DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Guwahati

Tutorial and practice problems on Single Variable Calculus

MA-101 : Mathematics-I Tutorial Problem Set - 11 November 06, 2013

PART-A (Tutorial)

1. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that $f'(c) = \lim_{n\to\infty}[n(f(c+1/n) - f(c))]$. Is the converse true?

2. Let $f : [-1/2, 1/2] \to \mathbb{R}$ be given by $f(x) := \begin{cases} \sqrt{2x - x^2}, & \text{if } 0 \le x \le 1/2, \\ \sqrt{-2x - x^2}, & \text{if } -1/2 \le x \le 0. \end{cases}$
   Show that $f(-1/2) = f(1/2)$ but $f'(x) \neq 0$ for all $0 < |x| < 1/2$. Does this contradict Rolle's theorem?

3. For $p, q \in \mathbb{R}$, show that the cubic $x^3 + px + q$ has three distinct real roots if and only if $4p^3 + 27q^2 < 0$.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$ then show that there exist distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

5. Let $f : [a, b] \to \mathbb{R}$ be continuous and twice differentiable on (a, b). If the line segment joining $(a, f(a))$ and $(b, f(b))$ intersects the graph of f at $(c, f(c))$ for some $c \in (a, b)$ then show that $f''(x_0) = 0$ for some $x_0 \in (a, b)$.

6. If $f''(c)$ exists then show that $\lim_{h\to 0^+} \frac{f(c+h)+f(c-h)-2f(c)}{h^2}$ exists and is equal to $f''(c)$. Give an example of a differentiable function on $(c - \delta, c + \delta)$, for some $\delta > 0$, for which this limit exists but f$''(c)$ does not exist.

7. Use MVT to prove the following.

   (i) $\frac{\pi}{15} < \tan(\pi/4) - \tan(\pi/5) < \frac{\pi}{10}$, $(ii) |\sin(a) - \sin(b)| \le |a - b|$.

PART-B (Homework/Practice problems)

1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) := 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

2. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ then show that $\lim_{h\to 0^+} \frac{f(c+h)-f(c-h)}{2h}$ exists and is equal to $f'(c)$. Is the converse true?

3. Let p and q be real and $p > 0$. Show that the cubic $x^3 + px + q$ has exactly one real root.

4. Let $n \in \mathbb{N}$ and $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous on $[a, b]$ and $f^{(n)}$ exists in (a, b). If f vanishes at $n + 1$ distinct points in $[a, b]$ then show that $f^{(n)}$ vanishes at least once in (a, b).

5. Let $\dfrac{a_0}{n+1} + \dfrac{a_1}{n} + \ldots + \dfrac{a_{n-1}}{2} + a_n = 0$. Show that the function $a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ vanishes at least once in $(0, 1)$.

6. In each case, find a function f satisfying all the given conditions or else show that no such function exists.

   (i) $f : \mathbb{R} \to \mathbb{R}$ such that $f''(x) > 0$ for all $x \in \mathbb{R}$ and $f'(0) = 1 = f'(1)$.

   (ii) $f : \mathbb{R} \to \mathbb{R}$ such that $f''(x) > 0$ for all $x \in \mathbb{R}$ and $f'(0) = 1, f'(1) = 2$.

   (iii) $f : \mathbb{R} \to \mathbb{R}$ such that $f''(x) \ge 0$ and $f'(0) = 1, f(x) \le 1$ for all $x < 0$.

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