PH 101
Tutorial-6
Date: 15/09/2017

1. A thin hoop of mass M and radius R rolls without slipping about the z-axis. It is supported by an axle of length R through its center, as shown in Fig. 1. The hoop circles around the z-axis with angular speed Ω.

(a) What is the instantaneous angular velocity ω of the hoop?

(b) What is the angular momentum **L** of the hoop? Is **L** parallel to ω? (The moment of inertia of a hoop for an axis along its diameter is $\frac{1}{2}MR^2$)

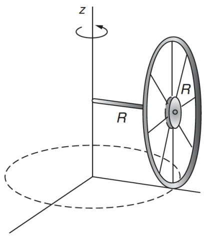

Fig. 1

2. A gyroscope wheel is at one end of an axle of length A. The other end of the axle is suspended from a string of length B. The wheel is set into motion so that it executes uniform precession in the horizontal plane. The wheel has mass M and moment of inertia about its center of mass I0. Its spin angular velocity is ωs. Neglect the masses of the shaft and string. Find the angle β that the string makes with the vertical. Assume that β is so small that approximations like sin β ≈ β are justified. (Refer to Fig. 2)

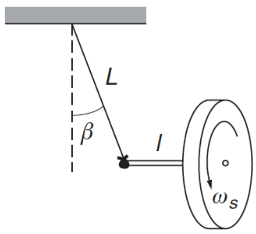

Fig. 2

3. A coin of radius b and mass M rolls on a horizontal surface at speed V. If the plane of the coin is vertical the coin rolls in a straight line. If the plane is tilted, the path of the coin is a circle of radius R. Find an expression for the tilt angle of the coin α in terms of the given quantities. (Because of the tilt of the coin the circle traced by its center of mass is slightly smaller than R but you can ignore the difference.)

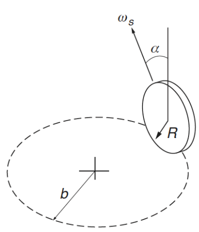

Fig. 3

4. As a bicycle changes direction, the rider leans inward creating a horizontal torque on the bike. Part of the torque is responsible for the change in direction of the spin angular momentum of the wheels. Consider a bicycle and rider system of total mass M with wheels of mass *m* and radius *b*, rounding a curve of radius R at speed V. The center of mass of the system is *1.5b* from the ground.

(a) Find an expression for the tilt angle α.

(b) Find the value of α, in degrees, if M = 70 kg, m = 2.5 kg, V = 30 km/hour and R = 30 m.

(c) What would be the percentage change in α if spin angular momentum were neglected?

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5. A particle of mass m is located at x = 2, y = 0, z = 3.
(a) Find its moments and products of inertia relative to the origin.
(b) The particle undergoes pure rotation about the z axis through a small angle α. Show that its moments and products of inertia are unchanged to first order in α if $\alpha \ll 1$. (Refer to Fig. 4)

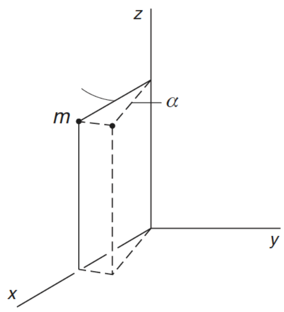


Fig. 4

6. Find the principal moments of inertia of a uniform circular disk of mass M and radius a (i) at its centre of mass, and (ii) at a point on the edge of the disk. (Refer to Fig. 5)

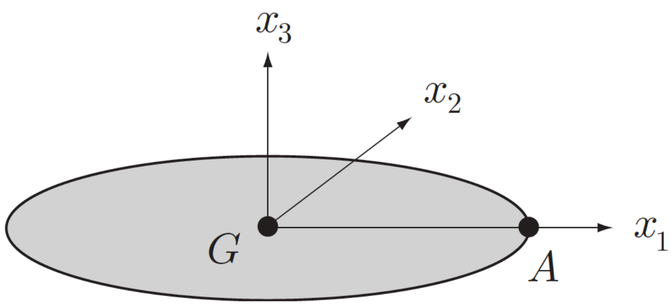
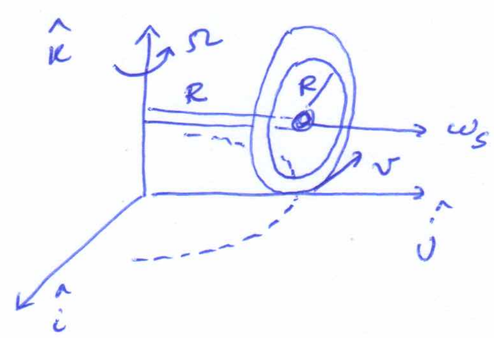


Fig. 5

7. Find the principal moments of inertia of a uniform cube of mass M and side 2a (i) at its centre of mass and (ii) at the centre of a face

Solution to Tutorial 6

1.



(a)

$$\omega_s = \frac{v}{R} = \frac{\Omega R}{R} = \Omega$$

$$\vec{\omega} = \vec{\omega}_s + \vec{\Omega}$$
$$= \Omega(\hat{j} + \hat{k})$$

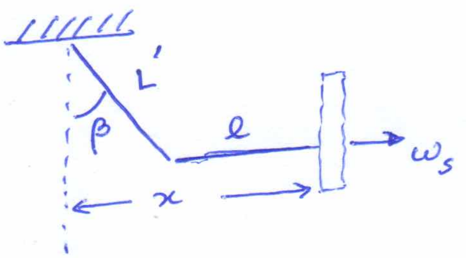
(b) $\vec{L} = \vec{L}_s + \vec{L}_\omega = I_s \vec{\omega}_s + I_z \vec{\Omega}$

$$I_s = MR^2, \quad I_z = I_0 + MR^2 = \frac{3}{2}MR^2$$

$$\vec{L} = MR^2\left(\vec{\omega}_s + \frac{3}{2}\vec{\Omega}\right) = MR^2\Omega\left(\hat{j} + \frac{3}{2}\hat{k}\right)$$

$\vec{\omega}$ and $\vec{L}$ are not parallel.

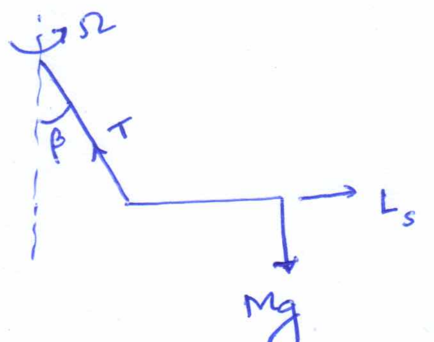
2.



Assume β to be small.

$$x = l + L'\sin\beta = l + L'\beta$$

Eq^n of motion:



$$Mg = T\cos\beta \approx T$$

$$M\Omega^2 x = T\sin\beta \approx T\beta$$

$$\Rightarrow \Omega^2 = \frac{T\beta}{Mx} = \frac{g\beta}{1 + L'\beta}$$

Torque: $T l = \dot{L}_s = \Omega I_0 \omega_s$

$$\Rightarrow \Omega = \frac{Mgl}{I_0\omega_s}$$

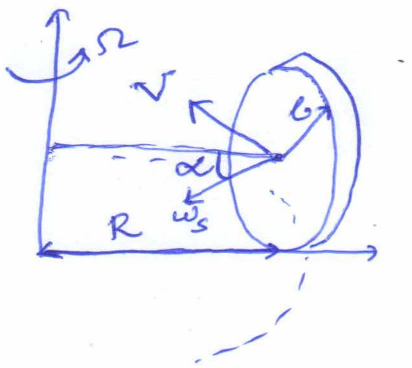
$$\frac{g\beta}{l + L'\beta} = \Omega^2 = \left(\frac{Mgl}{I_0\omega_s}\right)^2$$

$$\Rightarrow \beta = \frac{M^2 g l^3}{I_0^2\omega_s^2}\left(1 - \frac{M^2 g l^2 L'}{I_0^2 \omega_s^2}\right)^{-1}$$

//

3.

As the coin rolls with speed v around the circle of radius R, it rotates around the vertical at rate $\Omega = V/R$. This rotation is caused by precession of its spin angular momentum due to the torque induced by the tilt. For rolling without slipping, $V = b\omega_s$, so $\Omega = \omega_s (b/R)$



The coin is accelerating, so take torques about the centre of mass. From the force diagram:

$$\tau_{cm} = fb\cos\alpha - Nb\sin\alpha$$

$$N = Mg \quad , \quad f = \frac{MV^2}{R}$$

The equation of motion for L_s is

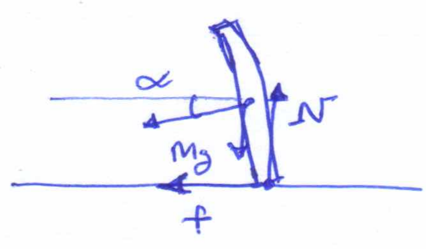
$$\tau_{cm} = \Omega L_s \cos\alpha = \Omega I_0 \omega_s \cos\alpha = \omega_s^2 \frac{b}{R} I_0 \cos\alpha$$

$$= \left(\frac{V}{b}\right)^2 \left(\frac{b}{R}\right)\left(\frac{1}{2}Mb^2\right)\cos\alpha$$

$$= \frac{1}{2} MV^2 \left(\frac{b}{R}\right)\cos\alpha$$

$$= MV^2 \left(\frac{b}{R}\right)\cos\alpha - Mgb\sin\alpha$$

Thus, $\tan\alpha = \dfrac{V^2}{2Rg}$

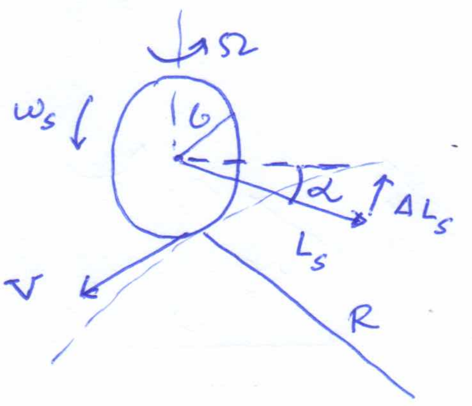


4.

The torque τ_h about the center of mass is into the paper.

$$\tau_h = N(1.5b)\tan\alpha - f(1.5b)$$

$$= Mg(1.5b)\tan\alpha - (1.5b)\frac{MV^2}{R}$$



The total spin angular momentum (two wheels) is

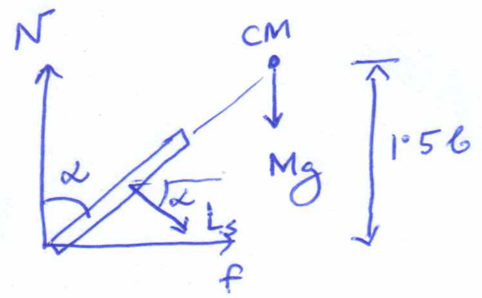
$$L_s = 2I_0\omega_s = 2mb^2\frac{V}{b} = 2mbV$$

$$\tau_h = L_h\Omega = L_s\cos\alpha\,\frac{V}{R}$$

Now,

$$Mg(1.5b)\tan\alpha - (1.5b)\frac{MV^2}{R} = 2mb\frac{V^2}{R}\cos\alpha$$

$$\Rightarrow \tan\alpha = \frac{V^2}{Rg}\left(1 + \frac{4}{3}\frac{m}{M}\cos\alpha\right)$$



As $m/M \ll 1$, the second term is a small correction So, Take $\cos\alpha \approx 1$.

(a) Thus, $\tan\alpha \approx \frac{V^2}{Rg}\left(1 + \frac{4}{3}\frac{m}{M}\right)$

(b) Putting the numbers one can obtain

$\alpha \approx 16^\circ$ //

(c) If spin is neglected, the term in m/M should be ~~om~~ omitted. Then $\alpha \approx 15^\circ$ //

5.

④

$$I_{xx} = m(y^2+z^2) = m(0^2+3^2) = 9m$$

$$I_{yy} = m(x^2+z^2) = \cancel{13m}\ m(2^2+3^2) = 13m$$

$$I_{zz} = m(x^2+y^2) = m(2^2+0^2) = 4m$$

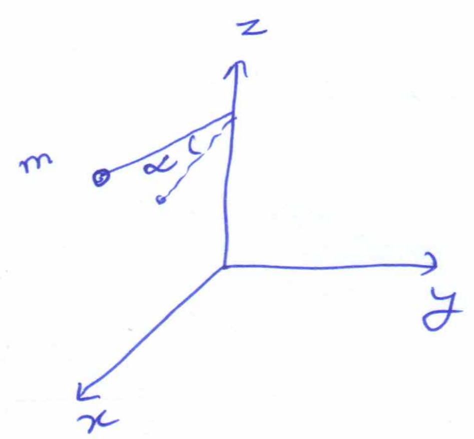
$$I_{xy} = I_{yx} = -mxy = 0$$

$$I_{yz} = I_{zy} = -myz = -m(0\times 3) = 0$$

$$I_{xz} = I_{zx} = -m(xz) = -m(2\times 3) = -6m$$

$$I = m\begin{pmatrix} 9 & 0 & -6 \\ 0 & 13 & 0 \\ -6 & 0 & 4 \end{pmatrix}$$

(b)



To order α^2

$$x = 2\cos\alpha \approx 2\left(1-\frac{\alpha^2}{2}\right)$$
$$y = 2\sin\alpha \approx 2\alpha$$
$$z = 3$$

$$I' = m\begin{pmatrix} 9+4\alpha^2 & -4\alpha & -6+3\alpha^2 \\ -4\alpha & 13-4\alpha^2 & -6\alpha \\ -6+3\alpha^2 & -6\alpha & 4 \end{pmatrix}$$

Comparing with part (a), note that the moments of inertia (along the main diagonal of the matrix) vary only as α^2, but some of the products of inertia (off-diagonal elements) can vary linearly with α. When making such approximations, be sure to include ~~all~~ all terms upto the highest order retained. For example, $I'_{zz} = m(x^2+y^2) = m[(2-\alpha^2)^2+(2\alpha)^2]$
$\approx m[4-4\alpha^2+4\alpha^2] = 4m$ //

6.

(i) $$I_{G,\hat{e}_3} = I_{G,\hat{e}_1} + I_{O,\hat{e}_2}$$

Because of rotational symmetry of the disk about the axis $\{G, \hat{e}_3\}$ we have

$$I_{\{G,\hat{e}_1\}} = I_{\{G,\hat{e}_2\}}$$

Now, $I_{G,\hat{e}_3} = \frac{1}{2}Ma^2$

Thus, the principal moments of inertia of the disk at G are:

$$I_{G,\hat{e}_1} = \frac{1}{4}Ma^2, \quad I_{G,\hat{e}_2} = \frac{1}{4}Ma^2, \quad I_{G,\hat{e}_3} = \frac{1}{2}Ma^2$$

(ii) Apply Parallel axes theorem;

$$I_{A,\hat{e}_1} = \frac{1}{4}Ma^2$$

$$I_{A,\hat{e}_2} = \frac{5}{4}Ma^2$$

$$I_{A,\hat{e}_3} = \frac{3}{2}Ma^2$$

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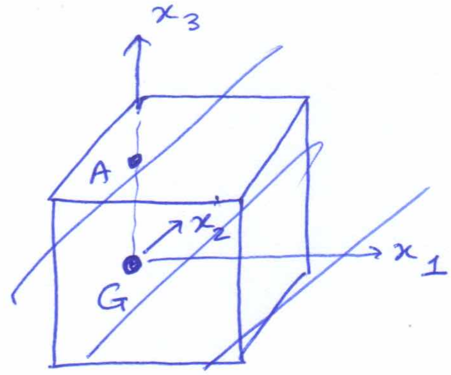
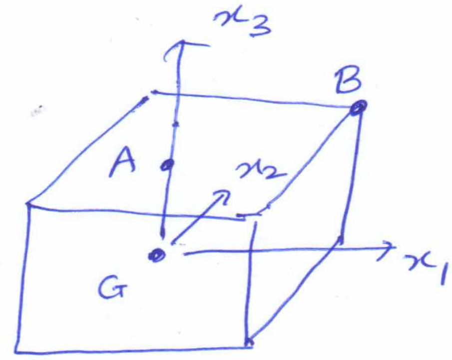
7.





(a) (i) consider the coordinate system $Gx_1x_2x_3$ shown in Fig. Since the cube has reflective symmetry in each of the three co-ordinate planes, this is a set of principal axes at G. The MI of the cube about ~~The principal moments of~~ the axis Gx_1 is the same as that of a uniform mass M occupying the region ~~$x_1=0$~~ $x_1 = 0, \quad -a \le x_2, x_3 \le a$.

$$I_{G,\hat{e}_1} = \int_{-a}^{a} dx_1 \int_{-a}^{a} dx_2 \int_{-a}^{a} dx_3 \, \rho \left(x_2^2 + x_3^2 \right), \quad \rho = \frac{M}{8a^3}$$

$$= \frac{2}{3} Ma^2$$

$$I_{G,\hat{e}_1} = I_{G,\hat{e}_2} = I_{G,\hat{e}_3} = \frac{2}{3} Ma^2 \quad //$$

(ii) Parallel axes theorem will lead us to the answer:

$$I_{A,\hat{e}_1} = \frac{5}{3} Ma^2, \quad I_{A,\hat{e}_2} = \frac{5}{3} Ma^2, \quad I_{A,\hat{e}_3} = \frac{2}{3} Ma^2$$