DEPARTMENT OF MATHEMATICS, IIT Guwahati
MA101: Mathematics I, July - November 2014
Summary of Lectures (Set - I)

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1 System of Linear Equations and Their Solutions

An Example for Motivation:

To solve the system of linear equations: $x - y - z = 2,\ 3x - 3y + 2z = 16,\ 2x - y + z = 9.$ We solve this system by eliminating the variables.

Step 1: Represent the given system of equations in the rectangular array form as follows.

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 2 \\ 3x & - & 3y & + & 2z & = & 16 \\ 2x & - & y & + & z & = & 9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

Step 2: Subtract 3 times the 1st equation from the 2nd equation; and ***subtract* 3 *times the 1st row from the 2nd row.***

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 2 \\ & & & & 5z & = & 10 \\ 2x & - & y & + & z & = & 9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

Step 3: Subtract 2 times the 1st equation from the 3rd equation; and ***subtract* 2 *times the 1st row from the 3rd row.***

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 2 \\ & & & & 5z & = & 10 \\ & & y & + & 3z & = & 5 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

Step 4: Interchange the 2nd and 3rd equation; and ***interchange the 2nd and 3rd row.***

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 2 \\ & & y & + & 3z & = & 5 \\ & & & & 5z & = & 10 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

Now by backward substitution, we find that $z = 2, y = -1, x = 3$ is a solution of the given system of equations.

Definition:

- A rectangular array of (complex) numbers is called a **matrix**. Formally, an $m \times n$ matrix $A = [a_{ij}]$ is an array of m rows and n columns as shown below:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

  The matrix A is called a matrix of **size** $m \times n$ or a matrix of **order** $m \times n$.

- The number a_{ij} is called the (i, j)-th entry of A.

- A $1 \times n$ matrix is called a **row matrix** (or *row vector*) and an $n \times 1$ matrix is called a **column matrix** (or *column vector*).

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** if they are of same size and $a_{ij} = b_{ij}$ for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

- A matrix whose all entries are 0 is called a **zero** matrix.

- The **transpose** A^t of an $m \times n$ matrix $A = [a_{ij}]$ is defined to be the $n \times m$ matrix $A^t = [a_{ji}]$, where the i-th row of A^t is the i-th column of A for all $i = 1, 2, \ldots, n$.
- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the **sum** $A + B$ is defined to be the matrix $C = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$. Similarly, the **difference** $A - B$ is defined to be the matrix $D = [d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}$.
- For a matrix $A = [a_{ij}]$ and $c \in \mathbb{C}$ (**set of complex numbers**), we define cA to be the matrix $[ca_{ij}]$.
- Let $A = [a_{ij}]$ and $B = [b_{jk}]$ be two $m \times n$ and $n \times r$ matrices, respectively. Then the **product** AB is defined to be the $m \times r$ matrix $AB = [c_{ik}]$, where

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}.$$

Row Echelon Form: A matrix A is said to be in row echelon form if it satisfies the following properties:

1. Any rows consisting entirely of 0's are at the bottom.
2. In each non-zero row, the first non-zero entry (called the **leading entry** or **pivot**) is in a column to the left (**strictly**) of any leading entry below it.

Notice that if a matrix A is in row echelon form, then in each column of A containing a leading entry, the entries below that leading entry are zero. For example, the following matrices are in row echelon form:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 3 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

However, the following matrices are not in row echelon form:

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -1 & 2 \\ 1 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Elementary Row Operations: The following row operations are called elementary row operation of a matrix:

1. Interchange of two rows R_i and R_j (shorthand notation $R_i \leftrightarrow R_j$).
2. Multiply a row R_i by a non-zero constant c (shorthand notation $R_i \to cR_i$).
3. Add a multiple of a row R_j to another row R_i (shorthand notation $R_i \to R_i + cR_j$).

Any given matrix can be reduced to a row echelon form by applying suitable elementary row operations on the matrix.

Example 1.1. *Transform the following matrix to row echelon form*

$$\begin{bmatrix} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{bmatrix}.$$

Note that if A is in row echelon form then for any $c \neq 0$, the matrix cA is also in row echelon form. Thus a given matrix can be reduced to several row echelon form.

Row Equivalent Matrices: Matrices A and B are said to be row equivalent if there is a sequence of elementary row operations that converts A into B.

Result 1.1. *Matrices A and B are row equivalent if and only if (**iff**) they can be reduced to the same row echelon form.*

Linear System of Equations:

A linear system of m equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots\ &\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned} \tag{1}$$

where $a_{ij}, b_i \in \mathbb{R}$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$. Letting

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

we can represent the above system of equations as $A\mathbf{x} = \mathbf{b}$.

- The matrix A is called the **coefficient matrix** of the system of equations $A\mathbf{x} = \mathbf{b}$.
- The matrix $[A|\ \mathbf{b}]$, as given below, is called the **augmented matrix** of the system of equations $A\mathbf{x} = \mathbf{b}$.

$$[A|\ \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

  The vertical bar is used in the augmented matrix $[A|\ \mathbf{b}]$ only to distinguish the column vector $\mathbf{b}$ from the coefficient matrix A.

- If $\mathbf{b} = \mathbf{0} = [0, 0, \dots, 0]^t$, i.e., if $b_1 = b_2 = \dots = b_m = 0$, the system $A\mathbf{x} = \mathbf{0}$ is called a **homogeneous** system of equations. Otherwise, if $\mathbf{b} \neq \mathbf{0}$ then $A\mathbf{x} = \mathbf{b}$ is called an **non-homogeneous** system of equations.
- A **solution** of the linear system $A\mathbf{x} = \mathbf{b}$ is a column vector $\mathbf{y} = [y_1, y_2, \dots, y_n]^t$ such that the linear system (1) is satisfied by substituting y_i in place of x_i. That is, $A\mathbf{y} = \mathbf{b}$ holds true.
- The solution $\mathbf{0}$ of $A\mathbf{x} = \mathbf{0}$ is called the **trivial solution** and any other solutions of $A\mathbf{x} = \mathbf{0}$ are called **non-trivial** solutions.

Result 1.2. *Let $Cx = d$ be the linear system obtained from the linear system $Ax = b$ by a single elementary operation. Then the linear systems $Ax = b$ and $Cx = d$ have the same set of solutions.*

Result 1.3. *Two equivalent system of linear equations have the same set of solutions.*

Leading and Free Variable: Consider the linear system $A\mathbf{x} = \mathbf{b}$ in n variables and m equations. Let $[R \mid \mathbf{r}]$ be the reduced row echelon form of the augmented matrix $[A \mid \mathbf{b}]$.

- Then the variables corresponding to the leading columns in the first n columns of $[R \mid \mathbf{r}]$ are called the **leading variables** or **basic variables**.
- The variables which are not leading are called **free variables**.

Gaussian Elimination Method: Use the following steps to solve a system of equations $A\mathbf{x} = \mathbf{b}$.

1. Write the augmented matrix $[A \mid \mathbf{b}]$.
2. Use elementary row operations to reduce $[A \mid \mathbf{b}]$ to row echelon form.
3. Use ***back substitution*** to solve the equivalent system that corresponds to the row echelon form.

Example 1.2. *Use Gaussian Elimination method to solve the system:*
(*a*) $y + z = 1, \quad x + y + z = 2, \quad x + 2y + 2z = 3$
(*b*) $y + z = 1, \quad x + y + z = 2, \quad x + 2y + 3z = 4$
(*c*) $y + z = 1, \quad x + y + z = 2, \quad x + 2y + 2z = 4.$

- If the system $A\mathbf{x} = \mathbf{b}$ has some solution then it is called a **consistent** system. Otherwise it is called an **inconsistent** system.

Reduced Row Echelon Form: A matrix A is said to be in *reduced row echelon form* (**RREF**) if it satisfies the following properties:

1. A is in row echelon form.

2. The leading entry in each non-zero row is a 1.

3. Each column containing a leading 1 has zeros everywhere else.

For example, the following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying elementary row operations, any given matrix can be transformed to a reduced row echelon form.

Method of Induction: [Version I] Let $P(n)$ be a mathematical statement based on all positive integers n. Suppose that $P(1)$ is true. If $k \geq 1$ and if the assumption that $P(k)$ is true gives that $P(k+1)$ is also true, then the statement $P(n)$ is true for all positive integers.

Method of Induction: [Version II] Let i be an integer and let $P(n)$ be a mathematical statement based on all integers n of the set $\{i, i+1, i+2, \ldots\}$. Suppose that $P(i)$ is true. If $k \geq i$ and if the assumption that $P(k)$ is true gives that $P(k+1)$ is also true, then the statement $P(n)$ is true for all integers of the set $\{i, i+1, i+2, \ldots\}$.

Result 1.4. *Every matrix has a* ***unique*** *reduced row echelon form.*

Proof. We will apply induction on number of columns.
Base Case: Consider a matrix with one column *i.e.,* $A_{m\times 1}$. The possibilities being $\mathbf{O}_{m\times 1}$ and e_1, its RREF is unique.
Induction Hypothesis: Let any matrix having $k-1$ columns have a unique RREF.
Inductive Case: We need to show that $A_{m\times k}$ also has unique RREF. Assume A has two RREF, say B and C. Note that because of induction hypothesis the submatrices B_1 and C_1 formed by the first $k-1$ columns of B and C, respectively, are identical. In particular, they have same number of nonzero rows.

We have that the systems $A\mathbf{x} = 0$, $B\mathbf{x} = 0$ and $C\mathbf{x} = 0$ have the same set of solutions. Let the i^{th} row of the k^{th} column be different in B and C i.e. $b_{ik} \neq c_{ik} \Rightarrow b_{ik} - c_{ik} \neq 0$.

Let $\mathbf{u}$ be an arbitrary solution of $A\mathbf{x} = 0$.
$\Rightarrow B\mathbf{u} = 0$, $C\mathbf{u} = 0$ and $(B - C)\mathbf{u} = 0$
$\Rightarrow (b_{ik} - c_{ik})u_k = 0 \Rightarrow u_k$ must be 0
$\Rightarrow x_k$ is not a free variable
$\Rightarrow$ there must be leading 1 in the k^{th} column of B and C
$\Rightarrow$ the location of 1 is different in the k^{th} column
$\Rightarrow$ numbers of nonzero rows in B_1 and C_1 are different, a contradiction.
This contradiction leads us to conclude that $B = C$.

Hence by the Method of Induction, we conclude that every matrix has a **unique** reduced row echelon form. $\square$

Method of transforming a given matrix to reduced row echelon form: Let A be an $m \times n$ matrix. Then the following step by step method is used to obtain the reduced row echelon form of the matrix A.

1. Let the i-th column be the left most non-zero column of A. Interchange rows, if necessary, to make the first entry of this column non-zero. Now use elementary row operations to make all the entries below this first entry equal to 0.

2. Forget the first row and first i columns. Start with the lower $(m-1) \times (n-i)$ sub matrix of the matrix obtained in the first step and proceed as in **Step 1**.

3. Repeat the above steps until we get a row echelon form of A.

4. Now use the leading term in each of the leading column to make (by elementary row operations) all other entries in that column equal to zero. Use this step starting form the rightmost leading column.

5. Scale all non-zero entries (leading entries) of the matrix obtained in the previous step, by multiplying the rows by suitable constants, to make all the leading entries equal to 1, ending with the unique reduced row echelon form of A.

Gauss-Jordan Elimination Method: Use the following steps to solve a system of equations $A\mathbf{x} = \mathbf{b}$.

1. Write the augmented matrix $[A \mid \mathbf{b}]$.

2. Use elementary row operations to transform $[A \mid \mathbf{b}]$ to reduced row echelon form.
3. Use back substitution to solve the equivalent system that corresponds to the reduced row echelon form. That is, solve for the leading variables in terms of the remaining free variables, if possible.

Example 1.3. *Solve the system* $w - x - y + 2z = 1,\ 2w - 2x - y + 3z = 3,\ -w + x - y = -3$ *using Gauss-Jordan elimination method.*

Example 1.4. *Solve the following systems using Gauss-Jordan elimination method:*
(a) $2y + 3z = 8, \quad 2x + 3y + z = 5, \quad x - y - 2z = -5$
(b) $x - y + 2z = 3, \quad x + 2y - z = -3, \quad 2y - 2z = 1.$

Rank: The rank of a matrix A, denoted $\text{rank}(A)$, is the number of non-zero rows in its row echelon form.

Result 1.5. *Let* $A\mathbf{x} = \mathbf{b}$ *be a consistent system of equations with* n *variables. Then number of free variables is equal to* $n - \text{rank}(A)$.

Result 1.6. *Let* $A\mathbf{x} = \mathbf{0}$ *be a homogeneous system of equations with* n *variables. If* $\text{rank}(A) < n$ *then the system has infinitely many solutions.*

Result 1.7. *Let* $A\mathbf{x} = \mathbf{b}$ *be a system of equations with* n *variables. Then*

1. *if* $\text{rank}(A) \neq \text{rank}([A \mid \mathbf{b}])$ *then the system* $A\mathbf{x} = \mathbf{b}$ *is inconsistent;*
2. *if* $\text{rank}(A) = \text{rank}([A \mid \mathbf{b}]) = n$ *then the system* $A\mathbf{x} = \mathbf{b}$ *has a unique solution; and*
3. *if* $\text{rank}(A) = \text{rank}([A \mid \mathbf{b}]) < n$ *then the system* $A\mathbf{x} = \mathbf{b}$ *has a infinitely many solutions.*