Higher Order Linear ODE: Existence and Uniqueness Results, Fundamental Solutions, Wronskian

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Recall that all solutions of $L(y) = g$ are given by

$$\mathrm{Ker}(L) + y_P,$$

where $L(y_P) = g$ is a particular solution.

Hence what we need to do is to find

- a basis $\{y_1, \ldots, y_n\}$ of $\mathrm{Ker}(L)$ and
- a particular solution y_P.

Then the general solution of $L(y) = g$ is given by

$$y := c_1 y_1 + \cdots + c_n y_n + y_P.$$

Definition: If $\{f_1, \ldots, f_n\} \subset C^n(I)$, then

$$W(f_1, \cdots, f_n) := \begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ & & \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of $f_1, \ldots, f_n$ on I.

Theorem: Let $y_1, y_2, \ldots, y_n \in C^n(I)$ be solutions of $L(y) = 0$, where

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y,$$

where $a_i : I \to \mathbb{R}$ are given functions, $a_i(x) \in C(I),\ i = 0, \ldots, n$, and $a_n(x) \neq 0$ on I. If $W(y_1, \ldots, y_n)(x_0) \neq 0$ for some $x_0 \in I$, then every solution $y(x)$ of $L(y) = 0$ on I can be expressed in the form

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x),$$

where $C_1, \ldots, C_n$ are constants.

Example: The functions $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ are both solutions of $y'' - 4y = 0$ on $(-\infty, \infty)$. The Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4 \neq 0.$$

The general solution is $y = c_1 e^{2x} + c_2 e^{-2x}$.

Theorem: (**Abel's formula**) Let $y_1, \dots, y_n$ be any n solutions to

$$Ly = y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y = 0$$

on I, where $p_1, \dots, p_n \in C(I)$. Then, for $x_0 \in I$, we have

$$W(y_1, \dots, y_n)(x) = W(y_1, \dots, y_n)(x_0) \exp\left(-\int_{x_0}^{x} p_1(t)dt \right)$$

for all $x \in I$.

Proof. Prove for $n = 2$ (See Theorem 8 in Chapter 3 of Coddington's book).

Corollary: The Wronskian of solutions $W(y_1, \ldots, y_n)(x)$ is either identically zero or never zero on I.

Definition: A set of n linearly independent solutions of $Ly = 0$ that spans $\mathrm{Ker}(L)$ are called fundamental solutions.

Fact: Let $y_1, y_2, \ldots, y_n \in C^n(I)$ be solutions of $L(y) = 0$ where $L(y)(x) = a_n(x)y^{(n)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x)$, $a_i \in C(I)$ and $a_n(x) \neq 0 \; \forall x \in I$. Then the following statements are equivalent:

- $\{y_1, y_2, \ldots, y_n\}$ is a fundamental solution set on I.
- $\{y_1, y_2, \ldots, y_n\}$ are linearly independent on I.
- $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ on I.

Proof. See Theorems 6 and 7 in Chapter 3 of Coddington's book.

Theorem: Let $y_p(x) \in C^n(I)$ be a particular solution to $L(y)(x) = g(x)$ on I and let $\{y_1, y_2, \ldots, y_n\} \in C^n(I)$ be a fundamental solution set of $L(y) = 0$ on I. Then every solution of $L(y) = g$ on I can be expressed in the form

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x) + y_p(x)$$

Example: Given that $y_p = x^2$ is a particular solution to $y'' - y = 2 - x^2$ and $y_1(x) = e^x$ and $y_2(x) = e^{-x}$ are solution to $y'' - y = 0$. A general solution is

$$y(x) = C_1 e^x + C_2 e^{-x} + x^2.$$

Homogeneous linear equations with constant coefficients

Aim: To find a basis for $Ker(L)$. That is, to find a set of fundamental solution to the homogeneous equation $L(y) = 0$, where

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

and $a_n \neq 0$, $a_{n-1}, \ldots, a_0$ are real constants.
For $y = e^{rx}$, we find

$$\begin{aligned} L(e^{rx}) &= a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_0 e^{rx} \\ &= e^{rx}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0) = e^{rx} P(r), \end{aligned}$$

where $P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$.
Thus $L(e^{rx}) = 0$ provided r is a root of the auxiliary equation

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0.$$

Case I (Distinct real roots): Let $r_1, \ldots, r_n$ be real and distinct roots. The n solutions are given by

$$y_1(x) = e^{r_1 x},\ y_2(x) = e^{r_2 x}, \ldots, y_n(x) = e^{r_n x}.$$

We need to show

$$c_1 e^{r_1 x} + \cdots + c_n e^{r_n x} = 0 \Longrightarrow c_1 = c_2 = \cdots = c_n = 0.$$

$P(r)$ can be factored as

$$P(r) = a_n (r - r_1)(r - r_2) \cdots (r - r_n).$$

Writing the operator L as

$$L = P(D) = a_n (D - r_1) \cdots (D - r_n).$$

Now, construct the polynomial $P_k(r)$ by deleting the factor $(r - r_k)$ from $P(r)$. Then

$$L_k := P_k(D) = a_n (D - r_1) \cdots (D - r_{k-1})(D - r_{k+1}) \cdots (D - r_n).$$

By linearity

$$L_k(\sum_{i=1}^{n} c_i e^{r_i x}) = L_k(0) \Rightarrow c_1 L_k(e^{r_1 x}) + \cdots + c_n L_k(e^{r_n x}) = 0.$$

Since $L_k = P_k(D)$, we find that $L_k(e^{rx}) = e^{rx} P_k(r)$ for all r. Thus

$$\sum_{i=1}^{n} c_i e^{r_i x} P_k(r_i) = 0 \Longrightarrow c_k e^{r_k x} P_k(r_k) = 0,$$

as $P_k(r_i) = 0$ for $i \neq k$. Since r_k is not a root of $P_k(r)$, then $P_k(r_k) \neq 0$. This yields $c_k = 0$. As k is arbitrary, we have

$$c_1 = c_2 = \cdots = c_n = 0.$$

Theorem: If $P(r) = 0$ has n distinct roots $r_1, r_2, \ldots, r_n$. Then the general solution of $L(y) = 0$ is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x},$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Example: Consider $y'' - 3y' + 2y = 0$. The auxiliary equation $P(r) = r^2 - 3r + 2 = 0$ has two roots $r_1 = 1, \ \ r_2 = 2$. The general solution is $y(x) = C_1 e^x + C_2 e^{2x}$..

Case II (Repeated roots): If r_1 is a root of multiplicity m. Then

$$P(r) = (r - r_1)^m \tilde{P}(r),$$

where $\tilde{P}(r) = a_n(r - r_{m+1}) \cdots (r - r_n)$ and $\tilde{P}(r_1) \neq 0$. Now

$$L(e^{rx}) = e^{rx}(r - r_1)^m \tilde{P}(r)$$

Setting $r = r_1$, we see that $e^{r_1 x}$ is a solution. To find other solutions, we note that $\frac{\partial^k}{\partial r^k} L(e^{rx}) = \frac{\partial^k}{\partial r^k}[e^{rx}(r - r_1)^m \tilde{P}(r)]$. Now,

$$\frac{\partial^k}{\partial r^k} L(e^{rx})|_{r=r_1} = 0 \quad \text{if } k \leq m - 1.$$

$$\implies \ L\left[\frac{\partial^k}{\partial r^k}(e^{rx})|_{r=r_1}\right] = 0.$$

Thus,

$$\frac{\partial^k}{\partial r^k}(e^{rx})\big|_{r=r_1} = x^k e^{r_1 x}$$

will be a solution to $L(y) = 0$ for $k = 0, 1, \ldots, m-1$.
So, m distinct solutions are

$$e^{r_1 x},\ xe^{r_1 x},\ \cdots,\ x^{m-1}e^{r_1 x}.$$

Theorem: If $P(r) = 0$ has the real root r_1 occurring m times and the remaining roots $r_{m+1}, r_{m+2}, \ldots, r_n$ are distinct, then the general solution of $L(y) = 0$ is

$$\begin{aligned} y(x) &= (C_1 + C_2 x + C_3 x^2 + \cdots + C_m x^{m-1})e^{r_1 x} \\ &\quad + C_{m+1}e^{r_{m+1}x} + \cdots + C_n e^{r_n x}, \end{aligned}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.