MA101 Mathematics I

Tutorial & Additional Problem Set - 4

SECTION - A (for Tutorial -4)

1. True or False? Give justifications.

   (a) If S is a subspace of $\mathbb{R}^n$ of dimension n, then $S = \mathbb{R}^n$.

   (b) For any two matrices A and B for which AB is defined, $\operatorname{rank}(AB) \leq \operatorname{rank}(A), \operatorname{rank}(B)$.

   (c) If $C = [A \,|\, B]$, then $\operatorname{rank}(C) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

   (d) If $C = \begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix}$, then $\operatorname{rank}(C) \geq \operatorname{rank}(A) + \operatorname{rank}(D)$.

Solution:

(a) True. If $y \in \mathbb{R}^n$ but not in S, then for any basis $\mathbb{B}$ of S, $\mathbb{B} \cup \{y\}$ is LI.

(b) True. $\operatorname{row}(AB) \subseteq \operatorname{row}(B)$ and $\operatorname{col}(AB) \subseteq \operatorname{col}(A)$.

(c) True. Let $\operatorname{rank}(A) = k$, $\operatorname{rank}(B) = r$ and let the columns $a_{i_1}, \ldots, a_{i_k}$ of A form a basis of $\operatorname{col}(A)$ and the columns $b_{j_1}, \ldots, b_{j_r}$ of B form a basis of $\operatorname{col}(B)$. Then $a_{i_1}, \ldots, a_{i_k}, b_{j_1}, \ldots, b_{j_r}$ spans $\operatorname{col}[A|B]$. Hence $\operatorname{rank}[A|B] \leq r + k$.

(d) True. If $\operatorname{rank}(A) = k$ and if the columns $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of A forms a basis of $\operatorname{col}(A)$ then the corresponding columns $(i_1, i_2, \ldots, i_k)$ in C are LI.
If $\operatorname{rank}(D) = r$ and the columns $d_{j_1}, d_{j_2}, \ldots, d_{j_r}$ of D forms a basis of $\operatorname{col}(D)$ then the corresponding columns $(m + j_1, m + j_2, \ldots, m + j_r)$ (if A has m columns) in C are LI.
It can be easily checked that the columns $i_1, i_2, \ldots, i_k, m + j_1, m + j_2, \ldots, m + j_r$ of C are LI. Hence $\operatorname{rank}(C) \geq r + k$.

2. If $\operatorname{rank}(A) = \operatorname{rank}(A^2)$ then show that $\operatorname{rank}(A^2) = \operatorname{rank}(A^3)$. Is $\operatorname{rank}(A^5) = \operatorname{rank}(A^6)$?

   Hint: Note that $\operatorname{col}(A^2) \subseteq \operatorname{col}(A)$, $\operatorname{rank}(A^2) = \operatorname{rank}(A)$ implies $\operatorname{col}(A^2) = \operatorname{col}(A)$. Again note that $\operatorname{col}(A^3) \subseteq \operatorname{col}(A^2)$, show $\operatorname{col}(A^3) = \operatorname{col}(A^2)$, and so on.

Solution: Note that $\operatorname{col}(A^2) \subseteq \operatorname{col}(A)$. $\operatorname{rank}(A^2) = \operatorname{rank}(A)$ implies $\operatorname{col}(A^2) = \operatorname{col}(A)$.
Again note that $\operatorname{col}(A^3) \subseteq \operatorname{col}(A^2)$. If $y \in \operatorname{col}(A^2)$, then $y = A^2 z = A(Az)$ for some $z \in \mathbb{R}^n$. Since $\operatorname{col}(A^2) = \operatorname{col}(A)$, $Az = A^2 u$ for some $u \in R^n$. Hence $y = A(A^2 u)$ for some $u \in R^n$, that is $y \in \operatorname{col}(A^3)$. Hence $\operatorname{col}(A^3) = \operatorname{col}(A^2)$.
By similar argument one can show that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$ for all $k \in \mathbb{N}$.
(One could have argued similarly by considering the row space.)

3. (a) Suppose $\mathbb{V}$ is a vector space over $\mathbb{R}$, and $A = [a_{ij}] \in \mathcal{M}_k(\mathbb{R})$ is invertible. Show that $\mathbf{u}_1, \cdots, \mathbf{u}_k \in \mathbb{V}$ are linearly independent if and only if $\sum_{i=1}^{k} a_{i1}\mathbf{u}_i, \cdots, \sum_{i=1}^{k} a_{ik}\mathbf{u}_i$ are linearly independent.

(b) Show that $\{\mathbf{u},\mathbf{v}\}$ is linearly independent iff $\{\mathbf{u}+\mathbf{v},\mathbf{u}-\mathbf{v}\}$ is linearly independent.

Solution:

(a) ($\Rightarrow$) Put $\mathbf{w}_r = \sum_{i=1}^{k} a_{ir}\mathbf{u}_i$. Then $\begin{bmatrix}\mathbf{w}_1\\ \vdots\\ \mathbf{w}_r\end{bmatrix} = \begin{bmatrix}a_{11} & \cdots & a_{k1}\\ \vdots & & \vdots\\ a_{1k} & \cdots & a_{kk}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_k\end{bmatrix} = A^T\begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_k\end{bmatrix}$.

Suppose $\mathbf{w}_1,\ldots,\mathbf{w}_k$ are linearly dependent. Then there exists $[\alpha_1 \quad \cdots \quad \alpha_k]^T \neq \mathbf{0}$ such that $[\alpha_1 \quad \cdots \quad \alpha_k]\begin{bmatrix}\mathbf{w}_1\\ \vdots\\ \mathbf{w}_k\end{bmatrix} = \mathbf{0}$. So

$$\mathbf{0} = [\alpha_1 \quad \cdots \quad \alpha_k]\begin{bmatrix}a_{11} & \cdots & a_{k1}\\ \vdots & & \vdots\\ a_{1k} & \cdots & a_{kk}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_k\end{bmatrix} = [\beta_1 \quad \cdots \quad \beta_k]\begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_k\end{bmatrix},$$

where $[\alpha_1 \quad \cdots \quad \alpha_k]A^T = [\beta_1 \quad \cdots \quad \beta_k] \neq \mathbf{0}$. Thus $\mathbf{u}_1,\ldots,\mathbf{u}_k$ are linearly dependent.

($\Rightarrow$) Similar.

(b) Take $A = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. Then $A^T\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = \begin{bmatrix}\mathbf{u}+\mathbf{v}\\ \mathbf{u}-\mathbf{v}\end{bmatrix}$.

4. Let $\mathbb{W},\mathbb{U}$ be subspaces of $\mathbb{V}$. Show that $\mathbb{W}\cup\mathbb{U}$ is a subspace iff either $\mathbb{W}\subseteq\mathbb{U}$ or $\mathbb{U}\subseteq\mathbb{W}$. What about union of three subspaces?

Solution: ($\Rightarrow$) Suppose that $\mathbb{U}\cup\mathbb{W}$ is a subspace. We claim that either $\mathbb{U}\subseteq\mathbb{W}$ or $\mathbb{W}\subseteq\mathbb{U}$. Assume our claim is not true. Then $\exists$ a $u\in\mathbb{U}\setminus\mathbb{W}$ and a $w\in\mathbb{W}\setminus\mathbb{U}$. Note that $u,w\in\mathbb{U}\cup\mathbb{W}$, a subspace. So $u+w\in\mathbb{U}\cup\mathbb{W}$, a union of two sets. So either $u+w\in\mathbb{U}$ or $u+w\in\mathbb{W}$. Let $u+w\in\mathbb{U}$. As u is already in $\mathbb{U}$, we get $w=(u+w)+(-1)u\in\mathbb{U}$, a contradiction. Similarly, $u+w\in\mathbb{W}$ leads to another contradiction. Hence our claim is valid and we are done.

($\Leftarrow$) Trivial.

2nd part Answer: If and only if one of the subspaces contains the other two (take the field as $\mathbb{R}$ or $\mathbb{C}$).

5. Extend

$$S = \left\{\begin{bmatrix}1\\2\\0\\-1\\0\\3\end{bmatrix}, \begin{bmatrix}2\\4\\1\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}3\\6\\2\\1\\2\\-1\end{bmatrix}\right\}$$

to a basis of $\mathbb{R}^6$ using GJE.

Solution: Consider $A = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 3 \\ 2 & 4 & 1 & 0 & 1 & -1 \\ 3 & 6 & 2 & 1 & 2 & -1 \end{bmatrix}$. Then $\tilde{A} = \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Note that $\text{span}(S) = \text{col}(A^T) = \text{col}(\tilde{A}^T)$. Since the 2nd, 4th and 5th columns in $\tilde{A}$ are non-leading, if we add rows $\mathbf{e}_2^T, \mathbf{e}_4^T, \mathbf{e}_5^T$ to $\tilde{A}$, then we get a 6×6 matrix of rank 6. Thus, $S \cup \{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}$ is a basis for $\mathbb{R}^6$.

6. For each of these, find a basis.

(a) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 2a - c - d = 0, a + 3b = 0, a, b, c, d \in \mathbb{R} \right\}$.

(b) $\{p(x) : p(x) = \mathbf{0}$ or $p(x)$ is a polynomial in x of degree at most 4 with real coefficients, $p(-2) = 0\}$.

Solution:

(a) Solving $2a - c - d = 0, a + 3b = 0$ we get $[a, b, c, d]^T = [\frac{s}{2} + \frac{t}{2},\ \frac{-s}{6} + \frac{-t}{6},\ s,\ t]^T$. Thus, $\left\{ \begin{bmatrix} \frac{1}{2} & \frac{-1}{6} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{-1}{6} \\ 0 & 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 0 & 6 \end{bmatrix} \right\}$ is a basis.

(b) Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ be such that $p(-2) = 0$. Then, $a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 = 0$. Therefore,

$$\begin{aligned} p(x) &= (2a_1 - 4a_2 + 8a_3 - 16a_4) + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\ &= a_1(2 + x) - a_2(4 - x^2) + a_3(8 + x^3) - a_4(16 - x^4). \end{aligned}$$

and we get a basis $\{2 + x,\ 4 - x^2,\ 8 + x^3,\ 16 - x^4\}$.

SECTION - B: ADDITIONAL PROBLEMS

1. (a) Show that for any two $m \times n$ matrices A and B, $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

Hint: $A + B = [A|B] \begin{bmatrix} I_n \\ I_n \end{bmatrix}$.

Solution: Since $A + B = [A|B] \begin{bmatrix} I_n \\ I_n \end{bmatrix}$, $\text{rank}(A + B) \leq rank[A|B] \leq \text{rank}(A) + \text{rank}(B)$.

(b) Hence show that if A is an $m \times n$ matrix and B is the matrix obtained by changing exactly k entries of A, then $\text{rank}(A) - k \leq \text{rank}(B) \leq \text{rank}(A) + k$.

Hint: $B = A + C$, where C has exactly k nonzero entries.

Solution: Since C has at most k nonzero rows, $\text{rank}(C) \leq k$, Hence $\text{rank}(B + C) \leq \text{rank}(A) + k$.
To show the other inequality, note that $A = B + (-C)$.

2. Let $\mathbb{V}$ be a vector space and S be a subset of $\mathbb{V}$. Let $L = \{\mathbb{U} | \;\; \mathbb{U} \preccurlyeq \mathbb{V},\, S \subseteq \mathbb{U}\}$. Then show that $\text{span}(S) = \bigcap\limits_{\mathbb{U}\in L} \mathbb{U}$ = the smallest subspace containing S.

> **Solution:** As each $\mathbb{U}$ contains S, it must contain $\text{span}(S)$. Hence $\bigcap_{\mathbb{U}\in L} \mathbb{U}$ contains $\text{span}(S)$. Further, as $\text{span}(S)$ is subspace, it must appear as one $\mathbb{U}$ on the right hand side. Thus $\bigcap_{\mathbb{U}\in L} \mathbb{U}$ cannot be larger than $\text{span}(S)$.

3. Give subspaces $\mathbb{W}_i$, $(1 \le i \le 5)$ of $\mathbb{R}^{[0,1]}$ such that $\mathbb{W}_5 \subsetneq \mathbb{W}_4 \subsetneq \cdots \subsetneq \mathbb{W}_1$.
   Hint: Take for example $\mathbb{W}_1 = \mathbb{R}[x]$, where $x \in [0,1]$.

4. Consider $\mathbb{W} = \{v \in \mathbb{R}^6 \,|\, v_1 + v_2 + v_3 = 0,\; v_2 + v_3 + v_4 = 0,\; v_5 + v_6 = 0\}$. Supply a basis for $\mathbb{W}$ and extend it to a basis of $\mathbb{R}^6$.

5. For each of these, find a basis.
   a) $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a - d = 0, a, b, c, d \in \mathbb{R}\right\}$.
   b) $\{a + bx + cx^3 : a, b, c \in \mathbb{R}, a - 2b + c = 0\}$.
   c) $\{A_{m\times n} : \text{row sums of } A \text{ are zero}\}$.

6. Give 2 bases for the trace 0 real symmetric matrices of size 3×3. Extend these bases to bases of the real symmetric matrices of size 3×3. Extend these bases to bases of the real matrices of size 3×3.

7. Consider the vector space of real polynomials in x. Let $S = \{1 + t, (1 + t)^2, 1 - t^2, 10\}$. Describe $\text{span}(S)$ and find its dimension.