Power Series Solutions to the Legendre Equation

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The Legendre equation

The equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \qquad (1)$$

where α is any real constant, is called Legendre's equation.

When $\alpha \in \mathbb{Z}^+$, the equation has polynomial solutions called Legendre polynomials. In fact, these are the same polynomial that encountered earlier in connection with the Gram-Schmidt process.

The Eqn. (1) can be rewritten as

$$[(x^2 - 1)y']' = \alpha(\alpha + 1)y,$$

which has the form $T(y) = \lambda y$, where $T(f) = (pf')'$, with $p(x) = x^2 - 1$ and $\lambda = \alpha(\alpha + 1)$.

Note that the nonzero solutions of (1) are eigenfunctions of T corresponding to the eigenvalue $\alpha(\alpha+1)$.

Since $p(1) = p(-1) = 0$, T is symmetric with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx.$$

Thus, eigenfunctions belonging to distinct eigenvalues are orthogonal.

Power series solution for the Legendre equation

The Legendre equation can be put in the form

$$y'' + p(x)y' + q(x)y = 0,$$

where

$$p(x) = -\frac{2x}{1-x^2} \quad \text{and} \quad q(x) = \frac{\alpha(\alpha+1)}{1-x^2}, \quad \text{if } x^2 \neq 1.$$

Since $\frac{1}{(1-x^2)} = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$, both $p(x)$ and $q(x)$ have power series expansions in the open interval $(-1, 1)$.

Thus, seek a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in (-1, 1).$$

Differentiating term by term, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Thus,

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n = \sum_{n=0}^{\infty} 2n a_n x^n,$$

and

$$\begin{aligned}
(1-x^2)y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n \\
&= \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - n(n-1) a_n] x^n.
\end{aligned}$$

Substituting in (1), we obtain

$$(n+2)(n+1)a_{n+2}-n(n-1)a_n-2na_n+\alpha(\alpha+1)a_n=0, \quad n\geq 0,$$

which leads to a recurrence relation

$$a_{n+2}=-\frac{(\alpha-n)(\alpha+n+1)}{(n+1)(n+2)}a_n.$$

Thus, we obtain

$$\begin{aligned}
a_2 &= -\frac{\alpha(\alpha+1)}{1\cdot 2}a_0,\\
a_4 &= -\frac{(\alpha-2)(\alpha+3)}{3\cdot 4}a_2=(-1)^2\frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}a_0,\\
&\vdots\\
a_{2n} &= (-1)^n\frac{\alpha(\alpha-2)\cdots(\alpha-2n+2)\cdot(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)}{(2n)!}a_0.
\end{aligned}$$

Similarly, we can compute $a_3, a_5, a_7, \ldots$, in terms of a_1 and obtain

$$\begin{aligned}
a_3 &= -\frac{(\alpha-1)(\alpha+2)}{2\cdot 3} a_1 \\
a_5 &= -\frac{(\alpha-3)(\alpha+4)}{4\cdot 5} a_3 = (-1)^2 \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!} a_1 \\
&\vdots \\
a_{2n+1} &= (-1)^n \frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2n)}{(2n+1)!} a_1
\end{aligned}$$

Therefore, the series for $y(x)$ can be written as

$$y(x) = a_0 y_1(x) + a_1 y_2(x), \quad \text{where}$$

$y_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\alpha(\alpha-2)\cdots(\alpha-2n+2)\cdot(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1)}{(2n)!} x^{2n}$, and
$y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)\cdot(\alpha+2)(\alpha+4)\cdots(\alpha+2n)}{(2n+1)!} x^{2n+1}$.

Note: The ratio test shows that $y_1(x)$ and $y_2(x)$ converges for $|x| < 1$. These solutions $y_1(x)$ and $y_2(x)$ satisfy the initial conditions

$$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

Since $y_1(x)$ and $y_2(x)$ are independent, the general solution of the Legendre equation over $(-1, 1)$ is

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

with arbitrary constants a_0 and a_1.

Observations

Case I. When $\alpha = 0$ or $\alpha = 2m$, we note that

$$\alpha(\alpha-2)\cdots(\alpha-2n+2) = 2m(2m-2)\cdots(2m-2n+2) = \frac{2^n m!}{(m-n)!}$$

and

$$\begin{aligned}(\alpha+1)(\alpha+3)\cdots(\alpha+2n-1) &= (2m+1)(2m+3)\cdots(2m+2n-1)\\ &= \frac{(2m+2n)!\, m!}{2^n(2m)!\,(m+n)!}.\end{aligned}$$

Then, in this case, $y_1(x)$ becomes

$$y_1(x) = 1 + \frac{(m!)^2}{(2m)!}\sum_{k=1}^{m}(-1)^k \frac{(2m+2k)!}{(m-k)!(m+k)!(2k)!}x^{2k},$$

which is a polynomial of degree $2m$. In particular, for $\alpha = 0, 2, 4 (m = 0, 1, 2)$, the corresponding polynomials are

$$y_1(x) = 1,\;\; 1-3x^2,\;\; 1-10x^2+\frac{35}{3}x^4.$$

Note that the series $y_2(x)$ is not a polynomial when α is even because the coefficients of x^{2n+1} is never zero.

Case II. When $\alpha = 2m+1$, $y_2(x)$ becomes a polynomial and $y_1(x)$ is not a polynomial.

In this case,

$$y_2(x) = x + \frac{(m!)^2}{(2m+1)!}\sum_{k=1}^{m}(-1)^k \frac{(2m+2k+1)!}{(m-k)!(m+k)!(2k+1)!}x^{2k+1}.$$

For example, when $\alpha = 1, 3, 5$ $(m = 0, 1, 2)$, the corresponding polynomials are

$$y_2(x) = x, \;\; x - \frac{5}{3}x^3, \;\; x - \frac{14}{3}x^3 + \frac{21}{5}x^5.$$

The Legendre polynomial

To obtain a single formula which contains both $y_1(x)$ and $y_2(x)$, let

$$P_n(x) = \frac{1}{2^n} \sum_{r=0}^{[n/2]} \frac{(-1)^r (2n-2r)!}{r!(n-r)!(n-2r)!} x^{n-2r},$$

where $[n/2]$ denotes the greatest integer $\leq n/2$.

- When n is even, it is a constant multiple of the polynomial $y_1(x)$.
- When n is odd, it is a constant multiple of the polynomial $y_2(x)$.

The first five Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

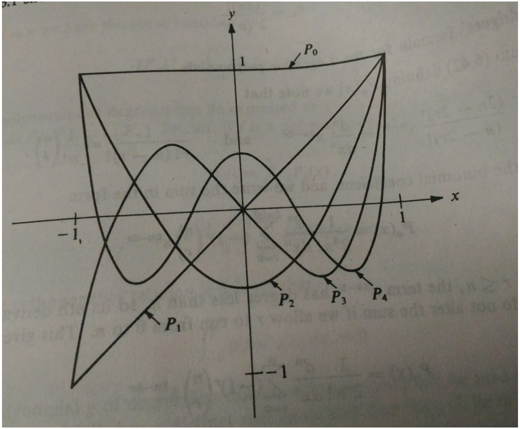

Figure : Legendre polynomial over the interval $[-1, 1]$

Rodrigues's formula for the Legendre polynomials

Note that

$$\frac{(2n-2r)!}{(n-2r)!}x^{n-2r} = \frac{d^n}{dx^n}x^{2n-2r} \text{ and } \frac{1}{r!(n-r)!} = \frac{1}{n!}\left(\begin{array}{c} n \\ r \end{array}\right).$$

Thus, $P_n(x)$ in (2) can be expressed as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\sum_{r=0}^{[n/2]}(-1)^r\left(\begin{array}{c} n \\ r \end{array}\right)x^{2n-2r}.$$

When $[n/2] < r \leq n$, the term x^{2n-2r} has degree less than n, so its nth derivative is zero. This gives

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\sum_{r=0}^{n}(-1)^r\left(\begin{array}{c} n \\ r \end{array}\right)x^{2n-2r} = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

which is known as Rodrigues' formula.

Properties of the Legendre polynomials $P_n(x)$

- For each $n \geq 0$, $P_n(1) = 1$. Moreover, $P_n(x)$ is the only polynomial which satisfies the Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

and $P_n(1) = 1$.

- For each $n \geq 0$, $P_n(-x) = (-1)^n P_n(x)$.

- $$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases}$$

- If $f(x)$ is a polynomial of degree n, we have

$$f(x) = \sum_{k=0}^{n} c_k P_k(x), \quad \text{where}$$

$$c_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx.$$

- It follows from the orthogonality relation that

$$\int_{-1}^{1} g(x) P_n(x) dx = 0$$

for every polynomial $g(x)$ with $\deg(g(x)) < n$.

*** End ***