MA101 Mathematics I

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Plan

- Similarity and Diagonalization
- Inner product, Norm and Orthogonality in $\mathbb{R}^n$
- Gram-Schmidt Process
- Orthogonal Complements

Similarity and Diagonalization

Suppose $A \in \mathcal{M}_n$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and $\mathbf{v}_i$ are eigenvectors w.r.t. λ_i, $1 \le i \le n$.

Recall: $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are LI and so form a basis for $\mathbb{R}^n$.

Let $P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]$. Then P is invertible. What is AP?

$$\begin{aligned} AP &= A[\mathbf{v}_1\,\mathbf{v}_2 \cdots \mathbf{v}_n] = [A\mathbf{v}_1\, A\mathbf{v}_2 \cdots A\mathbf{v}_n] = [\lambda_1\mathbf{v}_1\,\lambda_2\mathbf{v}_2 \cdots \lambda_n\mathbf{v}_n] \\ &= P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = P\,\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = PD. \end{aligned}$$

Thus, we get $P^{-1}AP = D$, a diagonal matrix.

Definition

If $P^{-1}AP = B$ for some P, then we say that A and B are similar. (Note that, in that case $PBP^{-1} = A$.) We then write, $A \approx B$.

If for some P, $P^{-1}AP = D$, a diagonal matrix, then A is said to be diagonalizable, and $P^{-1}AP = D$ a diagonalization of A.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$. Then $A \approx B$, since $AP = PB$, where $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Note: A has eigenvalues $\lambda = \pm 1$, and hence A is diagonalizable.

Find a diagonalization $Q^{-1}AQ = \mathrm{diag}\,(1, -1)$. Is there another diagonalization?

(Note: For any invertible diagonal matrix D_1, $D_1^{-1}DD_1 = D$.)

Is B diagonalizable? Yes, A and B have same eigenvalues. Moreover,

$$\mathrm{diag}\,(1, -1) = Q^{-1}AQ = Q^{-1}PBP^{-1}Q = (P^{-1}Q)^{-1}B(P^{-1}Q).$$

Exercise

Is the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ *diagonalizable? If so, find a diagonalization of* A.

Diagonalization

Suppose $A \in \mathcal{M}_n$ has n distinct eigenvalues. Then A is diagonalizable. Is the converse true? No. Consider $A = I_n$.

Suppose A is diagonalizable, and $P^{-1}AP = D = \text{diag}(\lambda_1, \ldots, \lambda_n)$, i.e., $AP = PD$. If $P = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$, then $\mathbf{v}_i \neq \mathbf{0}$ and

$$[\lambda_1\mathbf{v}_1\ \lambda_2\mathbf{v}_2 \cdots \lambda_n\mathbf{v}_n] = PD = AP = [A\mathbf{v}_1\ A\mathbf{v}_2 \cdots A\mathbf{v}_n],$$

i.e., λ_i is an eigenvalue and $\mathbf{v}_i$ is an eigenvector w.r.t. λ_i. Consequently, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{R}^n$ comprising of eigenvectors of A.

Converse: If $\mathbb{R}^n$ has a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ comprising of eigenvectors of A, then A is diagonalizable: $P^{-1}AP$ is diagonal, where $P = [\mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_n]$.

Note: The condition will hold if $\dim(E_{\lambda_i})$ equals the algebraic multiplicity of λ_i for each eigenvalue λ_i of A.

Exercise

The geometric multiplicity of each eigenvalue of a matrix is less than or equal to its algebraic multiplicity.

Exercise

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of a matrix A. Suppose $\mathcal{B}_i$ is a basis for the eigenspace E_{λ_i}. Then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_k$ is a linearly independent set.

Result (**The Diagonalization Theorem**)

For an $n \times n$ matrix A, the following statements are equivalent:

1. *A is diagonalizable.*
2. *A has n linearly independent eigenvectors.*
3. *The union $\mathcal{B}$ of the bases of the eigenspaces of A contains n vectors.*
4. *The algebraic multiplicity of each eigenvalue A equals its geometric multiplicity.*

Exercise

Is the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ diagonalizable? Yes. $1, 4, 6$ are (distinct) eigenvalues of A.

You can easily find: $1, 4, 6$ have eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}$, respectively.

Exercise

Is the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable? No. 1 is the only eigenvalue with algebraic multiplicity three. However, the eigenspace,

$$E_1 = \text{null}\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

Result

For a matrix $A \in \mathcal{M}_n$, sum of the eigenvalues is $\text{trace}(A)$ and the product of the eigenvalues is $\det(A)$.

[We will try to give a proof in the later part of the slides.]

Exercise

Is the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix}$ diagonalizable?

Note that $\text{rank}(A) = 1$. So, $\dim(E_0) = \text{nullity}(A) = 2$. Since sum of the eigenvalues is 11, the third eigenvalue must be 11. The algebraic and geometric multiplicities of the eigenvalues are equal. So, A is diagonalizable.

$$\left[\text{Eigenvalues } 0, 0, 11; \quad \text{Eigenvectors } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \text{ respectively.}\right]$$

Inner Products on $\mathbb{C}^n$ and $\mathbb{R}^n$, norm, angle

For $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{C}^n$, we define

- Conjugate transpose of $\mathbf{u}$: $\mathbf{u}^* := \overline{\mathbf{u}}^T = [\overline{u_1}, \overline{u_2}, \ldots, \overline{u_n}]$.
- The inner product or the dot product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \cdot \mathbf{v} := \mathbf{u}^* \mathbf{v} = \overline{u_1} v_1 + \overline{u_2} v_2 + \ldots + \overline{u_n} v_n.$$

  In case $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^* \mathbf{v} = \mathbf{u}^T \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

- The norm or length of $\mathbf{u}$:

$$\|\mathbf{u}\| := \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{|u_1|^2 + |u_2|^2 + \ldots + |u_n|^2}.$$

- $\mathbf{u}$ is called an unit vector if $\|\mathbf{u}\| = 1$.
- For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is defined to be θ,

  where $\cos\theta := \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$.

Result

For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$, $c \in \mathbb{C}$,

- $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$.
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
- $(c\,\mathbf{u}) \cdot \mathbf{v} = \overline{c}(u \cdot \mathbf{v}), \quad \mathbf{u} \cdot (c\,\mathbf{v}) = c(u \cdot \mathbf{v})$.
- $\|\mathbf{u}\| \geq 0$ *and* $\|\mathbf{u}\| = 0 \Leftrightarrow \mathbf{u} = 0$.

Definition

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

- Two elements $\mathbf{u}, \mathbf{v}$ in $\mathbb{F}^n$ are said to be orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$, i.e., if $\cos\theta = 0$.
- A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbb{F}^n$ is said to be orthogonal if

  $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.
- A basis $\mathcal{B}$ of $\mathbb{F}^n$ is an orthogonal basis, if it is an orthogonal set.

- The vector $\mathbf{0} \in \mathbb{F}^n$ is orthogonal to every vector in $\mathbb{F}^n$.
- The standard basis of $\mathbb{F}^n$ is orthogonal.

Example

- The set $\left\{\begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$ is orthogonal in $\mathbb{F}^3$.
- The set $\left\{\begin{bmatrix}1\\i\end{bmatrix}, \begin{bmatrix}i\\1\end{bmatrix}\right\}$ is orthogonal set in $\mathbb{C}^2$.

Result

Any orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of nonzero vectors is linearly independent.

PROOF. Suppose $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = 0$. Then, for each i $0 = \mathbf{v}_i \cdot (\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) = \alpha_i\mathbf{v}_i \cdot \mathbf{v}_i = \alpha_i\|\mathbf{v}_i\|^2$. Note that since $\mathbf{v} \neq \mathbf{0}$, $\|\mathbf{v}\| \neq 0$. Thus, $\alpha_i = 0$. ∎

Result

Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal basis for a subspace $\mathbb{W}$ of $\mathbb{F}^n$. Suppose $\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k \in \mathbb{W}$. Then,

$$\mathbf{v}_i \cdot \mathbf{w} = \alpha_i\mathbf{v}_i \cdot \mathbf{v}_i, \quad \text{i.e.,} \quad \alpha_i = \frac{\mathbf{v}_i \cdot \mathbf{w}}{\|\mathbf{v}_i\|^2} \quad \text{for} \quad i = 1, 2, \ldots, k.$$

In case $\|\mathbf{v}_i\|^2 = 1$ (i.e., $\mathbf{v}_i$ are unit vectors), then $\alpha_i = \mathbf{v}_i \cdot \mathbf{w}$.

Definition

- An orthogonal set of unit vectors is called an orthonormal set.
- A basis which is an orthonormal set is called an othonormal basis.

Example

The standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis of $\mathbb{F}^n$.

Does every subspace of $\mathbb{F}^n$ have an orthonormal basis? Yes. In fact, you can create one from any given basis of the subspace. The process is called Gram-Schmidt Process.

Gram-Schmidt Process- example

Suppose $\mathbb{W} = \text{span}\left(\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 11 \\ 4 \end{bmatrix}\right) \preccurlyeq \mathbb{R}^4$.

We find an orthonormal basis of $\mathbb{W}$ as follows: Put

- $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \\ 0 \end{bmatrix}$;

  $\mathbf{u}_2 = \mathbf{v}_2 - (\mathbf{w}_1 \cdot \mathbf{v}_2)\mathbf{w}_1 = \begin{bmatrix} -1 \\ 0 \\ 7 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix}$,

- $\mathbf{w}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{bmatrix} \frac{-4}{5} \\ 0 \\ \frac{3}{5} \\ 0 \end{bmatrix}$;

  $\mathbf{u}_3 = \mathbf{v}_3 - (\mathbf{w}_1 \cdot \mathbf{v}_3)\mathbf{w}_1 - (\mathbf{w}_2 \cdot \mathbf{v}_3)\mathbf{w}_2 = \mathbf{v}_3 - 10\mathbf{w}_1 + 5\mathbf{w}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \end{bmatrix}$.

- $\mathbf{w}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \begin{bmatrix} 0 \\ \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}$. Then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthonormal basis of $\mathbb{W}$.

Result (**The Gram-Schmidt Process**)

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$. Define

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1 & \mathbf{w}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}; \\
\mathbf{u}_2 &= \mathbf{v}_2 - (\mathbf{w}_1 \cdot \mathbf{v}_2)\mathbf{w}_1 & \mathbf{w}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}; \\
\mathbf{u}_3 &= \mathbf{v}_3 - (\mathbf{w}_1 \cdot \mathbf{v}_3)\mathbf{w}_1 - (\mathbf{w}_2 \cdot \mathbf{v}_3)\mathbf{w}_2 & \mathbf{w}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}; \\
&\vdots \quad \vdots & &\vdots \quad \vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{i=1}^{k-1} (\mathbf{w}_i \cdot \mathbf{v}_k)\mathbf{w}_i & \mathbf{w}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

Then

- $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ *is orthonormal.*
- *For* $1 \le i \le k$, *span*$(\mathbf{w}_1, \dots, \mathbf{w}_i) =$ *span*$(\mathbf{v}_1, \dots, \mathbf{v}_i)$.
- *Therefore,* $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ *is an orthonormal basis of* $\mathbb{W}$.

- Recall: For $A \in \mathcal{M}_n(\mathbb{C})$, the conjugate transpose of A, $A^* := \overline{A}^T$.

Result

Let A be a Hermitian (or real symmetric) matrix. Then

- *all eigenvalues of A are real.*
- *eigenvectors w.r.t. distinct eigenvalues of A are orthogonal.*

PROOF. Suppose A is a Hermitian matrix, i.e., $A^* = A$. Let λ be an eigenvalue of A. Then for an eigenvector $\mathbf{v}$ of A,

$$\begin{aligned}\lambda\|v\|^2 &= \lambda(\mathbf{v}\cdot\mathbf{v}) = \mathbf{v}\cdot(\lambda\mathbf{v}) = \mathbf{v}\cdot A\mathbf{v} = \mathbf{v}^*(A\mathbf{v}) = (\overline{\mathbf{v}}^T A)\mathbf{v} \\ &= (A^T\overline{\mathbf{v}})^T\mathbf{v} = (\overline{A^*\mathbf{v}})^T\mathbf{v} = (\overline{A\mathbf{v}})^T\mathbf{v} = (A\mathbf{v})^*\mathbf{v} \\ &= (\lambda\mathbf{v})\cdot\mathbf{v} = \overline{\lambda}(\mathbf{v}\cdot\mathbf{v}) = \overline{\lambda}\|v\|^2.\end{aligned}$$

Since $\mathbf{v} \neq 0$, we have $\|v\|^2 \neq 0$. Thus, $\lambda = \overline{\lambda}$, i.e., λ is real.

Next, suppose λ, μ are two distinct eigenvalues of A, and $\mathbf{u}$ and $\mathbf{v}$ are corresponding eigenvectors. Then

$$\begin{aligned}\mu(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{u}\cdot(\mu\mathbf{v}) = \mathbf{u}\cdot(A\mathbf{v}) = (\mathbf{u}^*A)\mathbf{v} = (A^*\mathbf{u})^*\mathbf{v} \\ &= (A\mathbf{u})^*\mathbf{v} = (\lambda\mathbf{u})\cdot\mathbf{v} = \overline{\lambda}(\mathbf{u}\cdot\mathbf{v}).\end{aligned}$$

Since λ is real we get $(\lambda-\mu)(\mathbf{u}\cdot\mathbf{v}) = 0$, i.e., $\mathbf{u}\cdot\mathbf{v} = 0$. ■

Definition

- A matrix $U \in \mathcal{M}_n(\mathbb{C})$ whose columns form an orthonormal set is called a unitary matrix.
- A real unitary matrix is called an orthogonal matrix.
- A matrix A is said to be unitarily diagonalizable if A has a diagonalization $U^{-1}AU = D$, where U is a unitary matrix.

Exercise

Show that $A \in \mathcal{M}_n(\mathbb{C})$ is unitary iff $A^{-1} = A^*$ and $A \in \mathcal{M}_n(\mathbb{R})$ is orthogonal iff $A^{-1} = A^*$

[Hint. If $A = [\mathbf{u}_1\, \mathbf{u}_2 \ldots \mathbf{u}_n]$, then the (i,j)-th entry of A^*A is $\mathbf{u}_i^*\mathbf{u}_j = \mathbf{u}_i \cdot \mathbf{u}_j$.]

Result (The Spectral Theorem of Hermitian matrices)

Every Hermitian matrix $A \in \mathcal{M}_n(\mathbb{C})$ is unitarily diagonalizable.

Sketch of the Proof.

Use induction on n. Result is obvious for $n = 1$. Suppose true for $n-1$. Let λ_1 be an eigenvalue with eigenvector $\mathbf{v}_1$. We can assume $\|v\| = 1$. Extend $\{\mathbf{v}_1\}$ to an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathbb{C}^n$. Let $U = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n]$. Then $U^{-1} = U^*$. Verify that $U^{-1}AU = \left[\begin{array}{c|c} \lambda_1 & \mathbf{0} \\ \hline \mathbf{0} & A_1 \end{array}\right]$ for some A_1 and that A_1 is Hermitian. By induction hypothesis, we have unitary diagonalization $U_1^{-1}A_1U_1 = D_1$ of A_1. Now verify that $U_2 = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & U_1 \end{array}\right]$ is unitary, and $(U_2U)^{-1}A(U_2U) = \left[\begin{array}{c|c} \lambda_1 & \mathbf{0} \\ \hline \mathbf{0} & D_1 \end{array}\right]$ is a unitarily diagonalization of A. ∎

The technique of the proof of the previous theorem can be used to prove the following:

Result (Schur Triangulization Theorem)

For every $A \in \mathcal{M}_n(\mathbb{C})$, there is a unitary matrix U such that $U^{-1}AU$ is triangular.

- Suppose $A \in \mathcal{M}_n(\mathbb{C})$ and $U^{-1}AU = Q$, where Q is triangular. Now note the following:
  - $A \approx Q$, and so they have same eigenvalues.
  - The eigenvalues of Q are its diagonal elements.
  - Sum of the eigenvalues of A equals the sum of the eigenvalues of Q, which equals the trace of Q.
  - Since $A \approx Q$, A and Q have same trace.
  - Sum of the eigenvalues of A is its trace.
  - Similarly (how?), the product of the eigenvalues of A is $\det(A)$.

Orthogonal Complements

Consider $\mathbb{W} = \text{span}(\mathbf{e}_1, \mathbf{e}_2) \preccurlyeq \mathbb{R}^3$. If $\mathbf{u}$ is orthogonal to every vector in $\mathbb{W}$, what can you say about $\mathbf{u}$? We must have $\mathbf{u} = \alpha\mathbf{e}_3$ for some $\alpha \in \mathbb{R}$.

Consider $\mathbb{W} = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) \preccurlyeq \mathbb{R}^n$. Then,

$$
\begin{aligned}
& w \cdot \mathbf{u} = 0 \text{ for every } \mathbf{w} \in \mathbb{W} \\
\text{iff} \quad & \mathbf{v}_i^T \mathbf{u} = 0 \text{ for } 1 \leq i \leq k, \\
\text{iff} \quad & A^T \mathbf{u} = \mathbf{0}, \text{ where } A = [\mathbf{v}_1 \, \mathbf{v}_2 \, \ldots \, \mathbf{v}_k], \\
\text{iff} \quad & \mathbf{u} \in \text{null}(A^T).
\end{aligned}
$$

- Let $\mathbb{W}$ be a subspace of $\mathbb{R}^n$.
  - $\mathbf{v} \in \mathbb{R}^n$ is said to be orthogonal to $\mathbb{W}$, if $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{w} \in \mathbb{W}$.
  - The orthogonal complement of $\mathbb{W}$:

$$\mathbb{W}^\perp := \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in \mathbb{W}\}.$$

- Clearly, $\mathbb{W}^\perp \preccurlyeq \mathbb{R}^n$, and $\mathbb{W} \cap \mathbb{W}^\perp = \{\mathbf{0}\}$.

- If $\mathbb{W} = \text{span}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k)$, then $\mathbf{v} \in \mathbb{W}^\perp$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, 2, \ldots, k$.

Result

Let A be an $m \times n$ matrix. Then $(\text{col}(A))^\perp = \text{null}(A^T)$ and $(\text{row}(A))^\perp = \text{null}(A)$.

Exercise

Find a basis for $\mathbb{W}^\perp$, where
$\mathbb{W} = \text{span}([1, -3, 5, 0, 5]^T, [-1, 1, 2, -2, 3]^T, [0, -1, 4, -1, 5]^T)$.

Result

Let $\mathbb{W} \preccurlyeq \mathbb{R}^n$ and let $\mathbf{v} \in \mathbb{R}^n$. Then there are unique vectors $\mathbf{w} \in \mathbb{W}$ and $\mathbf{w}^\perp \in \mathbb{W}^\perp$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$. That is, $\mathbb{R}^n = \mathbb{W} \bigoplus \mathbb{W}^\perp$.

[This result is known as the Orthogonal Decomposition Theorem.]

Proof. Take an orthonormal basis $\{u_1, \ldots, \mathbf{u}_k\}$ of $\mathbb{W}$. For $\mathbf{v} \in \mathbb{R}^n$ take $\mathbf{w} = \sum_{i=1}^{k} (\mathbf{v} \cdot \mathbf{u}_i)\mathbf{u}_i$. Put $\mathbf{w}^\perp = \mathbf{v} - \mathbf{w} \in \mathbb{W}^\perp$. Then $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ is the unique representation. ■

Exercises

- Let $A, B \in \mathcal{M}_n(\mathbb{F})$ be similar. Argue that the following hold:
  1. For some invertible P, $AP = PB$.
  2. $\det A = \det B$.
  3. A is invertible iff B is invertible.
  4. A and B have the same rank.
  5. A and B have the same characteristic polynomial.
  6. A and B have the same eigenvalues.

Exercises

- Check for the diagonalizablity of the following matrices. If they are diagonalizable, find invertible matrices P that diagonalizes them:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}.$$

- Show that for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$
  1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;
  2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$;
  3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$;
  4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
- Apply the Gram-Schmidt process to find an orthonormal basis of the subspace spanned by $\mathbf{u} = [1, -1, -1]^T$, $\mathbf{v} = [0, 3, 3]^T$ and $\mathbf{w} = [3, 2, 4]^T$.

Exercises

- Use Orthogonal Decomposition Theorem to show:
  (1) $\dim \mathbb{W} + \dim \mathbb{W}^{\perp} = n$,

  (2) $(\mathbb{W}^{\perp})^{\perp} = \mathbb{W}$.

  (3) For any matrix A, $\text{rank}(A) + \text{nullity}(A) = n$.

- Let $\mathbb{W}$ be a subspace of $\mathbb{R}^n$. Show that there exists a matrix A such that $\mathbb{W} = \text{null}(A)$.

  [Hint. Take $A = [\mathbf{a}_1 \ \ldots \ \mathbf{a}_k]$, where $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ is an orthonormal basis of $\mathbb{W}^{\perp}$.]

- Find a basis for $W^{\perp}$, where $W = \text{span}([1, -3, 5, 0, 5]^t, [-1, 1, 2, -2, 3]^t, [0, -1, 4, -1, 5]^t)$.