MA 102 (Mathematics II)

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Chain rule, tangent and normal, Jacobian matrix

(1) Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where m is a nonnegative integer. If f is differentiable then show that $\langle x, \nabla f(x)\rangle = mf(x)$.

Solution: Set $\phi(t) = f(tx)$. Then by chain rule $\phi'(t) = \nabla f(tx) \bullet x$. On the other hand, $\phi'(t) = mt^{m-1}f(x)$. Hence the result follows. $\blacksquare$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define $F, G : \mathbb{R}^2 \to \mathbb{R}$ by $F(x,y) := \int_0^{x+y} f(t)dt$ and $G(x,y) := \int_0^{xy} f(t)dt$. Show that F and G are differentiable and determine $DF(x,y)$ and $DG(x,y)$.

Solution: Since $F_x = f(x+y) = F_y$ are continuous, F is differentiable and $DF(x,y)(h,k) = f(x+y)(h+k)$ for $(h,k) \in \mathbb{R}^2$. Again since $G_x = yf(xy)$ and $G_y = f(xy)x$ are continuous, G is differentiable and $DG(x,y)(h,k) = f(xy)(hy+xk)$ for $(h,k) \in \mathbb{R}^2$. $\blacksquare$

(3) Let $f(x,y,z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of f at $(1,-1,3)$ and the equations of the tangent plane and the normal line to the surface $f(x,y,z) = 7$ at $(1,-1,3)$.

Solution: We have $\nabla f(1,-1,3) = \left(\frac{\partial f}{\partial x}(1,-2,3), \frac{\partial f}{\partial y}(1,-1,3), \frac{\partial f}{\partial z}(1,-1,3)\right) = (0,4,6)$. The tangent plane to the surface $f(x,y,z) = 7$ at the point $(1,-1,3)$ is given by

$$0 \times (x-1) + 4 \times (y+1) + 6 \times (z-3) = 0, \quad \text{i.e. } , 2y + 3z = 7.$$

The Normal Line to the surface $f(x,y,z) = 7$ at the point $(1,-1,3)$ is given by $(x,y,z) = (1,-1,3) + t(0,4,6)$ for $t \in \mathbb{R}$. Eliminating t, we have $x = 1, 3y - 2z + 9 = 0$. $\blacksquare$.

(4) Find $D_u f(2,2,1)$, where $f(x,y,z) = 3x - 5y + 2z$ and u is the unit vector in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2,2,1)$.

Solution: We have $u = \frac{(2,2,1)}{\sqrt{2^2+2^2+1^2}} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ and $\nabla f(2,2,1) = (3,-5,2)$. Therefore, $D_u f(2,2,1) = \nabla f(2,2,1) \bullet u = \frac{6}{3} - \frac{10}{3} + \frac{2}{3} = -\frac{2}{3}$. $\blacksquare$.

(5) Find the equation of the tangent plane to the graphs of the following functions at the given point:
(a) $f(x,y) := x^2 - y^4 + e^{xy}$ at the point $(1,0,2)$
(b) $f(x,y) = \tan^{-1}\dfrac{y}{x}$ at the point $(1, \sqrt{3}, \dfrac{\pi}{3})$.

Solution: The equation of tangent plane to the surface $z = f(x,y)$ at the point (x_0, y_0) is

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

(a) We have $f_x = 2x + ye^{xy}$ and $f_y = 4y^3 + xe^{xy}$. The equation of the tangent plane at $(1,0,2)$ is given by $z = 2(x-1) + 1(y-0) + 2 \Rightarrow z = 2x + y$.

(b) The equation of the tangent plane is given by

$$z = \frac{\pi}{3} - \frac{\sqrt{3}}{4}(x-1) + \frac{1}{4}(y - \sqrt{3}) \Rightarrow 3\sqrt{3}x - 3y + 12z - 4\pi = 0. \blacksquare$$

(6) Check the following functions for differentiability and Jacobian Matrix.
(a) $f(x,y) = (e^{x+y}+y, xy^2)$ (b) $f(x,y) = (x^2+\cos y, e^x y)$ (c) $f(x,y,z) = (ze^x, -ye^z)$.

Solution: (a) $Df(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y}+1 \\ y^2 & 2xy \end{bmatrix}$. (b) $Df(x,y) = \begin{bmatrix} 2x & -\sin y \\ ye^x & e^x \end{bmatrix}$.
(c) $Df(x,y,z) = \begin{bmatrix} ze^x & 0 & e^x \\ 0 & -e^z & -ye^z \end{bmatrix}$. ∎

(7) Let $z = x^2 + y^2$, and $x = 1/t, y = t^2$. Compute $\dfrac{dz}{dt}$ by (a) expressing z explicitly in terms of t and (ii) chain rule.

Solution: (a) By direct substitution we have $z = x^2 + y^2 = t^{-2} + t^4$ for $t \neq 0$. Therefore $\dfrac{dz}{dt} = -2t^{-3} + 4t^3$. ∎

(b) Note that $\dfrac{\partial z}{\partial x} = 2x, \dfrac{\partial z}{\partial y} = 2y, \dfrac{dx}{dt} = -t^{-2}, \dfrac{dy}{dt} = 2t$. Therefore by chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x)(-t^{-2}) + (2y)(2t) = -2t^{-3} + 4t^3.$$

(8) Let $w = 4x + y^2 + z^3$ and $x = e^{rs^2}, y = \log \dfrac{r+s}{t}, z = rst^2$. Find $\dfrac{\partial w}{\partial s}$.

Solution: By chain rule,

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= \left(\frac{\partial}{\partial x}(4x + y^2 + z^3)\right)\left(\frac{\partial}{\partial s}(e^{rs^2})\right) + \left(\frac{\partial}{\partial y}(4x + y^2 + z^3)\right)\left(\frac{\partial}{\partial s}\left(\log \frac{r+s}{t}\right)\right) \\
&\quad + \left(\frac{\partial}{\partial z}(4x + y^2 + z^3)\right)\left(\frac{\partial}{\partial s}(rst^2)\right) \\
&= 8rse^{rs^2} + 2y\left(\frac{t}{r+s}\right)\left(\frac{1}{t}\right) + 3rt^2z^2 = 8rse^{rs^2} + \frac{2}{r+s}\log\frac{r+s}{t} + 3rt^2z^2.
\end{aligned}$$

∎

(9) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0,0) := 0$ and, for $(x,y) \neq (0,0)$, $f(x,y) := xy\dfrac{x^2-y^2}{x^2+y^2}$.
(a) Show that $\frac{\partial f}{\partial y}(x,0) = x$ for $x \in \mathbb{R}$ and $\frac{\partial f}{\partial x}(0,y) = -y$ for $y \in \mathbb{R}$.
(b) Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Solution: We have $f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h} = -k$ and $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$. Hence

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{-k-0}{k} = -1.$$

Similarly $f_y(x,0) = x$ and $f_{yx}(0,0) = 1$. By directly computing f_{xy}, f_{yx} for $(x,y) \neq (0,0)$, one observes that these are not continuous at (0,0). ∎

(10) Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Show that

$$\lim_{h\to 0} \frac{f(x+h) - [f(x) + \langle \nabla f(x), h\rangle + \frac{1}{2}\langle H_f(x)h, h\rangle]}{\|h\|^2} = 0,$$

where $H_f(x)$ is the Hessian of f at x.

By EMVT $f(x+h) = f(x) + \langle \nabla f(x),\, h\rangle + \frac{1}{2}\langle H_f(x+\theta h)h,\, h\rangle$ for some $0 < \theta < 1$. Therefore $f(x+h) - [f(x) + \langle \nabla f(x),\, h\rangle + \frac{1}{2}\langle H_f(x)h,\, h\rangle] = \frac{1}{2}\langle [H_f(x+\theta h) - H_f(x)]h,\, h\rangle$. Since $\partial_i\partial_j f(x+\theta h) \to \partial_i\partial_j f(x)$ as $h \to 0$, it follows that

$$\lim_{h\to 0} \frac{\langle [H_f(x+\theta h) - H_f(x)]h,\, h\rangle}{\|h\|^2} = 0.$$

Hence the result follows. ■

(11) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable and $x = r\cos\theta, y = r\sin\theta$. Show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

Solution: Step-1: By chain rule $f_r = f_x x_r + f_y y_r = f_x\cos\theta + f_y\sin\theta$. Similarly $f_\theta = -r\sin\theta f_x + r\cos\theta f_y$. Then $f_x = f_r\cos\theta - f_\theta\frac{\sin\theta}{r}$ and $f_y = f_r\sin\theta + \frac{\cos\theta}{r}f_\theta$.

This shows that

$$\partial_x = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta \text{ and } \partial_y = \sin\theta\partial_r - \frac{\cos\theta}{r}\partial_\theta.$$

Step-2: Applying ∂_x to f_x, ∂_y to f_y and adding we get the desired result. ■

**** End ****