MA 102 (Mathematics II)

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Tutorial Sheet No. 4

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(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \qquad \text{for } (x,y) \neq (0,0).$$

(a) Find f_x and f_y at every $(x,y) \in \mathbb{R}^2$.

(b) Show that the partial derivatives of f are not bounded in any disc (howsoever small) around $(0,0)$.

(c) Examine the differentiability at every point (x,y).

Solution. $f_x(0,0) = f_y(0,0) = 0$.
Let $(x,y) \neq (0,0)$. We have

$$f_x(x,y) = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right)$$

and

$$f_y(x,y) = 2y\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$

Clearly, f_x and f_y are continuous at every $(x,y) \neq (0,0)$, and hence f is continuous at any $(x,y) \neq (0,0)$ (using sufficient condition for continuity).
The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at (0,0), while $\frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)$ is unbounded in any such disc. (Consider$(x,y) = \left(\frac{1}{\sqrt{n\pi}}, 0\right)$ for n a large positive integer.) Thus f_x is unbounded in any disc around $(0,0)$.
Differentiability at $(x,y) \neq (0,0)$ follows from the continuity of f_x and f_y.
Differentiability at $(0,0)$: We have

$$\begin{aligned}\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - \nabla f(0,0) \bullet (h,k)|}{\sqrt{h^2+k^2}} &= \lim_{(h,k)\to(0,0)} \frac{(h^2+k^2)\sin\frac{1}{h^2+k^2}}{\sqrt{h^2+k^2}} \\ &= \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\frac{1}{h^2+k^2} \\ &= 0.\end{aligned}$$

Hence, f is differentiable at $(0,0)$. Thus, f is differentiable everywhere. □

(2) Examine the differentiability of the following function at $(0,0)$:

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Solution.

$$f_x(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1\,.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1\,.$$

Now taking $h = \rho cos\theta$ and $k = \rho sin\theta$ we have

$$\frac{\Delta f - df}{\rho} = \frac{hk(h-k)}{(h^2+k^2)^3/2} = \frac{\rho^3 cos\theta sin\theta(cos\theta - sin\theta)}{\rho^3}\,.$$

Thus limit $\rho \to 0$ does not exist, and hence the function is not differentiable at $(0,0)$. □

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $f_x(x,y) = 0 = f_y(x,y)$ for all $(x,y) \in \mathbb{R}^2$ then show that f is a constant function.

Solution. We have $f(x,y) - f(0,0) = [f(x,y) - f(0,y)] + [f(0,y) - f(0,0)]$. By MVT, there exists $0 < \theta_i < 1$ for $i = 1,2$ such that

$$[f(x,y) - f(0,y)] + [f(0,y) - f(0,0)] = f_x(\theta_1 x, y)x + f_y(0, \theta_2 y)y = 0.$$

This shows $f(x,y) - f(0,0) = 0$ for all $(x,y) \in \mathbb{R}^2$. Hence f is constant. □

(4) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(0,0) = 0$ and, for $(x,y) \neq (0,0)$,

$$g(x,y) = \frac{\sin^2(x+y)}{|x|+|y|}.$$

Examine the existence of partial and directional derivatives of g at $(0,0)$.
Also, examine the differentiability of g at $(0,0)$.

Solution. We have $g_x(0,0) = \lim_{h\to 0} \frac{\sin^2(h)/|h|}{h} = \lim_{h\to 0} \frac{\sin^2(h)}{h|h|}$ which shows that the limit does not exist. Similarly, $g_y(0,0)$ does not exist. Hence g is not differentiable.
Let $U = (u,v)$ be a unit vector. Then

$$\begin{aligned} D_U g(0,0) &= \lim_{t\to 0} \frac{g(tu,tv)}{t} = \lim_{t\to 0} \frac{\sin^2(t(u+v))}{t|t|(|u|+|v|)} \\ &= \frac{(u+v)^2}{|u|+|v|} \lim_{t\to 0} \frac{t}{|t|}, \end{aligned}$$

which does not exist. □

(5) Find the directional derivative of $f(x,y) = y^3 - 2x^2 + 3$ at the point $(1,2)$ in the direction of $U := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Also, find the directional derivative of $f(x,y) = \log(x^2+y^2)$ at $(1,-3)$ in the direction of $V := (2,-3)$.

Solution. (i) We have $f_x(x,y) = -4x$, $f_y(x,y) = 3y^2$ which are continuous. Therefore, f is differentiable and

$$D_u f(1,2) = \nabla f(1,2) \bullet u = f_x(1,2)\frac{1}{2} + f_y(1,2)\frac{\sqrt{3}}{2} = -2 + 6\sqrt{3}.$$

(ii) Next, we have $f_x(x, y) = \frac{2x}{x^2+y^2}$ and $f_y(x, y) = \frac{2y}{x^2+y^2}$ which are continuous at $(1, -3)$. Therefore, f is differentiable at $(1, -3)$, and for $u = (\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}})$, we have

$$D_u f(1, -3) = \nabla f(1, -3) \bullet u = \frac{11}{5\sqrt{3}}.$$

□

(6) Find the directional derivative of $f(x,\ y) = x^2 - 3xy$ along the parabola $y = x^2 - x + 2$ (That is, in the parametric form $x(t) = t$ and $y(t) = t^2 - t + 2$) at the point $(1,\ 2)$. (Note: When a direction is given in terms of a curve, then one must take the direction as the (unit) tangent vector to the curve at that point).

Solution. Unit tangent vector to the parabola at $(1, 2)$ is $U = (1/\sqrt{2}, 1/\sqrt{2})$. Now,

$$D_U f(1, 2) = \lim_{t \to 0} \frac{f(t/\sqrt{2} + 1, t/\sqrt{2} + 2) - f(1, 2)}{t} = -\frac{7}{\sqrt{2}}.$$

□

(7) Discuss the differentiability of the following functions at $(0, 0)$.

$$(a) f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & x^2 + y^2 \neq 0, \\ 0 & x = y = 0 \end{cases} \qquad (b)\ g(x, y) = \begin{cases} \frac{x^6 - 2y^4}{x^2+y^2}, & x^2 + y^2 \neq 0 \\ 0, & x = 0, y = 0 \end{cases}$$

Solution. (a) Both the partial derivatives are 0 at the point (0,0). But,

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

which does not exist. Hence, f is not differentiable at $(0, 0)$.

(b) Both the partial derivatives are 0 at the point (0,0). Now,

$$\lim_{(x,y)\to(0,0)} \frac{g(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^6 - 2y^4}{(x^2 + y^2)\sqrt{x^2 + y^2}} = 0. \qquad \text{[Explain the estimate]}$$

Therefore, f is differentiable at $(0, 0)$. □

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