Multiple integrals and change of variables

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Riemann sum for Triple integral

Consider the rectangular cube $V := [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ and a bounded function $f : V \to \mathbb{R}$.

Let P be a partition of V into sub-cubes V_{ijk} and $\mathbf{c}_{ijk} \in V_{ijk}$ for $i = 1 : m$, $j = 1 : n$, $k = 1 : p$. Also let

$\Delta V_{ijk} := \mathrm{Volume}(V_{ijk}) = \Delta x_i \Delta y_j \Delta z_k$ and $\mu(P) := \max_{ijk} \Delta V_{ijk}$.

Consider the Riemann sum

$$S(P, f) := \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f(\mathbf{c}_{ijk}) \Delta V_{ijk}.$$

Triple integral

If $\lim_{\mu(P)\to 0} S(P,f)$ exists then f is said to be Riemann integrable and the (triple) integral of f over V is given by

$$\iiint_V f(x,y,z)dV = \iiint_V f(x,y,z)dxdydz = \lim_{\mu(P)\to 0} S(P,f).$$

Theorem: Let $f : V \to \mathbb{R}$ is continuous. Then

- f is Riemann integrable over V.
- Fubini's theorem holds, i.e, the iterated integrals exist and are equal to $\iiint_V f dV$.

Example

Evaluate $\iiint_V xyz^2 dV$ where $V = [0,1] \times [-1,2] \times [0,3]$.

By Fubini's theorem,

$$\iiint_V f dV = \int_0^3 \left(\int_{-1}^2 \left(\int_0^1 x dx \right) y dy \right) z^2 dz = \frac{27}{4}.$$

Triple integrals over general domains

Let $D \subset \mathbb{R}^3$ be bounded and $f : D \to \mathbb{R}$ be a bounded function. Then f is said to be integrable over D if for some rectangular cube V containing D the function

$$F(x, y, z) := \begin{cases} f(x, y, z) & \text{if } (x, y, z) \in D \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable over V. Then

$$\iiint_D f(x, y, z)dV := \iiint_V F(x, y, z)dV$$

and

$$\mathrm{Volume}(D) := \iiint_D dV.$$

Type-I domain:

A domain $V \subset \mathbb{R}^3$ is **Type-I** if

$$V = \{(x, y, z) : (x, y) \in D \text{ and } u_1(x, y) \leq z \leq u_2(x, y)\}$$

for some $D \subset \mathbb{R}^2$ and continuous functions $u_i : D \to \mathbb{R}$.

If $f : V \to \mathbb{R}$ be continuous and D is a special domain (e.g.,Type-I, Type-II, Type-III) then

$$\iiint_V f(x, y, z)dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz \right) dxdy.$$

Similar results hold for Type-II and Type-III domains.

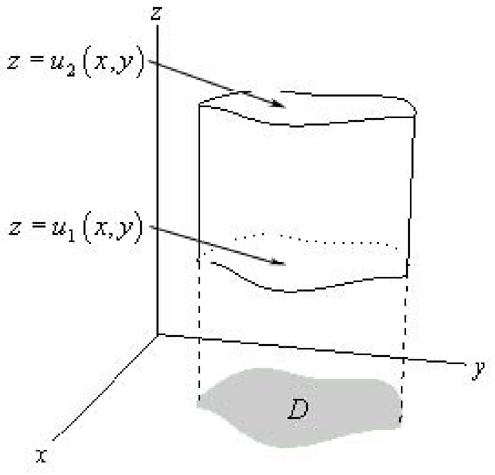

Figure: Type-I domain

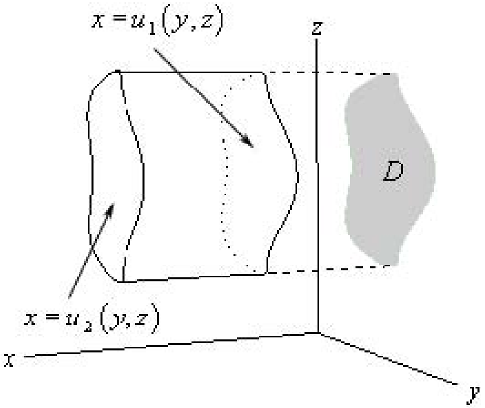

Figure: Type-II domain

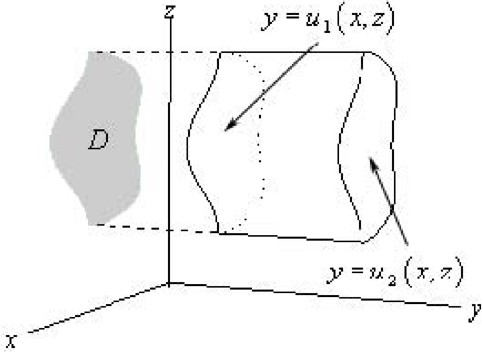

Figure: Type-III domain

Example:

Evaluate $\iiint_V 2x dV$ where V is the region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 3y + z = 6$.

Note that V is Type-I:

$$0 \leq z \leq 6 - 2x - 3y \text{ and } (x, y) \in D,$$

where D is special domain given by

$$0 \leq x \leq 3 \text{ and } 0 \leq y \leq -\frac{2}{3}x + 2.$$

Thus

$$\begin{aligned} \iiint_V 2x dV &= \iint_D \left(\int_0^{6-2x-3y} dz \right) 2x dx dy \\ &= \int_0^3 \int_0^{-\frac{2}{3}x+2} (6 - 2x - 3y) 2x dx dy = 9. \end{aligned}$$

Change of variable

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 given by $T(u, v) = (x(u, v), y(u, v))$. Then the Jacobian matrix $J(u, v)$ of T is given by

$$J(u, v) := \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}.$$

Define the Jacobian of T by

$$\frac{\partial(x, y)}{\partial(u, v)} := x_u y_v - x_v y_u = \det J(u, v).$$

Polar coordinates: Define $T(r, \theta) := (r\cos\theta, r\sin\theta)$. Then

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Change of variable for double integrals

Suppose T is injective and $J(u,v)$ is nonsingular. Let $D \subset \mathbb{R}^2$ and $G := T(D)$. Suppose that f is integrable on G. Then

$$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

and

$$\iint_G f(x,y)dxdy = \iint_D f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

Polar coordinates:

$$\iint_G f(x,y)dxdy = \iint_D f(r\cos\theta, r\sin\theta) r dr d\theta.$$

Example

Evaluate $\iiint_G \sqrt{x^2+z^2}\,dV$ where G is the region bounded by the paraboloid $y = x^2+z^2$ and $y = 4$.

We have

$$\iiint_G f(x,y,z)dV = \iint_D \left(\int_{x^2+z^2}^{4} dy\right) f(x,y,z)dxdz,$$

where $D = \{(x,z) : x^2+z^2 \leq 4\}$.

Setting $x = r\cos\theta$ and $z = r\sin\theta$ for $(r,\theta) \in [0,2] \times [0,2\pi]$,

$$\iiint_G f(x,y,z)dV = \int_0^{2\pi}\int_0^2 r(4-r^2)rdrd\theta = \frac{128\pi}{5}.$$

Change of variable for multiple integrals

Let $D \subset \mathbb{R}^n$ be open and bounded. Let $T : D \to \mathbb{R}^n$ be such that T is C^1, injective and the Jacobian $J(u)$ is nonsingular for $u \in D$.

Let $G := T(D)$ and $f : G \to \mathbb{R}$ be integrable over G. Then

$$dx_1 \cdots dx_n = \left| \frac{\partial(x_1, \cdots, x_n)}{\partial(u_1, \cdots, u_n)} \right| du_1 \cdots du_n$$

and

$$\begin{aligned} \int_G f(x) dx_1 \cdots dx_n &= \int_D f(x(u)) \left| \frac{\partial(x_1, \cdots, x_n)}{\partial(u_1, \cdots, u_n)} \right| du_1 \cdots du_n \\ &= \int_D f(x(u)) \left| \frac{dx}{du} \right| du. \end{aligned}$$

Cylindrical coordinates

Consider $T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$. Then

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Thus $dV = rdrd\theta dz$ and

$$\iiint_G f(x,y,z)dV = \iiint_D f(r\cos\theta, r\sin\theta, z)rdrd\theta dz.$$

Example

Evaluate $\iiint_G \sqrt{x^2+y^2}dV$, where G is the region bounded by $x^2+y^2=1, z=4$ and $z=1-x^2-y^2$.

Consider cylindrical coordinates
$D := \{(r,\theta,z) : (r,\theta) \in [0,1]\times[0,2\pi],\, 1-r^2 \leq z \leq 4\}$.

Then

$$\iiint_G f(x,y,z)dV = \int_0^1 \int_0^{2\pi} \left(\int_{1-r^2}^4 dz\right) r\, rdrd\theta = \frac{12\pi}{5}.$$

Spherical coordinates

Consider $T(r,\theta,\phi) = (r\sin\phi\cos\theta, r\sin\phi\cos\theta, r\cos\phi)$.
Then

$$\begin{aligned}\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| &= \begin{vmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} \\ &= -r^2\sin\phi.\end{aligned}$$

Thus $dV = r^2\sin\phi drd\theta d\phi$ and

$$\iiint_G f(x,y,z)dV =$$
$$\iiint_D f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)r^2\sin\phi drd\theta d\phi.$$

Example

Evaluate $\iiint_G e^{(x^2+y^2+z^2)^{3/2}} dV$, where
$G := \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$.

Using spherical coordinates we have

$$\begin{aligned} \iiint_D f(x, y, z) dV &= \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{r^3} r^2 \sin \phi \, dr d\theta d\phi \\ &= \frac{4}{3}\pi(e - 1). \end{aligned}$$

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