PH 102: Physics II

Lecture 17 (Spring 2018)

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LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	6-3-2018	Lorentz Force, Biot-Savart law	5.1, 5.2	1	I, II (5-5:55 pm)
Lec 15	7-3-2018	Lorentz Force, Biot-Savart law	5.1, 5.2	1	III, IV (9-9:55 am)
Tut 8	13-3-2018	Lec 15			
Lec 16	13-3-2018	Divergence & Curl of Magnetostatic Fields, Applications of Ampere's law	5.3	1	I, II (5-5:55 pm)
Lec 16	14-3-2018	Divergence & Curl of Magnetostatic Fields, Applications of Ampere's law	5.3	1	III, IV (9-9:55 am)
Lec 17	14-3-2018	Magnetic Vector Potential, Force & torque on a magnetic dipole	5.4	1	I, II (4-4:55 pm)
Lec 17	15-3-2018	Magnetic Vector Potential, Force & torque on a magnetic dipole	5.4	1	III, IV (10-10:55 am)
Lec 18	15-3-2018	Lec 16+Lec 17 Continues		1	I, II (3-3:55 pm)
Lec 18	16-3-2018	Lec 16+Lec 17 Continues		1	III, IV (11-11:55 am)
Tut 9	20-3-2018	Lec 16+Lec 17+Lec 18			
Lec 19	21-3-2018	Magnetic Materials, Magnetization	6.1	1	I, II, III, IV
Lec 20	23-3-2018	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II, III, IV
Tut 10	3-4-2018	Quiz II			
Lec 21	4-4-2018	Ohm's law, motional emf, electromotive force	7.1	1	I, II, III, IV
Lec 22	6-4-2018	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II, III, IV
Tut 11	10-4-2018	Lec 21+Lec 22			
Lec 23	11-4-2018	Maxwell's equations	7.3	1	I, II, III, IV
Lec 24	13-4-2018	Continuity equation, Poynting Theorem	8.1	1	I, II, III, IV
Tut 12	17-4-2018	Lec 23+Lec 24			
Lec 25	18-4-2018	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II, III, IV
Lec 26	20-4-2018	L11+Electromagnetic waves in matter	9.3	1	I, II, III, IV
Tut 13	24-4-2018	Lec 25+ Lec 26			
Lec 27	25-4-2018	Reflection and transmission: Normal & Oblique Incidence	9.3, 9.4	1	I, II, III, IV
Lec 28	27-4-2018	Lec 27+Discussions	9.3, 9.4	1	I, II, III, IV

Magnetic Vector Potential

Electrostatics $\vec{\nabla} \times \vec{E} = 0 \implies \vec{E} = -\vec{\nabla} V$

Magnetostatics $\vec{\nabla} \cdot \vec{B} = 0 \implies \vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Both electric and magnetic potentials have built-in ambiguities. For example,

$$V \to V + C, \vec{\nabla} C = 0; \vec{A} \to \vec{A} + \vec{\nabla} \lambda$$

Such redefinitions of the potentials do not change the fields.

One can always use this freedom to eliminate the divergence of magnetic vector potential (To simplify Ampere's law written above in terms of vector potential)!

Magnetic Vector Potential

- Let the original vector potential is not divergenceless.

$$\vec{A}_0 \to \vec{A} = \vec{A}_0 + \vec{\nabla}\lambda \quad \Longrightarrow \vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}_0 + \nabla^2 \lambda$$

- For the final vector potential to be divergenceless, we can choose the scalar function in such a way that

$$\nabla^2 \lambda = -\vec{\nabla} \cdot \vec{A}_0$$

- The above equation is similar to the Poisson's equation in electrostatics:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \Longrightarrow V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\mathfrak{r}} d\tau'$$

(If $\rho \to 0$ as $r' \to \infty$)

Magnetic Vector Potential

- Similarly, if $\vec{\nabla} \cdot \vec{A}_0$ goes to zero at infinity, one can always find a scalar function as

$$\lambda = \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \vec{A}_0}{\mathfrak{r}} d\tau'$$

- Therefore, it is always possible to make the magnetic vector potential divergenceless. Coulomb Gauge!

- For such a case, the Ampere's law in terms of vector potential simply becomes similar to the Poisson's equation $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ (Using $\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$)

whose solution (assuming the current goes to zero at infinity) is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'})}{\mathfrak{r}} d\tau'$$

Advantages!

Though it is a vector, but it is still simpler in many cases, to find the vector potential than the magnetic field itself.

The vector potential, typically, is parallel to the direction of the given current.

The freedom in choosing vector potential $\vec{A} \to \vec{A} + \vec{\nabla}\Phi$ without affecting magnetic field simplifies many calculations in electrodynamics: *Gauge Symmetry!*

Magnetic vector potential can also have observable consequences instead of just being a mathematical tool:

Aharanov-Bohm Effect!

Magnetic Vector Potential

- For line and surface currents, the vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{\mathfrak{r}}dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathfrak{r}}d\vec{l'}; \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{\mathfrak{r}}da'$$

- Example 5.11 (Introduction to Electrodynamics, D. J. Griffiths): A spherical shell, or radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω .Find the vector potential it produces at point r.

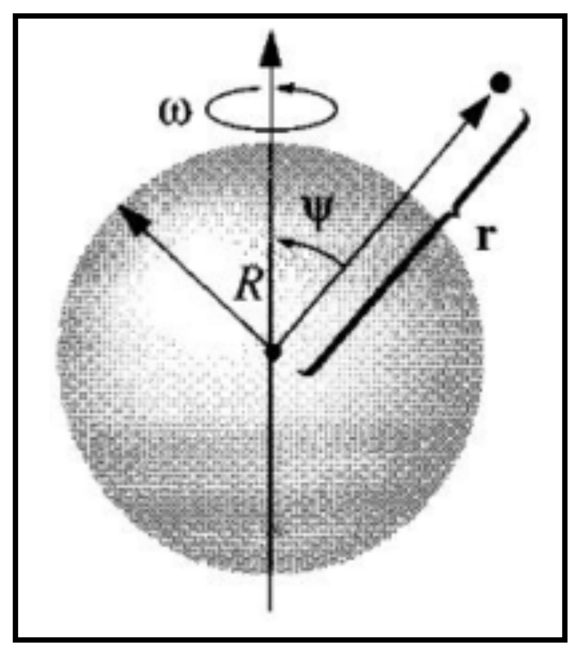


Figure 5.45, Introduction to Electrodynamics, D. J. Griffiths

The surface current for an elemental area of the spinning charged spherical shell is $\vec{K}(\vec{r'}) = \sigma\vec{v} = \sigma(\vec{\omega} \times \vec{r'})$

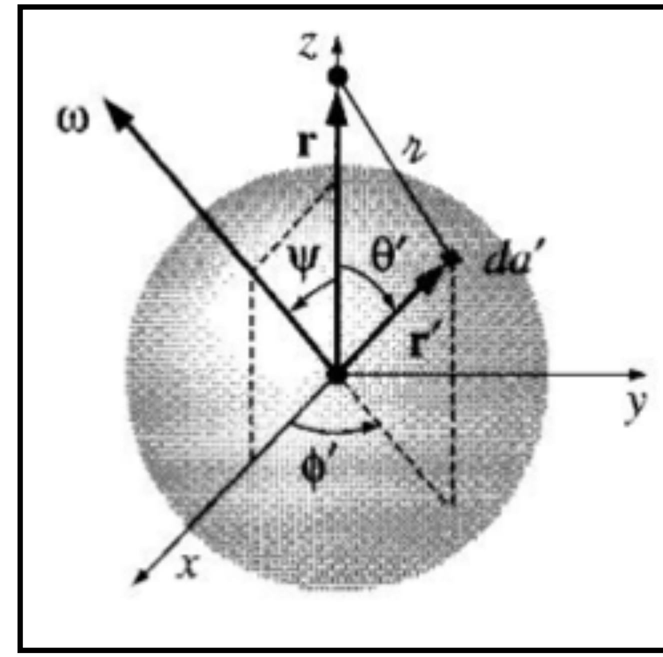


Figure 5.45, Introduction to Electrodynamics, D. J. Griffiths

$$\vec{\omega} = \omega \sin\psi \hat{x} + \omega \cos\psi \hat{z}$$

$$\vec{\omega} \times \vec{r'} = R\omega[-(\cos\psi \sin\theta' \sin\phi')\hat{x}$$
$$+ (\cos\psi \sin\theta' \cos\phi' - \sin\psi \cos\theta')\hat{y} + (\sin\psi \sin\theta' \sin\phi')\hat{z}]$$

Vector potential: $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r'})}{\mathfrak{r}} (R^2 \sin\theta' d\theta' d\phi')$

Ignoring the terms in $\vec{\omega} \times \vec{r'}$ which have $\sin\phi', \cos\phi'$ (as they identically vanish after integration), we get:

$$\vec{A}(\vec{r}) = -\frac{\mu_0 R^3 \sigma\omega \sin\psi}{2} \int_0^\pi \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} d\theta' \hat{y}$$

$$\mathfrak{r} = |\vec{r} - \vec{r'}|$$
$$= \sqrt{R^2 + r^2 - 2Rr\cos\theta}$$

Using $u = \cos\theta'$, the above integral becomes

$$\int_{-1}^{+1} \frac{u du}{\sqrt{R^2 + r^2 - 2Rru}} = -\frac{(R^2 + r^2 + Rru)}{3R^2 r^2} \sqrt{R^2 + r^2 - 2Rru}\Big|_{-1}^{+1}$$
$$= -\frac{1}{3R^2 r^2}\left[(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r)\right]$$

The integral has the following possible values:

$$-\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r|-(R^2+r^2-Rr)(R+r)\right]=\begin{cases}\frac{2r}{3R^2} & \text{for } r<R\\ \frac{2R}{3r^2} & \text{for } r>R\end{cases}$$

Also, using the fact that $-\omega r\sin\psi\hat{y}=(\vec{\omega}\times\vec{r})$, the vector potential is

$$\vec{A}(\vec{r})=\begin{cases}\frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}) & \text{for } r<R\\ \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) & \text{for } r>R\end{cases}$$

If the shell spins about the z axis then,

$$\vec{A}(r,\theta,\phi)=\begin{cases}\frac{\mu_0 R\sigma\omega}{3}r\sin\theta\hat{\phi} & \text{for } r<R\\ \frac{\mu_0 R^4\sigma\omega}{3r^3}r\sin\theta\hat{\phi} & \text{for } r>R\end{cases}$$

Calculate the magnetic field using this potential!

Tutorial 9

Magnetic Vector Potential

- Example 5.12 (Introduction to Electrodynamics, D. J. Griffiths): Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

- Since the current extends to infinity, the simple expressions mentioned in the last two slides are no longer applicable.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I}}{\mathfrak{r}} dl'$$

- Typically, the direction of vector potential will match the direction of the current.

Example 5.12: Although the formula for vector potential can not be used, we can find its line integral around a closed amperian loop around the axis of the solenoid.

$$\oint \vec{A} \cdot d\vec{l} = A(2\pi s)$$

A and I are typically in same direction i.e. circumferential

We know that: $$\oint \vec{A} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a}$$

Using the known values of B: $$\vec{B}_{\text{out}} = 0, \vec{B}_{\text{in}} = \mu_0 n I \hat{z}$$

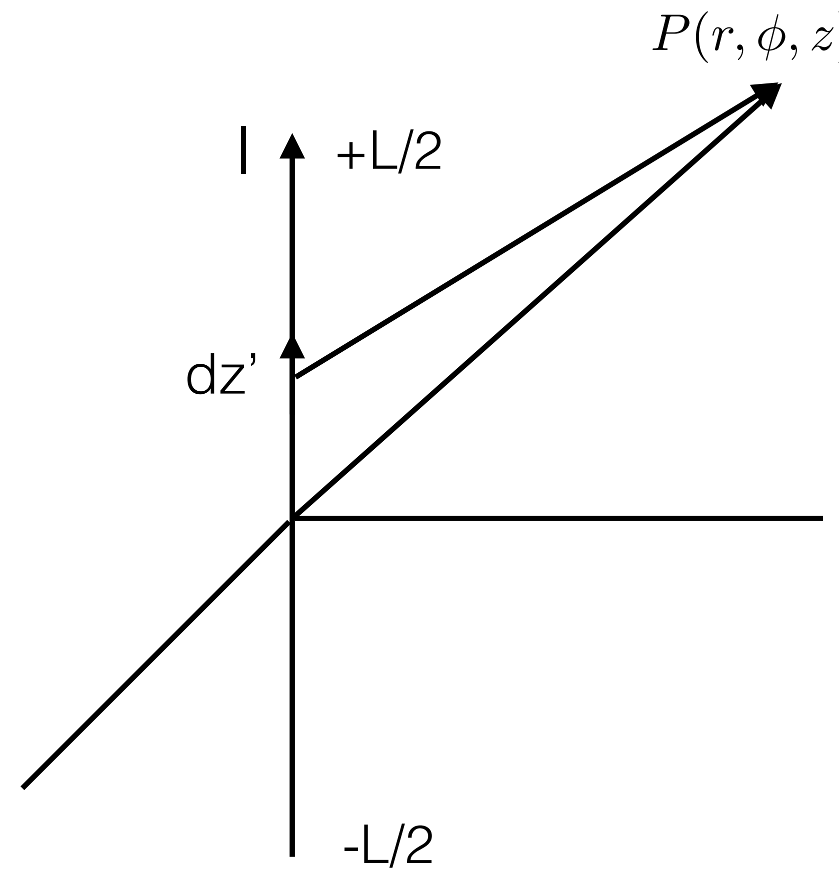
We can find A to be:

$$\vec{A}(s) = \begin{cases} \frac{\mu_0 n I(\pi s^2)}{2\pi s}\hat{\phi} = \frac{\mu_0 n I}{2} s\hat{\phi} \quad \text{for} \quad s < R \\ \frac{\mu_0 n I(\pi R^2)}{2\pi s}\hat{\phi} = \frac{\mu_0 n I}{2}\frac{R^2}{s}\hat{\phi} \quad \text{for} \quad s > R \end{cases}$$

Take curl of $\vec{A}$ & verify the known results for $\vec{B}$ in case of solenoid

Calculate magnetic vector potential for a current (I) carrying wire of length L. Using the answer, find the corresponding magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Verify your answer with the expression for field obtained using Biot-Savart law (Lecture 15).

Assuming current to be in z direction, it is straightforward to show that A is also in z direction.

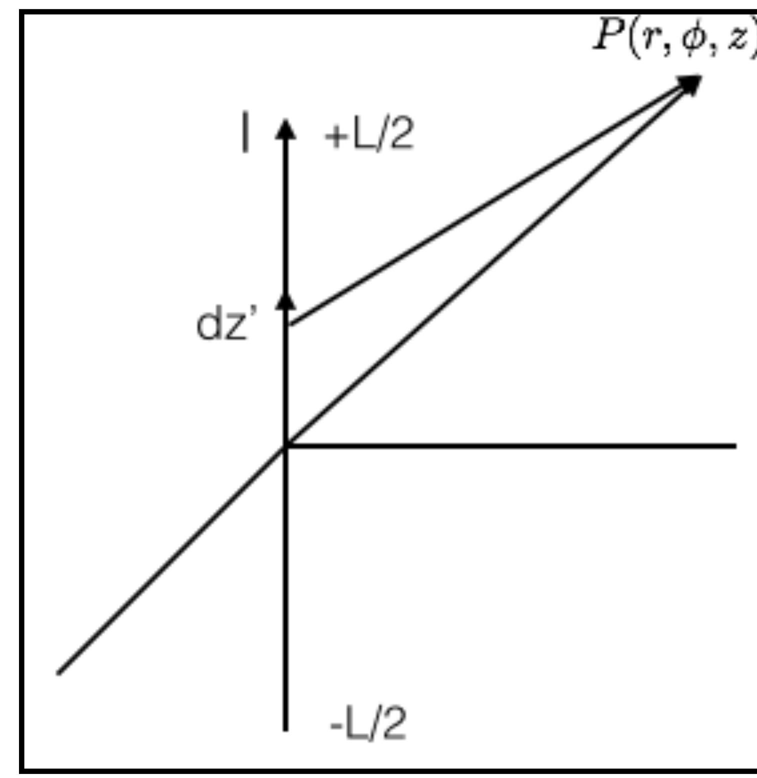


$$
\begin{aligned}
A_z &= \frac{\mu_0 I}{4\pi} \int_{-L/2}^{L/2} \frac{dz'}{\mathfrak{r}} \\
&= \frac{\mu_0 I}{4\pi} \int_{-L/2}^{L/2} \frac{dz'}{[(z-z')^2 + r^2]^{1/2}} \\
&= \frac{\mu_0 I}{4\pi} \left(\sinh^{-1}\left(\frac{-(z - L/2)}{r} \right) + \sinh^{-1}\left(\frac{z + L/2}{r} \right) \right)
\end{aligned}
$$

One can now find the magnetic field using the curl of vector potential

$$
\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
&= \frac{1}{r}\left[\left(\frac{\partial A_z}{\partial \phi} - \frac{\partial (rA_\phi)}{\partial z}\right)\hat{e_r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) r\hat{e_\phi} + \left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\hat{e_z}\right] \\
&= -\frac{\partial A_z}{\partial r}\hat{e_\phi} \\
&= \frac{\mu_0 I}{4\pi r}\left[\frac{z+L/2}{[r^2+(z+L/2)^2]^{1/2}} - \frac{z-L/2}{[r^2+(z-L/2)^2]^{1/2}}\right]\hat{e_\phi} \\
&= \frac{\mu_0 I}{4\pi r}(\sin\theta_2 - \sin\theta_1)\hat{e_\phi}
\end{aligned}
$$

Which is same as the result obtained in Lecture 15!



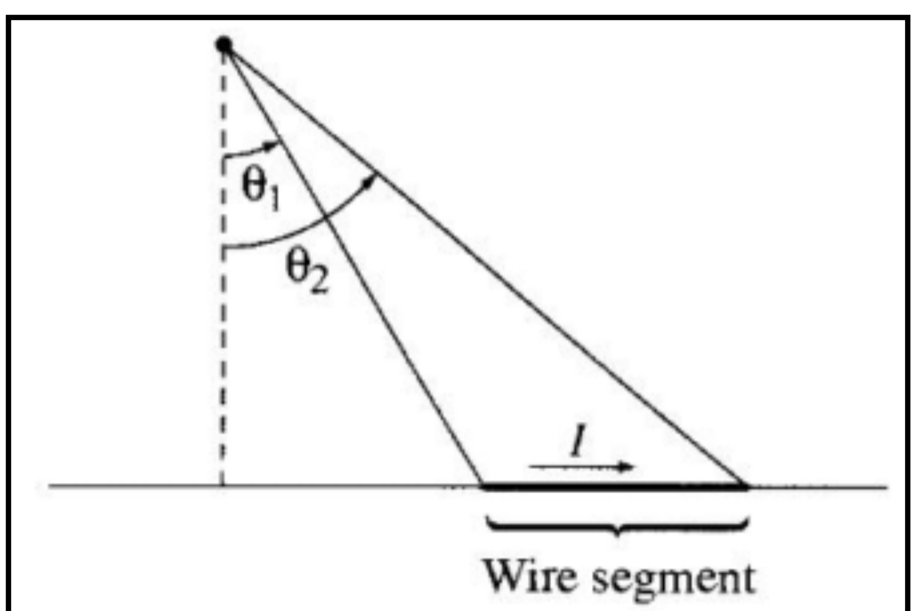


Fig. 5.19 (Introduction to Electrodynamics, D. J. Griffiths)

Exercise: Calculate the magnetic vector potential for a surface current $\vec{K} = K\hat{z}$ of finite width w. Use the result to find the corresponding magnetic field. Hint: Use the results for a wire obtained in the previous example.

To find the current density if the vector potential is given

Problem 5.23 (Introduction to Electrodynamics, D J Griffiths): What current density would produce the vector potential $\vec{A} = k\hat{\phi}$ in cylindrical coordinates?

Since $\vec{J} = (\vec{\nabla} \times \vec{B})/\mu_0$, we first find the magnetic field:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{s}\left[-\frac{\partial A_\phi}{\partial z}\hat{s} + \frac{\partial (sA_\phi)}{\partial s}\hat{z}\right] = \frac{1}{s}\frac{\partial (sk)}{\partial s}\hat{z} = \frac{k}{s}\hat{z}$$

Therefore,

$$\vec{J} = \frac{1}{\mu_0}\frac{1}{s}\left(-s\hat{\phi}\frac{\partial (k/s)}{\partial s}\right) = \frac{k}{\mu_0 s^2}\hat{\phi}$$

Summary: Magnetostatics

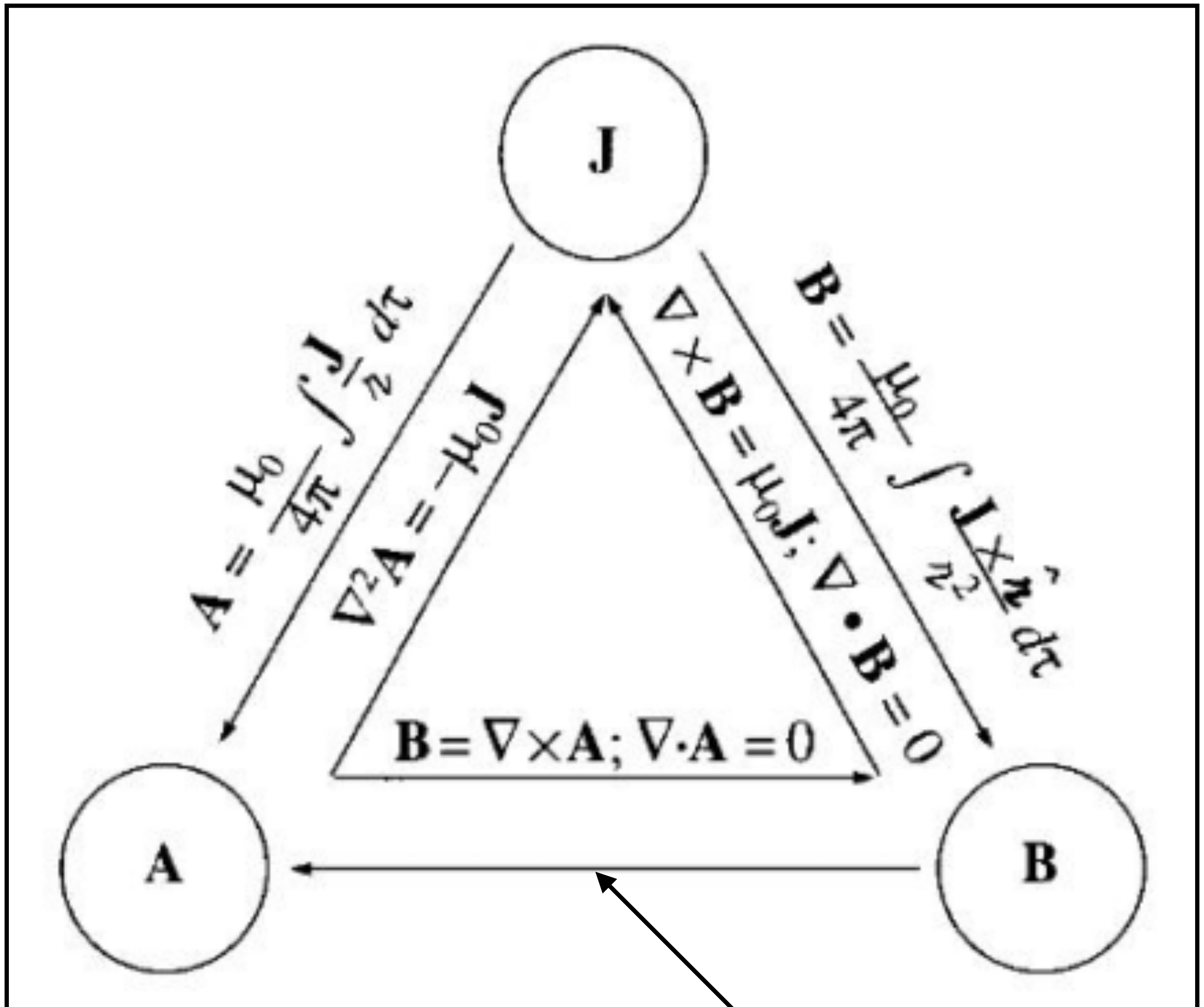


Figure 5.48, Introduction to Electrodynamics, D. J. Griffiths

Missing Link: No equation for A in terms of B

Magnetostatic Boundary Conditions

- Just like electric field suffers a discontinuity at a surface charge, so the magnetic field is discontinuous at a surface current.

- Using the integral form of $\vec{\nabla} \cdot \vec{B} = 0$ that is,

$$\oint \vec{B} \cdot d\vec{a} = 0$$

to a thin pillbox straddling the surface, we get

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

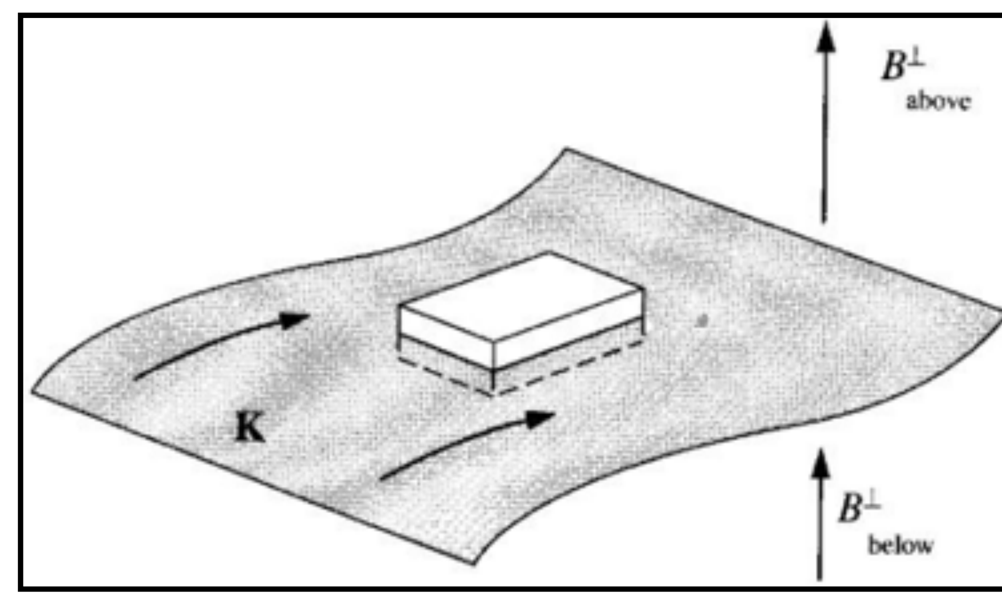


Figure 5.49, Introduction to Electrodynamics, D. J. Griffiths

Magnetostatic Boundary Conditions

- The boundary conditions for tangential components can be found by taking an Amperian loop running perpendicular to the current which gives

$$\oint \vec{B} \cdot d\vec{l} = (B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}})l = \mu_0 I_{\text{enc}} = \mu_0 K l$$

$$\implies B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$

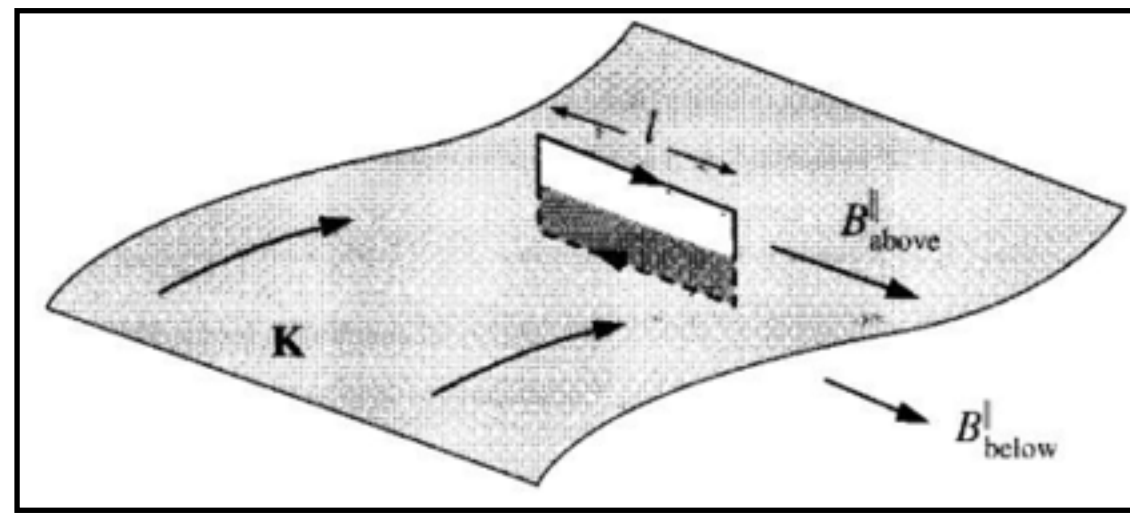


Figure 5.50, Introduction to Electrodynamics, D. J. Griffiths

- In general,

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{K} \times \hat{n})$$

where $\hat{n}$ is a unit vector perpendicular to the surface, pointing upward

Magnetostatic Boundary Conditions

- Magnetic vector potential is continuous across any boundary.

- Continuity of normal components is guaranteed by

$$\vec{\nabla} \cdot \vec{A} = 0 \implies \oint \vec{A} \cdot d\vec{a} = 0$$

- For tangential components, we can calculate

$$\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a} = \Phi$$

which is zero for an Amperian loop of vanishing thickness. Thus, tangential components are continuos.

- The derivative of vector potential however, is discontinuous

$$\frac{\partial \vec{A}_{\text{above}}}{\partial n} - \frac{\partial \vec{A}_{\text{below}}}{\partial n} = -\mu_0 \vec{K}$$

Since A is continuous across the boundary we have, at all points on the surface: $\vec{A}_{\text{above}} = \vec{A}_{\text{below}}$

If the boundary is the x-y plane, the above condition means

$\dfrac{\partial A}{\partial x}, \dfrac{\partial A}{\partial y}$ are same above and below.

Only normal derivatives can be discontinuous

Why?

From the boundary condition on magnetic field:

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{K} \times \hat{n})$$

The parallel components of B are $\left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right)\hat{x} + \left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)\hat{y}$

Using the continuity of x,y derivatives, we get:

$$\left(-\frac{\partial A_{y\text{above}}}{\partial z} + \frac{\partial A_{y\text{below}}}{\partial z}\right)\hat{x} + \left(\frac{\partial A_{x\text{above}}}{\partial z} - \frac{\partial A_{x\text{below}}}{\partial z}\right)\hat{y} = \mu_0(\vec{K} \times \hat{n})$$

Considering the surface current to be in x direction, the right hand side of the previous relation is $-\mu_0 K\hat{y}$

Equating x and y components on both sides:

$$\left(-\frac{\partial A_{y\text{above}}}{\partial z}+\frac{\partial A_{y\text{below}}}{\partial z}\right)=0,\ \left(\frac{\partial A_{x\text{above}}}{\partial z}-\frac{\partial A_{x\text{below}}}{\partial z}\right)=-\mu_0 K$$

Therefore, in general

$$\frac{\partial \vec{A}_{\text{above}}}{\partial n}-\frac{\partial \vec{A}_{\text{below}}}{\partial n}=-\mu_0\vec{K}$$