

The Method of Frobenius

Department of Mathematics
IIT Guwahati

If either $p(x)$ or $q(x)$ in

$$y'' + p(x)y' + q(x)y = 0$$

is **not analytic** near x_0 , power series solutions valid near x_0 **may or may not exist**.

Example: Try to find a power series solution of

$$x^2 y'' - y' - y = 0 \tag{1}$$

about the point $x_0 = 0$.

Assume that a solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

exists.

Substituting this series in (1), we obtain the recursion formula

$$a_{n+1} = \frac{n^2 - n - 1}{n + 1} a_n.$$

The ratio test shows that this power series converges only for $x = 0$. Thus, there is no power series solution valid in any open interval about $x_0 = 0$. This is because (1) has a singular point at $x = 0$.

The method of Frobenius is a useful method to treat such equations.

Cauchy-Euler equations revisited

Recall that a second order homogeneous Cauchy-Euler equation has the form

$$ax^2y''(x) + bxy'(x) + cy(x) = 0, \quad x > 0, \quad (2)$$

where $a(\neq 0)$, b , c are real constants. Writing (2) in the standard form as

$$y'' + p(x)y' + q(x)y = 0, \quad \text{where } p(x) = \frac{b}{ax}, \quad q(x) = \frac{c}{ax^2}.$$

Note that $x = 0$ is a singular point for (2). We seek solutions of the form

$$y(x) = x^r$$

and then try to determine the values for r .

Set

$$L(y)(x) := ax^2y''(x) + bxy'(x) + cy(x) \text{ and } w(r, x) := x^r.$$

Now

$$\begin{aligned} L(w)(x) &= ax^2r(r-1)x^{r-2} + bxx^rx^{r-1} + cx^r \\ &= \{ar^2 + (b-a)r + c\}x^r. \end{aligned}$$

Thus,

$$\begin{aligned} w = x^r \text{ is a solution} &\iff r \text{ satisfies} \\ ar^2 + (b-a)r + c &= 0. \end{aligned} \tag{3}$$

The equation (3) is known as the **auxiliary** or **indicial** equation for (2).

Case I: When (3) has two distinct roots r_1, r_2 . Then

$$L(w)(x) = a(r - r_1)(r - r_2)x^r.$$

The two linearly independent solutions are

$$y_1(x) = w(r_1, x) = x^{r_1}, \quad y_2(x) = w(r_2, x) = x^{r_2} \quad \text{for } x > 0.$$

Case II: When $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$. Then

$$\begin{aligned} x^{\alpha+i\beta} &= e^{(\alpha+i\beta)\ln x} = e^{\alpha\ln x} \cos(\beta\ln x) + i e^{\alpha\ln x} \sin(\beta\ln x) \\ &= x^{\alpha} \cos(\beta\ln x) + i x^{\alpha} \sin(\beta\ln x). \end{aligned}$$

Thus, two linearly independent real-valued solutions are

$$y_1(x) = x^{\alpha} \cos(\beta\ln x), \quad y_2(x) = x^{\alpha} \sin(\beta\ln x).$$

Case III: When $r_1 = r_2 = r_0$ is a repeated roots. Then

$$L(w)(x) = a(r - r_0)^2 x^r.$$

Setting $r = r_0$ yields the solution

$$y_1(x) = w(r_0, x) = x^{r_0}, \quad x > 0.$$

To find the second linearly independent solution, we note that

$$\frac{\partial}{\partial r} \{L(w)(x)\} \Big|_{r=r_0} = \{a(r - r_0)^2 x^r \ln x + 2a(r - r_0)x^r\} \Big|_{r=r_0} = 0.$$

Since $\frac{\partial}{\partial r} L(w) = L \left[\frac{\partial w}{\partial r} \right]$ ($\frac{\partial}{\partial r}$ and L commute), we obtain

$$L \left[\frac{\partial w}{\partial r} \right] \Big|_{r=r_0} = 0.$$

A second linearly independent solution is

$$y_2(x) = \frac{\partial w}{\partial r}(r_0, x) = \frac{\partial}{\partial r}(x^r) \Big|_{r=r_0} = x^{r_0} \ln x, \quad x > 0.$$

Example: Find a general solution to

$$4x^2 y''(x) + y(x) = 0, \quad x > 0.$$

Note that

$$L(w)(x) = (4r^2 - 4r + 1)x^r.$$

The indicial equation has repeated roots $r_0 = 1/2$. Thus, the general solution is

$$y(x) = c_1 \sqrt{x} + c_2 \sqrt{x} \ln x, \quad x > 0.$$

The Method of Frobenius

To motivate the procedure, recall the Cauchy-Euler equation in the standard form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \quad (4)$$

where

$$p(x) = \frac{p_0}{x}, \quad q(x) = \frac{q_0}{x^2} \text{ with } p_0 = b/a \quad q_0 = c/a.$$

The indicial equation is of the form

$$r(r-1) + p_0r + q_0 = 0. \quad (5)$$

If $r = r_1$ is a root of (5), then $w(r_1, x) = x^{r_1}$ is a solution to (4).

Assume that $xp(x)$ and $x^2q(x)$ (instead of being constants) are analytic functions. Then

$$xp(x) = p_0 + p_1x + p_2x^2 + \cdots = \sum_{n=0}^{\infty} p_n x^n, \quad (6)$$

$$x^2q(x) = q_0 + q_1x + q_2x^2 + \cdots = \sum_{n=0}^{\infty} q_n x^n \quad (7)$$

in some neighborhood of $x = 0$. Then, it follows that

$$\lim_{x \rightarrow 0} xp(x) = p_0 \quad \text{and} \quad \lim_{x \rightarrow 0} x^2q(x) = q_0.$$

Therefore, it is reasonable to expect that the solutions to (2) will behave (for x near 0) like the solutions to the Cauchy-Euler equation

$$x^2y''(x) + p_0xy'(x) + q_0y(x) = 0.$$

When $p(x)$ and $q(x)$ satisfy (6) and (7), we say that the singular point $x = 0$ is regular. This observation leads to the following definition.

Definition: A singular point x_0 of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

is said to be a **regular singular point** if both $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are analytic at x_0 . Otherwise x_0 is called an **irregular singular point**.

Example: Classify the singular points of the equation

$$(x^2 - 1)^2y''(x) + (x + 1)y'(x) - y(x) = 0.$$

The singular points are 1 and -1 . Note that $x = 1$ is an irregular singular point and $x = -1$ is a regular singular point.

Series solutions about a regular singular point

Assume that $x = 0$ is a regular singular point for

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

so that

$$p(x) = \sum_{n=0}^{\infty} p_n x^{n-1}, \quad q(x) = \sum_{n=0}^{\infty} q_n x^{n-2}.$$

In the method of Frobenius, we seek solutions of the form

$$w(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad x > 0.$$

Assume that $a_0 \neq 0$. We now determine r and a_n , $n \geq 1$.

Differentiating $w(r, x)$ with respect to x , we have

$$w'(r, x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1},$$

$$w''(r, x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Substituting w , w' , w'' , $p(x)$ and $q(x)$ into (4), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \\ & + \left(\sum_{n=0}^{\infty} p_n x^{n-1} \right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) \\ & + \left(\sum_{n=0}^{\infty} q_n x^{n-2} \right) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0. \end{aligned}$$

Group like powers of x , starting with the lowest power, x^{n-2} . We find that

$$[r(r-1) + p_0r + q_0]a_0x^{r-2} + [(r+1)ra_1 + (r+1)p_0a_1 + p_1ra_0 + q_0a_1 + q_1a_0]x^{r-1} + \dots = 0.$$

Considering the first term, x^{r-2} , we obtain

$$\{r(r-1) + p_0r + q_0\}a_0 = 0.$$

Since $a_0 \neq 0$, we obtain the indicial equation.

Definition: If x_0 is a regular singular point of $y'' + p(x)y' + q(x)y = 0$, then the **indicial equation** for this point is

$$r(r-1) + p_0r + q_0 = 0,$$

where

$$p_0 := \lim_{x \rightarrow x_0} (x - x_0)p(x), \quad q_0 := \lim_{x \rightarrow x_0} (x - x_0)^2 q(x).$$

Example: Find the indicial equation at the singularity $x = -1$ of

$$(x^2 - 1)^2 y''(x) + (x + 1)y'(x) - y(x) = 0.$$

Here $x = -1$ is a regular singular point. We find that

$$p_0 = \lim_{x \rightarrow -1} (x + 1)p(x) = \lim_{x \rightarrow -1} (x - 1)^{-2} = \frac{1}{4},$$

$$q_0 = \lim_{x \rightarrow -1} (x + 1)^2 q(x) = \lim_{x \rightarrow -1} [-(x - 1)^{-2}] = -\frac{1}{4}.$$

Thus, the indicial equation is given by

$$r(r - 1) + \frac{1}{4}r - \frac{1}{4} = 0.$$

The method of Frobenius

To derive a series solution about the singular point x_0 of

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \quad x > x_0. \quad (8)$$

Set $p(x) = a_1(x)/a_2(x)$, $q(x) = a_0(x)/a_2(x)$.

If both $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are analytic at x_0 , then x_0 is a regular singular point and the following steps apply.

Step 1: Seek solution of the form

$$w(r, x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r}.$$

Using termwise differentiation and substitute $w(r, x)$ into (8) to obtain an equation of the form

$$A_0(x - x_0)^{r+J} + A_1(x - x_0)^{r+J+1} + \dots = 0.$$

Step 2: Set $A_0 = A_1 = A_2 = \cdots = 0$. (Notice that $A_0 = 0$ is a constant multiple of the indicial equation $r(r-1) + p_0r + q_0 = 0$).

Step 3: Use the system of equations

$$A_0 = 0, \quad A_1 = 0, \quad \dots, \quad A_k = 0$$

to find a recurrence relation involving a_k and a_0, a_1, \dots, a_{k-1} .

Step 4: Take $r = r_1$, the larger root of the indicial equation, and use the relation obtained in Step 3 to determine a_1, a_2, \dots recursively in terms of a_0 and r_1 .

Step 5: A series expansion of a solution to (8) is

$$w(r_1, x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad x > x_0,$$

where a_0 is arbitrary and a_n 's are defined in terms of a_0 and r_1 .

Theorem: Let x_0 be a regular singular point for

$$y'' + p(x)y'(x) + q(x)y(x) = 0$$

and let r_1 and r_2 be the roots of the associated indicial equation, where $r_1 \geq r_2$ or $\operatorname{Re} r_1 \geq \operatorname{Re} r_2$.

Case a: If $r_1 - r_2$ is not an integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}; \quad a_0 \neq 0,$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \quad b_0 \neq 0.$$

Case b: If $r_1 = r_2$, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}; \quad a_0 \neq 0,$$

$$y_2(x) = y_1(x) \ln(x - x_0) + \sum_{n=1}^{\infty} b_n (x - x_0)^{n+r_1}.$$

Case c: If $r_1 - r_2$ is a positive integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}; \quad a_0 \neq 0,$$

$$y_2(x) = C y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \quad b_0 \neq 0,$$

where C is a constant that could be zero.

*** End ***