MA 102 (Mathematics II)

Department of Mathematics, IIT Guwahati

Tutorial Sheet No. 6

---

(1) Find all the critical points of $f(x, y) = \sin x \sin y$ in the domain $-2 \leq x \leq 2,\ -2 \leq y \leq 2$.

(2) Find all the local maxima, local minima and saddle points of the following functions:

(a) $f(x,\ y) = x^2 + xy + y^2 + 3x - 3y + 4$ (b) $f(x,\ y) = 6x^2 - 2x^3 + 3y^2 + 6xy$

(3) Let $f(x,\ y) = xy - x^2$, and let R be the square region given by $R = \{(x,\ y) \in \mathbb{R}^2\ :\ 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$. Find the extreme values of f on R.

(4) Verify that $f(x,\ y,\ z) = x^4 + y^4 + z^4 - 4xyz$ has a critical point $(1,\ 1,\ 1)$, and determine the nature of this critical point by computing the eigenvalues of its Hessian matrix.

(5) Using the method of Lagrange multipliers, find the extremum values of $f(x,\ y) = xy$ subject to the constraint $g(x,\ y) = x^2 + y^2 - 10 = 0$.

(6) Using the method of Lagrange multipliers, find the points on the curve $xy^2 = 54$ nearest to the origin.

(7) Evaluate the double integral $\iint_R f(x, y)\ dxdy$ for f and R given below.

(a) $f(x, y) := x^2 + y^2$ and $R = [-1, 1] \times [0, 1]$.

(b) $f(x, y) := x^2 + y$ and $R = [0, 1] \times [0, 1]$.

(c) $f(x, y) := \sin(x + y)$ and $R = [0, \pi] \times [0, \pi]$.

(8) Evaluate the following double integrals.

(a) $\displaystyle\int_0^3 \int_{-y}^{y} (x^2 + y^2) dxdy$ (b) $\displaystyle\int_0^{\pi} \int_0^{\pi} |\cos(x + y)| dxdy$

(c) $\displaystyle\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dy\ dx$ (d) $\displaystyle\int_0^{3/2} \int_0^{9-4x^2} 16x dy\ dx$

(9) Evaluate the following double integrals.

(a) $\displaystyle\iint_R \frac{dA}{\sqrt{xy - x^2}}$, where R is the region bounded by $x = 0, x = 1, y = x$ and $y = x + 1$.

(b) $\displaystyle\int_0^1 \int_0^{1-x} e^{\frac{x-y}{x+y}} dxdy$ (c) $\displaystyle\int_0^{1/\sqrt{2}} \int_y^{\sqrt{1-y^2}} (x + y) dx\ dy$

(d) $\displaystyle\iint_R \cos(9x^2 + 4y^2)\ dx\ dy$, where $R = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^2 \leq 1\}$.

(10) Find the volume of the following:

(a) Region under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x, x = 0$, and $x + y = 2$ in the xy plane.

(b) Region bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and line $y = x$ in the xy plane.

(c) Region bounded in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

(d) Solid cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

(e) Tetrahedron bounded by the planes $y = 0, z = 0, x = 0$ and $-x + y + z = 1$.

(11) Evaluate the following triple integrals:

(a) $\iiint_D (z^2x^2 + z^2y^2)\, dV$, where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$

(b) $\iiint_D xyz\, dV$ where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \leq 1,\ 0 \leq z \leq x^2 + y^2\}$

(c) $\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\, dV$ where $D = \{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 + z^2 \leq 1\}$

(12) Find the volume of the following regions using triple integrals:

(a) The region in the first octant bounded by the coordinate planes and the planes $x+z = 1, y + 2z = 2$.

(b) The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$.

(c) The tetrahedron in the first octant bounded by the coordinate planes and the plane $x + y/2 + z/3 = 1$.

(d) The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

(e) The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$.

(f) The region enclosed by $y = x^2, y = x + 2, 4z = x^2 + y^2$ and $z = x + 3$.

(g) The region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

(h) The solid bounded by the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$.

(13) Evaluate the line integral $\int_\Gamma F \bullet dr$ of the vector field F given below.

(a) $F(x, y) := (x^2 + 2xy,\ y^2 - 2xy)$ from $(-1, 1)$ to $(1, 1)$ along $y = x^2$.

(b) $F(x, y) := (x^2 - y^2,\ x - y)$ and $\Gamma : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the counterclockwise direction.

(14) Evaluate the line integral $\displaystyle\int_\Gamma \frac{(x + y)dx - (x - y)dy}{x^2 + y^2}$ along $\Gamma : x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.

(15) Evaluate the line integral $\displaystyle\int_\Gamma \frac{x^2ydx - x^3dy}{(x^2 + y^2)^2}$, where Γ is the square with vertices $(\pm1, \pm1)$ oriented in the counter clockwise direction.

(16) Verify Green's theorem in each of the following cases:

(a) $f(x, y) := -xy^2;\ g(x, y) := x^2y;$ the region R is given by $x \geq 0, 0 \leq y \leq 1 - x^2$.

(b) $f(x, y) := 2xy;\ g(x, y) := e^x + x^2;$ the region R is the triangle with vertices $(0, 0), (1, 0)$ and $(1, 1)$.

(17) Evaluate $\int_\Gamma (y^2dx + xdy)$ using Green's theorem, where Γ is boundary of R and

(a) R is the square with vertices $(0, 0), (0, 2), (2, 2), (2, 0)$.

(b) R is the square with vertices $(\pm1, \pm1)$.

(c) R is the disc of radius 2 and center $(0, 0)$.

(18) Determine which of the following vector fields F is conservative and find a scalar potential when it exists.

(a) $F(x, y) = (\cos(xy) - xy\sin(xy), x^2\sin(xy))$.

(b) $F(x, y) = (xy, xy)$.

(c) $F(x, y, z) = (x^2, xy, 1)$.