DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Guwahati

MA101: Mathematics I, July - November, 2014

Tutorial Sheet: LA - 4

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1. For all invertible matrices A and B of the same size, show that $\text{adj}(AB) = \text{adj}(B)\text{adj}(A)$. (The result is also true for non-invertible matrices, but the proof is beyond the present scope of this course.)

2. If A is an $n \times n$ matrix then prove that $\det(\text{adj}(A)) = (\det A)^{n-1}$.

3. Let A and B be two matrices, where $\text{rank}(A) = r$. Show that

   (a) if AB is defined then $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$;

   (b) there exist invertible matrices T, S such that
   $$TAS = \begin{bmatrix} I_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix};$$

   (c) there exist matrices $P_{m\times r}, Q_{r\times n}$ such that $A = PQ$; (This is known as Rank-Factorization Theorem)

   (d) A can be expressed as a sum of r rank one matrices;

   (e) $r = 1$ if and only if $A = \mathbf{u}\mathbf{v}^t$ for some $\mathbf{u}(\neq \mathbf{0}) \in \mathbb{R}^m$ and $\mathbf{v}\ (\neq \mathbf{0}) \in \mathbb{R}^n$;

   (f) if $A + B$ is defined then $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

4. Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 6 & 4 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. Determine all $\mathbf{b} \in \mathbb{R}^3$ for which the system $A\mathbf{x} = \mathbf{b}$ is consistent.

5. Let A be an $n \times n$ matrix. Find the eigenvalues of $A - 3I$ in terms of the eigenvalues of A. Also, show that their corresponding eigenspaces are equal.

6. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let $k \in \mathbb{R}$. Suppose that $\sum\limits_{j=1}^{n} a_{ij} = k$ for $i = 1, 2, \ldots, n$. Prove that k is an eigenvalue of A. Also, find an eigenvector of A corresponding to the eigenvalue k.

7. Find all real values of a, b, c, d, e, f for which the matrix $\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.

8. Let A be a 6×6 matrix with characteristic polynomial $p(\lambda) = (1 + \lambda)(1 - \lambda)^2(2 - \lambda)^3$.

   (a) Prove that it is not possible to find three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^6$ such that $A\mathbf{v}_1 = \mathbf{v}_1, A\mathbf{v}_2 = \mathbf{v}_2$ and $A\mathbf{v}_3 = \mathbf{v}_3$.

   (b) If A is diagonalizable, find the dimensions of the eigenspaces E_{-1}, E_1 and E_2?

9. Let A and B be two $n \times n$ matrices satisfying $AB = BA$ and let B have n distinct eigenvalues. Show that the matrix A is diagonalizable.
