MA101 Mathematics I

Tutorial & Additional Problem Set - 6

SECTION - A (for Tutorial -6)

1. True or False? Give justifications.

   (a) If $A \in \mathcal{M}_n(\mathbb{R})$ $(n > 2)$ has $rank(A) = 2$, then all its 3×3 submatrices are singular and it has at least one 2×2 submatrix which is nonsingular.

   (b) $\mathbf{x}$ is an eigenvector of A w.r.t eigenvalue λ if and only if $\mathbf{x}$ is an eigenvector of A^2 w.r.t eigenvalue λ^2.

   (c) Let A be a nonzero matrix such that $A^{31} = \mathbf{0}$ then A has all eigenvalues equal to 0 and A is not diagonalizable.

   (d) A real 2×2 matrix which gives reflection of $\mathbb{R}^2$ w.r.t a line $y = mx$ in $\mathbb{R}^2$ always has a real eigenvector and a real eigenvalue.

   (e) If A is diagonalizable then $rank(A - cI) = rank(A - cI)^2$ for all $c \in \mathbb{C}$.

Solution:

(a) True. Since $rank(A) = 2$, any set of 3 column or row vectors of A are LD and there exists a set of 2 columns which are LI. If B is the $n \times 2$ submatrix of A consisting of any two such LI columns, then B will have 2 rows which are LI.

(b) False. If $\mathbf{x}$ is an e.vector of A corresponding to e.value λ then $A\mathbf{x} = \lambda\mathbf{x}$ implies $A(A\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda^2(\mathbf{x})$, hence $\mathbf{x}$ is also an e.vector of A^2 corresponding to e.value λ^2. But the converse is not true, for example take
$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $E_0(A^2) = \mathbb{R}^2 \neq E_0(A) = span\{[1, 0]^T\}$.

(c) True. If $\lambda \neq 0$ is an eigenvalue of A then $\lambda^{31} \neq 0$ is an eigenvalue of A^{31}, which is a contradiction.
Since all eigenvalues of A is 0, if A is diagonalizable then A has to the $\mathbf{0}$ matrix, which is a contradiction.

(d) True. It has a real eigenvector which is the line of reflection and the corresponding e.value is 1.
Also, if a real matrix has a real eigenvector corresponding to some e.value λ, then the λ has to be real.

(e) True. If $P^{-1}AP = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \lambda_n \end{bmatrix}$, then $P^{-1}(A - cI)P = \begin{bmatrix} \lambda_1 - c & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \lambda_n - c \end{bmatrix}$ and
$P^{-1}(A - cI)^2P = \begin{bmatrix} (\lambda_1 - c)^2 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & (\lambda_n - c)^2 \end{bmatrix}$.

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Hence $rank(A-cI) = rank P^{-1}(A-cI)P = rank P^{-1}(A-cI)^2P = rank(A-cI)^2$, which is equal to the number of i's for which $\lambda_i \neq c$.

2. (a) If A is an $n \times n$ matrix with $nullity(A) = k$, then show that there exists an invertible matrix P such that $P^{-1}AP = \begin{bmatrix} \mathbf{0} & B \\ \mathbf{0} & D \end{bmatrix}$, where D is an $(n-k) \times (n-k)$ matrix.
Hint: Take a basis of $null(A)$ say $\{P_1, \ldots, P_k\}$, and extend it to a basis of $\mathbb{R}^n$, $\{P_1, \ldots, P_n\}$. Take $P = [P_1 \ldots P_n]$.
   (b) Hence show that for any eigenvalue λ of A, algebraic multiplicity of $\lambda \geq$ the geometric multiplicity of λ.
   (c) Deduce that $rank(A) \geq$ number of nonzero eigenvalues of A.

Solution:

(a) $AP = [AP_1 \ldots AP_k \ldots AP_n] = \begin{bmatrix} \mathbf{0} & C \\ \mathbf{0} & E \end{bmatrix}$, where E is an $(n-k) \times (n-k)$ matrix.
Hence $P^{-1}AP$ is of the form $\begin{bmatrix} \mathbf{0} & B \\ \mathbf{0} & D \end{bmatrix}$, for some $(n-k) \times (n-k)$ matrix D.

(b) Hence 0 is an eigenvalue of $P^{-1}AP$ with algebraic multiplicity $\geq k$. Since eigenvalues of A and $P^{-1}AP$ are equal (including multiplicity) it follows that algebraic multiplicity of 0 as an e.value of $A \geq k$, where k is the geometric multiplicity of 0 as an e.value of A.
Similarly by following the same procedure for $(A - \lambda I)$ where λ is an eigenvalue of A it follows that for any e.value λ of A, algebraic multiplicity of $\lambda \geq$ geometric multiplicity of λ.

(c) Since $rank(A) = n - nullity(A)$, and the number of nonzero e.values of $(A) = n-$ algebraic multiplicity of 0, the result follows.

3. Let $\mathbb{V}, \mathbb{W}$ be n dimensional vector spaces with ordered bases B and C, respectively, and $T : V \to W$ is a LT. Then T is invertible if and only if the matrix $[T]_{C \leftarrow B}$ is invertible. In that case,

$$([T]_{C \leftarrow B})^{-1} = [T^{-1}]_{B \leftarrow C}.$$

Solution: Since $rank(T) = rank[T]_{C \leftarrow B}$, the first part is obvious, hence $[T]_{C \leftarrow B}$ is invertible.
Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $C = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$. Let $D = \{T(v_1), \ldots, T(v_n)\}$, then note that D is also a basis of W, since T is invertible.
Since $[T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)] = [\mathbf{w}_1, \ldots, \mathbf{w}_n][T]_{C \leftarrow B}$, $P_{C \leftarrow D} = [T]_{C \leftarrow B}$.
Hence $([T]_{C \leftarrow B})^{-1} = P_{D \leftarrow C}$.
Since $[T^{-1}(T(\mathbf{v}_1)), \ldots, T^{-1}(T(\mathbf{v}_n))] = [\mathbf{v}_1, \ldots, \mathbf{v}_n]I$, $[T^{-1}]_{B \leftarrow D} = I$.

Since $([T^{-1}]_{B\leftarrow C})_i = [T^{-1}(\mathbf{w}_i)]_B = [T^{-1}]_{B\leftarrow D}[\mathbf{w}_i]_D = [T^{-1}]_{B\leftarrow D}(P_{D\leftarrow C})_i = (([T]_{C\leftarrow B})^{-1})_i$, the result follows.
Alternatively define $S : \mathbb{W} \to \mathbb{W}$ as the identity map and $T^{-1} : \mathbb{W} \to \mathbb{V}$.
Then $[T^{-1} \circ S]_{B\leftarrow C} = [T^{-1}]_{B\leftarrow D}[S]_{D\leftarrow C}$, where B, C and D are as defined above.
Since $[T^{-1}]_{B\leftarrow D} = I$ and $[S]_{D\leftarrow C} = P_{D\leftarrow C} = ([T]_{C\leftarrow B})^{-1}$, the result follows.

4. Find the eigenvalues of the $n \times n$ matrix A which has all diagonal entries equal to 3 and all other entries equal to 2. Find two eigenvectors $\mathbf{x}, \mathbf{y}$ corresponding to two distinct eigenvalues of A.
Hint: $A = B+I$, where B is the matrix having all entries equal to 2. Check that $\mathbf{x}$ is an eigenvector of B corresponding to eigenvalue λ if and only if $\mathbf{x}$ is an eigenvector of A corresponding to eigenvalue $\lambda + 1$.

Solution: Since B is a rank 1 matrix ($nullity(B) = n-1$) and for every row the sum of its entries is equal to $2n$, 0 is an eigenvalue of B with algebraic multiplicity $n-1$, the other eigenvalue of B being $2n$, with the corresponding e.vector $[1, \ldots, 1]^T$. An e.vector of B corresponding to e.value 0 is $[1, -1, 0, \ldots, 0]^T$, one can construct many others by using the same trick.
Hence by using the hint one can get the corr e.values and e.vectors for A.

5. If $A = \mathbf{u}\mathbf{u}^T$ where $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$, then find the eigenvalues of A and show that A is diagonalizable.
Hint: Check that $rank(A) = 1$ and $\mathbf{u}^T\mathbf{u}$ is the nonzero eigenvalue of A.

Solution: $nullity(A) = n-1$, hence there are $n-1$ LI e.vectors corr to e.value 0. Since $(\mathbf{u}\mathbf{u}^T)\mathbf{u} = (\mathbf{u}^T\mathbf{u})\mathbf{u}$, the nonzero e.value $(\mathbf{u}^T\mathbf{u})$ will give one more LI e.vector.

SECTION - B: ADDITIONAL PROBLEMS

1. True or False? Give justifications.
   (a) If both A and A^{-1} has only integer entries, then $det(A) = +1$ or -1.
   (b) If A and B are non square matrices such that both AB and BA are defined then either AB or BA has a zero eigenvalue.
   (c) If A is a 3×3 matrix with eigenvalues $0, 3, 4$ and D is a 2×2 matrix with eigenvalues $0, 3$ then the matrix $C = \begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix}$ has eigenvalues $0, 3, 4$ with algebraic multiplicities $2, 2, 1$, respectively.
   Hint: $det(C - \lambda I) = det(A - \lambda I)det(D - \lambda I)$.
   (d) An upper triangular matrix with all diagonal entries equal to a, is diagonalizable only if A is a diagonal matrix.
   Hint: Look at $null(A - aI)$.
   (e) Eigenvalues of real matrices occur in conjugate pairs (that is if $a+ib$ is an eigenvalue of A then $a - ib$ is also an eigenvalue of A).
   Hint: $\overline{det(A - cI)} = det(\overline{A - cI})$.

Solution:

(a) True. Since $det(A)det(A^{-1}) = 1$ and both A and A^{-1} has integer entries.

(b) True. If A is an $m \times n$ matrix, then in order that AB and BA are defined, B should be $n \times m$ matrix. If $m > n$, then $rank(AB) \leq rank(A), rank(B) \leq n < m$, hence AB should have a 0 e.value.

(c) True.

(d) True. Since $(A - aI)$ is upper triangular with all diagonal entries 0, all its eigenvalues are 0. But if A is not a diagonal matrix $nullity(A - aI) < n$ since $(A - aI)$ is not the $\mathbf{0}$ matrix(note that for an $n \times n$ matrix A, $nullity(A) = n$ iff $null(A) = \mathbb{R}^n$ iff $A = \mathbf{0}$).

(e) True. Since A and I are real matrices, from the hint it follows that $det(A - cI) = 0$ iff $det(A - \overline{c}I) = 0$

2. Let $\mathbb{V}, \mathbb{W}$ be n and m dimensional vector spaces and $T : V \to W$ is a LT. Let $[T]_{C\leftarrow B}$ be the matrix of T w.r.t ordered bases $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $C = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, respectively. If $D = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ and $E = \{\mathbf{w}'_1, \ldots, \mathbf{w}'_m\}$ are also ordered bases of V and W respectively, then write $[T]_{E\leftarrow D}$ in terms of $[T]_{C\leftarrow B}, P_{D\leftarrow B}$ and $P_{E\leftarrow C}$.

Solution: Since $[T(\mathbf{v}_1)\ldots T(\mathbf{v}_n)] = [\mathbf{w}_1 \ldots \mathbf{w}_m][T]_{C\leftarrow B}$, $[T(\mathbf{v}'_i)]_C = [T]_{C\leftarrow B}[\mathbf{v}'_i]_B$,
hence $[T(\mathbf{v}'_1)\ldots T(\mathbf{v}'_n)] = [\mathbf{w}_1 \ldots \mathbf{w}_m][T]_{C\leftarrow B}[[\mathbf{v}'_1]_B \ldots [\mathbf{v}'_n]_B] = [\mathbf{w}_1 \ldots \mathbf{w}_m][T]_{C\leftarrow B}P_{B\leftarrow D}$.
Also $[\mathbf{w}_1 \ldots \mathbf{w}_m] = [\mathbf{w}'_1 \ldots \mathbf{w}'_m]P_{E\leftarrow C}$,
hence $[T(\mathbf{v}'_1)\ldots T(\mathbf{v}'_n)] = [\mathbf{w}'_1 \ldots \mathbf{w}'_m]P_{E\leftarrow C}[T]_{C\leftarrow B}(P_{D\leftarrow B})^{-1}$,
and $[T]_{E\leftarrow D} = P_{E\leftarrow C}[T]_{C\leftarrow B}(P_{D\leftarrow B})^{-1}$.

3. Consider $\mathbb{U} = \mathbb{R}^3$, $\mathbb{V} = \mathcal{M}_2(\mathbb{R})$ and $\mathbb{W} = \mathbb{R}_2[x]$ with ordered bases $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $D = \{1, x, x^2\}$, respectively.
Let $T : \mathbb{U} \to \mathbb{V}$ be defined as $T(x, y, z)^T = \begin{bmatrix} 0 & x \\ y & y+z \end{bmatrix}$ and let $S : \mathbb{V} \to \mathbb{W}$ be defined as
$S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + (b+c)x + dx^2.$
Then determine $[S \circ T]_{D\leftarrow B}$, $[S]_{D\leftarrow C}$ and $[T]_{C\leftarrow B}$ and verify that $[S \circ T]_{D\leftarrow B} = [S]_{D\leftarrow C}[T]_{C\leftarrow B}$.

4. Find all the eigenvalues and the corresponding eigenspaces of the matrix
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

5. If A is a 3×3 matrix having eigenvalues $0, 2$ and 3 with eigenvectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, respectively, then show that $A\mathbf{x} = \mathbf{u}$ has no solution. Find all solutions of $A\mathbf{x} = \mathbf{v} + \mathbf{w}$.
Hint: (For the first part) Note that $null(A) = null(A^2)$ (why?).

Solution:

(a) Since A has 3 distinct e.values so A has 3 LI eigenvectors (hence A is diagonalizable) and $null(A) = null(A^2) = span\{\mathbf{u}\}$. But if $A\mathbf{x} = \mathbf{u}$ has a solution, then $A^2\mathbf{x} = A\mathbf{u} = \mathbf{0}$, hence $x \in null(A^2)$ but x does not belong to $null(A)$, since $\mathbf{u} \neq \mathbf{0}$.
Also $A(\frac{\mathbf{v}}{2} + \frac{\mathbf{w}}{3}) = \mathbf{v} + \mathbf{w}$, so $\frac{\mathbf{v}}{2} + \frac{\mathbf{w}}{3}$ is a particular solution of $A\mathbf{x} = \mathbf{v} + \mathbf{w}$. One can get all the solutions of this system by looking at $null(A)$.

6. If A is a real symmetric matrix then show that all the eigenvalues of A are real. If $\mathbf{x}, \mathbf{y}$ are real eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 then show that $\mathbf{x}^T\mathbf{y} = 0$.
Hint: If $A\mathbf{x} = \lambda\mathbf{x}$, then by taking conjugate transpose on both sides we get, $x^*A = \overline{\lambda}\mathbf{x}^*$, where $x^* = \overline{x}^T$. Hence show that $\lambda = \overline{\lambda}$.

Solution: By following the hint, we get $x^*Ax = \overline{\lambda}\mathbf{x}^*x$, but by pre-multiplying $Ax = \lambda x$ with x^* we get $x^*Ax = \lambda x^*x$, hence $\lambda x^*x = \overline{\lambda}\mathbf{x}^*x$, which gives $\lambda = \overline{\lambda}$ since $x^*x \neq 0$.
Since A is a real matrix with real e.values it should have atleast one real e.vector corresponding to every e.value (why?).
Let x and y be e.vectors corr to distinct e.values λ and μ, then $Ax = \lambda x$ gives $y^TAx = \lambda y^Tx$. Similarly $x^TAy = \mu x^Ty$. Since x^TAy is a number (real) $x^TAy = (x^TAy)^T = y^TAx$, which implies $\lambda y^Tx = \mu x^Ty$. Since $y^Tx = x^Ty$ and $\lambda \neq \mu$, $y^Tx = x^Ty = 0$.
PS: If $\mathbf{x} = [x_1 \dots x_n]^T$ then $\overline{\mathbf{x}} = [\overline{x_1} \dots \overline{x_n}]^T$.

7. Use Gram Schmidt procedure to change the ordered basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$ to an orthogonal basis. What happens if the order in which the vectors are taken changes, does the elements of the orthogonal basis remain same?