MA101 Mathematics I

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Slides 4

Plan

- Vector Space and Subspaces
- Linear Independence, Basis and Dimension

Fields

A field $\mathbb{F}$ is a set from which we choose our coefficients and scalars. Expected properties are:

1. $a+b$ and $a \cdot b$ should be defined in it. i.e., $a+b$ and $a \cdot b$ must be inside the field.
2. Both operations must be commutative: $a+b=b+a$; $a \cdot b = b \cdot a$.
3. Both operations must be associative: $(a+b)+c = a+(b+c)$; $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
4. There should be identity elements for both operations. Identity element for $+$ is called 0 and that for $\cdot$ is called 1.
5. Inverse for a w.r.t. '$+$': $\forall a \in \mathbb{F}, \exists b \in \mathbb{F}$ s.t. $a+b=0$.
6. Inverse for $a \neq 0$ w.r.t. '$\cdot$': $\forall a \in \mathbb{F} \setminus \{0\}, \exists b \in \mathbb{F}$ s.t. $a \cdot b = 1$.
7. '$\cdot$' distributes itself over '$+$': $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$.

Example

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, with usual addition and multiplication as $+$ and '$\cdot$'

What about $\mathbb{Z}$? No, since 2 does not have inverse w.r.t. '$\cdot$'. Take $\mathbb{Z}_5 = \{0,1,2,3,4\}$, and define $a+b := (a+b) \mod 5$ and $a \cdot b := (ab) \mod 5$. $\mathbb{Z}_5$ is a field. Here $3+4=2$, $4 \cdot 2 = 3$, etc.

Remark

Many concepts and results we have discussed (e.g., theory of linear systems, matrices), hold if $\mathbb{R}$ is replaced by any field $\mathbb{F}$.

Example

Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ over $\mathbb{Z}_5$. A is invertible, and $A^{-1} = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$.

(Computed using $A^{-1} = (ad-bc)^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.) The system $A\mathbf{x} = [3\ 4]^T$ has unique solution, $\mathbf{x} = A^{-1}[3\ 4]^T = [0\ 4]^T$.

Remark

For any field, usually one writes ab instead of $a \cdot b$.

Vector Spaces

A set $\mathbb{V} \neq \emptyset$ is a vector space (VS) over a field $\mathbb{F}$, if $\mathbf{u} + \mathbf{v}$ is defined in $\mathbb{V}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, and $\alpha \cdot \mathbf{u}$ is defined in $\mathbb{V}$ for all $\mathbf{u} \in \mathbb{V}, \alpha \in \mathbb{F}$ such that

- $+$ is commutative & associative.
- Identity element 0 exists in $\mathbb{V}$ for $+$.
- Each $\mathbf{u} \in \mathbb{V}$ has an inverse w.r.t $+$.
- $1 \cdot \mathbf{u} = \mathbf{u}$ holds $\forall \mathbf{u} \in \mathbb{V}$ (1 is the multiplicative identity of $\mathbb{F}$)
- $(\alpha\beta) \cdot \mathbf{u} = \alpha \cdot (\beta \cdot \mathbf{u}), (\alpha + \beta) \cdot \mathbf{u} = \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{u}$ hold $\forall \alpha, \beta \in \mathbb{F}, \mathbf{u} \in \mathbb{V}$.
- $\alpha \cdot (\mathbf{u} + \mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{v}$ holds $\forall \alpha \in \mathbb{F}, \forall \mathbf{u}, \mathbf{v} \in \mathbb{V}$.

The elements of $\mathbb{V}$ are called vectors and the elements of $\mathbb{F}$ scalars. We will mostly consider $\mathbb{F}$ as $\mathbb{R}$ and $\mathbb{C}$.

If there is no chance of any confusion, one writes $\alpha\mathbf{u}$ instead of $\alpha \cdot \mathbf{u}$.

Example

- $\mathbb{F}^n$ over $\mathbb{F}$, for $n \geq 1$, is a VS w.r.t. usual operations of addition and scalar multiplication induced from $\mathbb{F}$.
- $\mathbb{C}^n$ over $\mathbb{R}$, for $n \geq 1$, is a VS w.r.t. usual addition and scalar multiplication.
- $\mathbb{R}^n$ over $\mathbb{Q}$, for $n \geq 1$, is a VS w.r.t. usual operations of addition and scalar multiplication.
- $\mathcal{M}_{m,n}(\mathbb{F}) := \{A_{m \times n} : a_{ij} \in \mathbb{F}\}$ is a VS over $\mathbb{F}$, under matrix addition and scalar-matrix multiplication.

Exercise

- Are these vector spaces (under the usual operations)? All $n \times n$ (a) symmetric matrices? (b) skew symmetric matrices? (c) upper-triangular matrices? (d) matrices with $a_{11} = 0$? (e) matrices A such that $A^2 = A$?

Example

- $\mathbb{R}[x] := \{p(x) \mid p(x)$ is a real polynomial in $x\}$ is a VS over $\mathbb{R}$.
- $\mathbb{R}_m[x] := \{p(x) \in \mathbb{R}[x] \mid p(x) = 0$ or $\deg(p(x)) \leq m\}$ is a VS over $\mathbb{R}$.
- $\mathbb{R}^S := \{$ functions from S to $\mathbb{R}\}$ is a VS over $\mathbb{R}$, where
$$(f+g)(s) := f(s) + g(s), \quad (\alpha f)(s) = \alpha(f(s)).$$
- $\mathcal{C}((a,b),\mathbb{R}) := \{f : (a,b) \to \mathbb{R} \mid f \text{is continuous}\}$ is a VS over $\mathbb{R}$.
- $\{f : (a,b) \to \mathbb{R} \mid f'' - 3f' + 7f = 0\}$ is a VS over $\mathbb{R}$.

A real vector space: a VS $\mathbb{V}$ over $\mathbb{R}$;

A complex vector space: a VS $\mathbb{V}$ over $\mathbb{C}$.

Result

In any vector space $\mathbb{V}$ over $\mathbb{F}$, the following holds:

1. $0\mathbf{u} = \mathbf{0}, \ \mathbf{u} \in \mathbb{V}$;
2. $\alpha\mathbf{0} = \mathbf{0}, \ \alpha \in \mathbb{F}$;
3. $(-1)\mathbf{u} = -\mathbf{u}, \ \mathbf{u} \in \mathbb{V}$;
4. *If* $\alpha\mathbf{u} = \mathbf{0}$ *then either* $\alpha = 0$ *or* $\mathbf{u} = \mathbf{0}$.

Exercise

- Define addition and scalar mult. on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as follows:

  For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, \ \alpha \in \mathbb{R}$
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad \alpha.(x, y) = (\alpha x, 0).$$
  Is $\mathbb{R}^2$ a VS over $\mathbb{R}$ w.r.t. respect these operations?

  Is $1 \cdot (x, y) = (x, y)$?

  Is $(-1) \cdot (2, 3)$ the additive inverse of $(2, 3)$?

Subspace

Let $\mathbb{V}$ be a VS over $\mathbb{F}$ and $(\emptyset \neq)\ \mathbb{W} \subseteq \mathbb{V}$. Then $\mathbb{W}$ is a subspace of V (write $\mathbb{W} \preccurlyeq V$), if

$$\mathbf{u}+\mathbf{v} \in \mathbb{W},\ \alpha\mathbf{u} \in \mathbb{W} \text{ for all } \mathbf{u},\mathbf{v} \in \mathbb{W},\ \alpha \in \mathbb{F}.$$

- $\mathbb{W} \preccurlyeq \mathbb{V}$
  iff $\alpha\mathbf{u}+\beta\mathbf{v} \in \mathbb{W}$, for all $\mathbf{u},\mathbf{v} \in \mathbb{W},\ \alpha,\beta \in \mathbb{F}$
  iff $\alpha\mathbf{u}+\mathbf{v} \in \mathbb{W}$, for all $\mathbf{u},\mathbf{v} \in \mathbb{W},\ \alpha \in \mathbb{F}$
  iff $\mathbb{W}$ is a VS over same $\mathbb{F}$ and under same operations.
- If $W \preccurlyeq \mathbb{V}$, then $\mathbf{0} \in W$.
- $\{\mathbf{0}\} \preccurlyeq \mathbb{V}$ and $\mathbb{V} \preccurlyeq \mathbb{V}$, called the trivial subspaces.

Exercise

- Identify some subspaces of $\mathcal{M}_{m\times n}(\mathbb{R})$, $\mathcal{M}_n(\mathbb{C})$ and $\mathbb{R}^{[a,b]}$.

Linear Span

- Let $\mathbf{v}_i \in \mathbb{V},\ \alpha_i \in \mathbb{F},\ 1 \le i \le k$. Then $\sum_{i=1}^{k} \alpha_i\mathbf{v}_i$ is a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_k$. Clearly,
$$\text{span}(\mathbf{v}_1,\ldots,v_k) := \left\{\sum_{i=1}^{k} \alpha_i\mathbf{v}_i \mid \alpha_i \in \mathbb{F}\right\} \preccurlyeq \mathbb{V}.$$
- Let $S \subseteq \mathbb{V}$ (may be infinite!) The span of S is defined by
$$\text{span}(S) := \left\{\sum_{i=1}^{m} \alpha_i\mathbf{v}_i \mid \mathbf{v}_i \in S, \alpha_i \in \mathbb{F}, m \text{ a nonnegative integer}\right\}.$$
- S is a spanning set for $\mathbb{V}$ if $\text{span}(S) = \mathbb{V}$.
- Convention: $\text{span}(\emptyset) = \{\mathbf{0}\}$

Example

- $\mathbb{R}_2[x] = \text{span}(1, x, x^2) = \text{span}(1+x, 1-x, 1+x+x^2)$.
- $\mathbb{R}[x] = \text{span}(\{1, x, x^2, \ldots\})$.

Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a VS $\mathbb{V}$ over $\mathbb{F}$. Then S is linearly dependent (LD) if at least one of $v_i \in S$ is a linear combination of the rest of elements in S, i.e., if

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k = \mathbf{0}$$

for some $\mathbf{0} \neq [\alpha_1, \alpha_2, \ldots, \alpha_k]^T \in \mathbb{F}^k$.

Example

- Any finite set containing $\mathbf{0}$ is linearly dependent.
- In $\mathbb{R}_2[x]$, is $\{x^2,\ 1 - x^2,\ 1 + x^2\}$ linearly dependent?

$$\begin{aligned} & ax^2 + b(1 - x^2) + c(1 + x^2) = 0 \\ \Rightarrow \quad & (b + c) + (a - b + c)x^2 = 0 \\ \Rightarrow \quad & b + c = 0,\ a - b + c = 0, \end{aligned}$$

because, a polynomial is zero iff all of its coefficients are zero. The last system has nontrivial solutions. Thus, the set is LD.

Linear Independence

We say $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \in \mathbb{V}$ to be linearly independent (LI) if it is not linearly dependent, that is, if

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_k\mathbf{v}_k = \mathbf{0} \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \ldots = \alpha_k = 0.$$

An infinite set $S \subseteq \mathbb{V}$ is linearly independent (LI) if every finite subset of S is linearly independent.

Example

- The set $\{1, 1 + x, 1 + x + x^2\} \subseteq \mathbb{R}_3[x]$ is linearly independent. Use GJE.
- The set $\{1, x, x^2, \ldots\} \subseteq \mathbb{R}[x]$ is linearly independent.
- The set $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is linearly independent in $\mathcal{M}_2(\mathbb{R})$.

Basis

A subset B of a VS $\mathbb{V}$ is said to be a basis for $\mathbb{V}$ if $\text{span}(B) = \mathbb{V}$ and B is linearly independent.

Example

- $\mathbb{V} = \mathbb{F}^n$ over $\mathbb{F}$: the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.
- $\mathbb{V} = \mathbb{R}_n[x]$ over $\mathbb{R}$: $\{1, x, x^2, \ldots, x^n\}$, called the standard basis.
- $\{1 + x, x + x^2, 1 + x^2\}$ is a basis of $\mathbb{R}_2[x]$ over $\mathbb{R}$. (Check)
- $\mathbb{V} = \mathbb{R}[x]$ over $\mathbb{R}$: $\{1, x, x^2, \ldots\}$.
- $\mathbb{V} = \mathbb{C}$ over $\mathbb{R}$: $\{1, i\}$.
- $\mathbb{V} = \mathcal{M}_2(\mathbb{F})$ over $\mathbb{F}$: $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$.
- $\mathbb{V} = \mathcal{M}_n(\mathbb{F})$ over $\mathbb{F}$: $\{E_{ij} : 1 \le i, j \le n\}$, where $E_{ij} = [a_{kl}]$, given by $a_{kl} = 1$ if $k = i, l = j$ and 0, otherwise.

Result

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_\}$ be LI in $\mathbb{V}$ and $\mathbf{v} \notin span(S)$. Then $S \cup \{\mathbf{v}\}$ is LI.

Proof. Suppose $\alpha_1\mathbf{v}_1 + \cdots + \alpha_m\mathbf{v}_m + \alpha\mathbf{v} = \mathbf{0}$ for some $\alpha_1, \ldots, \alpha_m, \alpha \in \mathbb{F}$. If $\alpha \neq \mathbf{0}$, then $\mathbf{v} \in \text{span}(S)$, not true. Thus, $\alpha = 0$, and $\alpha_1\mathbf{v}_1 + \cdots + \alpha_m\mathbf{v}_m = \mathbf{0}$. S being LI, we have $\alpha_i = 0$. ■

Result

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subseteq \mathbb{V}$ and $\mathbb{U} = span(S)$. Then S contains a basis of $\mathbb{U}$.

Proof. If $\mathbf{v}_1 = \mathbf{0}$, replace S by $S \setminus \{\mathbf{v}_1\}$. Otherwise, for $1 \le k \le m$, check if $\mathbf{v}_k \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1})$. Whenever your answer is yes, replace S by $S \setminus \{\mathbf{v}_k\}$ and repeat the process. The process must end in at most m steps.
The set $B \subseteq S$ thus obtained spans $\mathbb{U}$ and is linearly independent.
Why? ■

Result

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathbb{V}$ and $T \subseteq span(S)$ be such that $m = |T| > r$. Then T is LD.

Proof. Let $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$. Write

$$\mathbf{u}_i = a_{i1}\mathbf{v}_1 + a_{i2}\mathbf{v}_2 + \cdots + a_{ir}\mathbf{v}_r, \ 1 \le i \le m.$$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mr} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$. So $u_i = \mathbf{a}_i^T \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_r \end{bmatrix}$.

Since $m > r$, the rows of A are linearly dependent. Suppose $\alpha_1 \mathbf{a}_1^T + \cdots + \alpha_m \mathbf{a}_m^T = \mathbf{0}^T$. Then

$$\sum_{i=1}^m \alpha_i \mathbf{u}_i = \sum_{i=1}^m \alpha_i \mathbf{a}_i^T \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_r \end{bmatrix} = \mathbf{0}^T \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_r \end{bmatrix} = \mathbf{0}^T. \quad \blacksquare$$

Result (The Basis Theorem)

Let $\mathbb{V}$ be a VS having a finite spanning set. Then $\mathbb{V}$ has a finite basis and any two bases of $\mathbb{V}$ has same number of elements.

Proof. Follows from the previous two results. $\blacksquare$

Dimension: If a VS $\mathbb{V}$ over $\mathbb{F}$ has a finite basis with $n \ge 0$ elements, then $\mathbb{V}$ is said to be finite dimensional and of dimension n. We then write $\dim(\mathbb{V}) = n$.

If $\mathbb{V}$ does not have a finite spanning set, then $\mathbb{V}$ is said to be infinite dimensional.

Dimension

Example

Finite dimensional:

- The zero space $\{\mathbf{0}\}$ has dimension 0.
- $\mathbb{F}^n$ over $\mathbb{F}$, dimension: n;
- $\mathbb{R}_n[x]$ over $\mathbb{R}$, dimension: $n+1$;
- $\mathbb{C}$ over $\mathbb{R}$, dimension: 2;
- $\mathcal{M}_n(\mathbb{F})$ over $\mathbb{F}$, dimension: n^2.

Infinite dimensional:

- $\mathbb{R}[x]$ over $\mathbb{R}$;
- $\mathbb{R}$ over $\mathbb{Q}$;
- $\mathcal{C}((0,1),\mathbb{R})$ over $\mathbb{R}$.

Exercise

Prove the following statements:

- $\mathcal{C}^2((a,b),\mathbb{R}) := \{f : (a,b) \to \mathbb{R} \mid f''\text{is continuous}\}$ is a subspace of the VS $\mathcal{C}((a,b),\mathbb{R})$ over $\mathbb{R}$.
- The VS $\mathbb{R}$ over $\mathbb{R}$ has no nontrivial subspaces?
- If $\mathbb{U} \preccurlyeq \mathbb{W}$ and $\mathbb{W} \preccurlyeq \mathbb{V}$, then $\mathbb{U} \preccurlyeq \mathbb{V}$.
- Let $\{\mathbb{U}_i \mid \mathbb{U}_i \preccurlyeq \mathbb{V}\}$ be nonempty. Then $\cap_i \mathbb{U}_i \preccurlyeq \mathbb{V}$.
- Let $\mathbb{U}, \mathbb{W} \preccurlyeq \mathbb{V}$. Then $\mathbb{U} \cup \mathbb{W} \preccurlyeq \mathbb{V}$ iff $\mathbb{U} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{U}$.
- Suppose $\mathbb{U}, \mathbb{W} \preccurlyeq \mathbb{V}$. Let $\mathbb{U} + \mathbb{W} := \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in \mathbb{U}, \mathbf{w} \in \mathbb{W}\}$. Then $\mathbb{U} + \mathbb{W} \preccurlyeq \mathbb{V}$. [$\mathbb{U} + \mathbb{W}$ is called an internal direct sum if $\mathbb{U} \cap \mathbb{W} = \{\mathbf{0}\}$, and then one writes $\mathbb{U} \oplus \mathbb{W}$.]
- Let $\mathbb{U}$, $\mathbb{W}$ be VS's over $\mathbb{F}$. Then $\mathbb{U} \times \mathbb{W}$ is a VS over $\mathbb{F}$:

  $(u_1, w_1) + (u_2, w_2) := (u_1 + u_2, w_1 + w_2), \alpha(u, w) := (\alpha u, \alpha w)$.
  [$\mathbb{U} \times \mathbb{V}$ is called the external direct sum of $\mathbb{U}$ and $\mathbb{W}$,
  Notation: $\mathbb{U} \oplus \mathbb{W}$.]

Exercise

Prove the following statements:

- Let $\mathbb{V} = \mathcal{M}_2(\mathbb{R})$, $\mathbb{U} = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & 0 \end{bmatrix} : x_i \in \mathbb{R} \right\}$, $\mathbb{W} = \left\{ \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} : x_i \in \mathbb{R} \right\}$.
  Then $\mathbb{U}, \mathbb{W} \preccurlyeq \mathbb{V}$, $\mathbb{V} = \mathbb{U} + \mathbb{W}$, but $\mathbb{V} \neq \mathbb{U} \oplus \mathbb{W}$.
  [Note: $\mathbb{U} \cap \mathbb{W} = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \in \mathbb{R} \right\}$.]
- Let $\mathbb{U}, \mathbb{W} \preccurlyeq \mathbb{V}$ and $\mathbb{V}' = \mathbb{U}_1 + \mathbb{U}_2$. Then $\mathbb{V}' = \mathbb{U}_1 \oplus \mathbb{U}_2$ iff every $\mathbf{v} \in \mathbb{V}'$ can be written in unique way as
  $\mathbf{v} = \mathbf{u} + \mathbf{w}, \ \mathbf{u} \in \mathbb{U}, \mathbf{w} \in \mathbb{W}$.
- For a VS $\mathbb{V}$ and $S \subseteq \mathbb{V}$, $\text{span}(S) = \bigcap \{\mathbb{U} \mid \mathbb{U} \preccurlyeq \mathbb{V}, S \subseteq \mathbb{U}\}$ = the smallest subspace of $\mathbb{V}$ containing S.

Exercise

Prove the following statement:
Let $\mathbb{V}$ be a VS and B a basis for $\mathbb{V}$. Then every nonzero vector $\mathbf{v}$ in $\mathbb{V}$ can be expressed uniquely as a linear combination of (finitely many) vectors in B with nonzero coefficients.

Exercise

Let $\mathbb{V}$ be a vector space with $\dim \mathbb{V} = n$. Prove that

- Any linearly independent set in $\mathbb{V}$ contains at most n vectors.
- Any spanning set for $\mathbb{V}$ contains at least n vectors.
- Any linearly independent set of exactly n vectors in $\mathbb{V}$ is a basis for $\mathbb{V}$.
- Any spanning set for $\mathbb{V}$ of exactly n vectors is a basis for $\mathbb{V}$.
- Any linearly independent set in $\mathbb{V}$ can be extended to a basis for $\mathbb{V}$.
- Any spanning set for $\mathbb{V}$ can be reduced to a basis for $\mathbb{V}$.