Final exam (objective part) MA 101, 21.11.2015

- For all questions, 2 marks if fully correct, 0 otherwise.
- For the first five questions only the correct options are mentioned here. These questions appeared in shuffled order in the various sheets.

Answers

1. Let $x_n \to x \in \mathbb{R}$. Let $\epsilon > 0$. Which of the following are true?
   $\boxed{\text{T}}$ There exists a tail of (x_n) that has all of its terms in $B_\epsilon(x)$.

2. Let (x_n) be a sequence in $\mathbb{R}$. Which of the following are true?
   $\boxed{\text{T}}$ If (x_n) is bounded then every tail of (x_n) is bounded.
   $\boxed{\text{T}}$ If (x_n) is unbounded then every tail of (x_n) is unbounded.
   $\boxed{\text{T}}$ If (x_n) is unbounded, it cannot be Cauchy.

3. Let $A = \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$. Which of the following are true?
   $\boxed{\text{T}}$ There exists a sequence such that every point of A is a limit point of the sequence (same as subsequential limit point).
   $\boxed{\text{T}}$ There exists a sequence such that every point of A is a limit point of the sequence , and 0 is also a limit point.

4. Let $h(x)$ be a polynomial such that $h(1) = h(3) = -1$ and $h(2) = h(4) = +1$. Which of the following are true?
   $\boxed{\text{T}}$ $h(x)$ is infinitely differentiable.
   $\boxed{\text{T}}$ $h^{(1)}(x)$ must have at least 2 zeroes.
   $\boxed{\text{T}}$ There must exist some n for which $h^{(n)}(1.5) = 0$.

5. Which of the following are true?
   $\boxed{\text{T}}$ There exists $g : \mathbb{R} \to \mathbb{R}$ such that g is differentiable everywhere but $|g|$ is not differentiable at 0.

6. Suppose U, W and $U \cup W$ are subspaces of a vector space V. If $\dim(V) = 3$ and $\dim(W) = 4$. Then $\dim(U \cup W)$ is

   $\boxed{\text{4. No marks for anything else.}}$

7. Let $S = \left\{\frac{\pi}{6}, a\right\}$ be a subset of $[0, 2\pi]$ such that the functions $\sin x, \cos x$ are linearly independent on S. Then all possible values of a are:

   $\boxed{[0, 2\pi] \backslash \{\pi/6, 7\pi/6\}.\ \text{Only the full set gets 2 marks. Missing even one value gets zero.}}$

8. Let $\mathcal{P}_2(R)$ denote the space of all polynomials of degree at most 2. Consider a map $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ given by $T(1) = 1 + x^2$, $T(x) = x + x^2$ and and $T(x^2 - 1) = x^2 + x$. Then compute a basis for $\text{Kernel(T)} = \{\text{v} \in \mathcal{P}_2(\mathbb{R}) \mid \text{T(v)} = 0\}$.

   $\boxed{c(1 + x - x^2) \text{ with } c \neq 0.\ c = -1 \text{ is OK. Appropriate row or column vector also OK.}}$

9. Let A, B be orthogonal matrices. Then write down **all** possible values of $\det(AB)$.

   $\boxed{\text{Correct is } \pm 1.\ \text{Writing } +1 \text{ alone, or } -1 \text{ alone, or } -1, 0, +1, \text{ all fail to get any marks.}}$

10. Let $W \subset \mathbb{R}^3$ be a subspace given by $x - y + 2z = 0$, and let $v = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ be a vector in $\mathbb{R}^3$. The $\mathrm{Proj_W(v)}$ is

$$\boxed{\begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix}.}$$

`Remark.` All three components of the projected vector need to be correct. An extra '$-$' sign, a missing '$-$' sign, a missing denominator, a permutation of the components, all get zero. Calculating only the norm of the projected vector also gets zero.