MA101 Mathematics I

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Slides 5

Plan

- Linear Transformation, Kernel and Range
- Matrix of a Linear Transformation

Linear Transformations

- Suppose $A \in \mathcal{M}_{m \times n}$. Take $\mathbf{v} \in \mathbb{R}^n$. Then $A\mathbf{v} \in \mathbb{R}^m$. Thus, we have a map (function) $F : \mathbb{R}^n \to \mathbb{R}^m$ given by $F(\mathbf{v}) = A\mathbf{v}$.
- Take $F : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $F(p(x)) = p'(x)$.
- Take $F : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ given by $(F(f))(x) = \int_0^x f(t)dt$.
- Take $F : \mathbb{R}[x] \to \mathbb{R}$ given by $F(p(x)) = p(3)$.

What is common in all of these? Well, they are maps (functions) with domains and codomains as VS's. What else? We have

$$F(\mathbf{u} + \mathbf{v}) = F(\mathbf{u}) + F(v), \quad F(\alpha\mathbf{v}) = \alpha\, F(\mathbf{v}),$$

or, equivalently, $F(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha F(\mathbf{u}) + \beta F(\mathbf{v})$. Such functions are called linear transformations (LT).

Definition

Let $\mathbb{V}$, $\mathbb{W}$ be VS's over $\mathbb{F}$. A map $T : \mathbb{V} \to \mathbb{W}$ is a linear transformation (LT) from V into W if $\forall \mathbf{u}, \mathbf{v} \in \mathbb{V}, \alpha \in \mathbb{F}$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \text{ and } T(\alpha\mathbf{v}) = \alpha\, T(\mathbf{v}),$$ or

equivalently, $T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\, T(\mathbf{u}) + \beta\, T(\mathbf{v}), \forall \mathbf{u}, \mathbf{v} \in \mathbb{V}, \alpha, \beta \in \mathbb{F}$.

Obvious LT's:

- $T_0 : \mathbb{V} \to \mathbb{W}, \quad T_0(\mathbf{v}) = \mathbf{0}, \ \forall\, \mathbf{v} \in \mathbb{V}$ (zero transformation). Here, $\mathbb{V}, \mathbb{W}$ are any VS's.
- $I_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}, \quad I_{\mathbb{V}}(\mathbf{v}) = \mathbf{v}, \ \forall\, \mathbf{v} \in \mathbb{V}$ (identity transformation). Here, $\mathbb{V}$ is any VS.

Exercise

Is $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T([x, y]^T) = [2x, x + y]^T$ an LT? Yes, check using definition, or observe that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Result

Let $T : \mathbb{V} \to \mathbb{W}$ be an LT. Then

1. $T(\mathbf{0}) = \mathbf{0}$; $[T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}) \Rightarrow \mathbf{0} = T(\mathbf{0}).]$
2. $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v} \in \mathbb{V}$; $[T(-\mathbf{v}) + T\mathbf{v} = \mathbf{0}.]$
3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ *for all* $\mathbf{u}, \mathbf{v} \in \mathbb{V}$.

Exercise

Is $T : \mathbb{R} \to \mathbb{R}$, where $T(x) = x + 1$ is an LT? No. $T(0) \neq 0$.

Exercise

Can there be an LT $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T([0,1]^T) = [2,3]^T$, $T([1,0]^T) = [3,2]^T$ and $T([1,1]^T) = [3,3]^T$?

No, $[3,3]^T = T([1,1]^T) = T(\mathbf{e}_1) + T(\mathbf{e}_2) = [5,5]^T$, not possible.

Exercise

Suppose $T : \mathbb{R}^2 \to \mathbb{R}_2[x]$ is an LT. Given that $T([1,0]^T) = 2 - 3x + x^2$ and $T([0,1]^T) = 1 - x^2$. What is $T([2,3]^T)$? $T([2,3]^T) = T(2\mathbf{e}_1 + 3\mathbf{e}_2) = 2T(\mathbf{e}_1) + 3T(\mathbf{e}_2) = 2(2 - 3x + x^2) + 3(1 - x^2) = 7 - 6x - x^2$. What is $T([a,b]^T)$?

Result

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{V}$ (dim$(\mathbb{V}) = n$). Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be arbitrarily chosen in $\mathbb{W}$. Then there is a unique LT $T : \mathbb{V} \to \mathbb{W}$ such that $T(\mathbf{v}_i) = \mathbf{u}_i$.

Proof. Let $\mathbf{v} \in \mathbb{V}$. Then $\mathbf{v}$ equals a unique linear combination $\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$. Define $T\mathbf{v} = \alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n \in \mathbb{W}$. The resulting map T is the LT we are looking for. ∎

Remark: To define (know) an LT, it is enough to define (know) the images of vectors in any basis of the domain.

Example

Consider $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{x}) = A\mathbf{x}$, where $A \in \mathcal{M}_{m\times n}$. What is the range of T? It is $\{T(\mathbf{x}) \in \mathbb{R}^m \mid \mathbf{x} \in \mathbb{R}^n\}$ $= \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} = \mathrm{col}(A)$. Note that the range is a subspace of $\mathbb{R}^m$. On the other hand,
$\mathrm{null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

Kernel and Range: For an LT $T : \mathbb{V} \to \mathbb{W}$ we define:

- Kernel (or null space) of T: $\ker(T) :=\{\mathbf{v} \in \mathbb{V} \mid T(\mathbf{v}) = \mathbf{0}\}$;
- range of T: $\mathrm{range}(T) :=\{T(\mathbf{v}) \in \mathbb{W} \mid \mathbf{v} \in \mathbb{V}\}$.

It is easy to see that $\ker(T) \preccurlyeq \mathbb{V}$ and $\mathrm{range}(T) \preccurlyeq \mathbb{W}$. Moreover,

If $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans V, then $T(B) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ spans $\mathrm{range}(T)$.

Exercise

Let $D : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ be defined by $D(p(x)) = p'(x)$. Find $\ker(D)$, $\mathrm{range}(D)$ and their dimensions.

Definition

For an LT $T : \mathbb{V} \to \mathbb{W}$ we define

- $\mathrm{rank}(T)$:= dimension of $\mathrm{range}(T)$; and
- $\mathrm{nullity}(T)$:= dimension of $\ker(T)$.

Example

For the previous example, $\mathrm{rank}(D) = 3$ and $\mathrm{nullity}(D) = 1$.

Result (The Rank-Nullity Theorem)

If $\mathbb{V}$ is finite dimensional, then for any LT $T : \mathbb{V} \to \mathbb{W}$,

$$rank(T) + nullity(T) = dim(V).$$

Proof. Take a basis $B = \{v_1, \ldots, \mathbf{v}_k\}$ of $\ker(T)$. Extend it to a basis $B \cup \{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ of $\mathbb{V}$. Then $\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ spans $\mathrm{range}(T)$. Moreover, $\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ is LI. ∎

Result

Let $T : \mathbb{V} \to \mathbb{W}$ be an LT. Then

- *T is one-one iff $ker(T) = \{\mathbf{0}\}$.*
- *$dim(\mathbb{V}) = dim(\mathbb{W}) = n$, then T is onto iff $ker(T) = \{\mathbf{0}\}$.*

Definition

For linear transformations $T : \mathbb{V} \to \mathbb{W}$, $S : \mathbb{V} \to \mathbb{W}$ and $\alpha \in \mathbb{F}$ we define $T + S : \mathbb{V} \to \mathbb{W}$, $\alpha T : \mathbb{V} \to \mathbb{W}$ by

$$(T + S)(\mathbf{v}) = T(\mathbf{v}) + S(\mathbf{v}), \ (\alpha T)(\mathbf{v}) = \alpha(T(\mathbf{v})), \quad \mathbf{v} \in \mathbb{V}.$$

Exercise

Show that $T + S$ and αT are linear transformations.

Result (Composition of Linear Transformations)

Suppose $T : \mathbb{U} \to \mathbb{V}$ and $S : \mathbb{V} \to \mathbb{W}$ are LT's. Then the composition $S \circ T : \mathbb{U} \to \mathbb{W}$ is also an LT.

Proof. Note: $(S \circ T)(\mathbf{u}) = S(T\mathbf{u})$ for all $\mathbf{u} \in \mathbb{U}$. Now,
$(S \circ T)(\alpha\mathbf{u} + \beta\mathbf{v}) = S(\alpha T\mathbf{u} + \beta T\mathbf{v}) = \alpha(S \circ T)\mathbf{u} + \beta(S \circ T)\mathbf{v}$. ∎

Coordinates

Suppose $\mathbb{V}$ is a VS of dimension n. Fix an ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ (i.e., a basis with a specific order of its elements). For $\mathbf{v} \in \mathbb{V}$ we have unique way of writing $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$, $\alpha_i \in \mathbb{F}$. Define $T(\mathbf{v}) = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$. Then, we have a map $T : \mathbb{V} \to \mathbb{F}^n$.

Definition

Consider the above map $T : \mathbb{V} \to \mathbb{F}^n$. $T(\mathbf{v})$ is called the coordinate vector of $\mathbf{v}$ w.r.t. the ordered basis B. $T(\mathbf{v})$ is denoted by $[\mathbf{v}]_B$. The scalars α_i are called the coordinates of $\mathbf{v}$ w.r.t. B.

Remark

- *$[\mathbf{v}]_B$ depends not only on B; but also on the order in which $\mathbf{v}_i$ chosen.*
- *$T : \mathbb{V} \to \mathbb{F}^n$, where $T(\mathbf{v}) = [\mathbf{v}]_B$ is an LT. Indeed,*

$$[\mathbf{u} + \mathbf{v}]_B = [\mathbf{u}]_B + [\mathbf{v}]_B \quad \textit{and} \quad [c\mathbf{u}]_B = c[\mathbf{u}]_B.$$

Isomorphism

Example

Take the standard (ordered) basis $B = \{1, x, x^2\}$ of $\mathbb{R}_2[x]$. Then, $[a + bx + cx^2]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Note that the LT $T : \mathbb{R}_2[x] \to \mathbb{R}^3$, $T(p(x)) = [p(x)]_B$ is one-one and onto (i.e., invertible).

Definition

A linear transformation $T : \mathbb{V} \to \mathbb{W}$ is called an isomorphism of $\mathbb{V}$ onto $\mathbb{W}$, if it is one-one and onto. In that case, we say that $\mathbb{V}$ is isomorphic to $\mathbb{W}$ and we write $\mathbb{V} \cong \mathbb{W}$.

Example

The LT $T : \mathbb{V} \to \mathbb{F}^n$, where $T(\mathbf{v}) = [\mathbf{v}]_B$, is an isomorphism. Thus, any VS of dimension n over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.

The Matrix of a Linear Transformation

Suppose $\dim(\mathbb{V}) = n$, $\dim(\mathbb{W}) = m$, $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ an ordered basis of $\mathbb{V}$, $C = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ an ordered basis of $\mathbb{W}$ and $T : V \to W$ an LT. Then the $m \times n$ matrix A defined by

$$A = \Big[[T(\mathbf{v}_1)]_C, [T(\mathbf{v}_2)]_C, \ldots, [T(\mathbf{v}_n)]_C\Big]$$

is called the matrix of T with respect to the bases B and C.

The matrix A is written as $[T]_{C \leftarrow B}$.

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T([x, y, z]^T) = [x - 2y, x + y - 3z]^T.$$

Consider the bases $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $C = \{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. Then

$$A = [T]_{C \leftarrow B} = \big[T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\big] = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \end{bmatrix}.$$

What is $T([1, 2, 3]^T)$? In general $T([x, y, z]^T) = A[x, y, z]^T$.

Remark

If $\mathbb{V} = \mathbb{W}$ and $B = C$, then $[T]_{C \leftarrow B}$ is written as $[T]_B$.

Example

Consider $D : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ defined by $D(p(x)) = p'(x)$. Take the standard (ordered) basis $B = \{1, x, x^2, x^3\}$ of $\mathbb{R}_3[x]$. Since $D(1) = 0,\ D(x) = 1,\ D(x^2) = 2x,\ D(x^3) = 3x^2$, we get

$$[D]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Then $D(p(x)) = a_1 + 2a_2x + 3a_3x^2$. Note that

$$[p(x)]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix},\ [D]_B[p(x)]_B = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix} = [D(p(x))]_B.$$

Result

Let $A = [T]_{C \leftarrow B}$. Then, for all $\mathbf{v} \in \mathbb{V}$,

$$A[\mathbf{v}]_B = [T(\mathbf{v})]_C, \quad i.e., \left\{ \begin{array}{ccc} \mathbf{v} \in \mathbb{V} & \xrightarrow{T} & T(\mathbf{v}) \in \mathbb{W} \\ \downarrow & & \downarrow \\ [\mathbf{v}]_B \in \mathbb{F}^n & \xrightarrow{T_A} & [T(\mathbf{v})]_C = A[\mathbf{v}]_B \in \mathbb{F}^m \end{array} \right\}$$

Here $T_A : \mathbb{F}^n \to \mathbb{F}^m$ is the LT given by $T_A(\mathbf{x}) = A\mathbf{x}$.

Remark

The above result means:

Suppose we know $[T]_{C \leftarrow B}$ w.r.t. given bases B and C. Then we know T in the following sense:

$$\text{If } \mathbf{v} = \sum_{i=1}^{n} \mathbf{a}_i \mathbf{v}_i \text{ and } [T]_{C \leftarrow B} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \text{ then } T(\mathbf{v}) = \sum_{j=1}^{m} b_j \mathbf{u}_j.$$

Result

Let $\mathbb{U}, \mathbb{V}$ and $\mathbb{W}$ be three vector spaces with bases B, C and D, respectively. Let $T : \mathbb{U} \to \mathbb{V}$ and $S : \mathbb{V} \to \mathbb{W}$ be linear transformations. Then $[S \circ T]_{D \leftarrow B} = [S]_{D \leftarrow C}[T]_{C \leftarrow B}$.

PROOF. We have

$$[S \circ T]_{D \leftarrow B} = \Big[[(S \circ T)(\mathbf{v}_1)]_D, [(S \circ T)(\mathbf{v}_2)]_D, \dots, [(S \circ T)(\mathbf{v}_n)]_D \Big].$$

Now, the i-th column of $[S \circ T]_{D \leftarrow B}$ is

$$\begin{aligned} [(S \circ T)(\mathbf{v}_i)]_D &= [(S(T(\mathbf{v}_i))]_D = [S]_{D \leftarrow C}[T(\mathbf{v}_i)]_C \\ &= [S]_{D \leftarrow C}[T]_{C \leftarrow B}[\mathbf{v}_i]_B = [S]_{D \leftarrow C}[T]_{C \leftarrow B}\mathbf{e}_i, \end{aligned}$$

the i-th column of $[S]_{D \leftarrow C}[T]_{C \leftarrow B}$. ∎

Result

Let $\mathbb{V}$ be VS with basis B, resp., and $T, S : \mathbb{V} \to \mathbb{V}$ are linear transformations. Then, $[S \circ T]_B = [S]_B[T]_B$.

Change of Basis:

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ and $C = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be two bases for a vector space $\mathbb{V}$ with dimension n. For $\mathbf{v} \in \mathbb{V}$, both $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$ are in $\mathbb{R}^n$. How are they related?

Let $P_{C \leftarrow B} := [[\mathbf{u}_1]_C, [\mathbf{u}_2]_C, \dots, [\mathbf{u}_n]_C]$, (called the change of basis matrix from B to C.) Then

1. $P_{C \leftarrow B}[\mathbf{x}]_B = [\mathbf{x}]_C$ for all $\mathbf{x} \in V$;
2. $P_{C \leftarrow B}$ is unique such matrix;
3. $P_{C \leftarrow B}$ is invertible and $(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C}$.

Example

Find the change of basis matrix $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ for the bases $B = \{1, x, x^2\}$ and $C = \{1 + x, x + x^2, 1 + x^2\}$ of $\mathbb{R}_2[x]$. Then find the coordinate vector of $p(x) = 1 + 2x - x^2$ w.r.t. respect to the basis C.

Exercises

- Check whether the following are linear transformations, one-one and onto.
  - $T : \mathbb{R} \to \mathbb{R}^2$ defined by $T(x) = [x, 0]^T$, $x \in \mathbb{R}$.
  - $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T[x, y]^T = x$, for $[x, y]^T \in \mathbb{R}^2$.
  - $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T[x, y]^T = [-x, -y]^T$, for $[x, y]^T \in \mathbb{R}^2$.
- Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be in $\mathbb{V}$ such that $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ are linearly independent. Can $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be linearly dependent? Justify?
- Let an LT $T : \mathbb{V} \to \mathbb{W}$ is given to be one-one. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an LI subset of $\mathbb{V}$ then show that $T(S) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is LI in $\mathbb{W}$.

Exercises

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
  $T[x, y]^T = [x - y, -3x + 4y]^T$ and
  $S[x, y]^T = [4x + y, 3x + y]^T$
  for $[x, y]^T \in \mathbb{R}^2$. Compute $T \circ S$ and $S \circ T$. What is your observation?
- Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$.
  - If $T : \mathbb{V} \to \mathbb{W}$ is an one-one and onto (i.e., invertible). linear transformation, then show that $T^{-1} : \mathbb{W} \to \mathbb{V}$ is an LT.
  - Argue that if $\mathbb{V}$ is isomorphic to $\mathbb{W}$, then $\mathbb{W}$ is isomorphic to $\mathbb{V}$.
- Let $\dim(\mathbb{V}) = \dim(\mathbb{W})$. Then show that a linear transformation $T : \mathbb{V} \to \mathbb{W}$ is one-one iff T is onto.
- Let $\dim(\mathbb{V}) = \dim(\mathbb{W})$. Then a one-one linear transformation $T : \mathbb{V} \to \mathbb{W}$ maps a basis for $\mathbb{V}$ onto a basis for $\mathbb{W}$.

Exercises

- Let $\mathbb{V}$ and $\mathbb{W}$ be two finite dimensional vector spaces. Then $\mathbb{V}$ is isomorphic to $\mathbb{W}$ iff $\dim(\mathbb{V}) = \dim(\mathbb{W})$.
- Show that
  - $\mathbb{R}^3$ and $\mathbb{R}_2[x]$ are isomorphic.
  - The vector spaces $\mathbb{R}^n$ and $\mathbb{R}_n[x]$ are not isomorphic.
- Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $\mathbb{V}$, and let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be vectors in $\mathbb{V}$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent in $\mathbb{V}$ if and only if $\{[\mathbf{u}_1]_B, [\mathbf{u}_2]_B, \ldots, [\mathbf{u}_k]_B\}$ is linearly independent in $\mathbb{R}^n$.
- Suppose $T, S : \mathbb{V} \to \mathbb{W}$ are LT's, and B and C are ordered bases of $\mathbb{V}$ and $\mathbb{W}$, resp. Show that

$$[T + S]_{C \leftarrow B} = [T]_{C \leftarrow B} + [S]_{C \leftarrow B},$$
$$[\alpha T]_{C \leftarrow B} = \alpha [T]_{C \leftarrow B}.$$

Exercises

- Let $\mathbb{V}, \mathbb{W}$ be n dimensional with bases B and C, resp., and $T : V \to W$ an LT. Then T is invertible if and only if the matrix $[T]_{C \leftarrow B}$ is invertible. In that case,

$$([T]_{C \leftarrow B})^{-1} = [T^{-1}]_{B \leftarrow C}.$$

- Let $T : \mathbb{R}^2 \to \mathbb{R}_1[x]$ be defined by $T([a, b]^T) = a + (a + b)x$ for $[a, b]^T \in \mathbb{R}^2$. Find $[T]_{C \leftarrow B}$ w.r.t. standard bases, show that T is invertible, and thus find T^{-1}.