Solution to Mid-Sem. Exam.

PH 102

PH

1.

$$(\vec{\nabla} \times \vec{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$$

$$= \frac{\partial}{\partial x}(Kx) - \frac{\partial}{\partial y}(Ky)$$

$$= K - K$$

$$= 0$$

$\boxed{(c)}$

2. (b) $\Phi_2 = \Phi_1$

Because the total charge enclosed in the system has not changed.

$\boxed{(b)}$

3. The force is repulsive, as expected at large distance from the sphere. However, the charge $+q$ induces an image charge $-q$ in the sphere. This charge exerts on $+q$, an attractive force which predominates when $+q$ is close to the sphere.

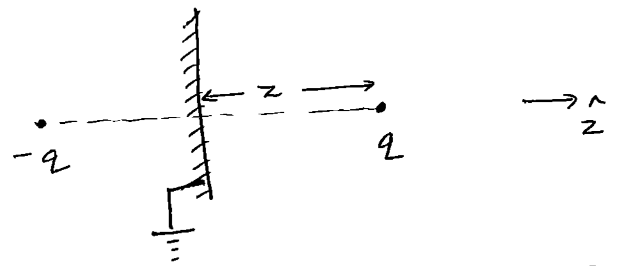
So, clearly the answer is $\boxed{(b)}$

4.

As the work done in a round trip in an electrostatic field is zero. Clearly, the work done in carrying a charge from C to A will be 1 J.

$\boxed{(b)}$

5.



Force $\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{-q^2}{(2z)^2} \hat{z}$

Energy (or work to be done) required to move the charge +q from a distance 'd' to infinity is

$$W = \int_d^\infty \vec{F} \cdot d\vec{l}$$

$$= -\frac{q^2}{4\pi\varepsilon_0} \int_d^\infty \frac{1}{4z^2} dz$$

$$= -\frac{q^2}{4\pi\varepsilon_0} \frac{1}{4} \left(-\frac{1}{z}\right)\Big|_d^\infty$$

$$= \frac{q^2}{4\pi\varepsilon_0} \frac{1}{4d}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q^2}{4d}$$

Thus the correct option is $\boxed{(d)}$

6. Centroid is equidistant from vertices. Hence zero.

$\boxed{(a)}$

7.

$$\frac{d^2\phi}{dx^2} = -\rho/\varepsilon_0$$

$$\phi = \phi_0 e^{-ax^2}$$

$$\frac{d\phi}{dx} = \phi_0 e^{-ax^2}(-2ax)$$

$$\frac{d^2\phi}{dx^2} = \phi_0 e^{-ax^2}(-2a) - 2ax\,\phi_0 e^{-ax^2}(-2ax)$$

$$= -2a\phi + 4a^2x^2\phi$$

$$\rho = -\varepsilon_0\left[4a^2x^2 - 2a\right]\phi$$

$$= 2a\varepsilon_0\phi\left(1 - 2ax^2\right)$$

$\boxed{(c)}$

8.

$$\underbrace{\int \vec{A}\cdot\hat{n}\,ds}_{\text{Flux}} = \int(\vec{\nabla}\cdot\vec{A})\,dV$$

Now,

$$\vec{\nabla}\cdot\vec{A} = 3x^2 + 3y^2 + 3z^2$$

$$= \cancel{3R^2}\ 3r^2$$

$$\therefore \text{Flux} = \cancel{\int 3R^2} \int_0^R 3r^2\, 4\pi r^2\,dr$$

$$= 12\pi\int_0^R r^4\,dr$$

$$= \frac{12\pi R^5}{5}$$

$\boxed{(a)}$

9. The electric field inside a conductor is zero irrespective of its shape.

$\boxed{c}$

10. $\boxed{b}$

11. $\boxed{d}$

12. $\boxed{b}$

13.

$$\phi = -\int_{\infty}^{2a} \frac{q}{4\pi\epsilon_0 r^2} dr$$

$$= \frac{q}{8\pi\epsilon_0 a}$$

$\boxed{(a)}$

14.

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

$$= -\left[\frac{\partial}{\partial x}(kx) + \frac{\partial}{\partial y}(ky) + \frac{\partial}{\partial z}(kz)\right]$$

$$= -3K$$

$$\sigma_b = \vec{P} \cdot \hat{n}$$

$$= KR$$

clearly the correct option is $\boxed{(c)}$

15. along $\hat{r}$: $dl_r = dr$

$\hat{\theta}$: $dl_\theta = r\,d\theta$

$\hat{\phi}$: $dl_\phi = r\sin\theta\,d\phi$

$\therefore\ d\tau = r^2 \sin\theta\,d\theta\,d\phi\,dr$

$\boxed{(a)}$

16.

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

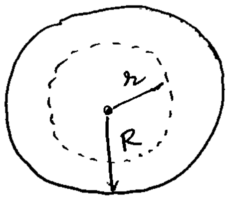
$$\vec{p}\cdot\vec{\nabla} = p_0 \frac{\partial}{\partial z}$$

$$\vec{F} = p_0 \frac{\partial}{\partial z}\left[(x^2 y)\,\hat{x} + (2xyz)\,\hat{y} + (3x)\,\hat{z}\right]$$

$$= 2p_0 xy\ \hat{y}$$

$\boxed{(d)}$

17.



$$E \times 4\pi r^2 = \frac{1}{\varepsilon_0} \int_0^r \rho \, 4\pi r'^2 dr'$$

$$= \frac{1}{\varepsilon_0} \rho \, 4\pi \frac{r^3}{3}$$

$$\Rightarrow \quad E = \frac{\rho r}{3\varepsilon_0} \quad //$$

$\boxed{(a)}$

18.

$$\phi = -E_0 \left(1 - \frac{R^3}{r^3}\right) r \cos\theta$$

$$E = \frac{\sigma}{\varepsilon_0}$$

$$E = -\vec{\nabla}\phi$$

$$\therefore \quad \sigma = -\varepsilon_0 \left.\frac{\partial \phi}{\partial r}\right|_{r=R}$$

$$= -\varepsilon_0 \left.\left[-E_0\cos\theta + E_0 R^3 \cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right]\right|_{r=R}$$

$$= -\varepsilon_0 \left.\left[-E_0\cos\theta - E_0 R^3 \cos\theta \frac{2}{r^3}\right]\right|_{r=R}$$

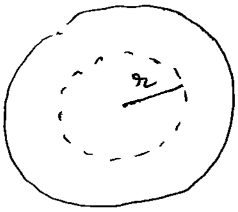
$$= \varepsilon_0 E_0 \cos\theta + 2\varepsilon_0 E_0 \cos\theta$$

$$= 3\varepsilon_0 E_0 \cos\theta$$

$\boxed{(a)}$

19.

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad r > a$$



$$E \times 4\pi r^2 = \frac{1}{\varepsilon_0}\frac{q}{4/3\,\pi a^3} 4/3\,\pi r^3$$

$$E = \frac{q}{4\pi\varepsilon_0}\frac{r}{a^3}, \quad r < a$$

$$\vec{E} = \frac{q\,r}{4\pi\varepsilon_0 a^3}\hat{r}, \quad r < a$$

clearly correct option is $\boxed{(c)}$

20. Laplace eq^n is $\nabla^2 V = 0$. Let us check all the ~~cases~~ options one by one!

(a) $V = 2x + 5$

$$\frac{\partial V}{\partial x} = 2, \quad \frac{\partial^2 V}{\partial x^2} = 0$$

(b) $V = \frac{10}{r}$, let us use ~~sph~~ spherical co-ordinates.

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial V}{\partial r}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)$$

$$\frac{\partial V}{\partial r} = -\frac{10}{r^2}$$

$$\therefore \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(-10) = 0$$

(c) $V = 10\,xy$

$$\frac{\partial V}{\partial x} = 10y, \quad \frac{\partial^2 V}{\partial x^2} = \cancel{10}\ 0$$

$$\frac{\partial V}{\partial y} = 10x, \quad \frac{\partial^2 V}{\partial y^2} = 0$$

$$\Rightarrow \nabla^2 V = 0$$

(d) $$V = x^2+y^2$$

$$\frac{\partial V}{\partial x} = 2x \quad , \quad \frac{\partial^2 V}{\partial x^2} = 2$$

$$\frac{\partial V}{\partial y} = 2y \quad ; \quad \frac{\partial^2 V}{\partial y^2} = 2$$

$$\nabla^2 V = 4$$

clearly the correct option is $\boxed{(d)}$

21.

Let the soap bubble carry a charge q. Its potential is $\phi = \frac{q}{4\pi\varepsilon_0 r}$ and energy $W = \frac{q^2}{8\pi\varepsilon_0 r}$

for $r = r_1 = 1\text{ cm} = 10^{-2}\text{ m}$

$\phi = \phi_1 = 100$ Volt

$\therefore\ q = 4\pi\varepsilon_0 r_1 \phi_1$

As the radius changes from r_1 to $r_2 = 1\text{ mm} = 10^{-3}\text{ m}$ the change in electrostatic energy is

~~ΔW~~ ~~ΔE =~~

$$\Delta W = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$= \frac{(4\pi\varepsilon_0 r_1 \phi_1)^2}{8\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$= 2\pi\varepsilon_0 (r_1\phi_1)^2 \left(\frac{r_1 - r_2}{r_1 r_2}\right)$$

$$= 2\pi \times 8.85\times10^{-12} \times (10^{-2}\times 10^2)^2 \left(\frac{10^{-2} - 10^{-3}}{10^{-2}\times 10^{-3}}\right)$$

$$= 6.28\times 8.85 \times 10^{-12} \times 10^5 \times \left(\frac{1}{100} - \frac{1}{1000}\right)$$

$$\cancel{= 6.28 \times}$$

$$\cancel{\approx 5.4 \times 10^{-12} \times 10^5 \times 10^{-3}} \times 9$$

$$= \cancel{55}\ 56 \times 10^{-7} \times 10^{-3} \times 9$$

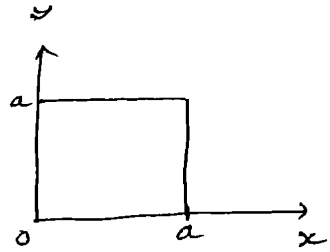
$$= 504 \times 10^{-10}$$

$$= 5.04 \times 10^{-8}\text{ J}$$

$$\approx 5\times10^{-8}\text{ J}$$

$\boxed{(c)}$

22.



$$q = \iint \sigma(x,y)\, dx\, dy$$

$$= \int_0^a \int_0^a \frac{\sigma_0}{a^2}\left(x^2 + xy + y^2\right) dx\, dy$$

$$= \frac{\sigma_0}{a^2}\left[\int_0^a \int_0^a x^2\, dx\, dy + \int_0^a \int_0^a xy\, dx\, dy + \int_0^a \int_0^a y^2\, dx\, dy\right]$$

$$= \frac{\sigma_0}{a^2}\left[a\,\frac{a^3}{3} + \frac{a^2}{2}\,\frac{a^2}{2} + a\,\frac{a^3}{3}\right]$$

$$= \frac{\sigma_0}{a^2}\left[\frac{a^4}{3} + \frac{a^4}{4} + \frac{a^4}{3}\right]$$

$$= \frac{\sigma_0}{a^2}\, a^4 \left[\frac{4+3+4}{12}\right]$$

$$= \frac{11}{12}\, \sigma_0\, a^2$$

$\boxed{(d)}$

23 and 24

$$V(r,\theta,\phi) = \frac{V_0}{2}\left(3 - \frac{r^2}{R^2}\right), \quad r < R$$

$$= \frac{V_0 R}{r}, \quad r > R$$

$$\frac{\partial V}{\partial r} = -\frac{V_0}{2}\frac{2r}{R^2} = -\frac{V_0 r}{R^2} \qquad r < R$$

$$\frac{\partial V}{\partial r} = -\frac{V_0 R}{r^2}, \quad r > R$$

$$\vec{E} = -\vec{\nabla} V = -\hat{r}\frac{\partial V}{\partial r}$$

For $r < R$

$$\vec{E} = +\hat{r}\frac{V_0 r}{R^2} \quad ; \quad r < R$$

For $r > R$

$$\vec{E} = \hat{r}\frac{V_0 R}{r^2}, \quad r > R$$

Again

$$\nabla^2 V = -\rho/\varepsilon_0$$

$$\Rightarrow \quad \rho = -\varepsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right]$$

Thus,

$$\rho = -\varepsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{V_0 r}{R^2}\right)\right]$$

$$= +\varepsilon_0 \frac{1}{r^2}\frac{V_0}{R^2} 3r^2 = \frac{3V_0\varepsilon_0}{R^2} \quad ; \quad r < R$$

$$\rho = -\varepsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{V_0 R}{r^2}\right)\right]$$

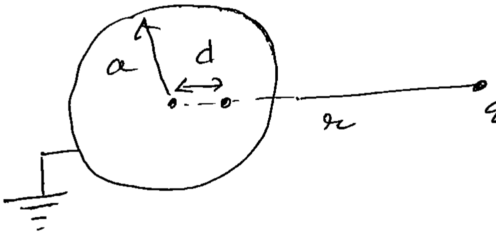
$$= 0 \quad ; \quad r > R$$

So correct option for 23 : $\boxed{(c)}$ correct option for 24 : $\boxed{(b)}$

25 and 26

You can do it the way it was done in lecture class. (See lecture 10 and 11 on method of images!)

Or just recall the results:



If a charge is kept at a distance r from the centre of the grounded conducting sphere of radius a,

The image charge is located at $d = \frac{a^2}{r}$

and the induced charge $q' = -q \frac{a}{r}$

In the given problem, $a = R$
$r = 2R$

So, $$d = \frac{R^2}{2R} = \frac{R}{2}$$

$$q' = -q \frac{R}{2R} = -\frac{q}{2}$$

Thus, correct option for 25 : $\boxed{(a)}$

correct ~~corect~~ option for 26 : $\boxed{(a)}$

27 and 28

The electric field outside the sphere is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \quad , \quad r > R$$

Potential at distance r is

$$V(r) = -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0 r'^2}\, dr'$$

$$= \frac{Q}{4\pi\epsilon_0 r}$$

So, the capacitance is

$$C = \frac{Q}{V(R)} = 4\pi\epsilon_0 R$$

Energy density is

$$u = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\epsilon_0 E^2$$

$$= \frac{1}{2}\epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2$$

$$= \frac{Q^2}{32\pi^2\epsilon_0 r^4}$$

So, correct option for 27 : $\boxed{(a)}$

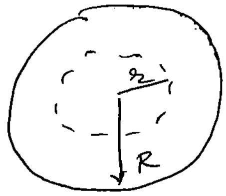
correct option for 28 : $\boxed{(b)}$

29 and 30

$$\rho = \rho_0 \quad , \quad 0 < r < R$$
$$= 0 \quad , \quad r > R$$

For $\quad r < R$

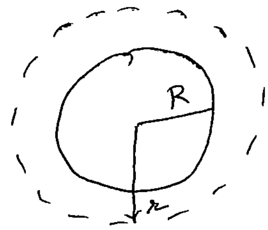
$$\oint \vec{D} \cdot \vec{ds} = q_{free}$$



$$\Rightarrow \quad \varepsilon E \, 4\pi r^2 = \rho_0 \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \vec{E} = \frac{\rho_0 r}{3\varepsilon}\hat{r} \, , \quad r < R$$

$r > R$



$$\varepsilon_0 E \, 4\pi r^2 = \rho_0 \frac{4}{3}\pi R^3$$

$$\Rightarrow \quad \vec{E} = \frac{\rho_0 R^3}{3\varepsilon_0 r^2}\hat{r} \, , \quad r > R$$

~~Potential at the centre of the sphere:~~

$$\cancel{V(r) = -\int_{\infty}^{R} E\Big|_{r>R} dr - \int_{R}^{r} E\Big|_{r<R} dr}$$

Potential at a point inside the sphere:

$$V(r) = -\int_{\infty}^{R} E\Big|_{r>R} dr - \int_{R}^{r} E\Big|_{r<R} dr$$

$$= -\int_{\infty}^{R} \frac{\rho_0 R^3}{3\varepsilon_0 r^2} dr - \int_{R}^{r} \frac{\rho_0 r'}{3\varepsilon} dr'$$

$$= \frac{\rho_0 R^3}{3\varepsilon_0}\left[\frac{1}{r}\Big|_{\infty}^{R}\right] - \frac{\rho_0}{3\varepsilon}\left[\frac{r'^2}{2}\Big|_{R}^{r}\right]$$

$$= \frac{\rho_0 R^2}{3\varepsilon_0} - \frac{\rho_0}{6\varepsilon}\left(r^2 - R^2\right)$$

Thus,

$$V(r) = \frac{\rho_0 R^2}{3\varepsilon_0} - \frac{\rho_0}{6\varepsilon}\left(r^2 - R^2\right) \longrightarrow (1)$$

At the centre

$$V(r=0) = \frac{\rho_0 R^2}{3\varepsilon_0} + \frac{\rho_0 R^2}{6\varepsilon}$$

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

Thus, $$V(\text{at centre}) = \frac{\rho_0 R^2}{6\varepsilon_0\varepsilon_r}\left[1 + 2\varepsilon_r\right]$$

Potential at the surface, i.e. at $r = R$

$$V(\text{at the surface}) = \frac{\rho_0 R^2}{3\varepsilon_0} \qquad \text{from (1)}$$

So, correct option for 29 : $\boxed{(a)}$

correct option for 30 : $\boxed{(c)}$