1 Mark**

Alternative

(Proof by contradiction) Define $g(x) = f(x + \frac{1}{3}) - f(x)$.

Suppose $g(x) \neq 0$ for all $x \in [0, \frac{2}{3}]$.

Then, either $g(x) > 0$ for all $x \in [0, \frac{2}{3}]$ or $g(x) < 0$ for all $x \in [0, \frac{2}{3}]$. [If it takes both negative and positive value then by IVT $g(x)$ must be 0 in the interval.]

Thus, either $f(x + \frac{1}{3}) > f(x)$ for all $x \in [0, \frac{2}{3}]$ or $f(x + \frac{1}{3}) < f(x)$ for all $x \in [0, \frac{2}{3}]$.

Consider $x = 0, \frac{1}{3}$ and $\frac{2}{3}$, then we have $f(1) > f(\frac{2}{3}) > f(\frac{1}{3} > f(0)$ which contradicts the given property that $f(0) = f(1)$.

Similarly for the other case.

(3 pts.) 9. Find the points of local extrema of the function $f(x) = 1 - (1-x)^{\frac{2}{3}}$ for $0 \leq x \leq 2$.

In $[0,2]$, $f(x) \geq 0$ and $f(x) = 0$ at $x = 0, 2$. Hence, f has (global) minimum at $x = 0, 2$. $\boxed{\textbf{1.5 Marks}}$

We have $f'(x) = \frac{2}{3}(1-x)^{-1/3}$ at $[0,2] \setminus \{1\}$. Moreover, $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$. Hence, f has local maxima at $x = 1$. $\boxed{\textbf{1.5 Marks}}$

Alternative

For local minima at 0 and 2, the argument that f is increasing on interval $[0,1]$ and decreasing on $[1,2]$ using property of derivative is also accepted.

Marks Deduction

No marks are awarded for merely stating that the function attains minima at 0 and 2.

(3 pts.) 10. Show that for $x > 0$, $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$.

Let $f(x) = \sqrt{1+x}$. Then,

By Taylor's theorem, $f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(c_x)$ for some $c_x \in (0, x)$. **1 Mark**

Now, $f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\qquad f''(x) = \frac{-1}{4}(1+x)^{-3/2}$.

Therefore, $f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2(1+c_x)^{-3/2}$ for some $c_x \in (0, x)$. **1 Mark**

Note that $-\frac{1}{8}x^2 < -\frac{1}{8}x^2(1+c)^{-3/2} < 0$ for any $x, c > 0$.

Therefore, $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$. **1 Mark**

Alternative

Similarly, $f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(d_x)$ for some $d_x \in (0, x)$.

Since $f'''(x) > 0$, we have $f(x) \geq f(0) + xf'(0) + \frac{x^2}{2} f''(0) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$ for all $x > 0$.

(2 pts.) 11. Let $f : [0, 2] \to \mathbb{R}$ be differentiable in $[0, 2]$ and let $f(0) = 0$, $f(1) = 2$ and $f(2) = 1$. Show that there exist $c, d \in (0, 2)$ such that $f'(c) = 2$ and $f'(d) = 1$.

By Mean Value Theorem (MVT), we have $f(1) - f(0) = f'(c)(1 - 0)$ for some $c \in (0, 1)$.

This gives $f'(c) = 2$. **1 Mark**

Again by MVT $f(2) - f(1) = f'(s)(2-1)$ for some $s \in (1, 2)$. Consequently, we have $f'(s) = -1$. Since $f'(s) = -1$ and $f'(c) = 2$, by Intermediate Value Theorem (IVT) of derivatives, there exits d between c and s such that $f'(d) = 1$.

Since s and c are contained in $(0, 2)$, it follows that $c, d \in (0, 2)$. **1 Mark**

(2 pts.) 12. Let $f : [0,1] \to \mathbb{R}$ be continuous. If $\int_0^x f(t)dt = \int_x^1 f(t)dt$ for all $x \in [0,1]$ then show that $f(x) = 0$ for all $x \in [0,1]$.

Let $F(x) = \int_0^x f(t)dt$.

Since f is continuous, F is differentiable, and $F'(x) = f(x)$ for $x \in [0,1]$. **1 Mark**

By the given condition, $F(x) = F(1) - F(x)$, that is, $2F(x) = F(1)$.
This gives $2F'(x) = 0$, that is, $f(x) = 0$ for $x \in [0,1]$. **1 Mark**

(3 pts.) 13. Let $f, g : [a,b] \to \mathbb{R}$ be continuous and let $g(x) \geq 0$ for $x \in [a,b]$. Show that there exists $c \in (a,b)$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

Let $m = f(x_1)$ and $M = f(x_2)$ be the global minimum and global maximum of f, respectively.

Since $g(x) \geq 0$, we have $m\,g(x) \leq f(x)g(x) \leq M\,g(x)$.

Then, $m\int_a^b g(t)dt \leq \int_a^b f(t)g(t)dt \leq M\int_a^b g(t)dt$. **1 Mark**

If $L := \int_a^b g(t)dt = 0$, then, $\int_a^b f(t)g(t)dt = 0$ and so the result holds for any $c \in [a,b]$. **0.5 Mark**

If $L := \int_a^b g(t)dt \neq 0$, then $L > 0$. Therefore, $m \leq \int_a^b f(t)g(t)dt/L \leq M$. Thus, by the IVT there exists $c \in [a,b]$ such that $f(c) = \int_a^b f(t)g(t)dt/L$. **1.5 Marks**

(2 pts.) 14. Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that

$$\lim_{a\to 0}\int_0^{\sqrt{a}} \frac{af(x)}{a^2+x^2}dx = \frac{\pi}{2}f(0).$$

[**Hint.** Use Question 13.]

By Question 13, we have

$$I(a) := \int_0^{\sqrt{a}} \frac{af(x)}{a^2+x^2}dx = f(c)\int_0^{\sqrt{a}} \frac{a}{a^2+x^2}dx = f(c)\tan^{-1}(1/\sqrt{a}),$$

for some $c \in (0, \sqrt{a})$. **1 Mark**

Now, $\tan^{-1}(1/\sqrt{a}) \to \pi/2$ as $a \to 0$. **Since** f **is continuous and** $0 < c < \sqrt{a}$**, it follows that** $f(c) \to f(0)$ **as** $a \to 0$**.** Hence $\lim_{a\to 0} I(a) = \frac{\pi}{2}f(0)$. **1 Mark**

(3 pts.) 15. Examine the convergence of the improper integral $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}dx$.

Since $0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ converges. Therefore, by comparison test, $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}dx$ converges. **1.5 Marks**

Also note that $\dfrac{\sin^2 x}{x^2}$ is continuous on $(0,1]$ and $\to 1$ as $x \to 0$. Thus, the integrand can be defined to be 1 at 0 to get a continuous (and so bounded) function on $[0,1]$. Thus, $\displaystyle\int_0^1 \frac{\sin^2 x}{x^2}dx$ exists in the sense of Riemann, **1.5 Marks**

Thus the improper integral converges.

Note: (1) $\displaystyle\int_0^1 \frac{1}{x^2}dx$ is not convergent, and therefore $\displaystyle\int_0^\infty \frac{1}{x^2}dx$ is not convergent. So, one does not get convergence of $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}dx$ comparing with $\displaystyle\int_0^\infty \frac{1}{x^2}dx$.

(2) Derichlet test is not applicable to $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}dx$ (with $\sin^2 x$ and $\frac{1}{x^2}$ as the two functions), because $\{\int_0^x \sin^2 t\,dt : x \in \mathbb{R}\}$ is not bounded.

(2 pts.) 16. Find the arc length of the curve: $y = \int_0^x \sqrt{\cos(2t)}\, dt$, $0 \le x \le \pi/4$.

$$\text{Length} = \int_0^{\pi/4} \sqrt{1+(y')^2}\, dx = \int_0^{\pi/4} \sqrt{1+\cos(2x)}\, dx \qquad \boxed{\textbf{1 Mark}}$$

$$= \sqrt{2} \int_0^{\pi/4} |\cos x| dx = 1. \qquad \boxed{\textbf{1 Mark}}$$

(2 pts.) 17. Let R be the region in the first quadrant bounded by the curve $y = x - x^2$ and the line $y = 0$. Find the volume generated by revolving the region R about the x-axis.

$$V = \int_0^1 \pi y^2 dx = \pi \int_0^1 (x - x^2)^2 dx \qquad \boxed{\textbf{1 Mark}}$$

$$\ldots = \frac{\pi}{30}. \qquad \boxed{\textbf{1 Mark}}$$