

The Annihilator and Operator Methods for Finding a Particular Solution y_p

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The Annihilator Method for Finding y_p

- This method provides a procedure for finding a particular solution (y_p) such that $L(y_p) = g$, where L is a linear differential operator with constant coefficients and $g(x)$ is a given function. The basic idea is to transform the given **nonhomogeneous equation** into a **homogeneous one**.

Definition: A linear differential operator Q is said to annihilate a function $f(x)$ in (a, b) if

$$Q(f)(x) = 0 \text{ for all } x \in (a, b).$$

Example:

- $f(x) = e^x$, $Q = D - 1$ (Q annihilates e^x).
- $f(x) = xe^x$, $Q = (D - 1)^2$.
- $f(x) = e^{2x} \sin(4x)$, $Q = (D^2 - 4D + 20)$.

Consider

$$L(y) = g(x), \quad L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y,$$

where a_i 's are constants.

Suppose $Q(g)(x) = 0$, then $Q(L(y))(x) = Q(g)(x) = 0$.

$$QL(y)(x) = 0 \implies y \in \text{Ker}(QL).$$

Determine $\text{Ker}(QL)$ and then compare with the general solution of $L(y) = 0$ (i.e., $\text{Ker}(L)$) to determine the form of the particular solution to $L(y) = g$.

$g(x)$ Annihilator of f x^{n-1} D^n $e^{\alpha x}$ $(D - \alpha)$ $x^{n-1}e^{\alpha x}$ $(D - \alpha)^n$ $\cos(\beta x)$ or $\sin(\beta x)$ $D^2 + \beta^2$ $x^{n-1} \cos(\beta x)$ or $x^{n-1} \sin(\beta x)$ $(D^2 + \beta^2)^n$

$g(x)$ Annihilator of f $e^{\alpha x} \cos(\beta x)$ or $e^{\alpha x} \sin(\beta x)$

$$D^2 - 2\alpha D + (\alpha^2 + \beta^2)$$

 $x^{n-1} e^{\alpha x} \cos(\beta x)$ or $x^{n-1} e^{\alpha x} \sin(\beta x)$ $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$

Note: If $g(x)$ has the form e^{x^2} , $\log x$, $\frac{1}{x}$, $\tan x$ or $\sin^{-1} x$ the annihilator method **will not work**.

Example: Find a particular solution of

$$Ly := y'' + y = e^{2x} + 1.$$

Note that $(D - 2)(e^{2x}) = 0$ and $D(1) = 0$. Hence,

$$D(D - 2)(e^{2x} + 1) = 0, \quad Q = D(D - 2).$$

Now,

$$QL(y) = Q(e^{2x} + 1) = 0 \implies D(D - 2)(D^2 + 1)(y) = 0.$$

Since $\text{Ker}(QL) = \text{span} \{ \cos x, \sin x, e^{2x}, 1 \}$, the general solution to $QL(y) = 0$ is

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 e^{2x} + c_4. \quad (*)$$

Since every solution of $L(y) = g$ is also a solution to $QL(y) = 0$ and the general solution of $L(y) = g$ is

$$y(x) = c_1 \cos x + c_2 \sin x + y_p(x),$$

where $\text{Ker}(L) = \text{span}\{\cos x, \sin x\}$ and $L(y_p) = e^{2x} + 1$.

Thus, comparing with (*), we obtain $y_p = c_3 e^{2x} + c_4$.

$$L(y_p) = e^{2x} + 1 \implies 5c_3 e^{2x} + c_4 = e^{2x} + 1 \implies c_3 = 1/5, c_4 = 1.$$

So, the particular solution is $y_p(x) = (1/5)e^{2x} + 1$.

Note: The general solution of $y'' + y = e^{2x} + 1$ is

$$y(x) = c_1 \cos x + c_2 \sin x + (1/5)e^{2x} + 1.$$

Operator Methods for Finding y_p

Writing $Ly = g$ as $P(D)y = g(x)$, where

$$L = P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0.$$

With each $P(D)$, associate a polynomial

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$$

called the **auxiliary** polynomial of $P(D)$.

If $P(r)$ can be factored as product of n linear factors, say

$$P(r) = a_n(r - r_1)(r - r_2) \cdots (r - r_n),$$

then the corresponding factorization of $P(D)$ has the form

$$P(D) = a_n(D - r_1)(D - r_2) \cdots (D - r_n),$$

where r_1, r_2, \dots, r_n are the roots of $P(r) = 0$.

Note that

- $Dy_p(x) = g(x) \Rightarrow y_p(x) = \int g(x)dx$. It is natural to define

$$\frac{1}{D}g(x) := \int g(x)dx.$$

- $(D - r)y_p = g(x)$, where r is a constant. Formally, we write

$$y_p = \frac{1}{D - r}g(x).$$

The solution of $(D - r)y_p = g(x)$ is

$$y_p(x) = e^{rx} \int e^{-rx} g(x) dx.$$

Thus, we define

$$\frac{1}{D - r}g(x) := e^{rx} \int e^{-rx} g(x) dx.$$

Operator like $\frac{1}{D}$, $\frac{1}{D-r}$ are called **inverse operators**.

Let $\frac{1}{P(D)}$ be the inverse of the operator $P(D)$. Then the particular solution to $P(D)y = g(x)$ is given by

$$y_p(x) = \frac{1}{P(D)}g(x).$$

Method 1:(Successive integrations)

If $P(D) = (D - r_1)(D - r_2) \cdots (D - r_n)$, then

$$\begin{aligned} y_p(x) &= \frac{1}{P(D)}g(x) = \frac{1}{(D - r_1)(D - r_2) \cdots (D - r_n)}g(x) \\ &= \frac{1}{(D - r_1)} \frac{1}{(D - r_2)} \cdots \frac{1}{(D - r_n)}g(x). \end{aligned}$$

Example: Find a particular solution of $y'' - 3y' + 2y = xe^x$.

Here $P(D)y = (D - 1)(D - 2)y = xe^x$. The particular solution y_p is

$$\begin{aligned} y_p(x) &= \frac{1}{D-1} \frac{1}{D-2} xe^x \\ &= \frac{1}{D-1} \left[e^{2x} \int e^{-2x} xe^x dx \right] = \frac{1}{D-1} [-(1+x)e^x] \\ &= -e^x \int e^{-x}(1+x)e^x dx = -\frac{1}{2}(1+x)^2 e^x. \end{aligned}$$

Note: The successive integrations are likely to become complicated and time-consuming.

Method 2:(Partial fractions)

If the factors of $P(D)$ are distinct, we can decompose operator $\frac{1}{P(D)}$ into partial fractions as

$$y_p = \frac{1}{P(D)}g(x) = \left[\frac{A_1}{(D-r_1)} + \frac{A_2}{(D-r_2)} + \cdots + \frac{A_n}{(D-r_n)} \right] g(x),$$

for suitable constants A_i 's.

Example: Find a particular solution of $y'' - 3y' + 2y = xe^x$.

$$\begin{aligned} y_p(x) &= \frac{1}{(D-1)(D-2)} = \left[\frac{1}{D-2} - \frac{1}{D-1} \right] xe^x \\ &= \frac{1}{D-2} xe^x - \frac{1}{D-1} xe^x \\ &= e^{2x} \int e^{-2x} xe^x dx - e^x \int e^{-x} xe^x dx \\ &= -(1+x + \frac{1}{2}x^2)e^x. \end{aligned}$$

Method 3:(Series expansions)

If $g(x) = x^n$, expand the inverse operator $\frac{1}{P(D)}$ in a power series in D so that

$$y_p(x) = \frac{1}{P(D)}g(x) = (a_0 + a_1D + a_2D^2 + \cdots + a_nD^n)g(x),$$

where $(a_0 + a_1D + a_2D^2 + \cdots + a_nD^n)$ is the expansion of $\frac{1}{P(D)}$ to $n + 1$ terms as $D^k x^n = 0$ if $k > n$.

Example: Find y_p of $y''' - 3y'' + 2y = x^4 + 2x + 5$.

$$\frac{1}{1 - 2D^2 + D^3} = 1 + 2D^2 - D^3 + 4D^4 - 4D^5 + \cdots$$

$$\begin{aligned} y_p(x) &= \frac{1}{1 - 2D^2 + D^3}(x^4 + 2x + 5) \\ &= (1 + 2D^2 - D^3 + 4D^4 - 4D^5 + \cdots)(x^4 + 2x + 5) \\ &= (x^4 + 2x + 5) + 2(12x^2) - (24x) + 4(24) \\ &= x^4 + 24x^2 - 22x + 101. \end{aligned}$$

Method 4: If $g(x) = e^{\alpha x}$, α a constant, then

$$(D - r)e^{\alpha x} = (\alpha - r)e^{\alpha x}.$$

Operating both sides of the above identity by $(\alpha - r)^{-1}(D - r)^{-1}$, we obtain

$$\frac{1}{(D - r)}e^{\alpha x} = \frac{1}{(\alpha - r)}e^{\alpha x},$$

provided $\alpha \neq r$. Similarly, if $P(D) = (D - r_1) \cdots (D - r_n)$ then

$$\begin{aligned} \frac{1}{P(D)}e^{\alpha x} &= \frac{1}{(D - r_1) \cdots (D - r_n)}e^{\alpha x} \\ &= \frac{1}{(\alpha - r_1) \cdots (\alpha - r_n)}e^{\alpha x}, \end{aligned}$$

provided r_1, \dots, r_n are distinct from α .

- If $P(D)$ is a polynomial in D such that $P(\alpha) \neq 0$, then

$$\frac{1}{P(D)}e^{\alpha x} = \frac{e^{\alpha x}}{P(\alpha)}.$$

Example: Find a particular solution of

$$y''' - y'' + y' + y = 3e^{-2x}.$$

$$\begin{aligned} y_p &= \frac{1}{P(D)} 3e^{-2x} \\ &= \frac{3e^{-2x}}{P(-2)} \\ &= \frac{3e^{-2x}}{(-2)^3 - (-2)^2 - 2 + 1} \\ &= -\frac{3}{13}e^{-2x}. \end{aligned}$$

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