MA 102 (Mathematics II)

Mid Semester Examination

Date: February 29, 2016 **Time:** 2 Hours **Maximum Marks:** 30

Answer ALL questions

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) := \sin(y^2/x) \cdot \sqrt{x^2 + y^2}$ if $x \neq 0$ and $f(x, y) = 0$ if $x = 0$. Show that f is continuous at $(0, 0)$ and has directional derivatives in every direction at $(0, 0)$. Is f differentiable at $(0, 0)$? **4 marks**

   Solution: We have $|f(x, y) - f(0, 0)| = |\sin(y^2/x)|\sqrt{x^2 + y^2} \leq \sqrt{x^2 + y^2} \to 0$ as $\|(x, y)\| = \sqrt{x^2 + y^2} \to 0$. Hence f is continuous at $(0, 0)$. **[1 mark]**

   Let $u = (u_1, u_2)$ be a unit vector. If $u_1u_2 = 0$ then it follows that

   $$D_uf(0, 0) = \lim_{t\to 0} \frac{f(tu) - f(0, 0)}{t} = 0.$$

   [1 mark]

   Now suppose that $u_1u_2 \neq 0$. Then

   $$D_uf(0, 0) = \lim_{t\to 0} \frac{f(tu) - f(0, 0)}{t} = \lim_{t\to 0} \frac{\sin(tu_2^2/u_1)|t|}{t} = 0.$$

   Thus $D_uf(0, 0)$ exists for all unit vector u. **[1 mark]**

   However, f is not differentiable. Indeed, we have

   $$\frac{|f(h, k) - f(0, 0) - (f_x(0, 0)h + f_y(0, 0)k)|}{\sqrt{h^2 + k^2}} = |\sin(k^2/h)| \to |\sin(1/m)| \not\to 0$$

   as $(h, k) \to (0, 0)$ along the path $h = mk^2$. **[1 mark]**

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x, y) = (x^3y + y^2, xy)$ and $g(x, y) = (x^2y, xy, x - 2y)$. Use chain rule to determine the Jacobian matrix $J_{g\circ f}(1, 2)$. **3 marks**

   Solution: By the chain rule, $J_{g\circ f}(1, 2) = J_g(f(1, 2)) \cdot J_f(1, 2)$. Now $J_f(x, y) = \begin{bmatrix} 3x^2y & x^3 + 2y \\ y & x \end{bmatrix}$ and $J_g(x, y) = \begin{bmatrix} 2xy & x^2 \\ y & x \\ 1 & -2 \end{bmatrix}$. **[1 mark]**

We have $f(1,2) = (6,2)$. So, $J_f(1,2) = \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix}$ and

$J_g(f(1,2)) = J_g(6,2) = \begin{bmatrix} 24 & 36 \\ 2 & 6 \\ 1 & -2 \end{bmatrix}$. **[1 mark]**

Hence $J_{g \circ f}(1,2) = \begin{bmatrix} 24 & 36 \\ 2 & 6 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 216 & 156 \\ 24 & 16 \\ 2 & 3 \end{bmatrix}$. **[1 mark]**

3. Show that the equation $xy - z \log y + e^{xz} = 1$ can be solved locally around the point $(0,1,1)$ as $y = f(x,z)$ for some C^1 function f. Determine $\nabla f(0,1)$.

3 marks

Solution: Let $F(x,y,z) := xy - z \log y + e^{xz} - 1$. Then $F_y = x - z/y \Rightarrow F_y(0,1,1) = -1 \neq 0$. Hence by the implicit function theorem $F(x,y,z) = 0$ can be solved locally as $y = f(x,z)$ for some C^1 function f. **[1 mark]**

Since $F(x, f(x,z), z) = 0$ in a neighborhood of $(0,1)$, differentiating w.r.t. x we have $F_x + F_y f_x = 0$. Hence $f_x(0,1) = -F_x(0,1,1)/F_y(0,1,1) = 2$. **[1 mark]**

Again differentiating $F(x, f(x,z), z) = 0$ w.r.t. z, we have
$f_z(0,1) = -F_z(0,1,1)/F_y(0,1,1) = 0$. **[1 mark]**

4. Find the maximum of $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) := x + z$ subject to the constraint $x^2 + y^2 + 2z^2 = 1$. **4 marks**

Solution: Consider the Lagrangian

$$L(x,y,z,\lambda) = x + z - \lambda(x^2 + y^2 + 2z^2 - 1)$$

Then

$$L_x = 1 - 2\lambda x = 0,\ L_y = -2\lambda y = 0,\ L_z = 1 - 4\lambda z,\ L_\lambda = -x^2 - y^2 - 2z^2 + 1 = 0.$$

Hence

$$1 - 2\lambda x = 0 \tag{1}$$

$$-2\lambda y = 0 \tag{2}$$

$$1 - 4\lambda z = 0 \tag{3}$$

$$x^2 + y^2 + 2z^2 = 1 \tag{4}$$

[1 mark]

It follows from (1) that $\lambda \neq 0$. Hence by (1), (2) and (3), we have

$$x = \frac{1}{2\lambda}, \qquad y = 0, \quad \text{and} \qquad z = \frac{1}{4\lambda} \tag{5}$$

respectively. Now using (5) in (4), we have

$$\lambda = \pm\sqrt{\frac{3}{8}} \tag{6}$$

[1 mark]

Finally, using (6) in (5), we have the critical points $\left(\sqrt{\frac{2}{3}}, 0, \frac{1}{\sqrt{6}}\right)$ and $\left(-\sqrt{\frac{2}{3}}, 0, -\frac{1}{\sqrt{6}}\right)$.

[1 mark]

At these critical points, the value of $f(x, y, z)$ is $\sqrt{\frac{3}{2}}$ and $-\sqrt{\frac{3}{2}}$ respectively. So the maximum value of $f(x, y, z)$ subject to the constraint $x^2 + y^2 + 2z^2 = 1$ is $\sqrt{\frac{3}{2}}$.

[1 mark]

5. Consider the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) := (x^2 + y^2,\ 2xy)$. Determine whether or not F is a conservative vector field and find a scalar potential, if it exists. **3 marks**

   Solution: Here $P = x^2 + y^2$ and $Q = 2xy$ which shows that $Q_x = P_y$ so that the necessary condition is satisfied. **[1 mark]**

   Now $f_x = P = x^2 + y^2 \Rightarrow f = x^3/3 + xy^2 + h(y)$ for some function $h(y)$.

   [1 mark]

   Hence $f_y = 2xy + h'(y) = Q = 2xy \Rightarrow h'(y) = 0 \Rightarrow h(y) = c$. This shows that $f(x, y) = x^3/3 + xy^2 + c$ is the scalar potential. **[1 mark]**

6. The force field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) := (xy, x^6y^2)$ moves a particle from $(0, 0)$ to the line $x = 1$ along $y = ax^b$, where $a > 0$ and $b > 0$. If the workdone is independent of b then find the value of a. **4 marks**

   Solution: Work done is given by

$$\begin{aligned} W &= \int_\Gamma F \bullet \mathbf{dr} = \int_\Gamma (xy, x^6y^2) \bullet (dx, dy) \\ &= \int_0^1 ax^{b+1}dx + \int_0^1 (a^2x^{2b+6})(abx^{b-1})dx \quad [\mathbf{1mark}] \\ &= \frac{a}{b+2} + \frac{a^3b}{3b+6} \quad [\mathbf{1mark}]. \end{aligned}$$

   Hence W is independent of b iff $\frac{dW}{db} = 0$ iff $0 = \frac{(b+2)a^2 - (3+a^2b)}{(b+2)^2}$ **[1 mark]**

   which gives $a = \sqrt{\frac{3}{2}}$ (as $a > 0$). **[1 mark]**

7. Evaluate $\iint_D xy\, dA$, where D is the region bounded by the parabola $y^2 = 2x + 6$ and the line $y = x - 1$. **3 marks**

Solution: Note that D is a Type-II domain given by

$$D := \{(x, y) : -2 \leq y \leq 4,\ y^2/2 - 3 \leq x \leq y + 1\} \quad [\textbf{1mark}].$$

Hence $\iint_D xy\, dA = \int_{-2}^{4}(\int_{y^2/2-3}^{y+1} x dx) y dy = 1/2 \int_{-2}^{4} y[(y + 1)^2 - (y^2/2 - 3)^2]dy$

[1 mark]

$= 1/2 \int_{-2}^{4}(-y^5/4 + 4y^3 + 2y^2 - 8y)dy = 1/2(-y^6/24 + y^4 + 2y^3/3 - 4y^2)|_{-2}^{4} = 36.$

[1 mark]

Aliter: If D is written as Type-I domain then

$D := \{-3 \leq x \leq -1,\ -\sqrt{2x + 6} \leq y \leq \sqrt{2x + 6}\} \cup \{-1 \leq x \leq 5,\ x-1 \leq y \leq \sqrt{2x + 6}\}.$

8. Let Γ be the circle $x^2 + y^2 = 9$ oriented positively. Use Green's theorem to evaluate the line integral $\oint_\Gamma \left((3y - e^{\cos x})dx + (7x + \sqrt{y^4 + 5})dy\right)$. **2 marks**

Solution: Let $D := \{(x, y) : x^2 + y^2 \leq 9\}$. We have $Q_x - P_y = 7 - 3 = 4$.

[1 mark]

Hence by Green's theorem $I = \iint_D 4dA = 4\text{Area}(D) = 36\pi$. **[1 mark]**

9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous satisfying $f(\mathbf{x}) > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $f(\alpha\mathbf{x}) = \alpha^2 f(\mathbf{x})$ for $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. Show that there is a real number $\beta > 0$ such that $f(\mathbf{x}) \geq \beta\|\mathbf{x}\|^2$ for $\mathbf{x} \in \mathbb{R}^n$. **4 marks**

Solution: Since f is continuous, f attains minimum on $S := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$. Let $\beta := \min_{\mathbf{x}\in S} f(\mathbf{x})$. Then $\beta = f(\mathbf{x}_0) > 0$ for some $\mathbf{x}_0 \in S$. **[2 marks]**

Let $\mathbf{x} \in \mathbb{R}^n$. If $\mathbf{x} = 0$ then $f(\mathbf{x}) = 0$ and hence $f(\mathbf{x}) \geq \beta\|\mathbf{x}\|^2$. On the other hand, if $\mathbf{x} \neq 0$ then $f(\mathbf{x}) = \|\mathbf{x}\|^2 f(\mathbf{x}/\|\mathbf{x}\|) \geq \beta\|\mathbf{x}\|^2$. **[2 marks]**

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