PH 101
Tutorial-5
Date: 11/09/2017

1. Which of the following forces is conservative?
   (a) $\vec{F} = k(x, \quad 2y, \quad 3z)$ where k is a constant.
   (b) (b) $\vec{F} = k(y, \quad x, \quad 0)$
   (c) (c) $\vec{F} = k(-y, \quad x, \quad 0)$
   For those, which are conservative, find the corresponding potential energy V, and verify by direct differentiation that $\vec{F} = -\vec{\nabla}V$.

2. A metal ball (mass m) with a hole through it is threaded on a frictionless vertical rod. A massless string (length l) attached to the ball runs over a massless, frictionless pulley and supports a block of mass M, as shown in Fig. 1. The positions of the two masses can be specified by the one angle θ.
   (a) Write down the potential energy U (θ). (The PE is given easily in terms of the heights shown as h and H. Eliminate these two variables in favor of θ and the constants b and l. Assume that the pulley and ball have negligible size.) (b) By differentiating U(θ) find whether the system has an equilibrium position, and for what values of m and M equilibrium can occur. Discuss the stability of any equilibrium positions.

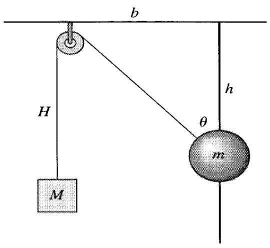


Fig. 1

3. A snooker ball of mass M and radius R is at rest on the table when it is projected forward with speed V and no angular velocity. Find the speed of the ball when it eventually begins to roll. What proportion of the original kinetic energy is lost in the process?

4. (a) A rigid body of general shape has mass M and can rotate freely about a fixed horizontal axis. The centre of mass of the body is distance h from the rotation axis, and the moment of inertia of the body about the rotation axis is I. (Refer to Fig. 2) Show that the period of small oscillations of the body about the downward equilibrium position is

$$2\pi\left(\frac{I}{Mgh}\right)^{1/2}$$

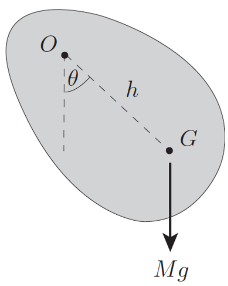


Fig. 2

Deduce the period of small oscillations of a uniform rod of length $2a$, pivoted about a horizontal axis perpendicular to the rod and distance b from its centre.

(b) A marble of radius b rolls back and forth in a shallow dish of radius R, where $R >> b$. Find the frequency of small oscillations.

5. A plank of length $2L$ leans against a wall (Refer to Fig. 3). It starts to slip downward without friction. Show that the top of the plank loses contact with the wall when it is at two-thirds of its initial height.

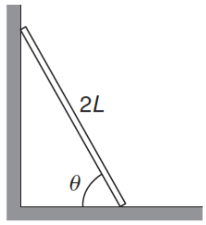


Fig. 3

6. The center of a long frictionless rod is pivoted at the origin and the rod is forced to rotate at a constant angular velocity Ω in a horizontal plane. Write down the equation of motion for a bead that is threaded on the rod, using the coordinates x and y of a frame that rotates with the rod (with x along the rod and y perpendicular to it). Solve for $x(t)$. What is the role of the centrifugal force? What of the Coriolis force?

7. The Coriolis force can produce a torque on a spinning object. To illustrate this, consider a horizontal hoop of mass m and radius r spinning with angular velocity ω about its vertical axis at colatitude θ. Show that the Coriolis force due to the earth's rotation produces a torque of magnitude $m\omega\Omega r^2 sin\theta$ directed to the west, where Ω is the earth's angular velocity. (This torque is the basis of the gyrocompass)

Solution to Tutorial 5

1.

(a) $\vec{F} = k\left(x\hat{i} + 2y\hat{j} + 3z\hat{k}\right)$

$\vec{\nabla}\times\vec{F} = (0,0,0) \Rightarrow \vec{F}$ is conservative

$$V = -\int \vec{F}\cdot d\vec{r}\,' = -k\int\left(x'dx' + 2y'dy' + 3z'dz\right)$$
$$= -k\left(\tfrac{1}{2}x^2 + y^2 + \tfrac{3}{2}z^2\right)$$

clearly, $-\vec{\nabla}V = \vec{F}$

(b) $\vec{F} = k\left(y\hat{i} + x\hat{j}\right)$

$\vec{\nabla}\times\vec{F} = (0,0,0) \Rightarrow \vec{F}$ is conservative

$\cancel{V = -\int \vec{F}\cdot d\vec{r}\,'}$

$$F_x = -\frac{\partial V}{\partial x} \Rightarrow V = -\int F_x\,dx + g(y)$$
$$= -\int ky\,dx + g(y) = -kxy + g(y)$$

$$\frac{\partial V}{\partial y} = -kx + \frac{\partial g}{\partial y}$$
$$= -F_y = -kx$$
$$\Rightarrow \frac{\partial g}{\partial y} = 0$$

Take $g = 0$

Thus, $V = -kxy$ //

(c) $\vec{F} = k\left(-\hat{i}y + \hat{j}x\right)$

$\vec{\nabla}\times\vec{F} \neq 0$

$\Rightarrow \vec{F}$ is not conservative.

2.

(a)

$$h = \frac{b}{\tan\theta}$$

$$H = l - \frac{b}{\sin\theta}$$

Thus,

$$U = -mgh - MgH = gb\left(\frac{M}{\sin\theta} - \frac{m}{\tan\theta}\right) - Mgl$$

$$= \frac{gb}{\sin\theta}\left(M - m\cos\theta\right) - Mgl$$

(b)

$$\frac{dU}{d\theta} = gb\left[\frac{m - M\cos\theta}{\sin^2\theta}\right]$$

- If $m > M$, $\frac{dU}{d\theta}$ never vanishes and there is no equilibrium points

- If $m = M$, it vanishes at $\theta = 0$, which is impossible (unless the string is infinitely long)

- If $m < M$, there is an equilibrium point at $\theta_0 = \cos^{-1}\left(\frac{m}{M}\right)$

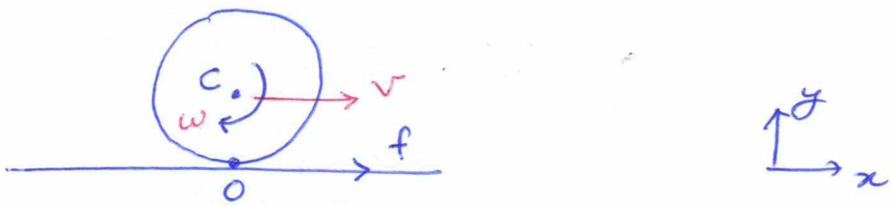
Since $\cos\theta$ decreases as θ increases, the factor $(m - M\cos\theta)$ is negative for $\theta < \theta_0$

i.e. $(m - M\cos\theta) < 0$ for $\theta < \theta_0$

while, $(m - M\cos\theta) > 0$ for $\theta > \theta_0$

$\Rightarrow$ $U(\theta)$ has a minimum at $\theta = \theta_0$ and the equilibrium is stable.

3.



Since the ball is moving horizontally, eq^n of motion is

$$\left.\begin{aligned} M\frac{d^2 v_x}{dt^2} &= F_x \\ I_c \frac{d\omega}{dt} &= \tau_c \end{aligned}\right\} \Rightarrow \begin{aligned} M\dot{v} &= f \quad \to (i) \\ \frac{2}{5}MR^2\dot{\omega} &= -Rf \to (ii) \end{aligned}$$

Eliminating ~~f~~ the unknown frictional force f, we find that

$$\dot{v} + \frac{2}{5}R\dot{\omega} = 0$$

Integrating with respect to t we obtain:

$$v + \frac{2}{5}R\omega = \alpha, \qquad \alpha \text{ is an integration constant.}$$

Initially, $v = V$ and $\omega = 0 \Rightarrow \alpha = V$. Hence,

$$\boxed{v + \frac{2}{5}R\omega = V} \to (iii)$$

, this holds in the subsequent motion whether the ball slides or rolls.

Say, the ball eventually rolls with speed v'. By the rolling condition, its angular velocity then will be v'/R. Thus from (iii): $V' + \frac{2}{5}R\left(\frac{V'}{R}\right) = V$

$$\Rightarrow \boxed{V' = \frac{5}{7}V}$$

The final KE of the ball,

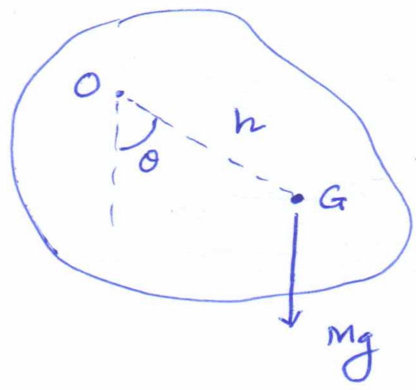
$$T' = \frac{1}{2}MV'^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{V'}{R}\right)^2 = \frac{5}{14}MV^2$$

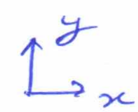
$$= \frac{5}{7}T, \quad T = \frac{1}{2}MV^2, \text{ the initial KE.}$$

$\Rightarrow$ The ball loses $\frac{2}{7}$ of its KE in transition from sliding to rolling.

4.

(a)





$$\boxed{\begin{aligned}\vec{\tau}_0 &= \vec{r} \times \vec{F} \\ &= (h\sin\theta\,\hat{x}) \times (-mg\hat{y}) \\ &= -\,mgh\sin\theta\,\hat{z}\end{aligned}}$$

$\frac{dL}{dt}$ about O, $\quad \frac{dL_0}{dt} = \tau_0 = (h\sin\theta)\,Mg$

$$L_0 = -I\dot{\theta}$$

{Here we are using the sign convention that for clockwise moments, angular velocities and angular momenta are positive}

where I is the moment of inertia of the body about the rotation axis.

Eq^n of motion:
$$\frac{d}{dt}\left(-I\dot{\theta}\right) = Mgh\sin\theta$$
$$\Rightarrow \quad \ddot{\theta} + \left(\frac{Mgh}{I}\right)\sin\theta = 0$$

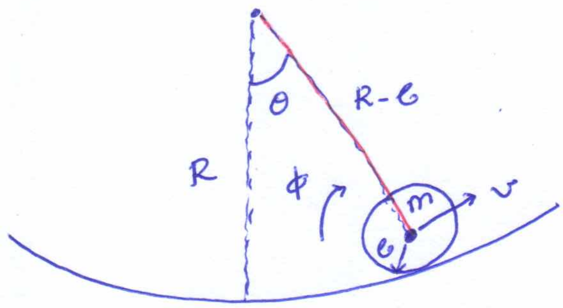
For oscillations of small amplitude:
$$\ddot{\theta} + \left(\frac{Mgh}{I}\right)\theta = 0$$

which is SHM with $\Omega^2 = Mgh/I$
$$\Rightarrow \quad T = 2\pi\sqrt{\frac{I}{Mgh}} \quad //$$

For the particular case of the rod,
$$h = b \quad \text{and}$$
$$I = \frac{1}{3}Ma^2 + Mb^2 = \frac{1}{3}M\left(a^2 + 3b^2\right)$$

Thus,
$$T = 2\pi\left(\frac{a^2 + 3b^2}{3gb}\right) \quad //$$

4. (b)



The gravitational potential energy of the marble

$$E_{pot} = mg(R-b)(1-\cos\theta)$$
$$\approx \frac{1}{2} mg R \theta^2$$

[Taking $E_{pot} = 0$ at the bottom of the dish, i.e. at $\theta = 0$]

[$R >> b$]

~~$E_{trans} =$~~

$$E_{trans} = \frac{1}{2} m v^2 = \frac{1}{2} m R^2 \dot{\theta}^2$$

$$E_{rot} = \frac{1}{2} I \dot{\phi}^2 = \frac{1}{2}\left(\frac{2}{5} m b^2\right)\left(\frac{R\dot{\theta}}{b}\right)^2 = \frac{1}{5} m R^2 \dot{\theta}^2$$

Thus,

$$E_{tot} = \frac{1}{2} m R \left[g\theta^2 + \frac{7}{5} R \dot{\theta}^2 \right] \rightarrow (1)$$

Now, it is guaranteed that, when we have a constant of motion, $E = \frac{1}{2}\alpha \dot{u}^2 + \beta u^2$, u undergoes simple harmonic motion with angular frequency, $\omega = \sqrt{\beta/\alpha}$.

[Lecture notes 7 and 8 by AKS]

Hence, ~~is~~ from Eq^n (1) we obtain:

$$\omega_{marble} = \sqrt{\frac{g}{7/5 R}} = \sqrt{\frac{5}{7} g/R}$$

//

5.

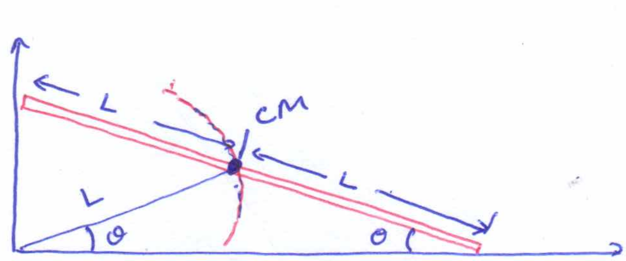


Fig. (i)

As long as the plank is in contact with the wall, the co-ordinates of its center of mass are

$$x = L\cos\theta, \quad y = L\sin\theta$$
$$x^2 + y^2 = L^2.$$

Until ~~the~~ contact with the wall is lost, the center of mass moves on a circular path of radius L, as indicated in Fig. (i).

Because the wall and floor are frictionless, the force F_W exerted by the wall on the plank and the force F_f exerted by the floor are normal to the surfaces, as shown in the sketch (ii).

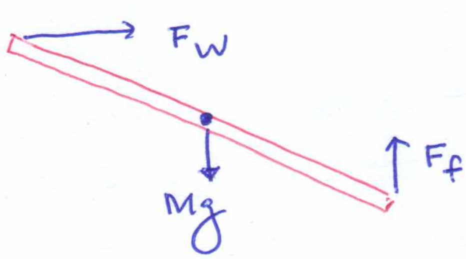


Fig. (ii)

The plank loses contact with the wall when $F_W = 0$, or equivalently,

$$M\ddot{x} = F_W = 0$$
$$\Rightarrow \ddot{x} = 0$$
$$\Rightarrow -L\sin\theta\,\ddot{\theta} - L\cos\theta\,\dot{\theta}^2 = 0$$
$$\Rightarrow \boxed{\dot{\theta}^2 = -\tan\theta\,\ddot{\theta}} \rightarrow (1)$$

As there is no dissipative force, mechanical energy E is conserved. Say y_0 be the initial height of the center of mass above the floor.

Then,
$$E_{initial} \equiv E_i = Mgy_0 = E_f = Mgy + \tfrac{1}{2}M(L\dot{\theta})^2 + \tfrac{1}{2}I_0\dot{\theta}^2$$

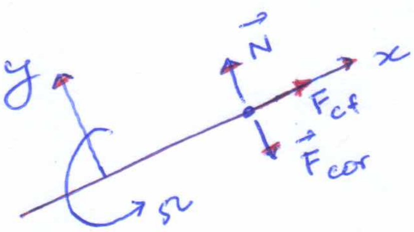
$$\Rightarrow \boxed{Mgy_0 = MgL\sin\theta + \tfrac{2}{3}ML^2\dot{\theta}^2} \rightarrow (2) \left\{ I_0 = \tfrac{1}{3}ML^2 \right\}$$

From (2), differentiating ~~we obtain~~ w.r.t. θ we obtain:

$$\boxed{\ddot{\theta} = -\tfrac{3}{4}\, {}^{g}/_{L}\cos\theta} \Rightarrow \boxed{\dot{\theta}^2 = \tfrac{3}{4}\, {}^{g}/_{L}\sin\theta} \text{ (From (1))}$$

From (2), $y_0 = \tfrac{3}{2}L\sin\theta = \tfrac{3}{2}y \Rightarrow \boxed{y = \tfrac{2}{3}y_0}$ //

6.



With axes fixed on the rotating rod as shown, the bead stays on the x-axis and its velocity $\vec{v} = \dot{x}\,\hat{x}$.

The three forces on the bead are: ~~be~~

- Normal force $\vec{N} = N\hat{y}$
- centrifugal force $\vec{F}_{cf} = m\Omega^2 x\,\hat{x}$
- Coriolis force $\vec{F}_{cor} = -2m\dot{x}\,\hat{y}$

Eq^n of motion:

$$m\ddot{x} = F_{cf} = m\Omega^2 x$$
$$N = F_{cor}$$

The solution is $x(t) = Ae^{\Omega t} + Be^{-\Omega t}$

The centrifugal force drives the bead out along the rod. The normal and coriolis forces just balance out.

7.

consider the axes, with x east, y north and z vertically up.

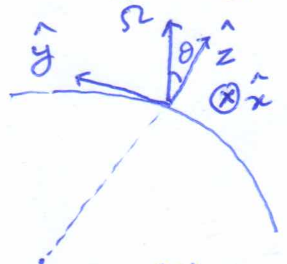


Fig. (i)

$$\vec{\Omega} = \Omega(0, \sin\theta, \cos\theta) \qquad (\text{Fig (i)})$$

Fig. (ii) shows the hoop as seen from above. consider first a small segment of hoop subtending an angle $d\alpha$ with polar angle α. The mass of this segment is $dm = \frac{m\,d\alpha}{2\pi}$, and Coriolis force on it is

$$d\vec{F}_{cor} = 2\,dm\,(\vec{v}\times\vec{\Omega}), \text{ where } \vec{v} = \omega r(-\sin\alpha, \cos\alpha, 0)$$

$\vec{r} = r(\cos\alpha, \sin\alpha, 0)$. The torque on it is

$$d\vec{\Gamma}_{cor} = \vec{r}\times d\vec{F}_{cor} = 2\,dm\,\vec{r}\times(\vec{v}\times\vec{\Omega})$$
$$= 2\,dm\,\omega r^2\Omega(-\sin^2\alpha, \sin\alpha\cos\alpha, 0)\sin\theta$$

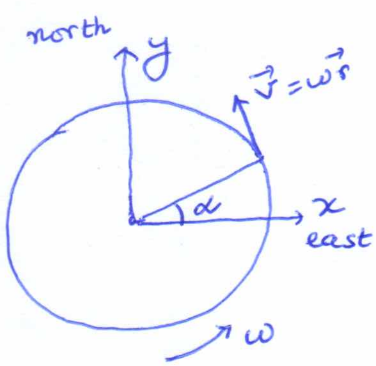


Fig. (ii)

To find total torque, we must replace dm by $m\,d\alpha/2\pi$ and integrate over α from 0 to 2π. Thus,

$$\vec{\Gamma}_{cor} = -(m\omega r^2\Omega\sin\theta)\,\hat{x} \quad //$$