DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
MA101 MATHEMATICS-I
Solutions to Problem 1 (SET A): Quiz - 1
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1. Let $A = \begin{bmatrix} 1 & -1 & 2 & -2 & 1 \\ 2 & -2 & 5 & -5 & 2 \\ 3 & -3 & 9 & -9 & 4 \\ 1 & -1 & 3 & -3 & 0 \end{bmatrix}$. Compute a basis for the null space of A.

Solution:
STEP 1: Let us compute the REF of A. (2 Marks)

$$A \xrightarrow[\substack{R_4 \leftarrow R_4 - R_1}]{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1}} \begin{bmatrix} 1 & -1 & 2 & -2 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -3 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_2]{R_3 \leftarrow R_3 - 3R_2} \begin{bmatrix} 1 & -1 & 2 & -2 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 \leftarrow R_4 + R_3} \begin{bmatrix} 1 & -1 & 2 & -2 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

▶ Compute REF(A) correctly. (2 Marks)
▶ Final Matrix is Not in REF (0 Mark)
▶ Final form is REF but Wrong REF
1 Mark if (no. of error $= 1$), 0 Mark if (no. of errors > 1)
(*error: not writing Row operation, each Incorrect entry etc.)

STEP 2: Express leading variables in terms of free variables. (1 Mark)
Free variables : $x_4 = s, x_2 = t,\ s, t \in \mathbb{R}$.
Then we have,

$$x_5 = 0, \quad x_3 = x_4 = s, \quad x_1 = x_2 - 2x_3 + 2x_4 - x_5 = t - 2s + 2s - 0 = t.$$

▶ If all equations in terms of free variables are correct w.r.t. final matrix. (1 Mark)
▶ If equations are wrong, $0 + 0$ Marks (No marks for basis as well)

STEP 3: Give a basis from the above equations. (1 Mark)

$$Null(A) = \left\{ \begin{bmatrix} t \\ t \\ s \\ s \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R} \right\}.$$

Therefore a required basis for $Null(A)$ is given by

$$\left\{ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^T, \quad \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T \right\}.$$

▶ If equations are correct, then
- For correct basis (1 mark)
- For incorrect basis (0 Mark).

NOTE: Exactly same Marking scheme has been followed for **Set B**.

Question 2B. Let $T : R^3 \to R^3$ be a map defined by

$$T([x, y, z]^t) = \begin{bmatrix} 3x + 4y + 5z \\ 2x + 3y + 2z \\ x + y + z \end{bmatrix}$$

.

Check whether T is invertible. If yes find the inverse. 4

Solution:
For

$$[T] = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 3 & 2 \\ x & y & z \end{bmatrix} = A - - - - - - - - - - - - - - - - > \mathbf{1}$$

.

For A is invertible. $- > \mathbf{2}$

You can show A is invertible by using use 1. det, $2.rank(A)$, $3.RREF(A) = I_3$.

1. $\det(A) = -1$. One mark is deducted for wrong $\det(A)$.

2. For wrong $rank(A) - - - - - - - -> \mathbf{0}$

3. For wrong calculations to find $RREF(A) = I_3 - - - - - - - > \mathbf{0}$

Any other correct solutions are also evaluated.

For correct $A^{-1} - > \mathbf{1}$

For wrong $A^{-1} - > \mathbf{0}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

Question 2A. Marking scheme is same as $2B$.

For this case $\det(A) = 1$ and

$$A = \begin{bmatrix} 5 & -4 & 5 \\ 3 & -1 & 4 \\ 1 & -1 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & -11 \\ 1 & 0 & -5 \\ -2 & 1 & 7 \end{bmatrix}$$

Question No-3 Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then show that $\boxed{\operatorname{rank}(AB) \le \operatorname{rank}(B)}$

Answer-1

$Bx = 0 \Rightarrow ABx = 0$ ——— $\boxed{1}$

$\Rightarrow \operatorname{Null}(B) \subseteq \operatorname{Null}(AB)$

$\Rightarrow \operatorname{nullity}(B) \le \operatorname{nullity}(AB)$ ——— $\boxed{1}$

$\therefore \operatorname{rank}(B) = p - \operatorname{nullity}(B)$ [from rank-nullity theorem]

$\ge p - \operatorname{nullity}(AB)$

$= \operatorname{rank}(AB)$ ——— $\boxed{1}$

$\square$

~~Answer~~

Answer-2 we have $\{ABx : x \in \mathbb{R}^p\} \subseteq \{Ay : y \in \mathbb{R}^n\}$

$\Rightarrow \operatorname{Col}(AB) \subseteq \operatorname{Col}(A)$ ——— [*]

Now $\operatorname{row}(AB) = \operatorname{Col}((AB)^T) = \operatorname{Col}(B^T A^T) \subseteq \operatorname{Col}(B^T)$ [from [*]]

$= \operatorname{row}(B)$ ——— $\boxed{1}$

$\Rightarrow \operatorname{row}(AB) \subseteq \operatorname{row}(B)$ ——— $\boxed{1}$

$\Rightarrow \dim(\operatorname{row}(AB)) \le \dim(\operatorname{row}(B))$

$\Rightarrow \operatorname{rank}(AB) \le \operatorname{rank}(B)$ ——— $\boxed{1}$

Answer-3

$\boxed{1}$ If $B = [b_1 | \cdots | b_p]$ b_i, $i = 1,2,\cdots p$ are the columns of B
then $AB = [Ab_1 | \cdots | Ab_p]$ where Ab_i, $i = 1,2,\cdots p$ are the columns of AB

$\boxed{1}$ It follows that if ith column of B is a linear combination of other columns of B, then ith column of AB also linear combination of others columns of AB. Since
If $b_i = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_{i-1} b_{i-1} + \alpha_{i+1} b_{i+1} + \cdots + \alpha_p b_p$
$\Rightarrow Ab_i = \alpha_1 Ab_1 + \cdots + \alpha_{i-1} Ab_{i-1} + \alpha_{i+1} Ab_{i+1} + \cdots + \alpha_p Ab_p$.

$\boxed{1}$ So, if there are K dependent columns ~~in~~ in B then there are atleast K dependent columns in AB.
In other words

$\text{rank}(AB) \leq \text{rank}(B)$ [as rank of a matrix is the no. of linearly independent colums in that matrix]

Answer-4

Let $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ where a_i, and b_j are rows of A and B respectively. $1 \leq i \leq m$ $1 \leq j \leq n$

$\boxed{1}$ ~~Now~~ ~~Answer~~ Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}$

Now $AB = \begin{bmatrix} a_{11} b_1 + a_{21} b_2 + \cdots + a_{1n} b_n \\ \vdots \\ a_{m1} b_1 + a_{m2} b_2 + \cdots + a_{mn} b_n \end{bmatrix}$

from AB, matrix we can say rows of AB is a linear combination of rows of B. In other words

$\boxed{1}$ $\text{row}(AB) \subseteq \text{row}(B)$

$\boxed{1}$ $\Rightarrow \dim(\text{row}(AB)) \leq \dim(\text{row}(B))$

$\Rightarrow \text{rank}(AB) \leq \text{rank}(B)$

Type-A

Q4. Let A be an invertible $n \times n$ matrix. Show that $\det A \neq 0$

Ans. Scheme 1.

$$RREF(A) = I_n$$

$$\Rightarrow E_n \cdots E_1 A = I_n \quad (E_i \text{ elementary matrices}) \quad \boxed{1}$$

$$\Rightarrow \det(E_n) \cdots \det(E_1) \det(A) = 1 \quad \boxed{1}$$

$$\Rightarrow \det A \neq 0 \quad \boxed{1}$$

Scheme 2

$$AB = I_n$$

$$\det A \det B = 1$$

$$\Rightarrow \det A \neq 0$$

$\boxed{3}$

Scheme 3

$$A^{-1} = \frac{adj(A)}{|A|}$$ (With Proof and Expression) $\boxed{2}$

Explanation why $\det A \neq 0$ $\boxed{1}$

Scheme 4

$$A^{-1} = \frac{adj(A)}{|A|}$$ (without proof)

Explanation why $\det A \neq 0$ $\boxed{1}$

Type (B) Q4 Let A be an $n \times n$ matrix such that $\det A \neq 0$. Show that A is invertible.

Soln.

Scheme (1)

Let $RREF(A) = R$

$\Rightarrow E_n \cdots E_1 A = R$ $\boxed{1}$

$\Rightarrow \det R = \det E_n \cdots \det E_1 \det A \neq 0$ $\boxed{1}$

$\therefore$ R has no zero rows.

$\Rightarrow R = I_n$

$\therefore$ A is invertible $\boxed{1}$

Scheme (2)

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|}$$

Explanation why A is invertible.

With proof $\boxed{3}$

Without proof $\boxed{1}$

Set - A

5)
$$\lambda x - 2y - 4z = 2$$
$$2x - \lambda y + 4z = 2$$
$$x - y + \lambda z = 2$$

$$\Delta = \begin{vmatrix} \lambda & -2 & -4 \\ 2 & -\lambda & 4 \\ 1 & -1 & \lambda \end{vmatrix} = \lambda(\lambda+2)(\lambda-2)$$

$$\Delta_1 = \begin{vmatrix} 2 & -2 & -4 \\ 2 & -\lambda & 4 \\ 2 & -1 & \lambda \end{vmatrix} = -2\lambda(\lambda+2)$$

$$\Delta_2 = \begin{vmatrix} \lambda & 2 & -4 \\ 2 & 2 & 4 \\ 1 & 2 & \lambda \end{vmatrix} = 2\lambda(\lambda+6)$$

$$\Delta_3 = \begin{vmatrix} \lambda & -2 & 2 \\ 2 & -\lambda & 2 \\ 1 & -1 & 2 \end{vmatrix} = -2\lambda(\lambda-2)$$

since $\Delta \neq 0$ at $\lambda \in \mathbb{R} - \{0, 2, -2\}$

$\therefore$ unique sol at $\lambda \in \mathbb{R} - \{0, 2, -2\}$

at $\lambda = 0$, $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$ $\therefore$ infinite many solution

at $\lambda = 2$, $\Delta = 0$ but $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$ $\therefore$ no solution

at $\lambda = -2$, $\Delta = 0$ but $\Delta_2 \neq 0$ and $\Delta_3 \neq 0$ $\therefore$ no solution