MA 102 (Mathematics II)
Department of Mathematics, IIT Guwahati

Tutorial Sheet No. 3 January 25, 2016

Continuity and limits of functions of several variables

(1) Examine whether the following limits exist and find their values if they exist.

$(a)\ \lim_{(x,y)\to(0,0)} \dfrac{x^3y}{x^4+y^2}$ $\quad (b)\ \lim_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2}$ $\quad (c)\ \lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$

$(d)\ \lim_{(x,y)\to(0,0)} \dfrac{|x|}{y^2}e^{-|x|/y^2}$ $\quad (e)\ \lim_{(x,y)\to(0,0)} \dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$ $\quad (f)\ \lim_{(x,y)\to(0,0)} \dfrac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$

(2) For the functions $f:\mathbb{R}^2\to\mathbb{R}$ given below examine continuity at $(0,0)$ and show that **exactly two** of the following limits exist and are equal:

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{x\to 0}\lim_{y\to 0} f(x,y), \qquad \lim_{y\to 0}\lim_{x\to 0} f(x,y).$$

(a) $f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$

(b) $f(x,y) := \begin{cases} y+x\sin(1/y) & \text{if } y\neq 0, \\ 0 & \text{if } y=0. \end{cases}$ (c) $f(x,y) := \begin{cases} x+y\sin(1/x) & \text{if } x\neq 0, \\ 0 & \text{if } x=0. \end{cases}$

(3) For the functions $f:\mathbb{R}^2\to\mathbb{R}$ given below show that **exactly one** of the following limits exists:

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{x\to 0}\lim_{y\to 0} f(x,y), \qquad \lim_{y\to 0}\lim_{x\to 0} f(x,y).$$

(a) $f(x,y) := \begin{cases} x\sin(1/y)+y\sin(1/x) & \text{if } xy\neq 0, \\ 0 & \text{if } xy=0. \end{cases}$

(b) $f(x,y) := \begin{cases} \frac{xy}{x^2+y^2}+x\sin(1/y) & \text{if } y\neq 0, \\ 0 & \text{if } y=0. \end{cases}$

(c) $f(x,y) := \begin{cases} \frac{xy}{x^2+y^2}+y\sin(1/x) & \text{if } x\neq 0, \\ 0 & \text{if } x=0. \end{cases}$

(4) Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y) := \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$ Show that the iterated limits $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ and $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ exist and are unequal.

(5) A function $f:A\subset\mathbb{R}^n\to\mathbb{R}$ is said to be **uniformly continuous** if the following holds: For any $\epsilon>0$ there is a $\delta>0$ such that $x,y\in A$ and $\|x-y\|<\delta \implies |f(x)-f(y)|<\epsilon$. Show that f is uniformly continuous if and only if for every $(x_k)\subset A$ and $(y_k)\subset A$ such that $\|x_k-y_k\|\to 0 \implies |f(x_k)-f(y_k)|\to 0$ as $k\to\infty$. If A is compact and f is continuous on A then show that f is uniformly continuous. Test uniform continuity of $f(x,y):=x^2+y^2$ on $\mathbb{R}^2$.

(6) A function $f:A\subset\mathbb{R}^n\to\mathbb{R}$ is said to be **Lipschitz** continuous if there a nonnegative number α such that $|f(x)-f(y)|\leq\alpha\|x-y\|$ for all $x,y\in A$. Show that $f(x):=\|x\|$ is Lipschitz continuous on $\mathbb{R}^n$. Show that $g(x):=\sqrt{x}$ is uniformly continuous on $[0,\infty)$ but is not Lipschitz continuous. Finally show that $h(x):=1/x$ is continuous on $(0,1)$ but is not uniformly continuous.

(7) Suppose that f is uniformly continuous on $A\subset\mathbb{R}^n$. If $(\mathbf{x}_k)$ is a Cauchy sequence in A, then show that $(f(\mathbf{x}_k))$ is a Cauchy sequence. Show by an example that if f is continuous on A then $(f(\mathbf{x}_k))$ may not be a Cauchy sequence.

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