Physics II: Electromagnetism (PH102)

Lecture 6

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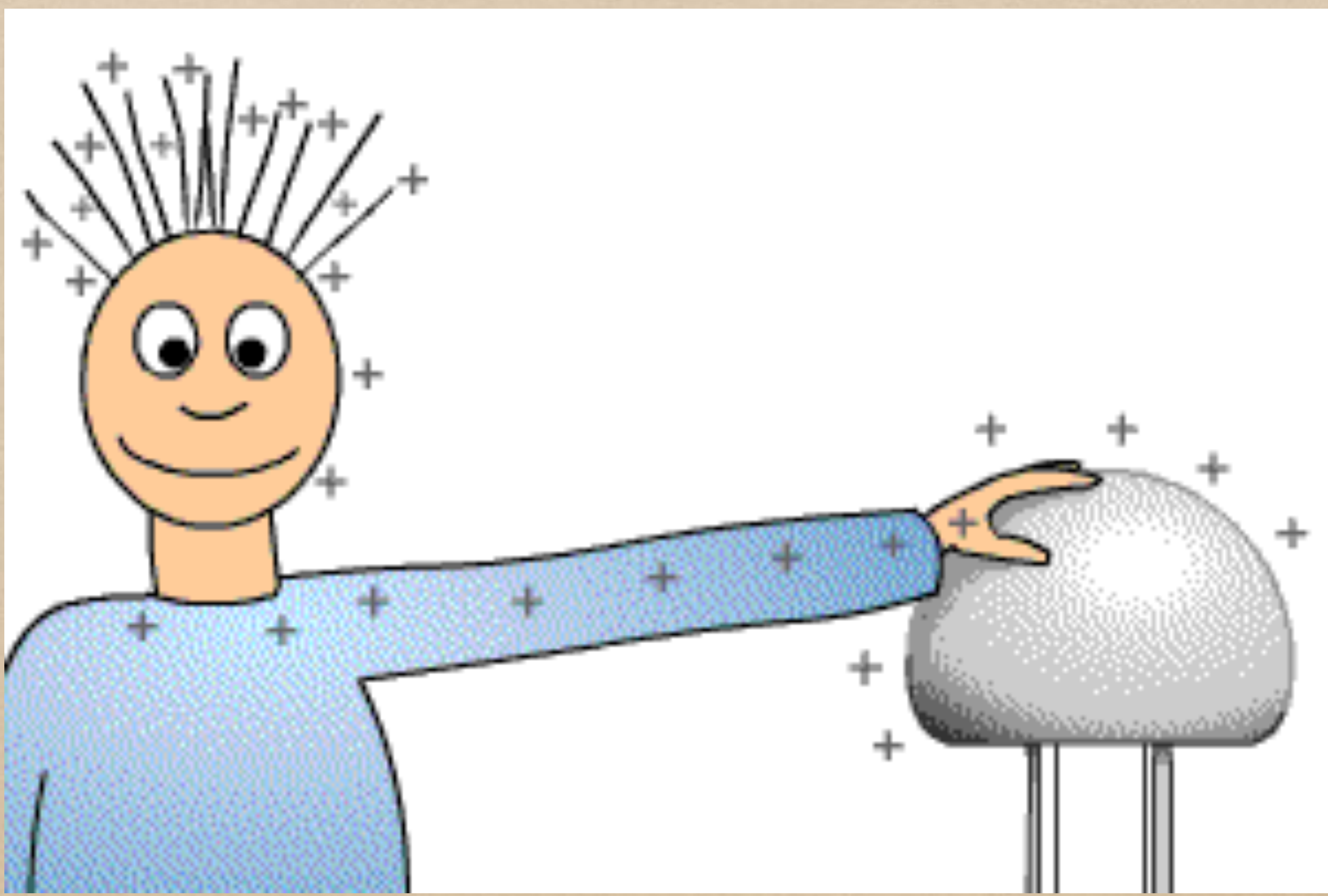
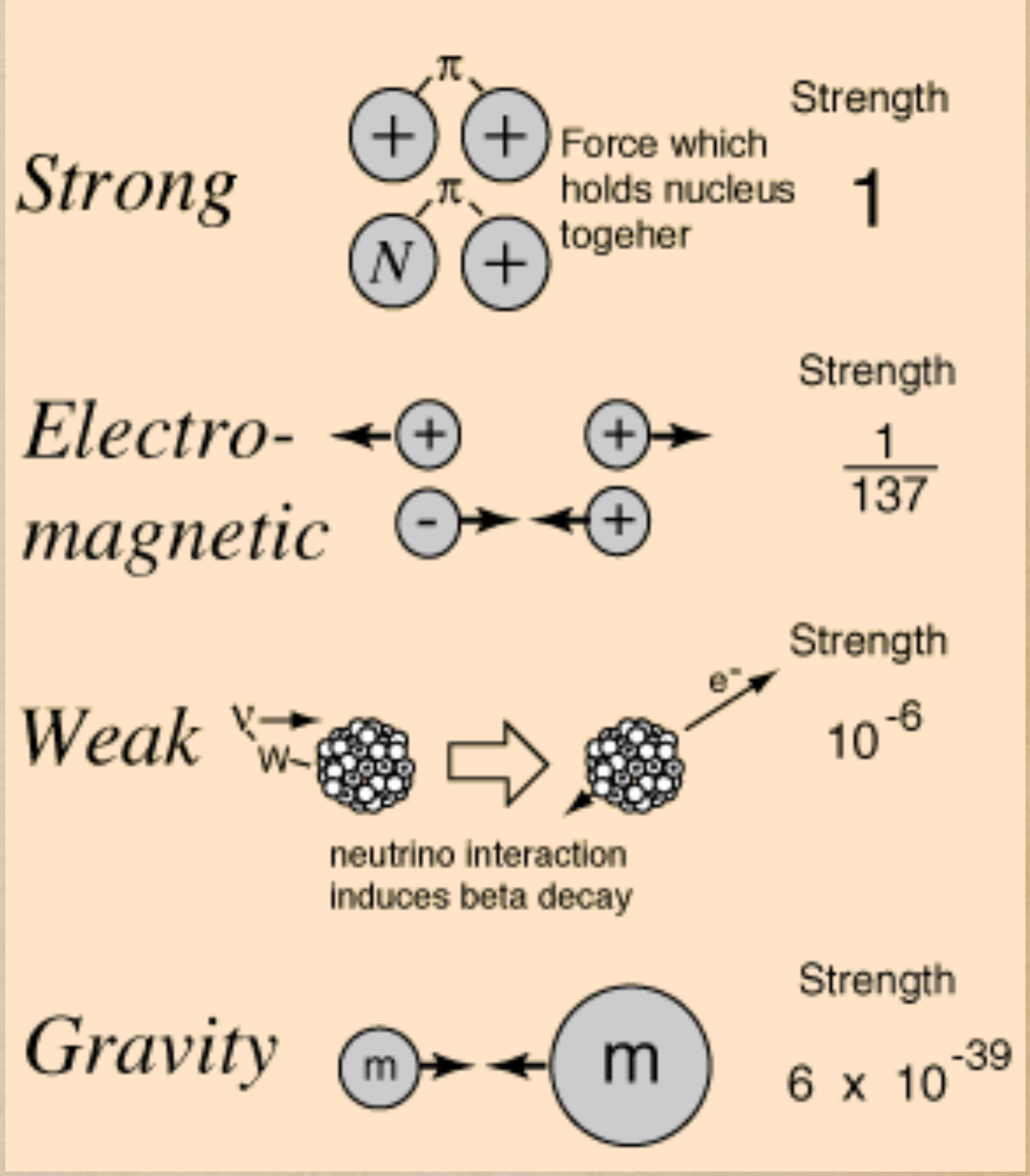
Electrostatics



Fundamental forces in nature

- Gravitational
- Electromagnetic
- Weak
- Strong

Electromagnetic forces are abundant in nature and can be treated classically at macroscopic level

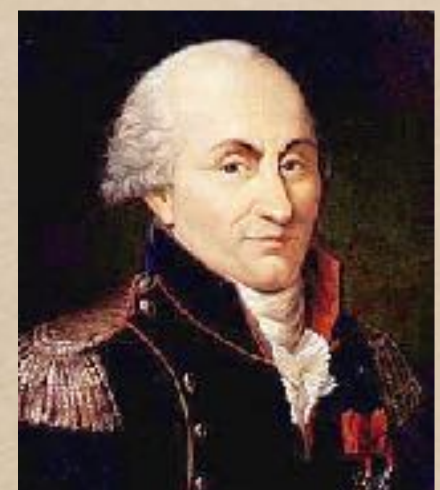


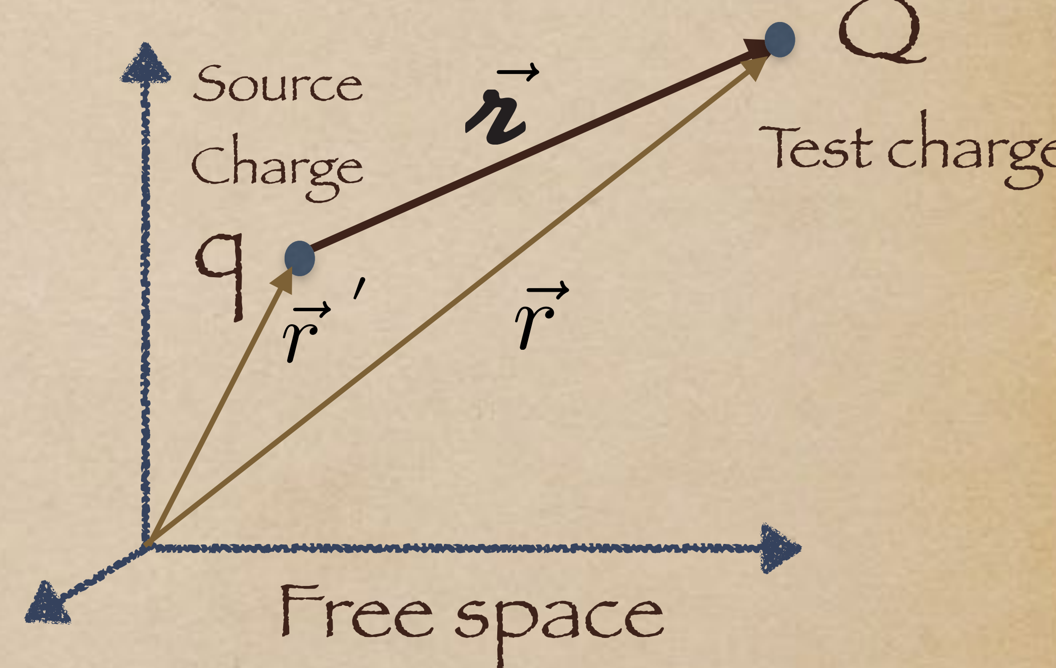
Electrostatics

- Charge is the fundamental property of forms of matter that exhibit electrostatic attraction or repulsion in the presence of other matter.
- Suppose, we have some electric charges, what force do they exert on another charge ?

Coulomb's law:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$$





$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' \longrightarrow$ Separation

$\epsilon_0 = 8.85 \times 10^{-12}\ \frac{C^2}{N\ m^2}$ is called permittivity of free space

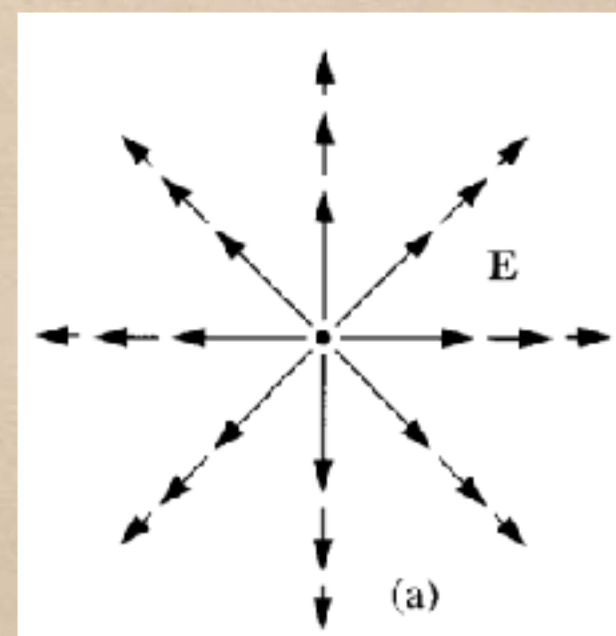
No body can derive Coulomb's law !

Electric Field

Electric field is the force per unit charge that would be exerted on a test charge Q

$$\vec{F} = Q\vec{E} \text{ where } \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$$

We also know how to draw such a vector field !



An example of a vector field with radial dependence !

using

$$\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

$$\vec{\nabla}.\vec{E} = \frac{q}{\epsilon_0}\delta^3(\vec{r})$$

$$\vec{\nabla} \times \vec{E} = 0$$

divergence and curl of electric field for a point charge sitting at the origin

Many source charges

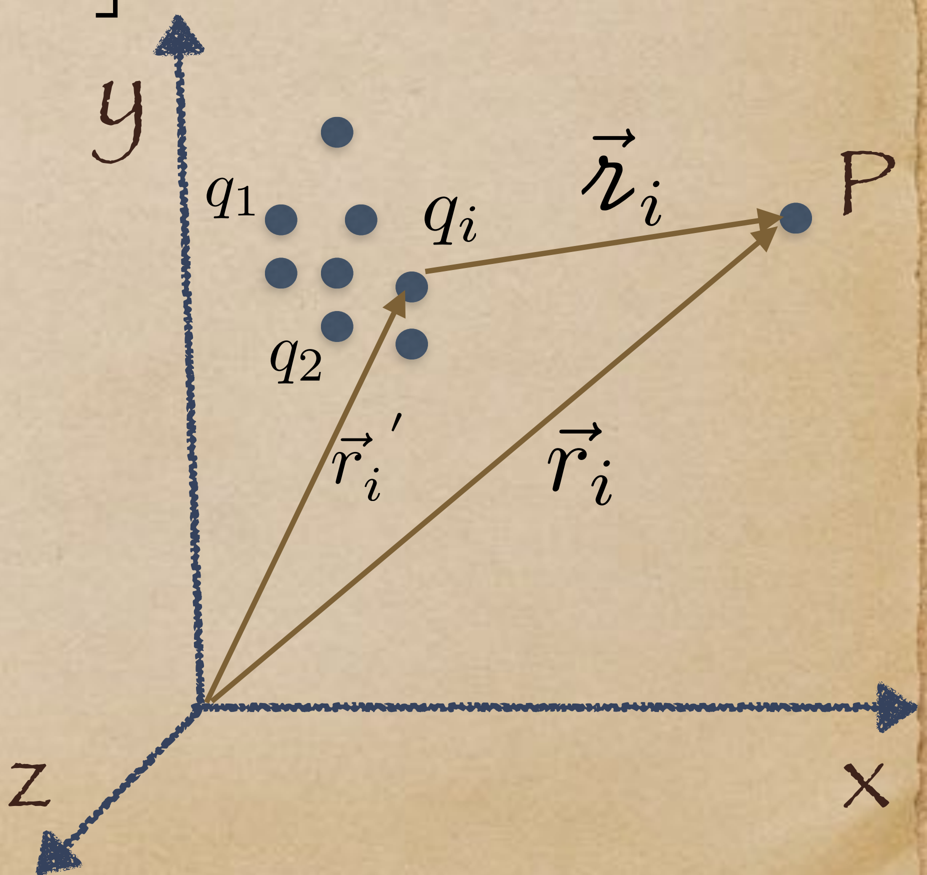
Principle of superposition: Interaction between any two charges is completely unaffected by the presence of other charges

If there are many point charges $q_1, \; q_2, \cdots, q_n$ at distances $r_1, \; r_2, \cdots, r_n$ from Q (test charge) then according to principle of superposition:

$$\vec{F} = \frac{Q}{4\pi\epsilon_0}\left[\frac{q_1\hat{\boldsymbol{\mathscr{r}}}_1}{\mathscr{r}_1^2} + \frac{q_2\hat{\boldsymbol{\mathscr{r}}}_2}{\mathscr{r}_2^2} + \frac{q_3\hat{\boldsymbol{\mathscr{r}}}_3}{\mathscr{r}_3^2} + \cdots\right] = Q\vec{E}$$

using $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots$

where $$\boxed{\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i}$$

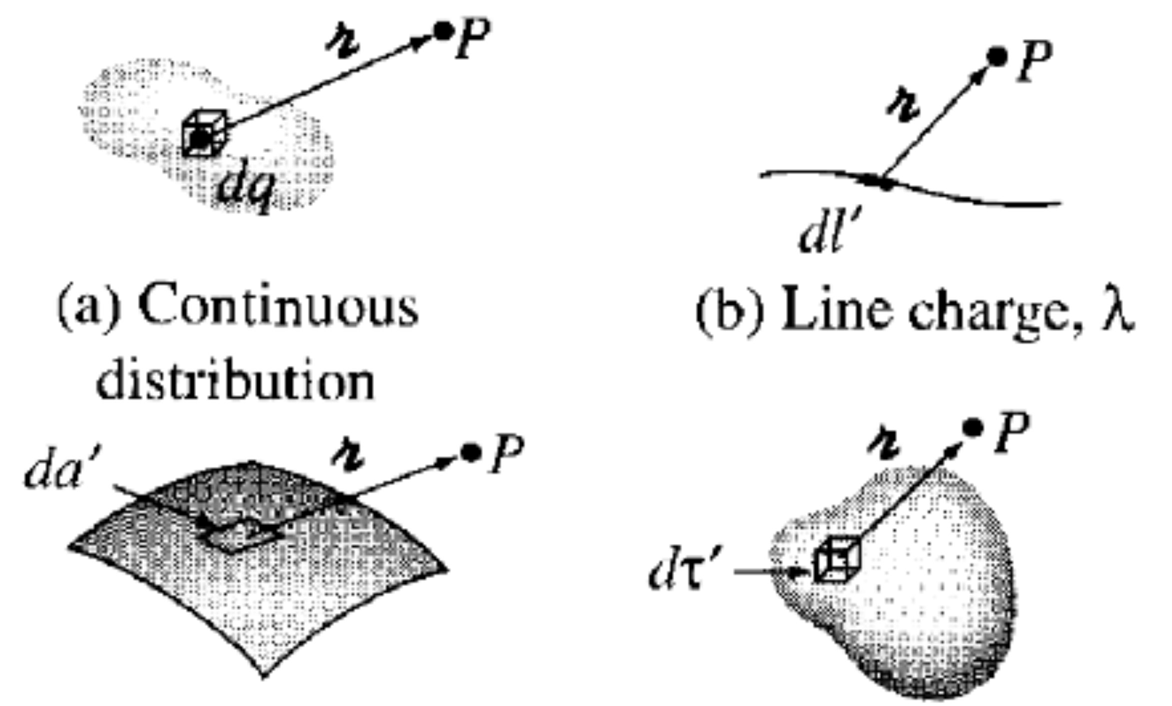


Continuous Charge Distributions

$$\sum_{i=1}^{n} (\,)\; q_i \rightarrow \sim \int_{line} (\,)\; \lambda dl$$

$$\sim \int_{sur} (\,)\; \sigma da$$

$$\sim \int_{vol} (\,)\; \rho d\tau$$



(a) Continuous distribution
(b) Line charge, λ
(c) Surface charge, σ
(d) Volume charge, ρ

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int_{line} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \lambda dl$$

$$= \frac{1}{4\pi\epsilon_0} \int_{sur} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \sigma da$$

$$= \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho d\tau$$

$$\vec{\mathscr{r}} = (x - x^{'})\hat{x} + (y - y^{'})\hat{y} + (z - z^{'})\hat{z}$$

$$\mathscr{r} = \left[(x - x^{'})^2 + (y - y^{'})^2 + (z - z^{'})^2\right]^{1/2}$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$$

Example of line, surface and volume integrals

Example of calculating Electric field…

Find the electric field a distance z above the midpoint of a straight line segment of length $2L$, which carries a uniform line charge λ (Fig. 2.6).

$$d\vec{E} = \frac{2}{4\pi\epsilon_0}\left(\frac{\lambda dx}{\mathscr{r}^2}\right)\cos\theta\,\hat{z}$$

Only the vertical component remains as the horizontal ones cancel

Here $\cos\theta = z/\mathscr{r}$, $\mathscr{r} = \sqrt{z^2 + x^2}$, and x runs from 0 to L:

$$\begin{aligned} E &= \frac{1}{4\pi\epsilon_0}\int_0^L \frac{2\lambda z}{(z^2+x^2)^{3/2}}dx \\ &= \frac{2\lambda z}{4\pi\epsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\Bigg|_0^L \\ &= \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}, \end{aligned}$$

For points far from the line ($z \gg L$), this result simplifies:

$$E \cong \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2},$$

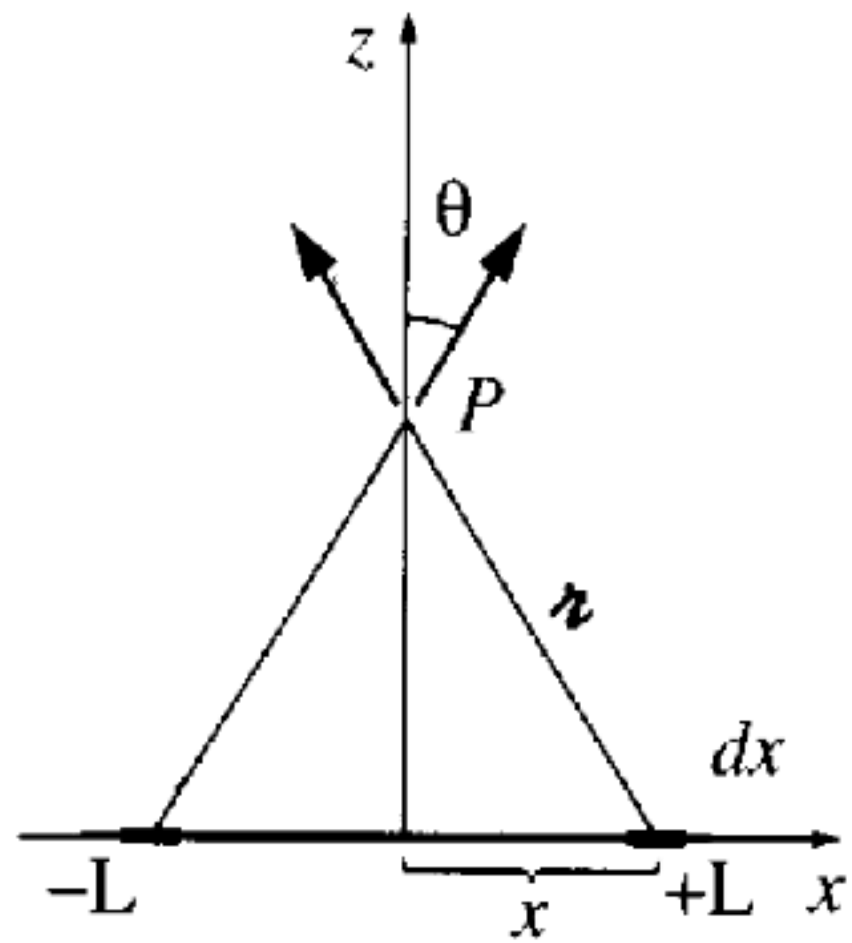
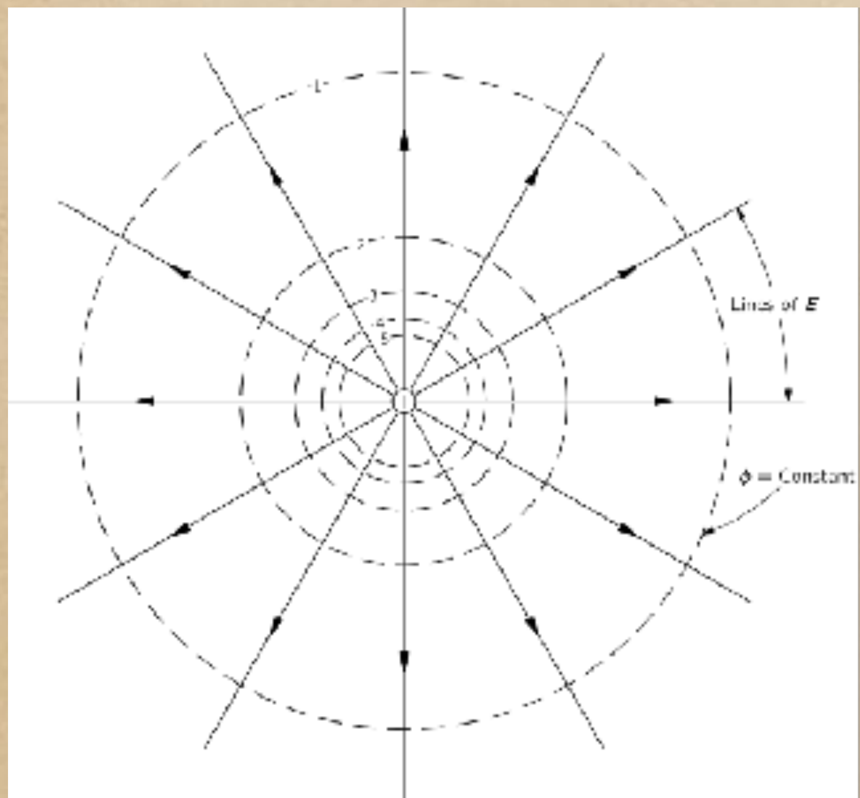


Figure 2.6

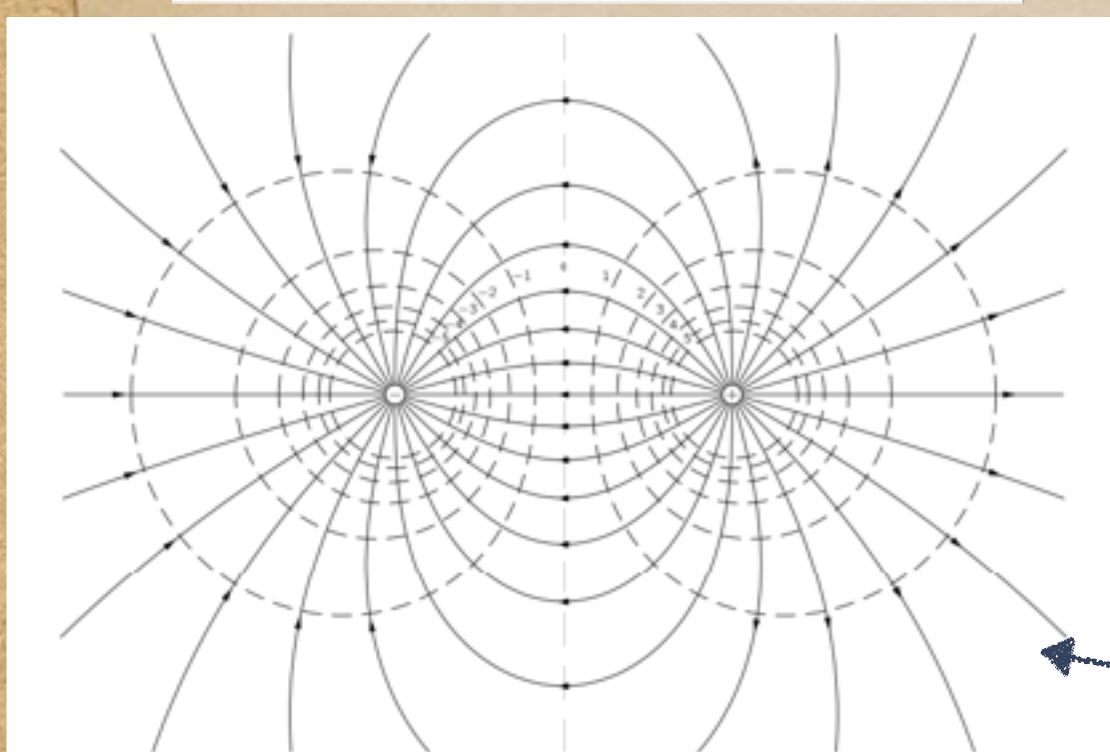
Electric field lines



Suppose we have +q charge at origin.
Then electric field at a distance r

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

- Electric Field is shown by the field lines which point radially outward for +q
- Strength of the field is indicated by the density of field lines: strong near the centre, weak further away
- The field lines emanate from a point charge symmetrically in all directions
- Field lines originate from positive charges and ends on a negative one



Opposite charges attract and same charges repel

Flux of Electric field

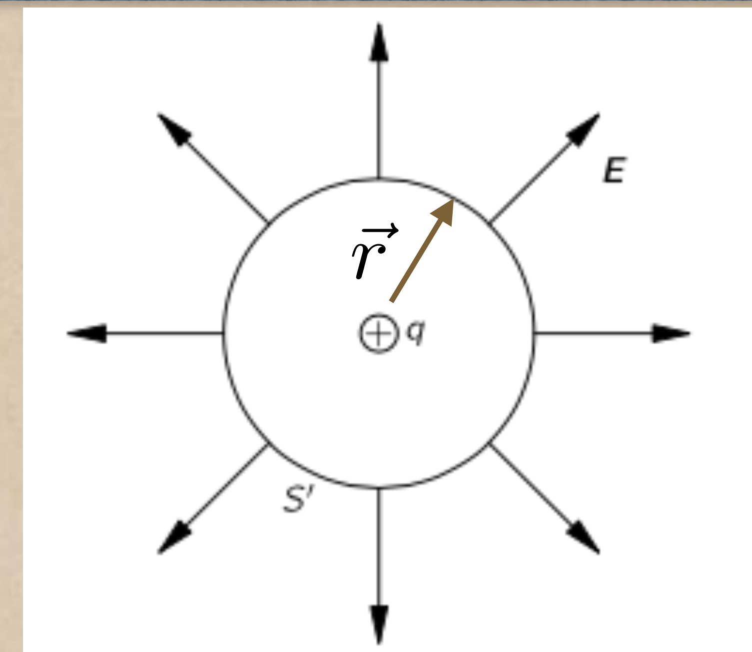
Field strength is proportional to the number of lines per unit area (Area perpendicular to the lines): Flux of electromagnetic field is proportional to the number of field lines passing through the surface

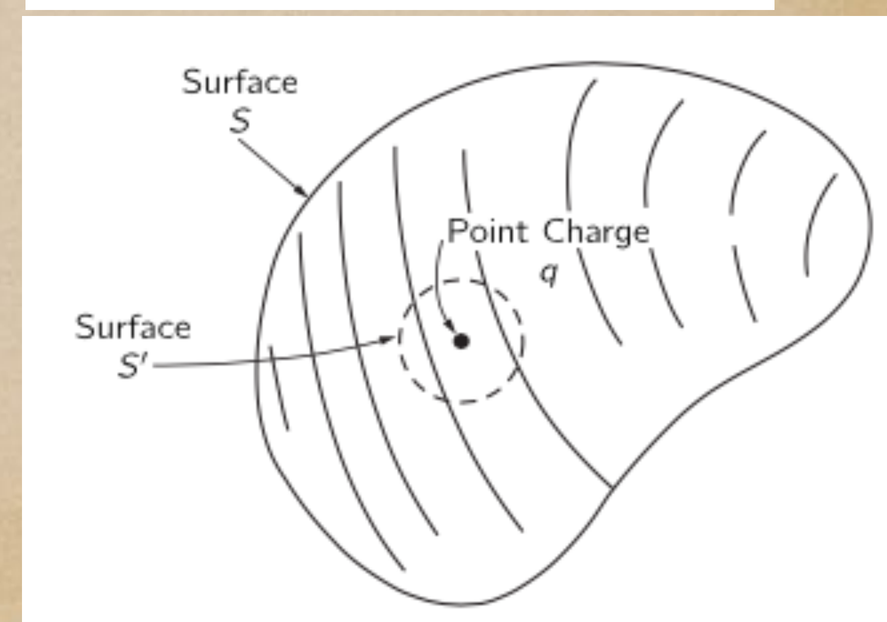
For a point charge at the origin the flux of E through a sphere of radius r:

$$\oint \vec{E}.\vec{da} = \frac{1}{4\pi\epsilon_0}\oint \frac{q}{r^2}\hat{r}.(r^2 \sin\theta d\theta d\phi)\hat{r} = \frac{q}{\epsilon_0}$$

The answer do not have any dependence on the enclosing surface !

The surface need not be a spherical one, any surface enclosing charge q will have the same flux !





Gauss's law

Flux through any surface that encloses bunch of electrostatic charges :

$$\oint \vec{E}.\vec{da} = \sum_{i=1}^{n}\left(\oint \vec{E}_i.\vec{da}\right) = \sum_{i=1}^{n}\frac{q_i}{\epsilon_0}$$

Using superposition principle

A charge outside the surface would contribute nothing to the total flux

$$\oint_{sur} \vec{E}.\vec{da} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Gauss's Law: Flux through an enclosed surface is proportional to the charge enclosed by the surface

Divergence of Electric field

Using Divergence Theorem, we can convert the integral form of Gauss's law to a differential form:

$$\oint_{sur} \vec{E}.\vec{da} = \int_{vol} (\vec{\nabla}.\vec{E}) d\tau = \frac{Q_{enc}}{\epsilon_0}$$

by Gauss' law

$$= \int_{vol} \left(\frac{\rho}{\epsilon_0}\right) d\tau \longrightarrow \vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$$

where $Q_{\text{enc}} = \int_{vol} \rho \, d\tau$

volume charge density

but, $\vec{\nabla}.\vec{E} = \frac{q}{\epsilon_0}\delta^3(\vec{r})$

So, everything is consistent !

$$\boxed{\text{Recall, } \rho(\vec{r}) = q(\vec{r}')\delta^3(\vec{r} - \vec{r}')}$$

charge density due to a point charge

Divergence of Electric Field for volume charge distribution

Recall $$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{vol}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\vec{r}^{\,\prime})d\tau^{\prime}$$

$$\vec{\mathscr{r}} = (x-x^{\prime})\hat{x} + (y-y^{\prime})\hat{y} + (z-z^{\prime})\hat{z}$$

$$\mathscr{r} = \left[(x-x^{\prime})^2 + (y-y^{\prime})^2 + (z-z^{\prime})^2\right]^{1/2}$$

$$\therefore\ \vec{\nabla}.\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{vol}\vec{\nabla}.\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\rho(\vec{r}^{\,\prime})d\tau^{\prime}$$

Divergence of the electric field only depends on the test position. Hence divergence operator only acts on $\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$ as it depend on $\vec{r}$

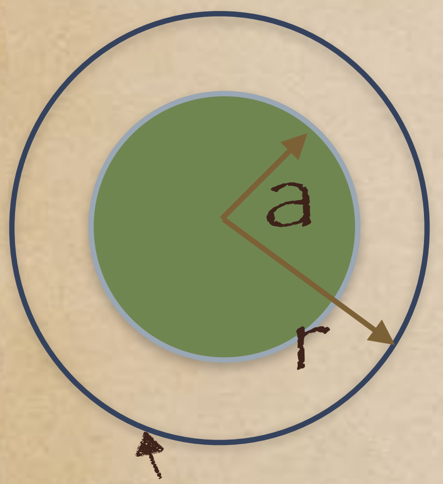
Also recall $$\boxed{\vec{\nabla}.\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})}$$

$$\therefore\ \vec{\nabla}.\vec{E} = \frac{1}{4\pi\epsilon_0}\int 4\pi\delta^3(\vec{r}-\vec{r}^{\,\prime})\rho(\vec{r}^{\,\prime})d\tau = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\therefore\ \int_{\mathrm{vol}}\vec{\nabla}.\vec{E}\,d\tau = \frac{1}{\epsilon_0}\int_{\mathrm{vol}}\rho(\vec{r})d\tau = \frac{Q_{\mathrm{enc}}}{\epsilon_0}$$

How to choose your Gaussian Surface?

Gauss law provides quickest way of calculating EM field



Gaussian surface for spherical symmetry

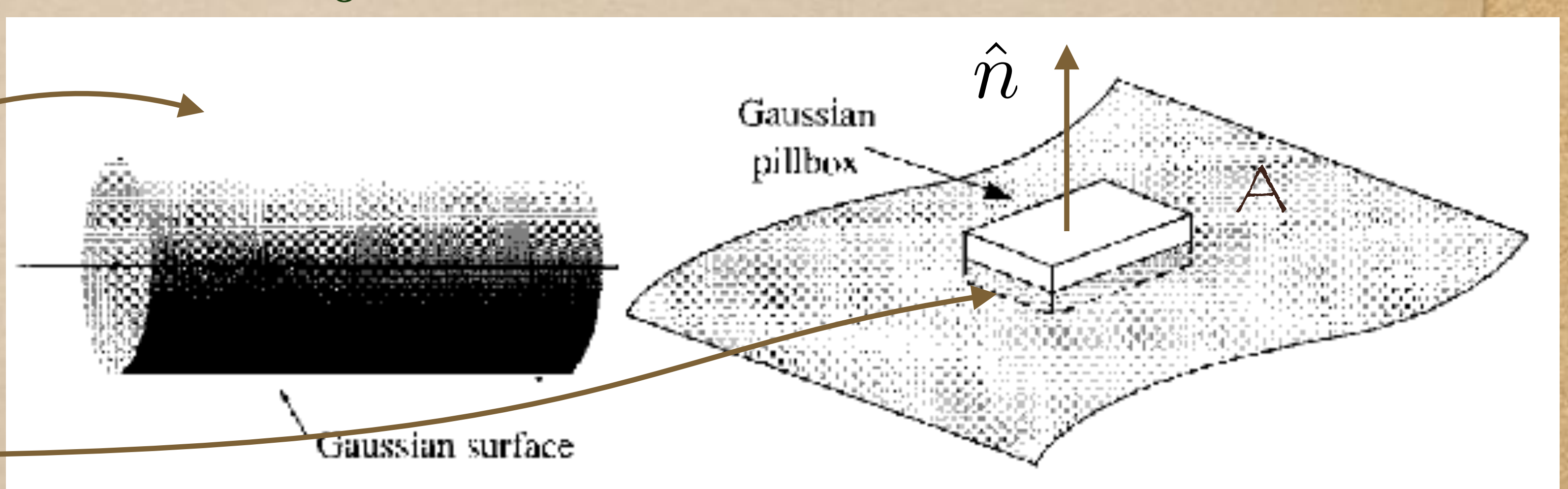
Field outside a uniformly charged sphere of radius a

$$\oint_{sur} \vec{E}.d\vec{a} = \oint_{sur} |\vec{E}|da = |\vec{E}| \oint_{sur} da = |\vec{E}|4\pi r^2 = \frac{Q_{enc}}{\epsilon_0}$$

$$\longrightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{(4/3)\pi a^3\rho}{r^2}\hat{r}$$

Cylindrical symmetry

Plane symmetry



For an infinite plane carrying uniform surface charge density

$$\oint \vec{E}.d\vec{a} = 2A|\vec{E}| = \frac{\sigma A}{\epsilon_0}$$

$$\implies \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

In summary ....

- Electric charge causes electrostatic interaction
- Coulomb's law estimates the amount of electrostatic force between two charged particles.
- Electric field (force per unit test charge) acts as fundamental measurable/observable.
- Gauss's law yields a quick estimate of electric field (subject to a symmetry of the charged object).