1. A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius **R**). Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

Sol^n:

$$\bar{\nabla}\cdot\bar{D} = \rho_f \Rightarrow \int \bar{\nabla}\cdot\bar{D}\, dv = \int \rho_f\, dv = Q_f$$

$$\oint_S \vec{D}\cdot d\bar{s} = Q_f \Rightarrow \bar{D} = \frac{q}{4\pi r^2}\hat{r} \; ; \; \bar{E} = \frac{\bar{D}}{\epsilon}$$

$$\boxed{\bar{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{(1+\chi_e)}\frac{\hat{r}}{r^2}}$$

$$\bar{P} = \epsilon_0\chi_e\bar{E} = \frac{q\chi_e}{4\pi(1+\chi_e)}\frac{\hat{r}}{r^2}$$

$$\rho_b = -\bar{\nabla}\cdot\bar{P} = -q\frac{\chi_e}{1+\chi_e}\delta^3(\bar{r})$$

$$\sigma_b = \bar{P}\cdot\hat{n} = \frac{q\chi_e}{4\pi(1+\chi_e)}\frac{1}{R^2}$$

$$Q_{surf} = 4\pi R^2\sigma_b = \frac{q\chi_e}{1+\chi_e}$$

The compensating negative charge is at the center

$$\int \rho_b\, dv = -\frac{q\chi_e}{1+\chi_e}$$

2. Given that $\vec{E}_1 = 2\hat{e}_x - 3\hat{e}_y + 5\hat{e}_z \, V/m$ at the charge-free dielectric interface of Figure 1. Find D_2 and the angles θ_1, and θ_2.

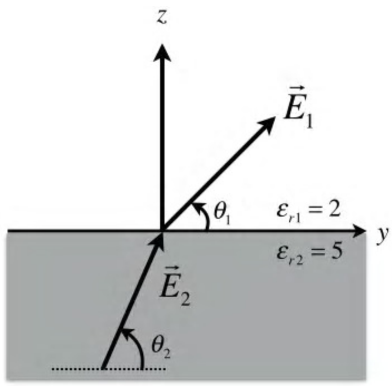


Figure 1: Problem 2

Sol^n:

$$\vec{E}_1 = 2\hat{e}_x - 3\hat{e}_y + 5\hat{e}_z \qquad \boxed{E_1^{\perp} = 5\hat{e}_z} \quad E_1^{\parallel} = 2\hat{e}_x - 3\hat{e}_y$$

$$\vec{D}_1 = \epsilon_0 \epsilon_{r1} \vec{E}_1 = 4\epsilon_0 \hat{e}_x - 6\epsilon_0 \hat{e}_y + 10\epsilon_0 \hat{e}_z$$

From the boundary condition

$$D_1^{\perp} - D_2^{\perp} = \rho_f = 0 \quad \text{(No free charge)}$$

$$D_2^{\perp} = 10\epsilon_0 \hat{e}_z$$

From continuity of tangential component of the electric vector across the surface

$$E_1^{\parallel} = E_2^{\parallel} \Rightarrow E_2^{\parallel} = 2\hat{e}_x - 3\hat{e}_y$$

$$\vec{D}_2 = \epsilon_0 \epsilon_{r2} \vec{E}_2 = \epsilon_0 [10\hat{e}_x - 15\hat{e}_y + 10\hat{e}_z]$$

$$\Rightarrow \vec{E}_2 = 2\hat{e}_x - 3\hat{e}_y + 2\hat{e}_z$$

The angle made with the plane of the interface are

$$\vec{E}_1 \cdot \hat{e}_z = |E_1| \, \mathrm{Cos}(90^\circ - \theta_1) \Rightarrow \theta_1 = 54.2^\circ$$

$$\vec{E}_2 \cdot \hat{e}_z = |E_2| \, \mathrm{Cos}(90^\circ - \theta_2) \Rightarrow \theta_2 = 29^\circ$$

3. A capacitor constituted of two vertical concentric cylinders of radii a and b has one end immersed in oil of dielectric constant $\kappa = \varepsilon/\varepsilon_0$. To what height does the oil rise in the space between the two cylinders when the potential difference V is applied between them? Ignore capillarity and fringing fields.

Sol: The electric field between the cylinders

$$E = \frac{V_a - V_b}{r \ln(a/b)} = \frac{V}{r \ln(a/b)}$$

$$W = \frac{1}{2}\int \epsilon E^2 \, dv \quad \Rightarrow dv = \rho \, d\rho \, d\varphi \, dz$$

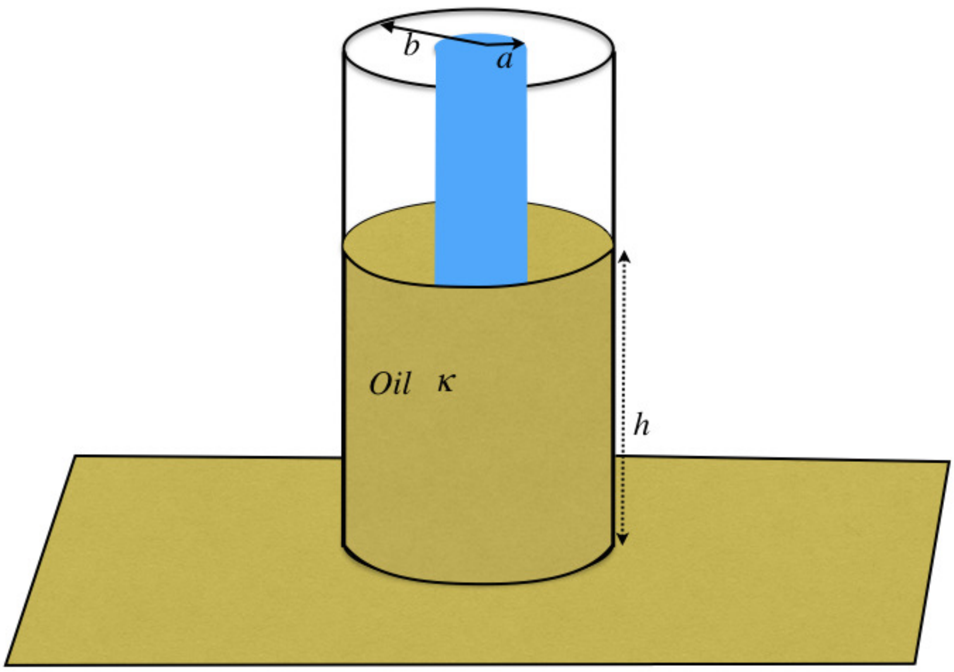
$$0 \le \rho \le a \qquad 0 \le \phi \le 2\pi$$

$$= \frac{\pi V^2}{\ln(a/b)}\left[\epsilon Z + \epsilon_0 (l - Z)\right] \qquad \boxed{\begin{array}{l} \epsilon = \epsilon_0 \text{ for } Z < z < l \\ = \epsilon \text{ for } 0 < z < Z \end{array}}$$

The vertical electrical force on the oil may be equated the gravitational force

$$F_z = \partial W/\partial Z = \frac{(\epsilon - \epsilon_0)\pi V^2}{\ln(a/b)} = \rho g Z \pi (b^2 - a^2)$$

$$Z = \frac{(\epsilon - \epsilon_0) V^2}{\rho g (b^2 - a^2) \ln(a/b)}$$



4. A parallel plate capacitor with plates of dimension $a \times b$ spaced at d has the corner of a large slab of dielectric of thickness $d/2$ and permittivity ε partly inserted to depth of Δx and Δy in the space between the plates. Find the force on the slab, ignoring fringing fields.

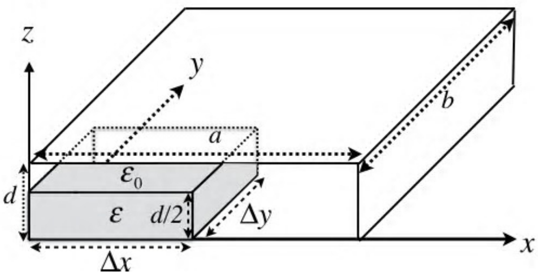


Figure 2: problem 4

Sol^n: Electric field vector in the region of the capacitor where the dielectric slab lies between plates

$$E_{air} = \frac{2\epsilon V}{(\epsilon+\epsilon_0)d} \qquad E_{diel} = \frac{2\epsilon_0 V}{(\epsilon+\epsilon_0)d}$$

$$0 \le x \le \Delta x \qquad 0 \le x \le \Delta x$$
$$0 \le y \le \Delta y \qquad 0 \le y \le \Delta y$$
$$d/2 \le z \le d \qquad 0 \le z \le d/2$$

The electric field vector in the remaining area

$$E_{rem} = V/d$$

The potential energy of the capacitor when charged to voltage V is then

$$W = \frac{1}{2}\int \epsilon E^2 dV = \frac{\epsilon_0(ab-\Delta x\Delta y)V^2}{2d} + \frac{\epsilon\epsilon_0^2\Delta x\Delta y V^2}{(\epsilon+\epsilon_0)^2 d} + \frac{\epsilon_0\epsilon^2\Delta x\Delta y V^2}{(\epsilon+\epsilon_0)^2 d}$$

The force in the x-direction (when V is held const) is

$$F_x = \partial W/\partial x = \frac{\epsilon_0(\epsilon-\epsilon_0)}{2(\epsilon+\epsilon_0)d}\Delta y V^2$$

$$F_y = \partial W/\partial y = \frac{\epsilon_0(\epsilon-\epsilon_0)}{2(\epsilon_0+\epsilon)d}\Delta x V^2$$

- Please clear student doubt on request for following take home exercise from D J Griffiths's 3rd edition book

  [1] G4.18 [2] G4.19 [3]G4.22 [4]G4.24 [5]G4.26 [6]G4.28 [7]G4.31 [8]G4.33