MA 102 (Ordinary Differential Equations)

IIT Guwahati

Tutorial Sheet No. 9 **Date:** March 21, 2016

Picard's Theorem, Integrating Factors and Exact Differential Equations.

(1) Determine the *order* and *degree* of the following differential equations. Also, state whether they are *linear* or *nonlinear*.
(a) $\frac{d^4y}{dx^4}+19\left(\frac{dy}{dx}\right)^2=11y$; (b) $\frac{d^2y}{dx^2}+x\sin y=0$; (c) $\frac{d^2y}{dx^2}+y\sin x=0$; (d) $(1+\frac{dy}{dx})^{\frac{1}{2}}=x\frac{d^2y}{dx^2}$;
(e) $\frac{d^6y}{dx^6}+\left(\frac{d^4y}{dx^4}\right)\left(\frac{d^3y}{dx^3}\right)+y=x$; (f) $x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}+y=e^x$.

(2) Eliminating the arbitrary constants c_1, c_2, obtain the differential equation satisfied by the following functions.
(a) $y=c_1e^{-x}+c_2e^{2x}$; (b) $x^2+c_1y^2=1$; (c) $y=c_1x-c_1^3$.

(3) Consider the equation $y'(x)=cy(x)$, $0<x<\infty$, where c is a real constant. Then
(a) Show that if ϕ is any solution and $\psi(x)=\phi(x)e^{-cx}$ then $\psi(x)$ is a constant.
(b) If $c<0$, show that every solution tends to zero as $x\to\infty$.
(c) If $c>0$, prove that the magnitude of every non-trivial solution tends to ∞ as $x\to\infty$.
(d) When $c=0$, what can be said about the magnitude of the solution?

(4) Find all real valued C^1 solutions $y(x)$ of the differential equation $xy'(x)+y(x)=x,\ x\in(-1,1)$.

(5) Under what conditions, the following differential equations are exact?
(a) $(ax+by)dx+(kx+ly)dy=0$; (b) $[f(x)+g(y)]dx+[h(x)+l(y)]dy=0$;
(c) $(x^3+xy^2)dx+(ax^2y+bxy^2)dy=0$.

(6) Are the following equations exact? If exact, obtain the general solution.
(a) $(2xy-\sec^2 x)dx+(x^2+2y)dy=0$. (b) $(x-2xy+e^y)dx+(y-x^2+xe^y)dy=0$.

(7) In each case find an integrating factor and solve:
(a) $y'-(2/x)y=x^2\cos x$, (b) $ydx+(x^2y-x)dy=0$, (c) $y(2x^2y^3+3)dx+x(x^2y^3-1)dy=0$

(8) Show that if $(N_x-M_y)/(xM-yM)=g(xy)$ then the equation $M(x,y)dx+N(x,y)dy=0$ has an integrating factor of the form $\mu(xy)$, where $\mu(u)=\exp\left(\int g(u)du\right)$.

(9) Discuss the existence and uniqueness of a solution of the following initial value problems (IVP) in the region $R:\ |x|\le 1\ |y|\le 1$.
(a) $\dfrac{dy}{dx}=3y^{2/3}$, $y(0)=0$; (b) $\dfrac{dy}{dx}=\sqrt{|y|}$, $y(0)=0$;
(c) $\dfrac{dy}{dx}=x^2+y^2$, $y(0)=0$.

(10) Show that the equation $|y'(x)|+|y(x)|+1=0$ has no real solutions.

(11) Find the particular solution of
(a) $xy'+3y=\frac{\sin x}{x^2}$, $x\neq 0$, $y(\pi/2)=1$.
(b) $y'+y=f(x)$, $y(0)=0$, where $f(x)=\begin{cases}2, & 0\le x<1,\\ 0, & x\ge 1.\end{cases}$
(c) $x^2y'+xy=\frac{y^3}{x}$, $y(1)=1$, $x\neq 0$.

(12) Given that $y_1(x)=x$ is a solution of $\dfrac{dy}{dx}=-y^2+xy+1$, obtain the general solution.

(13) Find the value of n such that the curves $x^n+y^n=c_1$ are the orthogonal trajectories of the family $y=\dfrac{x}{1-c_2x}$, where c_1 and c_2 are arbitrary constants.

(14) A point P is dragged along the xy plane by a string PT of length a. If T starts at the origin and moves along the positive y axis, and if P starts at $(a,0)$, what is the path of P?