Indian Institute of Technology Guwahati

Mid-Semester Examination, July-November 2012

MA 101 Mathematics-I

Time: 2 Hrs Marks: 25

- *Answer all the questions carefully, precisely and neatly.*
- *Unnecessary or unreadable statements will lead to negative marks.*
- *No queries shall be entertained during the examination.*
- *The paper has five questions:* $1, \ldots, 5$. *Each question has at most five parts:* $(a), \ldots, (e)$.
- *For each question, your answer should start in a fresh page and all the parts of the question should be answered at a stretch in one place, failing which the question may not be evaluated.*
- *It is mandatory to write the range of page numbers of your answer for each question on the cover page of the answer booklet,* ***failing which five marks may be deducted***.
- $\mathbb{R}$ denotes the set of real numbers.

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1. With proper justifications, prove or disprove the following statements.

   (a) The set $\{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + 5z = 3\}$ is a subspace of $\mathbb{R}^3$. [1]

   (b) If A is any $m \times n$ matrix, then $\text{rank}(A^T A) = \text{rank}(A)$. [1]

   (c) If $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for eigenspaces corresponding to two distinct eigenvalues of a matrix, then $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$. [1]

   (d) If A and B are 2×2 matrices such that $\det A = \det B$, then A must be similar to B. [1]

   (e) The basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for $\mathbb{R}^3$ is orthonormal. [1]

2. (a) Define all the three types of elementary matrices. [1]

   (b) Write the inverse of each of the three types of elementary matrices. [1]

   (c) Write the determinant of each of the three types of elementary matrices. [1]

   (d) Show that every invertible matrix is a product of elementary matrices. [2]

3. (a) Prove that the range of a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the column space of its standard matrix $[T]$. [1]

   (b) Prove that the linear transformation defined by an orthogonal matrix is angle-preserving in $\mathbb{R}^n$. [1]

   (c) If the null space of a matrix A is $\{\mathbf{0}\}$, then prove that the matrix transformation T_A is injective (one-one). [1]

   (d) Let A be a matrix. Prove that each vector in $\mathrm{row}(A)$ is orthogonal to every vector in $\mathrm{null}(A)$. [1]

   (e) If the algebraic multiplicity of an eigenvalue λ of a matrix is 1, then prove that any two eigenvectors corresponding to λ are parallel (a scalar multiple of one another). [1]

4. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

   (a) Write all the eigenvalues of A. [1]

   (b) Write an eigenvector corresponding to each of the eigenvalues of A. [1]

   (c) Write an invertible matrix P such that $PAP^{-1} = D$, where D is a diagonal matrix. [2]

   (d) For $k \geq 0$, compute A^k. [1]

5. (a) Use Cayley-Hamilton theorem to compute the inverse of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 0 \end{bmatrix}$. [2]

   (b) If A is an $n \times n$ matrix and B is the matrix obtained by interchanging any two rows of A, then prove that $\det B = -\det A$. [3]

   (The results which are proved using this result, in the class, should not be used.)

$* * *$ End $* * *$