Physics II: Electromagnetism (PH102)

Lecture 14

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Electric Field in dielectric materials

Quick recap on dielectric

A dielectric material placed in external field will produce a lot of tiny little dipoles along the direction of the field: Material is polarised

$\vec{P}$= dipole moment per unit volume

The field of a polarized object is identical to the field that would be produced by a certain distribution of 'bound charges', σ_b and ρ_b

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\vec{P}(\vec{r}\,').\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b(\vec{r}\,')}{\mathscr{r}} da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b(\vec{r}\,')}{\mathscr{r}} d\tau'$$

Gauss's law for dielectric

$$\vec{\nabla}.\vec{D} = \rho_f$$

Bound surface charge density

$$\boxed{\sigma_b = \vec{P}(\vec{r}).\hat{n}}$$

Bound volume charge density

$$\boxed{\rho_b = -\vec{\nabla}.P(\vec{r})}$$

- Here $\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$: **Electric displacement**.
- Integral form of Gauss's law $\oint \vec{D}.d\vec{a} = Q_{\mathrm{f_{enc}}}$ and $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$

Boundary conditions: $D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f \quad D^{\parallel}_{\text{above}} - D^{\parallel}_{\text{below}} = P^{\parallel}_{\text{above}} - P^{\parallel}_{\text{below}}$

Quick recap for linear dielectric

Linear dielectric

$$\boxed{\vec{P} = \epsilon_0 \chi_e \vec{E}}$$

Electric susceptibility

Electric displacement : $\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon \vec{E}$

$$\boxed{\epsilon = \epsilon_0 (1 + \chi_e)}$$

Permittivity of the material

$$\boxed{\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e}$$

Relative permittivity or Dielectric constant

$$\vec{E} = \frac{1}{\epsilon} \vec{D} = \frac{1}{\epsilon_r} \vec{E}_{vac}$$

Electric field is diminished by dielectric constant.

Electric field produced by a charged particle in linear dielectric:

$$\vec{E} = \frac{1}{4\pi\epsilon} \frac{q}{r^2} \hat{r} \quad (\epsilon \text{ not } \epsilon_0)$$

An Example of Capacitor

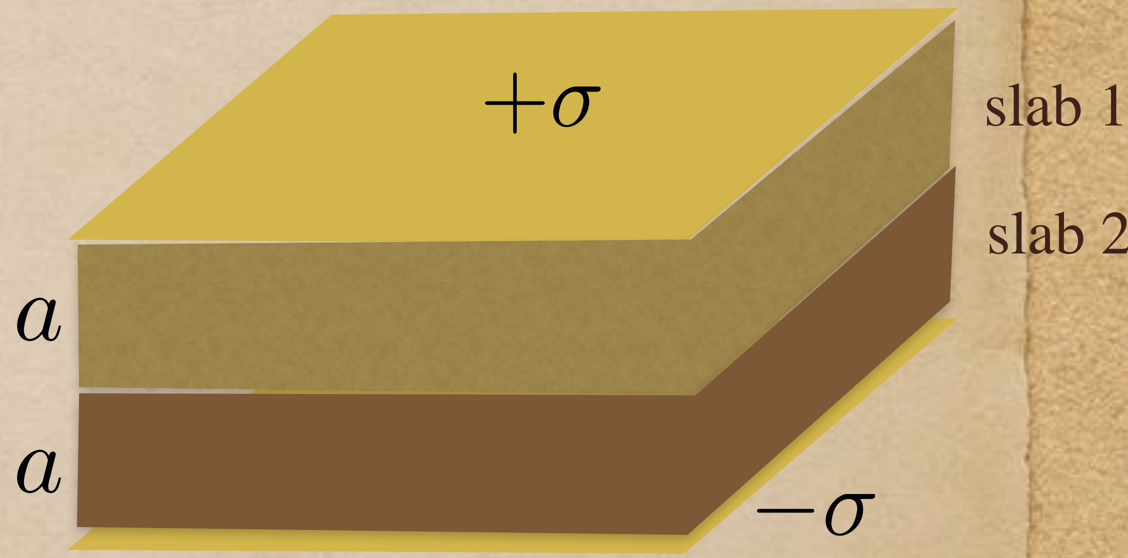
The space between the plates of a parallel plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness a. Slab 1 has dielectric constant of 2 and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on bottom plate is $-\sigma$.

Find electric displacement $\vec{D}$ in each slab.

Find electric field $\vec{E}$ in each slab.

Find the polarization $\vec{P}$ in each slab.

Find the potential difference between plates.



Find the location and amount of all bound charges.

- Apply $\oint \vec{D}.d\vec{a} = Q_{\mathrm{f_{enc}}}$ to the Gaussian surface shown.

$DA = \sigma A \implies D = \sigma \longrightarrow$ points down

$D = 0$ inside the conductor

Solution continued..

- $\vec{D} = \epsilon \vec{E} \implies \boxed{E = \sigma/\epsilon_1}$ in slab 1, $E = \sigma/\epsilon_2$ in slab 2. But, $\epsilon = \epsilon_0 \epsilon_r$, so $\epsilon_1 = 2\epsilon_0$; $\epsilon_2 = \frac{3}{2}\epsilon_0$. Hence $E_1 = \sigma/2\epsilon_0$ and $E_2 = 2\sigma/3\epsilon_0$.

- $\vec{P} = \epsilon_0 \chi_e \vec{E}$, so $P = \epsilon_0 \chi_e \sigma/(\epsilon_0 \epsilon_r) = (\chi_e/\epsilon_r)\sigma$; $\chi_e = \epsilon_r - 1 \implies P = (1 - \epsilon_r^{-1})\sigma$. Therefore $P_1 = \sigma/2$ and $P_2 = \sigma/3$

- $V = E_1 a + E_2 a = (\sigma a/6\epsilon_0)(3+4) = 7\sigma a/6\epsilon_0$

- $\rho_b = 0$

bound charges

$\sigma_b = +P_1$ at bottom of slab 1$=\sigma/2$, $\sigma_b = -P_1$ at top of slab 1$=-\sigma/2$.

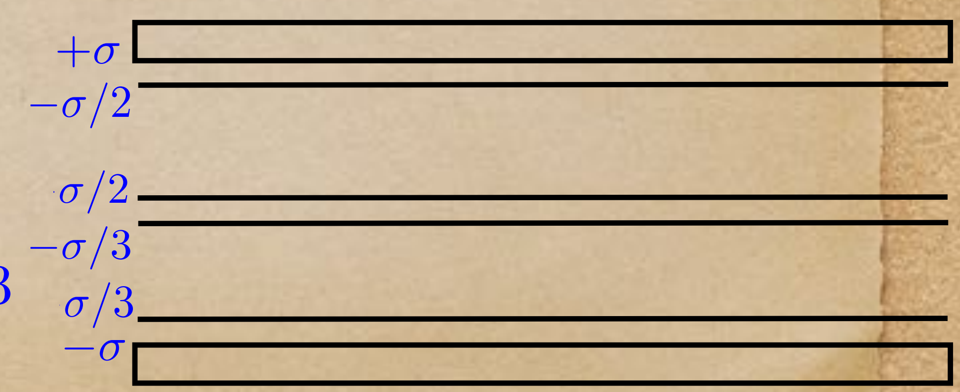
$\sigma_b = +P_2$ at bottom of slab 2$=\sigma/3$, $\sigma_b = -P_2$ at top of slab 2$=-\sigma/3$.

- In slab 1: Total surface charge above:$\sigma - \sigma/2 = \sigma/2$

Total surface charge below: $\sigma/2 - \sigma/3 + \sigma/3 - \sigma = -\sigma/2$

In slab 2: Total surface charge above: $\sigma - \sigma/2 + \sigma/2 - \sigma/3 = 2\sigma/3$

Total surface charge below: $\sigma/3 - \sigma = -2\sigma/3$



Another example on linear dielectric

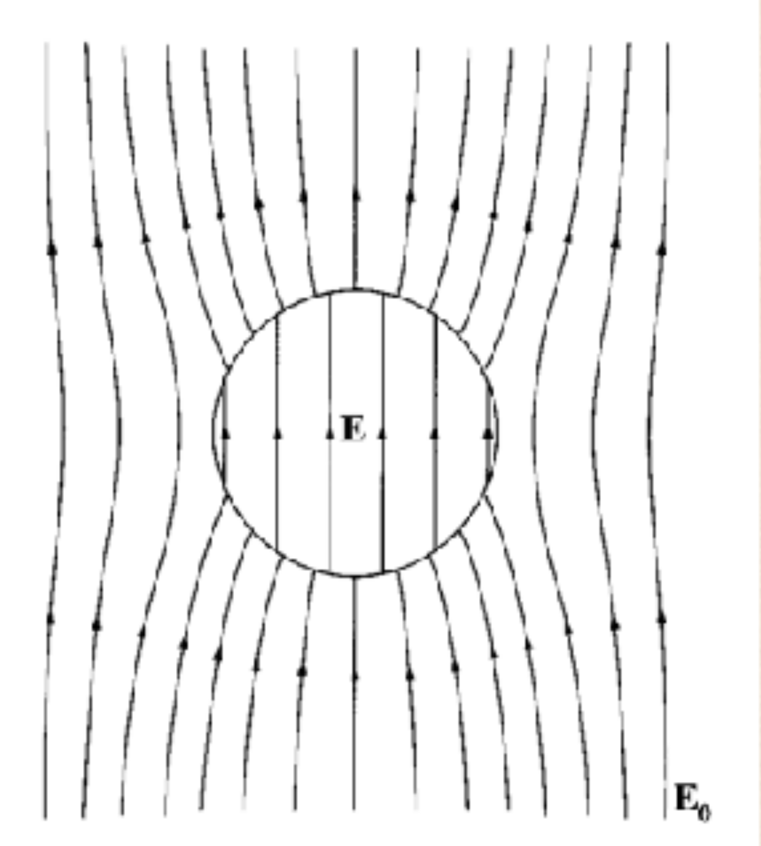
A sphere of homogeneous linear dielectric material is placed in uniform electric field. Find the electric field inside.

Polarisation created by external field $\vec{P}_0 = \epsilon_0 \chi_e \vec{E}_0$

- Field created by the polarisation $\vec{E}_1 = -\frac{1}{3\epsilon_0}\vec{P}_0 = -\frac{\chi_e}{3}\vec{E}_0$

Polarisation created by $\vec{E}_1$

$$\vec{P}_1 = \epsilon_0 \chi_e \vec{E}_1 = \epsilon_0 \chi_e (-\frac{\chi_e}{3})\vec{E}_0$$



Similarly it will further create a field..

$$\vec{E}_2 = -\frac{1}{3\epsilon_0}\vec{P}_1 = -(\frac{\chi_e}{3})^2\vec{E}_0$$

$$\vec{E}_n = -\left(\frac{\chi_e}{3}\right)^n \vec{E}_0$$

Total Electric field

$$\vec{E} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + ... = \sum_{n=0}^{\infty}\left(-\frac{\chi_e}{3}\right)^n \vec{E}_0$$

$$\vec{E} = \left(\frac{1}{1+\frac{\chi_e}{3}}\right) = \left(\frac{3}{2+\epsilon_r}\right)\vec{E}_0$$

Boundary value problems with linear dielectric

In a homogeneous linear dielectric the bound charge is proportional to free charge. One can easily check it as follows:

$$\rho_b = -\vec{\nabla}.\vec{P} = -\vec{\nabla}.\left(\epsilon_0 \frac{\chi_e}{\epsilon}\vec{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$$

Hence, unless free charge is actually embedded in the material, volume charge density is zero and hence the potential obeys Laplace's equation.

We will use the discontinuity of displacement due to surface charge density

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f \longrightarrow$$

$$\epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \sigma_f$$

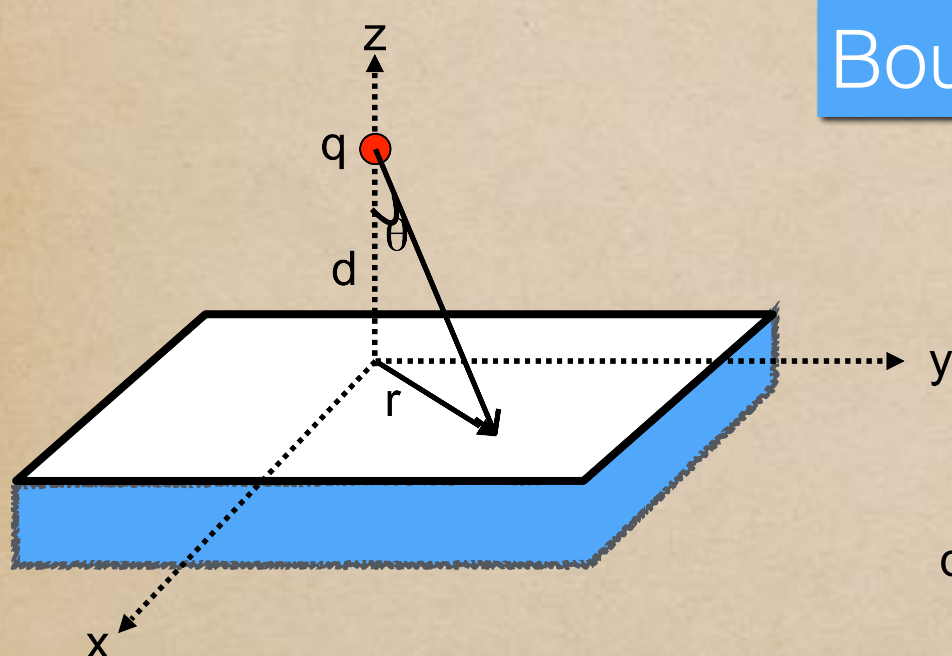
$$\epsilon_{above}\frac{\partial V_{above}}{\partial n} - \epsilon_{below}\frac{\partial V_{below}}{\partial n} = -\sigma_f$$

- The potential however is continuous still across the boundary

$$V_{above} = V_{below}$$

Boundary value problem with dielectric

GEx.4.8: Suppose the entire region below the plane z=0 is filled with uniform linear dielectric material of susceptibility χ_e Calculate the force on a point charge q situated a distance ***d*** above the origin.



Bound surface charge

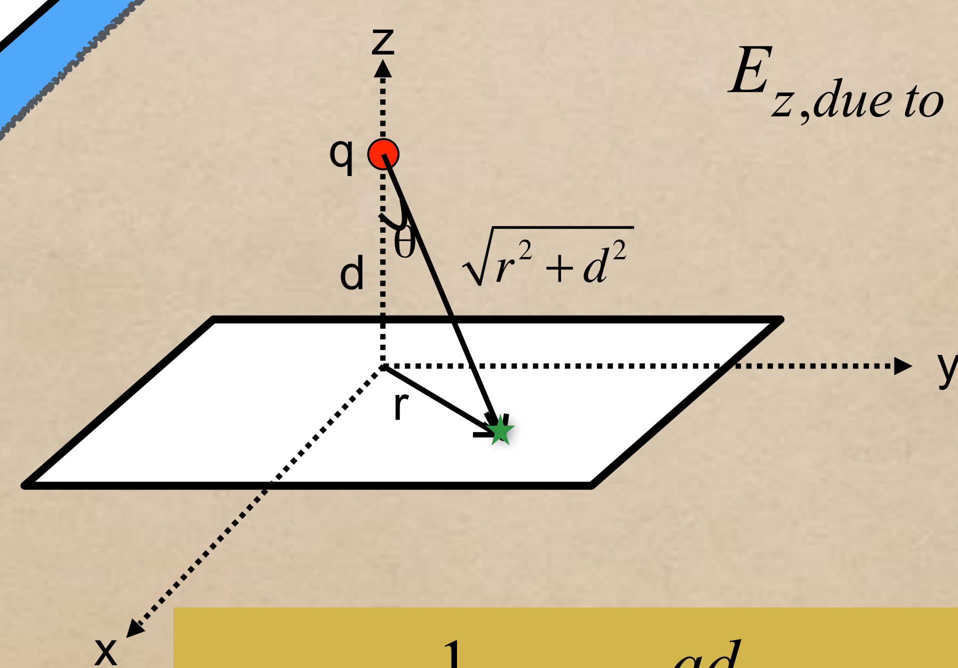
$$\sigma_b = \vec{P} \cdot \hat{n} = P_z = \varepsilon_0 x_e E_z$$

Total field

$$E_z = E_{z,due\ to\ q} + E_{z,due\ to\ \sigma_b}$$

$$E_{z,due\ to\ q} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{(r^2+d^2)}\cos\theta$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{qd}{(r^2+d^2)^{3/2}}$$



$$\boxed{E_{z,due\ to\ \sigma_b} = -\frac{\sigma_b}{2\varepsilon_0}}$$

Just below the surface.

There is no bound volume charge density as there is no free charge in the dielectric material

$$E_z = -\frac{1}{4\pi\varepsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\varepsilon_0}$$

Point charge above dielectric

One can then evaluate bound surface charge

$$\sigma_b = \vec{P}\cdot\hat{n} = P_z = \varepsilon_0 x_e E_z = -\varepsilon_0\chi_e\left(\frac{1}{4\pi\varepsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} + \frac{\sigma_b}{2\varepsilon_0}\right)$$

Bound surface charge

$$\sigma_b = \vec{P}\cdot\hat{n} = P_z = \varepsilon_0 x_e E_z$$

$$= -\frac{1}{2\pi}\frac{x_e}{x_e+2}\frac{qd}{(r^2+d^2)^{3/2}}$$

Total Bound charge

$$q_b = \int \sigma_b r\, dr d\theta$$

$$= -\frac{x_e}{x_e+2} q$$

Force

$$\vec{F}_q = \frac{1}{4\pi\varepsilon_0}\frac{q q_b}{(2d)^2}\hat{z}$$

$$= -\frac{1}{4\pi\varepsilon_0}\left(\frac{x_e}{x_e+2}\right)\frac{q^2}{4d^2}\hat{z}$$

Energy in dielectric system

$W = \frac{1}{2} C V^2$ to charge a capacitor up

$C = \varepsilon_r C_{vac}$ with linear dielectric

Energy stored in any electrostatic system (vacuum) $W = \frac{\varepsilon_0}{2} \int E^2 d\tau$

Energy stored in linear dielectric $W = \frac{\varepsilon_0}{2} \int \varepsilon_r E^2 d\tau = \frac{1}{2} \int \vec{D} \cdot \vec{E} d\tau$

- Suppose dielectric is fixed in position and we bring in free charges ρ_f, a bit at a time. i.e. ρ_f is increased by an amount $\Delta\rho_f$.

$$\begin{aligned}
\delta W &= \int (\delta\rho_f) V d\tau \\
&= \int (\vec{\nabla} \cdot \delta\vec{D}) V \, d\tau \\
&= \int \vec{\nabla} \cdot (\delta\vec{D} V) d\tau - \int \delta\vec{D} \cdot \vec{\nabla} V \, d\tau \\
&= \int_s \delta\vec{D} V \cdot d\vec{a} + \int \delta\vec{D} \cdot \vec{E} \, d\tau \\
&= \int \delta(\tfrac{1}{2} \vec{D} \cdot \vec{E}) d\tau \\
&= \delta[\tfrac{1}{2} \int \vec{D} \cdot \vec{E} d\tau] \\
W &= \tfrac{1}{2} \int \vec{D} \cdot \vec{E} d\tau
\end{aligned}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

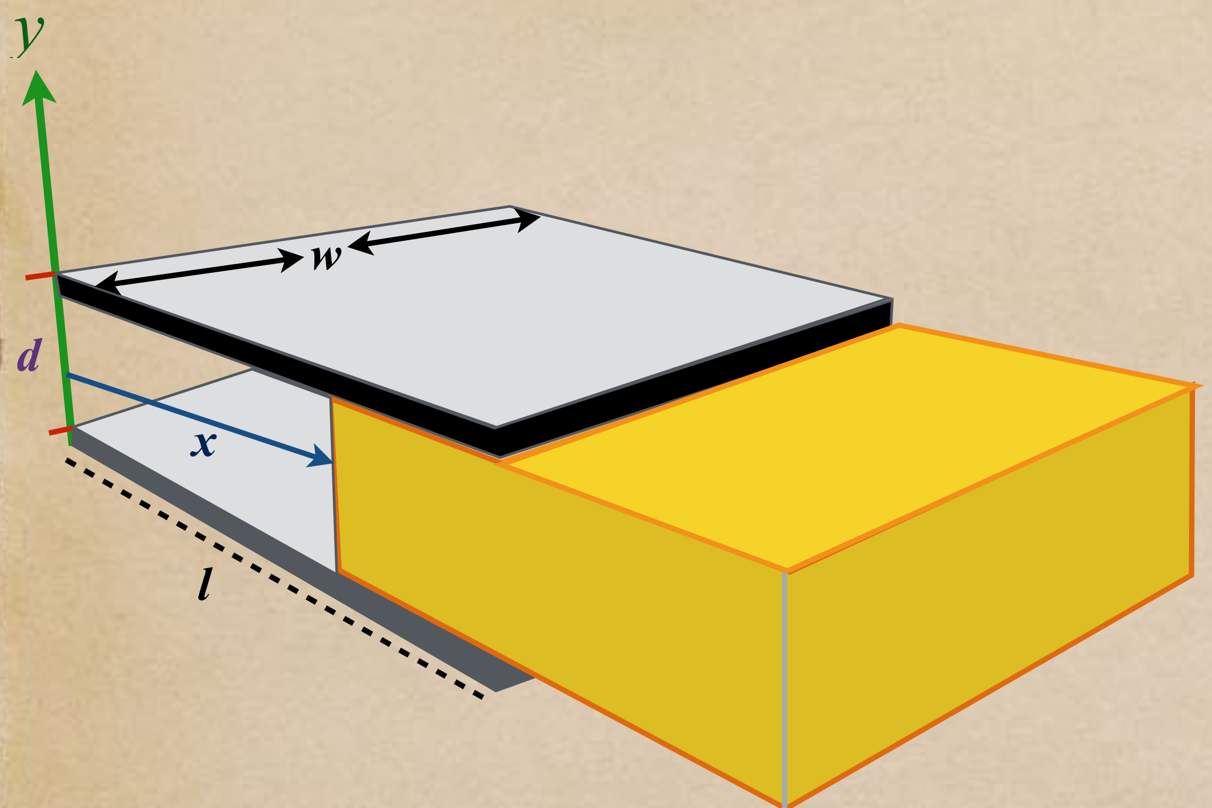
$$\vec{\nabla} \cdot (\delta\vec{D} V) = (\vec{\nabla} \cdot \delta\vec{D}) V + \delta\vec{D} \cdot (\vec{\nabla} V)$$

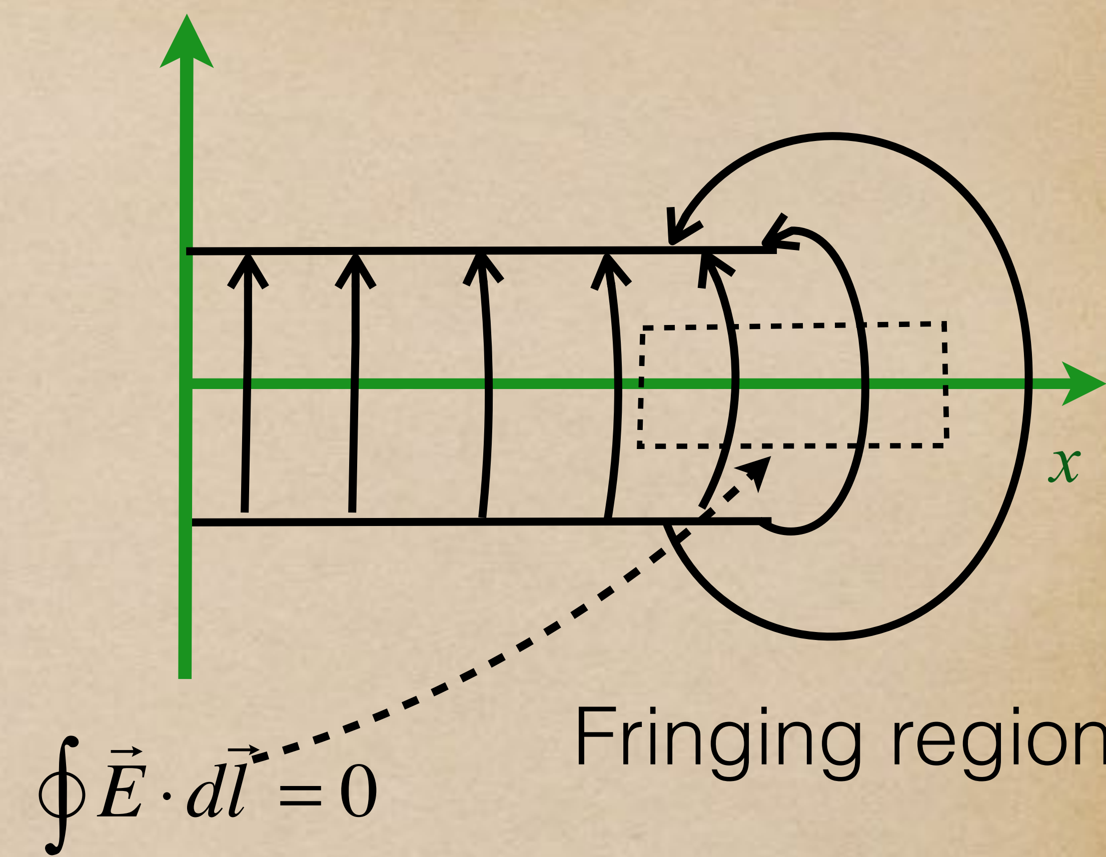
for linear dielectric $\vec{D} = \varepsilon \vec{E}$

$$\int_s \delta\vec{D} V \cdot d\vec{a} \Rightarrow 0 \quad as\ s \to \infty$$

$$\tfrac{1}{2} \delta(\vec{D} \cdot \vec{E}) = \tfrac{1}{2} \delta(\varepsilon E^2) = \varepsilon \delta\vec{E} \cdot \vec{E} = \delta\vec{D} \cdot \vec{E}$$

Forces on dielectric





- Slab of dielectric is partially inserted between the parallel plate capacitor.

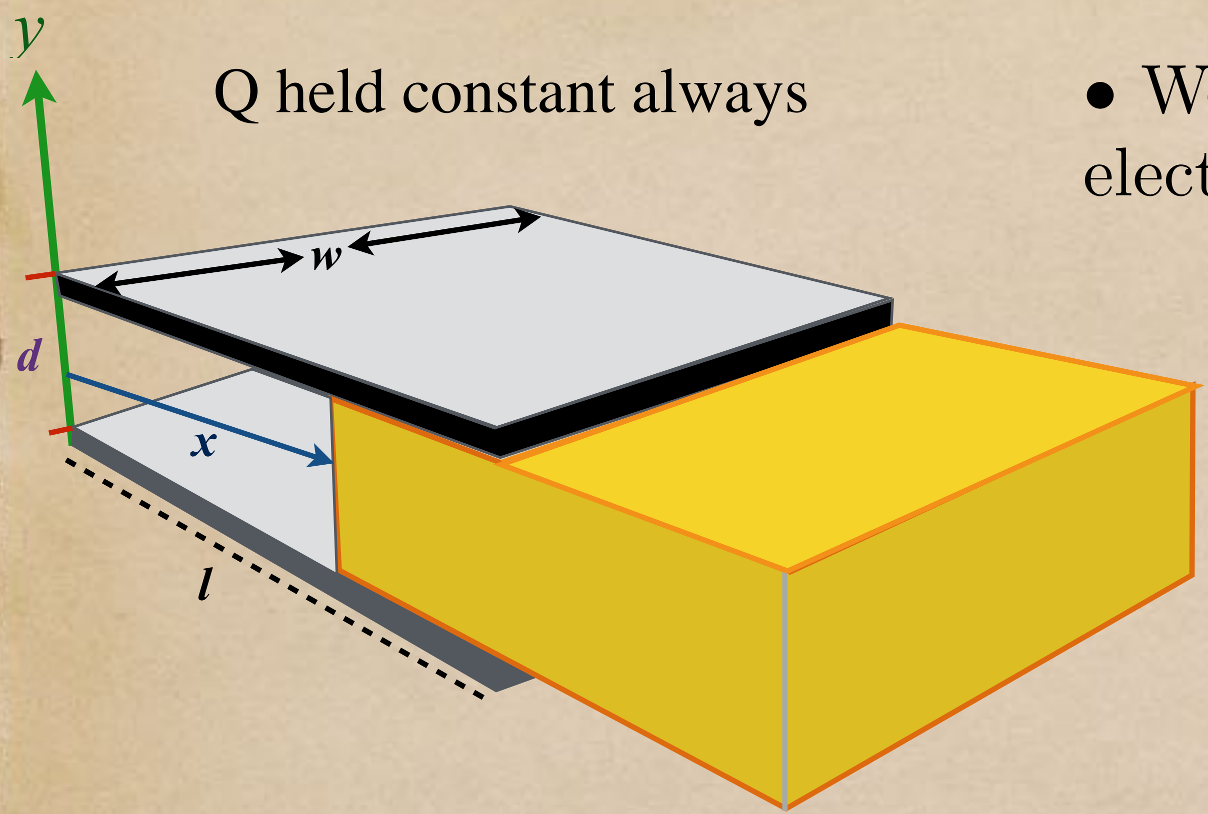
- Just as a conductor is attracted into an electric field, so too is a dielectric.

- The bound charge tends to accumulate near free charge of the opposite sign.

- A non-uniform fringing field near the edges of the capacitor is responsible for pulling the dielectric into the capacitor.

- If I pull the dielectric by an infinitesimal distance dx, the energy is changed by an amount equal to the work done: $dW = F_{\mathrm{me}} dx$

Forces on dielectric



- We must pull hard enough to overcome the electrical force F on dielectric: $F_{\mathrm{me}} = -F$.
  - Hence the electrical force on the slab
$$F = -\frac{dW}{dx}$$
  - Capacitance in this case is given by:
$$\boxed{\begin{aligned} C &= \frac{Q}{V} = \frac{\epsilon_0 wx}{d} + \frac{\epsilon(l-x)w}{d} \\ &= \frac{\epsilon_0 w}{d}(\epsilon_r l - \chi_e x) \end{aligned}}$$
$$\frac{dC}{dx} = -\frac{\epsilon_0 \chi_e w}{d}$$
- The energy stored in the capacitor is $W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$ The charge is fixed !
- So that $F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx} \Longrightarrow \boxed{F = -\frac{\epsilon_0 \chi_e w}{2d}V^2}$

Forces on dielectric

• It is of course possible to maintain capacitor at a fixed potential by connecting it up to a battery. But, in that case, the battery also does work as the dielectric moves:

$$dW = F_{\text{me}} dx + V dQ$$

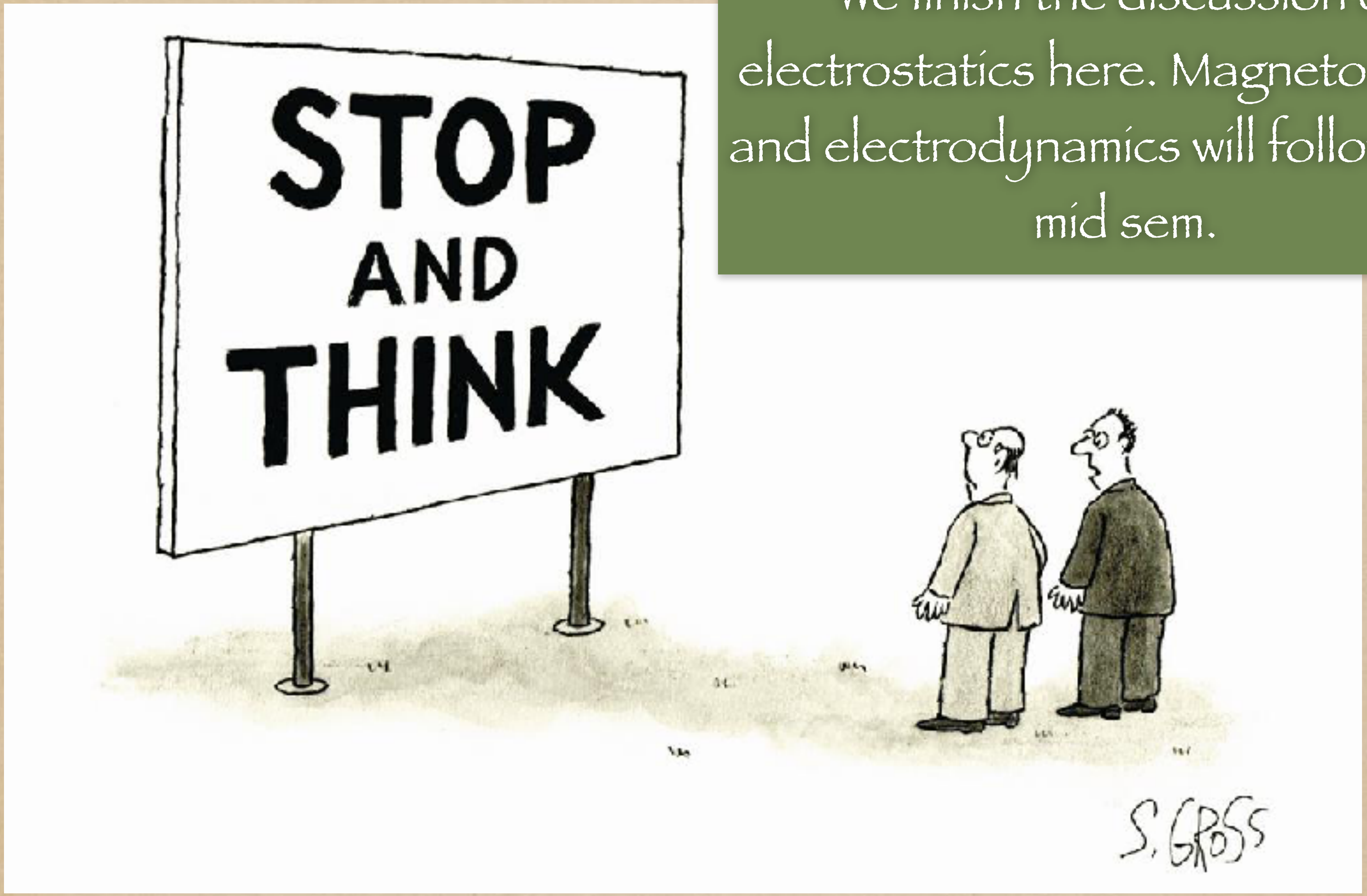
• Here $V dQ$ is the work done by the battery.

• Therefore it follows that $F = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$

• Same result as we obtained by keeping Q fixed. This is expected because force on a dielectric can not depend on whether you plan to hold Q constant or V constant - it is determined entirely by the distribution of charge, free and bound.

• We were able to determine the force without knowing the fringing field!

We finish the discussion on electrostatics here. Magnetostatics and electrodynamics will follow after mid sem.



Thank you !