Basic Definitions, Existence and Uniqueness Results for First-Order IVP

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Texts/References:

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Example: Let $\mathscr{R}\ :\ |x| \leq 5,\ |y| \leq 3$ be the rectangle. Consider the IVP

$$y' = 1 + y^2, \quad y(0) = 0$$

over $\mathscr{R}$.

Here, $a = 5,\ b = 3$. Then

$$\max_{(x,y)\in\mathscr{R}} |f(x,y)| = \max_{(x,y)\in\mathscr{R}} |1 + y^2| = 10 (= K),$$

$$\max_{(x,y)\in\mathscr{R}} \left|\frac{\partial f}{\partial y}\right| = \max_{(x,y)\in\mathscr{R}} 2|y| = 6 (= L).$$

$$h = \min\{a, \frac{b}{K}\} = \min\{5,\ \frac{3}{10}\} = 0.3 < 5.$$

Note that the solution of the IVP is $y = \tan x$. This solution is valid in the interval $|x| \leq 0.3$ instead of the entire interval $|x| \leq 5$.

Example **(Non-uniqueness)**: Consider the IVP:

$$y' = 3\, y^{\frac{2}{3}} \text{ for } x \in \mathbb{R}, \quad y(0) = 0.$$

For each real number $c \geq 0$, let

$$y_c(x) = \begin{cases} 0 & \text{if } \; 0 \leq x < c, \\ (x-c)^3 & \text{if } \; c \leq x < \infty, \end{cases} \qquad .$$

It is easy to verify that $y_c(x)$ is a solution to the IVP. Therefore, this IVP has infinitely many solutions.

The Method of Successive Approximations

Consider the IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0. \tag{1}$$

Key Idea: Replacing the IVP (1) by an the equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt. \tag{2}$$

Note that (1) and (2) are equivalent.

A first (rough) approximation to a solution is given by $y_0(x) = y_0$. A second approximation $y_1(x)$ is obtained as follows:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\, dt.$$

The next step is to use $y_1(x)$ to generate another approximation $y_2(x)$ in the same way:

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt.$$

At the nth step, we have

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt.$$

This procedure is called Picard's method of successive approximations.

Example: Consider IVP: $y' = y, \quad y(0) = 1$.

$$y_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Note that $y_n(x) \to e^x$ as $n \to \infty$.

Theorem: Let $\mathscr{R} \; : \; |x - x_0| \leq a, \; |y - y_0| \leq b$ be a rectangle where $a, b > 0$. Let $f \in C(\mathscr{R})$ and let $|f(x, y)| \leq M$ for all $(x, y) \in \mathscr{R}$. Further suppose that f satisfies Lipschitz condition w.r.t y with constant K in $\mathscr{R}$. Then the successive approximations

$$y_0(x) \equiv y_0$$

$$y_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_k(t))dt, \;\; k = 0, 1, 2, 3, ...$$

converge uniformly on the interval $I : |x - x_0| \leq h$ where $h = min\{a, \frac{b}{M}\}$ to a solution $y(x)$ of the IVP $y' = f(x, y), y(x_0) = y_0$.

Theorem **(Continuous dependence on initial data)**:

Let f, $\frac{\partial f}{\partial y} \in C(\mathscr{R})$ and $(x_0, y_0), (x_0, y_{0_m}) \in \mathscr{R}$. Let $\phi(x)$ be the solution of

$$y' = f(x, y), y(x_0) = y_0,$$

and let $\phi_m(x)$ be the solution of

$$y' = f(x, y), y(x_0) = y_{0_m},$$

in $\mathscr{R}$ for $|x - x_0| \leq h$. Then, for $|x - x_0| \leq h$, we have

$$|\phi(x) - \phi_m(x)| \leq |y_0 - y_{0_m}|e^{Lh},$$

where $|\frac{\partial f}{\partial y}(x, y)| \leq L$ for all $(x, y) \in \mathscr{R}$.

Further, as $y_{0_m} \to y_0$, $\phi_m \to \phi$ uniformly on $[x_0 - h, x_0 + h]$.

Theorem **(Continuous dependence on f)**:

Let f, f_m, $\frac{\partial f}{\partial y}, \frac{\partial f_m}{\partial y} \in C(\mathscr{R})$, and $(x_0, y_0) \in \mathscr{R}$. Let $\phi(x)$ be the solution of

$$y' = f(x, y),\ y(x_0) = y_0,$$

and $\phi_m(x)$ be the solution of

$$y' = f_m(x, y),\ \ y(x_0) = y_0.$$

Assume that both $\phi(x)$, $\phi_m(x)$ exist on $[x_0 - h, x_0 + h]$. Then, for $|x - x_0| \leq h$, we have

$$|\phi(x) - \phi_m(x)| \leq he^{\hat{L}h} \max_{(x,y)\in\mathscr{R}} \Big\{|f(x, y) - f_m(x, y)|\Big\},$$

$\hat{L} = \min\{L, L_m\}, |\frac{\partial f}{\partial y}(x, y)| \leq L, |\frac{\partial f_m}{\partial y}(x, y)| \leq L_m \forall (x, y) \in \mathscr{R}$. Further, as $f_m \to f$, $\phi_m \to \phi$ uniformly on $[x_0 - h, x_0 + h]$.

Separable Equations

Definition: A first-order equation $y'(x) = f(x, y)$ is separable if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y)$$

Method for solving separable equations: To solve the equation

$$\frac{dy}{dx} = g(x)p(y),$$

we write it as $h(y)dy = g(x)dx,$ where $h(y) := \frac{1}{p(y)}$.
Integrating both sides

$$\int h(y)dy = \int g(x)dx \implies H(y) = G(x) + C,$$

which gives an implicit solution to the differential equation.

Formal justification of method: Writing the equation in the form

$$h(y)\frac{dy}{dx} = g(x), \;\; h(y) := \frac{1}{p(y)}.$$

Let $H(y)$ and $G(x)$ be such that

$$H'(y) = h(y), \quad G'(x) = g(x).$$

Then

$$H'(y)\frac{dy}{dx} = G'(x).$$

Since $\frac{d}{dx}H(y(x)) = H'(y(x))\frac{dy}{dx}$(by chain rule), we obtain

$$\frac{d}{dx}H(y(x)) = \frac{d}{dx}G(x) \Rightarrow H(y(x)) = G(x) + C.$$

Remark: In finding a one-parameter family of solutions in the separation process, we assume that $p(y) \neq 0$. Then we must find the solutions $y = y_0$ of the equation $p(y) = 0$ and determine whether any of these are solutions of the original equation which were lost in the formal separation process.

Example: Consider $(x-4)y^4dx - x^3(y^2-3)dy = 0$. Separating the variable by dividing x^3y^4, we obtain

$$\frac{(x-4)dx}{x^3} - \frac{(y^2-3)dy}{y^4} = 0$$

The general solution is $-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = C, \quad y \neq 0$

Note: $y = 0$ is a solution of the original equation which was lost in the separation process.

First-Order Linear Equations

A linear first-order equation can be expressed in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \tag{3}$$

where $a_1(x), a_0(x)$ and $b(x)$ depend only on the independent variable x, not on y.

Examples:

$(1+2x)\frac{dy}{dx} + 6y = e^x$ (linear)
$\sin x\frac{dy}{dx} + (\cos x)y = x^2$ (linear)
$\frac{dy}{dx} + xy^3 = x^2$ (not linear)

Theorem**(Existence and Uniqueness)**:
Suppose $a_1(x), a_0(x)$, $b(x) \in C((a,b))$, $a_1(x) \neq 0$ and $x_0 \in (a,b)$. Then for any $y_0 \in \mathbb{R}$, there exists a unique solution $y(x) \in C^1((a,b))$ to the IVP

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \quad y(x_0) = y_0.$$