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1. (a) Show that the electric field of a 'pure dipole' can be written as

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\vec{p}.\hat{r})\hat{r} - \vec{p}]$$

Note that this form has the advantage of not committing to a particular coordinate system.

(b) Find the force and torque on a dipole in the field of a point charge. Let the charge q be at the origin and the dipole $\vec{p} = p_0(\sin\zeta_0\hat{x} + \cos\zeta_0\hat{z})$ be at the point $(0, 0, z_0)$. Also find the force on q due to the dipole and verify Newton's third law.

Solution:

(a) Given the expression of the electric field in the problem, we will show that it reduces to our familiar form of the electric field due to a dipole with dipole moment $\vec{p}$ in spherical polar coordinate. We can resolve the dipole moment in spherical polar coordinate as follows:

$$\vec{p} = (\vec{p}.\hat{r})\hat{r} + (\vec{p}.\hat{\theta})\hat{\theta} = p\cos\theta\hat{r} - p\sin\theta\hat{\theta}.$$

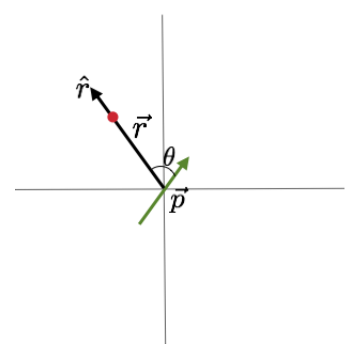

Figure 1

Hence

$$\begin{aligned}[3(\vec{p}.\hat{r})\hat{r} - \vec{p}] &= 3p\cos\theta\hat{r} - p\cos\theta\hat{r} + p\sin\theta\hat{\theta} \\ &= 2p\cos\theta\hat{r} + p\sin\theta\hat{\theta}\end{aligned}$$

This reduces the electric field in spherical polar coordinate as

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}(2p\cos\theta\hat{r} + p\sin\theta\hat{\theta}),$$

which we have already derived in the class.

(b) Given $\vec{p} = p_0(\sin\zeta_0\hat{x} + \cos\zeta_0\hat{z})$, we first need to calculate the electric field due to the dipole in order to find the force acting on the point charge q at origin. Following above formula in part (a), we can write

$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{1}{z_0^3}(3(\vec{p}.\hat{z})\hat{z} - \vec{p}) \\ &= \frac{1}{4\pi\epsilon_0}\frac{p_0}{z_0^3}(-\sin\zeta_0\hat{x} + 2\cos\zeta_0\hat{z})\end{aligned}$$

Hence the force on the charge q is

$$\vec{F} = q\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{p_0}{z_0^3}(-\sin\zeta_0\hat{x} + 2\cos\zeta_0\hat{z}).$$

Now we will calculate the force on the dipole due to the charge q. The force on a dipole in non uniform electric field was derived in the class as

$$\vec{F} = (\vec{p}.\vec{\nabla})\vec{E}.$$

Now,

$$\vec{p}.\vec{\nabla} = p_0\sin\zeta_0\frac{\partial}{\partial x} + p_0\cos\zeta_0\frac{\partial}{\partial z}.$$

We know that the electric field at a point (x, y, z) due to the charge q is given by

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{x} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{y} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{z}\right]$$

Hence the force on the dipole situated at $(0, 0, z_0)$ is

$$\begin{aligned}\vec{F}(0,0,z_0) &= \left(p_0\sin\zeta_0\frac{\partial}{\partial x} + p_0\cos\zeta_0\frac{\partial}{\partial z}\right)\vec{E} \\ &= \frac{p_0 q}{4\pi\epsilon_0}\left[\sin\zeta_0\hat{x} - 2\cos\zeta_0\hat{z}\right]\end{aligned}$$

which verifies that the forces are equal and opposite.

The torque on a dipole can once again be calculated by

$$\vec{N}(0,0,z_0) = \vec{p}\times\vec{E} = -\frac{p_0 q\sin\zeta_0}{4\pi\epsilon_0 z_0^2}\hat{y}$$

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2. Energy of a dipole:
   (a) Show that the energy of a dipole with dipole moment $\vec{p}$ in an electric field $\vec{E}$

   $$U = -\vec{p}.\vec{E}$$

   (b) Show that the interaction between two dipoles with dipole moments $\vec{p}_1$ and $\vec{p}_2$ separated by distance $\vec{r}$ is given by:

   $$U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[\vec{p}_1.\vec{p}_2 - 3(\vec{p}_1.\hat{r})(\vec{p}_2.\hat{r})]$$

Solution:

(a) We calculate the energy of a dipole in a electric field $\vec{E}$ by bringing the dipole from infinity and placing it such that $-q$ charge is at $\vec{r}$ and $+q$ at $\vec{r}+\vec{s}$, where s is the length of the dipole and $s << r$. Hence the work done is,

$$U = -qV(r) + qV(r+s)$$

As $s << r$, we can Taylor expand

$$V(r+s) = V(r) + \vec{s}.\vec{\nabla}V(r) + \cdots$$

Putting back the above expression in W,, we get

$$U = -qV(r) + qV(r) + q\vec{s}.\vec{\nabla}V(r) + \cdots = \vec{p}.\vec{\nabla}V(r) = -\vec{p}.\vec{E},$$

where $\vec{p} = q\vec{s}$ is the dipole moment.

We can also evaluate the energy of the dipole by the work done in rotating it:

$$U = \int_{\pi/2}^{\theta} N d\theta' = \int_{\pi/2}^{\theta} pE\sin\theta' d\theta' = -pE\cos\theta = -\vec{p}.\vec{E}$$

Here we have judiciously chosen the reference point to be at $\theta = \pi/2$.

(b) We need to calculate the field created by one dipole on the other. The field due to p_2 is

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}(3(\vec{p}_2.\hat{r})\hat{r} - \vec{p}_2).$$

Using the result of part (a) we can calculate the energy as

$$\begin{aligned} U = -\vec{p}_1.\vec{E}_2 &= -\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\vec{p}_1.(3(\vec{p}_2.\hat{r})\hat{r} - \vec{p}_2) \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}(-3(\vec{p}_1.\hat{r})(\vec{p}_2.\hat{r}) + \vec{p}_1.\vec{p}_2) \end{aligned}$$

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3. Suppose we have limited dielectric material of dielectric constant ϵ_r to only half fill a parallel plate capacitor. Compare the two cases depicted in Figure (a) and (b) commenting on which case the capacitance is more. Assuming a potential difference V between the plates, find $\vec{E}, \vec{D}, \vec{P}$ in each region and free and bound charges on all surfaces in both the cases.

Solution:

With no dielectric $C_0 = \epsilon_0 \frac{A}{d}$. In configuration (a), with $+\sigma$ on upper plate and $-\sigma$ on lower, $D = \sigma$ between the plates (derived in class as follows $\oint \vec{D}.d\vec{a} = Q_{\text{f}_{\text{enc}}} \implies DA =$

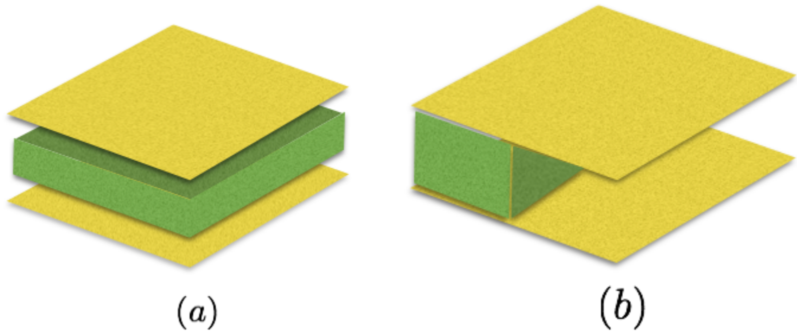

Figure 2

$\sigma A \implies D = \sigma$). Note that $\vec{D} = \vec{E} = \vec{P} = 0$ in conductors. Therefore $E_1 = \sigma/\epsilon_0$ in air and $E_2 = \sigma/\epsilon$ in dielectric. Hence the potential difference $V = E_1\frac{d}{2} + E_2\frac{d}{2} = \frac{\sigma}{\epsilon_0}\frac{d}{2} + \frac{\sigma}{\epsilon}\frac{d}{2} = \frac{Qd}{2\epsilon_0 A}(1 + \frac{\epsilon_0}{\epsilon}) = \frac{Qd}{2\epsilon_0 A}(1 + \frac{1}{\epsilon_r})$. Hence the capacitance:

$$\begin{aligned}
C_{(a)} &= \frac{Q}{V} = 2\epsilon_0\frac{A}{d}\left(\frac{1}{(1+\frac{1}{\epsilon_r})}\right) \\
\therefore \frac{Q}{A} &= \sigma = \sigma_f = \frac{2\epsilon_r\epsilon_0}{1+\epsilon_r}\frac{V}{d} \\
\text{and } \frac{C_{(a)}}{C_0} &= \frac{2\epsilon_r}{1+\epsilon_r}.
\end{aligned}$$

Let us choose the direction perpendicular to the plates as the z direction. Hence, **In air:** $\vec{E} = \frac{\sigma}{\epsilon_0} = \frac{2\epsilon_r}{1+\epsilon_r}\frac{V}{d}\hat{z}$, $\vec{D} = \frac{2\epsilon_r}{1+\epsilon_r}\frac{\epsilon_0 V}{d}\hat{z}$, $\vec{P} = 0$, $\sigma_b = 0$ (it has to be zero in air), $\sigma_f = \frac{2\epsilon_r\epsilon_0}{1+\epsilon_r}\frac{V}{d}$ on the top plate.

In dielectric: $\vec{E} = \frac{\sigma}{\epsilon} = \frac{2}{1+\epsilon_r}\frac{V}{d}\hat{z}$, $\vec{D} = \frac{2\epsilon_r}{1+\epsilon_r}\frac{\epsilon_0 V}{d}\hat{z}$, $\vec{P} = \vec{D} - \epsilon_0\vec{E} = \frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\hat{z}$, $\sigma_b = \vec{P}.\hat{n} = -\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$ and there is no free charge on the dielectric surface.

In configuration (b), with potential difference V: $E = V/d$, so, $\sigma = \epsilon_0 E = \epsilon_0 V/d$ in air. For the part filled with dielectric $P = \epsilon_0\chi_e E = \epsilon_0\chi_e V/d$, so $\sigma_b = \vec{P}.\hat{n} = -\epsilon_0\chi_e V/d$ at the top surface of the dielectric. Note, the negative sign is due to the fact that the polarization is opposite to the direction of the unit normal vector. We can evaluate σ_f in presence of the dielectric as follows:

$$\begin{aligned}
\oint \vec{D}.d\vec{a} = Q_{\text{fenc}} &= \sigma_f A \implies D = \sigma_f \\
\implies D &= \epsilon E = \sigma_f \\
\therefore \sigma_f &= \epsilon E = \epsilon\frac{V}{d} = \epsilon_0\epsilon_r\frac{V}{d}
\end{aligned}$$

The last line uses the fact that the potential difference between the plates in air and in dielectric is the same. Hence

$$\begin{aligned} C_{(b)} &= \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\epsilon_0\frac{V}{d} + \epsilon_0\epsilon_r\frac{V}{d}\right) \\ &= \frac{A\epsilon_0}{d}\left(\frac{1+\epsilon_r}{2}\right) \\ \therefore \frac{C_{(b)}}{C_0} &= \frac{1+\epsilon_r}{2} \end{aligned}$$

Note that

$$\begin{aligned} \frac{C_{(b)}}{C_0} - \frac{C_{(a)}}{C_0} &= \frac{1+\epsilon_r}{2} - \frac{2\epsilon_r}{1+\epsilon_r} = \frac{(1-\epsilon_r)^2}{2(1+\epsilon_r)} > 0 \\ &\implies C_{(b)} > C_{(a)} \end{aligned}$$

For this case, **in air** : $\vec{E} = \frac{V}{d}\hat{z}$, $\vec{D} = \frac{\epsilon_0 V}{d}\hat{z}$, $\vec{P} = 0$, $\sigma_b = 0$, $\sigma_f = \frac{\epsilon_0 V}{d}$.

in dielectric : $\vec{E} = \frac{V}{d}\hat{z}$, $\vec{D} = \frac{\epsilon_0\epsilon_r V}{d}\hat{z}$, $\vec{P} = (\epsilon_r - 1)\frac{\epsilon_0 V}{d}\hat{z}$, $\sigma_b = -(\epsilon_r - 1)\frac{\epsilon_0 V}{d}$, $\sigma_f = \frac{\epsilon_0\epsilon_r V}{d}$.

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4. Consider a conducting spherical shell with an inner radius a and outer radius c. Let the space between two surfaces be filled with two different dielectric materials of dielectric constant κ_1 between a and b, and κ_2 between b and c. Determine the capacitance of the system.

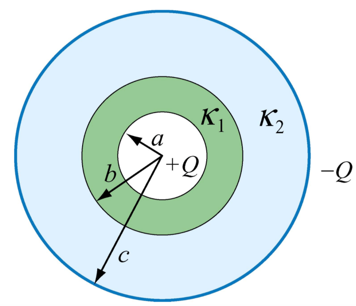

Figure 3

This problem may be omitted. Supply final answer only.

Solution:

Electric displacement can be found using Gauss's law within $a < r < c$

$$\oint \vec{D}.d\vec{a} = Q_{\text{enc}} \implies \vec{D} = \frac{1}{4\pi r^2}Q\hat{r}$$

Everywhere else it is zero. Let $\epsilon_1 = \epsilon_0\kappa_1,\ \epsilon_2 = \epsilon_0\kappa_2$. Therefore,

$$\begin{aligned} \vec{E}_{a<r<b} &= \frac{1}{4\pi\epsilon_1}\frac{Q}{r^2}\hat{r} \implies V_1 = -\frac{Q}{4\pi\epsilon_1}\int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_1}(\frac{1}{a} - \frac{1}{b}) \\ \vec{E}_{b<r<c} &= \frac{1}{4\pi\epsilon_2}\frac{Q}{r^2}\hat{r} \implies V_2 = -\frac{Q}{4\pi\epsilon_2}\int_b^c \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_2}(\frac{1}{b} - \frac{1}{c}) \end{aligned}$$

Hence,

$$\begin{aligned}\Delta V &= V_1 + V_2 = \frac{Q}{4\pi}\left[\frac{1}{\epsilon_1}(\frac{1}{a}-\frac{1}{b}) + \frac{1}{\epsilon_2}(\frac{1}{b}-\frac{1}{c})\right] \\ \therefore C &= \frac{Q}{\Delta V} \implies C = \frac{4\pi\epsilon_0\kappa_1\kappa_2 abc}{\kappa_2 c(b-a)+\kappa_1 a(c-b)}.\end{aligned}$$

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5. Two long coaxial cylindrical metal tubes (inner radius a and outer radius b) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at a potential V and the outer one is grounded. To what height does the oil rise in the space between the tubes ?

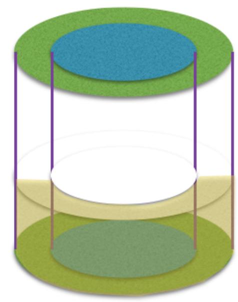

Figure 4

Solution:

Let σ be the surface charge density on the inner cylinder in the air part. Hence, electric field in air:

$$\oint \vec{E}.d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \implies E2\pi s\ell = \frac{\sigma 2\pi a\ell}{\epsilon_0} \implies E = \frac{\sigma a}{\epsilon_0 s}$$

Therefore the potential at the surface of the inner cylinder

$$V_a = -\int_b^a \vec{E}.d\vec{l} = \frac{\sigma a}{\epsilon_0}\ln\left(\frac{b}{a}\right) = V \text{ (given)}$$

Here we have chose the reference point at b as it is grounded.

Let, σ' be the free surface charge density and hence the displacement field in dielectric

$$\oint \vec{D}.d\vec{a} = Q_{\text{enc}} \implies D = \frac{\sigma' a}{s} \implies E = \frac{\sigma' a}{\epsilon s}$$

Therefore the potential at the surface of the inner cylinder

$$V_a = -\int_b^a \vec{E}.d\vec{l} = \frac{\sigma' a}{\epsilon}\ln\left(\frac{b}{a}\right) = V \text{ (given)}$$

Hence, equating the two expressions, we get

$$\frac{\sigma}{\epsilon_0} = \frac{\sigma'}{\epsilon} \implies \sigma' = \epsilon_r \sigma$$

Let the length of the cylinder be L and the oil rises to a height h. Hence, the total charge enclosed

$$Q = [\sigma' h + \sigma(L-h)]2\pi a = 2\pi a\sigma(\chi_e h + L)$$

Hence, the capacitance is

$$C = \frac{Q}{V} = 2\pi\epsilon_0 \frac{\chi_e h + L}{\ln\left(\frac{b}{a}\right)} \implies \frac{dC}{dh} = 2\pi\epsilon_0 \frac{\chi_e}{\ln\left(\frac{b}{a}\right)}.$$

The net upward electrostatic force, therefore, is $F = \frac{1}{2}V^2\frac{dC}{dh} = \frac{1}{2}V^2 2\pi\epsilon_0 \frac{\chi_e}{\ln\left(\frac{b}{a}\right)}$ (derived in class). The gravitational force acting downwards is $F = mg = \rho\pi(b^2 - a^2)gh$, which balances the electrostatic force.

$$\therefore h = \frac{\epsilon_0 \chi_e V^2}{\rho(b^2 - a^2)g\ln(b/a)}.$$

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6. A spherical conductor of radius a, carries a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration.

Solution:

Inside the conductor ($r < a$), $\vec{D} = 0$. Let us calculate electric displacement in $r > a$ using Gauss's formula:

$$\oint \vec{D}.d\vec{a} = Q_{f_{\text{enc}}} \implies D4\pi r^2 = Q \implies \vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

Therefore electric field is

$$\vec{E} = \begin{cases} 0 & \text{for} \quad r < a \\ \frac{Q}{4\pi\epsilon r^2}\hat{r} & \text{for} \quad a < r < b \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{for} \quad r > b \end{cases}$$

Hence, energy of the configuration

$$\begin{aligned} W &= \frac{1}{2}\int \vec{D}.\vec{E}d\tau = \frac{1}{2}\int \vec{D}.\vec{E}4\pi r^2 dr \\ &= \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left[\frac{1}{\epsilon}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2 dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{r^2}\frac{1}{r^2}r^2 dr\right] \\ &= \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a} + \frac{\chi_e}{b}\right). \end{aligned}$$
