1. Evaluate $\int \mathbf{A}\cdot\hat{\mathbf{n}}\, ds$, where $\mathbf{A} = 18z\,\hat{\mathbf{x}} - 12\,\hat{\mathbf{y}} + 3y\,\hat{\mathbf{z}}$ and S is that part of the plane $2x+3y+6z = 12$ which is located in the first octant.

   Solution:

   The surface S and its projection R on the xy plane are shown in the figure below.

$$\int \mathbf{A}\cdot\hat{\mathbf{n}}\, ds = \int\int_R \mathbf{A}\cdot\hat{\mathbf{n}}\frac{dxdy}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}|}$$

   To obtain $\hat{\mathbf{n}}$ note that a vector perpendicular to the surface $2x+3y+6z = 12$ is given by $\nabla(2x+3y+6z) = 2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$. Then the unit normal to S at any point is

$$\mathbf{n} = \frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\mathbf{i}+\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}$$

   Thus $\mathbf{n}\cdot\mathbf{k} = \left(\frac{2}{7}\mathbf{i}+\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}\right)\cdot\mathbf{k} = \frac{6}{7}$ and so $\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \frac{7}{6}dxdy$.
   Also

$$\mathbf{A}\cdot\mathbf{n} = \frac{36z-36+18y}{7} = \frac{36-12x}{7}$$

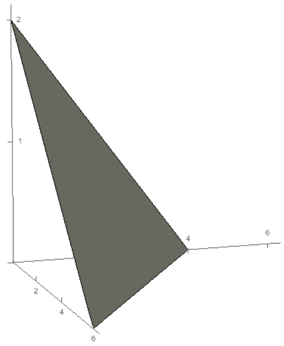
   using the fact that $z = \frac{12-2x-3y}{6}$ from the equation of S. Then

$$\int\int_R \mathbf{A}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \int\int_R\left(\frac{36-12x}{7}\right)\frac{7}{6}dxdy = \int\int_R(6-2x)\,dxdy$$

   The integral becomes

$$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3}(6-2x)\,dydx = \int_{x=0}^{6}\left(24-12x+\frac{4x^2}{3}\right)dx = 24$$

   If we had chosen the positive unit normal $\mathbf{n}$ opposite to that in the figure, we would have obtained the result -24.



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2. Evaluate $\int_S \mathbf{A}\cdot\hat{\mathbf{n}}\, ds$, where $A = z\hat{\mathbf{i}} + x\hat{\mathbf{j}} - 3y^2 z\hat{\mathbf{k}}$ and S is the surface of the cylinder $x^2+y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

   Solution:

   Method 1: We will project this cylindrical surface on XZ plane. (Why not XY plane?)

   The projected region is rectangular: $R = \{(x,0,z)\,|\,0 \le x \le 4,\ 0 \le z \le 5\}$.

   Let $\phi = x^2+y^2$. Normal to the surface $\hat{\mathbf{n}} = \nabla\phi/|\nabla\phi| = \frac{1}{4}(x\hat{\mathbf{x}}+y\hat{\mathbf{y}})$.

The required integral

$$\begin{aligned}\int_S \mathbf{A}\cdot\hat{\mathbf{n}}\,ds &= \int_R \mathbf{A}\cdot\hat{\mathbf{n}}\,\frac{dxdy}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{y}}|} = \int_R \frac{1}{4}\,(xz+xy)\,\frac{dxdz}{y/4}\\ &= \int_0^5 dz\int_0^4 dx\left(\frac{xz}{\sqrt{16-x^2}}+x\right)\\ &= 90\end{aligned}$$

Method 2: Use parametrization: $x = 4\cos\phi$, $y = 4\sin\phi$, and $z = z$.

The parameters are $\phi \in [0, \pi/2]$ and $z \in [0, 5]$.

So, at point $\mathbf{S} \equiv (x, y, z) \equiv (4\cos\phi, 4\sin\phi, z)$, first find the normal

$$\frac{\partial\mathbf{S}}{\partial\phi}\times\frac{\partial\mathbf{S}}{\partial z} = (-4\sin\phi, 4\cos\phi, 0)\times(0,0,1) = (4\cos\phi, 4\sin\phi, 0)\,.$$

Then the unit normal is $\hat{\mathbf{n}} = (\cos\phi, \sin\phi, 0)$ and elemental area is $\left|\frac{\partial\mathbf{S}}{\partial\phi}\times\frac{\partial\mathbf{S}}{\partial z}\right| d\phi dz = 4d\phi dz.$

Write $\mathbf{A}$ in terms of parameters. $\mathbf{A} = z\hat{\mathbf{i}} + x\hat{\mathbf{j}} - 3y^2z\hat{\mathbf{k}} = z\hat{\mathbf{i}} + 4\cos\phi\hat{\mathbf{j}} - 3\sin^2\phi\, z\hat{\mathbf{k}}$

$$\begin{aligned}\int_S \mathbf{A}\cdot\hat{\mathbf{n}}\,ds &= \int_R (z\cos\phi + x\sin\phi)\,4d\phi dz =\\ &= 4\int_0^5 dz\int_0^{\pi/2} d\phi\,(z\cos\phi + 4\cos\phi\sin\phi)\\ &= 90\end{aligned}$$

---

3. **[G 1.30]** Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

You can do the integrals in any order - here it is simplest to save z for last:

$$\int z^2\left[\int\left(\int dx\right)dy\right]dz.$$

The sloping surface is $x + y + z = 1$, so the x integral is

$$\int_0^{(1-y-z)} dx = 1 - y - z$$

For a given z, y ranges from 0 to $1 - z$, so the y integral is

$$\begin{aligned}\int_0^{(1-z)}(1-y-z)dy &= \Big[(1-z)y - (y^2/2)\Big]\Big|_0^{(1-z)}\\ &= (1-z)^2 - [(1-z)^2/2] = (1/2) - z + (z^2/2)\end{aligned}$$

Finally, the z integral is

$$\int_0^1 z^2\left(\frac{1}{2} - z + \frac{z^2}{2}\right)dz = \int_0^1\left(\frac{z^2}{2} - z^3 + \frac{z^4}{2}\right)dz = \boxed{1/60.}$$

---

4. **[G 1.31]** Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $\mathbf{a} = (0,0,0)$, $\mathbf{b} = (1,1,1)$, and the three paths in Fig.:

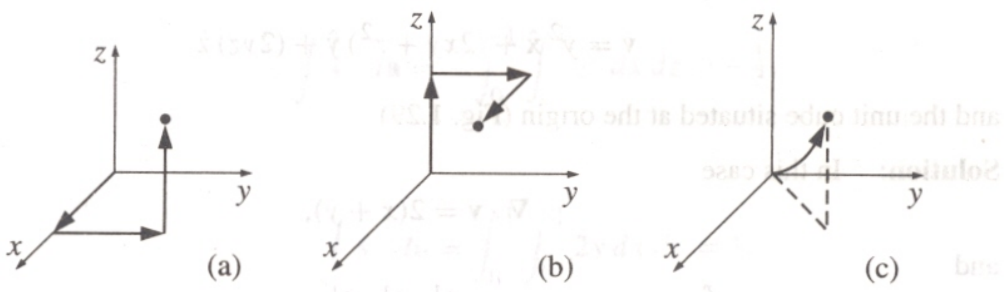

Figure 1: Problem 4

(a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$;

(b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$;

(c) the parabolic path $z = x^2$; $y = x$.

Solution:

First, $T(0,0,0) = 0$ and $T(1,1,1) = 7$. Thus we have to show that for each path given in question $\int \nabla T \cdot d\mathbf{r} = 7 - 0 = 7$. Now

$$\nabla T = (2x + 4y)\,\hat{\mathbf{x}} + \left(4x + 2z^3\right)\hat{\mathbf{y}} + 6yz^2\hat{\mathbf{z}}.$$

(a) For $(0,0,0) \to (1,0,0)$: Let $x : 0 \to 1$ be the parameter of the line. $y = z = 0$. Then $d\mathbf{r} = dx\hat{\mathbf{x}}$. $\nabla T = 2x\hat{\mathbf{x}} + 4x\hat{\mathbf{y}}$. Then

$$\int_{(0,0,0)\to(0,0,1)} \nabla T \cdot d\mathbf{r} = \int_0^1 2x\,dx = 1$$

Similarly, for $(1,0,0) \to (1,1,0)$: $x = 1$, $z = 0$, $y : 0 \to 1$. $d\mathbf{r} = dy\hat{\mathbf{y}}$. $\nabla T = (2 + 4y)\,\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$. Then $\int = 4$.
Finally for $(1,1,0) \to (1,1,1)$: $x = 1$, $y = 1$, $z : 0 \to 1$. $d\mathbf{r} = dz\hat{\mathbf{z}}$. $\nabla T = 6\hat{\mathbf{x}} + (4 + 2z^3)\,\hat{\mathbf{y}} + 6z^2\hat{\mathbf{z}}$. Then $\int = 2$.
Thus $\int_{(0,0,0)\to(1,1,1)} = 1 + 4 + 2 = 7$.

(b) Do the same

(c) Here, the parameter is already given, that is $x : 0 \to 1$. And $y = x$ and $z = x^2$. First,

$$d\mathbf{r} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}} = dx\,(\hat{\mathbf{x}} + \hat{\mathbf{y}} + 2x\hat{\mathbf{z}})$$

And

$$\nabla T = (2x + 4y)\,\hat{\mathbf{x}} + \left(4x + 2z^3\right)\hat{\mathbf{y}} + 6yz^2\hat{\mathbf{z}} = 6x\hat{\mathbf{x}} + \left(4x + 2x^6\right)\hat{\mathbf{y}} + 6x^5\hat{\mathbf{z}}.$$

$$\nabla T \cdot d\mathbf{r} = dx\left(6x + \left(4x + 2x^6\right) + 12x^6\right) = dx\left(10x + 14x^6\right)$$

Now the integral reduces to integral in one variable x. Integrate to get answer 7.

---

5. **[G 1.33]** Test Stokes' theorem for the function $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$, using the triangular shaded area of Fig.

Solution:

$\nabla \times \mathbf{v} = \hat{\mathbf{x}}(0 - 2y) + \hat{\mathbf{y}}(0 - 3z) + \hat{\mathbf{z}}(0 - x) = -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}$.

The area element $d\mathbf{a} = dydz\hat{\mathbf{x}}$, if we agree that the path integral shall run counterclockwise. So $(\nabla \times \mathbf{v}) \cdot d\mathbf{a} = -2ydydz$.

Then

$$\begin{aligned} \int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} &= \int_0^2 \left\{ \int_0^{2-z} (-2y)dy \right\} dz \\ &= -\frac{8}{3} \end{aligned}$$

Now, $\mathbf{v} \cdot d\mathbf{l} = (xy)dx + (2yz)dy + (3zx)dz$. There are three segments.

(a) $x = z = 0;\ y : 0 \to 2$. Then $\int \mathbf{v} \cdot \mathbf{dl} = 0$

(b) $x = 0;\ y = 2 - z;\ dy = -dz,\ z : 0 \to 2$
$\int \mathbf{v} \cdot \mathbf{dl} = \int 2yzdy = -\int_0^2 (4z - 2z^2)dz = -\frac{8}{3}$.

(c) $x = y = 0;\ dx = dy = 0; z : 2 \to 0. \mathbf{v} \cdot \mathbf{dl} = 0. \int \mathbf{v} \cdot \mathbf{dl} = 0$.

So $\oint \mathbf{v} \cdot \mathbf{dl} = -\frac{8}{3}$.

---

6. Consider a vector field $\mathbf{F}$, for which line integral in independent of path between **any** two points. Show that $\nabla \times \mathbf{F} = 0$.

Solution:

If line integral of $\mathbf{F}$ is independent of path between any two points, then line integral over any simple closed loop is also zero. Then by stokes theorem

$$\int_S \nabla \times \mathbf{F} ds = \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

Since this is true for **any arbitrary surface** clearly the integrad must be zero. Hence $\nabla \times \mathbf{F} = 0$

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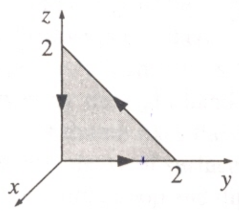
7. **[G 1.39]** Compute the divergence of the function

$$\mathbf{v} = (r\cos\theta)\hat{\mathbf{r}} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}.$$

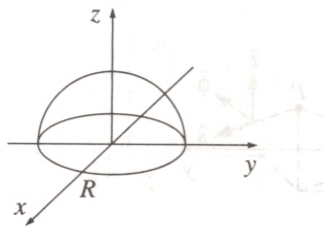
Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin (See fig).

Solution:

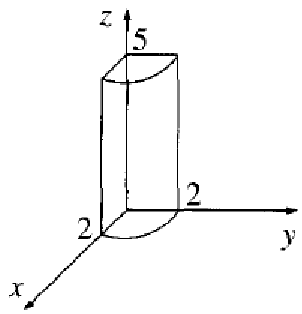
$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)$
$= \frac{1}{r^2}3r^2\cos\theta + \frac{1}{r\sin\theta}r2\sin\theta\cos\theta + \frac{1}{r\sin\theta}r\sin\theta(-\sin\phi)$
$= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi$
$\int(\nabla \cdot \mathbf{v})d\tau = \int(5\cos\theta - \sin\phi)r^2\sin\theta drd\theta d\phi = \int_0^R r^2dr \int_0^{\frac{\pi}{2}}[\int_0^{2\pi}(5\cos\theta - \sin\phi)d\phi]d\theta\sin\theta$
$= (\frac{R^3}{3})(10\pi)\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta$
$= \frac{5\pi}{3}R^3$.
Two surfaces - one the hemisphere: $d\mathbf{a} = R^2\sin\theta d\theta d\phi\hat{\mathbf{r}};\ r = R;\ \phi : 0 \to 2\pi,\ \theta : 0 \to \frac{\pi}{2}$.
$\int \mathbf{v} \cdot d\mathbf{a} = \int(r\cos\theta)R^2\sin\theta d\theta d\phi = R^3\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi = R^3(\frac{1}{2})(2\pi) = \pi R^3$.
other the flat bottom: $d\mathbf{a} = (dr)(r\sin\theta d\phi)(+\hat{\theta}) = rdrd\phi\hat{\theta}$ (here $\theta = \frac{\pi}{2}$). r: $0 \to R$, $\phi : 0 \to 2\pi$.
$\int \mathbf{v} \cdot d\mathbf{a} = \int(r\sin\theta)(rdrd\phi) = \int_0^R r^2dr\int_0^{2\pi} d\phi = 2\pi\frac{R^3}{3}$.
Total: $\int \mathbf{v} \cdot d\mathbf{a} = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3$.



(a) Problem 5



(b) Problem 7



(c) Problem 9

8. **[G 1.41]** Derive the relations for unit vectors of cylindrical coordinate system:

$$\begin{aligned}\hat{\mathbf{s}} &= \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}},\\ \hat{\phi} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}},\\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}.\end{aligned}$$

Invert the formulas to get $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ in terms of $\hat{\mathbf{s}}$, $\hat{\phi}$, $\hat{\mathbf{z}}$ (and ϕ).

Solution:

Let $\mathbf{r} = (x, y, z) = (s\cos\phi, s\sin\phi, z)$. Then, $\dfrac{\partial \mathbf{r}}{\partial s} = (\cos\phi, \sin\phi, 0)$ and

$$\hat{\mathbf{s}} = \frac{\partial \mathbf{r}}{\partial s} / \left|\frac{\partial \mathbf{r}}{\partial s}\right| = (\cos\phi, \sin\phi, 0)$$

similarly, $\dfrac{\partial \mathbf{r}}{\partial \phi} = s\,(-\sin\phi, \cos\phi, 0)$ and

$$\hat{\phi} = \frac{\partial \mathbf{r}}{\partial \phi} / \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = (-\sin\phi, \cos\phi, 0)$$

$\hat{\mathbf{z}}$ is obvious.

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9. **[G 1.42]**

   (a) Find the divergence of the function $\mathbf{v} = s(2 + \sin^2\phi)\hat{\mathbf{s}} + s\sin\phi\cos\phi\hat{\phi} + 3z\hat{\mathbf{z}}$.

   (b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig.

   (c) Find the curl of v.

Solution:

(a) To find divergence, we can use the divergence formula in cylindrical coordiantes:

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial}{\partial \phi}v_\phi + \frac{\partial v_z}{\partial z}\\ &= 2\left(2 + \sin^2\phi\right) + \cos 2\phi + 3\\ &= 8\end{aligned}$$

(b) Now $\int_v \nabla \cdot \mathbf{v}\, dv = 8 \int_v dv = 8 \times 5\pi = 40\pi$

There are five surfaces to the volume.

surface	parameters	area element	$\mathbf{v} \cdot d\mathbf{s}$	integral
$z = 0$	$s : 0 \to 2,\ \phi : 0 \to \pi/2$	$-sdsd\phi\hat{\mathbf{z}}$	$-3zsd\phi ds = 0$	0
$z = 5$	$s : 0 \to 2,\ \phi : 0 \to \pi/2$	$sdsd\phi\hat{\mathbf{z}}$	$15sd\phi ds$	15π
$\phi = 0$ (XZ plane)	$s : 0 \to 2,\ z : 0 \to 5$	$dsdz\left(-\hat{\phi}\right)$	0	0
$\phi = \pi/2$ (XZ plane)	$s : 0 \to 2,\ z : 0 \to 5$	$dsdz\left(\hat{\phi}\right)$	0	0
$s = 2$ (Curved)	$z : 0 \to 5;\ \phi : 0 \to \pi/2$	$2d\phi dz\hat{\mathbf{s}}$	$4(2 + \sin^2 \phi)d\phi dz$	25π

(c) You should use the curl formula given in the book.

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10. **[G 1.44]** Evaluate the following integrals:

(a) $\int_{-2}^{2}(2x + 3)\delta(3x)dx$.

(b) $\int_{0}^{2}(x^3 + 3x + 2)\delta(1 - x)dx$.

(c) $\int_{-1}^{1} 9x^2\delta(3x + 1)dx$.

(d) $\int_{-\infty}^{a} \delta(x - b)dx$.

Solutions:

(a) $\int_{-2}^{2}(2x + 3)\frac{1}{3}\delta(x)dx = \frac{1}{3}(0 + 3) = 1$.

(b) $\delta(1 - x) = \delta(x - 1)$, so $1 + 3 + 2 = 6$.

(c) $\int_{-1}^{1} 9x^2\frac{1}{3}\delta(x + \frac{1}{3})dx = 9(-\frac{1}{3})^2\frac{1}{3} = \frac{1}{3}$.

(d) 1 (if $a > b$), 0 (if $a < b$).
