Physics II: Electromagnetism (PH102)

Lecture 4

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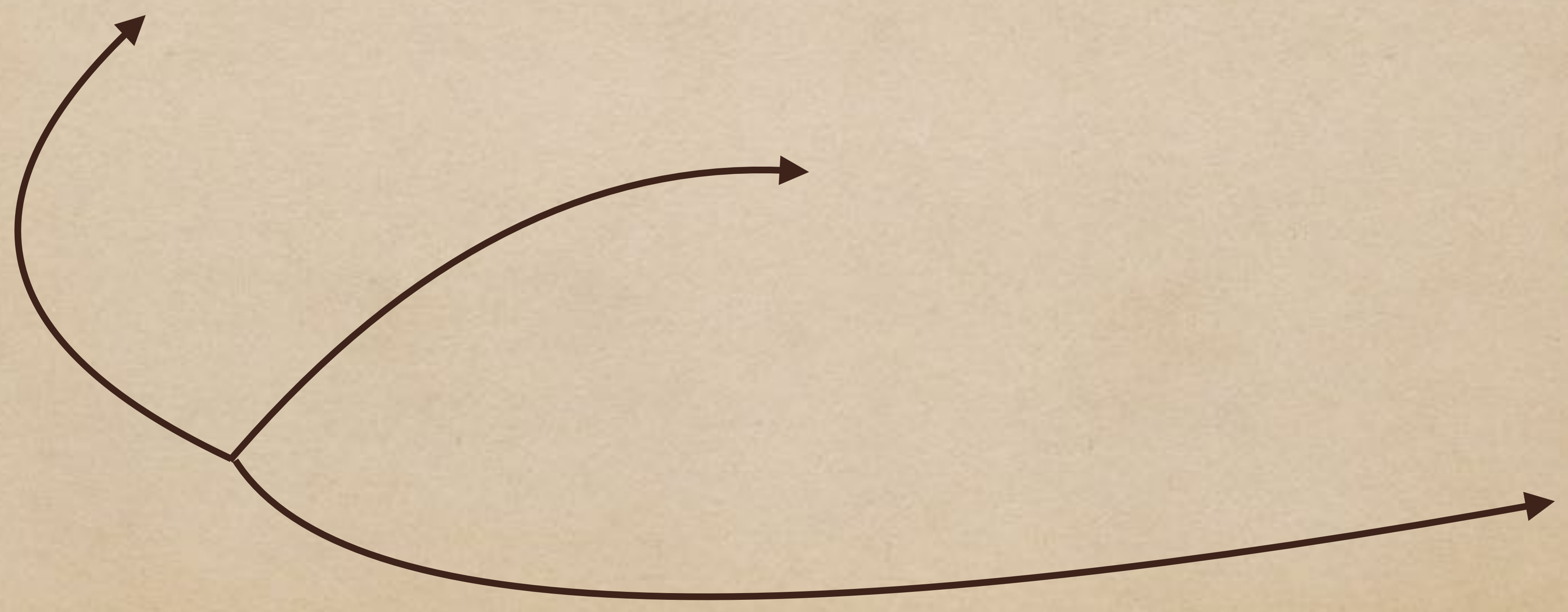
Curvilinear coordinates



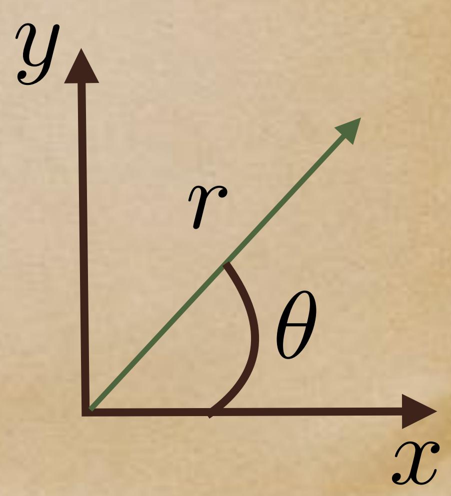
Curvilinear Coordinates

A coordinate system in which coordinate lines and surfaces are curved.
Example: Spherical polar coordinate or Cylindrical polar coordinate

For example, one recall the relations in cartesian (x,y) to polar coordinates (r, θ) in two dim:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ r = \sqrt{(x^2 + y^2)},\ \ \theta = \tan^{-1}(\frac{y}{x})$$

We will discuss generalisation of polar coordinates in 3d



Curvilinear Coordinates: generic form

Suppose Cartesian coordinate: (x, y, z) can be expressed as functions of: (u_1, u_2, u_3)

$$x = x(u_1, u_2, u_3),\ \ y = y(u_1, u_2, u_3),$$
$$z = z(u_1, u_2, u_3)$$

can be solved uniquely to

$$u_1 = u_1(x, y, z),\ \ u_2 = u_2(x, y, z),$$
$$u_3 = u_3(x, y, z)$$

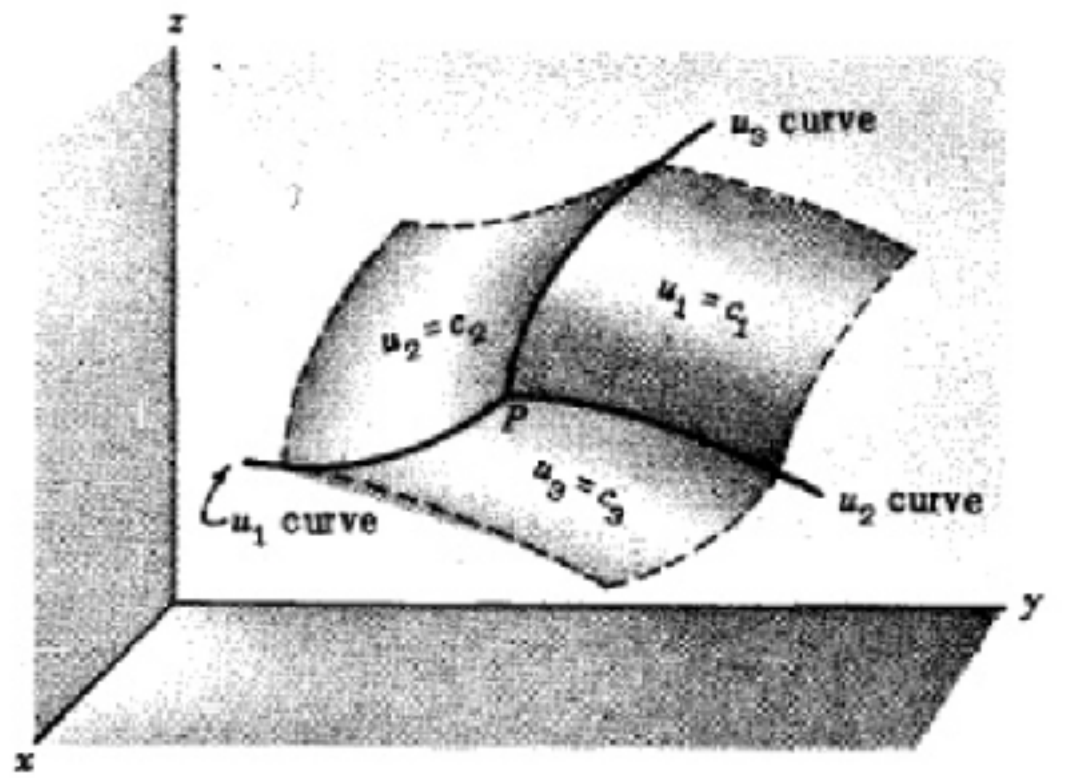


Fig. 1

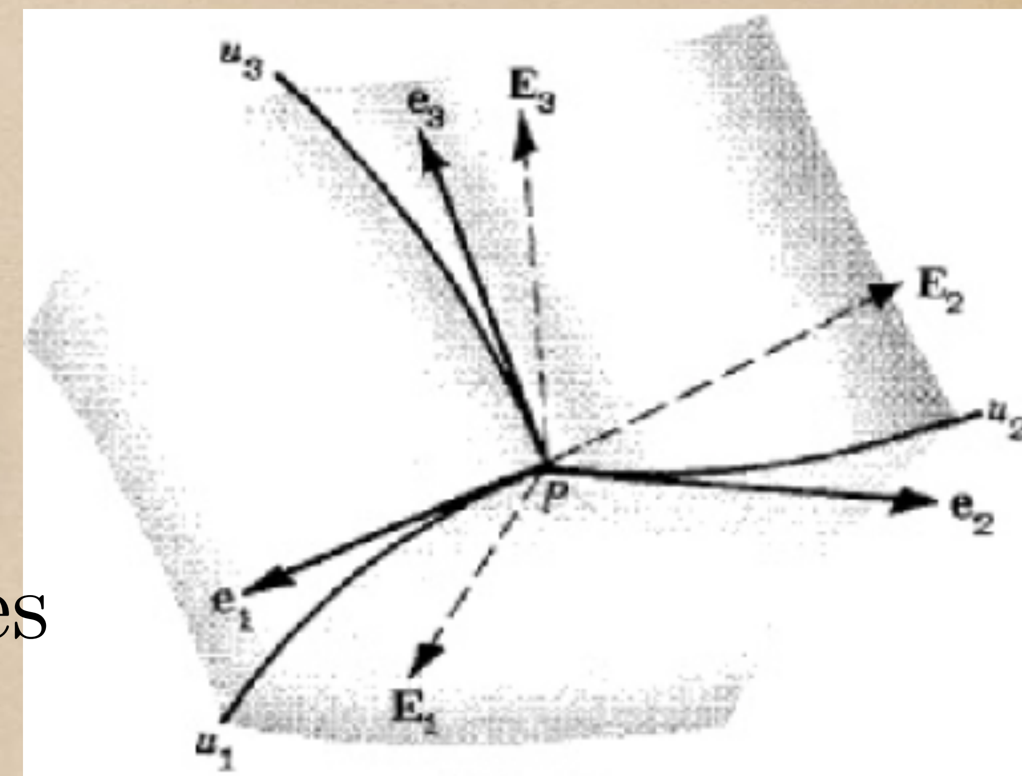
Surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ where c_1, c_2, c_3 are constants
$\rightarrow$ *Coordinate Surfaces*
Each pair of these surfaces intersect in
$\rightarrow$ *Coordinate Curves*

If the coordinate surfaces intersect at right angles, they are called orthogonal coordinate system

Ex: Spherical and Cylindrical polar coordinates

Unit Vectors in Curvilinear coordinates

$\hat{e_1}$: unit vector along curve u_1 at P.
$\hat{e_1} = (\partial\vec{r}/\partial u_1)/|\partial\vec{r}/\partial u_1|$



$$\partial\vec{r}/\partial u_1 = h_1\hat{e_1};\ \ h_1 = |\partial\vec{r}/\partial u_1|$$

Similarly unit vectors along curves u_2, u_3 are respectively $\hat{e}_2$ and $\hat{e}_3$

$$\partial\vec{r}/\partial u_2 = h_2\hat{e}_2;\ \ h_2 = |\partial\vec{r}/\partial u_2|$$

$$\left.\begin{aligned} \mathbf{r} &\neq u\hat{\mathbf{u}} + v\hat{\mathbf{v}} + w\hat{\mathbf{w}} \\ d\mathbf{r} &\neq du\hat{\mathbf{u}} + dv\hat{\mathbf{v}} + dw\hat{\mathbf{w}} \\ |d\mathbf{r}| = ds &\neq \sqrt{du^2 + dv^2 + dw^2} \end{aligned}\right\} \textbf{THESE ARE BAD}$$

$$\partial\vec{r}/\partial u_3 = h_3\hat{e}_3;\ \ h_3 = |\partial\vec{r}/\partial u_3|$$

Line Element : $\vec{dr} = \frac{\partial\vec{r}}{\partial u_1}du_1 + \frac{\partial\vec{r}}{\partial u_2}du_2 + \frac{\partial\vec{r}}{\partial u_3}du_3$
$= h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3$

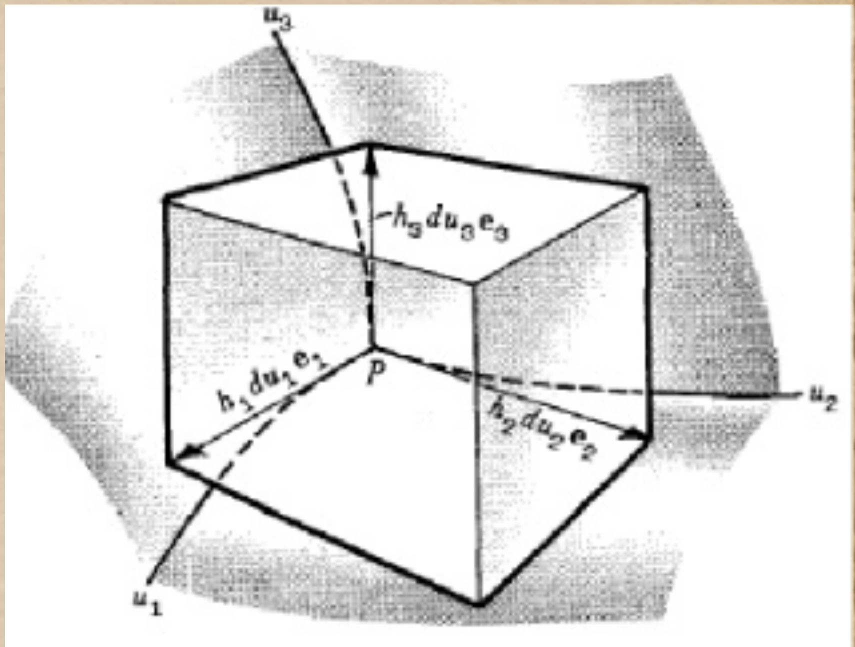
For orthogonal coordinate: $ds^2 = \vec{dr}.\vec{dr} = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$

Gradient, Divergence, Curl in Curvilinear coordinates

$\vec{da} = h_1 h_2 du_1 du_2 (\hat{e}_1 \times \hat{e}_2)$

For orthogonal coordinates: $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$

Hence, $\vec{da} = h_1 h_2 du_1 du_2 \hat{e}_3$



Volume element: $d\tau = h_1 h_2 h_3 du_1 du_2 du_3$

$$\vec{\nabla}\phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial u_3}\hat{e}_3$$

$$\vec{\nabla}.\vec{V} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 V_1)}{\partial u_1} + \frac{\partial(h_3 h_1 V_2)}{\partial u_2} + \frac{\partial(h_1 h_2 V_3)}{\partial u_3}\right)$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

For orthogonal coordinates

$$\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial u_3}\right)\right]$$

Spherical Polar Coordinates

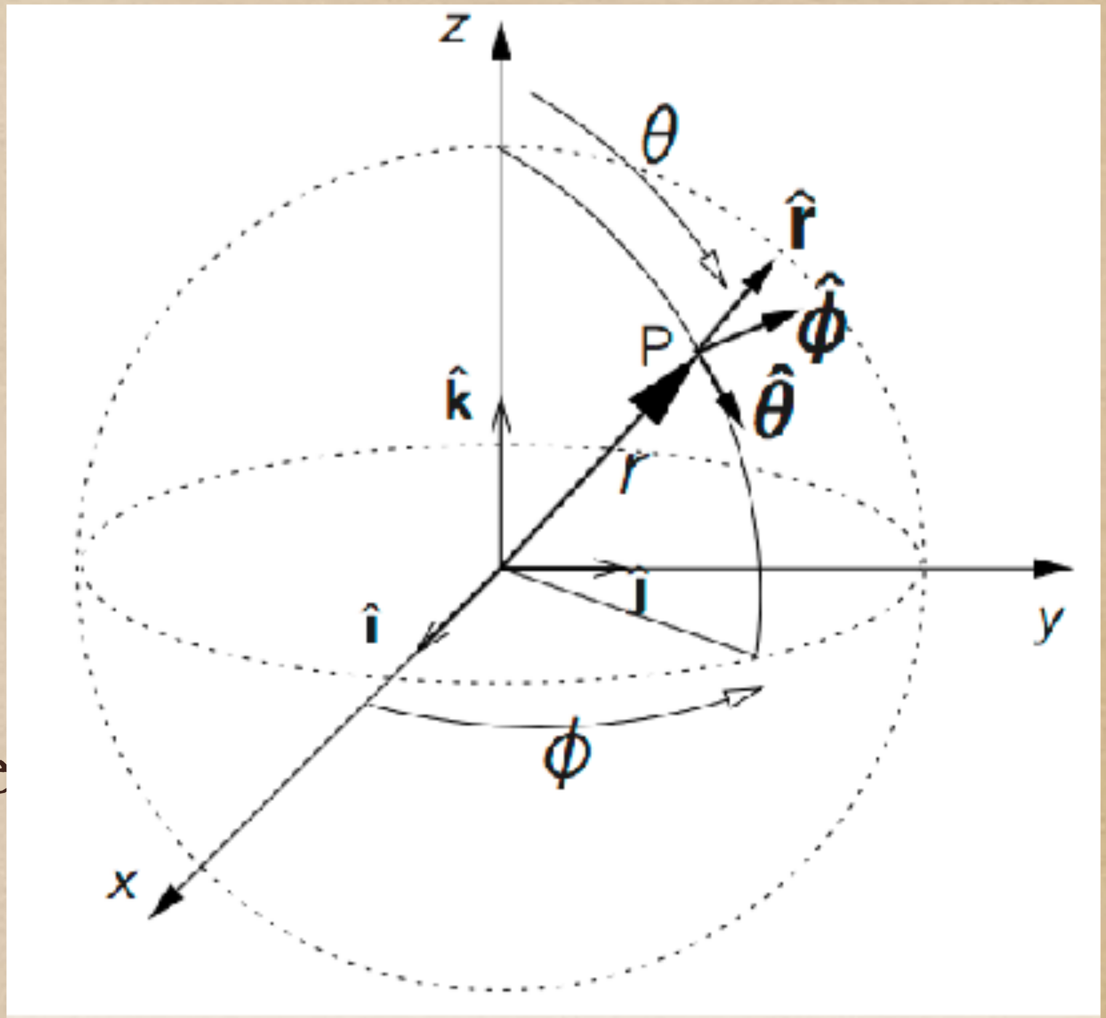
Radial coordinate $r \geq 0$

Polar Angle $0 \leq \theta \leq \pi$

Azimuthal angle $0 \leq \phi \leq 2\pi$

$$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$$

Note: Unit vectors always point in the direction of increase of the corresponding coordinates



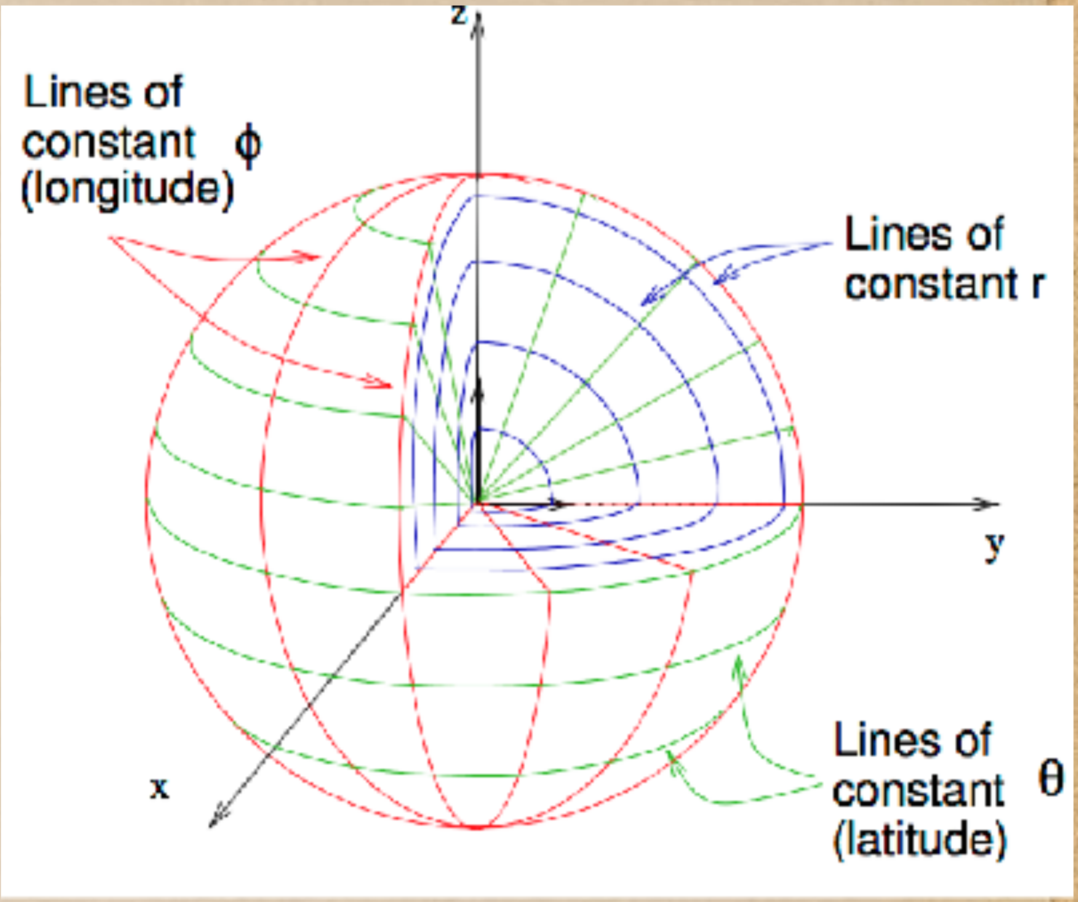
$$x = r\sin\theta\cos\phi, \;\; y = r\sin\theta\sin\phi, \;\; z = r\cos\theta,$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right) \quad r = \sqrt{(x^2+y^2+z^2)}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

Spherical Polar coordinates: unit vectors

Unit vectors change directions from point to point unlike cartesian unit vectors



In cartesian coordinates

$$\vec{r}(x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$$

In spherical polar coordinates: $\vec{r} = \vec{r}(r, \theta, \phi)$

Unit vector along r is unit tangent along r curve:

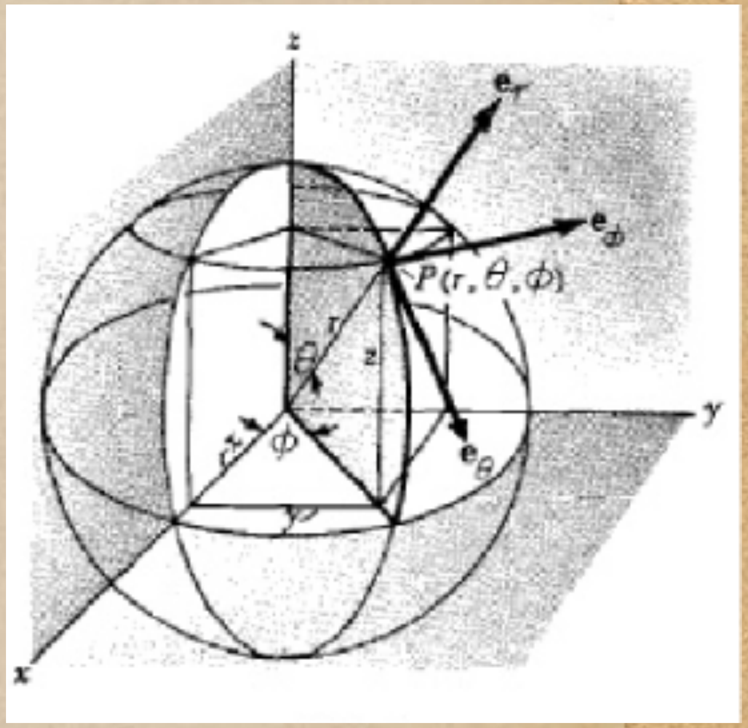
$$\hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{|\frac{\partial \vec{r}}{\partial r}|}$$

Similarly

$$\hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{|\frac{\partial \vec{r}}{\partial \theta}|} \qquad \hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{|\frac{\partial \vec{r}}{\partial \phi}|}$$

Unit vectors in Spherical polar coordinates

Using the transformations, position vector



$$\vec{r} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z}$$

$$\hat{r} = \frac{\frac{\partial\vec{r}}{\partial r}}{|\frac{\partial\vec{r}}{\partial r}|} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$

$$\hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{|\frac{\partial\vec{r}}{\partial\theta}|} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$

$$\longrightarrow \quad \frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$$

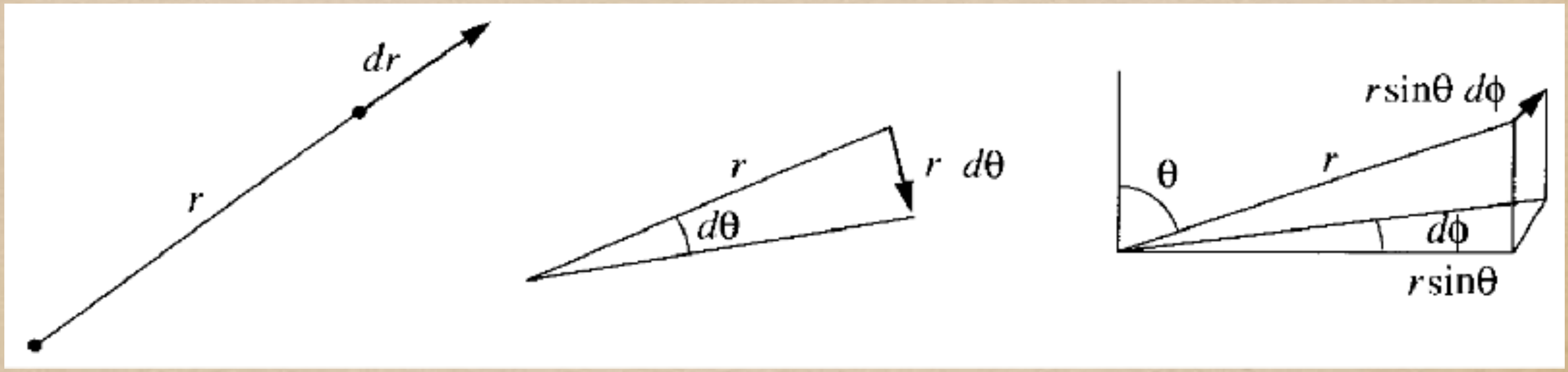
Unit vectors have non-zero derivatives unlike Cartesian coordinates

$$\hat{\phi} = \frac{\frac{\partial\vec{r}}{\partial\phi}}{|\frac{\partial\vec{r}}{\partial\phi}|} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

Line element in spherical polar coordinates

Line element in Cartesian coordinate:

$$\vec{dl} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$



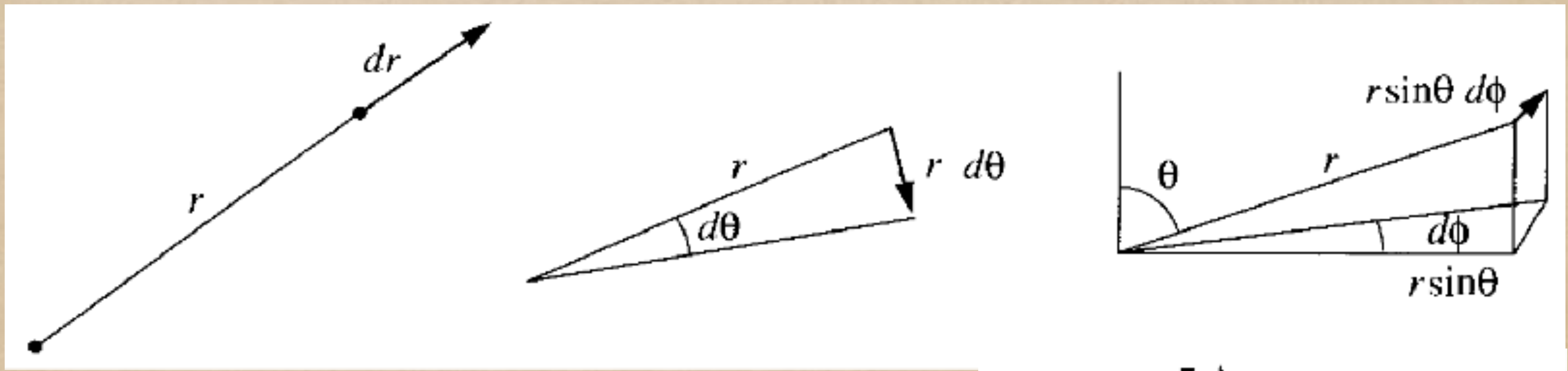
$$dl_r = dr; \;\; dl_\theta = rd\theta; \;\; dl_\phi = r\sin\theta d\phi$$

$$\vec{dl} = dl_r\hat{r} + dl_\theta\hat{\theta} + dl_\phi\hat{\phi}$$
$$= dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$$

$$\boxed{\begin{aligned}\vec{dl} &= h_r dr\hat{r} + h_\theta d\theta\hat{\theta} + h_\phi d\phi\hat{\phi} \\ h_r &= 1, \;\; h_\theta = r, \;\; h_\phi = r\sin\theta\end{aligned}}$$

This plays the role for line integral in this coordinate as played in cartesian coordinate

Surface element in spherical polar coordinates

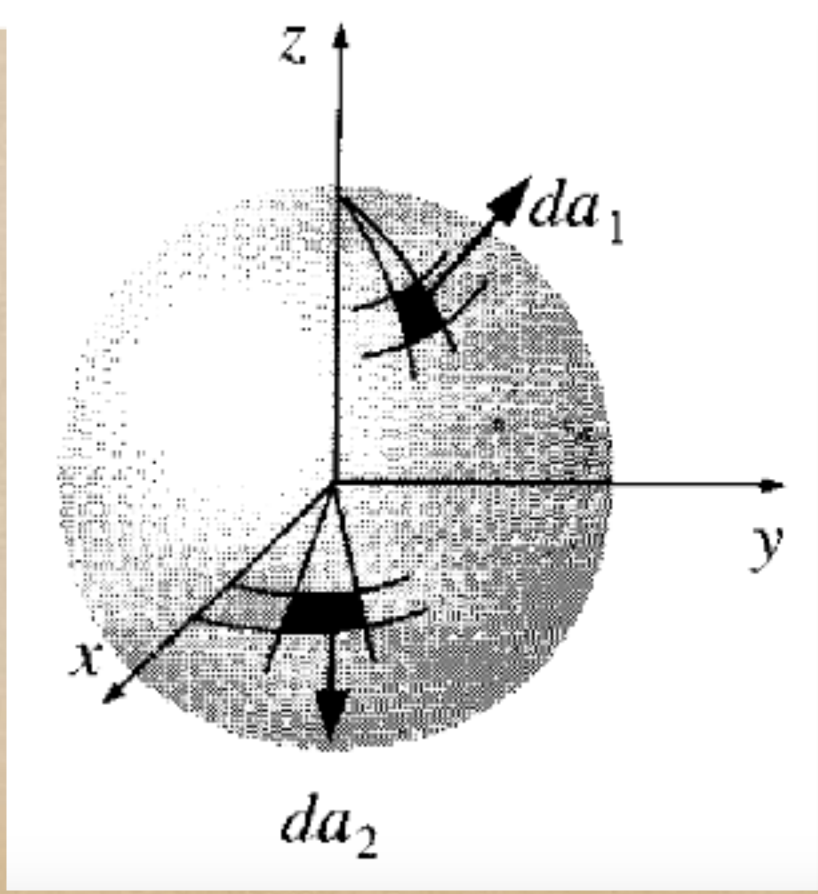


For r= constant surface

$$\vec{da_1} = dl_\theta dl_\phi \hat{r} = r^2 \sin\theta d\theta d\phi \hat{r}$$

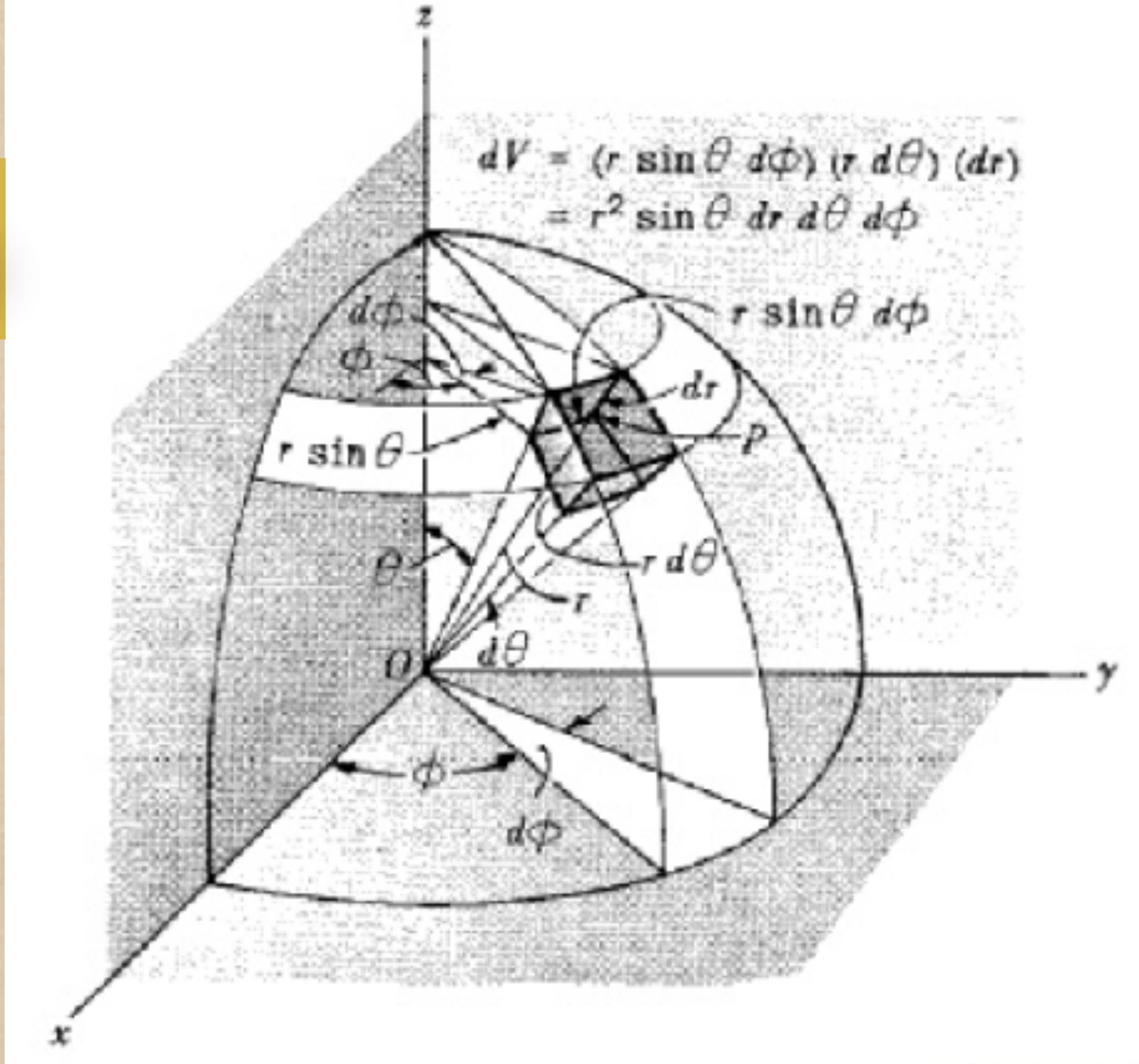
For θ = constant surface

$$\vec{da_2} = dl_r dl_\phi \hat{\theta} = r \sin\theta dr d\phi \hat{\theta}$$



Volume Element in Spherical polar coordinates

$$d\tau = dl_r dl_\theta dl_\phi = r^2 \sin\theta dr d\theta d\phi$$



Ex: Volume of a sphere of radius R

$$\int_{\text{Vol}} d\tau = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta dr d\theta d\phi$$

$$= \frac{4}{3}\pi R^3$$

Gradient, Divergence and Curl in Spherical polar coordinates

$u_1 \to r;\ u_2 \to \theta; u_3 \to \phi$

$\hat{e}_1 \to \hat{r};\ \hat{e}_2 \to \hat{\theta};\ \hat{e}_3 \to \hat{\phi}$

$h_1 = |\partial\vec{r}/\partial r| = 1;\ h_2 = |\partial\vec{r}/\partial\theta| = r;\ h_3 = |\partial\vec{r}/\partial\phi| = r\sin\theta$

$$\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\phi}$$

$$\vec{\nabla}.\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta V_\phi) - \frac{\partial V_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial}{\partial r}(rV_\phi)\right]\hat{\theta}$$
$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right]\hat{\phi}$$

$$\nabla^2T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

Example of a surface integral

Q Evaluate $\int_S \mathbf{a} \cdot d\mathbf{S}$, where $\mathbf{a} = z^3\hat{\mathbf{k}}$ and S is the sphere of radius A centred on the origin.

A On the surface of the sphere:

$$\mathbf{a} = A^3 cos^3\theta \hat{\mathbf{k}} \,, \qquad d\mathbf{S} = A^2 \sin\theta \; d\theta \; d\phi \hat{\mathbf{r}}$$

Hence

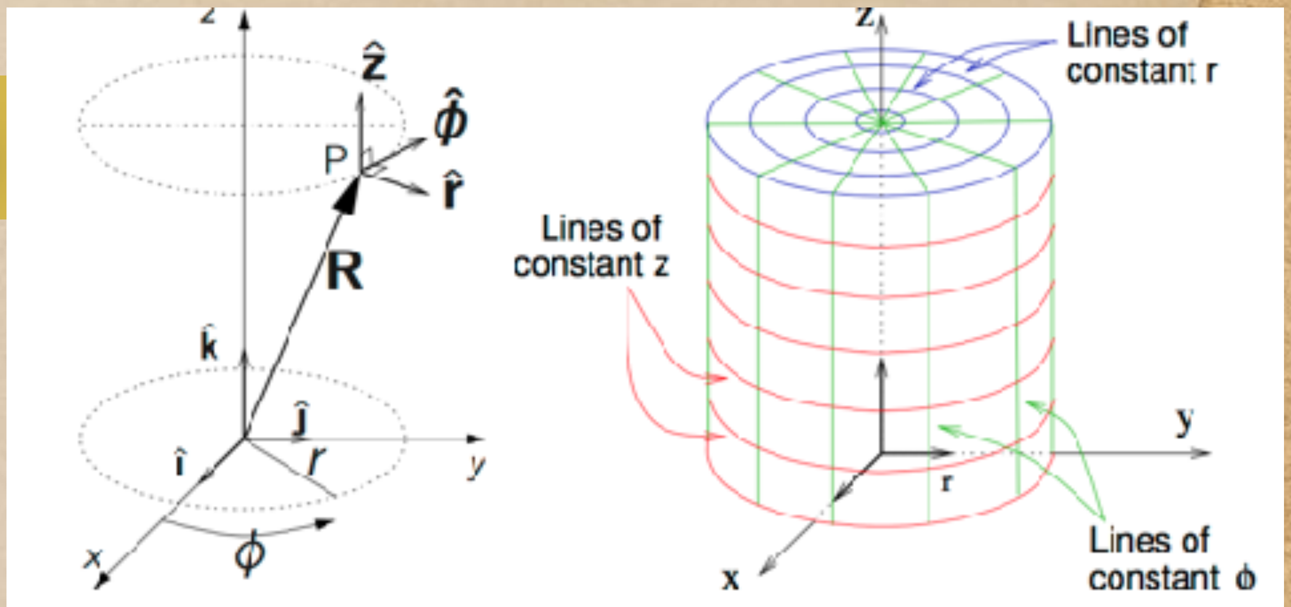
$$\begin{aligned}
\int_S \mathbf{a} \cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} A^3 cos^3\theta \; A^2 \sin\theta \; [\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}] \; d\theta d\phi \\
&\hphantom{=} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
&= A^5 \int_0^{2\pi} d\phi \; \int_0^{\pi} cos^3\theta \sin\theta [\cos\theta] \; d\theta \\
&= 2\pi A^5 \frac{1}{5} \left[-\cos^5\theta \right]_0^{\pi} \quad = \quad \frac{4\pi A^5}{5}
\end{aligned}$$

Cylindrical polar coordinates

$$x = s\cos\phi,\ \ y = s\sin\phi,\ \ z = z$$

$$s \geq 0,\ \ 0 \leq \phi \leq 2\pi,\ \ -\infty \leq z \leq \infty$$

$$\vec{A} = A_s\hat{s} + A_\phi\hat{\phi} + A_z\hat{z}$$



(a) Quantities
(b) Iso lines

Unit vectors in cylindrical polar coordinate

$$\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y}$$
$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$
$$\hat{z} = \hat{z}$$

$$h_s = \left|\frac{\partial\vec{r}}{\partial s}\right| = 1$$
$$h_\phi = \left|\frac{\partial\vec{r}}{\partial \phi}\right| = s$$
$$h_z = \left|\frac{\partial\vec{r}}{\partial z}\right| = 1$$

$$\begin{aligned} d\vec{r} &= h_s ds\hat{s} + h_\phi d\phi\hat{\phi} + h_z dz\hat{z} \\ &= ds\hat{s} + s d\phi\hat{\phi} + dz\hat{z} \end{aligned}$$

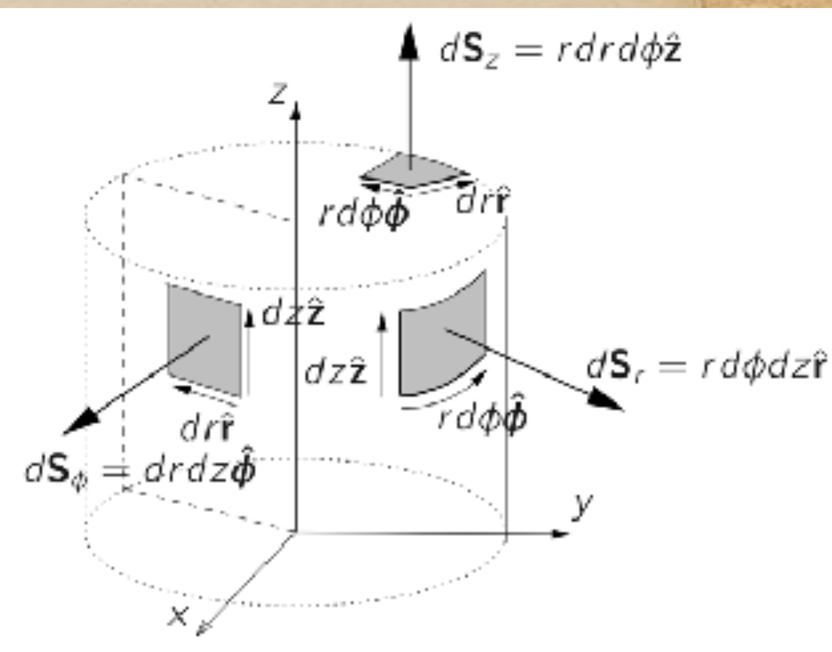
Line, surface and volume elements in cylindrical polar coordinates

Line element: $d\mathbf{R} = dr\,\hat{\mathbf{r}} + r d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$

Surface of constant r $d\mathbf{S}_r = h_\phi h_z d\phi dz(\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}) = r d\phi dz \hat{\mathbf{r}}$

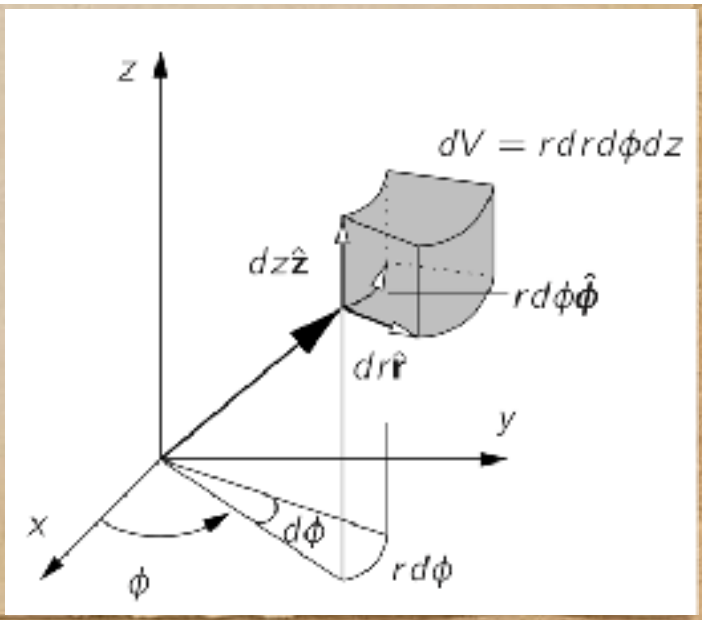
Surface of constant z $d\mathbf{S}_z = h_r h_\phi dr d\phi(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}}) = r dr d\phi \hat{\mathbf{z}}$

Surface of constant phi $d\mathbf{S}_\phi = h_z h_r dz dr(\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = dz dr \hat{\boldsymbol{\phi}}$



Volume element

$$dV = h_r dr \hat{\mathbf{r}} \cdot (h_\phi d\phi \hat{\boldsymbol{\phi}} \times h_z dz \hat{\mathbf{z}}) = r dr\, d\phi dz\,.$$



Gradient, divergence, curl in ~~spherical~~ polar coordinates

$$u_1 \rightarrow s,\ u_2 \rightarrow \phi,\ u_3 \rightarrow z$$

$$h_s = \left|\frac{\partial \vec{r}}{\partial s}\right| = 1,\ h_\phi = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = s,\ h_z = \left|\frac{\partial \vec{r}}{\partial z}\right| = 1$$

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}. \quad \nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}.$$

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}.$$

In summary.....

Curvilinear coordinates are often more useful than cartesian coordinates considering the symmetry of a given problem.

Spherical polar and cylindrical polar coordinates are two most useful examples of them.

Gradient, divergence and curl formulae gets mapped by the transformation with respect to Cartesian coordinates.

Line, surface and volume elements gets modified similarly.