# MA101 Mathematics I

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# Plan

- The Geometry and Algebra of Vectors (in $\mathbb{R}^n$)
- Matrices
  - Matrix operations
  - Transpose
  - Matrix Multiplication and powers
- Properties of matrix operations

# Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Recall:

- A vector is a directed line segment - corresponds to a displacement from one point $A$ to another $B$.
- A vector in $\mathbb{R}^2$:

  $\mathbf{v} = a\,\widehat{\mathbf{i}} + b\,\widehat{\mathbf{j}} \equiv$ the position vector of the point $(a, b) := [a, b]$.

- A vector in $\mathbb{R}^3$:

$$\begin{aligned} \mathbf{v} &= a\,\widehat{\mathbf{i}} + b\,\widehat{\mathbf{j}} + c\,\widehat{\mathbf{k}} \equiv \text{the position vector of the point } (a, b, c) \\ &:= [a, b, c]. \end{aligned}$$

- Thus, the vector $[a, b, c]$ is identified by the position vector of $(a, b, c)$ in $\mathbb{R}^3$.

Sometimes, we write the vector also as $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = [a, b, c]^T$.



# New Vectors from Old

- How do you add two vectors $\mathbf{u}$ and $\mathbf{v}$?

  Using the 'head-to-tail rule' (which is same as the 'parallelogram rule'), right?
- Suppose $\mathbf{u} = a_1\,\widehat{\mathbf{i}} + b_1\,\widehat{\mathbf{j}} + c_1\,\widehat{\mathbf{k}}$ and $\mathbf{v} = a_2\,\widehat{\mathbf{i}} + b_2\,\widehat{\mathbf{j}} + c_2\,\widehat{\mathbf{k}}$. Then it follows from the 'parallelogram rule' that

$$\mathbf{u} + \mathbf{v} = (a_1 + a_2)\,\widehat{\mathbf{i}} + (b_1 + b_2)\,\widehat{\mathbf{j}} + (c_1 + c_2)\,\widehat{\mathbf{k}}.$$

- Suppose $k$ is a constant. What is the vector which is $k$ times $\mathbf{u} = a_1\,\widehat{\mathbf{i}} + b_1\,\widehat{\mathbf{j}} + c_1\,\widehat{\mathbf{k}}$? Naturally,

$$k\mathbf{u} = ka_1\,\widehat{\mathbf{i}} + kb_1\,\widehat{\mathbf{j}} + kc_1\,\widehat{\mathbf{k}}.$$

# The Euclidean Space $\mathbb{R}^n$

Let $n \in \mathbb{N}$. Then $\mathbb{R}^n$, as a Cartesian Product of sets, is the set of all ordered $n$-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{R}$.
We can think the point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ as a vector, and write it as

- $[x_1, x_2, \ldots, x_n]$, when written as a row vector,
- $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, \ldots, x_n]^T$, when written as a column vector.

Definition: Let $\mathbf{u} = [u_1,\ u_2,\ \ldots,\ u_n]^T, \mathbf{v} = [v_1,\ v_2, \ldots,\ v_n]^T \in \mathbb{R}^n$ and $k \in \mathbb{R}$. We define

1. $\mathbf{u} + \mathbf{v} = [u_1 + v_1,\ u_2 + v_2,\ \ldots,\ u_n + v_n]^T$ (Vector Addition)
2. $k\mathbf{u} = [ku_1,\ ku_2,\ \ldots,\ ku_n]^T$ (Scalar Multiplication).
3. The concept of scalar product of vectors is also generalized to elements of $\mathbb{R}^n$, and is called inner product. (More later).



# The Euclidean Space $\mathbb{R}^n$

Definition: Let set

$$\mathbb{R}^n = \left\{ [x_1, x_2, \ldots, x_n]^T : x_1, x_2, \ldots, x_n \in \mathbb{R} \right\},$$

with the vector addition, the scalar multiplication and the inner product (to be defined later) is called the $n$-dimensional Euclidean space.

The vector $[0, 0, \ldots, 0]^T$ of $\mathbb{R}^n$, called the zero vector, is denoted by the symbol $\mathbf{0}$.

## Some Basic Properties:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity);
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity);
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, where $-u = (-1)u = [-u_1, -u_2, \cdots, -u_n]^T$;
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity over vector addition);
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity over scalar addition);
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$;
8. $1\mathbf{u} = \mathbf{u}$;
9. $0\mathbf{u} = \mathbf{0}$;



# Matrices

Matrix: A rectangular array of scalars (real or complex numbers, for us).

- A matrix $A$ is of order or size $m \times n$ if it has $m$ rows and $n$ columns.
- We write a matrix $A$ as $A = [a_{ij}]$, where $a_{ij}$ is the entry in $A$ on $i$-th row and $j$-th column.
- Matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if they are of same size and $a_{ij} = b_{ij}$ for each $i, j$.
- If $m = n$, then $A$ is called a square matrix.
- If $A$ is a square matrix, then the entries $a_{ii}$ are called the diagonal entries of $A$.
- If $A$ is a square matrix and if $a_{ij} = 0$ for all $i \neq j$, then $A$ is called a diagonal matrix.

- Identity matrix $\mathbf{I}_n$ of size $n$: the $n \times n$ diagonal matrix with all diagonal entries equal to 1. $\mathbf{I}$ means the identity matrix of some size $n$.

- Zero matrix of size $m \times n$: The $mn \times n$ matrix with all entries 0.Notation: $\mathbf{O}_{m\times n}$ (or simply by $\mathbf{O}$)

- A sub matrix $B$ of $A$: one obtained from $A$ by deleting some (may be 0 in number) rows and/or columns of $A$.

- The transpose $A^T$ of $A = [a_{ij}]$ is defined as $A^T = [b_{ji}]$, where $b_{ji} = a_{ij}$ for all $i, j$.

- The matrix $A$ is said to be symmetric if $A^T = A$.

- The matrix $A$ is said to be anti-symmetric (or skew-symmetric) if $A^T = -A$.



- If $A$ is a complex matrix, then $\overline{A} = [\overline{a}_{ij}]$ and $A^* = \overline{A}^T$.

- The matrix $A^*$ is called the conjugate transpose of $A$.

- The (complex) matrix $A$ is Hermitian if $A^* = A$, and skew-Hermitian if $A^* = -A$.

- A square matrix $A$ is upper triangular if $a_{ij} = 0$ for all $i > j$.

- A square matrix $A$ is lower triangular if $a_{ij} = 0$ for all $i < j$.

## Matrix Operations

- $\mathcal{M}_{m\times n} :=$ the set of a $m \times n$ matrices.
- To specify real (complex) matrices, write $\mathcal{M}_{m\times n}(\mathbb{R})$ $(\mathcal{M}_{m\times n}(\mathbb{R}))$.
- If $m = n$, we write $\mathcal{M}_n$ for $\mathcal{M}_{m\times n}$.
- For $A = [a_{ij}],\ B = [b_{ij}] \in \mathcal{M}_{m\times n}$ and a scalar $c$ (real or complex)
  1. Matrix Addition: $A + B := [a_{ij} + b_{ij}] \in \mathcal{M}_{m\times n}$.
  2. Multiplication by a Scalar: $c\,A := [c\,a_{ij}] \in \mathcal{M}_{m\times n}$.



## Properties of Addition and Scalar multiplication

**Result**

*Let $A, B, C \in \mathcal{M}_{m\times n}$ and $s, r$ be scalars. Then*

1. *Commutative Law:* $A + B = B + A$.
2. *Associative Law:* $(A + B) + C = A + (B + C)$.
3. $A + \mathbf{O} = A$, *where* $\mathbf{O} = \mathbf{O}_{m\times n} \in \mathcal{M}_{m\times n}$.
4. $A + (-A) = \mathbf{O}$, *where* $-A = (-1)A \in \mathcal{M}_{m\times n}$.
5. $s(A + B) = s\,A + s\,B$.
6. $(s + r)A = s\,A + r\,A$.
7. $s\,(r\,A) = (sr)A$.
8. $1A = A$.

# Matrix multiplication

Definition: For $A = [a_{ij}] \in \mathcal{M}_{m\times n}$, and $B = [b_{jk}] \in \mathcal{M}_{n\times p}$ the product of $A$ and $B$ is defined to be $AB = [c_{ik}] \in \mathcal{M}_{m\times p}$, where

$$c_{ik} := a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}.$$

- If $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{n\times p}$,, then
  - $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ where $\mathbf{a}_i$ are vectors in $\mathbb{R}^n$ ($\mathbf{a}_i^T$ is the $i$-th row),
  
    and
  - $B = [\mathbf{b}_1,\ \mathbf{b}_2,\ \ldots,\ \mathbf{b}_p]$ where $\mathbf{b}_j$ are vectors in $\mathbb{R}^n$ ($\mathbf{b}_j$ is the $j$-th column).
  - If $AB = [c_{ik}]$, then $c_{ik} = \mathbf{a}_i^T b_j$.



# Matrix multiplication

- Let $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$ and $B = [\mathbf{b}_1,\ \mathbf{b}_2,\ \ldots,\ \mathbf{b}_p]$. Then

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p] = \begin{bmatrix} \mathbf{a}_1^T B \\ \mathbf{a}_2^T B \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}.$$

- If $A = [\mathbf{a}_1,\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_n]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_n]$, then

$$A\mathbf{b} = b_1\mathbf{a}_1 + b_2\mathbf{a}_2 + \cdots + b_n\mathbf{a}_n,$$

  a linear combination of the columns of $A$.

# Matrix multiplication

## Result

*Let $A, B$ and $C$ be matrices, and let $s \in \mathbb{R}$. Then*

1. *Associative Law: $(AB)C = A(BC)$, if the respective matrix products are defined.*

2. *Distributive Law:*
   $A(B + C) = AB + AC, \quad (A + B)C = AC + BC,$
   *if the respective matrix sum and matrix products are defined.*

3. *$s(AB) = (sA)B = A(sB)$, if the respective matrix products are defined.*

4. *$I_m A = A = A I_n$, if $A$ is of size $m \times n$.*