Plan

- Determinant
- Determinants of Elementary Matrices
- Laplace Expansion Theorem
- Cramer's Rule
- Subspaces Associated with Matrices
- Rank Nullity Theorem
- The Fundamental Theorem of Invertible Matrices

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- Let A_{ij} be the submatrix of A obtained by deleting the i-th row and the j-th column of A.
- In general, $\det(A)$ is defined recursively as follows:

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \ldots + (-1)^{1+n} a_{1n}\det(A_{1n}).$$

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- Thus we can write

$$\det(A) = |A| = \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det(A_{1j}) = \sum_{j=1}^{n} a_{1j}C_{1j}.$$

- Take a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$.

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- In general, the coefficient of $|A_{1,2|i,j}|$ in the double expansion of $|A|$ is $(-1)^{1+2+i+j}\begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}$.

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- Hence $|A| = \sum_{i<j} (-1)^{1+2+i+j} \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} |A_{1,2|i,j}|$.

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★ A matrix A is said to be singular or non-singular according as $\det(A) = 0$ or $\det(A) \neq 0$.

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★ All the previous results based on row-wise expansion of determinant are also valid for column-wise expansion.

Result (**Laplace Expansion Theorem**)

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij},$$

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and also as

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Definition

Let A be an $n \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^n$. Then $A_i(\mathbf{b})$ denotes the matrix obtained by replacing the i-th column of A by $\mathbf{b}$. That is, if $A = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n]$, then $A_i(\mathbf{b}) = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1}\ \ldots\ \mathbf{a}_n]$.

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Result (**Cramer's Rule**)

Let A be an $n \times n$ invertible matrix and let $\mathbf{b} \in \mathbb{R}^n$. Then the unique solution $\mathbf{x} = [x_1, x_2, \ldots, x_n]^t$ of the system $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{det\,(A_i(\mathbf{b}))}{det(A)} \quad \text{for } i = 1, 2, \ldots, n.$$

The Adjoint of a Matrix: Let $A = [a_{ij}]$ be an $n \times n$ matrix and let C_{ij} be the (i, j)-cofactor of A. Then the adjoint of A, denoted $\mathrm{adj}(A)$, is defined as

$$\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} = [C_{ij}]^t.$$

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Exercise

Use the adjoint method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}.$$

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   [Here, elements of $\text{row}(A)$ are row vectors. How can they be elements of $\mathbb{R}^n$?

Subspaces Associated with Matrices

Definition

Let A be an $m \times n$ matrix.

1. The null space of A, denoted $\text{null}(A)$, is the subspace of $\mathbb{R}^n$ consisting of the solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. In other words, $\text{null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$
2. The column space of A, denoted $\text{col}(A)$, is the subspace of $\mathbb{R}^m$ spanned by the columns of A.
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   [Here, elements of $\text{row}(A)$ are row vectors. How can they be elements of $\mathbb{R}^n$? In strict sense, $\text{row}(A) := \text{col}(A^T)$.]

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4. Or, Use the columns of A that correspond to the columns of R containing the leading 1's to form a basis for $\text{col}(A)$.

Subspaces Associated with Matrices

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Example

Find bases for the row space, column space and null space of the following matrix:

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Result

Let $R = [\mathbf{b}_1 \ \mathbf{b}_2 \ \ldots \ \mathbf{b}_n]$ *be the reduced row echelen form of a matrix* $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n]$ *of rank* r*. Let* $\mathbf{b}_{j_1}, \mathbf{b}_{j_2}, \ldots, \mathbf{b}_{j_r}$ *be the columns of* R *such that* $\mathbf{b}_{j_k} = \mathbf{e}_k$ *for* $k = 1, \ldots, r$*. Then* $\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_r}\}$ *is a basis for* $col(A)$*.*

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Result (Rank Nullity Theorem)

Let A *be an* $m \times n$ *matrix. Then*

$$rank(A) + nullity(A) = n.$$

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Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. *A is invertible.*
2. *A^t is invertible.*
3. *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
4. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.*
5. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
6. *The reduced row echelon form of A is I_n.*
7. *The rows of A are linearly independent.*
8. *The columns of A are linearly independent.*
9. *$rank(A) = n$.*
10. *A is a product of elementary matrices.*

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Show that the vectors $[1, 2, 3]^t, [-1, 0, 1]^t$ *and* $[4, 9, 7]^t$ *form a basis for* $\mathbb{R}^3$.

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Let A *be an* $m \times n$ *matrix. Then*

1. $rank(A^tA) = rank(A)$.
2. *The* $n \times n$ *matrix* A^tA *is invertible if and only if* $rank(A) = n$.

Result

Let A, B, T, S be matrices, where T are S are invertible.

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3. *If $A + B$ is defined then* $rank(A + B) \leq rank(A) + rank(B)$.

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Result

Let A be an $n \times n$ matrix of rank r, where $1 \leq r < n$. Show that there exist elementary matrices $E_1, \ldots, E_p$ and $F_1, \ldots, F_q$ such that $E_1 \ldots E_p A F_1 \ldots F_q = \begin{bmatrix} I_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$.