MA 102 (Mathematics II)

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Tutorial Sheet No. 6

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(1) Find all the critical points of $f(x, y) = \sin x \sin y$ in the domain $-2 \leq x \leq 2,\ -2 \leq y \leq 2$.

Solution. Given $f(x, y) = \sin x \sin y$, $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$. $f_x = 0$ implies $\cos x \sin y = 0$ and $f_y = 0$ implies $\sin x \cos y = 0$. Thus $x = \pm(2n+1)\frac{\pi}{2}$ or $y = \pm n\pi$ and $x = \pm n\pi$ or $y = \pm(2n+1)\frac{\pi}{2}$ i.e. $(x, y) = (\pm(2n+1)\frac{\pi}{2}, \pm(2n+1)\frac{\pi}{2})$ and $(\pm n\pi, \pm n\pi)$. Thus critical points in the domain are given by $(0, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, -\frac{\pi}{2})$ and $(-\frac{\pi}{2}, -\frac{\pi}{2})$. $\square$

(2) Find all the local maxima, local minima and saddle points of the following functions:
(a) $f(x,\ y) = x^2 + xy + y^2 + 3x - 3y + 4$ (b) $f(x,\ y) = 6x^2 - 2x^3 + 3y^2 + 6xy$

Solution. (a) $(-3, 3)$ is the only critical point, which is a local minimum.

(b) Critical points are $(0, 0)$ and $(1, -1)$. We have $(0, 0)$ is a local minimum and $(1, -1)$ is a saddle point.

$\square$

(3) Let $f(x,\ y) = xy - x^2$, and let R be the square region given by $R = \{(x,\ y) \in \mathbb{R}^2\ :\ 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$. Find the extreme values of f on R.

Solution. Solving $f_x = 0$ and $f_y = 0$, we find that there is no critical point in the interior of R. We now consider the functions $f(x, 0), f(x, 1), f(0, y)$ and $f(1, y)$ on the boundary $y = 0, y = 1, x = 0, x = 1$, respectively. We easily find that the maximum value $1/4$ is attained at $(1/2, 1)$ and the minimum value -1 is attained at $(1, 0)$. $\square$

(4) Verify that $f(x,\ y,\ z) = x^4 + y^4 + z^4 - 4xyz$ has a critical point $(1,\ 1,\ 1)$, and determine the nature of this critical point by computing the eigenvalues of its Hessian matrix.

Solution. Easy to check that $(1, 1, 1)$ is a crtical point. The Hessian at $(1, 1, 1)$ is

$$H = \begin{bmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{bmatrix}.$$

The eigenvalues of H are $4, 16, 16$. Hence, H is a positive definite matrix and f has a local minimum at $(1, 1, 1)$. $\square$

(5) Using the method of Lagrange multipliers, find the extremum values of $f(x,\ y) = xy$ subject to the constraint $g(x,\ y) = x^2 + y^2 - 10 = 0$.

Solution. Solving $f_x = \lambda g_x, f_y = \lambda g_y$ and $g(x, y) = 0$, we find eligible solutions $(x, y, \lambda) = (\pm\sqrt{5}, \pm\sqrt{5}, 1/2)$ and $(x, y, \lambda) = (\pm\sqrt{5}, \mp\sqrt{5}, -1/2)$. Hence, maximum value of f is 5 and the minimum value is -5. $\square$

(6) Using the method of Lagrange multipliers, find the points on the curve $xy^2 = 54$ nearest to the origin.

Solution. We must minimize $f(x,y) = x^2 + y^2$ subject to $g(x,y) := xy^2 - 54 = 0$. Solving $f_x = \lambda g_x, f_y = \lambda g_y$ and $g(x,y) = 0$, we find eligible solutions $(x, y, \lambda) = (3, \pm 3\sqrt{2}, 1/3)$. Hence, $(3, \pm 3\sqrt{2})$ are the points on the curve nearest to the origin. □

(7) Evaluate the double integral $\iint_R f(x,y)\ dxdy$ for f and R given below.

(a) $f(x,y) := x^2 + y^2$ and $R = [-1,1] \times [0,1]$.
(b) $f(x,y) := x^2 + y$ and $R = [0,1] \times [0,1]$.
(c) $f(x,y) := \sin(x+y)$ and $R = [0,\pi] \times [0,\pi]$.
(d) $f(x,y) = \sin x \cos y$ and $R = [0,\pi/2] \times [0,\pi/2]$.

Solution.

(a) By Fubini's theorem, $\iint_R (x^2+y^2)\ dxdy = \int_0^1 \left(\int_{-1}^1 (x^2+y^2)\ dx\right)\ dy.$

Now $\int_{-1}^1 (x^2+y^2)\ dx = \left[\frac{x^3}{3} + xy^2\right]_{x=-1}^1 = \frac{2}{3} + 2y^2.$

Consequently, $\int_0^1 \left(\frac{2}{3} + 2y^2\right)\ dy = \left[\frac{2y}{3} + \frac{2y^3}{3}\right]_{y=0}^1 = \frac{4}{3}.$

(b) Fubini's theorem,

$$\iint_R (x^2+y)\ dA = \int_0^1 \int_0^1 (x^2+y)\ dxdy = \int_0^1 \left(\int_0^1 (x^2+y)\ dx\right)\ dy.$$

Now integrating w.r.t x we have $\int_0^1 (x^2+y)\ dx = \left[\frac{x^3}{3} + yx\right]_{x=0}^1 = \frac{1}{3} + y.$

Thus $\iint_R (x^2+y)\ dA = \int_0^1 \left(\frac{1}{3} + y\right)\ dy = \left[\frac{y}{3} + \frac{y^2}{2}\right]_{x=0}^1 = \frac{5}{6}.$

(c) Once again by Fubini's theorem, we have

$$\begin{aligned}\iint_R \sin(x+y)\ dxdy &= \int_0^\pi \left(\int_0^\pi \sin(x+y)\ dx\right)\ dy = \int_0^\pi \left(\cos(x+y)|_{x=0}^\pi\right)\ dy \\ &= \int_0^\pi \big(\cos y - \cos(y+\pi)\big)\ dy = \big[\sin y - \sin(y+\pi)\big]_{y=0}^{2\pi} = 0.\end{aligned}$$

(d) Easy. □

(8) Evaluate the following double integrals.

$(a)\ \displaystyle\int_0^3 \int_{-y}^y (x^2+y^2) dxdy \qquad (b)\ \int_0^\pi \int_0^\pi |\cos(x+y)| dxdy$

$(c)\ \displaystyle\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy\ dx \qquad (d)\ \int_0^{3/2} \int_0^{9-4x^2} 16xdy\ dx$

Solution. (a) Easy.

(b) The integral may be split into two pieces where $\cos(x+y)$ is positive and negative respectively. In the domain bounded by $x = 0, y = 0$ and $x + y = \pi/2$ we have $\cos(x+y)$ is non-negative. In the domain bounded below by $x+y = \pi/2, 0 \le x \le \pi/2$ and above by $y = \pi$, $\cos(x+y)$ is non-positive.

Similarly, in the domain $\pi/2 \leq x \leq \pi,\ 0 \leq y \leq \frac{3\pi}{2} - x$, $\cos(x+y) \leq 0$ and in the domain $\pi/2 \leq x \leq \pi,\ \frac{3\pi}{2} - x \leq y \leq \pi$, $\cos(x+y) \geq 0$. Hence the given integral may be written as

$$\begin{aligned}
&\int_0^\pi \int_0^\pi |\cos(x+y)| dx dy \\
&= \int_{x=0}^{\pi/2} \left(\int_{y=0}^{\frac{\pi}{2}-x} \cos(x+y) dy + \int_{\frac{\pi}{2}-x}^{\pi} -\cos(x+y) dy \right) dx \\
&\quad + \int_{x=\pi/2}^{\pi} \left(\int_0^{\frac{3\pi}{2}-x} -\cos(x+y) dy + \int_{\frac{3\pi}{2}-x}^{\pi} \cos(x+y) dy \right) dx
\end{aligned}$$

(c) The domain is bounded by the lines $x = 0, y = \pi$ and $y = x$. Hence

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx = \int_{y=0}^{\pi} \left(\int_{x=0}^{y} \frac{\sin y}{y} dx \right) dy = \int_{y=0}^{\pi} \sin y \ dy = 2.$$

(d) 81.

□

(9) Evaluate the following double integrals.

(a) $\iint_R \frac{dA}{\sqrt{xy - x^2}}$, where R is the region bounded by $x = 0, x = 1, y = x$ and $y = x + 1$.

(b) $\int_0^1 \int_0^{1-x} e^{\frac{x-y}{x+y}} dx dy$ (c) $\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} (x+y) dx\ dy$

(d) $\iint_R \cos(9x^2 + 4y^2)\ dx\ dy$, where $R = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^2 \leq 1\}$.

Solution. (a) Take $u = x$ and $v = y - x$. Then $y = x \Rightarrow v = 0$ and $y = x + 1 \Rightarrow v = 1$. We have $J = 1$ and

$$\iint_R \frac{dA}{\sqrt{xy - x^2}} = \iint_D \frac{1}{\sqrt{uv}} du dv,$$

where D is the region in the uv-plane bounded by the lines $u = 0, u = 1, v = 0$ and $v = 1$.

(b) Consider $u = x - y$ and $v = x + y$. Then

$$\int_0^1 \int_0^{1-x} e^{\frac{x-y}{x+y}} dx dy = \frac{1}{2} \int_{v=0}^{1} \int_{u=-v}^{v} e^{\frac{u}{v}} du dv = \frac{1}{2}(e - e^{-1}) \int_{v=0}^{1} v dv = \frac{1}{4}(e - e^{-1}).$$

(c) The region is bounded by the lines $y = 0, y = x$ and the circle $x^2 + y^2 = 1$. Using polar coordinates, we have

$$\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} (x+y) dx\ dy = \int_0^{\pi/4} \int_0^1 (r\cos\theta + r\sin\theta) r dr d\theta$$

(d) Take $x = \frac{r}{3}\cos\theta$ and $y = \frac{r}{2}\sin\theta$. Then $J = \frac{r}{6}$. Therefore,

$$\iint_R \cos(9x^2 + 4y^2)\ dx\ dy = \frac{1}{12} \int_0^{2\pi} \int_0^1 \cos u\ du d\theta.$$

$\square$

(10) Find the volume of the following:

(a) Region under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x, x = 0$, and $x + y = 2$ in the xy plane.

(b) Region bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy plane.

(c) Region bounded in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

(d) Solid cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

(e) Tetrahedron bounded by the planes $y = 0, z = 0, x = 0$ and $-x + y + z = 1$.

Solution. (a) We use the formula $V = \iint_R f(x,y)dA$ where $f(x,y) \geq 0$ is a continuous real valued function defined over the domain R of the plane.

Here $f(x,y) = x^2 + y^2$ and the domain is bounded by $x = 0, y = 2 - x$ and $y = x$. So in this domain drawing a line parallel to y axis, it is easy to see that the domain may be described as $x \leq y \leq 2 - x,\ 0 \leq x \leq 1$. In otherwords,the upper curve is $y = 2 - x$ and lower curve is $y = x$ between $x = 0$ and $x = 1$. Hence

$$V = \iint_R (x^2 + y^2)dA = \int_0^1 \left(\int_x^{2-x} (x^2 + y^2)dy \right) dx = \frac{4}{3}$$

(b) The domain R is bounded by $y = 2 - x^2$ and $y = x$ in the xy plane. The function $f(x,y) = x^2$. So the upper curve is $y = 2 - x^2$ and the lower curve is $y = x$. Hence

$$V = \iint_R x^2 dA = \int_{x=-2}^1 \left(\int_{y=x}^{2-x^2} x^2 dy \right) dx = \frac{63}{20}$$

(c) The domain is bounded by $x = 0, y = 0$ and $x^2 + y^2 = 4$. The non-negative function is $f(x,y) = 3 - y$. The upper curve is $y = \sqrt{4 - x^2}$ and the lower curve is $y = 0$. Hence

$$V = \iint_R (3 - y)dA = \int_0^2 \left(\int_0^{\sqrt{4-x^2}} (3 - y)dy \right) dx = \frac{9\pi - 8}{3}$$

(d) The domain is bounded by $y = 2 - x, y = 0$ and $x = 0$. The non-negative function $f(x,y) = 12 - 3y^2$. Hence

$$V = \iint_R (12 - 3y^2)dA = \int_0^2 \left(\int_0^{2-x} (12 - 3y^2)dy \right) dx = 20$$

(e) The non-negative function is $f(x,y) = 1 + x - y$ over the domain R bounded by $y = 0, x = 0$ and $y = 1 + x$. Hence

$$V = \iint_R (1 - y + x)dA = \int_{x=-1}^0 \left(\int_0^{1+x} (1 - y + x)dy \right) dx = \frac{1}{6}$$

$\square$

(11) Evaluate the following triple integrals:

(a) $\iiint_D (z^2x^2 + z^2y^2)\; dV$, where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$

(b) $\iiint_D xyz\; dV$ where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \leq 1,\; 0 \leq z \leq x^2 + y^2\}$

(c) $\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\; dV$ where $D = \{(x, y, z) \in \mathbb{R}^3 :\; x^2 + y^2 + z^2 \leq 1\}$

Solution. (a) Using the cylindrical coordinates $x = r\cos\theta, y = r\sin\theta$ and $z = z$, the given domain may be represented as

$$D = \{(r, \theta, z) :\; 0 \leq r \leq 1,\; 0 \leq \theta \leq 2\pi,\; -1 \leq z \leq 1\}.$$

Hence the integral becomes

$$\iiint_D z^2(x^2 + y^2)dV = \int_{z=-1}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(z^2r^2)r\,dr\,d\theta\,dz$$
$$\int_{z=-1}^{1}\int_{0}^{2\pi} z^2\frac{1}{4}d\theta\,dz = \frac{\pi}{3}$$

(b) Again using the cylindrical coordinates

$$D = \{(r, \theta, z) :\; 0 \leq z \leq r^2,\; 0 \leq r \leq 1,\; 0 \leq \theta \leq 2\pi\}.$$

Hence

$$\iiint_D xyz\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{r^2} zr^3\cos\theta\sin\theta\,dz\,dr\,d\theta = 0$$

(c) In this case, using the spherical coordinates, the domain is represented as

$$D = \{(r, \theta, \phi) : 0 \leq r \leq 1, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

Hence

$$\iiint_D e^{(x^2+y^2+z^2)^{3/2}}dV = \int_{\theta=0}^{2\pi}\int_{0}^{\pi}\int_{0}^{1} e^{r^3}r^2\sin\phi\,dr\,d\phi\,d\theta = \frac{4\pi}{3}(e-1)$$

□

(12) Find the volume of the following regions using triple integrals:

(a) The region in the first octant bounded by the coordinate planes and the planes $x+z = 1, y + 2z = 2$.

(b) The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$.

(c) The tetrahedron in the first octant bounded by the coordinate planes and the plane $x + y/2 + z/3 = 1$.

(d) The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

(e) The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$.

(f) The region enclosed by $y = x^2, y = x + 2, 4z = x^2 + y^2$ and $z = x + 3$.

(g) The region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

(h) The solid bounded by the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$.

Solution. (a) This is like tetrahedron with base on the xz-plane. The limits of y for the solid are $y = 0$ to $y = 2 - 2z$. The triangle on the xz-plane is bounded by $x = 0, y = 0$ and $x + z = 1$. On this triangle $0 \le z \le 1 - x$ and $0 \le x \le 1$. Hence volume is equal to the iterated integral

$$V = \int_{x=0}^{1} \int_{z=0}^{1-x} \int_{y=0}^{2-2z} dydzdx = 2/3.$$

(b) Imagine the solid as the piece of parabolic cylinder on over the region R of the xy-plane bounded by $x = 4 - y^2$ in the first quadrant. Over the solid z varies from 0 to $2 - y$ and $R = \{(x, y) : 0 \le y \le \sqrt{4 - x}, 0 \le x \le 4\}$. Hence

$$V = \int_{x=0}^{4} \int_{y=0}^{\sqrt{4-x}} \int_{0}^{2-y} dzdydx = \int_{y=0}^{2} \int_{x=0}^{4-y^2} \int_{0}^{2-y} dzdxdy = 10/3.$$

(c) $V = \int_{x=0}^{1} \int_{0}^{2-2x} \int_{0}^{3-3x-\frac{3y}{2}} dzdydx = 1.$

(d) $V = 8 \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2}} dzdydx = 16/3.$

(e) $V = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{3-x} dzdydx = 12\pi.$

(f) $V = \int_{x=-1}^{2} \int_{y=x^2}^{x+2} \int_{z=(x^2+y^2)/4}^{x+3} dzdydx = \frac{783}{70}$

(g) The region is bounded above by the surface $z^2 = 2 - x^2 - y^2$ and the lower surface $z = x^2 + y^2$. Hence the limits of z are $x^2 + y^2 \le z \le 2 - x^2 - y^2$.

The given surfaces intersect on the plane $z = 1$ ($z^2 + z - 2 = 0 \implies z = 1, -2$). So the solid base is the circle $x^2 + y^2 = 1$. The limits of y are $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$ and that of x are $-1 \le x \le 1$.

Now going to cylindrical coordinates

$$V = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \int_{r^2}^{\sqrt{2-r^2}} r\ dzdrd\theta$$

(h) In cylindrical coordinates, the given solid is bounded by $z = r$ and $z = r^2$. The solid is obtained by rotating the area between $z = r$ and $z = r^2$. Hence

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{r} rdzdrd\theta = \frac{\pi}{6}$$

$\square$

(13) Evaluate the line integral $\int_\Gamma F \bullet dr$ of the vector field F given below.

(a) $F(x, y) := (x^2 + 2xy,\ y^2 - 2xy)$ from $(-1, 1)$ to $(1, 1)$ along $y = x^2$.

(b) $F(x, y) := (x^2 - y^2,\ x - y)$ and $\Gamma : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the counterclockwise direction.

Solution. (a) Parametrize the curve $\Gamma : y = x^2$ as $r(t) = (t, t^2)$. Then $r'(t) = (1, 2t)$. Thus

$$\int_\Gamma F \bullet dr = \int_{-1}^{1} \left(t^2 + 2t^3, t^4 - 2t^3\right) \cdot (1, 2t) dt = \int_{-1}^{1} (t^2 + t^3) + 2t(t^4 - 2t^3) dt = \frac{-14}{15}.$$

(b) Parametrization of ellipse is given by $r(\theta) = (a\cos\theta, b\sin\theta), \theta \in [0, 2\pi]$. Thus, $F(a\cos\theta, b\sin\theta) \bullet r'(\theta) = \big((a^2\cos^2\theta - b^2\sin^2\theta), (a\cos\theta - b\sin\theta)\big) \bullet (-a\sin\theta, b\cos\theta)$.
Hence, $\int_\Gamma F \bullet dr = \int_0^{2\pi} \big((-a^3\cos^2\theta\sin\theta + ab^2\sin^3\theta) + (ab\cos^2\theta - b^2\sin\theta\cos\theta)\big)\, d\theta = \pi ab.$

□

(14) Evaluate the line integral $\displaystyle\int_\Gamma \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$ along $\Gamma : x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.

Solution. A parametrization of the curve is given by $r(\theta) = (a\cos\theta, a\sin\theta), \theta \in [0, 2\pi]$. Therefore $r'(\theta) = (-a\sin\theta, a\cos\theta)$. Consequently, the line integral equals

$$\int_0^{2\pi} \frac{a^2(\cos\theta + \sin\theta)(-\sin\theta) + a^2(\sin\theta - \cos\theta)(\cos\theta)}{a^2}\, d\theta = \int_0^{2\pi} \frac{-a^2}{a^2}\, d\theta = -2\pi.$$

□

(15) Evaluate the line integral $\displaystyle\int_\Gamma \frac{x^2y dx - x^3 dy}{(x^2+y^2)^2}$, where Γ is the square with vertices $(\pm1, \pm1)$ oriented in the counter clockwise direction.

Solution. We have

$$\begin{aligned}
\int_{\Gamma_1} (Pdx + Qdy) &= -\int_{-1}^{1} \frac{x^2 dx}{(1+x^2)^2} = \int_{-\pi/4}^{\pi/4} \sin^2\theta\ d\theta = -\pi/4 + 1/2,\\
\int_{\Gamma_2} (Pdx + Qdy) &= -\int_{-1}^{1} \frac{dy}{(1+y^2)^2} = \int_{-\pi/4}^{\pi/4} \cos^2\theta\ d\theta = -\pi/4 - 1/2,\\
\int_{\Gamma_3} (Pdx + Qdy) &= \int_{-1}^{1} \frac{-x^2 dx}{(1+x^2)^2} = -\pi/4 + 1/2,\\
\int_{\Gamma_4} (Pdx + Qdy) &= \int_{-1}^{1} \frac{-dy}{(1+y^2)^2} = -\pi/4 - 1/2.
\end{aligned}$$

Hence $\int_\Gamma (Pdx + Qdy) = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} = -\pi.$ □

(16) Verify Green's theorem in each of the following cases:
(a) $f(x, y) := -xy^2$; $g(x, y) := x^2y$; the region R is given by $x \geq 0, 0 \leq y \leq 1 - x^2$.
(b) $f(x, y) := 2xy$; $g(x, y) := e^x + x^2$; the region R is the triangle with vertices $(0, 0), (1, 0)$ and $(1, 1)$.

Solution. We have to show that $\iint_R (g_x - f_y)\ dxdy = \oint_{\partial R} (f dx + g dy)$.

(a) We have $\iint_R 4xy\ dxdy = \int_0^1 \left(\int_0^{1-x^2} 4xy\ dy\right)\ dx = \frac{1}{3}$.

The boundary of R consists of 3 smooth curves: a segment of x-axis, a part of parabola $y = 1 - x^2$ and a segment of y-axis. The integrand on RHS vanishes on both the axes. Consider the parametrization $t \mapsto (t, 1-t^2)\ t \in [0,1]$, for the part of the parabola traced in the opposite direction. This gives

$$RHS = \int_{\partial R} (-xy^2\ dx + x^2y\ dy) = -\int_0^1 \left(-t(1-t^2)^2 + t^2(1-t^2)(-2t)\right)\ dt = \frac{1}{3}.$$

(b) We have $\iint_R (e^x + 2x - 2x)\ dxdy = \int_0^1 \left(\int_0^x e^x\ dy\right)\ dx = 1$. On the other hand, $\int_{\partial R} (2xy\ dx + (e^x + x^2)\ dy) = \int_0^1 (e+1)\ dy + \int_1^0 (3t^2 + e^t)\ dt = e + 1 - e = 1$. □

(17) Evaluate $\int_\Gamma (y^2 dx + x dy)$ using Green's theorem, where Γ is boundary of R and
   (a) R is the square with vertices $(0,0), (0,2), (2,2), (2,0)$.
   (b) R is the square with vertices $(\pm 1, \pm 1)$.
   (c) R is the disc of radius 2 and center $(0,0)$.

Solution. (a) We have $f(x,y) = y^2, \quad g(x,y) = x$. Therefore, the given path integral is

$$\iint_R (1-2y)\ dxdy = \int_0^2 \int_0^2 (1-2y)\ dydx = 4 - 4\int_0^2 dx = 4 - 8 = -4.$$

(b) We have

$$\iint_R (1-2y)\ dxdy = \iint_R dxdy + \int_{-1}^1 \int_{-1}^1 (-2y)\ dydx = 4 + 0 = 4.$$

(c) We have

$$\iint_R (1-2y)\ dxdy = \iint_R dxdy + \int_{-2}^2 \left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-2y)\ dy\right)\ dx = 4\pi + 0 = 4\pi.$$

□

(18) Determine which of the following vector fields F is conservative and find a scalar potential when it exists.
   (a) $F(x,y) = (\cos(xy) - xy\sin(xy), x^2\sin(xy))$.
   (b) $F(x,y) = (xy, xy)$.
   (c) $F(x,y,z) = (x^2, xy, 1)$.

Solution. These vector fields are all C^1. □