1. [2 Marks] Prove following identities:

   (a) $\nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi (\nabla \times \mathbf{A})$

   (b) $\nabla \times (\phi \nabla \phi) = 0$.

   Solution:

   (a) Consider x component of LHS

$$\begin{aligned} [\nabla \times (\phi \mathbf{A})]_x &= \partial_y (\phi A_z) - \partial_z (\phi A_y) \\ &= (\partial_y \phi) A_z - (\partial_z \phi) A_y + \phi \partial_y A_z - \phi \partial_z A_y \\ &= [(\nabla \phi) \times \mathbf{A}]_x + \phi [\nabla \times \mathbf{A}]_x \end{aligned}$$

   Similarly, we can prove for y and z component and add to get the identity.

   (b) Using previous identity:

$$\begin{aligned} \nabla \times (\phi \nabla \phi) &= (\nabla \phi) \times (\nabla \phi) + \phi \nabla \times (\nabla \phi) \\ &= 0 \end{aligned}$$

   because the first term is cross product of same vector and the second term is the curl of a grad which is always zero.

2. [5 Marks] A vector field is given by

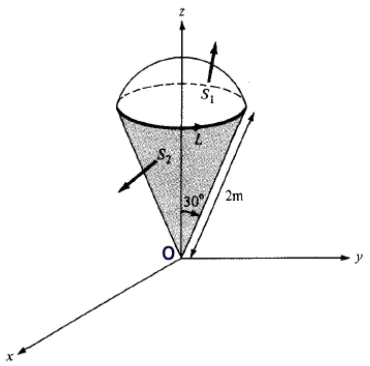
$$\mathbf{F} = r \left(\sin\theta \hat{\mathbf{r}} + \cos\theta \hat{\theta} + \hat{\phi} \right)$$

   Evaluate the following integrals:

   (a) $\oint_L \mathbf{F} \cdot d\mathbf{r}$ where L is the circular edge of the volume in the form of an ice-cream cone shown in Figure.

   (b) $\int_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{s}$ where S_1 is the top surface of the volume. (Note: S_1 is a spherical cap of radius 2m centered at the origin O.)

   (c) $\int_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{s}$ where S_2 is the slanting surface of the volume.



   Solution:

   (a) The parametric curve with ϕ as the parameter is given by $\mathbf{r}(\phi) = 2\hat{\mathbf{r}} \left(\theta = \frac{\pi}{6}, \phi\right)$ with $\phi : 0 \to 2\pi$. Then, $d\mathbf{r} = 2\sin(\pi/6) \hat{\phi} d\phi = \hat{\phi} d\phi$. And

$$\oint_L \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 2 d\phi = 4\pi$$

   (b) First,

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ r\sin\theta & r^2\cos\theta & r^2\sin\theta \end{vmatrix} \\ &= \cot\theta \hat{\mathbf{r}} - 2\hat{\theta} + \cos\theta\hat{\phi} \end{aligned}$$

   Now, on S_1, $d\mathbf{s} = \hat{\mathbf{r}} 4 \sin\theta d\theta d\phi$. The required integral is

$$\begin{aligned} \int_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{s} &= 4 \int_0^{2\pi} \int_0^{\pi/6} \cot\theta \sin\theta d\theta d\phi \\ &= 4 \cdot 2\pi \cdot \frac{1}{2} = 4\pi \end{aligned}$$

(c) And on S_2, $d\mathbf{s} = r\sin\theta drd\phi\hat{\theta}$. The required integral is

$$\begin{aligned}\int_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{s} &= \frac{1}{2}\int_0^{2\pi}\int_0^2 (-2)\, rdrd\phi \\ &= -4\pi\end{aligned}$$

3. [3 Marks] Elliptic cylindrical coordinates (u, v, z) are defined by the transformations

$$x = a\cosh u \cos v, \qquad y = a \sinh u \sin v, \qquad z = z$$

where a is a positive constant.

(a) Prove that the coordinate curves obtained by keeping z and u constant are ellipses. What are the coordinates of the foci?

(b) Prove that the coordinate curves obtained by keeping z and v constant are hyperbolas.

(c) Express unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{z}}$ in terms of cartesian unit vectors. Is this an orthogonal coordinate system?

Solution:

(a) Say, $z = z_0$ and $u = u_0$. Then, eliminating v, we get equation of the curve:

$$\frac{x^2}{a^2\cosh^2 u_0} + \frac{y^2}{a^2\sinh^2 u_0} = 1, \qquad z = z_0,$$

This is an ellipse in $z = z_0$ plane with center at $(0, 0, z_0)$ and focus at $(\pm a, 0, z_0)$.

(b) Similarly, let $v = v_0$, then

$$\frac{x^2}{a^2\cos^2 v_0} - \frac{y^2}{a^2\sin^2 v_0} = 1, \qquad z = z_0$$

is an equation of hyperbola in $z = z_0$ plane.

(c) Now, $\mathbf{r} = (a\cosh u\cos v,\ a\sinh u \sin v,\ z)$.

$$\begin{aligned}\hat{\mathbf{u}} &= \frac{\partial \mathbf{r}}{\partial u}\Big/\left|\frac{\partial \mathbf{r}}{\partial u}\right| = (a\sinh u\cos v,\ a\cosh u\sin v,\ 0)\Big/\sqrt{(a\sinh u\cos v)^2 + (a\cosh u\sin v)^2 + z^2} \\ \hat{\mathbf{v}} &= \frac{\partial \mathbf{r}}{\partial v}\Big/\left|\frac{\partial \mathbf{r}}{\partial v}\right| = (-a\cosh u\sin v,\ a\sinh u\cos v,\ 0)\Big/\sqrt{(a\sinh u\cos v)^2 + (a\cosh u\sin v)^2 + z^2} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}.\end{aligned}$$

This is an orthogonal coordinate system since all three unit vectors are mutually orthogonal.

Useful Formulae:

1. $\nabla \times \mathbf{F} = \dfrac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{r}} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ F_r & rF_\theta & r\sin\theta F_\phi\end{vmatrix}$

2. $\cosh x = \frac{1}{2}(e^x + e^{-x})$; $\sinh x = \frac{1}{2}(e^x - e^{-x})$