Lecture Slides 3: Limit and Continuity of Functions

Department of Mathematics
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Topology of $\mathbb{R}^n$

Open Ball: Let $\epsilon > 0$ and $\mathbf{a} \in \mathbb{R}^n$. Then

$$B(\mathbf{a}, \epsilon) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < \epsilon\}$$

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Examples:

1. $B(\mathbf{a}, \epsilon) \subset \mathbb{R}^n$ is an open set.
2. $O := (a_1, b_1) \times \cdots \times (a_n, b_n)$ is open in $\mathbb{R}^n$.
3. $\mathbb{R}^n$ is open.

Closed set: $S \subset \mathbb{R}^n$ is closed if $S^c := \mathbb{R}^n \setminus S$ is open.

Examples:

1. $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ is closed set.
2. $E := [a_1, b_1] \times \cdots \times [a_n, b_n]$ is closed in $\mathbb{R}^n$.
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3. $\mathbb{R}^n$ is closed.

Fact: Let $S \subset \mathbb{R}^n$. Then the following are equivalent:

1. S is closed.
2. If $(\mathbf{x}_k) \subset S$ and $\mathbf{x}_k \to \mathbf{x} \in \mathbb{R}^n$ then $\mathbf{x} \in S$.

Limit point: Let $A \subset \mathbb{R}^n$ and $\mathbf{a} \in \mathbb{R}^n$. Then $\mathbf{a}$ is a limit point of A if $A \cap (B(\mathbf{a}, \epsilon) \setminus \{\mathbf{a}\}) \neq \emptyset$ for any $\epsilon > 0$.

Examples:

1. Each point in $B(\mathbf{a}, \epsilon)$ is a limit point.
2. Each $\mathbf{x} \in \mathbb{R}^n$ such that $\|\mathbf{x} - \mathbf{a}\| = \epsilon$ is a limit point of $B(\mathbf{a}, \epsilon)$.

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Fact: Let $S \subset \mathbb{R}^n$. Then S is closed $\iff$ S contains all of its limit points.

Limit of a function

Definition:

- Let $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^n$ and $L \in \mathbb{R}$. Then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ if for any $\epsilon > 0$ there is $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \Longrightarrow |f(\mathbf{x}) - L| < \epsilon.$$

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- Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $L \in \mathbb{R}$. Let $\mathbf{a} \in \mathbb{R}^n$ be a limit point of A. Then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if for any $\epsilon > 0$ there is $\delta > 0$ such that

$$\mathbf{x} \in A \text{ and } 0 < \|\mathbf{x} - \mathbf{a}\| < \delta \implies |f(\mathbf{x}) - L| < \epsilon.$$

Sequential characterization

Theorem: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$, $L \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$ be a limit point of A. Then the following are equivalent:

- $\lim\limits_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$
- If $(\mathbf{x}_k) \subset A \setminus \{\mathbf{a}\}$ and $\mathbf{x}_k \to \mathbf{a}$ then $f(\mathbf{x}_k) \to L$.

Proof: Exercise.

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Proof: Exercise.

Remark:

- Limit, when exists, is unique.
- Sum, product and quotient rules hold.

Examples:

1. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) := 0$ and $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

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2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) := \begin{cases} x\sin(1/y) + y\sin(1/x) & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

Then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0.$

Iterated limit

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. Then $\lim_{x \to a} \lim_{y \to b} f(x, y)$, when exists, is called an iterated limit of f at (a, b).

Ditto for $\lim_{y \to b} \lim_{x \to a} f(x, y)$ when it exists.

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Remark:

- Iterated limits are defined similarly for $f : A \subset \mathbb{R}^n \to \mathbb{R}$.
- Existence of limit does not guarantee existence of iterated limits and vice-versa.
- Iterated limits when exist may be unequal. However, if limit and iterated limits exist then they are all equal.

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3. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) := 0$ and $f(x,y) := \frac{x^2-y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Then iterated limits exist at $(0,0)$ and are unequal.

Continuous function and limit

Fact: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in A$ be a limit point. Then f is continuous at $\mathbf{a} \iff \lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x})$ exists and equals $f(\mathbf{a})$.

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- If f is continuous at $\mathbf{a}$ then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ may or may not be defined.
- If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exits then f may or may not be continuous at $\mathbf{a}$.

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Example: Consider $f : [0,1] \cup \{3\} \to \mathbb{R}$ given by $f(x) := 2x$ for $x \in [0,1]$ and $f(3) := 10$.

Continuous function and compact set

Compact set: $A \subset \mathbb{R}^n$ is compact if $(\mathbf{x}_k) \subset A$ then $(\mathbf{x}_k)$ has a subsequence $(\mathbf{x}_{k_p})$ that converges to some $\mathbf{x} \in A$.

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Bounded set: $S \subset \mathbb{R}^n$ is bounded if $S \subset B(0, \alpha)$ for some $\alpha > 0$.

Theorem (Heine-Borel):
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Extreme Value Theorem

Theorem: Let $f : K \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous. If K is compact then $f(K)$ is compact.

Proof: $(\mathbf{y}_n) \subset f(K) \Rightarrow \mathbf{y}_n = f(\mathbf{x}_n)$.
K compact $\Rightarrow \mathbf{x}_{k_p} \to \mathbf{x} \in K \Rightarrow \mathbf{y}_{k_p} = f(\mathbf{x}_{k_p}) \to f(\mathbf{x}) \in f(K)$.

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Theorem (EVT): Let $f : K \subset \mathbb{R}^n \to \mathbb{R}$ be continuous and K compact. Then

- there is $\mathbf{x}_{\min} \in K$ such that $f(\mathbf{x}_{\min}) = \inf\{f(\mathbf{x}) : \mathbf{x} \in K\}$,
- there is $\mathbf{x}_{\max} \in K$ such that $f(\mathbf{x}_{\max}) = \sup\{f(\mathbf{x}) : \mathbf{x} \in K\}$.

*** End ***