The Annihilator and Operator Methods for Finding a Particular Solution y_p

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The Annihilator Method for Finding y_p

- This method provides a procedure for finding a particular solution (y_p) such that $L(y_p) = g$, where L is a linear differential operator with constant coefficients and $g(x)$ is a given function. The basic idea is to transform the given nonhomogeneous equation into a homogeneous one.

Definition: A linear differential operator Q is said to annihilate a function $f(x)$ in (a, b) if

$$Q(f)(x) = 0 \text{ for all } x \in (a, b).$$

Example:

1. $f(x) = e^x$, $Q = D - 1$ (Q annihilates e^x).
2. $f(x) = xe^x$, $Q = (D-1)^2$.
3. $f(x) = e^{2x}\sin(4x)$, $Q = (D^2 - 4D + 20)$.

Consider

$$L(y) = g(x),\ L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y,$$

where a_i's are constants.

Suppose $Q(g)(x) = 0$, then $Q(L(y))(x) = Q(g)(x) = 0$.

$$QL(y)(x) = 0 \implies y \in Ker(QL).$$

Determine $Ker(QL)$ and then compare with the general solution of $L(y) = 0$ (i.e., $Ker(L)$) to determine the form of the particular solution to $L(y) = g$.

$g(x)$	Annihilator of g
x^{n-1}	D^n
$e^{\alpha x}$	$(D-\alpha)$
$x^{n-1}e^{\alpha x}$	$(D-\alpha)^n$
$\cos(\beta x)$ or $\sin(\beta x)$	$D^2+\beta^2$
$x^{n-1}\cos(\beta x)$ or $x^{n-1}\sin(\beta x)$	$(D^2+\beta^2)^n$

$g(x)$	Annihilator of g
$e^{\alpha x}\cos(\beta x)$ or $e^{\alpha x}\sin(\beta x)$	$D^2 - 2\alpha D + (\alpha^2 + \beta^2)$
$x^{n-1}e^{\alpha x}\cos(\beta x)$ or $x^{n-1}e^{\alpha x}\sin(\beta x)$	$[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$

Note: If $g(x)$ has the form e^{x^2}, $\log x$, $\frac{1}{x}$, $\tan x$ or $\sin^{-1} x$ the annihilator method will not work.

Example: Find a particular solution of

$$Ly := y'' + y = e^{2x} + 1.$$

Note that $(D-2)(e^{2x}) = 0$ and $D(1) = 0$. Hence,

$$D(D-2)(e^{2x}+1) = 0, \;\; Q = D(D-2).$$

Now,

$$QL(y) = Q(e^{2x}+1) = 0 \Longrightarrow D(D-2)(D^2+1)(y) = 0.$$

Since $Ker(QL) =$ span $\{\cos x, \sin x, e^{2x}, 1\}$, the general solution to $QL(y) = 0$ is

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 e^{2x} + c_4. \qquad (*)$$

Since every solution of $L(y) = g$ is also a solution to $QL(y) = 0$ and the general solution of $L(y) = g$ is

$$y(x) = c_1 \cos x + c_2 \sin x + y_p(x),$$

where $Ker(L) =$ span$\{\cos x, \sin x\}$ and $L(y_p) = e^{2x} + 1$.

Thus, comparing with $(*)$, we obtain $y_p = c_3 e^{2x} + c_4$.

$$L(y_p) = e^{2x}+1 \Longrightarrow 5c_3 e^{2x}+c_4 = e^{2x}+1 \Longrightarrow c_3 = 1/5,\ c_4 = 1.$$

So, the particular solution is $y_p(x) = (1/5)e^{2x} + 1$.

Note: The general solution of $y'' + y = e^{2x} + 1$ is

$$y(x) = c_1 \cos x + c_2 \sin x + (1/5)e^{2x} + 1.$$

Operator Methods for Finding y_p

Writing $Ly = g$ as $P(D)y = g(x)$, where

$$L = P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0.$$

With each $P(D)$, associate a polynomial

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$$

called the auxiliary polynomial of $P(D)$.

If $P(r)$ can be factored as product of n linear factors, say

$$P(r) = a_n (r - r_1)(r - r_2) \cdots (r - r_n),$$

then the corresponding factorization of $P(D)$ has the form

$$P(D) = a_n (D - r_1)(D - r_2) \cdots (D - r_n),$$

where $r_1, r_2, \ldots, r_n$ are the roots of $P(r) = 0$.

Note that

- $Dy_p(x) = g(x) \Rightarrow y_p(x) = \int g(x)dx$. It is natural to define
$$\frac{1}{D}g(x) := \int g(x)dx.$$
- $(D-r)y_p = g(x)$, where r is a constant. Formally, we write
$$y_p = \frac{1}{D-r}g(x).$$
The solution of $(D-r)y_p = g(x)$ is
$$y_p(x) = e^{rx}\int e^{-rx}g(x)dx.$$
(Because $e^{\int P(x)dx}$ is an integrating factor for the ODE $\frac{dy}{dx} + P(x)y = q(x)$.) Thus, we define $\frac{1}{D-r}g(x) := e^{rx}\int e^{-rx}g(x)dx$. Operators like $\frac{1}{D}$, $\frac{1}{D-r}$ are called inverse operators.

Let $\frac{1}{P(D)}$ be the inverse of the operator $P(D)$. Then the particular solution to $P(D)y = g(x)$ is given by

$$y_p(x) = \frac{1}{P(D)} g(x).$$

Method 1:(Successive integrations)
If $P(D) = (D - r_1)(D - r_2) \cdots (D - r_n)$, then

$$\begin{aligned} y_p(x) &= \frac{1}{P(D)} g(x) = \frac{1}{(D - r_1)(D - r_2) \cdots (D - r_n)} g(x) \\ &= \frac{1}{(D - r_1)} \frac{1}{(D - r_2)} \cdots \frac{1}{(D - r_n)} g(x). \end{aligned}$$

Example: Find a particular solution of $y'' - 3y' + 2y = xe^x$.

Here $P(D)y = (D-1)(D-2)y = xe^x$. The particular solution y_p is

$$
\begin{aligned}
y_p(x) &= \frac{1}{D-1}\frac{1}{D-2}xe^x \\
&= \frac{1}{D-1}\left[e^{2x}\int e^{-2x}xe^x dx\right] = \frac{1}{D-1}\left[-(1+x)e^x\right] \\
&= -e^x\int e^{-x}(1+x)e^x dx = -\frac{1}{2}(1+x)^2e^x.
\end{aligned}
$$

Note: The successive integrations are likely to become complicated and time-consuming.

Method 2:(Partial fractions)

If the factors of $P(D)$ are distinct, we can decompose operator $\frac{1}{P(D)}$ into partial fractions as

$$y_p = \frac{1}{P(D)}g(x) = \left[\frac{A_1}{(D-r_1)} + \frac{A_1}{(D-r_2)} + \cdots + \frac{A_n}{(D-r_n)}\right]g(x),$$

for suitable constants A_i's.

Example: Find a particular solution of $y'' - 3y' + 2y = xe^x$.

$$\begin{aligned} y_p(x) &= \frac{1}{(D-1)(D-2)} = \left[\frac{1}{D-2} - \frac{1}{D-1}\right]xe^x \\ &= \frac{1}{D-2}xe^x - \frac{1}{D-1}xe^x \\ &= e^{2x}\int e^{-2x}xe^x dx - e^x \int e^{-x}xe^x dx \\ &= -(1 + x + \frac{1}{2}x^2)e^x. \end{aligned}$$

Method 3:(Series expansions)

If $g(x) = x^n$, expand the inverse operator $\frac{1}{P(D)}$ in a power series in D so that

$$y_p(x) = \frac{1}{P(D)} g(x) = (a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n) g(x),$$

where $(a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n)$ is the expansion of $\frac{1}{P(D)}$ to $n+1$ terms as $D^k x^n = 0$ if $k > n$.

Example: Find y_p of $y''' - 3y'' + 2y = x^4 + 2x + 5$.

$$\frac{1}{1 - 2D^2 + D^3} = 1 + 2D^2 - D^3 + 4D^4 - 4D^5 + \cdots .$$

$$\begin{aligned} y_p(x) &= \frac{1}{1 - 2D^2 + D^3}(x^4 + 2x + 5) \\ &= (1 + 2D^2 - D^3 + 4D^4 - 4D^5 + \cdots)(x^4 + 2x + 5) \\ &= (x^4 + 2x + 5) + 2(12x^2) - (24x) + 4(24) \\ &= x^4 + 24x^2 - 22x + 101. \end{aligned}$$

Method 4: If $g(x) = e^{\alpha x}$, α a constant, then

$$(D - r)e^{\alpha x} = (\alpha - r)e^{\alpha x}.$$

Operating both sides of the above identity by $(\alpha - r)^{-1}(D - r)^{-1}$, we obtain

$$\frac{1}{(D - r)}e^{\alpha x} = \frac{1}{(\alpha - r)}e^{\alpha x},$$

provided $\alpha \neq r$. Similarly, if $P(D) = (D - r_1) \cdots (D - r_n)$ then

$$\begin{aligned} \frac{1}{P(D)}e^{\alpha x} &= \frac{1}{(D - r_1) \cdots (D - r_n)}e^{\alpha x} \\ &= \frac{1}{(\alpha - r_1) \cdots (\alpha - r_n)}e^{\alpha x}, \end{aligned}$$

provided $r_1, \ldots, r_2$ are distinct from α.

- If $P(D)$ is a polynomial in D such that $P(\alpha) \neq 0$, then

$$\frac{1}{P(D)}e^{\alpha x} = \frac{e^{\alpha x}}{P(\alpha)}.$$

Example: Find a particular solution of

$$y''' - y'' + y' + y = 3e^{-2x}.$$

$$\begin{aligned} y_p &= \frac{1}{P(D)} 3e^{-2x} \\ &= \frac{3e^{-2x}}{P(-2)} \\ &= \frac{3e^{-2x}}{(-2)^3 - (-2)^2 - 2 + 1} \\ &= -\frac{3}{13} e^{-2x}. \end{aligned}$$

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