Physics II: Electromagnetism (PH102)

Lecture 13

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Potential due to polarised object

Potential due to polarised volume with polarisation P

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\vec{P}(\vec{r}\,').\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' \qquad \text{Using, } \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\implies V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}} \vec{\nabla}'.\left(\frac{\vec{P}(\vec{r}\,')}{\mathscr{r}}\right) d\tau' - \int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\vec{\nabla}'.\vec{P}(\vec{r}\,'))d\tau'\right]$$

$$= \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b(\vec{r}\,')}{\mathscr{r}} da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b(\vec{r}\,')}{\mathscr{r}} d\tau'$$

Bound surface charge density $\boxed{\sigma_b = \vec{P}(\vec{r}).\hat{n}}$

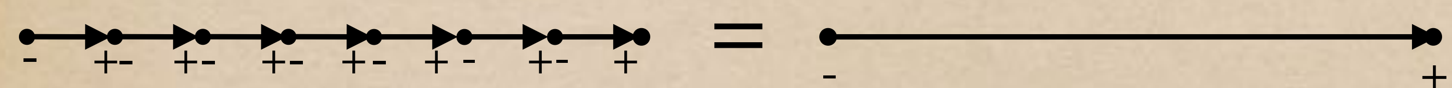
Bound volume charge density $\boxed{\rho_b = -\vec{\nabla}.P(\vec{r})}$

Potential of a polarised object can be identified as created by bound surface charge density and bound volume charge density

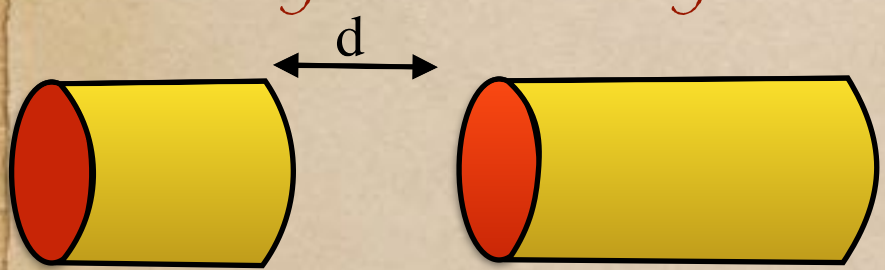
How to see bound charges in action ?

The field of a polarized object is identical to the field that would be produced by a certain distribution of 'bound charges', σ_b and ρ_b

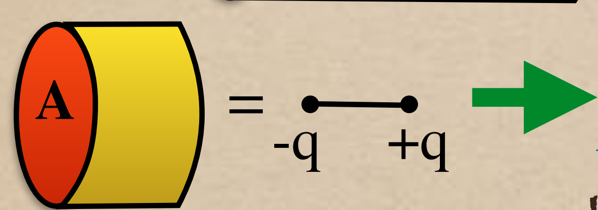
genuine accumulation of charges



- Long string of dipoles, essentially cancel the head and tail of the neighbour -> <u>bound charges</u>
- Why bound ? They can't be removed



Dielectric with polarisation $\vec{P}$



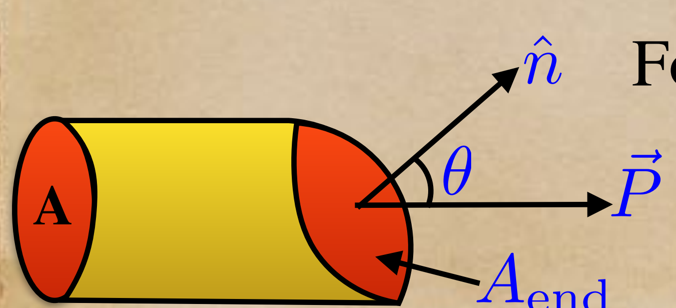
This chunk of dielectric has the dipole moment $P(Ad)$, where A is the area of cross section and d is the length of the chunk.

In terms of charges q, dipole moment $p = qd$

Hence, bound charges q=PA

If the ends are cut perpendicularly, then the surface charge density is: $\sigma_b = \dfrac{q}{A} = P$

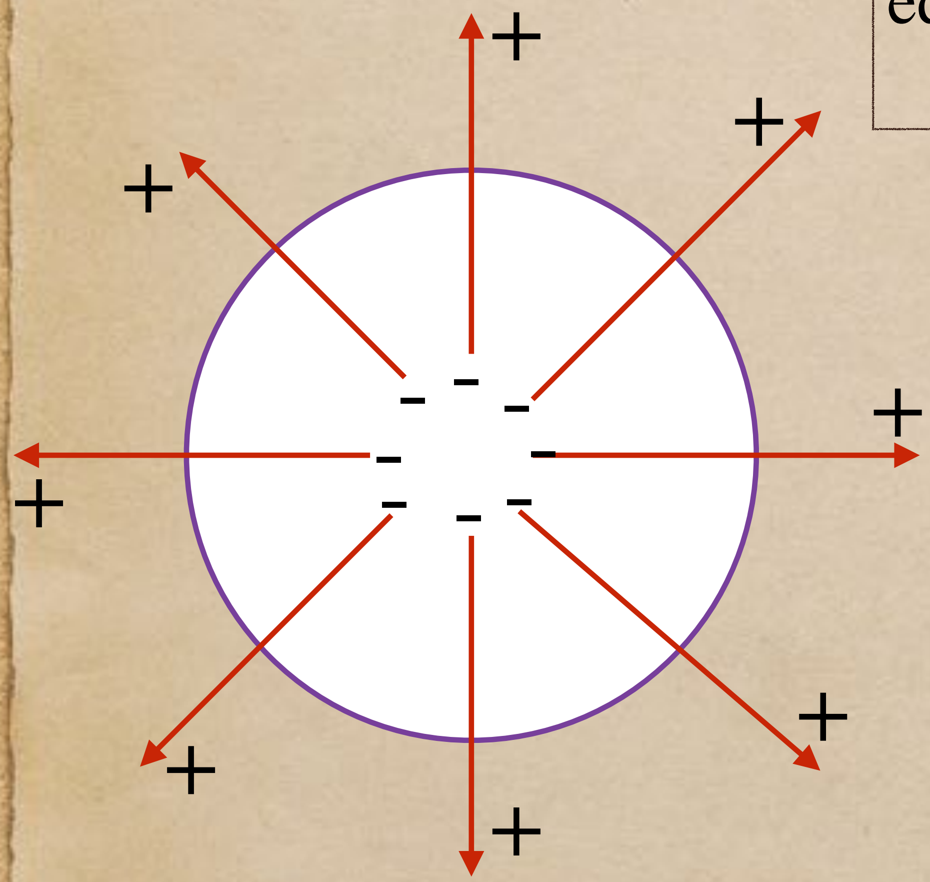
For the oblique cut, the charge is still the same but A is $A_{\rm end}\cos\theta$



$$\sigma_b = \frac{q}{A_{\rm end}} = \frac{q}{A}\cos\theta = P\cos\theta = \vec{P}.\hat{n}$$

What about bound volume charge density ?

If the polarisation is non-uniform charges start accumulating in the volume of the dielectric as well -> Bound volume charge



The net bound charge $\int \rho_b d\tau$ in a given volume is equal and opposite to the amount that has been pushed out through the surface.

The latter is $\vec{P}.\hat{n}$ per unit area.

$$\int_{\mathcal{V}} \rho_b d\tau = -\oint_{\mathcal{S}} \vec{P}.d\vec{a} = -\int_{\mathcal{V}} (\vec{\nabla}.\vec{P}) d\tau.$$

Hence we have $\rho_b = -\vec{\nabla}.\vec{P}$

Example..

Find bound charges in a spherical dielectric of radius R, centred at the origin with polarisation $\vec{P}(\vec{r}) = k\vec{r}$ with K a constant. What is the net charge of the sphere ?

- Bound surface charge density is given by $\sigma_b = \vec{P}.\hat{n} = kR\hat{r}.\hat{r} = kR.$
- Total bound surface charge is $4\pi kR^3$.
- Bound volume charge density is $\rho_b = -\vec{\nabla}.\vec{P} = -3k.$
- Total bound volume charge is $\frac{4}{3}\pi R^3\rho_b = -4\pi kR^3$.
- Net charge in material is zero.

What will be the electric field both inside and outside this polarized sphere?

- for $r < R$: Enclosed charge is $\frac{4}{3}\pi r^3\rho_b$.
- Applying Gauss's law inside the sphere: $\vec{E} = \frac{\rho_b r}{3\epsilon_0}\hat{r} \implies \vec{E} = -\frac{k}{\epsilon_0}\vec{r}.$
- for $r > R$: Enclosed charge is zero. Therefore field outside will be zero.

Gauss's law for dielectric

• We just saw that effect of polarization is to produce accumulation of bound charge: $\rho_b = -\vec{\nabla}.\vec{P}$ within the dielectric and $\sigma_b = \vec{P}.\hat{n}$ on the surface.

• Field due to polarization of medium is just the field due to this bound charge.

• We also want to accommodate fields due to everything else (excluding the field due to polarization) into the picture.

• Call "this everything else" ρ_f: free charge density.

• Total charge density: $\rho = \rho_b + \rho_f$.

• Therefore the Gauss's law reads: $\epsilon_0 \vec{\nabla}.\vec{E} = \rho = \rho_b + \rho_f$

$$\implies \vec{\nabla}.(\epsilon_0 \vec{E} + \vec{P}) = \rho_f \implies \vec{\nabla}.\vec{D} = \rho_f$$

Gauss's law for dielectric

• Here $\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$: **Electric displacement**.

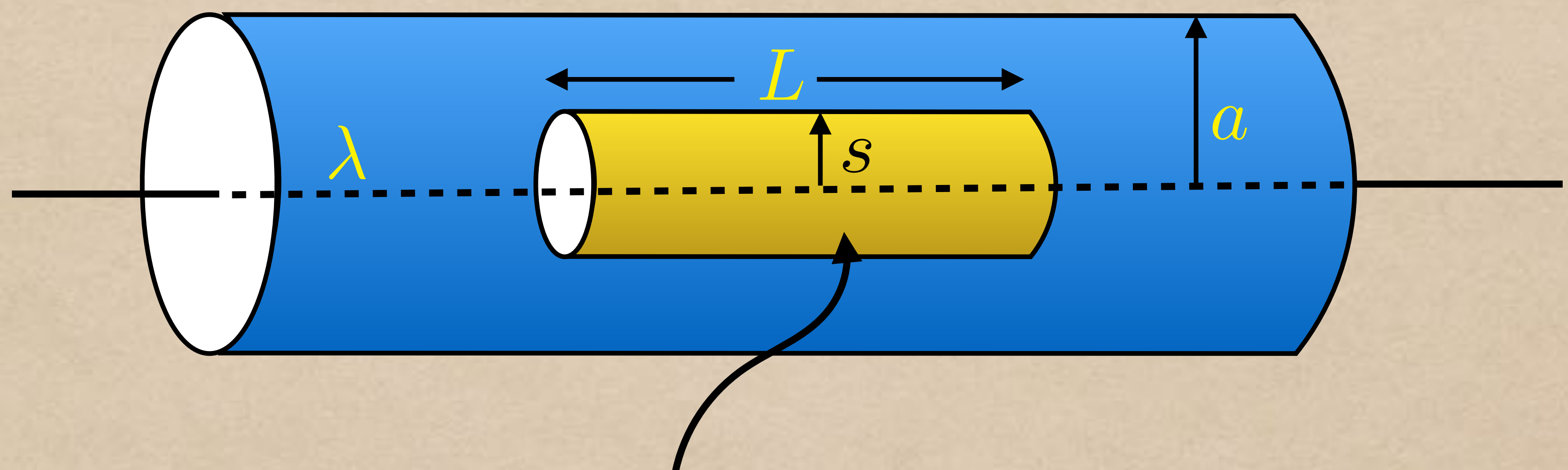
• Integral form: $\oint \vec{D}.d\vec{a} = Q_{\mathrm{f_{enc}}}$.

Total free charges enclosed in volume

Gauss's law in dielectrics involving displacement refers to the free charges only

Example...

A long straight wire, carrying uniform line charge density λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement.



Draw a cylindrical Gaussian surface of radius s and length L:

$$D(2\pi s L) = \lambda L$$

$$\boxed{\vec{D} = \frac{\lambda}{2\pi s}\hat{s}}$$

Note: for insulators we will use the Gauss's law involving displacement, not fields

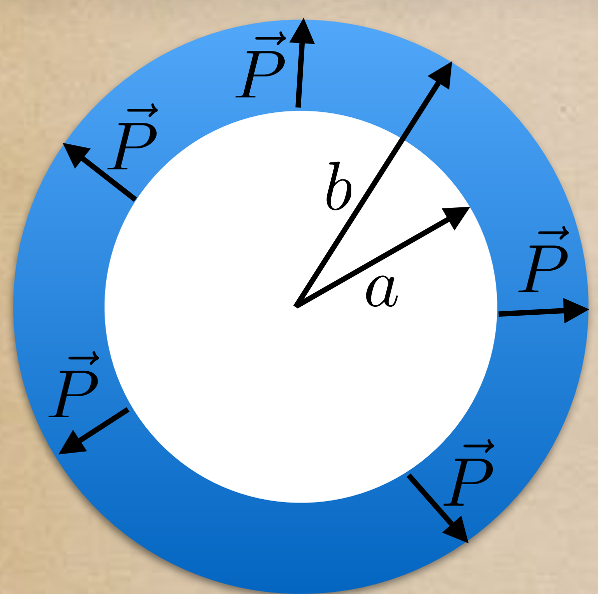
This formula is valid everywhere. Particularly note: outside the rubber, $\vec{P} = 0$:

$$\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s} \quad \text{for } s > a$$

Inside rubber, we do not know $\vec{E}$ since we do not know $\vec{P}$.

Example...

A thick spherical shell of inner radius a and outer radius b is made of dielectric material with a "frozen in" polarization $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$, where k is a constant and r is the distance from the centre. There is no free charge in the problem. Find the electric field in all three regions.



$\oint \vec{D}.d\vec{a} = Q_{\mathrm{f_{enc}}} = 0$, since there is no free charge.

$\Longrightarrow$ $\vec{D} = 0$ everywhere.

$\Longrightarrow$ $\vec{D} = \epsilon_0 \vec{E} + \vec{P} = 0 \Longrightarrow \vec{E} = -\frac{1}{\epsilon_0}\vec{P}$

In $a < r < b$, $\vec{E} = -\frac{k}{\epsilon_0 r}\hat{r}$

In $r < a$ and $r > b$, $\vec{E} = 0$ since $\vec{P} = 0$ there.

Bound charges

$$\boxed{\rho_b = -\vec{\nabla}.\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}}$$

$$\sigma_b = \vec{P}.\hat{n} = +\vec{P}.\hat{r} = \frac{k}{b} \quad (\text{at } r = b)$$

$$= -\vec{P}.\hat{r} = -\frac{k}{a} \quad (\text{at } r = a).$$

In region $a < r < b$

$Q_{\mathrm{enc}} = (-\frac{k}{a})4\pi a^2 + \int_a^r (-\frac{k}{\bar{r}^2})4\pi \bar{r}^2 d\bar{r} = -4\pi k r.$

Therefore: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\mathrm{enc}}}{r^2}\hat{r} = -(\frac{k}{\epsilon_0 r})\hat{r}$

In $r < a$ and $r > b$, $\vec{E} = 0$

D versus E !

• Note that $\vec{\nabla}.\vec{D} = \rho_f$ just looks like Gauss's law, only the total charge density ρ has been replaced by free charge density ρ_f.

• Here $\vec{D}$ is the electric displacement vector $\epsilon_0\vec{E} + \vec{P}$. But **do not** conclude that $\vec{D}$ is just like $\vec{E}$!

• In particular, there is no Coulomb's law for $\vec{D}$:

• Curl of the electric field is **always** zero! But curl of $\vec{D}$ is not always zero.

$$\vec{\nabla} \times \vec{D} = \epsilon_0(\vec{\nabla} \times \vec{E}) + (\vec{\nabla} \times P) = \vec{\nabla} \times \vec{P}$$

usually not zero !

We will talk more on the curl of polarisation in a moment !

• Because $\vec{\nabla} \times \vec{D} \neq 0$, moreover, $\vec{D}$ can not be expressed as gradient of a scalar - there is no potential for $\vec{D}$.

Boundary conditions

• The electrostatic boundary conditions can be represented in terms of $\vec{D}$. We already know:

$$\oint \vec{D}.d\vec{a} = Q_{\mathrm{f_{enc}}} \quad \text{and} \quad \vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$$

Talks about discontinuity in component perpendicular to an interface.

$$\boxed{D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f}$$

Talks about discontinuity in parallel component along the interface.

$$\boxed{D^{||}_{\text{above}} - D^{||}_{\text{below}} = P^{||}_{\text{above}} - P^{||}_{\text{below}}}$$

• In presence of dielectrics, these are sometimes more useful that the corresponding boundary conditions on $\vec{E}$:

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \quad \text{and} \quad E^{||}_{\text{above}} - E^{||}_{\text{below}} = 0$$

Linear Dielectric

What causes polarisation in a material ?

Electric field that lines up atomic or molecular dipoles

- For many substances, polarisation is linearly proportional to electric field -

  > **Linear Dielectric**

- For non-linear media, there may be a cubic term of the field

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

Electric susceptibility

- Susceptibility of a material depends on the microscopic structure
- dimensionless quantity

- Note: The electric field above is not only the external field, but also includes the contribution due to polarisation .
- Easier way to parametrise it, is to identify the electric displacement

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon \vec{E}$$

$$\epsilon = \epsilon_0 (1 + \chi_e)$$ → Permittivity of the material

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$ → Relative permittivity or Dielectric constant

Polarisation of crystal

Although, the material may be still linear dielectric, polarisation of some materials are different in different direction.

Recall, we had $\vec{P} = \epsilon_0 \chi_e \vec{E}$

Instead of polarisation being proportional to Electric field in any direction subject to a generic susceptibility, we will have :

$$\boxed{\begin{aligned} P_x &= \epsilon_0(\chi_{xx}E_x + \chi_{xy}E_y + \chi_{xz}E_z) \\ P_y &= \epsilon_0(\chi_{yx}E_x + \chi_{yy}E_y + \chi_{yz}E_z) \\ P_z &= \epsilon_0(\chi_{zx}E_x + \chi_{zy}E_y + \chi_{zz}E_z) \end{aligned}}$$

χ_{ij} forms Susceptibility tensor

Curl of Polarisation in linear dielectric

We might now say, that as polarisation and displacement is proportional to field in linear media, the curl of them should vanish $\vec{\nabla}\times\vec{P}=0, \quad \vec{\nabla}\times\vec{D}=0$?

If the medium is homogeneously filled with dielectric of one kind, then, indeed so, otherwise not. $\vec{\nabla}\times\vec{P}=0, \quad \vec{\nabla}\times\vec{D}=0 \quad \vec{\nabla}.\vec{D}=\rho_f$

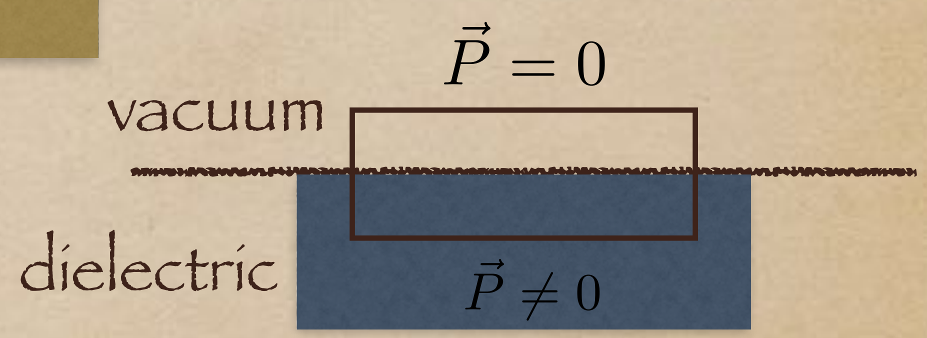
- So, D can be found out from the free charges as if the dielectric is not there

$$\vec{D}=\epsilon_0\vec{E}_{vac}$$

$$\boxed{\vec{E}=\frac{1}{\epsilon}\vec{D}=\frac{1}{\epsilon_r}\vec{E}_{vac}}$$

Hence, when the space is filled with homogeneous dielectric the field is just reduced by a factor of dielectric constant $\epsilon_r=\dfrac{\epsilon}{\epsilon_0}$

- Consider, the interface of two different medium, eg: dielectric with vacuum:



$$\oint\vec{P}.d\vec{l}\neq 0 \longrightarrow \vec{\nabla}\times\vec{P}\neq 0$$

$$\longrightarrow \vec{\nabla}\times\vec{D}=\vec{\nabla}\times(\epsilon_0\vec{E}+\vec{P})\neq 0$$

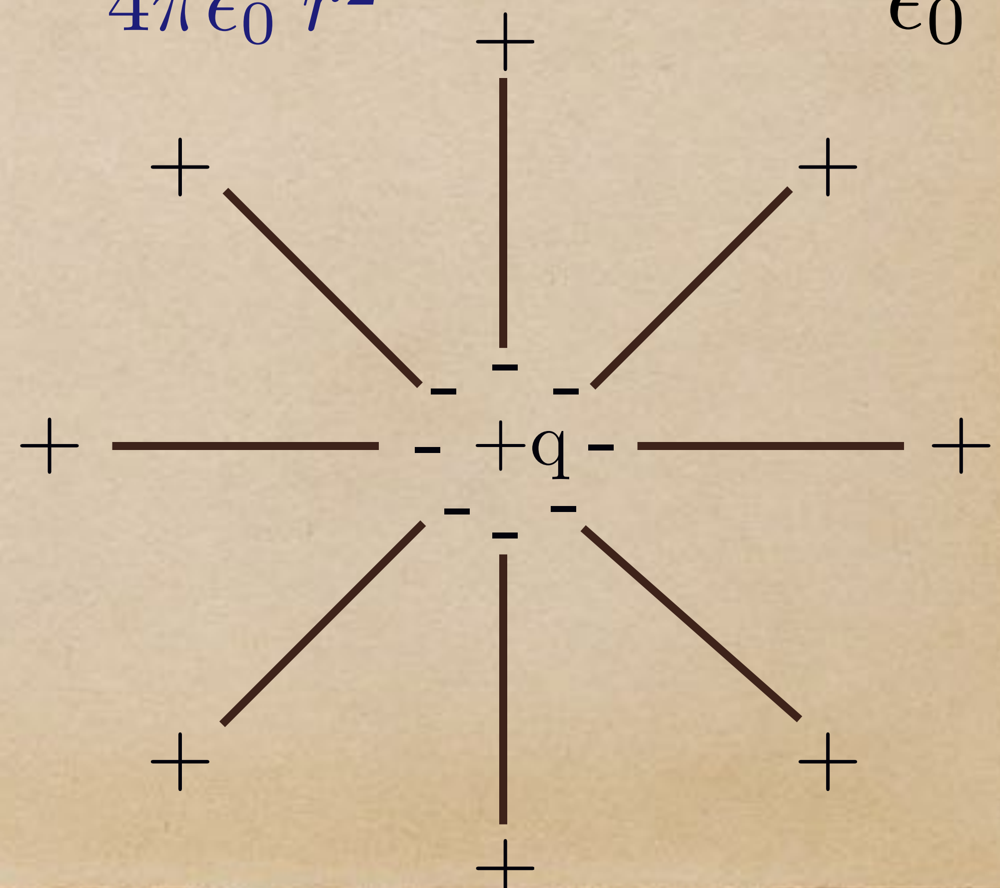
What does a free charge do in dielectric medium ?

It produces a field: $\vec{E} = \dfrac{1}{4\pi\epsilon}\dfrac{q}{r^2}\hat{r}$ $(\epsilon \text{ not } \epsilon_0)$

It can be simply obtained as follows:

$$\boxed{\vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{1}{\epsilon_r}\vec{E}_{vac}} \text{ with } \vec{E}_{vac} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \text{ and } \epsilon_r = \frac{\epsilon}{\epsilon_0}$$

Physically: Polarisation shields the charge by surrounding it with bound charges of opposite signs



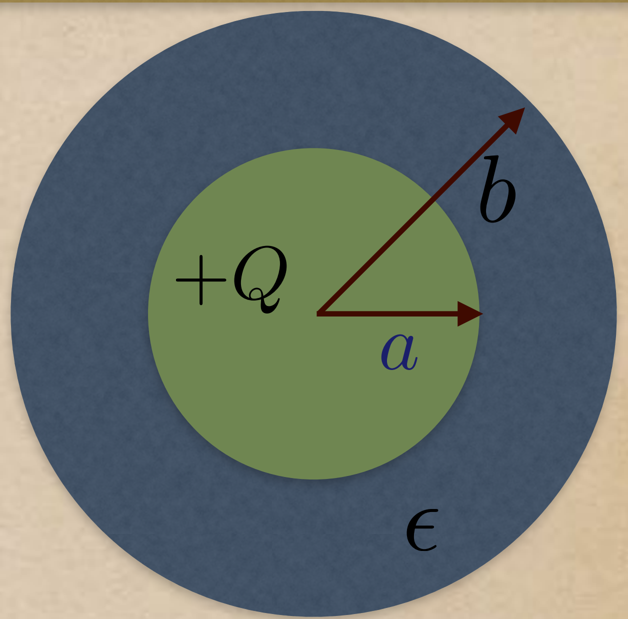
This causes reduction in the field in the dielectric material

Example

A metal sphere of charge +Q surrounded by a linear dielectric material. What is the potential at centre ? What are the bound charges ?

- First calculate displacement using Gauss's law:

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2}\hat{r}} \quad \text{for all points } r > a$$



- Now one can calculate electric field:

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon r^2}\hat{r} & \text{for} \quad a < r < b \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{for} \quad r > b. \end{cases}$$

Hence, Potential

$$V = -\int_\infty^0 \vec{E}.d\vec{l} = -\int_\infty^b \frac{Q}{4\pi\epsilon_0 r^2}dr - \int_b^a \frac{Q}{4\pi\epsilon r^2}dr - \int_a^0 0\ dr$$
$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right)$$

- Polarisation

$$\vec{P} = \epsilon_0\chi_e\vec{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{r}$$

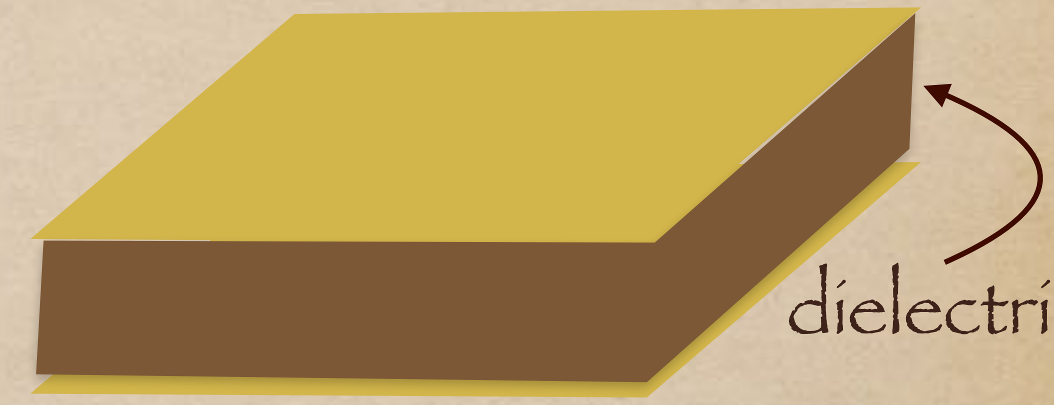
$$\rho_b = -\vec{\nabla}.\vec{P} = 0 \qquad \text{(Bound charges)}$$

$$\sigma_b = \vec{P}.\hat{n} = \begin{cases} \frac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2} & \text{outer surface} \\ \frac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2} & \text{inner surface.} \end{cases}$$

Consider capacitor filled with linear dielectric material

We can simply use the relation of electric field inside the dielectric material compared to the case of vacuum

$$\boxed{\vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{1}{\epsilon_r}\vec{E}_{vac}}$$



$$\longrightarrow \quad V = \frac{V_{vac}}{\epsilon_r}$$

$$\boxed{C = \frac{Q}{V} = \epsilon_r \frac{Q}{V_{vac}} = \epsilon_r C_{vac}}$$

Capacitance increases by the factor of dielectric constant of the material $\epsilon_r = \frac{\epsilon}{\epsilon_0}$