MA101 Mathematics I

July-November, 2013

Tutorial & Additional Problem Set - 2

SECTION - A (for Tutorial - 2)

1. True or False? Give justifications.
   (a) If A' is the matrix obtained from A by replacing the ith column a_i of A by $2a_i$ then the systems $A'\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$ are equivalent.
   (b) If the RREF of $A \in \mathcal{M}_5(R)$ has the third column as $[1, 2, 0, 0, 0]^T$ then $[-1, -2, 1, 0, 0]^T$ is a solution of $A\mathbf{x} = \mathbf{0}$.
   (c) For an $n \times n$ matrix A, the systems $A\mathbf{x} = \mathbf{0}$ and $A^T\mathbf{x} = \mathbf{0}$ are equivalent.

Solution: a) False. Take, for example, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $[1, -1]^T$ is a solution of $Ax = \mathbf{0}$ but not of $A'x = \mathbf{0}$.
b)True. Observation: The first two columns are leading columns and the third is not, hence the first three rows of the RREF of A are $[1, 0, 1, *, *], [0, 1, 2, *, *], [0, 0, 0, *, *]$.
c)False. Consider, for example, $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

2. For an $\mathbf{x} \in \mathbb{R}^n$ and an $S \subseteq \mathbb{R}^n$ define $\mathbf{x} + S = \{\mathbf{x} + \mathbf{s} \,|\, \mathbf{s} \in S\}$.
   (a) Draw and illustrate the following in $\mathbb{R}^2$, where e_i gives the ith column of I_2.
   i) $e_1 + e_2$ ii) $e_1 + \{e_2, 2e_2, 3e_2\}$ iii) $e_1 + \{\alpha e_2 | \alpha \in \mathbb{R}\}$ iv) $e_1 + \{\alpha[1, 2]^T | \alpha \in \mathbb{R}\}$.
   (b) Find systems $A\mathbf{x} = \mathbf{b}$ the solutions of which are given by the sets in parts iii) and iv).

Solution: Answer :(b): (iii) $x = 1$. (iv) $2x - y = 2$.

3. Take a consistent system $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.
   Imagine the planes corresponding to the equations in $\mathbb{R}^3$. The line of intersection is the solution set S. Take its associated homogeneous system. Imagine planes for that. Their line of intersection is the solution set S_h. By what vector do you translate (shift) S_h to get S? Can you give another vector? Five more?

sb, 27 Aug 2013, 12:42

Solution: The RREF of $[A|b]$ is given by $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Hence the line which gives the solution set is given by $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + z = 1, y - z = 0 \right\} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left\{ z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$.
For any $\mathbf{u} \in S_h$, if S_h is shifted by $\mathbf{u}$ then we get S. Take six different particular solutions $\mathbf{u}$'s of the homogeneous system, and then shift S_h by $\mathbf{u}$.

4. Given any $A_{4\times 3}$ matrix of rank 3, then show that there exists a $B_{3\times 4}$ such that $BA = I_3$.

Solution: The RREF$(A) = [I_3, \mathbf{0}]^T$. Hence there exists an invertible P such that $PA = [I_3, \mathbf{0}]^T$. Take $B = [I_3, \mathbf{0}]P$, then $BA = I_3$.

5. Let $A_{5\times 5}$ be invertible with row sums 1. Show that the sum of all the elements of A^{-1} is 5.

Solution: Let $\mathbf{1} = [1, 1, 1, 1, 1]^T$. Then $A\mathbf{1} = [1, 1, 1, 1, 1]^T = \mathbf{1}$, which gives $A^{-1}\mathbf{1} = \mathbf{1}$, i.e., A^{-1} has row sums 1 and the result follows.

6. If A and B are $m \times n$ matrices such that $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are equivalent, then show that A and B are row equivalent.

Solution: Let $\tilde{A}$ and $\tilde{B}$ be the RREF's of A and B, respectively. Then $\tilde{A}x = 0$ and $\tilde{B}x = 0$ are equivalent. We show that $\tilde{A} = \tilde{B}$. This will immediately imply that A and B are row equivalent.

Let us assume $\tilde{A}x = 0$ and $\tilde{B}x = 0$ are equivalent, to show $\tilde{A} = \tilde{B}$.
If the first column of $\tilde{A}$ is not equal to that of $\tilde{B}$, then one of $\tilde{A}$ or $\tilde{B}$, say $\tilde{A}$ must have the first column as the zero column and for the other it will be $[1, 0, ..., 0]^T$.
Then $[1, 0, ..., 0]^T$ will be a solution of $\tilde{A}x = 0$ but not of $\tilde{B}x = 0$, which is not possible. So let us assume that the first k columns of $\tilde{A}$ and $\tilde{B}$ are equal and $\tilde{A}_{(k+1)} \neq \tilde{B}_{(k+1)}$ where $\tilde{A}_{(k+1)}$ and $\tilde{B}_{(k+1)}$ are the $(k+1)$ th columns of $\tilde{A}$ and $\tilde{B}$ respectively. Then both $\tilde{A}_{(k+1)}$ and $\tilde{B}_{(k+1)}$ cannot be leading columns, WLOG let $\tilde{A}_{(k+1)}$ not be a leading column.
Let s be the number of leading columns in the first k columns of $\tilde{A}$ and $\tilde{B}$. If the $(k+1)$th column of either $\tilde{A}$ or $\tilde{B}$ is the zero column, then by the previous argument, we get a contradiction. Hence assume that the $(k+1)$ th column is nonzero for both $\tilde{A}$ and $\tilde{B}$. Note that $-\tilde{A}_{(1,k+1)}e_1 - \tilde{A}_{(2,k+1)}e_2 - \ldots - \tilde{A}_{(s,k+1)}e_s + \begin{bmatrix} \tilde{A}_{(1,k+1)} \\ \cdots \\ \tilde{A}_{(s,k+1)} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$, where e_i is the i th

column of I_m.
Take $\mathbf{u} = [u_1, u_2, ..u_k, 1, 0, .., 0]^T$, where for $i = 1, 2, .., k$,

$$u_i = \begin{cases} -\tilde{A}_{r,k+1}, & \text{if the } i\text{-th column of } \tilde{A} \text{ has the leading entry of the } r\text{-th row of } \tilde{A}. \\ 0, & \text{if the } i\text{-th column of } \tilde{A} \text{ is nonleading.} \end{cases}$$

Then check that $\tilde{A}\mathbf{u} = \mathbf{0}$ but $\tilde{B}\mathbf{u} \neq \mathbf{0}$.

SECTION - B: ADDITIONAL PROBLEMS

7. True or False? Give justifications.

   (a) If for all $A \in \mathcal{M}_n(R)$, $AB = A$ then $B = I_n$.

   (b) If A and B are square matrices of order n with $AB = I_n$ then A and B are invertible and $BA = I_n$.
   Hint: If P is invertible then rank$(P) = n$. $AB = I$ implies there exists an invertible P such that $PAB = P$, where PA is in RREF.

   (c) If A is an $m \times n$ matrix with at least one nonzero row (at least one entry of this row is nonzero) then A is row equivalent to a matrix B, with all nonzero rows.

   (d) If all the columns of an $n \times m$ nonzero matrix (it has at least one nonzero entry) A are equal then $rank(A) = 1$.

   (e) If A is an $m \times n$ matrix with a zero column (all entries of the column is zero) then the RREF of A will again have a zero column.

   (f) If P is any invertible matrix such that PA is defined then, $Ax = b$ and $PAx = Pb$ are equivalent.

Solution:

(a) True, take $A = I_n$.

(b) True. Observation: If P is invertible then $Rank(P) = n$.
$AB = I$ implies there exists an invertible P such that $PAB = P$, where PA is in RREF. Since P is invertible, PAB cannot have a zero row, hence PA cannot have a zero row. So $PA = I_n$ or $A = P^{-1}$ and $B = P$. $AB = I$ implies $B(AB)B^{-1} = I = BA(BB^{-1}) = BA$.

(c) True. If the RREF of A has a zero row, say $\tilde{a}_i$, then replace $\tilde{a}_i$ with $\tilde{a}_i + \tilde{a}_j$, where $\tilde{a}_j$ is some nonzero row of the RREF.

(d) True. (Each row of A is a multiple of some nonzero row of A.)

(e) True.

(f) True.

8. Give examples of three matrices $A_{3\times 4}$ in RREF with different number of nonzero rows. How many different types (with respect to different number of nonzero rows and different positions of the leading 1's) are possible? Discuss about the number of solutions of the corresponding system $Ax = b$ by considering different types of b s.
Hint: For example take b s with zero or nonzero entries corresponding to the zero rows of A.

Solution: Answer of only second part, rest are exercises.

3 leading columns: $\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ $\begin{bmatrix}1&*&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ $\begin{bmatrix}1&0&*&0\\0&1&*&0\\0&0&0&1\end{bmatrix}$ $\begin{bmatrix}1&0&0&*\\0&1&0&*\\0&0&1&*\end{bmatrix}$: $C(4,3)$ in number.

2 leading columns: $\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$ $\begin{bmatrix}0&1&*&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$ and so on: $C(4,2)$ in number.

1 leading column: $C(4,1)$ in number.

0 leading column: $C(4,0)$ in number.

Total is 15.

9. Using Gauss Jordan elimination prove that

$$\left\{\alpha\begin{bmatrix}2\\1\\1\end{bmatrix} : \alpha\in\mathbb{R}\right\} + \left\{\alpha\begin{bmatrix}1\\1\\0\end{bmatrix} : \alpha\in\mathbb{R}\right\} + \left\{\alpha\begin{bmatrix}0\\1\\1\end{bmatrix} : \alpha\in\mathbb{R}\right\} = \mathbb{R}^3.$$

Solution: Check that the RREF of A is I_3. Therefore, for any $\mathbf{b}\in\mathbb{R}^3$, the system $A\mathbf{x} = \mathbf{b}$ is consistent, where the columns of A are given by $[2,1,1]^T, [1,1,0]^T$ and $[0,1,1]^T$. Thus, $\mathbf{b}$ is a linear combination of $[2,1,1]^T, [1,1,0]^T$ and $[0,1,1]^T$, and therefore, $\mathbb{R}^3$ is a subset of the set in the left. That the set in the left is a subset of $\mathbb{R}^3$ is obvious.

10. If A is upper triangular and B is any matrix such that $AB = I$, then show that each diagonal entry of A is nonzero.

Solution: Note that A is square, suppose of order n. Suppose $R = \text{RREF}(A) = PA$, where P is invertible. Now, if A has at least one zero diagonal entry, consider the least i such that $a_{ii} = 0$, then the corresponding column of R is a nonleading column. Thus, R has less than n leading columns, and so has a zero row. Consequently, $RB = PAB = PI = P$ has a zero row, which is not possible because P is invertible.

11. Use Gauss Jordan elimination to find the inverse of $A = \begin{bmatrix}1&x&x^2\\1&y&y^2\\1&z&z^2\end{bmatrix}$, when $x \neq y \neq z$.

Solution: Exercise. You will get entries of A^{-1} as algebraic expressions in x, y and z.