

Variation of Parameters, Use of a Known Solution to Find Another and Cauchy-Euler Equation

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Variation of Parameters

- The variation of parameter is a more general method for finding a particular solution (y_p). The method applies even when the coefficients of the differential equation are functions of x .

Consider $L(y) = g(x)$, where

$$L(y) := a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y,$$

where $a_n(x), a_{n-1}(x), \dots, a_0(x) \in C((a, b))$ and $a_n(x) \neq 0$.

We know the general solution to $L(y) = g$ is given by

$$y(x) = y_h(x) + y_p(x),$$

where y_h is the general solution to $Ly = 0$ and $y_p(x)$ is a particular solution to $L(y) = g$.

Suppose we know a fundamental solution set $\{y_1, \dots, y_n\}$ for $L(y) = 0$. Then

$$y_h(x) = C_1 y_1(x) + \dots + C_n y_n(x).$$

In this method, seek a particular solution (y_p) of the form

$$y_p(x) = v_1(x)y_1(x) + \dots + v_n(x)y_n(x),$$

and try to determine the functions v_1, \dots, v_n .

Differentiating y_p ,

$$y_p' = \sum_{i=1}^n v_i y_i' + \sum_{i=1}^n v_i' y_i.$$

To avoid second and higher-order derivatives of v_i 's, we impose the condition

$$\sum_{i=1}^n v_i' y_i = 0. \tag{1}$$

Therefore,

$$y'_p = \sum_{i=1}^n v_i y'_i.$$

Again, differentiating y'_p , we obtain

$$y''_p = \sum_{i=1}^n v_i y''_i + \sum_{i=1}^n v'_i y'_i.$$

For the same reason, we impose the condition

$$\sum_{i=1}^n v'_i y'_i = 0. \quad (2)$$

Likewise, on computing $y'''_p, \dots, y_p^{(n-1)}$, we impose $(n-3)$ additional conditions involving v'_i 's as

$$\sum_{i=1}^n v'_i y''_i = 0, \quad \dots, \quad \sum_{i=1}^n v'_i y_i^{(n-2)} = 0. \quad (3)$$

Finally, for the n th condition, use the derived conditions and the fact that $L(y_h) = 0$ and $L(y_p) = g$ to obtain

$$\sum_{i=1}^n v_i' y_i^{(n-1)} = g. \quad (4)$$

Therefore, we seek v_1', \dots, v_n' that satisfy the system

$$\begin{aligned} y_1 v_1' + \cdots + y_n v_n' &= 0, \\ y_1 v_1' + \cdots + y_n v_n' &= 0, \\ \vdots + \vdots + \vdots &= \vdots \\ y_1^{(n-1)} v_1' + \cdots + y_n^{(n-1)} v_n' &= g. \end{aligned}$$

The existence of v'_1, \dots, v'_n requires

$$\begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W(y_1, \dots, y_n)(x) \neq 0$$

on (a, b) , which is true as $\{y_1, \dots, y_n\}$ is a fundamental solution set.

On solving for v'_1, \dots, v'_n , we find that

$$v'_k(x) = \frac{g(x)W_k(x)}{W(y_1, \dots, y_n)(x)}, \quad k = 1, \dots, n,$$

where $W_k(x)$ is obtained from $W(y_1, \dots, y_n)(x)$ by replacing k th column by $[0, \dots, 0, 1]^T$.

We can express $W_k(x)$ as

$$W_k(x) = (-1)^{(n-k)} W(y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n)(x)$$

for $k = 1, \dots, n$. Integrating $v_k'(x)$ yields

$$v_k(x) = \int \frac{g(x)W_k(x)}{W(y_1, \dots, y_n)(x)} dx, \quad k = 1, \dots, n.$$

Finally, substituting the v_k 's back into y_p , we obtain

$$y_p(x) = v_1(x)y_1(x) + \dots + v_n(x)y_n(x)$$

we obtain

$$y_p(x) = \sum_{k=1}^n y_k(x) \int \frac{g(x)W_k(x)}{W(y_1, \dots, y_n)(x)} dx.$$

For $n = 2$, v_1' and v_2' are given by

$$v_1'(x) = \frac{-g(x)y_2(x)}{W(y_1, y_2)(x)}, \quad v_2'(x) = \frac{g(x)y_1(x)}{W(y_1, y_2)(x)},$$

where $W(y_1, y_2)(x) \neq 0$. Integrating these equations, we obtain

$$v_1(x) = \int \frac{-g(x)y_2(x)}{W(y_1, y_2)(x)} dx, \quad v_2(x) = \int \frac{g(x)y_1(x)}{W(y_1, y_2)(x)} dx.$$

Thus, the particular solution is given by

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x).$$

Example: Consider $y'' + y = \operatorname{cosec} x$.

$$y_h(x) = c_1 \sin x + c_2 \cos x.$$

The two linearly independent solutions are $y_1(x) = \sin x$ and $y_2(x) = \cos x$ and $W(y_1, y_2) = -1 \neq 0$.

$$v_1(x) = \int \frac{-g(x)y_2(x)}{W(y_1, y_2)(x)} dx = \int \frac{-\cos x \operatorname{cosec} x}{-1} dx = \log(\sin x).$$

$$v_2(x) = \int \frac{g(x)y_1(x)}{W(y_1, y_2)(x)} dx = \int \frac{\sin x \operatorname{cosec} x}{-1} dx = -x.$$

$$y_p = \sin x \log(\sin x) - x \cos x.$$

The general solution is

$$y(x) = c_1 \sin x + c_2 \cos x + \sin x \log(\sin x) - x \cos x.$$

Use of a known solution to find another

Assume that $y_1(x) \neq 0$ is a known solution of $L(y) = 0$, where

$$L(y) = y'' + p(x)y' + q(x)y.$$

We know $L((cy_1)) = 0$, where c is any arbitrary constant. The basic idea is to replace c by an unknown function $v(x)$ so that $L(y_2) = 0$, where $y_2 = v(x)y_1(x)$.

Suppose $L(y_2) = L(vy_1) = 0$. Then, we have

$$v(y_1'' + py_1' + qy_1) + v''y_1 + v'(2y_1' + py_1) = 0.$$

Since $L(y_1) = 0$, we have

$$v''y_1 + v'(2y_1' + py_1) = 0 \Rightarrow \frac{v''}{v'} = -2\frac{y_1'}{y_1} - p.$$

Integrating

$$v' = \frac{1}{y_1^2} e^{-\int p dx} \Rightarrow v(x) = \int \frac{1}{y_1^2} e^{-\int p dx} dx.$$

Thus, the second solution is $y_2(x) = v(x)y_1(x)$.

Example: Given that $y_1 = e^x$ is a solution to $y'' - 2y' + y = 0$. Determine the second linear independent solution y_2 .

Note that $v(x) = x$. The second linearly dependent solution is

$$y_2(x) = vy_1 = xe^x.$$

Cauchy-Euler Equation

An equation of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x),$$

where a_i 's are constants is called **Cauchy-Euler equation**.

The substitution $x = e^t$ transform the above equation into an equation with constant coefficients. For simplicity, take $n = 2$.

Assume that $x > 0$ and let $x = e^t$. By the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx} e^t = x \frac{dy}{dx},$$

hence

$$x \frac{dy}{dx} = \frac{dy}{dt}.$$

Differentiating $x \frac{dy}{dx} = \frac{dy}{dt}$ with respect to t , we find that

$$\begin{aligned} \frac{d^2 y}{dt^2} &= \frac{d}{dt} \left(x \frac{dy}{dx} \right) = \frac{dx}{dt} \frac{dy}{dx} + x \frac{d}{dt} \left(\frac{dy}{dx} \right) \\ &= \frac{dy}{dt} + x \frac{d^2 y}{dx^2} \frac{dx}{dt} = \frac{dy}{dt} + x \frac{d^2 y}{dx^2} e^t \\ &= \frac{dy}{dt} + x^2 \frac{d^2 y}{dx^2}. \end{aligned}$$

Thus

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}.$$

Substituting into the equation we obtain the constant coefficient ODE

$$a_2 \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + a_1 \frac{dy}{dt} + a_0 y = g(e^t),$$

which may be written as

$$a_2 \frac{d^2 y}{dt^2} + (a_1 - a_2) \frac{dy}{dt} + a_0 y = g(e^t).$$

Note: Observe that in the proof it is assumed that $x > 0$. If $x < 0$, the substitution $x = -e^t$ will reduced the Cauchy-Euler equation to constant coefficients ODE. The method can be applied to higher-order Cauchy-Euler equation.

Example: Consider $x^2y'' - 2xy' + 2y = x^3$, $x > 0$.

Setting $x = e^t$, we obtain

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = e^{3t},$$

or

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}.$$

The GS to the homogeneous equation is

$$y_h(x) = c_1e^t + c_2e^{2t} = c_1x + c_2x^2.$$

To find a particular solution, let $y_p = Ae^{3t}$. Then, $A = \frac{1}{2}$
hence, $y_p = \frac{1}{2}e^{3t} = \frac{1}{2}x^3$. The GS is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1x + c_2x^2 + \frac{1}{2}x^3, \quad x > 0. \end{aligned}$$

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