DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
MA101 MATHEMATICS-I

First Semester of Academic Year 2015 - 2016

Solutions to Tutorial Sheet - 1

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Row-echelon Form, System of Linear Equations & Gaussian Elimination

Recall:

- A matrix is in row echelon form if it satisfies the following properties:
  1. Any rows consisting entirely of zeros are at the bottom,
  2. In each non-zero row, the first non-zero entry (called the leading entry) is in a column to the left of any leading entries below it.
- A matrix is in reduced row echelon form if it satisfies the following properties:
  1. It is in row echelon form.
  2. The leading entry in each non-zero row is a 1 (called a leading 1),
  3. Each column containing a leading 1 has zeros everywhere else
- Two system of linear equations are equivalent if and only if they have the same set of solutions.
- $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ is the set of remainders modulo 3.

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1. Check whether the following systems of linear equations are equivalent, by reducing the corresponding matrices to the row-echelon form.

$$System - I$$

$$\begin{aligned} 7x + 2y - 3z &= 33 \\ 3x - y + 5z &= -14 \\ 2x + 4y - z &= 27 \end{aligned}$$

$$System - II$$

$$\begin{aligned} 3x + 4y + 2z &= 20 \\ x + 3y - z &= 20 \\ x + 2y + 3z &= 3 \\ 4y + z &= 17 \end{aligned}$$

Soln. Consider the augmented matrix for System I:

$$\begin{bmatrix} 7 & 2 & -3 & | & 33 \\ 3 & -1 & 5 & | & -14 \\ 2 & 4 & -1 & | & 27 \end{bmatrix} \xrightarrow{R_2 \leftarrow 7R_2 - 3R_1} \begin{bmatrix} 7 & 2 & -3 & | & 33 \\ 0 & -13 & 44 & | & -197 \\ 2 & 4 & -1 & | & 27 \end{bmatrix} \xrightarrow{R_3 \leftarrow 7R_2 - 2R_1} \begin{bmatrix} 7 & 2 & -3 & | & 33 \\ 0 & -13 & 44 & | & -197 \\ 0 & 24 & -1 & | & 123 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow 24R_2 + 13R_3} \begin{bmatrix} 7 & 2 & -3 & | & 33 \\ 0 & -13 & 44 & | & -197 \\ 0 & 0 & 1043 & | & -3129 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3/1043} \begin{bmatrix} 7 & 2 & -3 & | & 33 \\ 0 & -13 & 44 & | & -197 \\ 0 & 0 & 1 & | & -3 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow 13R_1 + 2R_2} \begin{bmatrix} 91 & 0 & 49 & | & 35 \\ 0 & -13 & 44 & | & -197 \\ 0 & 0 & 1 & | & -3 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 49R_3, R_2 \leftarrow R_2 - 44R_3} \begin{bmatrix} 91 & 0 & 0 & | & 182 \\ 0 & -13 & 0 & | & -65 \\ 0 & 0 & 1 & | & -3 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1/91, R_2 \leftarrow R_2/(-13)} \begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & -3 \end{bmatrix}$$

Similarly we can compute the rref of the augmented matrix corresponding to System II. Since the rref of the augmented matrices of the corresponding systems are given by

$$\underset{\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}}{System - I} \qquad \underset{\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}}{System - II}$$

Hence these are equivalent systems.

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2. List all possible reduced row echelon form of each of a 2×2 and a 3×3 matrix.
Soln. The required list is as follows:

(a) Case I : 2×2 matrices.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Case II : 3×3 matrices.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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3. For what values of $\lambda \in \mathbb{R}$, the following system of equations has (i) no solution, (ii) a unique solution, and infinitely many solutions ?

$$(5-\lambda)x + 4y + 2z = 4, \quad 4x + (5-\lambda)y + 2z = 4, \quad 2x + 2y + (2-\lambda)z = 2.$$

Soln. Consider the augmented matrix of the above systems,

$$\begin{bmatrix} 5-\lambda & 4 & 2 & | & 4 \\ 4 & 5-\lambda & 2 & | & 4 \\ 2 & 2 & 2-\lambda & | & 2 \end{bmatrix} \to \begin{bmatrix} 4 & 5-\lambda & 2 & | & 4 \\ 2 & 2 & 2-\lambda & | & 2 \\ 5-\lambda & 4 & 2 & | & 4 \end{bmatrix} \to \begin{bmatrix} 4 & 5-\lambda & 2 & | & 4 \\ 0 & \frac{\lambda-1}{2} & 1-\lambda & | & 0 \\ 0 & \frac{16-(5-\lambda)^2}{4} & \frac{\lambda-1}{2} & | & \lambda-1 \end{bmatrix}$$

If $\lambda = 1$, then the matrix becomes $\begin{bmatrix} 4 & 4 & 2 & | & 4 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$, so we have $2x + 2y + z = 2$ as the set of solution. So $\lambda = 1$ implies infinite solutions.

If $\lambda \neq 1$, then

$$\begin{bmatrix} 4 & 5-\lambda & 2 & | & 4 \\ 0 & \frac{\lambda-1}{2} & 1-\lambda & | & 0 \\ 0 & \frac{16-(5-\lambda)^2}{4} & \frac{\lambda-1}{2} & | & \lambda-1 \end{bmatrix} \to \begin{bmatrix} 4 & 5-\lambda & 2 & | & 4 \\ 0 & 1 & -2 & | & 0 \\ 0 & \frac{16-(5-\lambda)^2}{\lambda-1} & 2 & | & 4 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & 12-2\lambda & | & 4 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & 2\left(1+\frac{16-(5-\lambda)^2}{\lambda-1}\right) & | & 4 \end{bmatrix}$$

So unless $\frac{\lambda^2-10\lambda+9}{1-\lambda} = -1$, the given system has a unique solution, and otherwise it is inconsistent. So it has no solution if and only if

$$\lambda^2 - 10\lambda + 9 = \lambda - 1 \Leftrightarrow \lambda^2 - 11\lambda + 10 = 0 \Leftrightarrow (\lambda - 10)(\lambda - 1) = 0 \Leftrightarrow \lambda = 10.$$

(since we have already assumed $\lambda \neq 1$)

Then the system has

(a) no solution if $\lambda = 10$,

(b) a unique solution if $\lambda \neq 1, 10$, and

(c) infinitely many solution if $\lambda = 1$.

Gaussian Elimination

4. Solve the following system of linear equation using Gaussian elimination:

$$\begin{aligned} A + 3B + C + 3D &= 14 \\ 4A - 2B - 3C + D &= 20 \\ 2A + B - C - D &= 9 \\ A + 2B - C - 2D &= 3 \end{aligned}$$

Soln. Consider the augmented matrix

$$\begin{bmatrix} 1 & 3 & 1 & 3 & | & 14 \\ 4 & -2 & -3 & 1 & | & 20 \\ 2 & 1 & -1 & -1 & | & 9 \\ 1 & 2 & -1 & -2 & | & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 5 \\ 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Therefore the required solution is,

$$A = 5, B = 1, C = 0, \text{ and } D = 2.$$

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5. Consider three planes in $\mathbb{R}^3$. List all possibilities of their intersection. Give example of each case (equations, not diagrams).
Ans. Here are the list of all possible intersections with examples:

(a) Empty intersection. (e.g. Disjoint parallel planes)

$$x = 0, \quad x = 1, \quad x = 2.$$

(b) A single point. (e.g. Consider xy, yz and zx-plane)

$$x = 0, \quad y = 0 \quad z = 0.$$

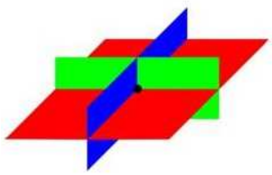
(c) A straight line. (e.g. Three planes passing through y-axis)

$$x = 0, \quad z = 0, \quad x - z = 0.$$

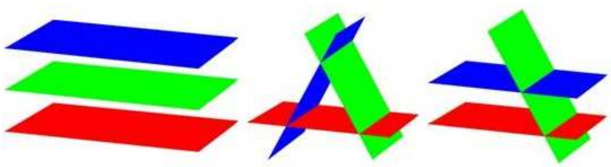
(d) A plane. (e.g. Three copies of the same plane)

$$x = 0, \quad x = 0, \quad x = 0.$$

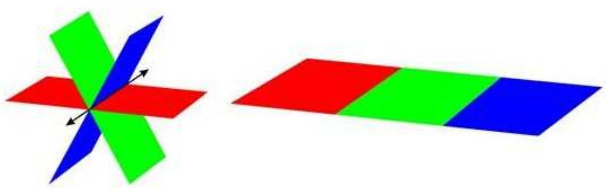
Case 1: There is one solution. In order for three equations with three variables to have one solution, the planes must intersect in a single point.



Case 2: There is no solution. The three planes do not have any points in common. (Note that two of the equations may have points in common with each other, but not all three.)



Case 3: There are an infinite number of solutions. This occurs when the three planes intersect in a line. And this can also occur when the three equations graph as the same plane.



System of Equations over $\mathbb{Z}_3$

6. Solve the following system over $\mathbb{Z}_3$:

$$\begin{aligned} x + 2y + z &= 0 \\ x + z &= 2 \\ y + 2z &= 1 \end{aligned}$$

Ans. Consider the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 1 & 0 & 1 & | & 2 \\ 0 & 1 & 2 & | & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 0 & | & 2 \\ 0 & 1 & 2 & | & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & | & 2 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 2 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & | & 2 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}.$$

(Note that in $\mathbb{Z}_3$, $-1 = 2$, $-2 = 1$, and $\pm 3 = 0$)
Hence the required solution is,

$$x = 1, \quad y = 2 \quad \text{and} \quad z = 1.$$

Theoretical

7. Let $A =$ be a matrix and $\overrightarrow{b} \in \mathbb{R}^2$ be a vector. Consider the system represented by $AX = b$. Then show that

   (a) If A is a zero matrix, then any vector $\overrightarrow{x} \in \mathbb{R}^2$ is a solution to the system $AX = 0$.

   (b) If $\alpha\delta - \beta\gamma \neq 0$, then the system $AX = 0$ has only the trivial solution $[0, 0]^t \in \mathbb{R}^2$.

Proof. (a) Suppose A is a zero matrix. Then for any vector $\overrightarrow{x} \in \mathbb{R}^2$ we have

$$Ax = 0,$$

i.e. $\overrightarrow{x}$ is a solution to the system $AX = 0$.

(b) Suppose $\alpha\delta - \beta\gamma \neq 0$, then A is invertible, so for any $\overrightarrow{x} = [x_1, x_2]^t \in \mathbb{R}^2$ we have

$$Ax = 0 \Leftrightarrow A^{-1}Ax = A^{-1}0 \Leftrightarrow x = 0.$$

Hence $x = [0, 0]^t \in \mathbb{R}^2$ is the only solution.

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