DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
MA101 MATHEMATICS-I
Solutions to Tutorial - 6
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Topics Covered:
Vector spaces, subspace, basis, change-of-basis, linear transformations, matrix of linear transformation, kernel and range of linear transformation.

Recall:

- Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u}+\mathbf{v}$, and if c is a scalar, the *scalar multiple* of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a ***vector space*** and its elements are called ***vectors***.

  1. $\mathbf{u}+\mathbf{v}$ is in V. (Closure under addition)
  2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$. (Commutativity)
  3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$. (Associativity)
  4. There exists an element $\mathbf{0}$ in V, called a ***zero vector***, such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
  5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
  6. $c\mathbf{u}$ is in V. (Closure under scalar multiplication)
  7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$. (Distributivity)
  8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$. (Distributivity)
  9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
  10. $1\mathbf{u}=\mathbf{u}$.

- A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space with the same scalars, addition, and scalar multiplication as V.

  As in $\mathbb{R}^n$ to see whether a subset W of a vector space V is a subspace of V involves only two of the ten vector space axioms. Those two axioms is given in the following theorem.

  Theorem 1. *Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold:*
  (a) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.
  (b) If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W.

- Let $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{v}$ be a vector in V, and write $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n$. Then $c_1,c_2,\cdots,c_n$ are called the ***coordinates of*** $\mathbf{v}$ ***with respect to*** $\mathcal{B}$, and the column vector

$$[\mathbf{v}]_{\mathcal{B}}=\begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix}$$

  is called ***coordinate vector of*** $\mathbf{v}$ ***with respect to*** $\mathcal{B}$.

- Let $\mathcal{B}=\{\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_n\}$ and $\mathcal{C}=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ be bases for a vector space V. The $n\times n$ matrix whose columns are the coordinate vectors $[\mathbf{u}_1]_{\mathcal{C}},[\mathbf{u}_2]_{\mathcal{C}},\cdots,[\mathbf{u}_n]_{\mathcal{C}}$ of the vectors in $\mathcal{B}$ with respect to $\mathcal{C}$ is denoted by $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and is called the ***change-of-basis matrix*** from $\mathcal{B}$ to $\mathcal{C}$. That is,

$$P_{\mathcal{C}\leftarrow\mathcal{B}}=[\ [\mathbf{u}_1]_{\mathcal{C}}\ [\mathbf{u}_2]_{\mathcal{C}}\ \cdots\ [\mathbf{u}_n]_{\mathcal{C}}\ ]$$

- If $T : U \to V$ and $S : V \to W$ are linear transformations, then the ***composition of*** S with T is the mapping $S \circ T$, defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

  where $\mathbf{u}$ is in U.

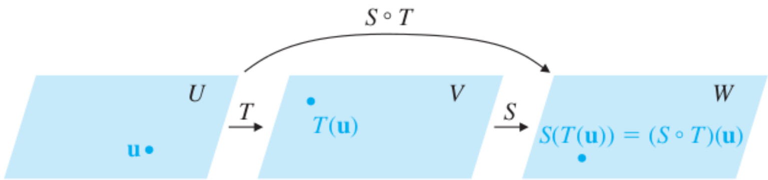

Figure 1: Composition of linear transformations.

- Let $T : V \to W$ be a linear transformation. The ***kernel*** of T, denoted $\ker(T)$, is the set of vectors in V that are mapped by T to $\mathbf{0}$ in W. That is,

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}.$$

  The ***range*** of T, denoted range(T), is the set of all vectors in W that are images of vectors in V under T. That is,

$$\text{range(T)} = \{T(\mathbf{v}) : \mathbf{v} \in V\} = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}.$$

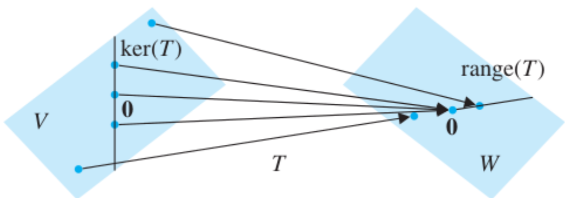

Figure 2: The kernel and range of $T : V \to W$.

- Let $T : V \to W$ be a linear transformation. The ***rank*** of T, is the dimension of the range of T and is denoted by rank(T). The ***nullity*** of T, is the dimension of the kernel of T and is denoted by nullity(T).

  Theorem 2. (The Rank Theorem)
  Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V into a vector space W. Then

$$rank(T) + nullity(T) = dim(V)$$

- A linear transformation $T : V \to W$ is called ***one-to-one*** if T maps distinct vectors in V to distinct vectors in W. If range$(T) = W$, then T is called ***onto***.

The above definition can be written more formally as follows:

$T : V \to W$ is one-to-one if, for all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} \neq \mathbf{v} \implies T(\mathbf{u}) \neq T(\mathbf{v})$. Equivalently, $T : V \to W$ is one-to-one if, for all $\mathbf{u}, \mathbf{v} \in V$, $T(\mathbf{u}) = T(\mathbf{v}) \implies \mathbf{u} = \mathbf{v}$.

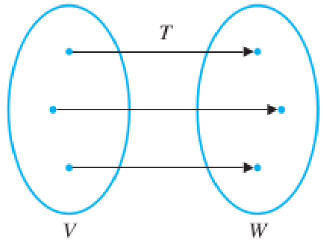

Figure 3: T is one-to-one.

Another way to write the definition of onto is as follows:

$T : V \to W$ is onto if, for all $\mathbf{w} \in W$, there is at least one $\mathbf{v}$ in V such that $\mathbf{w} = T(\mathbf{v})$.

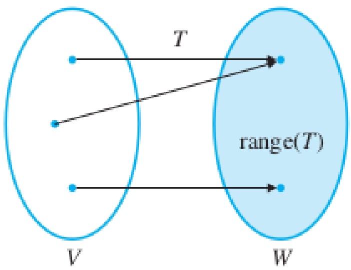

Figure 4: T is onto.

- A linear transformation $T : V \to W$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.
- Let $\dim(V) = \dim(W) = n$. Then a linear transformation $T : V \to W$ is one-to-one if and only if it is onto.
- A linear transformation $T : V \to W$ is **invertible** if and only if it is one-to-one and onto.
- Let V and W be two finite-dimensional vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$, respectively, where $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$. If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix A defined by

$$A = [\ [T(\mathbf{v}_1)]_{\mathcal{C}} \mid [T(\mathbf{v}_2)]_{\mathcal{C}} \mid \cdots \mid [T(\mathbf{v}_n)]_{\mathcal{C}}\ ]$$

satisfies $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$ for every vector $\mathbf{v}$ in V. The matrix A is called the ***matrix of*** T ***with respect to the bases*** $\mathcal{B}$ and $\mathcal{C}$.

$$\begin{tikzcd}
\mathbf{v} \arrow[r, "T"] \arrow[d] & T(\mathbf{v}) \arrow[d] \\
{[\mathbf{v}]_{\mathcal{B}}} \arrow[r, "T_A"] & A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}
\end{tikzcd}$$

- Let U, V, and W be finite-dimensional vector spaces with bases $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, respectively. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then

$$[S \circ T]_{\mathcal{D}\leftarrow\mathcal{B}} = [S]_{\mathcal{D}\leftarrow\mathcal{C}}[T]_{\mathcal{C}\leftarrow\mathcal{B}}$$

.

- Let $T : V \to W$ be a linear transformation between n-dimensional vector spaces V and W and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively. Then T is invertible if and only if the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible. In this case

$$([T]_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = [T^{-1}]_{\mathcal{B}\leftarrow\mathcal{C}}$$

Matrix of Linear Transformation & Change-of-Basis Matrices

1. In the following examples,

   (i) Find the co-ordinate vectors $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{C}}$ with respect to given bases $\mathcal{B}$ and $\mathcal{C}$.

   (ii) Find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$, and then find $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

   (iii) Use the answer in (ii) to compute $[v]_{\mathcal{C}}$ from $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{B}}$ from $[v]_{\mathcal{C}}$. Compare with answers in (i):

   (a) $v = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \mathcal{B} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} \qquad \mathcal{C} = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}.$

   (b) $v = 1 + x^2$, $B = \{1, x, x^2\}$, $C = \{1 + x + x^2, x + x^2, x^2\}$ in $\mathcal{P}_2(R)$.

Solution:

(i) (a) We need to find c_1, c_2, c_3 such that $v = c_1u_1 + c_2u_2 + c_3u_3$ where,

$$u_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, u_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, u_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

Equivalently, $v = [u_1 \quad u_2 \quad u_3] \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \implies \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$

Now consider the augmented matrix and convert into a REF

$$\begin{bmatrix}1&0&1&|1\\1&1&0&|2\\0&1&1&|3\end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix}1&0&1&|1\\0&1&-1&|1\\0&1&1&|3\end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix}1&0&1&|1\\0&1&-1&|1\\0&0&2&|2\end{bmatrix}$$

Therefore the solution of the system is $c_1 = 0, c_2 = 2$ and $c_3 = 1$. So required coordinate vector

$$[v]_{\mathcal{B}} = \begin{bmatrix}0\\2\\1\end{bmatrix}$$

Similarly, We need to find c_1, c_2, c_3 such that $v = c_1w_1 + c_2w_2 + c_3w_3$ where,

$$w_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, w_2 = \begin{bmatrix}1\\1\\-1\end{bmatrix}, w_3 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

Equivalently, $v = [w_1 \quad w_2 \quad w_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \implies \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

Now consider the augmented matrix and convert into a REF

$$\begin{bmatrix} 1 & 1 & 1 & |1 \\ 1 & 1 & -1 & |2 \\ 1 & -1 & 1 & |3 \end{bmatrix} \begin{matrix} R_3 \leftarrow R_3 - R_1 \\ - - - - \rightarrow \\ R_2 \leftarrow R_2 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & |1 \\ 0 & 0 & -2 & |1 \\ 0 & -2 & 0 & |2 \end{bmatrix} \begin{matrix} R_2 \leftrightarrow R_3 \\ - - - - \rightarrow \end{matrix} \begin{bmatrix} 1 & 1 & 1 & |1 \\ 0 & -2 & 0 & |2 \\ 0 & 0 & -2 & |1 \end{bmatrix}$$

Therefore the solution of the system is $c_1 = \frac{5}{2}, c_2 = -1$ and $c_3 = -\frac{1}{2}$. So required coordinate vector

$$[v]_{\mathcal{C}} = \begin{bmatrix} \frac{5}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix}$$

(b) We need to find c_1, c_2, c_3 such that $v = c_1(1) + c_2(x) + c_3(x^2) \implies 1 + x^2 = c_1 + c_2x + c_3x^2$. Therefore $c_1 = 1, c_2 = 0$ and $c_3 = 1$. So required coordinate vector

$$[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Similarly, We need to find c_1, c_2, c_3 such that $1 + x^2 = c_1(1 + x + x^2) + c_2(x + x^2) + c_3x^2$. Equivalently, to solve the system

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Therefore the solution of the system is $c_1 = 1, c_2 = -1$ and $c_3 = 1$. So required coordinate vector

$$[v]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(ii) (a) Apply Gauss-Jordan elimination on $[\ C|B\ ]$ to get $[\ I|P_{\mathcal{C}\leftarrow\mathcal{B}}\ ]$ from which we obtain $P_{\mathcal{C}\leftarrow\mathcal{B}}$ where B and C are the basis matrices (columns of the matrices are basis vectors) corresponding to bases $\mathcal{B}$ and $\mathcal{C}$ respectively.

$$[\ C|B\ ] = \begin{bmatrix} 1 & 1 & 1 & |1 & 0 & 1 \\ 1 & 1 & -1 & |1 & 1 & 0 \\ 1 & -1 & 1 & |0 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & |\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & |\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & |0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The required change-of-basis matrix is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

To get $P_{\mathcal{B}\leftarrow\mathcal{C}}$ use the formula $P_{\mathcal{B}\leftarrow\mathcal{C}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$. Therefore,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

(b)

$$[\,C|B\,] = \begin{bmatrix} 1 & 0 & 0 & |1 & 0 & 0 \\ 1 & 1 & 0 & |0 & 1 & 0 \\ 1 & 1 & 1 & |0 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & |1 & 0 & 0 \\ 0 & 1 & 0 & |-1 & 1 & 0 \\ 0 & 0 & 1 & |0 & -1 & 1 \end{bmatrix}$$

The required change-of-basis matrix is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Therefore,

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(iii) (a) To obtain $[v]_{\mathcal{C}}$ from $[v]_{\mathcal{B}}$ we use the formula $[v]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[v]_{\mathcal{B}}$. Similarly to compute $[v]_{\mathcal{B}}$ from $[v]_{\mathcal{C}}$ we use the formula $[v]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[v]_{\mathcal{C}}$. Therefore,

$$[v]_{\mathcal{C}} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad [v]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{5}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

(b)

$$[v]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad [v]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

2. In each of the following, find the matrix of T with respect to the standard bases. Is the matrix invertible? If so, find the inverse matrix and hence the inverse linear map of T.

(i) $T\colon \mathbb{R}^2 \to \mathcal{P}_1$, $\quad \begin{bmatrix} a \\ b \end{bmatrix} \mapsto a + (a-b)x.$

(ii) $D\colon \mathrm{Span}(e^x, xe^x, x^2e^x) \to \mathcal{F}$ where D is the differential operator and $\mathcal{F}$ is the space of all real valued functions from $\mathbb{R}$ to $\mathbb{R}$.

Solution:

1. Standard basis for $\mathbb{R}^2$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and the standard basis for $\mathcal{P}_1$ is given by $\{1, x\}$. Now we have

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= 1 + x &= [1 \quad x]\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \text{and } T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= 0 + (-1)x &= [1 \quad x]\begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{aligned}$$

Therefore matrix of T w.r.t. standard bases is given by

$$[T] = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.$$

T is invertible with

$$[T^{-1}] = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Therefore, $T^{-1} : \mathcal{P}_1 \to \mathbb{R}^2$ is given by

$$T^{-1}(a + bx) = \begin{bmatrix} a \\ a - b \end{bmatrix}.$$

2. A basis for domain(D) is given by $\{e^x, xe^x, x^2e^x\}$ and a basis for image(D) is given by $\{e^x, xe^x, x^2e^x\}$, which can be extended to a basis of the space ($\mathcal{F}$) of all real valued functions from $\mathbb{R}$ to $\mathbb{R}$.

$$D(e^x) = e^x, \qquad D(xe^x) = e^x + xe^x, \qquad D(x^2e^x) = 2xe^x + x^2e^x.$$

Therefore matrix of D w.r.t. above bases is given by

$$[D] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

Since matrix $[D]$ has zero rows, $[D]$ is not invertible which implies D is not invertible.

3. Let V be a vector space and $T\colon V \to V$ be a linear transformation such that $T \circ T = I$ and let $v \in V$ be a non-zero vector. Then show that the set $\{v, T(v)\}$ is linearly dependent if and only if $T(v) = \pm v$.

Solution: Let set $\{v, T(v)\}$ is linearly dependent, which means $T(v)$ can be written as scalar multiple of v.

$$T(v) = \lambda v \Rightarrow T(T(v)) = \lambda T(v) \Rightarrow (T \circ T)(v) = \lambda(\lambda v) \Rightarrow Iv = \lambda^2 v$$

$$(1 - \lambda^2)v = 0$$

Since v is non zero hence $1 - \lambda^2 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow T(v) = \pm v$

Conversely if $T(v) = \pm v$ this implies that $T(v)$ is the scalar multiple of v. Hence set $\{v, T(v)\}$ is linearly dependent.

4. Let V, W be two vector spaces over $\mathbb{R}$, and $T : V \to W$ be a linear map. Let $w_1, \cdots, w_n$ be linearly independent vectors in W, and let $v_1, \cdots, v_n$ be vectors in V such that $T(v_i) = w_i$. Show that $v_1, \cdots, v_n$ are linearly independent.

Solution: Consider

$$\begin{aligned}
c_1v_1 + c_2v_2 + \cdots + c_nv_n &= 0 \\
T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) &= T(0) = 0 && \text{(T is linear map)} \\
c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) &= 0 && (T(v_i) = w_i) \\
c_1w_1 + c_2w_2 + \cdots + c_nw_n &= 0 && (w_1, \dots w_n \text{ are l.i}) \\
c_1 = c_2 = \cdots = c_n &= 0
\end{aligned}$$

Hence $v_1, \cdots, v_n$ are linearly independent.

5. Let V, W be finite dimensional vector spaces over the field $\mathbb{F}$ and $T\colon V \to W$ be a linear map.
   (a) Show that
   $$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$
   (b) Assume that $\dim V = \dim W$, then show that T is injective if and only if T is surjective.
   (c) If $\dim(V) < \dim(W)$ then show that T is not surjective.
   (d) If $\dim(V) > \dim(W)$ then show that T is not injective.

Solution:

(a) Let $\dim(V) = n$ and let $\{v_1, \dots v_k\}$ be a basis for $\ker(T)$ (so that $\operatorname{nullity}(T) = \dim(\ker(T)) = k$). Since $\{v_1, \dots v_k\}$ is a linearly independent set, it can be extended to a basis for V, Let $\mathcal{B} = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ be such a basis. If we can show that the set $\mathcal{C} = \{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for range(T), then we will have $\operatorname{rank}(T) = \dim(\operatorname{range}(T)) = n - k$ and thus
$$\operatorname{rank}(T) + \operatorname{nullity}(T) = (n - k) + k = n = \dim(V).$$
as required.
It is clear that $\mathcal{C}$ is contained in the range of T. To show that $\mathcal{C}$ spans the range of T, let $T(v)$ be a vector in the range of T. Then v is in V, and since $\mathcal{B}$ is a basis for V, we can find scalars $c_1, c_2, \dots, c_n$ such that

$$v = c_1v_1 + \dots c_kv_k + c_{k+1}v_{k+1} + \cdots + c_nv_n$$

Since $v_1, \dots, v_k$ are in the kernal of T, we have $T(v_1) = \cdots = T(v_k) = 0$, So

$$\begin{aligned}
T(v) &= T(c_1v_1 + \dots c_kv_k + c_{k+1}v_{k+1} + \cdots + c_nv_n) \\
&= c_1T(v_1) + \dots c_kT(v_k) + c_{k+1}T(v_{k+1}) + \cdots + c_nT(v_n) \\
&= c_{k+1}T(v_{k+1}) + \cdots + c_nT(v_n)
\end{aligned}$$

This shows that the range of T is spanned by $\mathcal{C}$.
To show that $\mathcal{C}$ is linearly independent, suppose that there are scalars $c_{k+1}, \dots, c_n$ such that

$$c_{k+1}T(v_{k+1}) + \cdots + c_nT(v_n) = 0$$

$$T(c_{k+1}v_{k+1} + \cdots + c_nv_n) = 0$$

which means that $c_{k+1}v_{k+1}+\cdots+c_nv_n$ is in the $\ker(T)$ and expressible as a linear combination of the basis vectors of $\ker(T)$,

$$c_{k+1}v_{k+1}+\cdots+c_nv_n = c_1v_1+\cdots+c_kv_k$$

$$-c_1v_1-\cdots-c_kv_k+c_{k+1}v_{k+1}+\cdots+c_nv_n = 0$$

which implies that $c_1 = c_2 = \cdots = c_k = c_{k+1} = \cdots = c_n = 0$. because of basis $\mathcal{B}$ of V.
We have shown that $\mathcal{C}$ is a basis for the range of T, so proof is complete.

(b) Let T is injective then $\ker(T) = \{0\} \Rightarrow \text{nullity}(T) = 0$ by part(a) $\text{rank}(T) = \dim(V) = \dim(W) \Rightarrow \dim(\text{Range}(T)) = \dim(W)$, hence T is surjective.
Conversely let T is surjective then $\dim(\text{Range}(T)) = \text{rank}(T) = \dim(W)$, by part ($a$), $\dim(W)+\text{nullity}(T) = \dim(V) \Rightarrow \text{nullity}(T) = 0$, (because $\dim V = \dim W$) So $\ker(T) = \{0\}$ hence T is injective.

(c) Given that $\dim(V) < \dim(W)$. If possible let T is surjective which implies that $\dim(\text{Range}(T)) = \text{rank}(T) = \dim(W)$ by part (a), $\dim(W)+\text{nullity}(T) = \dim(V) \Rightarrow \dim(W) \leq \dim(V)$ which is the contradiction.

(d) Given that $\dim(V) > \dim(W)$. If possible let T is injective which means $\ker(T) = \{0\} \Rightarrow \text{nullity}(T) = 0$ So by part (a), $\text{rank}(T) = \dim(V)$, but we know that range(T) is the subspace of W. so $\dim(\text{range}(T)) \leq \dim(W) \Rightarrow \text{rank(T)} \leq \dim(W) \Rightarrow \dim(V) \leq \dim(W)$, which is the contradiction.

6. Let V be a vector space over a field $\mathbb{F}$ and $T\colon V \to V$ be a linear map such that $T^n = 0$ for some $n \in \mathbb{N}$. Show that the linear map $(1-T)$ is one-one and onto.

Solution: Let $v(\neq 0) \in \ker(1-T)$. Therefore,

$$\begin{aligned}
(1-T)v = 0 \;&\Rightarrow\; v - Tv = 0\\
&\Rightarrow\; Tv = v \qquad \text{for all } v \in \ker(1-T)\\
&\Rightarrow\; T^nv = T(T(\cdots T(v))\cdots)) = v \neq 0 \qquad -\text{ a contradiction}\\
&\Rightarrow\; \ker(1-T) = \{0\} \quad\Rightarrow\quad (1-T) \text{ is injective or one-one.}
\end{aligned}$$

From 5(b) T is surjective. (as domain(T) = codomain(T) = V).
This completes the proof.

7. Show that there is no linear transformation $T : \mathbb{R}^3 \to \mathcal{P}_2$ for which

$$T\left(\begin{bmatrix}2\\-1\\0\end{bmatrix}\right) = 1+2x+x^2, \quad T\left(\begin{bmatrix}3\\0\\-2\end{bmatrix}\right) = 1-x^2, \quad T\left(\begin{bmatrix}0\\3\\-4\end{bmatrix}\right) = 2x+3x^2.$$

Solution: Since

$$(-3)\begin{bmatrix}2\\-1\\0\end{bmatrix} + 2\begin{bmatrix}3\\0\\-2\end{bmatrix} = \begin{bmatrix}0\\3\\-4\end{bmatrix}$$

but

$$
\begin{aligned}
T\left((-3)\begin{bmatrix}2\\-1\\0\end{bmatrix}+2\begin{bmatrix}3\\0\\-2\end{bmatrix}\right) &= (-3)T\left(\begin{bmatrix}2\\-1\\0\end{bmatrix}\right)+2T\left(\begin{bmatrix}3\\0\\-2\end{bmatrix}\right)\\
&= (-3)(1+2x+x^2)+2(1-x^2)\\
&= -1-6x-5x^2\\
&\neq 2x+3x^2\\
&= T\left(\begin{bmatrix}0\\3\\-4\end{bmatrix}\right)
\end{aligned}
$$

8. For each of the maps given below, show that T is a linear transformation. Describe $\ker(T)$ and range(T) of the following. Hence find the rank or the nullity of the following

   (a) $T : M_{2,2} \to \mathbb{R}$ defined by $T(A) = tr(A)$.

   (b) $T : \mathcal{P}_3 \to \mathbb{R}^3$, $T(a+bx+cx^2+dx^3) = \begin{bmatrix}a\\b-a\\c-a+b\end{bmatrix}$.

   (c) $T : M_{2,2} \to M_{2,2}$ defined by $T\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = \begin{bmatrix}a & 0\\0 & d\end{bmatrix}$.

   (d) $T : M_{2,2} \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = \begin{bmatrix}a-b\\c-d\end{bmatrix}$.

   (e) $T : \mathcal{P}_2 \to \mathbb{R}^2$ defined by $T(\mathsf{p}(x)) = \begin{bmatrix}\mathsf{p}(0)\\\mathsf{p}(1)\end{bmatrix}$.

Solution:

(a) So, $\ker(T)$ is given by

$$
\begin{aligned}
\ker(T) &= \left\{\begin{bmatrix}a & b\\c & d\end{bmatrix} : a+d=0,\ a,b,c,d \in \mathbb{R}\right\}\\
&= \left\{a\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}+b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}+c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} : a,b,c \in \mathbb{R}\right\}\\
\Rightarrow nullity(T) &= \dim(\ker T) = 3.
\end{aligned}
$$

Similarly range(T) is given by,

$$
\begin{aligned}
\text{range}(T) &= \left\{a+d\colon \begin{bmatrix}a & b\\c & d\end{bmatrix} \in M_{2,2}\right\} = \mathbb{R}.\\
\Rightarrow rank(T) &= \dim(\text{range}(T)) = 1.
\end{aligned}
$$

(b) $\ker(T) = \{\mathsf{p}(x) \in \mathcal{P}_3\colon T(\mathsf{p}(x)) = 0\}$. Therefore,

$$
\begin{aligned}
\ker(T) &= \left\{a+bx+cx^2+dx^3\colon \begin{bmatrix}a\\b-a\\c-a+b\end{bmatrix} = 0\right\}\\
&= \{a+bx+cx^2+dx^3\colon a=b=c=0\}\\
&= \{dx^3 : d \in \mathbb{R}\}.\\
\Rightarrow nullity(T) &= \dim(\ker T) = 1.
\end{aligned}
$$

$$
\begin{aligned}
\text{range}(T) &= \left\{ \begin{bmatrix} a \\ b-a \\ c-a+b \end{bmatrix} : a+bx+cx^2+dx^3 \in \mathcal{P}_3 \right\} \\
& \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x=a, y=b-a, z=c-a+b \quad \text{for some } a,b,c \in \mathbb{R} \right\} \\
&= \mathbb{R}^3. \\
\Rightarrow rank(T) &= \dim(\text{range}(T)) = 3.
\end{aligned}
$$

(c) So, $\ker(T)$ is given by

$$
\begin{aligned}
\ker(T) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a=d=0,\ a,b,c,d \in \mathbb{R} \right\} \\
&= \left\{ b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} : b,c \in \mathbb{R} \right\} \\
\Rightarrow nullity(T) &= \dim(\ker T) = 2.
\end{aligned}
$$

Similarly range(T) is given by,

$$
\begin{aligned}
\text{range}(T) &= \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a,d \in \mathbb{R} \right\} \\
&= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a,d \in \mathbb{R} \right\} \\
\Rightarrow rank(T) &= \dim(\text{range}(T)) = 2.
\end{aligned}
$$

(d) So, $\ker(T)$ is given by

$$
\begin{aligned}
\ker(T) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a-b=0, c-d=0,\ a,b,c,d \in \mathbb{R} \right\} \\
&= \left\{ a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} : b,c \in \mathbb{R} \right\} \\
\Rightarrow nullity(T) &= \dim(\ker T) = 2.
\end{aligned}
$$

Similarly range(T) is given by,

$$
\begin{aligned}
\text{range}(T) &= \left\{ \begin{bmatrix} a-b \\ c-d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\} \\
&= \mathbb{R}^2 \\
\Rightarrow rank(T) &= \dim(\text{range}(T)) = 2.
\end{aligned}
$$

(e) $\ker(T) = \{\mathsf{p}(x) \in \mathcal{P}_2 \colon T(\mathsf{p}(x)) = 0\}$. Therefore,

$$
\begin{aligned}
\ker(T) &= \left\{ a+bx+cx^2 \colon \begin{bmatrix} \mathsf{p}(0) \\ \mathsf{p}(1) \end{bmatrix} = 0 \right\} \\
&= \left\{ a+bx+cx^2 \colon a=0, b+c=0 \right\} \\
&= \{ b(x-x^2) : b \in \mathbb{R} \}. \\
\Rightarrow nullity(T) &= \dim(\ker T) = 1.
\end{aligned}
$$

$$
\begin{aligned}
\text{range}(T) &= \left\{ \begin{bmatrix} \mathsf{p}(0) \\ \mathsf{p}(1) \end{bmatrix} : \mathsf{p}(x) = a + bx + cx^2 \in \mathcal{P}_2 \right\} \\
& \left\{ \begin{bmatrix} a \\ a+b+c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \\
&= \mathbb{R}^2. \\
\Rightarrow rank(T) &= \dim(\text{range}(T)) = 2.
\end{aligned}
$$

9. Let V be a vector space and $T\colon V \to V$ be a linear transformation. Suppose $\dim V = n$. If there exists a vector $x \in V$ such that $T^n(x) = 0$ but $T^{n-1}(x) \neq 0$, then show that the set $\{x, T(x), \cdots, T^{n-1}(x)\}$ is a basis for V. Also, find the matrix representation of T with respect to this basis.

Solution: Consider the equation,

$$c_1 x + c_2 T(x) + \cdots + c_n T^{n-1}(x) = 0.$$

Then

$$T^{n-1}\left(c_1 x + c_2 T(x) + \cdots + c_n T^{n-1}(x)\right) = 0 \Rightarrow c_1 T^{n-1}(x) = 0.$$

(since, $T^k(x) = 0$ for all $k \geq n$)

Similarly, we can show $c_j = 0$ for all $j = 1, 2, \cdots, n$.
Therefore the set $\{x, T(x), \cdots, T^{n-1}(x)\}$ of n vectors in V is linearly independent, and also $\dim V = n$.
Hence $\{x, T(x), \cdots, T^{n-1}(x)\}$ has to be a basis of V.
Consider $\mathcal{B} = \{x, T(x), \cdots, T^{n-1}(x)\}$. Then image of basis elements is

$$\{T(x), T^2(x), \cdots, T^{n-1}(x), T^n(x)\} = \{T(x), T^2(x), \cdots, T^{n-1}(x), 0\} \qquad [\because T^n(x) = 0]$$

. The matrix representation of T with respect to the basis $\mathcal{B}$ is given by

$$[T]_{\mathcal{B}} = \left[[T(x)]_{\mathcal{B}} \quad [T^2(x)]_{\mathcal{B}} \quad \cdots \quad [T^{n-1}(x)]_{\mathcal{B}} \quad [\,0\,]_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Practice Problems

1. If $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{R}^3$ such that the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is given by $\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$. If $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ find $\mathcal{B}$.

Solution: $[\mathcal{B}]_\epsilon = \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}}[\mathcal{C}]_\epsilon$ where ϵ is a standard basis of $\mathbb{R}^3$

$$[\mathcal{B}]_\epsilon = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 6 \\ -5 & 8 & 9 \\ -6 & 0 & 6 \end{bmatrix}$$

hence $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 9 \\ 6 \end{bmatrix} \right\}$

2. Let $T\colon \mathbb{R}^2 \to \mathcal{P}_2(\mathbb{R})$ be a linear map for which $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 1 + 2x + x^2$ and $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 3 + 4x^2$. Find $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$.

Solution: Consider $\begin{bmatrix} a \\ b \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ on solving this system we get $c_1 = -b$ and $c_2 = a + b$

Now $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = c_1 T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + c_2 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = (-b)(1 + 2x + x^2) + (a + b)(3 + 4x^2) = (3a + 2b) + (-2b)x + (4a + 3b)x^2$

from above we get $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = (13) + (-4)x + (18)x^2$

3. Find the matrix $[S]$, $[T]$ and $[S \circ T]$ with respect to the standard bases and then verify that $[S \circ T] = [S][T]$, where

$$T\colon \mathbb{R}^2 \to \mathcal{P}_2, \qquad \begin{bmatrix} a \\ b \end{bmatrix} \mapsto a + (a - b)x + bx^2$$
$$S\colon \mathcal{P}_2 \to \mathcal{P}_1, \qquad a + bx + cx^2 \mapsto (3a + 2b + c) + (a + b)x.$$

Solution:

1. Standard basis for $\mathbb{R}^2$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and the standard basis for $\mathcal{P}_2$ is given by $\{1, x, x^2\}$. Now we have

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= 1 + x + 0.x^2 &&= [1 \quad x \quad x^2] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \text{and } T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= 0 + (-1)x + x^2 &&= [1 \quad x \quad x^2] \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore matrix of T w.r.t. standard bases is given by

$$[T] = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

2. The standard basis for $\mathcal{P}_2$ is given by $\{1, x, x^2\}$ and the standard basis for $\mathcal{P}_1$ is given by $\{1, x\}$. Now from given definition of S we have

$$S(1) = 3 + x = [1 \quad x] \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad S(x) = 2 + x = [1 \quad x] \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad S(x^2) = 1 + 0.x = [1 \quad x] \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore matrix of S w.r.t. standard bases is given by

$$[S] = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

3. **Verification:**

$$S \circ T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = S(a+(a-b)x+bx^2) = 3a+2(a-b)+b+(a+a-b)x = (5a-b)+(2a-b)x.$$

Therefore,

$$S \circ T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 5+2x = [1 \quad x]\begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \text{and} \quad S \circ T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -1-x = [1 \quad x]\begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Therefore matrix of $S \circ T$ w.r.t. standard bases is given by

$$[S \circ T] = \begin{bmatrix} 5 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = [S][T].$$

4. Show that a linear map $T\colon V \to W$ is uniquely determined by its values on the elements of a basis of V.

Solution: Let $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a basis of V and $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ an arbitrary list of vectors in W. We need to show $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, \cdots, n$. Take any $\mathbf{v} \in V$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for V, $\exists$ unique scalars $c_1, c_2, \cdots, c_n$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. By linearity we must have

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) \\ &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n \end{aligned}$$

Hence $T(\mathbf{v})$ is completely determined.

Now for uniqueness, let us assume $\exists\, S\colon V \to W$ such that $S(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, \cdots, n$. Now for any $\mathbf{u} \in V$ we have

$$\begin{aligned} S(\mathbf{u}) &= S(d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n) \\ &= d_1S(\mathbf{v}_1) + \cdots + d_nS(\mathbf{v}_n) \\ &= d_1\mathbf{w}_1 + \cdots + d_n\mathbf{w}_n \\ &= d_1T(\mathbf{v}_1) + \cdots + d_nT(\mathbf{v}_n) \\ &= T(d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n) \\ &= T(\mathbf{u}) \end{aligned}$$

Since $\mathbf{u} \in V$, is arbitrary we have $S = T$.

5. Let W be a subspace of a finite dimensional vector space V. Show that W is a finite dimensional vector space and $\dim W \leq \dim V$.

Solution: Let $\dim V = n$. If $W = \{\mathbf{0}\}$, then $\dim W = 0 \leq n = \dim V$. If W is nonzero, then any basis $\mathcal{B}$ for V (containing n vectors) certainly spans W, as W is contained in V. But $\mathcal{B}$ can be reduced to a basis $\mathcal{B}'$ for W (containing atmost n vectors). Hence, W is finite-dimensional and $\dim W \leq n = \dim V$.

[**NOTE:** Any spanning set for a vector space V with $\dim V = n$ can be reduced to a basis for V.]

6. Let $\mathcal{B}$ be a set of vectors in a vector space V with the property that every vector in V is a unique linear combination of vectors in $\mathcal{B}$. Prove that $\mathcal{B}$ is a basis for V.

Solution: Since every vector in V is a linear combination of vectors in $\mathcal{B}$, so of $\text{Span}(\mathcal{B}) = V$. So it is enough to show that vectors in $\mathcal{B}$ are linearly independent.
If not, then there exists a non-trivial linear combination of vectors in $\mathcal{B}$. Let us assume

$$\mathcal{B} = \{v_1, v_2, \cdots\}, \qquad \text{and} \qquad \sum_i c_i v_i = 0 \text{ for some } c_i \neq 0.$$

Without loss of generality, we can assume $c_1 \neq 0$. Then we have

$$v_1 = \sum_{i \geq 2} -\frac{c_i}{c_1} v_i,$$

which contradicts the uniqueness of the representation of $v_1 \in V$.
Therefore, vectors in $\mathcal{B}$ must be linearly independent and hence $\mathcal{B}$ is a basis for V.

7. Let V and W be vector spaces over $\mathbb{R}$. Prove that the set $\mathcal{L}(V, W)$ of all linear transformations forms a vector space over $\mathbb{R}$, with the vector addition and scalar multiplication as defined over $\mathcal{F}$, the space of functions from $\mathbb{R}$ to $\mathbb{R}$.

Solution: Firstly, note that the null map $\mathbf{0} : V \to W$ is in $\mathcal{L}(V, W)$. So $\mathcal{L}(V, W)$ is non-empty.
It is enough to check that it is closed under $+$(addition) and $\cdot$(scalar multiplication).
Let $S, T \in \mathcal{L}(V, W)$ and $u, v \in V$ $\alpha \in \mathbb{R}$. Enough to show: $S + T, \alpha \cdot S \in \mathcal{L}(V, W)$.
Note that $(S + T)(v) = S(v) + T(v)$ and $(\alpha \cdot S)(v) = \alpha S(v)$.

$$(S + T)(u + v) = S(u + v) + T(u + v) = S(u) + T(u) + S(v) + T(v) = (S + T)(u) + (S + T)(v).$$

Also for $c \in \mathbb{R}$,

$$(S + T)(cu) = S(cu) + T(cu) = cS(u) + cT(u) = c(S + T)(u).$$

Therefore, $S + T \in \mathcal{L}(V, W)$. ...(1)
Again, for $u, v \in V$ and $c \in \mathbb{R}$,

$$(\alpha \cdot S)(u + v) = \alpha S(u + v) = \alpha[S(u) + S(v)] = (\alpha \cdot S)(u) + (\alpha \cdot S)(v)$$

$$(\alpha \cdot S)(cu) = \alpha S(cu) = c(\alpha \cdot S)(u)$$

Therefore, $\alpha \cdot S \in \mathcal{L}(V, W)$. ...(2)
From (1) & (2) the result follows.

8. Let W_1 and W_2 be two subspaces of a finite dimensional vector space V. Show that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Solution: Let $(u_1, \ldots, u_m)$ be a basis of $W_1 \cap W_2$, thus $\dim(W_1 \cap W_2) = m$. Because $(u_1, \ldots, u_m)$ is a basis of $W_1 \cap W_2$, it is linearly independent in W_1 and hence can be extended to a basis $(u_1, \ldots, u_m, v_1, \ldots, v_j)$ of W_1. Thus $\dim(W_1) = m + j$. Also extend $(u_1, \ldots, u_m)$ to a basis $(u_1, \ldots, u_m, w_1, \ldots, w_k)$ of W_2, thus $\dim(W_2) = m + k$.
We will show that $(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ is a basis of $W_1 + W_2$. This will complete the proof because then we will have

$$\dim(W_1 + W_2) = m + j + k = (m + j) + (m + k) - m = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Clearly $span(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ contains W_1 and W_2 and hence contains $W_1 + W_2$. So to show that this list is a basis of $W_1 + W_2$ we need only show that it is linearly independent. To prove this, suppose

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_k w_k = 0$$

where all the a's, b's, and c's are scalars. We need to prove that all the a's, b's, and c's equal 0. The equation above can be rewritten as

$$c_1 w_1 + \cdots + c_k w_k = -a_1 u_1 - \cdots - a_m u_m - b_1 v_1 - \cdots - b_j v_j$$

which shows that $c_1 w_1 + \cdots + c_k w_k \in W_1$. All the w's are in W_2, so this implies that $c_1 w_1 + \cdots + c_k w_k \in W_1 \cap W_2$. Because $(u_1, ..., u_m)$ is a basis of $W_1 \cap W_2$, we can write

$$c_1 w_1 + \cdots + c_k w_k = d_1 u_1 + \cdots + d_m u_m$$

for some choice of scalars $d_1, \ldots, d_m$. But $(u_1, \ldots, u_m, w_1, \ldots, w_k)$ is linearly independent, so the last equation implies that all the c's (and d's) equal 0. Thus our original equation involving the a's, b's, and c's becomes

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j = 0.$$

This equation implies that all the a's and b's are 0 because the list $(u_1, \ldots, u_m, v_1, \ldots, v_j)$ is linearly independent. We now know that all the a's, b's, and c's equal 0, as desired.