DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati
MA101: Mathematics I
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Practice Problem Set: Linear Algebra (Contains 275 Problems)

1. What are the possible reduced row echelon form of each of a 2×2 and a 3×3 matrix?

2. Find the reduced row echelon form of each of the following matrices.

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 6 & 2 \\ -1 & 2 & 4 & 3 \\ 1 & 2 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 5 & -6 \\ 2 & 3 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 5 & -5 & 5 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 2 & 2 & 2 & 4 \\ 2 & -2 & 4 & 0 & 8 \\ 4 & 2 & 5 & 6 & 10 \end{bmatrix}.$$

3. Solve the following systems of equations using Gaussian elimination method as well as Gauss-Jordan elimination method:

   (a) $4x + 2y - 5z = 0, \quad 3x + 3y + z = 0, \quad 2x + 8y + 5z = 0;$

   (b) $-x + y + z + w = 0, \quad x - y + z + w = 0, \quad -x + y + 3z + 3w = 0, \quad x - y + 5z + 5w = 0;$

   (c) $x + y + z = 3, \quad x - y - z = -1, \quad 4x + 4y + z = 9;$

   (d) $x + y + 2z = 3, \quad -x - 3y + 4z = 2, \quad -x - 5y + 10z = 11;$

   (e) $x - 3x - 2z = 0, \quad -x + 2y + z = 0, \quad 2x + 4y + 6z = 0;$ and

   (f) $2w + 3x - y + 4z = 0, \quad 3w - x + z = 1, \quad 3w - 4x + y - z = 2.$

4. Consider the system $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove the following:

   (a) If each entry of A is 0 then each vector $\mathbf{x} = [x, y]^t$ is a solution of $A\mathbf{x} = \mathbf{0}$.

   (b) If $ad - bc \neq 0$ then the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0} = [0, 0]^t$.

   (c) If $ad - bc = 0$ and some entry of A is non-zero, then there is a solution $\mathbf{x}_0 = [x_0, y_0]^t$ such that $\mathbf{x} = [x, y]^t$ is a solution if and only if $x = kx_0, y = ky_0$ for some constant k.

5. For what values of $c \in \mathbb{R}$ and $k \in \mathbb{R}$, the following systems of equations has (i) no solution, (ii) a unique solution, and (iii) infinitely many solutions?

   (a) $x + y + z = 3, \quad x + 2y + cz = 4, \quad 2x + 3y + 2cz = k;$

   (b) $x + y + 2z = 3, \quad x + 2y + cz = 5, \quad x + 2y + 4z = k;$ and

   (c) $x + 2y - z = 1, \quad 2x + 3y + kz = 3, \quad x + ky + 3z = 2.$

   Also, find the solutions whenever they exist.

6. For what values of $a \in \mathbb{R}$ and $b \in \mathbb{R}$, the following system of equations in unknowns x, y and z, has (i) no solution, (ii) a unique solution, and (iii) infinitely many solutions:

$$ax + by + 2z = 1, \quad ax + (2b-1)y + 3z = 1, \quad ax + by + (b+3)z = 2b - 1?$$

   Also, find the solutions whenever they exist.

7. Solve the following system of equations applying Gaussian elimination method:

$$\begin{aligned} (1-n)x_1 + x_2 + \ldots + x_n &= 0 \\ x_1 + (1-n)x_2 + x_3 + \ldots + x_n &= 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots & \quad \vdots \\ x_1 + x_2 + \ldots + x_{n-1} + (1-n)x_n &= 0. \end{aligned}$$

8. Prove that if $r < s$, then the r-th and the s-th rows of a matrix can be interchanged by performing $2(s-r)-1$ interchanges of adjacent rows.

9. If $\mathbf{x}_1$ is a solution of the non-homogeneous system $A\mathbf{x} = \mathbf{b}$ and if $\mathbf{y}_1$ is a solution of the system $A\mathbf{x} = \mathbf{0}$ then show that $\mathbf{x}_1 + \mathbf{y}_1$ is a solution of $A\mathbf{x} = \mathbf{b}$. Moreover, if $\mathbf{x}_0$ is any solution of the system $A\mathbf{x} = \mathbf{b}$ then show that there is a solution $\mathbf{y}_0$ of the system $A\mathbf{x} = \mathbf{0}$ such that $\mathbf{x}_0 = \mathbf{x}_1 + \mathbf{y}_0$.

10. Let A be an $m \times n$ matrix. Prove that the system of equations $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$ if and only if each row of the row echelon form of A contains a leading term.

11. Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Prove that the system of equations $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if there is a leading term in the last column of the row echelon form of its augmented matrix.

12. Let $A = \begin{bmatrix} i & -1-i & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix}$. Determine the reduced row echelon form of A over $\mathbb{C}$. Hence solve the system $A\mathbf{x} = \mathbf{0}$ over $\mathbb{C}$.

13. Show that $\mathbf{x} = \mathbf{0}$ is the only solution of the system of equations $A\mathbf{x} = \mathbf{0}$ if and only if the rank of A equals the number of unknowns.

14. Show that a consistent system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if the rank of A equals the number of unknowns.

15. Prove that if two homogeneous systems of linear equations in two unknowns have the same solution, then these two systems are equivalent. Is the same true for more than two unknowns? Justify.

16. (**Optional**) Solve the following systems of equations over the indicated $\mathbb{Z}_p$.
    (a) $x + 2y = 1,\ x + y = 2$ over $\mathbb{Z}_3$.
    (b) $x + y = 1,\ y + z = 0, x + z = 1$ over $\mathbb{Z}_2$.
    (c) $3x + 2y = 1,\ x + 4y = 1$ over $\mathbb{Z}_5$.
    (d) $2x + 3y = 4,\ 4x + 3y = 2$ over $\mathbb{Z}_6$.

17. (**Optional**) Let A be an $m \times n$ matrix of rank r with entries in $\mathbb{Z}_p$, where p is a prime number. Prove that every consistent system of equations with coefficient matrix A has exactly p^{n-r} solutions over $\mathbb{Z}_p$.

18. Let A be an $m \times n$ matrix with complex entries. Show that the system $A^*A\mathbf{x} = A^*\mathbf{b}$ is consistent for each $\mathbf{b} \in \mathbb{C}^n$.

19. Suppose that the non-homogeneous system $A\mathbf{x} = \mathbf{b}$ has solutions $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$. Show that a linear combination $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_k\mathbf{u}_k$ is a solution of $A\mathbf{x} = \mathbf{b}$ if and only if $a_1 + a_2 + \ldots + a_k = 1$. Also, show that $b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \ldots + b_n\mathbf{u}_n = \mathbf{0}$ implies $b_1 + b_2 + \ldots + b_n = 0$.

20. Find a basis for the solution space of the following system of $n+1$ linear equations in $2n$ unknowns:

$$\begin{aligned} x_1 + x_2 + \ldots + x_n &= 0 \\ x_2 + x_3 + \ldots + x_{n+1} &= 0 \\ \vdots \quad \vdots \quad \vdots \quad & \vdots \ \vdots \\ x_{n+1} + x_{n+2} + \ldots + x_{2n} &= 0. \end{aligned}$$

21. Examine whether the following sets are subspaces of $\mathbb{R}^2$.

$$\{[x,y]^t \in \mathbb{R}^2 : x^2 + y^2 = 0\},\quad \{[x,y]^t \in \mathbb{R}^2 : x + y = 0\},\quad \{[x,y]^t \in \mathbb{R}^2 : x, y \in \mathbb{Z}\},$$
$$\{[x,y]^t \in \mathbb{R}^2 : x - y = 1\},\quad \{[x,y]^t \in \mathbb{R}^2 : xy \geq 0\} \quad \text{and} \quad \{[x,y]^t \in \mathbb{R}^2 : \frac{x}{y} = 1\}.$$

22. Examine whether the following sets are subspaces of $\mathbb{R}^3$.

$$\{[x,y,z]^t \in \mathbb{R}^3 \mid x \geq 0\},\quad \{[x,y,z]^t \in \mathbb{R}^3 \mid x + y = z\} \quad \text{and} \quad \{[x,y,z]^t \in \mathbb{R}^3 \mid x = y^2\}.$$

23. Examine whether the following sets are subspaces of $\mathbb{R}^n$.

$$S_1 = \{[x_1, x_2, \ldots, x_n]^t : x_1 \geq 0, n \geq 1\}, \qquad S_2 = \{[x_1, x_2, \ldots, x_n]^t) : x_2 = x_1^2, n \geq 2\}$$

$$\text{and} \quad S_3 = \{[x_1, x_2, \ldots, x_n]^t : 3x_1 - x_2 + x_3 + 2x_4 = 0, n \geq 4\}.$$

24. Determine the subspaces of $\mathbb{R}^3$ spanned by each of the following sets:

$$\{[1,1,1]^t, [0,1,2]^t, [1,0,-1]^t\}, \quad \{[1,2,3]^t, [1,3,5]^t\}, \quad \{[2,1,0]^t, [2,0,-2]^t\},$$

$$\{[1,2,3]^t, [1,3,5]^t, [1,2,4]^t\} \text{ and } \{[1,2,0]^t, [1,0,5]^t, [1,2,3]^t\}.$$

25. Show that $S = \{[1,0,0]^t, [1,1,0]^t, [1,1,1]^t\}$ is a linearly independent set in $\mathbb{R}^3$. In general, if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in some $\mathbb{R}^n$ then prove that $\{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is also a linearly independent set.

26. Show, without using the following result, that any set of k vectors in $\mathbb{R}^3$ is linearly dependent if $k \geq 4$.

    Result: Any set of m vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$.

27. Let W be a subspace of $\mathbb{R}^3$. Show that $\{[1,0,0]^t, [0,1,0]^t, [0,0,1]^t\} \subseteq W$ if and only if $W = \mathbb{R}^3$. Determine which of the following sets span $\mathbb{R}^3$:

$$\{[0,0,2]^t, [2,2,0]^t, [0,2,2]^t\}, \quad \{[3,3,1]^t, [1,1,0]^t, [0,0,1]^t\} \text{ and } \{[-1,2,3]^t, [0,1,2]^t, [3,2,1]^t\}.$$

28. Let $t \in \mathbb{R}$. Discuss the linear independence of the following three vectors over $\mathbb{R}$:

$$\mathbf{u} = [1,1,0]^t, \quad \mathbf{v} = [1,3,-1]^t, \quad \mathbf{w} = [5,3,t]^t.$$

29. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and $\mathbf{x}$ be four linearly independent vectors in $\mathbb{R}^n$, where $n \geq 4$. Answer true or false for each of the following statements, with proper justification:

    (a) The vectors $\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{x}$ and $\mathbf{x}+\mathbf{u}$ are linearly independent.

    (b) The vectors $\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{w}-\mathbf{x}$ and $\mathbf{x}-\mathbf{u}$ are linearly independent.

    (c) The vectors $\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{x}$ and $\mathbf{x}-\mathbf{u}$ are linearly independent.

    (d) The vectors $\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}-\mathbf{x}$ and $\mathbf{x}-\mathbf{u}$ are linearly independent.

30. Let $S = \{[x_1, x_2, \ldots, x_n]^t \in \mathbb{R}^n : \sum\limits_{i=1}^{n} x_i = 0\}$. Show that S is a subspace of $\mathbb{R}^n$. Find a basis and the dimension of S.

31. Prove that the row vectors of a matrix form a linearly dependent set if and only if there is a zero row in any row echelon form of that matrix.

32. Prove that the column vectors of a matrix form a linearly dependent set if and only if there is a column containing no leading term in any row echelon form of that matrix.

33. Prove that the column vectors of a matrix form a linearly independent set if and only if each column contains a leading term in any row echelon form of that matrix.

34. Find two different 2×2 real matrices A and B such that $A^2 = \mathbf{O} = B^2$ but $A \neq \mathbf{O}, B \neq \mathbf{O}$.

35. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 4 \\ 4 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}$. Compute $A^2 + 2AB + B^2$ and $(A+B)^2$. Are they equal? If not, give reasons.

36. Let A and B be two $n \times n$ matrices. If $AB = BA$ then show that $(AB)^m = A^m B^m$ and $(A+B)^m = \sum\limits_{i=1}^{m} \binom{m}{i} A^{m-i} B^i$ for every $m \in \mathbb{N}$. If $AB \neq BA$ then show that these two results need not be true.

37. Show that every square matrix can be written as a sum of a symmetric and an anti-symmetric matrix. Further, show that if A and B are symmetric, then AB is symmetric if and only if $AB = BA$. Give an example to show that if A and B are symmetric then AB need not be symmetric.

38. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}$. Is there a matrix C such that $CA = B$?

39. Let $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. Show that there exist two 2×2 matrices A and B satisfying $C = AB - BA$ if and only if $a + d = 0$.

40. Find conditions on the numbers a, b, c and d such that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with every 2×2 matrix.

41. Let $A_1, A_2, \ldots, A_n$ be matrices of the same size, where $n \geq 1$. Using mathematical induction, prove that

    (a) $(A_1 + A_2 + \ldots + A_n)^t = A_1^t + A_2^t + \ldots + A_n^t$; and

    (b) $(A_1 A_2 \ldots A_n)^t = A_n^t A_{n-1}^t \ldots A_1^t$.

42. For an $n \times n$ matrix $A = [a_{ij}]$, the trace is defined as $\mathrm{tr}(A) = a_{11} + \ldots + a_{nn}$. If A and B are two $n \times n$ matrices then prove that $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\mathrm{tr}(kA) = k.\mathrm{tr}(A)$, where $k \in \mathbb{R}$. To what number is $\mathrm{tr}(AA^t)$ equal?

43. Let A and B be two $n \times n$ matrices. If $AB = \mathbf{O}$ then show that for any positive integer k,
$$\mathrm{tr}((A + B)^k) = \mathrm{tr}(A^k) + \mathrm{tr}(B^k).$$

44. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Show that for any positive integer k,
$$\mathrm{tr}(A^k) = \mathrm{tr}(A^{k-1}) + \mathrm{tr}(A^{k-2}).$$

45. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that $a_{ij} = \begin{cases} 1 & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$ Show that $A^n = \mathbf{O}$ but $A^l \neq \mathbf{O}$ for $1 \leq l \leq n - 1$.

46. Show that the product of two lower triangular matrices is a lower triangular matrix. (A similar statement also holds for upper triangular matrices.)

47. Let A and B be two anti-symmetric matrices such that $AB = BA$. Is the matrix AB symmetric or anti-symmetric?

48. Let A and B be two $m \times n$ matrices. Prove that if $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$ then A is the zero matrix. Further, prove that if $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ then $A = B$.

49. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$. Show that there exist infinitely many matrices B such that $BA = I_2$. Also, show that there does not exist any matrix C such that $AC = I_3$.

50. Show that if A is a complex triangular matrix and $AA^* = A^*A$, then A is a diagonal matrix. [Recall that if $A = [a_{ij}]$ then $A^* = [b_{ij}]$, where $b_{ij} = \overline{a_{ji}}$ for $1 \leq i, j \leq n$.]

51. Using mathematical induction, prove that if $A_1, A_2, \ldots, A_n$ are invertible matrices of the same size then the product $A_1 A_2 \ldots A_n$ is also invertible and that $(A_1 A_2 \ldots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \ldots A_1^{-1}$ for all $n \geq 1$.

52. Give a counterexample to show that $(AB)^{-1} \neq A^{-1}B^{-1}$ in general, where A and B are two invertible matrices of the same size. Find a necessary and sufficient condition such that $(AB)^{-1} = A^{-1}B^{-1}$.

53. Give a counterexample to show that $(A + B)^{-1} \neq A^{-1} + B^{-1}$ in general, where A and B are two matrices of the same size such that each of A, B and $A + B$ are invertible.

54. Show that if A is a square matrix that satisfy the equation $A^2 - 2A + I = \mathbf{O}$, then A is invertible and $A^{-1} = 2I - A$.

55. Solve each of the following matrix equations for X:

$$XA^2 = A^{-1},\ (A^{-1}X)^{-1} = A(B^{-2}A)^{-1} \text{ and } ABXA^{-1}B^{-1} = I + A.$$

56. Let A be a matrix such that $AA^t = \mathbf{O}$. Show that $A = \mathbf{O}$. Is the same true if A is a complex matrix?

57. Let A be an invertible matrix. Show that no row or column of A can be entirely zero.

58. Find the inverse of the following matrices, whenever they exist, preferably using Gauss-Jordan method:

$$\begin{bmatrix} 2+i & 6 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 2 \\ -6 & 3 & 1 \\ -7 & -2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & a & 1 & 0 \\ a^3 & a^2 & a & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & -6 & -2 \end{bmatrix},$$

where x, y, z are distinct real numbers.

59. Find the inverse of the following matrices, where a, b, c, d are all non-zero real numbers:

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \end{bmatrix}.$$

60. Find the inverse of the following matrices using adjoint method:

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

where x, y, z are distinct real numbers.

61. Find a 3×3 real matrix A such that $A\mathbf{u}_1 = \mathbf{u}_1, A\mathbf{u}_2 = 2\mathbf{u}_2$ and $A\mathbf{u}_3 = 3\mathbf{u}_3$, where

$$\mathbf{u}_1 = [1, 2, 2]^t,\ \mathbf{u}_2 = [2, -2, 1]^t \text{ and } \mathbf{u}_3 = [-2, -1, 2]^t.$$

62. If A is an idempotent matrix (*i.e.*, $A^2 = A$), then find all possible values of $\det(A)$.

63. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that $a_{ij} = \max\{i, j\}$. Find $\det(A)$.

64. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that $a_{ij} = j^{i-1}$. Show that

$$\det(A) = (n-1)(n-2)^2(n-3)^3 \ldots 3^{n-3}2^{n-2}.$$

65. Let $A = [\mathbf{a}, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4]$ and $B = [\mathbf{b}, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4]$ be two 4×4 matrices, where $\mathbf{a}, \mathbf{b}, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4 \in \mathbb{R}^4$. If $\det(A) = 4$ and $\det(B) = 1$, find $\det(A + B)$. What is $\det(C)$, where $C = [\mathbf{r}_4, \mathbf{r}_3, \mathbf{r}_2, \mathbf{a} + \mathbf{b}]$?

66. Let A be an $n \times n$ matrix.

    (a) Show that if $A^2 + I = \mathbf{O}$ then n must be an even integer.

    (b) Does (a) remain true for complex matrices?

67. Let A be an $n \times n$ matrix. If $AA^t = I$ and $\det(A) < 0$, then find $\det(A + I)$.

68. Let A, B, C, D be $n \times n$ matrices such that $ABCD = I$. Show that

$$ABCD = DABC = CDAB = BCDA = I.$$

69. If A is a matrix satisfying $A^3 = 2I$ then show that the matrix B is invertible, where $B = A^2 - 2A + 2I$.

70. Show that if $a \neq b$ then $\det(A) = \frac{a^{n+1}-b^{n+1}}{a-b}$, where A is an $n \times n$ matrix given as follows:

$$A = \begin{bmatrix} a+b & ab & 0 & \dots & 0 & 0 \\ 1 & a+b & ab & \dots & 0 & 0 \\ 0 & 1 & a+b & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a+b & ab \\ 0 & 0 & 0 & \dots & 1 & a+b \end{bmatrix}.$$

If $a = b$ then what happens to the value of $\det(A)$?

71. The position of the (i, j)-th entry a_{ij} of an $n \times n$ matrix $A = [a_{ij}]$ is called even or odd according as $i+j$ is even or odd.

    (a) Let B be the matrix obtained from multiplying all the entries of A in odd positions by -1. That is, if $B = [b_{ij}]$ then $b_{ij} = a_{ij}$ or $b_{ij} = -a_{ij}$ according as $i + j$ is even or odd. Show that $\det(B) = \det(A)$.

    (b) Let $C = [c_{ij}]$ be the matrix such that $c_{ij} = -a_{ij}$ or $c_{ij} = a_{ij}$ according as $i + j$ is even or odd. Show that $\det(C) = \det(A)$ or $\det(C) = -\det(A)$ according as n is even or odd.

72. Let $\lambda \neq 0$. Determine $|\lambda I - A|$ for the 10×10 matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 10^{10} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

73. Show that an upper triangular (square) matrix is invertible if and only if every entry on its main diagonal is non-zero.

74. Let A and B be two $n \times n$ matrices. Show that if $AB = A \pm B$ then $AB = BA$.

75. Let A and B be two non-singular matrices of the same size. Are $A + B, A - B$ and $-A$ non-singular? Justify.

76. Each of the numbers $1375, 1287, 4191$ and 5731 is divisible by 11. Show, without calculating the actual value, that the determinant of the matrix $\begin{bmatrix} 1 & 1 & 4 & 5 \\ 3 & 2 & 1 & 7 \\ 7 & 8 & 9 & 3 \\ 5 & 7 & 1 & 1 \end{bmatrix}$ is also divisible by 11.

77. Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $a_{ij} = \frac{1}{i+j}$ for all i, j. Show that A is invertible.

78. Let A be the $n \times n$ matrix all of whose main diagonal entries are 0, and elsewhere 1, *i.e.*, $a_{ii} = 0$ for $1 \leq i \leq n$ and $a_{ij} = 1$ for $i \neq j$. Show that A is invertible and find A^{-1}.

79. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that $\sum_{j=1}^{n} a_{ij} = 1$ for all $i = 1, 2, \dots, n$. Is it true that $\det(A - I_n) = 0$? Justify your answer.

80. Let A be an $n \times n$ matrix. Show that there exists an $n \times n$ non-zero matrix B such that $AB = \mathbf{O}$ if and only if $\det(A) = 0$.

81. Let a_{ij} be integers, where $1 \leq i, j \leq n$. If for any set of integers $b_1, b_2, \dots, b_n$, the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_j \quad \text{for} \quad i = 1, 2, \dots, n,$$

has integer solution $[x_1, x_2, ..., x_n]^t$ then show that $\det(A) = \pm 1$, where $A = [a_{ij}]$.

82. For a complex matrix $A = [a_{ij}]$, let $\overline{A} = [\overline{a_{ij}}]$ and $A^* = \overline{A}^t$. Show that $\det(\overline{A}) = \det(A^*) = \overline{\det(A)}$. Hence show that if A is Hermitian (*i.e.*, $A^* = A$) then $\det(A)$ is a real number.

83. A matrix A is said to be orthogonal if $AA^t = I = A^tA$. Show that if A is orthogonal then $\det(A) = \pm 1$.

84. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

   (a) Find elementary matrices E_1 and E_2 such that $E_2E_1A = I_2$.

   (b) Write A and A^{-1} as a product of elementary matrices.

85. Let A and B be two $n \times n$ matrices and let B be invertible. If $\mathbf{b} \in \mathbb{R}^n$ then show that the system of equations $A\mathbf{x} = \mathbf{b}$ and $BA\mathbf{x} = B\mathbf{b}$ are equivalent.

86. Let $A = [a_{ij}]$ be a 5×5 invertible matrix such that $\sum\limits_{j=1}^{5} a_{ij} = 1$ for $i = 1, 2, 3, 4, 5$. Show that the sum of all the entries of A^{-1} is 5.

87. Prove that if a matrix A is row equivalent to B, then there exists a non-singular matrix P such that $B = PA$. Further, show that

   (a) if A is $n \times n$ and non-singular then P is unique; and

   (b) if A is $n \times n$ and singular then P need not be unique.

88. Find non-singular matrices P and Q such that $B = PAQ$, where

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 1 & 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

89. Let $A = [a_{ij}]$ be an $n \times n$ complex matrix. Show that

   (a) if A is skew-Hermitian (*i.e.*, $\overline{A}^t = -A$) and n is even then $\det(A)$ is real;

   (b) if A is anti-symmetric (*i.e.*, $A^t = -A$) and n is odd then $\det(A) = 0$;

   (c) if A is invertible and $A^{-1} = A^t$ then $\det(A) = \pm 1$; and

   (d) if A is invertible and $A^{-1} = \overline{A}^t$ then $|\det(A)| = \pm 1$.

90. Show that the inverse of an invertible Hermitian matrix (*i.e.*, $A = A^*$) is Hermitian. Also, show that the product of two Hermitian matrices is Hermitian if and only if they commute.

91. Suppose A is a 2×1 matrix and B is an 1×2 matrix. Prove that the matrix AB is not invertible. When is the matrix BA invertible?

92. Let A be an $m \times n$ matrix. Show that, by means of a finite number of elementary row and/or column operations, one can transform A to a matrix $R = [r_{ij}]$ which is both 'reduced row echelon form' and 'reduced column echelon form', *i.e.*, $r_{ij} = 0$ if $i \neq j$, and there is a $k \in \{1, 2, \ldots, n\}$ such that $r_{ii} = 1$ if $1 \leq i \leq k$ and $r_{ii} = 0$ if $i > k$. Show that $R = PAQ$, where P and Q are invertible matrices of sizes $m \times m$ and $n \times n$, respectively.

93. Prove or disprove: The matrix A, as given below, is invertible and all entries of A^{-1} are integers.

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}.$$

94. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 2 & \ldots & 2 \\ 1 & 2 & 3 & \ldots & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \ldots & n \end{bmatrix}.$$

95. Find the determinant and the inverse of the $n \times n$ matrix

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}.$$

96. Assuming that all matrix inverses involved below exist, show that

$$(A - B)^{-1} = A^{-1} + A^{-1}(B^{-1} - A^{-1})^{-1}A^{-1}.$$

In particular, show that

$$(I + A)^{-1} = I - (A^{-1} + I)^{-1} \text{ and } |(I + A)^{-1} + (I + A^{-1})^{-1}| = 1.$$

97. Let S be the backward identity matrix; that is,

$$S = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Show that $S^{-1} = S^t = S$. Find $\det(S)$ and SAS for $A = [a_{ij}] \in M_n(\mathbb{R})$.

98. Determine the reduced row echelon form for each of the following matrices. Hence, find a basis for each of the corresponding row spaces, column spaces and the null spaces.

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 6 & 2 \\ -1 & 2 & 4 & 3 \\ 1 & 2 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 5 & -6 \\ 2 & 3 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 5 & -5 & 5 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 2 & 2 & 2 & 4 \\ 2 & -2 & 4 & 0 & 8 \\ 4 & 2 & 5 & 6 & 10 \end{bmatrix}.$$

99. Prove that if R is an echelon form of a matrix A, then the non-zero rows of R form a basis of $\text{row}(A)$.

100. Give examples to show that the column space of two row equivalent matrices need not be the same.

101. Find a matrix whose row space contains the vector $[1, 2, 1]^t$ and whose null space contains the vector $[1, -2, 1]^t$, or prove that there is no such matrix.

102. If a matrix A has rank r then prove that A can be written as the sum of r matrices, each of which has rank 1.

103. If the rank of the matrix $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & t & 0 \\ 0 & -4 & 5 & 2 \end{bmatrix}$ is 2, find all the possible values of t.

104. For what values of t is the rank of the matrix $\begin{bmatrix} t & 1 & 1 & 1 \\ 1 & t & 1 & 1 \\ 1 & 1 & t & 1 \\ 1 & 1 & 1 & t \end{bmatrix}$ equal to 3?

105. Let A and B be two $n \times n$ matrices. Show that if $AB = \mathbf{O}$ then $\text{rank}(A) + \text{rank}(B) \leq n$.

106. Let A be an $n \times n$ matrix such that $A^2 = A^3$ and $\text{rank}(A) = \text{rank}(A^2)$. Show that $A = A^2$. Also, show that the condition $\text{rank}(A) = \text{rank}(A^2)$ cannot be dropped even for a 2×2 matrix.

107. If B is a sub matrix of a matrix A obtained by deleting s rows and t columns from A, then show that $\text{rank}(A) \leq s + t + \text{rank}(B)$.

108. Let A be an $n \times n$ matrix. Show that $A^2 = A$ if and only if $\text{rank}(A) + \text{rank}(A - I) = n$.

109. Find the values of $\lambda \in \mathbb{R}$ for which $\beta = [0, \lambda, \lambda^2]^t$ belongs to the column space of A, where

$$A = \begin{bmatrix} 1+\lambda & 1 & 1 \\ 1 & 1+\lambda & 1 \\ 1 & 1 & 1+\lambda \end{bmatrix}.$$

110. Let A be a square matrix. If $\text{rank}(A) = \text{rank}(A^2)$, show that the linear systems of equations $A\mathbf{x} = \mathbf{0}$ and $A^2\mathbf{x} = \mathbf{0}$ have the same solution space.

111. Let A be a $p \times n$ matrix and B be a $q \times n$ matrix. If $\text{rank}(A) + \text{rank}(B) < n$ then show that there exists an $\mathbf{x}\ (\neq \mathbf{0}) \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$.

112. Let A and B be $n \times n$ matrices such that $\text{nullity}(A) = l$ and $\text{nullity}(B) = m$. Show that $\text{nullity}(AB) \geq \max(l, m)$. When does it happen that every $\mathbf{x} \in \mathbb{R}^n$ is either in $\text{null}(A)$ or $\text{null}(B)$?

113. Let A be an $m \times n$ matrix of rank r. Show that A can be expressed as $A = BC$, where each of B and C have rank r, B is a matrix of size $m \times r$ and C is a matrix of size $r \times n$.

114. Let A and B be two matrices such that AB is defined and $\text{rank}(A) = \text{rank}(AB)$. Show that $A = ABX$ for some matrix X. Similarly, if BA is defined and $\text{rank}(A) = \text{rank}(BA)$ then show that $A = YBA$ for some matrix Y.

115. Let A be an $n \times n$ matrix and S be an $n \times n$ invertible matrix. Show that the eigenvalues of A and $S^{-1}AS$ are the same. Are their corresponding eigenvectors same?

116. Find the eigenvalues and the corresponding eigenvectors of the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & i \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ 2 & 4 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -2 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

117. Show that the following matrices A, B and C are diagonalizable. Also, find invertible matrices S_1, S_2 and S_3 such that $S_1^{-1}AS_1, S_2^{-1}BS_2$ and $S_3^{-1}CS_3$ are all diagonal matrices.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & -2 \\ 0 & 3 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

118. Prove that the following matrices A and B are similar by showing that they are similar to the same diagonal matrix. Also, find an invertible matrix P such that $P^{-1}AP = B$.

(a) $A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$

(b) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 2 & -5 \\ 1 & 2 & -1 \\ 2 & 2 & -4 \end{bmatrix}.$

119. Let A and B be invertible matrices of the same size. Show that the matrices AB and BA are similar.

120. Let A and B be two similar matrices and let λ be an eigenvalue of A and B. Prove that the algebraic (geometric) multiplicity of λ in A is equal to the algebraic (geometric) multiplicity of λ in B.

121. Show that the matrices $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & i \\ i & 0 \end{bmatrix}$ are not diagonalizable.

122. Let A be a symmetric matrix. Show that the eigenvalues of A are real numbers.

123. Let A be an anti-symmetric matrix of odd order. Prove that 0 is an eigenvalue of A.

124. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that $\det(A) = \lambda_1.\lambda_2.\ldots.\lambda_n$ and $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. Further, show that A is invertible if and only if none of the eigenvalues of A are zero.

125. Let A and B be two $n \times n$ matrices. Prove that the sum of all the eigenvalues of $A + B$ is the sum of all the eigenvalues of A and B individually. Also, prove that the product of all the eigenvalues of AB is the product of all the eigenvalues of A and B individually.

126. Prove or disprove: If $A = \begin{bmatrix} 5 & 4 & 14 & 0 \\ 4 & 13 & 14 & 0 \\ 14 & 14 & 49 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ then there exists a symmetric matrix B such that $A = B^{52}$.

127. Let A and B be two $n \times n$ matrices with eigenvalues λ and μ, respectively.

(a) Give an example to show that $\lambda + \mu$ need not be an eigenvalue of $A + B$.

(b) Give an example to show that $\lambda\mu$ need not be an eigenvalue of AB.

(c) Suppose that λ and μ correspond to the same eigenvector $\mathbf{x}$. Show that $\lambda + \mu$ is an eigenvalue of $A + B$ and $\lambda\mu$ is an eigenvalue of AB.

128. Let A be an $n \times n$ matrix and let c $(\neq 0)$ be a constant. Show that λ is an eigenvalue of A if and only if $c\lambda$ is an eigenvalue of cA.

129. Let A be an $n \times n$ matrix. Show that A and A^t have the same eigenvalues. Are their corresponding eigenspaces same?

130. Let A be a $n \times n$ matrix. Show that the eigenvalues of A are either real numbers or complex conjugates occurring in pairs. Further, show that if the order of A is odd then A has at least one real eigenvalue.

131. Let A be an $n \times n$ complex matrix. Show that

(a) if A is Hermitian (*i.e.*, $A^* = \overline{A}^t = A$) then all eigenvalues of A are real numbers; and

(b) if A is skew-Hermitian (*i.e.*, $A^* = \overline{A}^t = -A$) and $A \neq \mathbf{O}$ then all the eigenvalues of A are purely imaginary numbers.

132. Let A be an $n \times n$ matrix. Show that

(a) if A is idempotent (*i.e.*, $A^2 = A$) then all the eigenvalues of A are either 0 or 1; and

(b) if A is nilpotent (*i.e.*, $A^m = \mathbf{0}$ for some $m \geq 1$) then all the eigenvalues of A are 0.

133. Let A be an $n \times n$ matrix. Prove that

(a) if λ $(\neq 0)$ is an eigenvalue of A then $\frac{1}{\lambda}\det(A)$ is an eigenvalue of $\operatorname{adj}(A)$; and

(b) if $\mathbf{v}$ is an eigenvector of A then $\mathbf{v}$ is also an eigenvector of $\operatorname{adj}(A)$.

134. Let A and B be two $n \times n$ matrices and let A be invertible. Show that the matrices BA^{-1} and $A^{-1}B$ have the same eigenvalues.

135. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (which are not necessarily real). Denote $\lambda_k = x_k + iy_k$ for $k = 1, 2, \ldots, n$. Show that

(a) $y_1 + y_2 + \ldots + y_n = 0$;

(b) $x_1y_1 + x_2y_2 + \ldots + x_ny_n = 0$; and

(c) $\operatorname{tr}(A^2) = (x_1^2 + x_2^2 + \ldots + x_n^2) - (y_1^2 + y_2^2 + \ldots + y_n^2)$.

136. For each of the following matrices, find the eigenvalues and the corresponding eigenspaces over $\mathbb{C}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}, \begin{bmatrix} i & 1+i \\ -1+i & i \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

137. Using **Cayley Hamilton Theorem**, find the inverse of the following matrices, whenever they exist:

$$\begin{bmatrix} 1 & i \\ i & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

138. Find all real values of k for which the following matrices are diagonalizable.

$$\begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix}, \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & k \\ 1 & 1 & k \\ 1 & 1 & k \end{bmatrix}.$$

139. Prove that if A and B are similar matrices then $\text{tr}(A) = \text{tr}(B)$.

140. For any real numbers a, b and c, show that the matrices

$$A = \begin{bmatrix} b & c & a \\ c & a & b \\ a & b & c \end{bmatrix}, \quad B = \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix} \text{ and } C = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

are similar to each other. Moreover, if $BC = CB$ then show that A has two zero eigenvalues.

141. Prove that if the matrix A is similar to B, then A^t is similar to B^t.

142. Prove that if the matrix A is diagonalizable, then A^t is also diagonalizable.

143. Let A be an invertible matrix. Prove that if A is diagonalizable, then A^{-1} is also diagonalizable.

144. Let A be a diagonalizable matrix and let $A^m = \mathbf{O}$ for some $m \geq 1$. Show that $A = \mathbf{O}$.

145. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

    (a) Prove that A is diagonalizable if $(a-d)^2 + 4bc > 0$ and A is not diagonalizable if $(a-d)^2 + 4bc < 0$.

    (b) Find two examples to demonstrate that if $(a-d)^2 + 4bc = 0$, then A may or may not be diagonalizable.

146. Let A be a 2×2 matrix with eigenvectors $v_1 = [1, -1]^t$ and $v_2 = [1, 1]^t$ and corresponding eigenvalues $\frac{1}{2}$ and 2, respectively. Find $A^{10}\mathbf{x}$ and $A^k\mathbf{x}$ for $k \geq 1$, where $\mathbf{x} = [5, 1]^t$. What happens if k becomes large (i.e., $k \to \infty$.)

Definition: Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ be a polynomial. Then the companion matrix of $p(x)$ is the $n \times n$ matrix

$$C(p) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

147. Show that the companion matrix $C(p)$ of $p(x) = x^2 + ax + b$ has characteristic polynomial $\lambda^2 + a\lambda + b$. Also, show that if λ is an eigenvalue of this companion matrix then $[\lambda, 1]^t$ is an eigenvector of $C(p)$ corresponding to the eigenvalue λ.

148. Show that the companion matrix $C(p)$ of $p(x) = x^3+ax^2+bx+c$ has characteristic polynomial $-(\lambda^3+a\lambda^2+b\lambda+c)$. Further, show that if λ is an eigenvalue of this companion matrix then $[\lambda^2, \lambda, 1]^t$ is an eigenvector of $C(p)$ corresponding to λ.

149. Use mathematical induction to show that for $n \geq 2$, the companion matrix $C(p)$ of $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ has characteristic polynomial $(-1)^n p(\lambda)$. Further, show that if λ is an eigenvalue of this companion matrix then $[\lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda, 1]^t$ is an eigenvector of $C(p)$ corresponding to λ.

150. Let A be an $n \times n$ non-singular matrix and let $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$ be its characteristic polynomial. Show that $A^{-1} = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1 I)$.

151. Let A and B be two 2×2 real matrices for which $\det(A) = \det(B)$ and $\text{tr}(A) = \text{tr}(B)$.

(a) Do A and B have the same set of eigenvalues?

(b) Give examples to show that the matrices A and B need not be similar.

152. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and let $a, b \in \mathbb{R}$. Show that $(a\mathbf{x} + b\mathbf{y}).\mathbf{z} = a(\mathbf{x}.\mathbf{z}) + b(\mathbf{y}.\mathbf{z})$ and $\mathbf{x}.(a\mathbf{y} + b\mathbf{z}) = a(\mathbf{x}.\mathbf{y}) + b(\mathbf{x}.\mathbf{z})$.

153. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, prove the following:

(a) $\mathbf{x}.\mathbf{y} = 0$ if and only if $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$;

(b) $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$;

(c) $\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 = 4(\mathbf{x}.\mathbf{y})$;

(d) $\|\mathbf{x}\| = \|\mathbf{y}\|$ if and only if $(\mathbf{x} + \mathbf{y}).(\mathbf{x} - \mathbf{y}) = 0$; and

(e) $\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$ if and only if $\|s\mathbf{x} + t\mathbf{y}\| = s\|\mathbf{x}\| + t\|\mathbf{y}\|$ for all $s, t \geq 0$.

154. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal set of vectors in $\mathbb{R}^n$. Prove that

$$\left\|\sum_{i=1}^{k} \mathbf{v}_i\right\|^2 = \sum_{i=1}^{k} \|\mathbf{v}_i\|^2.$$

155. Find the orthogonal complements $S^\perp, M^\perp$ and $W^\perp$, and find a basis for each of them, where

$$S = \{[x, y, z]^t \in \mathbb{R}^3 : 2x - y + 3z = 0\}, \quad M = \{[x, y, z]^t \in \mathbb{R}^3 : x = 2t = y, z = -t, t \in \mathbb{R}\}$$

$$\text{and } \; W = \{[x, y, z]^t \in \mathbb{R}^3 : x = t, y = -t, z = 3t, t \in \mathbb{R}\}.$$

156. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Prove that the matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ is invertible, and compute its inverse.

157. Let M and N be two subspaces of $\mathbb{R}^n$. Show that $(M + N)^\perp = M^\perp \cap N^\perp$ and $(M \cap N)^\perp = M^\perp + N^\perp$.

158. Let A be a symmetric matrix. Show that the eigenvectors corresponding to distinct eigenvalues of A are orthogonal to each other.

159. If Q is an orthogonal matrix, prove that any matrix obtained by rearranging the rows of Q is also orthogonal.

160. Prove that the columns of an $m \times n$ matrix Q form an orthonormal set if and only if $Q^tQ = I_n$.

161. Prove that a square matrix Q is orthogonal if and only if $Q^{-1} = Q^t$.

162. Prove that if an upper triangular matrix is orthogonal, then it must be a diagonal matrix.

163. Let Q be an $n \times n$ matrix. Prove that the following statements are equivalent:

(a) Q is orthogonal.

(b) $(Q\mathbf{x}).(Q\mathbf{y}) = \mathbf{x}.\mathbf{y}$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(c) $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{R}^n$.

164. Prove that if $n > m$, then there is no $m \times n$ matrix A such that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

165. Apply Gram-Schmidt process to the following sets to obtain an orthonormal set in the corresponding spaces:

(a) $\{[1, 1, 0]^t, [3, 4, 2]^t\}$ in $\mathbb{R}^3$;

(b) $\{[-1, 0, 1]^t, [1, -1, 0]^t, [0, 0, 1]^t\}$ in $\mathbb{R}^3$;

(c) $\{[2, -1, 1, 2]^t, [3, -1, 0, 4]^t\}$ in $\mathbb{R}^4$; and

(d) $\{[1, 1, 1, 1]^t, [0, 2, 0, 2]^t, [-1, 1, 3, -1]^t\}$ in $\mathbb{R}^4$.

166. Use Gram-Schmidt process to find an orthogonal basis for the column space of each of the following matrices:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}.$$

167. Find an orthogonal basis for $\mathbb{R}^3$ containing the vector $[3, 1, 5]^t$.

168. Find an orthogonal basis for $\mathbb{R}^4$ containing the vectors $[2, 1, 0, -1]^t$ and $[1, 0, 3, 2]^t$.

169. Find an orthogonal basis for the subspace spanned by $[1, 1, 0, 1]^t, [-1, 1, 1, -1]^t$, $[0, 2, 1, 0]^t$ and $[1, 0, 0, 0]^t$.

170. Find the orthogonal projection of $\mathbf{v}$ onto the subspace W spanned by the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, where

$$\mathbf{v} = [1, 2, 3]^t,\ \mathbf{u}_1 = [2, -2, 1]^t \text{ and } \mathbf{u}_2 = [-1, 1, 4]^t.$$

171. Find the orthogonal projection of $\mathbf{v}$ onto the subspace W spanned by the vectors $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$, where

$$\mathbf{v} = [3, -2, 4, -3]^t,\ \mathbf{u}_1 = [1, 1, 0, 0]^t, \mathbf{u}_2 = [1, -1, -1, 1]^t \text{ and } \mathbf{u}_3 = [0, 0, 1, 1]^t.$$

172. Let M be a subspace of $\mathbb{R}^m$ and let $\dim(M) = k$. How many linearly independent vectors can be orthogonal to M? Justify your answer.

173. Let A be an $n \times n$ orthogonal matrix. Show that the rows of A form an orthonormal basis for $\mathbb{R}^n$. Similarly, the columns of A also form an orthonormal basis for $\mathbb{R}^n$.

174. Let $\mathbf{u} = [1, 0, 0, 0]^t$ and $\mathbf{v} = [0, \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}]^t$. Find vectors $\mathbf{w}$ and $\mathbf{x}$ in $\mathbb{R}^4$ such that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ form an orthonormal basis for $\mathbb{R}^4$.

175. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthogonal basis for $\mathbb{R}^n$ and let $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ for some $1 \leq k < n$. Is it necessarily true that $W^\perp = \text{span}(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$? Either prove that it is true or find a counterexample.

176. Let W be a subspace of $\mathbb{R}^n$ and let $\mathbf{x} \in \mathbb{R}^n$. Prove that
   (a) $\mathbf{x} \in W$ if and only if $\text{proj}_W(\mathbf{x}) = \mathbf{x}$;
   (b) $\mathbf{x}$ is orthogonal to W if and only if $\text{proj}_W(\mathbf{x}) = \mathbf{0}$; and
   (c) $\text{proj}_W(\text{proj}_W(\mathbf{x})) = \text{proj}_W(\mathbf{x})$.

177. Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$. Give an example of a subset of $\mathbb{R}^2$ which is
   (a) closed under addition but not closed under scalar multiplication; and
   (b) closed under scalar multiplication but not closed under addition.

178. Let S be a non-empty set and let V be the set of all functions from S to $\mathbb{R}$. Show that V is a vector space with respect to addition $(f + g)(x) = f(x) + g(x)$ for $f, g \in V$ and scalar multiplication $(c.f)(x) = cf(x)$ for $c \in \mathbb{R}, f \in V$.

179. Show that the space of all real (respectively complex) matrices is a vector space over $\mathbb{R}$ (respectively $\mathbb{C}$) with respect to usual addition and scalar multiplication of matrices.

180. A set V and the operation of vector addition and scalar multiplication are given below. Examine whether V is a vector space over $\mathbb{R}$.
   (a) $V = \mathbb{R}^2$ and for $[x, y]^t, [z, w]^t \in V, a \in \mathbb{R}$ define $[x, y]^t + [z, w]^t = [x + z, y + w]^t$ and $a.[x, y]^t = [ax, 0]^t$.
   (b) $V = \mathbb{R}^2$ and for $[x, y]^t, [z, w]^t \in V, a \in \mathbb{R}$ define $[x, y]^t + [z, w]^t = [x + z, 0]^t$ and $a.[x, y]^t = [ax, 0]^t$.
   (c) $V = \mathbb{R}$ and for $x, y \in V, a \in \mathbb{R}$ define $x \oplus y = x - y$ and $a \odot x = -ax$.
   (d) $V = \mathbb{R}^2$ and for $[x, y]^t, [z, w]^t \in V, a \in \mathbb{R}$ define $[x, y]^t + [z, w]^t = [x + z, y + w]^t$ and $a.[x, y]^t = [ax, y]^t$.
   (e) $V = \mathbb{R}^2$ and for $[x, y]^t, [z, w]^t \in V, a \in \mathbb{R}$ define $[x, y]^t + [z, w]^t = [x - z, y - w]^t$ and $a.[x, y]^t = [-ax, -ay]^t$.
   (f) $V = \{[x, 3x + 1]^t : x \in \mathbb{R}\}$ with usual addition and scalar multiplication in $\mathbb{R}^2$.
   (g) $V = \{[x, mx + c]^t : x \in \mathbb{R}\}$, where m and c are some fixed real numbers, with usual addition and scalar multiplication in $\mathbb{R}^2$.
   (h) $V = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is a function such that } f(-t) = \overline{f(t)} \text{ for all } t \in \mathbb{R}\}$, with usual addition and scalar multiplication of functions (as defined in Problem 178). Also, find a function in V whose range is contained in $\mathbb{R}$.

(i) $V = M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$ with usual addition and scalar multiplication of matrices.

181. Let $V = \mathbb{R}^+$ (the set of positive real numbers). This is **not** a vector space under usual operations of addition and scalar multiplication (why?). Define a new vector addition and scalar multiplication as

$$v_1 \oplus v_2 = v_1.v_2 \quad \text{and} \quad \alpha \odot v = v^\alpha,$$

for all $v_1, v_2, v \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$. Show that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$ with 1 as the additive identity.

182. Let $V = \mathbb{R}^2$. Define $[x_1, x_2]^t \oplus [y_1, y_2]^t = [x_1 + y_1 + 1, x_2 + y_2 - 3]^t, \alpha \odot [x_1, x_2]^t = [\alpha x_1 + \alpha - 1, \alpha x_2 - 3\alpha + 3]^t$ for $[x_1, x_2]^t, [y_1, y_2]^t \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Verify that the vector $[-1, 3]^t$ is the additive identity, and V is indeed a vector space over $\mathbb{R}$.

183. Let V and W be vector spaces over $\mathbb{R}$ with binary operations $(+, \bullet)$ and $(\oplus, \odot)$, respectively. Consider the following operations on the set $V \times W$: for $(x_1, y_1), (x_2, y_2) \in V \times W$ and $\alpha \in \mathbb{R}$, define

$$(x_1, y_1) \ominus (x_2, y_2) = (x_1 + x_2, y_1 \oplus y_2) \quad \text{and} \quad \alpha \circ (x_1, y_1) = (\alpha \bullet x_1, \alpha \odot y_1).$$

With the above definitions, show that $V \times W$ is also a vector space over $\mathbb{R}$.

184. Show that $\mathbb{R}$ is a vector space over $\mathbb{R}$ with additive identity 1 with respect to the addition $\oplus$ and scalar multiplication $\otimes$ defined as follows:

$$x \oplus y = x + y - 1 \text{ for } x, y \in \mathbb{R} \quad \text{and} \quad a \otimes x = a(x - 1) + 1 \text{ for } a, x \in \mathbb{R}.$$

185. Let $V = \mathbb{C}[x]$ be the set of all polynomials with complex coefficients. Show that V is a vector space over $\mathbb{C}$ with respect to the following addition and scalar multiplication: If $p(x) = \sum\limits_{i=0}^{n} a_i x^i, q(x) = \sum\limits_{i=0}^{n} b_i x^i \in V$ and $c \in \mathbb{C}$, then

$$p(x) + q(x) = \sum_{i=0}^{n} (a_i + b_i) x^i \text{ and } c.p(x) = \sum_{i=0}^{n} (c a_i) x^i.$$

186. Prove that every vector space has a **unique** zero vector.

187. Prove that for every vector $\mathbf{v}$ in a vector space V, there is a **unique** $\mathbf{v}' \in V$ such that $\mathbf{v} + \mathbf{v}' = \mathbf{0}$.

188. Determine whether $s(x) = 3 - 5x - x^2$ is in $\text{span}(p(x), q(x), r(x))$, where $p(x) = 1 - 2x$, $q(x) = x - x^2$ and $r(x) = -2 + 3x + x^2$.

189. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a linearly independent set of vectors in a vector space V. If $\mathbf{v} \in V$ such that $\mathbf{v} \notin \text{span}(S)$ then show that the set $S' = \{\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is also linearly independent.

190. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a spanning set for a vector space V. Show that if $\mathbf{u}_k \in \text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_{k-1})$ then $\text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}) = V$.

191. Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$) and let $\emptyset \neq S \subseteq V$. Let $\mathcal{A} = \{W \mid S \subseteq W, W \text{ is a subspace of V}\}$. Show that $\text{span}(S) = \bigcap\limits_{W \in \mathcal{A}} W$ (*i.e.*, $\text{span}(S)$ is the smallest subspace of V containing S).

192. Prove or disprove:

(a) If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ are linearly dependent vectors in a vector space then anyone of these vectors is a linear combination of the other vectors.

(b) If any $r - 1$ vectors of the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ are linearly independent in a vector space then the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ is also linearly independent.

(c) If $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and if every $\mathbf{u}_i$ $(1 \leq i \leq n)$ is a linear combination of no more than r vectors in $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \setminus \{\mathbf{u}_i\}$ then $\dim V \leq r$.

193. If $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ are linearly independent vectors in a vector space V and the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{v}$ are linearly dependent, then show that $\mathbf{v}$ can be uniquely expressed as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$.

194. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for a vector space V, where $n \geq 2$. Show that $\{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \ldots, \mathbf{u}_1 + \ldots + \mathbf{u}_n\}$ is also a basis for V. How about the converse?

195. Discuss the linear independence/linear dependence of the following sets. For those that are linearly dependent, express one of the vector as a linear combination of the others.

   (a) $\{[1,0,0]^t, [1,1,0]^t, [1,1,1]^t\}$ of $\mathbb{R}^3$.

   (b) $\{[1,0,0,0]^t, [1,1,0,0]^t, [1,2,0,0]^t, [1,1,1,1]^t\}$ of $\mathbb{R}^4$.

   (c) $\{[1,i,0]^t, [1,0,1]^t, [i+2,-1,2]^t\}$ of $\mathbb{C}^3(\mathbb{C})$.

   (d) $\{[1,i,0]^t, [1,0,1]^t, [i+2,-1,2]^t\}$ of $\mathbb{C}^3(\mathbb{R})$.

   (e) $\{1, i\}$ of $\mathbb{C}(\mathbb{C})$ and $S = \{1, i\}$ of $\mathbb{C}(\mathbb{R})$.

   (f) $\{1+x, 1+x^2, 1-x+x^2\}$ of $\mathbb{R}_2[x]$.

   (g) $\left\{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\right\}$ of $M_2(\mathbb{R})$.

196. Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a linearly independent set of vectors in a vector space.

   (a) Is $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$ linearly independent? Either prove that it is or give a counterexample to show that it is not.

   (b) Is $\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{u}-\mathbf{w}\}$ linearly independent? Either prove that it is or give a counterexample to show that it is not.

197. Let A be a lower triangular matrix such that none of the diagonal entries are zero. Show that the row (or column) vectors of A are linearly independent.

198. Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Let $W = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ and $\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - 2\mathbf{v}_2 \in \operatorname{span}(\mathbf{v}_3, \mathbf{v}_4, \ldots, \mathbf{v}_k)$. Prove that $W = \operatorname{span}(\mathbf{v}_3, \mathbf{v}_4, \ldots, \mathbf{v}_k)$.

199. Examine whether the sets $\{1-x, 1-x^2, x-x^2\}$ and $\{1, 1+2x+3x^2\}$ are bases for $\mathbb{R}_2[x]$.

200. Find all values of a for which the set $\{[a^2,0,1]^t, [0,a,2]^t, [1,0,1]^t\}$ is a basis for $\mathbb{R}^3$.

201. Let V be a vector space over $\mathbb{R}$ and let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ such that $\mathbf{x}+2\mathbf{y}+7\mathbf{z} = \mathbf{0}$. Show that $\operatorname{span}(\mathbf{x}, \mathbf{y}) = \operatorname{span}(\mathbf{y}, \mathbf{z}) = \operatorname{span}(\mathbf{z}, \mathbf{x})$.

202. Show that the set $\{[1,0]^t, [i,0]^t\}$ is linearly independent over $\mathbb{R}$ but is linearly dependent over $\mathbb{C}$.

203. Under what conditions on the complex number α are the vectors $[1+\alpha, 1-\alpha]^t$ and $[\alpha-1, 1+\alpha]^t$ in $\mathbb{C}^2(\mathbb{R})$ linearly independent?

204. Let V be a vector space over $\mathbb{C}$ and let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a linearly independent subset of V. Show that the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, i\mathbf{x}_1, i\mathbf{x}_2, \ldots, i\mathbf{x}_k\}$ is linearly independent over $\mathbb{R}$.

205. Examine whether the following sets are subspaces of each of $\mathbb{C}^3(\mathbb{R})$ and $\mathbb{C}^3(\mathbb{C})$:

$$\{[z_1, z_2, z_3]^t \in \mathbb{C}^3 : z_1 \text{ is real}\}, \quad \{[z_1, z_2, z_3]^t \in \mathbb{C}^3 : z_1 + z_2 = 10z_3\},$$

$$\{[z_1, z_2, z_3]^t \in \mathbb{C}^3 : |z_1| = |z_2|\} \quad \text{and} \quad \{[z_1, z_2, z_3]^t \in \mathbb{C}^3 : z_1 + z_2 = 2\overline{z}_3\}.$$

206. Let U and W be two subspaces of a vector space V. Show that $U \cap W$ is also a subspace of V. Give examples to show that $U \cup W$ need not be a subspace of V.

207. Let U and W be subspaces of a vector space V. Define the *linear sum* of U and W to be

$$U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}.$$

   Show that $U+W$ is a subspace of V. If $V = \mathbb{R}^3$, U is the x-axis and W is the y-axis, what is $U+W$ geometrically?

208. Let W be a subspace of a vector space V. Show that $\Delta = \{(\mathbf{w}, \mathbf{w}) : \mathbf{w} \in W\}$ is a subspace of $V \times V$ and that $\dim \Delta = \dim W$.

209. Let U and V be two finite-dimensional vector spaces. Find a formula for $\dim (U \times V)$ in terms of $\dim U$ and $\dim V$.

210. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for a vector space V. Show that the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, c\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n\}$ is also a basis for V, where c is a non-zero scalar.

211. Find a basis for $\text{span}(1-x, x-x^2, 1-x^2, 1-2x+x^2)$ in $\mathbb{R}_2[x]$.

212. Extend $\{1+x, 1+x+x^2\}$ to a basis for $\mathbb{R}_2[x]$.

213. Extend $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ to a basis for $M_2(\mathbb{R})$.

214. Prove or disprove:

(a) Every lineally independent set of a vector space V is a basis for some subspace of V.

(b) If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a lineally dependent subset of a vector space V, then $\mathbf{x}_n \in \text{span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1})$.

(c) Let W_1 and W_2 be two subspaces of a vector space V. If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$ is a basis for W_1 and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a basis for W_2, then $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $W_1 \cap W_2$.

(d) $B = \{x^3, x^3+2, x^2+1, x+1\}$ is a basis for $\mathbb{R}_3[x]$.

215. Find a basis for each of the following subspaces.

(a) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) : a - d = 0 \right\}$.

(b) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) : 2a - c - d = 0, a + 3b = 0 \right\}$.

(c) $\{a + bx + cx^3 \in \mathbb{R}_3[x] : a - 2b + c = 0\}$.

(d) $\{A = [a_{ij}] \in M_{m \times n}(\mathbb{R}) : \sum_{j=1}^{n} a_{ij} = 0 \text{ for } i = 1, \ldots, m\}$.

216. Recall that $M_n(\mathbb{R})$ denote the space of all $n \times n$ real matrices. Show that the following sets are subspaces of $M_n(\mathbb{R})$. Also, find a basis and the dimension of each of these subspaces.

(a) $U_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A \text{ is upper triangular}\}$.

(b) $L_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A \text{ is lower triangular}\}$.

(c) $D_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A \text{ is diagonal}\}$.

(d) $sl(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \text{tr}(A) = 0\}$.

(e) $A_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A + A^t = 0\}$.

217. Let $W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$. Show that W is a subspace of $M_2(\mathbb{R})$ and that the following set form a basis for W:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Find the coordinate of the matrix $\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$ with respect to this basis.

218. Find a basis and the dimension for each of the following vector spaces:

(a) $M_n(\mathbb{C})$ over $\mathbb{C}$;

(b) $M_n(\mathbb{C})$ over $\mathbb{R}$;

(c) $M_n(\mathbb{R})$ over $\mathbb{R}$.

(d) $H_n(\mathbb{C})$ (the space of all $n \times n$ Hermitian matrices) over $\mathbb{R}$; and

(e) $S_n(\mathbb{C})$ (the space of all $n \times n$ skew-Hermitian matrices) over $\mathbb{R}$.

219. Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}) : a + d = 0 \right\}$ and $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V : c = -\bar{b} \right\}$. Show that V is a vector space over $\mathbb{R}$ and find a basis for V. Also, show that W is a subspace of V, and find the dimension of W.

220. Show that $W = \{[x, y, z, t]^t \in \mathbb{R}^4 : x + y + z + 2t = 0 = x - y + z\}$ is a subspace of $\mathbb{R}^4$. Find a basis for W, and extend it to a basis for $\mathbb{R}^4$.

221. Show that $W = \{[v_1, \ldots, v_6]^t \in \mathbb{R}^6 \mid v_1 + v_2 + v_3 = 0, v_2 + v_3 + v_4 = 0, v_5 + v_6 = 0\}$ is a subspace of $\mathbb{R}^6$. Find a basis for W, and extend it to a basis for $\mathbb{R}^6$.

222. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$. Show that $\{\mathbf{v}_1, \alpha\mathbf{v}_1 + \mathbf{v}_2, \alpha\mathbf{v}_1 + \mathbf{v}_3, \ldots, \alpha\mathbf{v}_1 + \mathbf{v}_n\}$ is also a basis of $\mathbb{R}^n$ for every $\alpha \in \mathbb{R}$.

223. Show that the set $\{[1, 0, 1]^t, [1, i, 0]^t, [1, 1, 1 - i]^t\}$ is a basis for $\mathbb{C}^3(\mathbb{C})$.

224. Show that $\{[1, 0, 0]^t, [1, 1, 1]^t, [1, 1, -1]^t\}$ is a basis for $\mathbb{C}^3(\mathbb{C})$. Is it a basis for $\mathbb{C}^3(\mathbb{R})$ as well? If not, extend it to a basis for $\mathbb{C}^3(\mathbb{R})$.

225. Show that $\{1, (x - 1), (x - 1)(x - 2)\}$ is a basis for $\mathbb{R}_2[x]$, and that $W = \{p(x) \in \mathbb{R}_2[x] : p(1) = 0\}$ is a subspace of $\mathbb{R}_2[x]$. Also, find $\dim W$.

226. For each of the following statements, answer true or false with proper justification:

(a) $\{p(x) \in \mathbb{R}_3[x] : p(x) = ax + b\}$ is a subspace of $\mathbb{R}_3[x]$.

(b) $\{p(x) \in \mathbb{R}_3[x] : p(x) = ax^2\}$ is a subspace of $\mathbb{R}_3[x]$.

(c) $\{p(x) \in \mathbb{R}[x] : p(0) = 1\}$ is a subspace of $\mathbb{R}[x]$.

(d) $\{p(x) \in \mathbb{R}[x] : p(0) = 0\}$ is a subspace of $\mathbb{R}[x]$.

(e) $\{p(x) \in \mathbb{R}[x] : p(x) = p(-x)\}$ is a subspace of $\mathbb{R}[x]$.

227. Let B be a set of vectors in a vector space V with the property that every vector in V can be uniquely expressed as a linear combination of the vectors in B. Prove that B is a basis for V.

228. Let W_1 be the set of all real matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and W_2 be the set of all real matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$. Show that W_1 and W_2 are subspaces of $M_2(\mathbb{R})$. Find the dimension for each of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

229. Let W_1 and W_2 be two subspaces of $\mathbb{R}^8$ and $\dim(W_1) = 6, \dim(W_2) = 5$. What are the possible values for $\dim(W_1 \cap W_2)$?

230. Does there exist subspaces M and N of $\mathbb{R}^7$ such that $\dim(M) = 4 = \dim(N)$ and $M \cap N = \{\mathbf{0}\}$? Justify.

231. Let M be an m-dimensional subspace of an n-dimensional vector space V. Prove that there exists an $(n - m)$-dimensional subspace N of V such that $M + N = V$ and $M \cap N = \{\mathbf{0}\}$.

232. Let $W_1 = \{[x, y, z]^t \in \mathbb{R}^3 : 2x + y + 4z = 0\}$ and $W_2 = \{[x, y, z]^t \in \mathbb{R}^3 : x - y + z = 0\}$. Show that W_1 and W_2 are subspaces of $\mathbb{R}^3$, and find a basis for each of $W_1, W_2, W_1 \cap W_2$ and $W_1 + W_2$.

233. Let $W_1 = \text{span}([1, 1, 0]^t, [-1, 1, 0]^t)$ and $W_2 = \text{span}([1, 0, 2]^t, [-1, 0, 4]^t)$. Show that $W_1 + W_2 = \mathbb{R}^3$. Give an example of a vector $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{v}$ can be expressed in two different ways in the form $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2$.

234. Are the vector spaces $M_{6\times4}(\mathbb{R})$ (the space of all 6×4 real matrices) and $M_{3\times8}(\mathbb{R})$ (the space of all 3×8 real matrices) isomorphic? Justify your answer.

235. Let $P = \text{span}([1, 0, 0]^t, [1, 1, 0]^t)$ and $Q = \text{span}([1, 1, 1]^t)$. Show that $\mathbb{R}^3 = P + Q$ and $P \cap Q = \{\mathbf{0}\}$. For an $\mathbf{x} \in \mathbb{R}^3$, find $\mathbf{x}_p \in P$ and $\mathbf{x}_q \in Q$ such that $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_q$. Is the choice of $\mathbf{x}_p$ and $\mathbf{x}_q$ unique? Justify.

236. Let $V = \{f \mid f : \mathbb{R} \to \mathbb{R}$ is a function$\}$, $V_e = \{f \in V : f(-x) = f(x)$ for all $x \in \mathbb{R}\}$ and $V_o = \{f \in V : f(-x) = -f(x)$ for all $x \in \mathbb{R}\}$. Prove that V_e and V_o are subspaces of V, $V_e \cap V_o = \{\mathbf{0}\}$ and $V = V_e + V_o$.

237. Let V_1 and V_2 be two subspaces of a vector space V such that $V = V_1 + V_2$ and $V_1 \cap V_2 = \{\mathbf{0}\}$. Prove that for each vector $\mathbf{v} \in V$, there are unique vectors $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$ such that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. (In such a situation, V is called the direct sum of V_1 and V_2, and is written as $V = V_1 \oplus V_2$.)

238. Find the coordinate of $p(x) = 2 - x + 3x^2$ with respect to the basis $\{1 + x, 1 - x, x^2\}$ of $\mathbb{R}_2[x]$.

239. Find the coordinate of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to the basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ of $M_2(\mathbb{R})$.

240. Consider the bases $B = \{1 + x + x^2, x + x^2, x^2\}$ and $C = \{1, x, x^2\}$ for $\mathbb{R}_2[x]$, and let $p(x) = 1 + x^2$. Find the coordinate vectors $[p(x)]_B$ and $[p(x)]_C$. Also, find the change of basis matrices $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$. Finally, compute $[p(x)]_B$ and $[p(x)]_C$ using the change of basis matrices and compare the result with the initially computed coordinates of $p(x)$.

241. Show that the set $\mathbb{R}_n[x]$ of all real polynomials of degree at most n is a subspace of the vector space $\mathbb{R}[x]$ of all real polynomials. Find a basis for $\mathbb{R}_n[x]$. Also, show that the set of all real polynomials of degree exactly n is not a subspace of $\mathbb{R}[x]$. Further, show that $\{x + 1, x^2 + x - 1, x^2 - x + 1\}$ is a basis for $\mathbb{R}_2[x]$. Finally, find the coordinates of the elements $2x - 1, x^2 + 1$ and $x^2 + 5x - 1$ of $\mathbb{R}_2[x]$ with respect to the above basis.

242. For $1 \le i \le n$, let $\mathbf{x}_i = [0, \ldots, 0, 1, 1, \ldots, 1]^t \in \mathbb{R}^n$ (*i.e.*, the first $(i-1)$ entries of $\mathbf{x}_i$ are 0 and the rest are 1). Show that $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$. Also, find the coordinates of the vectors $[1, 1, \ldots, 1]^t$, $[1, 2, 3, \ldots, n]^t$ and $[0, \ldots, 0, 1, 0, \ldots, 0]^t$ with respect to the basis B.

243. Consider the bases $B = \{[1, 2, 0]^t, [1, 3, 2]^t, [0, 1, 3]^t\}$ and $C = \{[1, 2, 1]^t, [0, 1, 2]^t, [1, 4, 6]^t\}$ for $\mathbb{R}^3$. Find the change of basis matrix P from B to C. Also, find the change of basis matrix Q from C to B. Verify that $PQ = I_3$.

244. Show that the vectors $\mathbf{u}_1 = [1, 1, 0, 0]^t, \mathbf{u}_2 = [0, 0, 1, 1]^t, \mathbf{u}_3 = [1, 0, 0, 4]^t$ and $\mathbf{u}_4 = [0, 0, 0, 2]^t$ form a basis for $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors of $\mathbb{R}^4$ with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

245. Let W be the subspace of $\mathbb{C}^3$ spanned by $\mathbf{u}_1 = [1, 0, i]^t$ and $\mathbf{u}_2 = [1 + i, 1, -1]^t$. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for W. Also, show that $\mathbf{v}_1 = [1, 1, 0]^t$ and $\mathbf{v}_2 = [1, i, 1 + i]^t$ are in W, and $\{\mathbf{v}_1, \mathbf{v}_2\}$ form a basis for W. Finally, find the coordinates of $\mathbf{u}_1$ and $\mathbf{u}_2$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

246. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for an n-dimensional vector space V. Show that $\{a_1\mathbf{u}_1, a_2\mathbf{u}_2, \ldots, a_n\mathbf{u}_n\}$ is also a basis for V, for any non-zero scalars $a_1, a_2, \ldots, a_n$. If the coordinate of a vector $\mathbf{v}$ with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is $\mathbf{x} = [x_1, x_2, \ldots, x_n]^t$, what is the coordinate of $\mathbf{v}$ with respect to the basis $\{a_1\mathbf{u}_1, a_2\mathbf{u}_2, \ldots, a_n\mathbf{u}_n\}$? What are the coordinates of $\mathbf{w} = \mathbf{u}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_n$ with respect to each of the bases $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\{a_1\mathbf{u}_1, a_2\mathbf{u}_2, \ldots, a_n\mathbf{u}_n\}$?

247. Let $A = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a set of vectors in an n-dimensional vectors space V, and let B be a basis for V. Let $S = \{[\mathbf{u}_1]_B, [\mathbf{u}_2]_B, \ldots, [\mathbf{u}_m]_B\}$ be the set of coordinate vectors of A with respect to the basis B. Prove that $\text{span}(A) = V$ if and only if $\text{span}(S) = \mathbb{R}^n$.

248. Consider two bases B and C for $\mathbb{R}_2[x]$. Find C, if $B = \{x, 1 + x, 1 - x + x^2\}$ and the change of basis matrix from B to C is

$$P_{C \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

249. Let V be an n-dimensional vector space with a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let $P = [p_{ij}]$ be an $n \times n$ invertible matrix, and set

$$\mathbf{u}_j = p_{1j}\mathbf{v}_1 + p_{2j}\mathbf{v}_2 + \ldots + p_{nj}\mathbf{v}_n \quad \text{for all } j = 1, 2, \ldots, n.$$

Prove that $C = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for V and $P = P_{B \leftarrow C}$.

250. Examine whether the following maps $T : V \to W$ are linear transformations.

(a) $V = W = \mathbb{R}^3$ and $T[x, y, z]^t = [3x + y, z, |x|]^t$ for all $[x, y, z]^t \in \mathbb{R}^3$.

(b) $V = W = M_2(\mathbb{R})$ and for every $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$, define **(i)** $T(A) = A^t$, **(ii)** $T(A) = A + I_2$, **(iii)** $T(A) = A^2$, **(iv)** $T(A) = \det A$, **(v)** $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$, **(vi)** $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix}$.

(c) $V = W = M_n(\mathbb{R})$ and for every $A = [a_{ij}] \in M_n(\mathbb{R})$, define **(i)** $T(A) = \mathrm{tr}(A)$, **(ii)** $T(A) = \mathrm{rank}(A)$ and **(iii)** $T(A) = a_{11}a_{22}\dots a_{nn}$.

(d) $V = W = \mathbb{R}_2[x]$ and $T(a + bx + cx^2) = a + b(x+1) + c(x+1)^2$ for all $a + bx + cx^2 \in \mathbb{R}_2[x]$.

251. Let $T : \mathbb{R}^2 \to \mathbb{R}_2[x]$ be a linear transformation for which $T([1,1]^t) = 1 - 2x$ and $T([3,-1]^t) = x + 2x^2$. Find $T([-7,9]^t)$ and $T([a,b]^t)$ for $[a,b]^t \in \mathbb{R}_2[x]$.

252. Let $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ be a linear transformation for which $T(1+x) = 1 + x^2$, $T(x + x^2) = x - x^2$ and $T(1 + x^2) = 1 + x + x^2$. Find $T(4 - x + 3x^2)$ and $T(a + bx + cx^2)$ for $a + bx + cx^2 \in \mathbb{R}_2[x]$.

253. Consider the linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T[x,y]^t = [0,x]^t$ and $S[x,y]^t = [y,x]^t$ for all $[x,y]^t \in \mathbb{R}^2$. Compute $T \circ S$ and $S \circ T$. What do you observe?

254. Let $T : \mathbb{C} \to \mathbb{C}$ be the map defined by $T(z) = \overline{z}$ for all $z \in \mathbb{C}$. Show that T is a linear transformation on $\mathbb{C}(\mathbb{R})$ but not a linear transformation on $\mathbb{C}(\mathbb{C})$.

255. Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be a basis for a vector space V and $T : V \to V$ be a linear transformation. Prove that if $T(\mathbf{u}_i) = \mathbf{u}_i$ for all $i = 1, 2, \dots, n$, then T is the identity transformation on V.

256. Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$) of dimension n and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V. If W is another vector space over $\mathbb{R}$ (or $\mathbb{C}$) and $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \in W$, then show that there exists a unique linear transformation $T : V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for all $i = 1, 2, \dots, n$.

257. Examine the linearity of the following maps. Also, find bases for their range spaces and null spaces, whenever they are linear.

(a) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$.

(b) $T : \mathbb{R}_2[x] \to \mathbb{R}^2$ defined by $T(a + bx + cx^2) = [a - b, b + c]^t$ for $a + bx + cx^2 \in \mathbb{R}_2[x]$.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T[x,y]^t = [x, x+y, x-y]^t$ for all $[x,y]^t \in \mathbb{R}^2$.

(d) $T : \mathbb{R}_2[x] \to \mathbb{R}_3[x]$ defined by $T(f(x)) = x.f(x)$ for all $f(x) \in \mathbb{R}_2[x]$.

(e) $T : M_2(\mathbb{R}) \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [a - b, c - d]^t$ for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$.

258. Examine whether the following linear transformations are one-one and onto.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T[x,y]^t = [2x - y, x + 2y]^t$ for all $[x,y]^t \in \mathbb{R}^2$.

(b) $T : \mathbb{R}_2[x] \to \mathbb{R}^3$ defined by $T(a + bx + cx^2) = [2a - b, a + b - 3c, c - a]^t$ for all $a + bx + cx^2 \in \mathbb{R}_2[x]$.

259. Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$(a)\ \ \mathrm{range}(T) = \mathrm{span}([1,1,1]^t), \qquad (b)\ \ \mathrm{range}(T) = \mathrm{span}([1,2,3]^t, [1,3,2]^t).$$

260. Let $S : V \to W$ and $T : U \to V$ be two linear transformations.

(a) If $S \circ T$ is one-one then prove that T is also one-one.

(b) If $S \circ T$ is onto then prove that S is also onto.

261. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T[1,0,0]^t = [1,0,0]^t$, $T[1,1,0]^t = [1,1,1]^t$ and $T[1,1,1]^t = [1,1,0]^t$. Find $T[x,y,z]^t$, nullity(T) and rank(T), where $[x,y,z]^t \in \mathbb{R}^3$. Also, show that $T^2 = T$.

262. Let $z_1, z_2, \dots, z_k$ be k distinct complex numbers. Let $T : \mathbb{C}_n[z] \to \mathbb{C}^k$ be defined by $T(f(z)) = [f(z_1), f(z_2), \dots, f(z_k)]^k$ for all $f(z) \in \mathbb{C}_n[z]$. Show that T is a linear transformation, and find the dimension of range(T).

263. Show that each of the following linear transformations is an isomorphism.

(a) $T : \mathbb{R}_3[x] \to \mathbb{R}^4$ defined by $T(a + bx + cx^2 + dx^3) = [a, b, c, d]^t$ for all $a + bx + cx^2 + dx^3 \in \mathbb{R}_3[x]$.

(b) $T : M_2(\mathbb{C}) \to \mathbb{C}^4$ defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [a, b, c, d]^t$ for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C})$.

(c) $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(X) = A^{-1}XA$ for all $X \in M_n(\mathbb{R})$, where A is a given $n \times n$ invertible matrix.

(d) $T : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ defined by $T(p(x)) = p(x) + \frac{d}{dx}(p(x))$ for all $p(x) \in \mathbb{R}_n[x]$.

264. Examine whether the following vector spaces V and W are isomorphic. Whenever they are isomorphic, find an explicit isomorphism $T : V \to W$.

(a) $V = \mathbb{C}$ and $W = \mathbb{R}^2$.

(b) $V = \{A \in M_2(\mathbb{R}) : \text{tr}(A) = 0\}$ and $W = \mathbb{R}^2$.

(c) $V =$ the vector space of all 3×3 diagonal matrices and $W = \mathbb{R}^3$.

(d) $V =$ the vector space of all 3×3 symmetric matrices and $W =$ the vector space of all 3×3 anti-symmetric matrices.

Result 0.1. *Let $T : V \to W$ be a linear transformation. If V is finite-dimensional then ker(T) and range(T) are also finite-dimensional.*

265. Let A be an $m \times n$ real matrix. Using the previous result, show that if $m < n$ then the system of equations $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, and if $m > n$ then there exists a non-zero vector $\mathbf{b} \in \mathbb{R}^m$ such that the system of equations $A\mathbf{x} = \mathbf{b}$ does not have any solution.

266. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ be a basis for a vector space V of dimension 4. Define a linear transformation $T : V \to V$ such that

$$T(\mathbf{u}_1) = T(\mathbf{u}_2) = T(\mathbf{u}_3) = \mathbf{u}_1, T(\mathbf{u}_4) = \mathbf{u}_2.$$

Describe each of the spaces $\ker(T)$, $\text{range}(T)$, $\ker(T) \cap \text{range}(T)$ and $\ker(T) + \text{range}(T)$.

267. Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear transformation. If $\text{rank}(T) = \text{rank}(T^2)$ then prove that $\text{range}(T) \cap \ker(T) = \{\mathbf{0}\}$.

268. Let U and W be subspaces of a finite-dimensional vector space V. Define $T : U \times W \to V$ by $T(\mathbf{u}, \mathbf{w}) = \mathbf{u} - \mathbf{w}$ for all $(\mathbf{u}, \mathbf{w}) \in U \times W$.

(a) Show that T is a linear transformation.

(b) Show that $\text{range}(T) = U + W$.

(c) Show that $\ker(T) \cong U \cap W$.

(d) Prove **Grassmann's identity**: $\dim\,(U + W) = \dim U + \dim W - \dim\,(U \cap W)$.

269. Find the matrix $[T]_{C \leftarrow B}$ for each of the following linear transformations $T : V \to W$ with respect to the given bases B and C for V and W, respectively.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T[x, y, z]^t = [x-y+z, y-z]^t$ for all $[x, y, z]^t \in \mathbb{R}^3$, and $B = \{[1, 1, 1]^t, [1, 1, 0]^t, [1, 0, 0]^t\}$, $C = \{[1, 1]^t, [1, -1]^t\}$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T[x, y, z]^t = [x + y, y + z, z + x]^t$ for all $[x, y, z]^t \in \mathbb{R}^3$, and $B = C = \{[1, 1, 0]^t, [0, 1, 1]^t, [1, 0, 1]^t\}$.

(c) $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T[x_1, x_2, x_3, \ldots, x_n]^t = [x_2, x_3, \ldots, x_n, 0]^t$ for all $[x_1, x_2, x_3, \ldots, x_n]^t \in \mathbb{R}^n$, and $B = C =$ the standard basis for $\mathbb{R}^n$.

(d) $T : \mathbb{R}_3[x] \to \mathbb{R}_4[x]$ defined by $T(f(x)) = x.f(x)$ for all $f(x) \in \mathbb{R}_3[x]$, and $B = \{1, x, x^2, x^3\}$, $C = \{1, x, x^2, x^3, x^4\}$.

(e) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T[z_1, z_2]^t = [z_1 + z_2, iz_2]^t$ for all $[z_1, z_2]^t \in \mathbb{C}^2$, and $B =$ the standard basis for $\mathbb{C}^2$, $C = \{[1, 1]^t, [1, 0]^t\}$.

(f) $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by $T(A) = A + iA^t$ for all $A \in M_2(\mathbb{C})$, and
$B = C = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

270. Let A be an $m \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that the matrix of T with respect to the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ is A.

271. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the standard basis B for $\mathbb{R}^3$ is given by

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & -3 & 5 \\ 1 & 0 & 1 \end{bmatrix}.$$

Let $[x, y, z]^t \in \mathbb{R}^3$. Determine $T[x, y, z]^t$ and show that T is invertible. Also, find $T^{-1}[x, y, z]^t$.

272. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the basis $B = \{[1, 1, 1]^t, [1, 0, 1]^t, [1, -1, -1]^t\}$ for $\mathbb{R}^3$ is given by

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & -3 & 5 \\ 1 & 0 & 1 \end{bmatrix}.$$

Determine $T[x, y, z]^t$, where $[x, y, z]^t \in \mathbb{R}^3$.

273. Consider the bases $B = \{[1, 1, 1]^t, [0, 1, 1]^t, [0, 0, 1]^t\}$ and $C = \{1 - t, 1 + t\}$ for $\mathbb{R}^3$ and $\mathbb{R}_1[t]$, respectively. If $[T]_{C \leftarrow B} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ is the matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}_1[t]$, determine $T[x, y, z]^t$, where $[x, y, z]^t \in \mathbb{R}^3$.

274. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ be a basis for a vector space V of dimension 4, and let T be a linear transformation on V whose matrix representation with respect to this basis is given by

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & 2 & 1 & 3 \\ 1 & 2 & 5 & 5 \\ 2 & -2 & 1 & -2 \end{bmatrix}.$$

(a) Describe each of the spaces $\ker(T)$ and $\operatorname{range}(T)$.

(b) Find a basis for $\ker(T)$, extend it to a basis for V, and then find the matrix representation of T with respect to this basis.

275. Let T and S be two linear transformations on $\mathbb{R}^2$. Suppose the matrix representation of T with respect to the basis $\{\mathbf{u}_1 = [1, 2]^t, \mathbf{u}_2 = [2, 1]^t\}$ is $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, and the matrix representation of S with respect to the basis $\{\mathbf{v}_1 = [1, 1]^t, \mathbf{v}_2 = [1, 2]^t\}$ is $\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}$. Let $\mathbf{u} = [3, 3]^t \in \mathbb{R}^2$.

(a) Find the matrix of $T + S$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) Find the matrix of $T \circ S$ with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.

(c) Find the coordinate of $T(\mathbf{u})$ with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.

(d) Find the coordinate of $S(\mathbf{u})$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.