Q1.

(a) $V = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \in \mathbb{R}^2 : x \in \mathbb{R} \right\}$.

Let $\begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$ & $\begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$ be elements of V.

$$\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ x_1+x_2 \end{bmatrix}$$ is in V.

— Closure

For $\begin{bmatrix} x_i \\ x_i \end{bmatrix} \in V$, $i = 1, 2, 3$,

$$\left(\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \right) + \begin{bmatrix} x_3 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+x_2 \\ x_1+x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1+x_2)+x_3 \\ (x_1+x_2)+x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+(x_2+x_3) \\ x_1+(x_2+x_3) \end{bmatrix}$$ (Using associativity of real numbers)

$$= \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} + \left(\begin{bmatrix} x_2+x_3 \\ x_2+x_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} + \left(\begin{bmatrix} x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_3 \end{bmatrix} \right)$$ — Associativity

$$\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ x_1+x_2 \end{bmatrix} = \begin{bmatrix} x_2+x_1 \\ x_2+x_1 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

— Commutativity

* $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V.$

* For $\begin{bmatrix} x \\ x \end{bmatrix} \in V$, we have $\begin{bmatrix} -x \\ -x \end{bmatrix}$ such that

$$\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} -x \\ -x \end{bmatrix} = \begin{bmatrix} x-x \\ x-x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

* For $\alpha \in \mathbb{R}$ & $\begin{bmatrix} x \\ x \end{bmatrix} \in V$,

$$\alpha \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha x \end{bmatrix} \in V.$$

* $$\alpha \left(\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} \right) = \alpha \begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} \alpha x + \alpha y \\ \alpha x + \alpha y \end{bmatrix}$$
$$= \begin{bmatrix} \alpha x \\ \alpha x \end{bmatrix} + \begin{bmatrix} \alpha y \\ \alpha y \end{bmatrix}.$$

* $$(\alpha + \beta) \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} (\alpha+\beta) x \\ (\alpha+\beta) x \end{bmatrix} = \begin{bmatrix} \alpha x + \beta x \\ \alpha x + \beta x \end{bmatrix}$$
$$= \begin{bmatrix} \alpha x \\ \alpha x \end{bmatrix} + \begin{bmatrix} \beta x \\ \beta x \end{bmatrix}$$
$$= \alpha \begin{bmatrix} x \\ x \end{bmatrix} + \beta \begin{bmatrix} x \\ x \end{bmatrix}.$$

* $$(\alpha \beta) \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} (\alpha\beta) x \\ (\alpha\beta) x \end{bmatrix} = \begin{bmatrix} \alpha(\beta x) \\ \alpha(\beta x) \end{bmatrix} = \alpha \begin{bmatrix} \beta x \\ \beta x \end{bmatrix} = \alpha \left(\beta \begin{bmatrix} x \\ x \end{bmatrix} \right)$$

* $$1 \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} 1 \cdot x \\ 1 \cdot x \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}.$$

Q.1 (b)

$V = \left\{ \begin{bmatrix} z \\ z \end{bmatrix} \in \mathbb{C}^2 : z \in \mathbb{C} \right\}.$

Field of scalars $= \mathbb{C}$.

$\rightarrow$ Exactly same as as Q.1 (a).

Q.1 (c)

$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \geq y \right\}.$

Note that the set V is closed wrt usual addition. Commutativity & associativi also hold. $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$.

However, for $\begin{bmatrix} x \\ y \end{bmatrix} \in V$ the vector $\begin{bmatrix} -x \\ -y \end{bmatrix} \notin V$

as $\{ x \geq y \Rightarrow -x \leq -y$.

Thus only the vectors of the type $\begin{bmatrix} x \\ x \end{bmatrix}$ have additive inverse.

Thus V is not a vector space.

Q.1 (d).

$$V=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\&\; a.d=0 \;\; a,b,c,d\in\mathbb{R}\right\}.$$

Note that $ad=0 \Rightarrow a=0$ or $d=0$.

Thus every element of V has atleast one diagonal entry to be zero.

Consider $A=\begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}$ & $B=\begin{bmatrix} 0 & b' \\ c' & 1 \end{bmatrix}$.

Then $A, B \in V$. But $A+B=\begin{bmatrix} 1 & b+b' \\ c+c' & 1 \end{bmatrix} \notin V$.

Thus V is not closed wrt usual addition.

$\therefore$ V is not a vector space.

Q-2

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}.$$
$$i^2 = j^2 = k^2 = ijk = -1$$

Remark:

Note this set is in one-to-one correspondence with $\mathbb{R}^4 = \{(a,b,c,d) : a,b,c,d \in \mathbb{R}\}$.

The relations $i^2 = j^2 = k^2 = ijk$ do not play any role in the vector space structure of $\mathbb{H}$.

Check that $\mathbb{H}$ satisfies all the axioms of a vector space.

The operations are defined as:

$$(a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k)$$

$$= (a_1 + a_2) + (b_1 + b_2) i + (c_1 + c_2) j + (d_1 + d_2) k.$$

For $\alpha \in \mathbb{R}$,

$$\alpha (a + bi + cj + dk) = (\alpha a) + (\alpha b) i + (\alpha c) j + (\alpha d) k$$

Q.3 (a)

$V = M_{2,2}$, & $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad \geq bc \right\}$.

Consider

$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$

Note $A, B \in W$.

But $A + B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \notin W$.

Thus W is not closed wrt usual additi

Q.3 (b) $V = M_{r,r}$. $W = \{ A \in V : A \text{ is invertible} \}$.

Note 0 (zero matrix of V) is not invertibl

$\therefore$ W is not a subspace of V.

Q.3 (c) $V = \mathcal{F}$, $W = \{ f \in \mathcal{F} : f(0) = 0 \}$.

Let $f, g \in W \Rightarrow f(0) = 0$ & $g(0) = 0$

then $(f+g)(0) = f(0) + g(0) = 0 \Rightarrow f + g \in W$.

$(\alpha f)(0) = \alpha \cdot f(0) = \alpha \cdot 0 = 0 \Rightarrow \alpha f \in W$.

$\therefore$ W is a subsp. of V.

Q.3(d) $V = \mathcal{F}$. $W = \{f \in \mathcal{F} : f(1) = 0\}$.

SAME as Q.3(c).

Q.3(e). $V = \mathcal{F}$ $W = \{f \in \mathcal{F} : f(0) = 1\}$.

$f, g \in W. \Rightarrow f(0) = 1$, $g(0) = 1$

$(f+g)(0) = f(0) + g(0) = 1 + 1 = 2$

$\Rightarrow f+g \notin W$.

Thus W is not a subsp. of V.

Q.3(f). $V = \mathcal{F}$, $V = \{f \in V : f \text{ is cont.}\}$.

Use the fact that addition of continuous functions is continuous. What about scalar multiplication?

Q.3(g) SAME AS Q.3(f).

Q.3(h) $f \in W \Rightarrow \int_{-\infty}^{\infty} f < \infty$.

Use arithmetic of integrable function

Q. 3(i) $V = \mathbb{C}$, $W = \mathbb{R}$.

Usual properties of real numbers!

Note that an element of W is written as $a + 0i$, which is equal to the real number a.

Q. 4 (a) To prove zero element of V is unique

Suppose 0 & $0'$ are two zero elements of V.

Then by the property of additive identity of a vector space, we have,

$$\left.\begin{aligned} 0 + 0' &= 0' \\ \& \quad 0 + 0' &= 0 \end{aligned}\right\} \Rightarrow 0 = 0'.$$

Q. 4 (b) Can you apply same arguments as above?

Q.5 (a) $V = P_1$.

$S = \{x, 1+x\}$

$\alpha \cdot x + \beta(1+x) = 0$

$\Rightarrow \beta + (\alpha+\beta)x = 0.$

Comparing coefficients, we get

$\beta = 0 \Rightarrow \alpha = 0$

Thus S is linarly independent.

Q.5 (b) $V = P_2$

$S = \{x, 2x - x^2, 3x + 2x^2\}$

$\alpha x + \beta(2x - x^2) + \gamma(3x + 2x^2) = 0$

$\Rightarrow (\alpha + 2\beta + 3\gamma)x + (-\beta + 2\gamma)x^2 = 0$

$\Rightarrow \alpha + 2\beta + 3\gamma = 0$
$\quad -\beta + 2\gamma = 0$

Solve this system! (for α, β, γ).

~~What can you say about~~

What is the relation of soln of this system & linear independence of S?

Q.5(c)

~~If~~ Note that

$$\beta_1 \subseteq \beta_2 \subseteq \beta_3 \subseteq \cdots \subseteq \beta_n \subseteq \cdots$$

~~Then if a~~

Que If $W \subseteq V$ is a vector subspace, & ~~B~~ & $S \subseteq W$ is linearly independent in W then what happens to the linear independence of S when treated as a subset of V. ?

Q.5(d)

$V = \mathcal{F}$ $\quad S = \{1, \sin x, \cos x\}$.

$$\alpha \cdot 1(x) + \beta \sin x + \gamma \cos x = 0(x)$$

$$\therefore \alpha + \beta \sin x + \gamma \cos x = 0 \quad \forall x \in \mathbb{R}$$

Note here $1 \in \mathcal{F}$ means $1(x) = 1 \ \forall x$ & $0 \in \mathcal{F}$ means $0(x) = 0 \ \forall x \in \mathbb{R}$.

~~For x~~ As this holds for all real numbers x, take ~~$x=0$~~ special values of x & get relations between α, β, γ.

Then solve this system!

Q.5(e) $V = \mathcal{F}$, $S = \{\sin x, \sin 2x, \sin 3x\}$.

Express $\sin 2x$ & $\sin 3x$ in terms of $\sin x$.

Now is the question similar to Q. 5(d)?

Q.6 ~~Pf B~~

To prove: $\mathcal{B}$ is a basis for V.

It is already given that every vector of V can be written as a linear combination of elements of $\mathcal{B}$.

$\therefore$ span $(\mathcal{B}) = V$.

To check/prove: $\mathcal{B}$ is linearly independent.

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$.

Suppose $a_1 v_1 + \cdots + a_n v_n = 0$.

But $0 \in V$ & the property that such a linear combination has to be unique.

$\Rightarrow a_i = 0 \quad \forall i$

$\Rightarrow \{v_1, \ldots, v_n\}$ is linearly independent.

$\therefore$ $\mathcal{B}$ is a basis.

7. (a) let $p(x) = a + bx + cx^2$ s.t. $p(1) = 0$

$\therefore \quad a + b + c = 0 \quad \Rightarrow c = -(a+b)$

$\therefore \quad p(x) = \cancel{a+b} \; a + bx - (a+b)x^2$

$= a(1 - x^2) + b(x - x^2)$

clearly, $1 - x^2$ and $x - x^2$ span $\{p(x) \mid p(1) = 0\} \in P_2$

Also, $c_1(1-x^2) + c_2(x - x^2) = 0$

$\Rightarrow \quad c_1 + c_2 x - (c_1 + c_2) x^2 = 0$

$\Rightarrow \quad c_1 = c_2 = 0$

$\therefore \quad \{1 - x^2, \; x - x^2\}$ is a basis

(b) let $U = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$

clearly, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

and $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \quad c_1 = c_2 = c_3 = 0$

$\therefore \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ spans U and is lin ind

$\therefore$ it is a basis.

7. (c) Let $D = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$

Clearly, $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{= D_1} + b \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{= D_2} + c \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= D_3}$ (span)

and $c_1 D_1 + c_2 D_2 + c_3 D_3 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0$ (lin ind)

$\therefore \{D_1, D_2, D_3\}$ is a basis.

(d) Consider $z = a + ib \in \mathbb{C}$.

Then $z = a.1 + b.i$ $\quad (i = \sqrt{-1})$

ie, $\{1, i\}$ spans $\mathbb{C}$ over $\mathbb{R}$.

Also, $c_1 . 1 + c_2 i = 0$ $\quad (c_1, c_2 \in \mathbb{R})$

$\Rightarrow c_1 = 0, \; c_2 = 0$ (by equating real and imaginary parts)

$\therefore \{1, i\}$ is lin ind

$\therefore \{1, i\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$.

8. (a) $T\left(\lambda\begin{bmatrix} a & b \\ c & d\end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1\end{bmatrix}\right)$

$= T\left(\begin{bmatrix} \lambda a + a_1 & \lambda b + b_1 \\ \lambda c + c_1 & \lambda d + d_1\end{bmatrix}\right)$

$= \begin{bmatrix} \lambda a + a_1 + \lambda b + b_1 & 0 \\ 0 & \lambda c + c_1 + \lambda d + d_1\end{bmatrix}$

$= \lambda \begin{bmatrix} a+b & 0 \\ 0 & c+d\end{bmatrix} + \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & c_1 + d_1\end{bmatrix}$

$= \lambda\, T\begin{bmatrix} a & b \\ c & d\end{bmatrix} + T\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1\end{bmatrix}$

$\therefore$ T is a lin tran.

(b) $T\left(2\begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix}\right) = T\begin{bmatrix} 2 & 2 \\ 2 & 2\end{bmatrix} = 2^4 \neq 2\,T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix}\right)$

$\therefore$ T is not a L.T.

(c) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}\right) = T\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} = 1$

But $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}\right) + T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}\right) = 0 + 0 = 0$

$\therefore$ T is not a lin trans.

(d) $T(\lambda A_1 + A_2) = (\lambda A_1 + A_2)B = \lambda A_1 B + A_2 B$

$= \lambda T(A_1) + T(A_2)$

$\therefore$ T is a LT

(e) $T(2A) = 2A + B \neq 2(A+B) = 2T(A)$

unless $B = 0$.

If $B = 0$ $T(A) = A$ is a L.T.

(f) $T(0) = 1 + x + x^2 \neq 0$ $\therefore$ T is not a L.T.

9. $T(1+x) + T(x+x^2) + T(1+x^2) = 1+x^2 + x - x^2 + 1+x+x^2$

$\Rightarrow \quad T(2+2x+2x^2) = 2+2x+x^2$

$\Rightarrow \quad T(1+x+x^2) = 1+x+\frac{x^2}{2}$

$\therefore \quad T(1) = T(1+x+x^2 - (x+x^2))$

$= 1+x+\frac{x^2}{2} - (x-x^2)$

$= 1+\frac{3x^2}{2}$

$T(x) = T(1+x+x^2 - (1+x^2))$

$= 1+x+\frac{x^2}{2} - (1+x+x^2)$

$= -\frac{x^2}{2}$

$T(x^2) = T(1+x+x^2 - (1+x))$

$= 1+x+\frac{x^2}{2} - (1+x^2)$

$= x - \frac{x^2}{2}$

$\therefore \quad T(4-x+3x^2) = 4T(1) - T(x) + 3T(x^2)$

$= 4.\left(1+\frac{3x^2}{2}\right) + \frac{x^2}{2} + 3\left(x - \frac{x^2}{2}\right)$

$= 4+3x-5x^2$

$T(a+bx+cx^2) = a.\left(1+\frac{3x^2}{2}\right) + b\left(-\frac{x^2}{2}\right) + c\left(x-\frac{x^2}{2}\right)$

$= a + cx + \frac{3a-b-c}{2}x^2$

10. $\mathcal{L}(V,W) = \{T : V \to W,\ T \text{ a lin trans}\}$

Define — $(T_1+T_2)(v) = T_1 v + T_2 v \quad \forall v \in \cancel{W}$

and $(cT)(v) = c\,T(v) \quad \forall v \in V, \forall c \in F$

(0) Clearly $(T_1+T_2)(cv+w) = c_1 (T_1+T_2)v + (T_1+T_2)(w) \quad \therefore T_1+T_2 \in \mathcal{L}(V,W)$
Similarly, $cT \in \mathcal{L}(V,W)$

(i) Now, the map $T_0 : V \to W$

$$v \mapsto 0 \quad \forall v \in V$$

is the identity of $\mathcal{L}(V,W)$, :

$$\begin{aligned}(T+T_0)(v) &= T(v) + T_0(v)\\ &= T(v) + 0\\ &= T(v)\end{aligned}$$

Similarly $(T_0+T)(v) = T(v) \ \forall v$ and $T_0 + T = T$.

(ii)
$$\begin{aligned}\big((T_1+T_2)+T_3\big)(v) &= (T_1+T_2)(v) + T_3(v)\\ &= (T_1(v) + T_2(v)) + T_3(v)\\ &= T_1(v) + (T_2(v) + T_3(v))\\ &= T_1(v) + (T_2+T_3)(v)\\ &= (T_1 + (T_2+T_3))(v) \quad \forall v \in V\end{aligned}$$

$\therefore$ + is associative on $\mathcal{L}(V,W)$

(iii) $(T_1+T_2)(v) = T_1 v + T_2 v = T_2 v + T_1 v = (T_2+T_1)v$

$$\Rightarrow T_1 + T_2 = T_2 + T_1$$

(iv) Clearly, $(-T)(v) = -Tv$ defines inverse of $-T$:

$$(T + (-T))(v) = T_0(v) \quad \forall v \in V.$$

(v) $(c(T_1+T_2))v = \cancel{c(T_1 v + T_2 v) = (cT_1 + cT_2)(v)}$

$= c(T_1+T_2)v = cT_1(v) + cT_2(v) = (cT_1 + cT_2)(v)$

$\Rightarrow c(T_1+T_2) = cT_1 + cT_2$

(vi) $(c_1 + c_2)\, T(v) = c_1 T v + c_2 T v$

$$= (c_1 T + c_2 T)(v) \quad \forall\, v \in V$$

$$\therefore (c_1 + c_2) T = c_1 T + c_2 T$$

(vii) $((c_1 c_2) T)(v) = c_1 c_2\, T(v)$

$$= c_1 (c_2 (T v))$$

$$= c_1 (c_2 T)(v)$$

$$\therefore (c_1 c_2) T = c_1 (c_2 T)$$

(viii) $1 . T(v) = T(v)$

$$\Rightarrow 1 T = T \quad \forall\, T \in \mathcal{L}(V, W)$$

$\therefore$ $\mathcal{L}(V, W)$ is a vector space.

11. (a) $T : M_{2,2} \to M_{2,2}$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

$$\therefore \ker T = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \,\middle|\, b, c \in \mathbb{R} \right\}$$

and it has basis $\left\{ M_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

$$\text{range}(T) = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \,\middle|\, a, d \in \mathbb{R} \right\}$$

has basis $\left\{ M_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\therefore$ $\text{rk}(T) = 2$, $\text{nullity}(T) = 2$

and $\dim(M_{2,2}) = 4 = 2 + 2 = \text{rk}(T) + \text{nullity}(T)$

(b) $T : M_{2,2}(\mathbb{R}) \to \mathbb{R}$

$$T(A) = \text{tr}(A)$$

ie $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto a + d$

$$\therefore A \in \ker(T) \iff A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{A_1} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_2} + c\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A_3}$$

$\therefore \ker(T)$ has dim 3. ($\{A_1, A_2, A_3\}$ span $\ker T$, and lin ind)

$\text{range}(T) = \mathbb{R}$ $\left(\text{eg } \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \xmapsto{\text{tr}} x \quad \forall x \in \mathbb{R} \right)$

and $\text{rank}(T) + \text{nullity}(T) = 1 + 3 = 4 = \dim M_{2,2}(\mathbb{R})$

11. (c) ~~ker~~. $T: M_{2,2}(\mathbb{R}) \to \mathbb{R}^2$

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix}$$

$\therefore \ker(T) = \left\{ \begin{bmatrix} a & a \\ c & c \end{bmatrix} \right\}$ has dim 2

$\left(\text{basis } \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} \right)$

and range$(T) \ni \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\therefore$ range(T) has dim 2.

$\therefore$ rank(T) + nullity(T) $= 2 + 2 = 4 = \dim M_{2,2}(\mathbb{R})$

(d) $T: P_2 \to \mathbb{R}^2$ $\quad T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

$T(a + bx + cx^2) = 0 \Rightarrow a = 0, \quad a + b + c = 0$

ie $c = -b$

$\therefore a + bx + cx^2 \in \ker T \Rightarrow a + bx + cx^2 = b(x - x^2)$

$\therefore$ nullity$(T) = 1$.

range(T) contains $T(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

lin ind

$\therefore$ rank$(T) = 2$.

$\therefore$ rk(T) + nullity(T) $= 1 + 2 = 3 = \dim P_2$

(a) $T : \mathbb{C} \to \mathbb{R}$

$T(2) = 2 \quad , \quad T(1+i) = 1$

$a+ib \in \ker T \Rightarrow T(a+ib) = 0$

$\Rightarrow T\big(a-b + (1+i)b\big) = 0$

$\Rightarrow \cancel{T(a-b)} + (a-b)\,T(1) + b\,T(1+i) = 0$

$\Rightarrow \frac{a-b}{2} \cdot T(2) + b \cdot 1 = 0$

$\Rightarrow a-b+b = 0$

$\Rightarrow a = 0$

$\therefore \ker T = \operatorname{span}\{i\} \quad \therefore \text{nullity}(T) = 1$

$\text{range}(T) \neq 0 \quad \therefore \text{rank}(T) = 1$

$\text{rank}(T) + \text{nullity}(T) = 1+1 = 2 = \dim_{\mathbb{R}} \mathbb{C}.$

(b) $T(A) = 0 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-b & -a+b \\ c-d & -c+d \end{bmatrix} - \begin{bmatrix} a-c & b-d \\ -a+c & -b+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\cancel{a-b = a-c}$ $\cancel{-a+b = b-d}$ $\cancel{-c+d = -b+d}$

$\Rightarrow \cancel{b=c} \Rightarrow \cancel{b=c}$

$\cancel{-a+b = b-d}$ $\cancel{c-d = -a+c}$

$\Rightarrow \cancel{a=d} \Rightarrow \cancel{a=d}$

$\therefore A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore \text{nullity}(T) = 2.$

$\text{range}(T)$ contains $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & - \end{bmatrix}$

lin. ind.

and $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (c-b)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + (a-d)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\therefore \text{rank}(T) = 2$

$= 2+2$

12. Linear independence of $\{x, Tx, \quad , T^{n-1}x\}$:

Consider $c_0 x + c_1 Tx + \quad + c_{n-1}T^{n-1}x = 0$ (*) $(c_i \in \mathbb{R})$

Applying T^{n-1} on both sides,

$$T^{n-1}(c_0 x + c_1 Tx + \quad + c_{n-1}T^{n-1}x) = T^{n-1}(0)$$

Noting that T^{n-1} is a linear transformation

we get $c_0 T^{n-1}x + c_1 T^n x + \cdots + c_n T^{2n-2}x = 0$

$\Rightarrow \quad c_0 T^{n-1}x = 0$

$\Rightarrow \quad c_0 = 0$ as $T^{n-1}x \neq 0$

~~Since~~ Next, apply T^{n-2} to obtain $c_1 = 0$ (from *)

$\therefore \{x, Tx, \ , \ , T^{n-1}x\}$ is lin ind.

Since $\dim V = n$, $\{x, \ , T^{n-1}x\}$ must be a basis of V.