MA 102 (Mathematics II)

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Mid-Term Solution

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(1) a. Let U be an open subset of $\mathbb{R}^n$ and let V be any subset of $\mathbb{R}^n$. Prove that $U + V = \{X + Y : X \in U, Y \in V\}$ is also open.

b. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(\tan^{-1}(x))$ is uniformly continuous. [Marks 2+2=4]

Solution. a. Since U is open we have

$$U + X \text{ is open for all } X \in \mathbb{R}^n\,,$$

since translate of an open set is open. Now

$$U + V = \bigcup_{X \in V} U + X\,.$$

Since arbitrary union of open sets is again open we have $U + V$ is open.

b. For any $x, y \in \mathbb{R}$ we have by Lagrange's MVT

$$|\sin(\tan^{-1} x) - \sin(\tan^{-1} y)| = |\frac{1}{1+z^2}\cos(\tan^{-1} z)||x - y|\,,$$

for some z between x and y. Thus

$$|\sin(\tan^{-1} x) - \sin(\tan^{-1} y)| \le |x - y|\,.$$

Thus $f(x) = \sin(\tan^{-1}(x))$ is Lipschitz continuous and hence uniformly continuous. □

(2) Examine the limit of the following function at $(0, 0)$:

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0)\,. \end{cases}$$

[Marks 3]

Solution. Along $y = 0$ we have $\lim_{x\to 0} f(x, 0) = 0$. Along $y = x$ we have,

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^4}{x^4+0} = 1\,.$$

Thus we get two different limits along two different curves and hence the double limit does not exist.

Alternative Solution: We have,

$$\lim_{n\to\infty} f(1/n, 0) = 0, \quad \lim_{n\to\infty} f(1/n, 1/n) = 1\,.$$

Thus we get two sequences $\{(1/n, 0)\}$ and $\{(1/n, 1/n)\}$ both of which converges to $(0, 0)$ but the functional limits are different. Hence limit does not exist. □

(3) Consider the function

$$g(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x=0, y=0\,. \end{cases}$$

Examine the continuity of g_x at $(0,0)$. [Marks 3]

Solution. Clearly $g_x(0,0) = 0$. For $(x,y) \neq (0,0)$ we have

$$g_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}\,.$$

Thus

$$\begin{aligned} |g_x(x,y) - g_x(0,0)| = |g_x(x,y)| &= \frac{|x^4y + 4x^2y^3 - y^5|}{(x^2+y^2)^2} \\ &\leq \frac{6(x^2+y^2)^{5/2}}{(x^2+y^2)^2} = 6\sqrt{x^2+y^2}\,. \end{aligned}$$

Thus for any $\epsilon > 0$ if we choose $\delta = \epsilon/6$, then we get,

$$|g_x(x,y) - g_x(0,0)| < \epsilon \text{ whenever } \sqrt{x^2+y^2} < \delta\,.$$

□

(4) Consider the function

$$f(x,y) = \begin{cases} \frac{x^3y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0)\,. \end{cases}$$

Find all the directional derivatives of f at $(0,0)$. Investigate the differentiability of f at $(0,0)$. [Marks 5]

Solution. Clearly $f_x(0,0) = f_y(0,0) = 0$. For $U = (u_1, u_2)$ with $u_1u_2 \neq 0$ we have

$$\lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} = \lim_{t\to 0} \frac{t^4u_1^3u_2}{t(t^4u_1^4 + t^2u_2^2)} = \lim_{t\to 0} \frac{tu_1^3u_2}{t^2u_1^4 + u_2^2} = 0\,.$$

Thus all directional derivatives at $(0,0)$ exist and are equal to 0. Now for differentiability we must have

$$\frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = \frac{h^3k}{h^4 + k^2(\sqrt{h^2+k^2})} \to 0\,,$$

as $(h,k) \to (0,0)$. Fix $m \neq 0$. Consider the curve $k = mh^2$. Then

$$\lim_{h\to 0+} \frac{mh^5}{h^4(1+m^2)h\sqrt{1+m^2h^2}} = \frac{m}{1+m^2}\,.$$

Thus the limit depends on m and hence is different along different curves. So the limit does not exist and hence f is not differentiable at $(0,0)$. □

(5) a. Find the maximum value and minimum value of the function f given by $f(x, y) = xy$ on the unit circle $x^2 + y^2 = 1$.

b. Find and classify the critical points(as local maximum, local minimum or saddle point) of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = 2x^3 + 9xy^2 + 15x^2 + 27y^2$.

[Marks 3+4=7]

Solution. a. $f(x, y) = xy$, $g(x, y) = x^2 + y^2 - 1$. Thus the Lagrange multiplier equations are

$$y = 2\lambda x,\ x = 2\lambda y \quad \text{and} \quad x^2 + y^2 - 1 = 0.$$

So the possible solutions are:

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right).$$

Also note that

$$f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2} \quad \text{and} \quad f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -\frac{1}{2}.$$

Thus the maximum value is $\frac{1}{2}$ and minimum value is $-\frac{1}{2}$.

b. First we find the partial derivatives:

$$\frac{\partial f}{\partial x} = 6x^2 + 9y^2 + 30x \quad \text{and} \quad \frac{\partial f}{\partial y} = 18xy + 54y.$$

Equating them to zero we get,

$$2x^2 + 3y^2 + 10x = 0, \tag{1}$$

and

$$xy + 3y = 0. \tag{2}$$

From equation (2) it follows that either $y = 0$ or $x = -3$. If $y = 0$, then putting in equation (1) we get, $x^2 + 5x = 0$ which gives $x = 0$ or $x = -5$. Thus we get the points $(0, 0)$ and $(-5, 0)$. If $x = -3$, then putting in equation (1) we get, $y^2 = 4$, which gives $y = \pm 2$. Thus we get the points $(-3, 2)$ and $(-3, -2)$. Thus the critical points are: $(0, 0), (-5, 0), (-3, 2)$ and $(-3, -2)$. On the other hand the second order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = 12x + 30, \quad \frac{\partial^2 f}{\partial x \partial y} = 18y \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = 18x + 54.$$

Thus the Hessian matrix is

$$H = \begin{pmatrix} 12x + 30 & 18y \\ 18y & 18x + 54 \end{pmatrix}.$$

$$\text{For} \quad (0, 0) \quad \text{we get} \quad H = \begin{pmatrix} 30 & 0 \\ 0 & 54 \end{pmatrix} \quad \text{and hence} \quad det\ (H) = 1620 > 0.$$

Thus $(0,0)$ is a local minimum.

For $(-5,0)$ we get $H = \begin{pmatrix} -30 & 0 \\ 0 & -36 \end{pmatrix}$ and hence $det\ (H) = 1080 > 0$.

Thus $(-5,0)$ is a local maximum.

For $(-3,2)$ we get $H = \begin{pmatrix} -6 & 36 \\ 36 & 0 \end{pmatrix}$ and hence $det\ (H) = -(36)^2 < 0$.

Thus $(-3,2)$ is a saddle point.

For $(-3,-2)$ we get $H = \begin{pmatrix} -6 & -36 \\ -36 & 0 \end{pmatrix}$ and hence $det\ (H) = -(36)^2 < 0$.

Thus $(-3,-2)$ is also a saddle point. $\square$

(6) a. Evaluate $\int\int_D \cos\left(\frac{y-x}{y+x}\right) dxdy$ where D is the inside of the triangle with vertices $(0,0),(1,0)$ and $(0,1)$.

b. Let W be the region bounded by the planes $x = 0, y = 0, z = 2$ and the surface $z = x^2 + y^2$ and lying in the quadrant $x \geq 0, y \geq 0$. Find

$$\int\int\int_W x dxdydz\,.$$

[Marks 4+3=7]

Solution. a. Let $u = y - x$ and $v = y + x$. Then $x = \frac{v-u}{2}$ and $y = \frac{u+v}{2}$. Thus

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2} \quad \text{and hence} \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

Now under the transformation the triangle with vertices $(0,0),(1,0)$ and $(0,1)$ gets transformed to the triangle with vertices $(0,0),(-1,1)$ and $(1,1)$. Thus

$$\int\int_D \cos\left(\frac{y-x}{y+x}\right) dxdy = \frac{1}{2}\int_{v=0}^{1}\int_{u=-v}^{v} \sin\left(\frac{u}{v}\right) dudv$$
$$= \frac{1}{2}\int_0^1 v\ \left[\sin(1) - \sin(-1)\right]\ dv = \frac{1}{4}\ 2\sin(1) = \frac{\sin(1)}{2}.$$

b.

$$\int\int\int_W x\, dxdydz = \int_{x=0}^{\sqrt{2}}\int_{y=0}^{\sqrt{2-x^2}}\int_{z=x^2+y^2}^{2} x\, dzdydx.$$

Changing to cylindrical co-ordinates $x = r\cos\theta, y = r\sin\theta, z = z$ we get,

$$\int_{r=0}^{\sqrt{2}}\int_{\theta=0}^{\frac{\pi}{2}}\int_{z=r^2}^{2} r^2\cos\theta\, dzd\theta dr = \int_{r=0}^{\sqrt{2}}\int_{\theta=0}^{\frac{\pi}{2}} r^2(2-r^2)\cos\theta\, d\theta dr$$
$$= \int_{r=0}^{\sqrt{2}} (2r^2 - r^4) dr = \frac{2}{3}r^3|_{r=0}^{\sqrt{2}} - \frac{r^5}{5}|_{r=0}^{\sqrt{2}} = \frac{2}{3}2\sqrt{2} - \frac{4}{5}\sqrt{2} = \frac{20\sqrt{2} - 12\sqrt{2}}{15} = \frac{8\sqrt{2}}{15}.$$

$\square$

(7) a. Evaluate $\displaystyle\int_\Gamma ydx + xdy$ where Γ is the curve parametrized by $r(t) = (t^9, \sin^9(\frac{\pi t}{2}))$, $0 \le t \le 1$.

b. Using Green's Theorem find the area of the region enclosed by the curve $x^{2/3}+y^{2/3} = a^{2/3}$ using the parametrization $r(t) = (a\cos^3 t, a\sin^3 t)$, $0 \le t \le 2\pi$. [Marks 3+3=6]

Solution. a. Note that

$$\int_\Gamma ydx + xdy = \int_\Gamma F \cdot dr,$$

where $F(x, y) = (y, x)$. Notice that $F = \nabla f$, where $f(x, y) = xy$. Thus by Fundamental theorem

$$\int_\Gamma F \cdot dr = f(r(1)) - f(r(0)) = 1$$

b. Note that using Green's theorem the required formula of the area is

$$A = \frac{1}{2}\int_\Gamma xdy - ydx,$$

where Γ is parametrized by $r(t) = (a\cos^3 t, a\sin^3 t)$. Note that $r'(t) = (-3a\cos^2 t\sin t, 3a\sin^2 t\cos t)$. Thus for $F(x, y) = (-y, x)$ we have

$$\begin{aligned}
A &= \frac{1}{2}\int_\Gamma F \cdot dr = \frac{1}{2}\int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt \\
&= \frac{1}{2}\int_0^{2\pi} (-a\sin^3 t)(-3a\cos^2 t\sin t) + (a\cos^3 t)(3a\sin^2 t\cos t)\, dt \\
&= \frac{3a^2}{2}\int_0^{2\pi} (\sin^4 t\cos^2 t + \cos^4 t\sin^2 t)\, dt = \frac{3a^2}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\, dt \\
&= \frac{3a^2}{8}\int_0^{2\pi} \sin^2 2t\, dt = \frac{3a^2}{8}\int_0^{2\pi} \frac{1-\cos 4t}{2}\, dt = \frac{3a^2}{16}\cdot 2\pi = \frac{3\pi a^2}{8}
\end{aligned}$$

$\square$

(8) For this question write only the answer in your answer booklet. You are not required to show the calculations.

a. Evaluate the limit:

$$\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}.$$

b. Consider the function $f(x, y) = 6x^2y + y\cos x$. Find $D_U f(0, 1)$ for $U = (1/\sqrt{2}, 1/\sqrt{2})$.

c. Find a scalar potential of the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = (y^2 + 6x^2y, 2xy + 2x^3).$$

d. Let $U = (1, 0, 1)$. Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(X) = \langle X, U\rangle$. Then find $DF(U)$.

e. Find the quadratic approximation of the function $f(x, y) = \sin(xy)$ near $(0, 0)$.

[Marks 1+1+1+1+1=5]

Solution. a. $\frac{1}{3}$, b. $\frac{1}{\sqrt{2}}$, c. $f(x, y) = xy^2 + 2x^3y + \text{constant}$, d. $\|U\|^2 = 2$, e. xy. $\square$