First-Order ODE: Separable Equations, Exact Equations and Integrating Factor

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REMARK: In the last theorem of the previous lecture, you can change the open interval (a, b) to any interval I (I may be open or closed or semi-closed, it does not matter). The theorem is given in its correct form on the next page...see below.

First-Order Linear Equations

A linear first-order equation can be expressed in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \tag{1}$$

where $a_1(x), a_0(x)$ and $b(x)$ depend only on the independent variable x, not on y.

Examples:
$(1+2x)\frac{dy}{dx} + 6y = e^x$ (linear)
$\sin x \frac{dy}{dx} + (\cos x)y = x^2$ (linear)
$\frac{dy}{dx} + xy^3 = x^2$ (not linear)

Theorem **(Existence and Uniqueness)**:
Let I be an interval. Suppose $a_1(x), a_0(x)$, $b(x) \in C(I)$, $a_1(x) \neq 0$ and $x_0 \in I$. Then for any $y_0 \in \mathbb{R}$, there exists a unique solution $y(x) \in C^1(I)$ to the IVP

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \quad y(x_0) = y_0.$$

Exact Differential Equation

Definition: Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x, y) = F_x(x, y)dx + F_y(x, y)dy$$

for all $(x, y) \in D$.

Definition: The expression $M(x, y)dx + N(x, y)dy$ is called an exact differential in a domain D if there exists a function F such that

$$F_x(x, y) = M(x, y) \; and \; F_y(x, y) = N(x, y)$$

for all $(x, y) \in D$.

Definition: If $M(x, y)dx + N(x, y)dy$ is an exact differential, then the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is called an exact differential equation.

Definition: If an equation

$$F(x, y) = c$$

can be solved for $y = \phi(x)$ or for $x = \psi(y)$ in a neighbourhood of each point (x, y) satisfying $F(x, y) = c$, and if the corresponding function ϕ or ψ satisfies

$$M(x, y)dx + N(x, y)dy = 0,$$

then $F(x, y) = c$ is said to be an **Implicit solution** of $M(x, y)dx + N(x, y)dy = 0$.

Theorem: Let $\mathscr{R}$ be a rectangle in $\mathbb{R}^2$. Let $M(x,y), N(x,y) \in C^1(\mathscr{R})$. Then

$$M(x,y) + N(x,y)y' = 0 \text{ is exact } \iff M_y(x,y) = N_x(x,y)$$

for $(x,y) \in \mathscr{R}$.

Example: Consider $4x + 3y + 3(x + y^2)y' = 0$.

Note that $M, N \in C^1(\mathscr{R})$ and $M_y = 3 = N_x$. Thus, there exists $f(x,y)$ such that $f_x = 4x + 3y$ and $f_y = 3x + 3y^2$.

$f_x = 4x + 3y \Rightarrow f(x,y) = 2x^2 + 3xy + \phi(y)$. Now,

$$3x + 3y^2 = f_y(x,y) = 3x + \phi'(y).$$

$$\Rightarrow \phi'(y) = 3y^2 \Rightarrow \phi(y) = y^3.$$

Thus, $f(x,y) = 2x^2 + 3xy + y^3$ and the general solution is given by

$$2x^2 + 3xy + y^3 = C$$

Definition: If the equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{2}$$

is not exact, but the equation

$$\mu(x,y)\{M(x,y)dx + N(x,y)dy\} = 0 \tag{3}$$

is exact then $\mu(x,y)$ is called an integrating factor of (2).

Example: The equation $(y^2 + y)dx - xdy = 0$ is not exact. But, when we multiply by $\frac{1}{y^2}$, the resulting equation

$$(1 + \frac{1}{y})dx - \frac{x}{y^2}dy = 0, \quad y \neq 0$$

is exact.

Remark: While (2) and (3) have essentially the same solutions, it is possible to lose solutions when multiplying by $\mu(x,y)$.

Theorem: If $\frac{(M_y - N_x)}{N}$ is continuous and depends only on x, then

$$\mu(x) = \exp\left(\int \left\{\frac{M_y - N_x}{N}\right\} dx\right)$$

is an integrating factor for $Mdx + Ndy = 0$.

Proof. If $\mu(x, y)$ is an integrating factor, we must have

$$\frac{\partial}{\partial y}\{\mu M\} = \frac{\partial}{\partial x}\{\mu N\} \Rightarrow M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu.$$

If $\mu = \mu(x)$ then $\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$, where $(M_y - N_x)/N$ is just a function of x.

Example: Solve $(2x^2 + y)dx + (x^2y - x)dy = 0$.
The equation is not exact as $M_y = 1 \neq (2xy - 1) = N_x$. Note that

$$\frac{M_y - N_x}{N} = \frac{2(1 - xy)}{-x(1 - xy)} = \frac{-2}{x},$$

which is a function of only x, so an I.F $\mu(x) = x^{-2}$ and the solution is given by $2x - 2yx^{-1} + \frac{y^2}{2} = C$.

Remark. Note that the solution $x = 0$ was lost in multiplying $\mu(x) = x^{-2}$.
Theorem: If $\frac{N_x - M_y}{M}$ is continuous and depends only on y, then

$$\mu(y) = \exp\left(\int \left\{\frac{N_x - M_y}{M}\right\} dy\right)$$

is an integrating factor for $Mdx + Ndy = 0$.

Homogeneous Functions

If $M(x, y)dx + N(x, y)dy = 0$ is not a separable, exact, or linear equation, then it may still be possible to transform it into one that we know how to solve.

Definition: A function $f(x, y)$ is said to be homogeneous of degree n if

$$f(tx, ty) = t^n f(x, y),$$

for all suitably restricted x, y and t, where $t \in \mathbb{R}$ and n is a constant.

Example:

1. $f(x, y) = x^2 + y^2 \log(y/x),\ x > 0,\ y > 0$
   (homogeneous of degree 2)
2. $f(x, y) = e^{y/x} + \tan(y/x)\ x > 0\ y > 0$(homogeneous of degree 0)

- If $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree then the substitution $y = vx$ transforms the equation into a separable equation.

Writing $Mdx + Ndy = 0$ in the form $\frac{dy}{dx} = -M/N = f(x, y)$. Then, $f(x, y)$ is a homogeneous function of degree 0. Now, substitution $y = vx$ transform the equation into

$$v + x\frac{dv}{dx} = f(1, v) \Rightarrow \frac{dv}{f(1, v) - v} = \frac{dx}{x},$$

which is in variable separable form.
Example: Consider $(x + y)dx - (x - y)dy = 0$.
Put $y = vx$ and separate the variable to have

$$\frac{(1 - v)dv}{1 + v^2} = \frac{dx}{x}$$

Integrating and replacing $v = y/x$, we obtain

$$tan^{-1}\frac{y}{x} = \log\sqrt{x^2 + y^2} + C.$$

Substitutions and Transformations

- A first-order equation of the form

$$y' + p(x)y = q(x)y^\alpha,$$

where $p(x), q(x) \in C((a,b))$ and $\alpha \in \mathbb{R}$, is called a Bernoulli equation.

The substitution $v = y^{1-\alpha}$ transforms the Bernoulli equation into a linear equation

$$\frac{dv}{dx} + p_1(x)v = q_1(x),$$

where $p_1(x) = (1-\alpha)p(x),\ q_1(x) = (1-\alpha)q(x)$.

Example: Consider $y' + y = xy^3$. The general solution is given by $\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$.

- An equation of the form

$$y' = p(x)y^2 + q(x)y + r(x)$$

  is called Riccati equation.

  If it's one solution, say $u(x)$ is known then the substitution $y = u + 1/v$ reduces to a linear equation in v.

Remark: Note that if $p(x) = 0$ then it is a linear equation. If $r(x) = 0$ then it is a Bernoulli equation.

A DE of the form $M(x, y)dx + N(x, y)dy = 0$ is called a **homogeneous** DE if $M(x, y)$ and $N(x, y)$ are both homogeneous functions of the same degree.

- A DE of the form $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$, where a_i's, b_i's and c_i's are constants, can be transformed into the homogeneous equation by substituting

$$x = u + h \quad \text{and} \quad y = v + k,$$

where h, k are solutions (provided solution exists) of $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$.
If $a_2/a_1 = b_2/b_1 = k$, then substitution $z = a_1x + b_1y$ reduces the above DE to a separable equation in x and z.

Orthogonal Trajectories

Suppose

$$\frac{dy}{dx} = f(x, y)$$

represents the DE of the family of curves. Then, the slope of any orthogonal trajectory is given by

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} \quad \text{or} \quad -\frac{dx}{dy} = f(x, y),$$

which is the DE of the orthogonal trajectories.

Example: Consider the family of circles $x^2 + y^2 = r^2$. Differentiate w.r.t x to obtain $x + y\frac{dy}{dx} = 0$. The differential equation of the orthogonal trajectories is $x + y\left(-\frac{dx}{dy}\right) = 0$. Separating variable and integrating we obtain $y = c\,x$ as the equation of the orthogonal trajectories.

*** End ***