MA 101 (Mathematics I)

Hints/Explanations for Examples in Lectures

Sequence

Example: The sequence $(\frac{n+1}{2n+3})$ is convergent with limit $\frac{1}{2}$.
Proof: Let $\varepsilon > 0$. For all $n \in \mathbb{N}$, we have $|\frac{n+1}{2n+3} - \frac{1}{2}| = \frac{1}{4n+6} < \frac{1}{4n}$. There exists $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{4\varepsilon}$. Hence $|\frac{n+1}{2n+3} - \frac{1}{2}| < \frac{1}{4n_0} < \varepsilon$ for all $n \geq n_0$ and so the given sequence is convergent with limit $\frac{1}{2}$.

Example: The sequence $(1, 2, 1, 2, ...)$ is not convergent.
Proof: If possible, let the given sequence (x_n) (say) be convergent with limit ℓ. Then there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \frac{1}{2}$ for all $n \geq n_0$. Hence $|x_{2n_0} - \ell| < \frac{1}{2}$ and $|x_{2n_0+1} - \ell| < \frac{1}{2}$ and so $|2 - \ell| < \frac{1}{2}$ and $|1 - \ell| < \frac{1}{2}$. This gives $1 = |(2 - \ell) - (1 - \ell)| \leq |2 - \ell| + |1 - \ell| < 1$, which is a contradiction. Therefore the given sequence is not convergent.

Example: The sequence $(n^3 + 1)$ is not convergent.
Proof: If possible, let $(n^3 + 1)$ be convergent. Then there exist $\ell \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $|n^3 + 1 - \ell| < 1$ for all $n \geq n_0 \Rightarrow n^3 < \ell$ for all $n \geq n_0$, which is not true. Therefore the given sequence is not convergent.

Example: The sequence $(\frac{3n+2}{2n+5})$ is bounded.
Proof: For all $n \in \mathbb{N}$, $|\frac{3n+2}{2n+5}| = \frac{3n+2}{2n+5} < \frac{3n+2}{2n} = \frac{3}{2} + \frac{1}{n} \leq \frac{5}{2}$. Hence the given sequence is bounded.

Example: The sequence $(1, 2, 1, 3, 1, 4, ...)$ is unbounded.
Proof: If possible, let the given sequence (x_n) (say) be bounded. Then there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. This gives $n \leq M$ for all $n \in \mathbb{N}$, which is not true. Therefore the given sequence is unbounded.

Example: The sequence $(\frac{2n^2-3n}{3n^2+5n+3})$ is convergent with limit $\frac{2}{3}$.
Proof: We have $\frac{2n^2-3n}{3n^2+5n+3} = \frac{2-\frac{3}{n}}{3+\frac{5}{n}+\frac{3}{n^2}}$ for all $n \in \mathbb{N}$. Since $\frac{1}{n} \to 0$, the limit rules for algebraic operations on sequences imply that the given sequence is convergent with limit $\frac{2-0}{3+0+0} = \frac{2}{3}$.

Example: The sequence $(\sqrt{n+1} - \sqrt{n})$ is convergent with limit 0.
Proof: For all $n \in \mathbb{N}$, $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\frac{1}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}+1}$. Since $\frac{1}{n} \to 0$, the limit rules for algebraic operations on sequences imply that the given sequence is convergent with limit $\frac{0}{\sqrt{1+0}+1} = 0$.

Example: If $|\alpha| < 1$, then the sequence (α^n) converges to 0.
Proof: If $\alpha = 0$, then $\alpha^n = 0$ for all $n \in \mathbb{N}$ and so (α^n) converges to 0. Now we assume that $\alpha \neq 0$. Since $|\alpha| < 1$, $\frac{1}{|\alpha|} > 1$ and so $\frac{1}{|\alpha|} = 1 + h$ for some $h > 0$. For all $n \in \mathbb{N}$, we have $(1 + h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \cdots + h^n > nh \Rightarrow |\alpha|^n = \frac{1}{(1+h)^n} < \frac{1}{nh}$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ satisfying $n_0 > \frac{1}{h\varepsilon}$. Then $|\alpha^n - 0| = |\alpha|^n < \frac{1}{n_0 h} < \varepsilon$ for all $n \geq n_0$ and hence (α^n) converges to 0.

Alternative proof: Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ satisfying $n_0 > \frac{\log \varepsilon}{\log |\alpha|}$. Then for all $n \geq n_0$, we have $|\alpha^n - 0| = |\alpha|^n \leq |\alpha|^{n_0} < \varepsilon$ and hence (α^n) converges to 0. (This proof assumes the definition of logarithm.)

Example: If $\alpha > 0$, then the sequence $(\alpha^{\frac{1}{n}})$ converges to 1.
Proof: We first assume that $\alpha \geq 1$ and let $x_n = \alpha^{\frac{1}{n}} - 1$ for all $n \in \mathbb{N}$. Then $x_n \geq 0$ and $\alpha = (1 + x_n)^n = 1 + nx_n + \frac{n(n-1)}{2!}x_n^2 + \cdots + x_n^n > nx_n$ for all $n \in \mathbb{N}$. So $0 \leq x_n < \frac{\alpha}{n}$ for all $n \in \mathbb{N}$. Since $\frac{\alpha}{n} \to 0$, by sandwich theorem, it follows that $x_n \to 0$. Consequently $\alpha^{\frac{1}{n}} \to 1$. If $\alpha < 1$, then $\frac{1}{\alpha} > 1$ and as proved above, $(\frac{1}{\alpha})^{\frac{1}{n}} \to 1$. It follows that $\alpha^{\frac{1}{n}} \to 1$.

Alternative proof: We first assume that $\alpha \geq 1$. For each $n \in \mathbb{N}$, applying the $A.M. \geq G.M.$ inequality for the numbers $1, ..., 1, \alpha$ (1 is repeated $n-1$ times), we get $1 \leq \alpha^{\frac{1}{n}} \leq 1 + \frac{\alpha - 1}{n}$. Since $\frac{\alpha-1}{n} \to 0$, by sandwich theorem, it follows that $\alpha^{\frac{1}{n}} \to 1$. The case for $\alpha < 1$ is same as given in the above proof.

Example: The sequence $(n^{\frac{1}{n}})$ converges to 1.
Proof: For all $n \in \mathbb{N}$, let $a_n = n^{\frac{1}{n}} - 1$. Then for all $n \in \mathbb{N}$, $n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2!}a_n^2 + \cdots + a_n^n > \frac{n(n-1)}{2!}a_n^2 \Rightarrow 0 \leq a_n^2 < \frac{2}{n-1}$ for all $n \in \mathbb{N}$. Since $\frac{2}{n-1} \to 0$, by sandwich theorem, it follows that $a_n^2 \to 0$ and so $a_n \to 0$. Consequently $n^{\frac{1}{n}} \to 1$.

Example: The sequence $((2^n + 3^n)^{\frac{1}{n}})$ converges to 3.
Proof: We have $3^n < 2^n + 3^n < 2.3^n$ for all $n \in \mathbb{N} \Rightarrow 3 < (2^n + 3^n)^{\frac{1}{n}} < 2^{\frac{1}{n}}.3$ for all $n \in \mathbb{N}$. Also, both the sequences $(3, 3, ...)$ and $(2^{\frac{1}{n}}.3)$ converge to 3. (Note that $2^{\frac{1}{n}} \to 1$.) Hence by sandwich theorem, the given sequence converges to 3.

Alternative proof: Since $(2^n+3^n)^{\frac{1}{n}} = 3[1+(\frac{2}{3})^n]^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, we have $3 < (2^n+3^n)^{\frac{1}{n}} \leq 3[1+(\frac{2}{3})^n]$ for all $n \in \mathbb{N}$. Also, both the sequences $(3, 3, ...)$ and $(3[1 + (\frac{2}{3})^n])$ converge to 3. (Note that $(\frac{2}{3})^n \to 0$.) Hence by sandwich theorem, the given sequence converges to 3.

Example: The sequence $(\frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}})$ converges to 1.
Proof: We have $\frac{n}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq \frac{n}{\sqrt{n^2+1}}$ for all $n \in \mathbb{N}$. Also, $\frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+\frac{1}{n}}} \to 1$ and $\frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \to 1$. Hence by sandwich theorem, the given sequence converges to 1.

Example: If $\alpha \in \mathbb{R}$, then the sequence $(\frac{\alpha^n}{n!})$ is convergent.
Proof: Let $x_n = \frac{\alpha^n}{n!}$ for all $n \in \mathbb{N}$. If $\alpha = 0$, then $x_n = 0$ for all $n \in \mathbb{N}$ and so (x_n) converges to 0. If $\alpha \neq 0$, then $\lim\limits_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim\limits_{n\to\infty} \frac{|\alpha|}{n+1} = 0 < 1$ and so (x_n) converges to 0.

Example: The sequence $(\frac{2^n}{n^4})$ is not convergent.
Proof: If $x_n = \frac{2^n}{n^4}$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim\limits_{n\to\infty} \frac{2}{(1+\frac{1}{n})^4} = 2 > 1$. Therefore the sequence (x_n) is not convergent.

Example: The sequence $(1 - \frac{1}{n})$ is increasing.
Proof: For all $n \in \mathbb{N}$, $\frac{1}{n+1} < \frac{1}{n}$ and so $1 - \frac{1}{n+1} > 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore the given sequence is increasing.

Example: The sequence $(n + \frac{1}{n})$ is increasing.
Proof: For all $n \in \mathbb{N}$, $(n + 1 + \frac{1}{n+1}) - (n + \frac{1}{n}) = 1 - \frac{1}{n(n+1)} > 0 \Rightarrow n + 1 + \frac{1}{n+1} > n + \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore the given sequence is increasing.

Example: The sequence $(\cos \frac{n\pi}{3})$ is not monotonic.
Proof: Since $\cos \frac{\pi}{3} = \frac{1}{2}$, $\cos \frac{3\pi}{3} = -1$ and $\cos \frac{6\pi}{3} = 1$, we have $\cos \frac{\pi}{3} > \cos \frac{3\pi}{3} < \cos \frac{6\pi}{3}$ and hence the given sequence is neither increasing nor decreasing. Consequently the given sequence is not monotonic.

Example: Let $x_1 = 1$ and $x_{n+1} = \frac{1}{3}(x_n + 1)$ for all $n \in \mathbb{N}$. Then the sequence (x_n) is convergent and $\lim_{n\to\infty} x_n = \frac{1}{2}$.
Proof: For all $n \in \mathbb{N}$, we have $x_{n+1} - x_n = \frac{1}{3}(1 - 2x_n)$. Also, $x_1 > \frac{1}{2}$ and if we assume that $x_k > \frac{1}{2}$ for some $k \in \mathbb{N}$, then $x_{k+1} = \frac{1}{3}(x_k + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2}$. Hence by the principle of mathematical induction, $x_n > \frac{1}{2}$ for all $n \in \mathbb{N}$. So (x_n) is bounded below. Again, from above, we get $x_{n+1} - x_n < 0$ for all $n \in \mathbb{N} \Rightarrow x_{n+1} < x_n$ for all $n \in \mathbb{N} \Rightarrow (x_n)$ is decreasing. Therefore (x_n) is convergent. Let $\ell = \lim_{n\to\infty} x_n$. Then $\lim_{n\to\infty} x_{n+1} = \ell$ and since $x_{n+1} = \frac{1}{3}(x_n + 1)$ for all $n \in \mathbb{N}$, we get $\ell = \frac{1}{3}(\ell + 1) \Rightarrow \ell = \frac{1}{2}$.

Alternative proof for showing that (x_n) is decreasing: We have $x_2 = \frac{2}{3} < 1 = x_1$ and if we assume that $x_{k+1} < x_k$ for some $k \in \mathbb{N}$, then $x_{k+2} = \frac{1}{3}(x_{k+1} + 1) < \frac{1}{3}(x_k + 1) = x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} < x_n$ for all $n \in \mathbb{N}$.

Example: Let $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ for all $n \in \mathbb{N}$. Then the sequence (x_n) is convergent.
Proof: For all $m, n \in \mathbb{N}$ with $m > n$, we have $|x_m - x_n| = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} = \frac{2}{2^n}(1 - \frac{1}{2^{m-n}}) < \frac{2}{2^n} < \frac{2}{n}$. Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ satisfying $n_0 > \frac{2}{\varepsilon}$. Then for all $m, n \geq n_0$, we get $|x_m - x_n| < \frac{2}{n_0} < \varepsilon$. Consequently (x_n) is a Cauchy sequence in $\mathbb{R}$ and hence (x_n) is convergent.

Example: Let $0 < \alpha < 1$ and let the sequence (x_n) satisfy the condition $|x_{n+1} - x_n| \leq \alpha^n$ for all $n \in \mathbb{N}$. Then (x_n) is a Cauchy sequence.
Proof: For all $m, n \in \mathbb{N}$ with $m > n$, we have $|x_m - x_n| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \leq \alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1} = \frac{\alpha^n}{1-\alpha}(1 - \alpha^{m-n}) < \frac{\alpha^n}{1-\alpha}$. Since $0 < \alpha < 1$, $\alpha^n \to 0$ and so given any $\varepsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such that $\frac{\alpha^{n_0}}{1-\alpha} < \varepsilon$. Hence for all $m, n \geq n_0$, we have $|x_m - x_n| < \frac{\alpha^{n_0}}{1-\alpha} < \varepsilon$. Therefore (x_n) is a Cauchy sequence.

Example: Let $0 < \alpha < 1$ and let the sequence (x_n) satisfy the condition $|x_{n+2} - x_{n+1}| \leq \alpha|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. Then (x_n) is a Cauchy sequence.
Solution: For all $m, n \in \mathbb{N}$ with $m > n$, we have $|x_m - x_n| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \leq (\alpha^{n-1} + \alpha^n + \cdots + \alpha^{m-2})|x_2 - x_1| = \frac{\alpha^{n-1}}{1-\alpha}(1 - \alpha^{m-n})|x_2 - x_1| \leq \frac{\alpha^{n-1}}{1-\alpha}|x_2 - x_1|$. Since $0 < \alpha < 1$, $\alpha^{n-1} \to 0$ and so given any $\varepsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such that $\frac{\alpha^{n_0-1}}{1-\alpha}|x_2 - x_1| < \varepsilon$. Hence for all $m, n \geq n_0$, we have $|x_m - x_n| \leq \frac{\alpha^{n_0-1}}{1-\alpha}|x_2 - x_1| < \varepsilon$. Therefore (x_n) is a Cauchy sequence.

Example: Let $x_1 = 1$ and let $x_{n+1} = \frac{1}{x_n+2}$ for all $n \in \mathbb{N}$. Then the sequence (x_n) is convergent and $\lim_{n\to\infty} x_n = \sqrt{2} - 1$.
Proof: For all $n \in \mathbb{N}$, we have $|x_{n+2} - x_{n+1}| = |\frac{1}{x_{n+1}+2} - \frac{1}{x_n+2}| = \frac{|x_{n+1}-x_n|}{|x_{n+1}+2||x_n+2|}$. Now, $x_1 > 0$ and if we assume that $x_k > 0$ for some $k \in \mathbb{N}$, then $x_{k+1} = \frac{1}{x_k+2} > 0$. Hence by the principle of mathematical induction, $x_n > 0$ for all $n \in \mathbb{N}$. Using this, we get $|x_{n+2} - x_{n+1}| \leq \frac{1}{4}|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. It follows that (x_n) is a Cauchy sequence in $\mathbb{R}$ and hence (x_n) is convergent. Let $\ell = \lim_{n\to\infty} x_n$. Then $\lim_{n\to\infty} x_{n+1} = \ell$ and since $x_{n+1} = \frac{1}{x_n+2}$ for all $n \in \mathbb{N}$, we get $\ell = \frac{1}{\ell+2} \Rightarrow \ell = -1 \pm \sqrt{2}$. Since $x_n > 0$ for all $n \in \mathbb{N}$, we have $\ell \geq 0$ and so $\ell = \sqrt{2} - 1$.

Example: If $x_n = (-1)^n(1 - \frac{1}{n})$ for all $n \in \mathbb{N}$, then $x_n \not\to 1$. In fact, (x_n) is not convergent.
Proof: Since $x_{2n-1} = (-1)^{2n-1}(1 - \frac{1}{2n-1}) = \frac{1}{2n-1} - 1 \to -1$, $x_n \not\to 1$. Again, since $x_{2n} = (-1)^{2n}(1 - \frac{1}{2n}) = 1 - \frac{1}{2n} \to 1 \neq -1$, (x_n) is not convergent.

Remark: Let (x_n) be a sequence in $\mathbb{R}$ such that $x_{2n} \to \ell \in \mathbb{R}$ and $x_{2n-1} \to \ell$. Then $x_n \to \ell$.
Proof: Let $\varepsilon > 0$. Since $x_{2n} \to \ell$ and $x_{2n-1} \to \ell$, there exist $n_1, n_2 \in \mathbb{N}$ such that $|x_{2n} - \ell| < \varepsilon$ for all $n \geq n_1$ and $|x_{2n-1} - \ell| < \varepsilon$ for all $n \geq n_2$. Taking $n_0 = \max\{2n_1, 2n_2 - 1\} \in \mathbb{N}$, we find that $|x_n - \ell| < \varepsilon$ for all $n \geq n_0$. Hence $x_n \to \ell$.

Example: The sequence $(1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, ...)$ converges to 1.
Proof: If (x_n) denotes the given sequence, then $x_{2n} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$ and $x_{2n-1} = 1 \to 1$. Therefore (x_n) converges to 1.

Example: If $x \in \mathbb{R}$, then there exists a sequence (r_n) of rationals converging to x. Similarly, if $x \in \mathbb{R}$, then there exists a sequence (t_n) of irrationals converging to x.
Proof: For each $n \in \mathbb{N}$, there exist $r_n \in \mathbb{Q}$ and $t_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $x - \frac{1}{n} < r_n < x + \frac{1}{n}$ and $x - \frac{1}{n} < t_n < x + \frac{1}{n}$. Since $x - \frac{1}{n} \to x$ and $x + \frac{1}{n} \to x$, by sandwich theorem, the sequence (r_n) of rationals converges to x and the sequence (t_n) of irrationals also converges to x.

Series

Example: The geometric series $\sum\limits_{n=1}^{\infty} ar^{n-1}$ (where $a \neq 0$) converges iff $|r| < 1$.
Proof: If $r = 1$, then the given series becomes $a + a + \cdots$, which is not convergent, since $(s_n) = (na)$ does not converge as $a \neq 0$. We now assume that $r \neq 1$. Then $s_n = \sum\limits_{i=1}^{n} ar^{i-1} = \frac{a}{1-r}(1 - r^n)$ for all $n \in \mathbb{N}$. If $|r| < 1$, then $\lim\limits_{n\to\infty} r^n = 0$ and so (s_n) converges to $\frac{a}{1-r}$. Therefore the given series converges (with sum $\frac{a}{1-r}$) if $|r| < 1$. If $|r| \geq 1$, then the sequence (r^n) does not converge and since $a \neq 0$, it follows that (s_n) does not converge. Hence in this case the given series is not convergent.

Example: The series $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent with sum 1.
Proof: Here $s_n = \sum\limits_{k=1}^{n} \frac{1}{k(k+1)} = \sum\limits_{k=1}^{n} (\frac{1}{k} - \frac{1}{k+1}) = 1 - \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} (1 - \frac{1}{n+1}) = 1$, the given series is convergent with sum 1.

Example: The series $1 - 1 + 1 - 1 + \cdots$ is not convergent.
Proof: Here $s_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$
and so the sequence (s_n) is not convergent. Therefore the given series is not convergent.

Example: The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.
Proof: For all $n \geq 2$, we have $s_n = \sum\limits_{k=1}^{n} \frac{1}{k^2} \leq 1 + \sum\limits_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \sum\limits_{k=2}^{n} (\frac{1}{k-1} - \frac{1}{k}) = 2 - \frac{1}{n} < 2$. Hence the sequence (s_n) is bounded above and consequently by monotonic criterion for series, the given series is convergent.

Example: The series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is not convergent.
Proof: If possible, let the given series be convergent. Then by Cauchy criterion for series, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m} < \frac{1}{2}$ for all $m > n \geq n_0$. In particular, we get $\frac{1}{n_0+1} + \frac{1}{n_0+2} + \cdots + \frac{1}{2n_0} < \frac{1}{2}$. But $\frac{1}{n_0+1} + \frac{1}{n_0+2} + \cdots + \frac{1}{2n_0} \geq \frac{1}{2n_0} + \frac{1}{2n_0} + \cdots + \frac{1}{2n_0} = \frac{1}{2}$, and so we

get a contradiction. Hence the given series is not convergent.

Alternative proof: Let $s_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ for all $n \in \mathbb{N}$. Then $s_{2^n} = 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+\cdots+(\frac{1}{2^{n-1}+1}+\cdots+\frac{1}{2^n}) > 1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+\cdots+(\frac{1}{2^n}+\cdots+\frac{1}{2^n}) = 1+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2} = 1+\frac{n}{2}$ for all $n \in \mathbb{N}$. This shows the sequence (s_n) is not bounded. Hence (s_n) is not convergent and consequently $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is not convergent.

Example: The series $\sum\limits_{n=1}^{\infty}\frac{n^2+1}{(n+3)(n+4)}$ is not convergent.

Proof: Since $\frac{n^2+1}{(n+3)(n+4)} = \frac{1+\frac{1}{n^2}}{(1+\frac{3}{n})(1+\frac{4}{n})} \to 1$, we have $\frac{n^2+1}{(n+3)(n+4)} \not\to 0$ and so the given series is not convergent.

Example: The series $\sum\limits_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$ is not convergent.

Proof: Since $(-1)^{2n}\frac{2n}{2n+2} = \frac{1}{1+\frac{1}{n}} \to 1$, we have $(-1)^n\frac{n}{n+2} \not\to 0$ and so the given series is not convergent.

Example: The series $\sum\limits_{n=1}^{\infty}\frac{1+\sin n}{1+n^2}$ is convergent.

Proof: We have $0 \le \frac{1+\sin n}{1+n^2} \le \frac{2}{n^2}$ for all $n \in \mathbb{N}$. Since $\sum\limits_{n=1}^{\infty}\frac{2}{n^2}$ is convergent, by comparison test, the given series is convergent.

Example: The series $\sum\limits_{n=1}^{\infty}\frac{1}{2^n+n}$ is convergent.

Proof: We have $0 < \frac{1}{2^n+n} < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Since $\sum\limits_{n=1}^{\infty}\frac{1}{2^n}$ is convergent, by comparison test, the given series is convergent.

Example: The series $\sum\limits_{n=2}^{\infty}\frac{1}{\sqrt{n(n-1)}}$ is not convergent.

Proof: Since $\frac{1}{\sqrt{n(n-1)}} > \frac{1}{n} > 0$ for all $n \ge 2$ and since $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is not convergent, by comparison test, the given series is not convergent.

Example: The series $\sum\limits_{n=1}^{\infty}\frac{n}{4n^3-2}$ is convergent.

Proof: Let $x_n = \frac{n}{4n^3-2}$ and $y_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then $x_n, y_n > 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty}\frac{x_n}{y_n} = \lim\limits_{n\to\infty}\frac{n^3}{4n^3-2} = \lim\limits_{n\to\infty}\frac{1}{4-\frac{2}{n^3}} = \frac{1}{4} \neq 0$. Since $\sum\limits_{n=1}^{\infty}y_n$ is convergent, by limit comparison test, $\sum\limits_{n=1}^{\infty}x_n$ is convergent.

Example: For $p \in \mathbb{R}$, the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ is convergent iff $p > 1$.

Proof: If $p \le 0$, then $\frac{1}{n^p} \not\to 0$ and so the given series is not convergent. Now, let $p > 0$. Then $(\frac{1}{n^p})$ is a decreasing sequence of non-negative real numbers. Also, $\sum\limits_{n=1}^{\infty}2^n\cdot\frac{1}{(2^n)^p} = \sum\limits_{n=1}^{\infty}(\frac{1}{2^{p-1}})^n$, being a geometric series, converges iff $\frac{1}{2^{p-1}} < 1$, *i.e.* iff $p > 1$. Hence by Cauchy's condensation test, the given series converges iff $p > 1$.

Example: For $p \in \mathbb{R}$, the series $\sum\limits_{n=2}^{\infty}\frac{1}{n(\log n)^p}$ is convergent iff $p > 1$.

Proof: Let $f(x) = \frac{1}{x(\log x)^p}$ for all $x > 1$. Then $f : (1,\infty) \to \mathbb{R}$ is differentiable and

$f'(x) = -\frac{(\log x)^{p-1}(\log x+p)}{x^2(\log x)^{2p}} \leq 0$ for all $x > \max\{1, e^{-p}\} = a$ (say). Hence f is decreasing on (a,∞) and so $f(n+1) \leq f(n)$ for all $n \geq n_0$, where $n_0 \in \mathbb{N}$ is chosen to satisfy $n_0 > a$. Thus the sequence $\left(\frac{1}{n(\log n)^p}\right)_{n=n_0}^{\infty}$ of non-negative real numbers is decreasing. Since the series $\sum\limits_{n=n_0}^{\infty} 2^n \cdot \frac{1}{2^n(\log 2^n)^p} = \sum\limits_{n=n_0}^{\infty} \frac{1}{(\log 2)^p n^p}$ is convergent iff $p > 1$, by Cauchy's condensation test, $\sum\limits_{n=n_0}^{\infty} \frac{1}{n(\log n)^p}$ is convergent iff $p > 1$. Consequently the given series is convergent iff $p > 1$.

Example: The series $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$ is convergent.
Proof: Taking $x_n = \frac{n}{2^n}$ for all $n \in \mathbb{N}$, we find that $\lim\limits_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim\limits_{n\to\infty} \frac{1}{2}(1+\frac{1}{n}) = \frac{1}{2} < 1$. Hence by the ratio test, the given series is convergent.

Example: The series $\sum\limits_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ is not convergent.
Proof: Taking $x_n = \frac{(2n)!}{(n!)^2}$ for all $n \in \mathbb{N}$, we find that $\lim\limits_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim\limits_{n\to\infty} \frac{4n+2}{n+1} = 4 > 1$. Hence by the ratio test, the given series is not convergent.

Example: The series $\sum\limits_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$ is convergent.
Proof: Taking $x_n = \frac{(n!)^n}{n^{n^2}}$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{n!}{n^n} = 0 < 1$ (since $\lim\limits_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$). Hence by the root test, the given series is convergent.

Example: The series $\sum\limits_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$ is not convergent.
Proof: Taking $x_n = \frac{5^n}{3^n+4^n}$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{5}{(3^n+4^n)^{\frac{1}{n}}} = \frac{5}{4} > 1$ (since $\lim\limits_{n\to\infty} (3^n+4^n)^{\frac{1}{n}} = 4$, as shown earlier). Hence by the root test, the given series is not convergent.

Example: For $p \in \mathbb{R}$, the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^p}$ is convergent iff $p > 0$.
Proof: For $p \leq 0$, $|(-1)^{n+1}\frac{1}{n^p}| = \frac{1}{n^p} \not\to 0$ and so $(-1)^{n+1}\frac{1}{n^p} \not\to 0$. Hence the given series is not convergent if $p \leq 0$. If $p > 0$, then $(\frac{1}{n^p})$ is a decreasing sequence of positive real numbers with $\frac{1}{n^p} \to 0$ and hence the given series converges by Leibniz's test.

Example: The series $\sum\limits_{n=1}^{\infty} (-1)^{n+1}\frac{n}{n^3+1}$ is convergent.
Proof: Since $(n+1)^2 + \frac{1}{n+1} = n^2 + 1 + 2n + \frac{1}{n+1} > n^2 + \frac{1}{n}$ for all $n \in \mathbb{N}$, we get $\frac{n+1}{(n+1)^3+1} = \frac{1}{(n+1)^2+\frac{1}{n+1}} < \frac{1}{n^2+\frac{1}{n}} = \frac{n}{n^3+1}$ for all $n \in \mathbb{N}$. Hence $(\frac{n}{n^3+1})$ is a decreasing sequence of positive real numbers. Also, $\frac{n}{n^3+1} = \frac{\frac{1}{n^2}}{1+\frac{1}{n^3}} \to 0$. Therefore by Leibniz's test, the given alternating series is convergent.

Alternative proof: Since $0 < \frac{n}{n^3+1} < \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test, the series $\sum\limits_{n=1}^{\infty} |(-1)^{n+1}\frac{n}{n^3+1}| = \sum\limits_{n=1}^{\infty} \frac{n}{n^3+1}$ converges. Thus $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n}{n^3+1}$ is an absolutely convergent series and hence it is convergent.

Example: If $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = s$, then $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \frac{3}{2}s$.
Proof: We first note that by Leibniz's test, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.
Let $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = s$. (i)
Then the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots = \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \cdots)$ converges to $\frac{1}{2}s$. It follows that the series

$0+\frac{1}{2}-0-\frac{1}{4}+0+\frac{1}{6}-0-\frac{1}{8}+\cdots$ (ii)
also converges to $\frac{1}{2}s$. Hence the series $(1+0)+(-\frac{1}{2}+\frac{1}{2})+(\frac{1}{3}-0)+(-\frac{1}{4}-\frac{1}{4})+(\frac{1}{5}+0)+\cdots$, which is the sum of the series (i) and (ii), converges to $s+\frac{1}{2}s=\frac{3}{2}s$. Therefore it follows that $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\cdots=\frac{3}{2}s$.

Continuity

Example: $\lim\limits_{n\to\infty}\frac{\sin(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}-\sqrt{n}}=1$
Proof: Since $\sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}\to 0$, using the fact that $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$, we obtain $\lim\limits_{n\to\infty}\frac{\sin(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}-\sqrt{n}}=1$.

Example: The function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=\begin{cases}3x+2 & \text{if } x<1,\\ 4x^2 & \text{if } x\geq 1,\end{cases}$
is not continuous at 1.
Proof: Since $\lim\limits_{x\to 1-}f(x)=\lim\limits_{x\to 1-}(3x+2)=5\neq 4=f(1)$, f is not continuous at 1.

Example: The function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=\begin{cases}x\sin\frac{1}{x} & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$
is continuous at 0.
Proof: For all $x(\neq 0)\in\mathbb{R}$, $|f(x)-f(0)|=|x\sin\frac{1}{x}|\leq|x|$ and hence given any $\varepsilon>0$, choosing $\delta=\varepsilon>0$, we get $|f(x)-f(0)|<\varepsilon$ for all $x\in\mathbb{R}$ satisfying $|x-0|<\delta$. Therefore f is continuous at 0.

Example: The function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=\begin{cases}\sin\frac{1}{x} & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$
is not continuous at 0.
Proof: If $x_n=\frac{2}{(4n+1)\pi}$ for all $n\in\mathbb{N}$, then the sequence (x_n) converges to 0, but $f(x_n)=\sin(4n+1)\frac{\pi}{2}=1$ for all $n\in\mathbb{N}$ and so $f(x_n)\to 1\neq 0=f(0)$. Therefore f is not continuous at 0.

Example: $\lim\limits_{x\to 0}\sin\frac{1}{x}$ does not exist (in $\mathbb{R}$).
Proof: If $x_n=\frac{2}{(4n+1)\pi}$ and $y_n=\frac{1}{n\pi}$ for all $n\in\mathbb{N}$, then $x_n\to 0$ and $y_n\to 0$. However, since $\sin\frac{1}{x_n}=1$ and $\sin\frac{1}{y_n}=0$ for all $n\in\mathbb{N}$, we get $\sin\frac{1}{x_n}\to 1$ and $\sin\frac{1}{y_n}\to 0$. Therefore by the sequential criterion for limit, $\lim\limits_{x\to 0}\sin\frac{1}{x}$ does not exist (in $\mathbb{R}$).

Example: The function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q},\\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q},\end{cases}$
is not continuous at any point of $\mathbb{R}$.
Proof: If $x_0\in\mathbb{Q}$, then there exists a sequence (t_n) in $\mathbb{R}\setminus\mathbb{Q}$ such that $t_n\to x_0$. Since $f(t_n)=0$ for all $n\in\mathbb{N}$, $f(t_n)\to 0\neq 1=f(x_0)$. Hence f is not continuous at x_0. Again, if $x_0\in\mathbb{R}\setminus\mathbb{Q}$, then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n\to x_0$. Since $f(r_n)=1$ for all $n\in\mathbb{N}$, $f(r_n)\to 1\neq 0=f(x_0)$. Hence f is not continuous at x_0.

Example: The function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=\begin{cases}x & \text{if } x\in\mathbb{Q},\\ -x & \text{if } x\in\mathbb{R}\setminus\mathbb{Q},\end{cases}$
is continuous only at 0.
Proof: Given any $\varepsilon>0$, choosing $\delta=\varepsilon>0$, we have $|f(x)-f(0)|=|x|<\varepsilon$ for all $x\in\mathbb{R}$ satisfying $|x-0|<\delta$. Therefore f is continuous at 0. If $x_0(\neq 0)\in\mathbb{Q}$, then there exists a sequence (t_n) in $\mathbb{R}\setminus\mathbb{Q}$ such that $t_n\to x_0$. Since $f(t_n)=-t_n$ for all $n\in\mathbb{N}$, $f(t_n)\to -x_0\neq x_0=f(x_0)$.

Hence f is not continuous at x_0. Again, if $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to x_0$. Since $f(r_n) = x_0$ for all $n \in \mathbb{N}$, $f(r_n) \to x_0 \neq -x_0 = f(x_0)$. Hence f is not continuous at x_0.

Example: The equation $x^2 = x \sin x + \cos x$ has at least two real roots.
Proof: Let $f(x) = x^2 - x \sin x - \cos x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(-\pi) = \pi^2 + 1 > 0$, $f(0) = -1 < 0$ and $f(\pi) = \pi^2 + 1 > 0$. Hence by the intermediate value theorem, the equation $f(x) = 0$ has at least one root in $(-\pi, 0)$ and at least one root in $(0, \pi)$. Thus the equation $f(x) = 0$ has at least two real roots.

Example: If $f : [0, 1] \to [0, 1]$ is continuous, then there exists $c \in [0, 1]$ such that $f(c) = c$.
Proof: Let $g(x) = f(x) - x$ for all $x \in [0, 1]$. Since f is continuous, $g : [0, 1] \to \mathbb{R}$ is continuous. If $f(0) = 0$ or $f(1) = 1$, then we get the result by taking $c = 0$ or $c = 1$ respectively. Otherwise $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$ (since it is given that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$). Hence by the intermediate value theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$, *i.e.* $f(c) = c$.

Example: Let $f : [0, 2] \to \mathbb{R}$ be continuous such that $f(0) = f(2)$. Then there exist $x_1, x_2 \in [0, 2]$ such that $x_1 - x_2 = 1$ and $f(x_1) = f(x_2)$.
Proof: Let $g(x) = f(x + 1) - f(x)$ for all $x \in [0, 1]$. Since f is continuous, $g : [0, 1] \to \mathbb{R}$ is continuous. Also, $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1) = -g(0)$, since $f(0) = f(2)$. If $g(0) = 0$, then $f(1) = f(0)$ and we get the result by taking $x_1 = 1$ and $x_2 = 0$. If $g(0) \neq 0$, then $g(0)$ and $g(1)$ are of opposite signs and hence by the intermediate value theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$, *i.e.* $f(c+1) = f(c)$. We get the result by taking $x_1 = c+1$ and $x_2 = c$.

Example: There does not exist any continuous function from $[0, 1]$ onto $(0, \infty)$.
Proof: If $f : [0, 1] \to (0, \infty)$ is continuous, then f must be bounded. Since $(0, \infty)$ is not a bounded set in $\mathbb{R}$, it follows that f cannot be onto.

Differentiation

Example: Let $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x (\neq 0) \in \mathbb{R}, \\ 0 & \text{if } x = 0. \end{cases}$
Then $f : \mathbb{R} \to \mathbb{R}$ is not differentiable at 0.
Proof: Since $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \sin \frac{1}{x}$ does not exist, f is not differentiable at 0.

Example: Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x (\neq 0) \in \mathbb{R}, \\ 0 & \text{if } x = 0. \end{cases}$
Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable but $f' : \mathbb{R} \to \mathbb{R}$ is not continuous at 0.
Proof: Clearly f is differentiable at all $x (\neq 0) \in \mathbb{R}$ and $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ for all $x (\neq 0) \in \mathbb{R}$. Also, for each $\varepsilon > 0$, choosing $\delta = \varepsilon > 0$, we find that $|\frac{f(x) - f(0)}{x - 0}| = |x \sin \frac{1}{x}| \leq |x| < \varepsilon$ for all $x \in \mathbb{R}$ satisfying $0 < |x| < \delta$. Hence $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$ and consequently f is differentiable at 0 with $f'(0) = 0$. Thus $f : \mathbb{R} \to \mathbb{R}$ is differentiable.
Again, since $\frac{1}{2n\pi} \to 0$ but $f'(\frac{1}{2n\pi}) \to -1 \neq f'(0)$, $f' : \mathbb{R} \to \mathbb{R}$ is not continuous at 0.

Example: Let $f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x (\neq 0) \in \mathbb{R}, \\ 0 & \text{if } x = 0. \end{cases}$
Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f' : \mathbb{R} \to \mathbb{R}$ is continuous, but f' is not differentiable at 0.
Proof: Clearly f is differentiable at all $x (\neq 0) \in \mathbb{R}$ and $f'(x) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$ for all $x (\neq 0) \in \mathbb{R}$. Also, for each $\varepsilon > 0$, choosing $\delta = \sqrt{\varepsilon} > 0$, we find that $|\frac{f(x) - f(0)}{x - 0}| = |x^2 \sin \frac{1}{x}| \leq |x|^2 < \varepsilon$ for all

$x \in \mathbb{R}$ satisfying $0 < |x| < \delta$. Hence $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ and consequently f is differentiable at 0 with $f'(0) = 0$. Thus $f : \mathbb{R} \to \mathbb{R}$ is differentiable.
Clearly $f' : \mathbb{R} \to \mathbb{R}$ is continuous at all $x(\neq 0) \in \mathbb{R}$. Also, since $\lim\limits_{x\to 0} x^2 \sin \frac{1}{x} = 0$ and $\lim\limits_{x\to 0} x \cos \frac{1}{x} = 0$ (similar to what we have shown earlier), we obtain $\lim\limits_{x\to 0} f'(x) = 0 = f'(0)$, which shows that f' is continuous at 0. Thus $f' : \mathbb{R} \to \mathbb{R}$ is continuous.
Again, $\lim\limits_{x\to 0} \frac{f'(x)-f'(0)}{x-0} = \lim\limits_{x\to 0}(3x \sin \frac{1}{x} - \cos \frac{1}{x})$ does not exist, because if $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+1)\pi}$ for all $n \in \mathbb{N}$, then $x_n \to 0$ and $y_n \to 0$, but $\lim\limits_{n\to\infty}(3x_n \sin \frac{1}{x_n} - \cos \frac{1}{x_n}) = -1$ and $\lim\limits_{n\to\infty}(3y_n \sin \frac{1}{y_n} - \cos \frac{1}{y_n}) = 1$. Therefore f' is not differentiable at 0.

Example: Let $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable only at 0 and $f'(0) = 0$.
Proof: If $x_0(\neq 0) \in \mathbb{Q}$, then there exists a sequence (t_n) in $\mathbb{R}\setminus\mathbb{Q}$ such that $t_n \to x_0$. Since $f(t_n) = 0$ for all $n \in \mathbb{N}$, $f(t_n) \to 0 \neq x_0^2 = f(x_0)$. Hence f is not continuous at x_0. Also, if $u_0 \in \mathbb{R} \setminus \mathbb{Q}$, then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to u_0$. Since $f(r_n) = r_n^2 \to u_0^2 \neq 0 = f(u_0)$, f is not continuous at u_0. Thus f is not continuous at any $x(\neq 0) \in \mathbb{R}$ and therefore f cannot be differentiable at any $x(\neq 0) \in \mathbb{R}$.
Again, for each $\varepsilon > 0$, choosing $\delta = \varepsilon > 0$, we find that $|\frac{f(x)-f(0)}{x-0}| \leq |x| < \varepsilon$ for all $x \in \mathbb{R}$ satisfying $0 < |x| < \delta$. Hence $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ and consequently f is differentiable at 0 with $f'(0) = 0$.

Example: The equation $x^2 = x \sin x + \cos x$ has exactly two (distinct) real roots.
Proof: Let $f(x) = x^2 - x \sin x - \cos x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable (and hence continuous) with $f'(x) = x(2 - \cos x)$ for all $x \in \mathbb{R}$. Since $\cos x \neq 2$ for any $x \in \mathbb{R}$, the equation $f'(x) = 0$ has exactly one real root, *viz.* $x = 0$. As a consequence of Rolle's theorem, it follows that the equation $f(x) = 0$ has at most two real roots. Also, since $f(-\pi) = \pi^2 + 1 > 0$, $f(0) = -1 < 0$ and $f(\pi) = \pi^2 + 1 > 0$, by the intermediate value property of continuous functions, the equation $f(x) = 0$ has at least one root in $(-\pi, 0)$ and at least one root in $(0, \pi)$. Thus the equation $f(x) = 0$ has exactly two (distinct) real roots and so the given equation has exactly two (distinct) real roots.

Example: Find the number of (distinct) real roots of the equation $x^4 + 2x^2 - 6x + 2 = 0$.
Solution: Taking $f(x) = x^4 + 2x^2 - 6x + 2$ for all $x \in \mathbb{R}$, we find that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable with $f'(x) = 4x^3 + 4x - 6$ and $f''(x) = 12x^2 + 4$ for all $x \in \mathbb{R}$. Since $f''(x) \neq 0$ for all $x \in \mathbb{R}$, as a consequence of Rolle's theorem, it follows that the equation $f'(x) = 0$ has at most one real root and hence the equation $f(x) = 0$ has at most two real roots. Again, since $f(0) = 2 > 0$, $f(1) = -2 < 0$ and $f(2) = 14 > 0$, by the intermediate value property of continuous functions, the equation $f(x) = 0$ has at least one real root in $(0, 1)$ and at least one real root in $(1, 2)$. Therefore the given equation has exactly two (distinct) real roots.

Example: $\sin x \geq x - \frac{x^3}{6}$ for all $x \in [0, \frac{\pi}{2}]$.
Proof: Let $f(x) = \sin x - x + \frac{x^3}{6}$ for all $x \in [0, \frac{\pi}{2}]$. Then $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ is infinitely differentiable and $f'(x) = \cos x - 1 + \frac{x^2}{2}$, $f''(x) = \sin x + x$ and $f'''(x) = 1 - \cos x$ for all $x \in [0, \frac{\pi}{2}]$. Since $f'''(x) \geq 0$ for all $x \in [0, \frac{\pi}{2}]$, f'' is increasing on $[0, \frac{\pi}{2}]$. Hence $f''(x) \geq f''(0) = 0$ for all $x \in [0, \frac{\pi}{2}]$. This shows that f' is increasing on $[0, \frac{\pi}{2}]$ and so $f'(x) \geq f'(0) = 0$ for all $x \in [0, \frac{\pi}{2}]$. Thus f is increasing on $[0, \frac{\pi}{2}]$ and so $f(x) \geq f(0) = 0$ for all $x \in [0, \frac{\pi}{2}]$. Therefore $\sin x \geq x - \frac{x^3}{6}$ for all $x \in [0, \frac{\pi}{2}]$.

Example: If $f(x) = x^3 + x^2 - 5x + 3$ for all $x \in \mathbb{R}$, then f is one-one on $[1, 5]$ but not one-one on $\mathbb{R}$.
Proof: $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) = 3x^2 + 2x - 5$ for all $x \in \mathbb{R}$. Clearly $f'(x) \neq 0$ for all $x \in (1, 5)$ and hence f is one-one on $[1, 5]$. Again, since $f(0) = 3$, $f(1) = 0$ and $f(2) = 5$, by the intermediate value property of continuous functions, there exist $x_1 \in (0, 1)$ and $x_2 \in (1, 2)$ such that $f(x_1) = 1 = f(x_2)$. Therefore f is not one-one on $\mathbb{R}$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(-1) = 5$, $f(0) = 0$ and $f(1) = 10$. Then there exist $c_1, c_2 \in (-1, 1)$ such that $f'(c_1) = -3$ and $f'(c_2) = 3$.
Proof: By the mean value theorem, there exist $\alpha \in (-1, 0)$ and $\beta \in (0, 1)$ such that $f'(\alpha) = \frac{f(0)-f(-1)}{0-(-1)} = -5$ and $f'(\beta) = \frac{f(1)-f(0)}{1-0} = 10$. Hence by the intermediate value property of derivatives, there exist $c_1, c_2 \in (\alpha, \beta)$ (and so $c_1, c_2 \in (-1, 1)$) such that $f'(c_1) = -3$ and $f'(c_2) = 3$.

Example: If $f(x) = 1 - x^{2/3}$ for all $x \in \mathbb{R}$, then f has no local maximum or local minimum at any nonzero $x \in \mathbb{R}$. Further, f has a local maximum at 0.
Proof: $f : \mathbb{R} \to \mathbb{R}$ is differentiable at all $x(\neq 0) \in \mathbb{R}$ and $f'(x) = -\frac{2}{3}x^{-1/3} \neq 0$ for all $x(\neq 0) \in \mathbb{R}$. Hence f does not have local maximum or local minimum at any $x(\neq 0) \in \mathbb{R}$. Again, since $f(x) \leq 1 = f(0)$ for all $x \in \mathbb{R}$, f has a local maximum at 0 (and the local maximum value is $f(0) = 1$).

Alternative method for showing local maximum at 0: Since $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, f has a local maximum at 0.

Example: $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$
Proof: Applying (first version of) L'Hôpital's rule, we obtain $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \frac{\frac{d}{dx}(\sqrt{1+x}-1)|_{x=0}}{\frac{d}{dx}(x)|_{x=0}} = \frac{1}{2}$.

Alternative proof: Applying (second version of) L'Hôpital's rule, we obtain $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim\limits_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1}$ $= \frac{1}{2}$.

Example: $\lim\limits_{x \to \frac{\pi}{2}} \frac{1-\sin x}{1+\cos 2x} = \frac{1}{4}$
Proof: Applying L'Hôpital's rule twice, we obtain $\lim\limits_{x \to \frac{\pi}{2}} \frac{1-\sin x}{1+\cos 2x} = \lim\limits_{x \to \frac{\pi}{2}} \frac{-\cos x}{-2\sin 2x} = \lim\limits_{x \to \frac{\pi}{2}} \frac{\sin x}{-4\cos 2x} = \frac{1}{4}$.

Example: $\lim\limits_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = 0$
Proof: For all $x(\neq 0) \in \mathbb{R}$, we have $0 \leq |x \sin \frac{1}{x}| \leq |x|$. Since $\lim\limits_{x \to 0} |x| = 0$, by sandwich theorem (for limit of functions), we get $\lim\limits_{x \to 0} |x \sin \frac{1}{x}| = 0$ and hence $\lim\limits_{x \to 0} x \sin \frac{1}{x} = 0$. It follows that $\lim\limits_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = \lim\limits_{x \to 0} \frac{x \sin \frac{1}{x}}{\frac{\sin x}{x}} = \frac{\lim\limits_{x \to 0} x \sin \frac{1}{x}}{\lim\limits_{x \to 0} \frac{\sin x}{x}} = \frac{0}{1} = 0$.

Example: $\lim\limits_{x \to 0} (\frac{\sin x}{x})^{\frac{1}{x}} = 1$
Proof: Let $f(x) = (\frac{\sin x}{x})^{\frac{1}{x}}$ for all $x(\neq 0) \in \mathbb{R}$. Then $f(x) > 0$ for all $x \in (-1, 1) \setminus \{0\}$ and we have $\lim\limits_{x \to 0} \log f(x) = \lim\limits_{x \to 0} \frac{\log(\frac{\sin x}{x})}{x} = \lim\limits_{x \to 0} \frac{x \cos x - \sin x}{x \sin x}$ (applying L'Hôpital's rule) $= \lim\limits_{x \to 0} \frac{-x \sin x}{\sin x + x \cos x}$ (applying L'Hôpital's rule again) $= \lim\limits_{x \to 0} \frac{-\sin x}{\frac{\sin x}{x} + \cos x} = 0$ (since $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$). By the continuity of the exponential function, it follows that $\lim\limits_{x \to 0} f(x) = e^0 = 1$.

Example: $\lim\limits_{x \to \infty} \frac{x-\sin x}{2x+\sin x} = \frac{1}{2}$
Proof: Since $|\frac{\sin x}{x}| \leq \frac{1}{x}$ for all $x > 0$ and since $\lim\limits_{x \to \infty} \frac{1}{x} = 0$, we get $\lim\limits_{x \to \infty} \frac{\sin x}{x} = 0$. Consequently $\lim\limits_{x \to \infty} \frac{x-\sin x}{2x+\sin x} = \lim\limits_{x \to \infty} \frac{1-\frac{\sin x}{x}}{2+\frac{\sin x}{x}} = \frac{1}{2}$.

Example: The sequence $(\frac{\log n}{n})$ is convergent with $\lim\limits_{n \to \infty} \frac{\log n}{n} = 0$.
Proof: Let $f(x) = \frac{\log x}{x}$ for all $x > 0$. Then applying L'Hôpital's rule, we obtain $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \frac{1/x}{1} = 0$. Therefore by the sequential criterion of limit, the sequence $(f(n)) = (\frac{\log n}{n})$ converges to 0.

Example: $1+\frac{x}{2}-\frac{x^2}{8} \le \sqrt{1+x} \le 1+\frac{x}{2}$ for all $x>0$.
Proof: Let $x>0$ and let $f(t)=\sqrt{1+t}$ for all $x\in[0,x]$. Then $f:[0,x]\to\mathbb{R}$ is twice differentiable and $f'(t)=\frac{1}{2\sqrt{1+t}}$, $f''(t)=-\frac{1}{4(1+t)^{3/2}}$ for all $t\in[0,x]$. By Taylor's theorem, there exists $c\in(0,x)$ such that $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(c)=1+\frac{x}{2}-\frac{x^2}{8}\cdot\frac{1}{(1+c)^{3/2}}$. Since $0<\frac{1}{(1+c)^{3/2}}<1$, we get $1+\frac{x}{2}-\frac{x^2}{8}\le\sqrt{1+x}\le 1+\frac{x}{2}$.

Example: For the power series $\sum\limits_{n=0}^{\infty}\frac{x^n}{n^2}$, the radius of convergence is 1 and the interval of convergence is $[-1,1]$
Proof: If $x=0$, then the given series becomes $0+0+\cdots$, which is clearly convergent. Let $x(\neq 0)\in\mathbb{R}$ and let $a_n=\frac{x^n}{n^2}$ for all $n\in\mathbb{N}$. Then $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}|=|x|$. Hence by ratio test, $\sum\limits_{n=1}^{\infty}a_n$ is convergent (absolutely) if $|x|<1$, *i.e.* if $x\in(-1,1)$ and is not convergent if $|x|>1$, *i.e.* if $x\in(-\infty,-1)\cup(1,\infty)$. Therefore the radius of convergence of the given power series is 1. Again, if $|x|=1$, then $\sum\limits_{n=1}^{\infty}|a_n|=\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ is convergent and hence $\sum\limits_{n=1}^{\infty}a_n$ is also convergent. Therefore the interval of convergence of the given power series is $[-1,1]$.

Example: For the power series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 4^n}(x-1)^n$, the radius of convergence is 4 and the interval of convergence is $(-3,5]$.
Proof: If $x=1$, then the given series becomes $0+0+\cdots$, which is clearly convergent. Let $x(\neq 1)\in\mathbb{R}$ and let $a_n=\frac{(-1)^n}{n.4^n}(x-1)^n$ for all $n\in\mathbb{N}$. Then $\lim\limits_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\frac{1}{4}|x-1|$. Hence by ratio test, $\sum\limits_{n=1}^{\infty}a_n$ is convergent (absolutely) if $\frac{1}{4}|x-1|<1$, *i.e.* if $x\in(-3,5)$ and is not convergent if $\frac{1}{4}|x-1|>1$, *i.e.* if $x\in(-\infty,-3)\cup(5,\infty)$. Therefore the radius of convergence of the given power series is 4. Again, if $x=-3$, then $\sum\limits_{n=1}^{\infty}a_n=\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is not convergent. If $x=5$, then $\sum\limits_{n=1}^{\infty}a_n=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}$ is convergent by Leibniz test, since $(\frac{1}{n})$ is a decreasing sequence of positive real numbers and $\lim\limits_{n\to\infty}\frac{1}{n}=0$. Therefore the interval of convergence of the given power series is $(-3,5]$.

Example: The Maclaurin series for e^x converges to e^x for all $x\in\mathbb{R}$.
Proof: If $f(x)=e^x$ for all $x\in\mathbb{R}$, then $f:\mathbb{R}\to\mathbb{R}$ is infinitely differentiable and $f^{(n)}(x)=e^x$ for all $x\in\mathbb{R}$ and for all $n\in\mathbb{N}$. Hence the Maclaurin series for e^x is the series $1+\sum\limits_{n=1}^{\infty}\frac{x^n}{n!}$, where $x\in\mathbb{R}$. For $x=0$, the Maclaurin series of e^x becomes $1+0+0+\cdots$, which clearly converges to $e^0=1$. Let $x(\neq 0)\in\mathbb{R}$. The remainder term in the Taylor expansion of e^x about the point 0 is given by $R_n(x)=\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c_n)=\frac{x^{n+1}}{(n+1)!}e^{c_n}$, where c_n lies between 0 and x. Since $e^{c_n}<e^x$ if $x>0$ and $e^{c_n}<1$ if $x<0$, we get $|R_n(x)|\le\frac{|x|^{n+1}}{(n+1)!}e^x$ if $x>0$ and $|R_n(x)|\le\frac{|x|^{n+1}}{(n+1)!}$ if $x<0$. Also, since $\lim\limits_{n\to\infty}\frac{|x|^{n+2}}{(n+2)!}\cdot\frac{(n+1)!}{|x|^{n+1}}=\lim\limits_{n\to\infty}\frac{|x|}{n+2}=0<1$, we get $\lim\limits_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!}=0$ and hence it follows that $\lim\limits_{n\to\infty}R_n(x)=0$. Therefore the Maclaurin series of e^x converges to e^x.

Example: The Maclaurin series for $\sin x$ converges to $\sin x$ for all $x\in\mathbb{R}$.
Proof: If $f(x)=\sin x$ for all $x\in\mathbb{R}$, then $f:\mathbb{R}\to\mathbb{R}$ is infinitely differentiable and $f^{(2n-1)}(x)=(-1)^{n+1}\cos x$, $f^{(2n)}(x)=(-1)^n\sin x$ for all $x\in\mathbb{R}$ and for all $n\in\mathbb{N}$. Hence the Maclaurin series for $\sin x$ is the series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$, where $x\in\mathbb{R}$. For $x=0$, the Maclaurin series of $\sin x$ becomes $0-0+0-\cdots$, which clearly converges to $\sin 0=0$. Let $x(\neq 0)\in\mathbb{R}$. The remainder term in the Taylor expansion of $\sin x$ about the point 0 is given by $R_n(x)=\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c_n)$, where c_n lies between 0 and x. Since $|\sin c_n|\le 1$ and $|\cos c_n|\le 1$, we get $|R_n(x)|\le\frac{|x|^{n+1}}{(n+1)!}$. Also,

since $\lim\limits_{n\to\infty} \frac{|x|^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{|x|^{n+1}} = \lim\limits_{n\to\infty} \frac{|x|}{n+2} = 0 < 1$, we get $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ and hence it follows that $\lim\limits_{n\to\infty} R_n(x) = 0$. Therefore the Maclaurin series of $\sin x$ converges to $\sin x$.

Example: The Maclaurin series for $\cos x$ converges to $\cos x$ for all $x \in \mathbb{R}$.
Proof: If $f(x) = \cos x$ for all $x \in \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and $f^{(2n-1)}(x) = (-1)^n \sin x$, $f^{(2n)}(x) = (-1)^n \cos x$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Hence the Maclaurin series for $\cos x$ is the series $1 + \sum\limits_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, where $x \in \mathbb{R}$. For $x = 0$, the Maclaurin series of $\cos x$ becomes $1 - 0 + 0 - \cdots$, which clearly converges to $\cos 0 = 1$. Let $x (\neq 0) \in \mathbb{R}$. The remainder term in the Taylor expansion of $\sin x$ about the point 0 is given by $R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c_n)$, where c_n lies between 0 and x. Since $|\sin c_n| \leq 1$ and $|\cos c_n| \leq 1$, we get $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$. Also, since $\lim\limits_{n\to\infty} \frac{|x|^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{|x|^{n+1}} = \lim\limits_{n\to\infty} \frac{|x|}{n+2} = 0 < 1$, we get $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ and hence it follows that $\lim\limits_{n\to\infty} R_n(x) = 0$. Therefore the Maclaurin series of $\cos x$ converges to $\cos x$.

Example: If $f(x) = x^5 - 5x^4 + 5x^3 + 12$ for all $x \in \mathbb{R}$, then f has a local maximum only at 1 and a local minimum only at 3.
Proof: $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and $f'(x) = 5x^2(x-1)(x-3)$, $f''(x) = 10x(2x^2-6x+3)$, $f'''(x) = 30(2x^2 - 4x + 1)$ for all $x \in \mathbb{R}$. Since $f'(x) = 0$ iff $x = 0$, 1, or 3, f has neither a local maximum nor a local minimum at any point of $\mathbb{R} \setminus \{0, 1, 3\}$. Again, since $f''(1) = -10 < 0$, $f''(3) = 90 > 0$, $f''(0) = 0$ and $f'''(0) = 30 \neq 0$, f has a local maximum at 1 (with local maximum value $f(1) = 13$), f has a local minimum at 3 (with local minimum value $f(3) = -15$) and f has neither a local maximum nor a local minimum at 0.

Integration

Example: Let $f(x) = x^4 - 4x^3 + 10$ for all $x \in [1, 4]$. Then for the partition $P = \{1, 2, 3, 4\}$ of $[1, 4]$, $U(f, P) = 11$ and $L(f, P) = -40$.
Proof: Since $f'(x) = 4x^2(x - 3)$ for all $x \in [1, 4]$, we have $f'(x) < 0$ for all $x \in (1, 3)$ and $f'(x) > 0$ for all $x \in (3, 4)$. Hence f is strictly decreasing on $[1, 3]$ and strictly increasing on $[3, 4]$. Consequently $\sup\{f(x) : x \in [1, 2]\} = f(1) = 7$, $\sup\{f(x) : x \in [2, 3]\} = f(2) = -6$, $\sup\{f(x) : x \in [3, 4]\} = f(4) = 10$ and $\inf\{f(x) : x \in [1, 2]\} = f(2) = -6$, $\inf\{f(x) : x \in [2, 3]\} = f(3) = -17$, $\inf\{f(x) : x \in [3, 4]\} = f(3) = -17$. Therefore $U(f, P) = 7(2 - 1) + (-6)(3 - 2) + 10(4 - 3) = 11$ and $L(f, P) = (-6)(2 - 1) + (-17)(3 - 2) + (-17)(4 - 3) = -40$.

Example: Let $k \in \mathbb{R}$ and let $f(x) = k$ for all $x \in [0, 1]$. Then $f : [0, 1] \to \mathbb{R}$ is Riemann integrable on $[0, 1]$ and $\int\limits_0^1 f(x)\, dx = k$.
Proof: Clearly f is bounded on $[0, 1]$. Let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[0, 1]$. Clearly $M_i = k = m_i$ for $i = 1, ..., n$ and hence $U(f, P) = L(f, P) = \sum\limits_{i=1}^{n} k(x_i - x_{i-1}) = k$. Consequently $\int\limits_0^{\overline{1}} f(x)\, dx = k = \int\limits_{\underline{0}}^1 f(x)\, dx$. Therefore f is Riemann integrable on $[0, 1]$ and $\int\limits_0^1 f(x)\, dx = k$.

Example: Let $f(x) = \begin{cases} 0 & \text{if } x \in (0, 1], \\ 1 & \text{if } x = 0. \end{cases}$

Then $f : [0, 1] \to \mathbb{R}$ is Riemann integrable on $[0, 1]$ and $\int\limits_0^1 f(x)\, dx = 0$.
Proof: Clearly f is bounded on $[0, 1]$. Let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[0, 1]$. Then

$m_i = 0$ and $M_i \geq 0$ for $i = 1, ..., n$ and so $L(f, P) = 0$ and $U(f, P) \geq 0$. Hence $\underline{\int_0^1} f(x)\,dx = 0$ and $\overline{\int_0^1} f(x)\,dx \geq 0$. Again, if $0 < \varepsilon < 1$, then considering the partition $P_1 = \{0, \frac{\varepsilon}{2}, 1\}$ of $[0,1]$, we get $0 \leq \overline{\int_0^1} f(x)\,dx \leq U(f, P_1) = \frac{\varepsilon}{2} < \varepsilon$ and consequently $\overline{\int_0^1} f(x)\,dx = 0$. Therefore f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = 0$.

Example: Let $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Then $f : [0,1] \to \mathbb{R}$ is not Riemann integrable on $[0,1]$.
Proof: Clearly f is bounded on $[0,1]$. Let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[0,1]$. Since every interval contains a rational as well as an irrational number, we get $M_i = 1$ and $m_i = 0$ for $i = 1, ..., n$ and hence $U(f, P) = \sum_{i=1}^{n}(x_i - x_{i-1}) = 1$ and $L(f, P) = 0$. Consequently $\overline{\int_0^1} f(x)\,dx = 1$ and $\underline{\int_0^1} f(x)\,dx = 0$. Since $\overline{\int_0^1} f(x)\,dx \neq \underline{\int_0^1} f(x)\,dx$, f is not Riemann integrable on $[0,1]$.

Example: Let $f(x) = x$ for all $x \in [0,1]$. Then $f : [0,1] \to \mathbb{R}$ is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = \frac{1}{2}$.
Proof: Clearly f is bounded on $[0,1]$. For each $n \in \mathbb{N}$, $P_n = \{0, \frac{1}{n}, ..., \frac{n}{n} = 1\}$ is a partition of $[0,1]$. Also, $L(f, P_n) = \frac{1}{n}(0 + \frac{1}{n} + \cdots + \frac{n-1}{n}) = \frac{1}{2} - \frac{1}{2n} \to \frac{1}{2}$ and $U(f, P_n) = \frac{1}{n}(\frac{1}{n} + \cdots + \frac{n}{n}) = \frac{1}{2} + \frac{1}{2n} \to \frac{1}{2}$. Hence f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = \frac{1}{2}$.

Example: Let $f(x) = x^2$ for all $x \in [0,1]$. Then $f : [0,1] \to \mathbb{R}$ is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = \frac{1}{3}$.
Proof: Clearly f is bounded on $[0,1]$. For each $n \in \mathbb{N}$, $P_n = \{0, \frac{1}{n}, ..., \frac{n}{n} = 1\}$ is a partition of $[0,1]$. Also, $L(f, P_n) = \frac{1}{n}(0 + \frac{1}{n^2} + \cdots + \frac{(n-1)^2}{n^2}) = (1 - \frac{1}{n})(\frac{1}{3} - \frac{1}{6n}) \to \frac{1}{3}$ and $U(f, P_n) = \frac{1}{n}(\frac{1}{n^2} + \cdots + \frac{n^2}{n^2}) = (1 + \frac{1}{n})(\frac{1}{3} + \frac{1}{6n}) \to \frac{1}{3}$. Hence f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = \frac{1}{3}$.

Example: $\frac{1}{3\sqrt{2}} \leq \int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx \leq \frac{1}{3}$
Proof: Since $1 \leq \sqrt{1+x} \leq \sqrt{2}$ for all $x \in [0,1]$, we get $\frac{x^2}{\sqrt{2}} \leq \frac{x^2}{\sqrt{1+x}} \leq x^2$ for all $x \in [0,1]$. Since all the given functions are continuous and hence Riemann integrable on $[0,1]$, we get $\int_0^1 \frac{x^2}{\sqrt{2}}\,dx \leq \int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx \leq \int_0^1 x^2\,dx \Rightarrow \frac{1}{3\sqrt{2}} \leq \int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx \leq \frac{1}{3}$.

Example: $\lim\limits_{n\to\infty} [\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}] = \log 2$
Proof: Let $f(x) = \frac{1}{1+x}$ for all $x \in [0,1]$. Considering the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$ of $[0,1]$ for each $n \in \mathbb{N}$ (and taking $c_i = \frac{i}{n}$ for $i = 1, ..., n$), we find that
$S(f, P_n) = \sum_{i=1}^{n} f(\frac{i}{n})(\frac{i}{n} - \frac{i-1}{n}) = \sum_{i=1}^{n} \frac{1}{n+i}$. Since $f : [0,1] \to \mathbb{R}$ is continuous, f is Riemann integrable on $[0,1]$ and hence $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \lim\limits_{n\to\infty} S(f, P_n) = \int_0^1 f(x)\,dx = \log(1+x)|_{x=0}^{1} = \log 2$.

Example: $\int_1^\infty \frac{1}{t^p}\,dt$ converges iff $p > 1$.

Proof: For all $x > 1$, we have $\int_1^x \frac{1}{t^p}\,dt = \frac{1}{1-p}(x^{1-p}-1)$ if $p \neq 1$ and $\int_1^x \frac{1}{t}\,dt = \log x$. Hence $\lim_{x\to\infty}\int_1^x \frac{1}{t^p}\,dt = \frac{1}{1-p}$ if $p > 1$ and $\lim_{x\to\infty}\int_1^x \frac{1}{t^p}\,dt = \infty$ if $p \leq 1$. Therefore $\int_1^\infty \frac{1}{t^p}\,dt$ converges iff $p > 1$.

Example: The improper integral $\int_{-\infty}^{\infty} e^t\,dt$ is not convergent.

Proof: In order that the improper integral $\int_{-\infty}^{\infty} e^t\,dt$ converges, both $\int_{-\infty}^{0} e^t\,dt$ and $\int_0^\infty e^t\,dt$ must converge. However, $\int_0^\infty e^t\,dt$ does not converge, because $\lim_{x\to\infty}\int_0^x e^t\,dt = \lim_{x\to\infty}(e^x-1) = \infty$. Hence $\int_{-\infty}^{\infty} e^t\,dt$ is not convergent.

Example: The improper integral $\int_0^\infty \frac{1}{1+t^2}\,dt$ converges.

Proof: Since $\lim_{x\to\infty}\int_0^x \frac{1}{1+t^2}\,dt = \lim_{x\to\infty}\tan^{-1}x = \frac{\pi}{2}$, the given improper integral converges.

Example: The improper integral $\int_1^\infty \frac{\sin^2 t}{t^2}\,dt$ converges.

Proof: Since $0 \leq \frac{\sin^2 t}{t^2} \leq \frac{1}{t^2}$ for all $t \geq 1$ and since $\int_1^\infty \frac{1}{t^2}\,dt$ converges, by the comparison test, $\int_1^\infty \frac{\sin^2 t}{t^2}\,dt$ converges.

Example: The improper integral $\int_1^\infty \frac{dt}{t\sqrt{1+t^2}}$ converges.

Proof: Let $f(t) = \frac{1}{t\sqrt{1+t^2}}$ and $g(t) = \frac{1}{t^2}$ for all $t \geq 1$. Then $\lim_{t\to\infty}\frac{f(t)}{g(t)} = \lim_{t\to\infty}\frac{1}{\sqrt{1+\frac{1}{t^2}}} = 1$. Since $\int_1^\infty g(t)\,dt$ converges, by the limit comparison test, $\int_1^\infty f(t)\,dt$ also converges.

Example: the improper integral $\int_0^\infty \frac{\cos t}{1+t^2}\,dt$ converges.

Proof: Since $\int_0^1 \frac{\cos t}{1+t^2}\,dt$ exists (in $\mathbb{R}$) as a Riemann integral, $\int_0^\infty \frac{\cos t}{1+t^2}\,dt$ converges iff $\int_1^\infty \frac{\cos t}{1+t^2}\,dt$ converges. Now $\left|\frac{\cos t}{1+t^2}\right| \leq \frac{1}{t^2}$ for all $t \geq 1$ and $\int_1^\infty \frac{1}{t^2}\,dt$ converges. Hence by comparison test, $\int_1^\infty \left|\frac{\cos t}{1+t^2}\right|\,dt$ converges and consequently $\int_1^\infty \frac{\cos t}{1+t^2}\,dt$ converges. By our remark at the beginning, $\int_0^\infty \frac{\cos t}{1+t^2}\,dt$ converges.

Alternative proof: We have $\left|\frac{\cos t}{1+t^2}\right| \leq \frac{1}{1+t^2}$ for all $t \geq 0$. Also, since $\lim_{x\to\infty}\int_0^x \frac{1}{1+t^2}\,dt = \lim_{x\to\infty}\tan^{-1}x = \frac{\pi}{2}$, $\int_0^\infty \frac{1}{1+t^2}\,dt$ converges. Hence by comparison test, $\int_0^\infty \left|\frac{\cos t}{1+t^2}\right|\,dt$ converges and consequently $\int_0^\infty \frac{\cos t}{1+t^2}\,dt$ converges.

Example: The improper integral $\int_1^\infty \frac{\sin t}{t}\,dt$ converges.

Proof: Let $f(t) = \frac{1}{t}$ and $g(t) = \sin t$ for all $t \geq 1$. Then $f : [1,\infty) \to \mathbb{R}$ is decreasing and $\lim_{t\to\infty} f(t) = 0$. Also, for all $x \geq 1$, we have $\left|\int_1^x g(t)\,dt\right| = |\cos 1 - \cos x| \leq |\cos 1| + |\cos x| \leq 2$. Hence by Dirichlet's test, $\int_1^\infty f(t)g(t)\,dt$ converges.

Example: $\int_0^1 \frac{1}{t^p}\,dt$ converges iff $p < 1$.

Proof: $\int_0^1 \frac{1}{t^p}\,dt$ exists (in $\mathbb{R}$) as a Riemann integral if $p \le 0$. So let $p > 0$. Then for $0 < x < 1$, we have $\int_x^1 \frac{1}{t^p}\,dt = \frac{1}{1-p}(1 - x^{1-p})$ if $p \ne 1$ and $\int_x^1 \frac{1}{t}\,dt = -\log x$. Hence $\lim\limits_{x\to 0+} \int_x^1 \frac{1}{t^p}\,dt = \frac{1}{1-p}$ if $p < 1$ and $\lim\limits_{x\to 0+} \int_x^1 \frac{1}{t^p}\,dt = \infty$ if $p \ge 1$. Therefore $\int_0^1 \frac{1}{t^p}\,dt$ converges iff $p < 1$.

Example: The length of the curve $y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$ from $x = 0$ to $x = 3$ is 12.
Proof: Since $\frac{dy}{dx} = x(x^2+2)^{\frac{1}{2}}$ for all $x \in [0,3]$, the length of the given curve from $x = 0$ to $x = 3$ is $\int_0^3 \sqrt{1 + x^2(x^2+2)}\,dx = \int_0^3 (x^2+1)\,dx = 12$.

Example: The perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$.
Proof: The parametric equations of the given ellipse are $x = a\cos t$, $y = b\sin t$, where $0 \le t \le 2\pi$. Since $\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$ for all $t \in [0, 2\pi]$, the perimeter of the given ellipse is $\int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$. (This integral does not have a simple expression in terms of a and b.)

Example: The length of the curve $x = e^t\sin t$, $y = e^t\cos t$, $0 \le t \le \frac{\pi}{2}$.
Proof: Since $\frac{dx}{dt} = e^t\cos t + e^t\sin t$ and $\frac{dy}{dt} = e^t\cos t - e^t\sin t$ for all $t \in [0, \frac{\pi}{2}]$, the required length is $\int_0^{\frac{\pi}{2}} \sqrt{(e^t\cos t + e^t\sin t)^2 + (e^t\cos t - e^t\sin t)^2}\,dt = \sqrt{2}\int_0^{\frac{\pi}{2}} e^t\,dt = \sqrt{2}(e^{\frac{\pi}{2}} - 1)$.

Example: The length of the cardioid $r = 1 - \cos\theta$ is 8.
Proof: Since $\frac{dr}{d\theta} = \sin\theta$ for all $\theta \in [0, \pi]$, by symmetry, the length of the given cardioid is $2\int_0^{\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\,d\theta = 4\int_0^{\pi} \sin\frac{\theta}{2}\,d\theta = 8$.

Example: The area above the x-axis which is included between the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 2ax$, where $a > 0$, is $(\frac{3\pi-8}{12})a^2$.
Proof: Solving $y^2 = ax$ and $x^2 + y^2 = 2ax$, we obtain the x-coordinates of the common points on the given parabola and the circle as 0 and a. Therefore the required area is
$\int_0^a (\sqrt{2ax - x^2} - \sqrt{ax})\,dx = (\frac{3\pi-8}{12})a^2$. (The integral $\int_0^a \sqrt{2ax - x^2}\,dx$ can be evaluated by the substitution $x = 2a\sin^2\theta$.)

Example: The area of the region that is inside the cardioid $r = a(1+\cos\theta)$ and also inside the circle $r = \frac{3}{2}a$ is $(\frac{7\pi}{4} - \frac{9\sqrt{3}}{8})a^2$.
Proof: At a point of intersection of the cardioid $r = a(1+\cos\theta)$ and the circle $r = \frac{3}{2}a$, we have $a(1+\cos\theta) = \frac{3}{2}a$. So $\theta = \frac{\pi}{3}$ corresponds to a point of intersection. Hence by symmetry, the area of the region that is inside the cardioid $r = a(1+\cos\theta)$ and inside the circle $r = \frac{3}{2}a$ is

$$2\left[\frac{1}{2}\int_0^{\pi/3} (\tfrac{3}{2}a)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi} a^2(1+\cos\theta)^2\,d\theta\right] = (\tfrac{7\pi}{4} - \tfrac{9\sqrt{3}}{8})a^2.$$

Example: A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Then the volume of the solid is 16.
Proof: The length of the diagonal of the cross-sectional square at a distance x from the origin is $2\sqrt{x}$ and hence the cross-sectional area at a distance x from the origin is $2x$. Therefore the volume

of the solid is $\int_{0}^{4} 2x\,dx = 16$.

Example: The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.
Proof: The volume of a sphere of radius r is same as the volume of the solid generated by revolving the semi-circular area bounded by the curve $y = \sqrt{r^2 - x^2}$ between $x = -r$ and $x = r$ about the x-axis. Hence the required volume is $\int_{-r}^{r} \pi(r^2 - x^2)\,dx = \frac{4}{3}\pi r^3$.

Example: A round hole of radius $\sqrt{3}$ is bored through the centre of a solid sphere of radius 2. Then the volume of the portion bored out is $\frac{28}{3}\pi$.
Proof: The required volume is $V_1 - V_2$, where V_1 is the volume of the solid sphere of radius 2 and V_2 is the volume of the solid generated by revolving the plane region common to $x^2 + y^2 \leq 4$ and $y \geq \sqrt{3}$ about the x-axis. We know that $V_1 = \frac{32}{3}\pi$. Also, solving $x^2 + y^2 = 4$ and $y = \sqrt{3}$, we get $x = -1, 1$ and so $V_2 = \int_{-1}^{1} \pi(4 - x^2 - 3)\,dx = \frac{4}{3}\pi$. Therefore the required volume is $\frac{28}{3}\pi$.
Example: The volume and area of the curved surface of a paraboloid of revolution formed by revolving the parabola $y^2 = 4ax$ about the x-axis, and bounded by the section $x = x_1$ are $2\pi a x_1^2$ and $\frac{8}{3}\pi\sqrt{a}((a + x_1)^{\frac{3}{2}} - a^{\frac{3}{2}})$ respectively.
Proof: The required volume is $\int_{0}^{x_1} 4\pi a x\,dx = 2\pi a x_1^2$ and the required surface area is
$\int_{0}^{x_1} 2\pi\sqrt{4ax}\sqrt{1 + \frac{a}{x}}\,dx$ (since $\frac{dy}{dx} = \frac{2a}{y}$) $= \frac{8}{3}\pi\sqrt{a}((a + x_1)^{\frac{3}{2}} - a^{\frac{3}{2}})$.