MA-102, Tutorial Group-7 January 15, 2018

♣ Tutorial, Solution ♣

Rakesh Jana Department of Mathematics, IIT Guwahati

Problem 1. Consider the Euclidean norm $\|X\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and show that $\big| \|X\| - \|Y\| \big| \leq \|X - Y\|$ for $X, Y \in \mathbb{R}^n$. Show that the vectors X and Y are orthogonal if and only if $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$.

Remark (How to think)**.** Note that to show $|a| \leq b$ for $a, b \in \mathbb{R}$, we need to show $-b \leq a \leq b$.

Solution. Notice that $\|X\| = \|(X - Y) + Y\| \leq \|X - Y\| + \|Y\|$ and similarly we can get $\|Y\| = \|(Y - X) + X\| \leq \|Y - X\| + \|X\| = \|X - Y\| + \|X\|$. Thus we have

$$- \|X - Y\| \leq (\|X\| - \|Y\|) \leq \|X - Y\| \implies \big| \|X\| - \|Y\| \big| \leq \|X - Y\| .$$

Notice the following

$$\|X + Y\|^2 = \langle X + Y, X + Y, =\rangle \langle X, X\rangle + \langle X, Y\rangle + \langle Y, X\rangle + \langle Y, Y\rangle = \|X\|^2 + \|Y\|^2 + 2\langle X, Y\rangle.$$

Last equality hold because $\langle X, Y\rangle = \langle Y, X\rangle$ for $X, Y \in \mathbb{R}^n$. Thus if X and Y are orthogonal then $\langle X, Y\rangle = 0$ and so $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$. Again if $\|X + Y\|^2 = \|X\|^2 + \|Y\|^2$ then $\langle X, Y\rangle = 0$ and so the vectors X and Y are orthogonal.

Problem 2. Let $(X_k) \subset \mathbb{R}^n$ and $X \in \mathbb{R}^n$. Show that $X_k \to X$ in $\mathbb{R}^n$ if and only if for every $Y \in \mathbb{R}^n$ the sequence $(\langle X_k, Y\rangle) \subset \mathbb{R}$ converges to $\langle X, Y\rangle$, that is, $\langle X_k, Y\rangle \to \langle X, Y\rangle$ in $\mathbb{R}$.

Remark (Cauchy-Schwarz inequality)**.** If $X, Y \in \mathbb{R}^n$ then $|\langle X, Y\rangle| \leq \|X\| \, \|Y\|$, equality holds if and only if X and Y are linearly dependent (meaning they are parallel).

If $X = (x_1, \cdots, x_n) \in \mathbb{R}^n$ then $x_i = \langle X, e_i\rangle$, where $e_i = (0, \cdots, 0, \underbrace{1}_{\text{i-th place}}, 0, \cdots, 0) \in \mathbb{R}^n$.

Solution. First suppose that $X_k \to X$ in $\mathbb{R}^n$ and $Y \in \mathbb{R}^n$. Then Cauchy-Schwarz inequality yields that

$$|\langle X_k, Y\rangle - \langle X, Y\rangle| = |\langle X_k - X, Y\rangle| \leq \|X_k - X\| \, \|Y\|$$

Since $\|Y\|$ is fixed and $X_k \to X$, that means $\|X_k - X\| \to 0$ gives us $\langle X_k, Y\rangle \to \langle X, Y\rangle$.

Conversely let us assume $\langle X_k, Y\rangle \to \langle X, Y\rangle$ for any $Y \in \mathbb{R}^n$. Let us assume $X_k = (x_{k,1}, \cdots, x_{k,n})$ and $X = (x_1, \cdots, x_n)$. For $i = 1, 2, \cdots, n$, taking $Y = e_i$ we have $\langle X_k, Y\rangle = \langle X_k, e_i\rangle = x_{k,i}$ and $\langle X, Y\rangle = \langle X, e_i\rangle = x_i$. Thus $x_{k,i} \to x_i$ in $\mathbb{R}$ for $i = 1, \cdots, n$ and so $X_k \to X$ in $\mathbb{R}^n$.

Problem 3. Let $(X_k) \subset \mathbb{R}^n$ be such that $X_k \to X$ for some $X \in \mathbb{R}^n$. Show that the sequence $(\|X_k\|) \subset \mathbb{R}$ converges to $\|X\|$. Additionally suppose that $X \neq 0$ and $X_k \neq \mathbf{0}$ for all k, and define $Y_k := X_k / \|X_k\|$ and $Y := X / \|X\|$. Show that $Y_k \to Y$.

Solution. Let $X_k \to X$ for some $X \in \mathbb{R}^n$, that is, $\|X_k - X\| \to 0$. From Problem-(1) we have $|\,\|X_k\| - \|X\|\,| \leq \|X_k - X\| \to 0$ and so $\|X_k\| \to \|X\|$.

Conversely, since $X \neq 0$ and $X_k \neq 0$ for all k, gives $\|X\| \neq 0$ and $\|X_k\| \neq 0$ for all k. Again $\|X_k\| \to \|X\|$ gives us $\dfrac{1}{\|X_k\|} \to \dfrac{1}{\|X\|}$. Now applying limit rule we get $\dfrac{X_k}{\|X_k\|} \to \dfrac{X}{\|X\|}$.

Problem 4. Let $(X_k) \subset \mathbb{R}^n$ and $X, Y \in \mathbb{R}^n$. Suppose that $X_k \to X$ and that $\langle X_k, Y\rangle = 0$ for all k. Show that $\langle X, Y\rangle = 0$.

Solution. Let $Y \in \mathbb{R}^n$ such that $\langle X_k, Y\rangle = 0$ for all k. From Problem-(2) we have $\langle X_k, Y\rangle \to \langle X, Y\rangle$ as $X_k \to X$. Now $(\langle X_k, Y\rangle)$ is a zero sequence and so $\langle X, Y\rangle = 0$.

Remark. The relation between spherical coordinates (ρ, ϕ, θ), cylindrical coordinates (r, θ, z) and rectangular coordinate (x, y, z) are given by

$$r = \rho \sin\phi, \quad z = \rho\cos\phi, \quad \theta = \theta \quad | \quad x = r\cos\theta, \quad y = r\sin\theta$$

where $r = \sqrt{x^2 + y^2} \geq 0$, $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} \geq 0$, and $\tan\theta = \dfrac{y}{x}$ with $0 \leq \theta < 2\pi$ and $\tan\phi = \dfrac{r}{z}$ with $0 \leq \phi \leq \pi$.

Note: Depending on position of (x, y) we have to choose θ accordingly. Value of θ will as follows:

$$\theta \in \begin{cases} \left[0, \frac{\pi}{2}\right] & \text{if } x \geq 0 \text{ and } y \geq 0 \quad \text{(first quadrant)} \\ \left(\frac{\pi}{2}, \pi\right] & \text{if } x < 0 \text{ and } y \geq 0 \quad \text{(second quadrant)} \\ \left(\pi, \frac{3\pi}{2}\right] & \text{if } x < 0 \text{ and } y \leq 0 \quad \text{(third quadrant)} \\ \left(\frac{3\pi}{2}, 2\pi\right] & \text{if } x > 0 \text{ and } y < 0 \quad \text{(fourth quadrant)} \end{cases}$$

Problem 5. Convert from rectangular coordinates (x, y, z) to spherical coordinates (ρ, ϕ, θ).

(a) $(1, \sqrt{3}, -2)$ (b) $(1, -1, \sqrt{2})$

Solution. (a) Given that $x = 1, y = \sqrt{3}, z = -2$. Then $\rho = \sqrt{1+3+4} = 2\sqrt{2}$ and $\tan\phi = \frac{2}{-2} = -1$. Thus $\phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. Again $\tan\theta = \frac{\sqrt{3}}{1}$ gives $\theta = \pi/3$. Thus $(\rho, \phi, \theta) = (2\sqrt{2}, 3\pi/4, \pi/3)$.

(b) Given that $x = 1, y = -1, z = \sqrt{2}$. Then $\rho = \sqrt{1+1+2} = 2$ and $\tan\phi = \frac{\sqrt{2}}{\sqrt{2}} = 1$. Thus $\phi = \frac{\pi}{4}$. Again $\tan\theta = -1$ gives $\theta = 2\pi - \pi/4 = \frac{7\pi}{4}$, as (x, y) in fourth quadrant. Thus $(\rho, \phi, \theta) = (2, \pi/4, \frac{7\pi}{4})$.

Problem 6. Convert from spherical coordinates (ρ, ϕ, θ) to rectangular coordinates (x, y, z).

(a) $(5, \pi/6, \pi/4)$ (b) $(7, \pi/2, \pi/2)$

Solution. (a) Given that $\rho = 5, \phi = \pi/6, \theta = \pi/4$. Using the relation between spherical coordinates and cartesian coordinates we get $x = r\cos\theta = \rho\cos\theta\sin\phi = 5 \cdot \cos\pi/4 \cdot \sin\pi/6 = \frac{5}{2\sqrt{2}}$ and $y = r\sin\theta = \rho\sin\theta\sin\phi = 5 \cdot \sin\pi/4 \cdot \sin\pi/6 = \frac{5}{2\sqrt{2}}$. Again $z = \rho\cos\phi = \frac{5\sqrt{3}}{2}$. Thus

Solution (Cont.)

$(x, y, z) = \left(\frac{5}{2\sqrt{2}}, \frac{5}{2\sqrt{2}}, \frac{5\sqrt{3}}{2}\right)$.

(b) Given that $\rho = 7, \phi = \pi/2, \theta = \pi/2$. Using the relation between spherical coordinates and cartesian coordinates we get $x = \rho\cos\theta\sin\phi = 7 \cdot 1 \cdot 0 = 0$ and $y = \rho\sin\theta\sin\phi = 7 \cdot 1 \cdot 1 = 7$. Again $z = \rho\cos\phi = 0$. Thus $(x, y, z) = (0, 7, 0)$.

Problem 7. Convert from cylindrical coordinates (r, θ, z) to spherical coordinates (ρ, ϕ, θ).

(a) $\left(\sqrt{3}, \pi/6, 3\right)$ (b) $(1, \pi/4, -1)$

Solution. (a) Given that $r = \sqrt{3}, \theta = \pi/6, z = 3$. Then we have $\phi = \sqrt{r^2 + z^2} = \sqrt{3 + 9} = 2\sqrt{3}$ and $\tan\phi = \frac{r}{z} = \frac{\sqrt{3}}{3}$ gives $\phi = \pi/6$. Thus $(\rho, \phi, \theta) = (2\sqrt{3}, \pi/6, \pi/6)$.

(b) Given that $r = 1, \theta = \pi/4, z = -1$. Then we have $\phi = \sqrt{r^2 + z^2} = \sqrt{2}$ and $\tan\phi = \frac{r}{z} = -1$ gives $\phi = \pi - \pi/4 = \frac{3\pi}{4}$. Thus $(\rho, \phi, \theta) = (\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{4})$.

Problem 8. Convert from spherical coordinates (ρ, ϕ, θ) to cylindrical coordinates (r, θ, z).

(a) $(5, \pi/4, 2\pi/3)$ (b) $(1, \pi/2, 7\pi/6)$

Solution. (a) Given that $\rho = 5, \phi = \pi/4, \theta = 2\pi/3$. Using the relation between spherical coordinates and cylindrical coordinates we get $r = \rho\sin\phi = \frac{5}{\sqrt{2}}$ and $z = \rho\cos\phi = \frac{5}{\sqrt{2}}$. Thus $(r, \theta, z) = \left(\frac{5}{\sqrt{2}}, \frac{2\pi}{3}, \frac{5}{\sqrt{2}}\right)$.

(b) Given that $\rho = 1, \phi = \pi/2, \theta = 7\pi/6$. Using the relation between spherical coordinates and cylindrical coordinates we get $r = \rho\sin\phi = 1$ and $z = \rho\cos\phi = 0$. Thus $(r, \theta, z) = \left(1, \frac{7\pi}{6}, 0\right)$.

Problem 9. For each of the following sets in their mentioned spaces, identify (i) interior points, (ii) limit points, (iii) boundary points, (iv) Closure of the set.

(a) Space $= \mathbb{R}$, $S = \{1, 2, 3, 4\}$

(b) Space $= \mathbb{R}$, $S = \mathbb{Q}$

(c) Space $= \mathbb{R}$, $S = \{x \in \mathbb{R} : 0 < x < 1\}$

(d) Space $= \mathbb{R}^2$, $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } y = 0\}$

(e) Space $= \mathbb{R}^2$, $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } y \in \mathbb{R}\}$

(f) Space $= \mathbb{R}^2$, $S = \mathbb{Q}^c \times \mathbb{Q}$

(g) Space $= \mathbb{R}^2$, $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 100\}$

(h) Space $= \mathbb{R}^3$, $S = \left\{\left(\frac{1}{n}, 0, 0\right) \in \mathbb{R}^3 : n \in \mathbb{N}\right\}$

(i) Space $= \mathbb{R}^n$, $S = \left\{(k, 0, 0, \cdots, 0) \in \mathbb{R}^n : k = 1, 2, \cdots, 10^{13}\right\}$

Solution. (a) Open intervals are only open balls in $\mathbb{R}$. Notice that S does not contain any open ball and hence $int(S) = \emptyset$. Notice that for any element x, take $\epsilon = \frac{1}{2}\min\{|x-1|, |x-2|, |x-3|, |x-4|\}$ if $x \notin S$ and $\epsilon = \frac{1}{2}$ if $x \in S$ then $S \cap (B(x, \frac{1}{2}) \setminus \{x\}) = \emptyset$ and so there is no limits point of S. Clearly $bd(S) = S$, as for $x \in S$ and $\epsilon > 0$, $x \in B(x, \epsilon)$ and $B(x, \epsilon) \setminus S \neq \emptyset$ and for $x \notin S$ take $\epsilon = \frac{1}{2}\min\{|x-1|, |x-2|, |x-3|, |x-4|\}$ then $B(x, \epsilon) \cap S = \emptyset$. Clearly $cl(S) = S$.

(b) Notice that S does not contain any open ball and hence $int(S) = \emptyset$. Since $\mathbb{Q}$ dense in $\mathbb{R}$. Thus every points in $\mathbb{R}$ is limit points of S. Again $bd(S) = \mathbb{R}$, as $\mathbb{R} \setminus \mathbb{Q}$ dense in R. Clearly $cl(S) = \mathbb{R}$.

(c) $int(S) = (0, 1)$, limit point $(S) = [0, 1]$, $bd(S) = \{0, 1\}$ and $cl(S) = [0, 1]$.

(d) Since S does not contain any open disk thus $int(S) = \emptyset$, limit point $(S) = [0, 1] \times \{0\}$, $bd(S) = [0, 1] \times \{0\}$ and $cl(S) = [0, 1] \times \{0\}$.

(e) $int(S) = S$, limit point $(S) = [0, 1] \times \mathbb{R}$, $bd(S) = \{0, 1\} \times \mathbb{R}$ and $cl(S) = [0, 1] \times \mathbb{R}$.

(f) $int(S) = \emptyset$, limit point $(S) = \mathbb{R}^2$, $bd(S) = \mathbb{R}^2$ and $cl(S) = \mathbb{R}^2$. (Answer is same as (b))

(g) $int(S) = S$, limit point $(S) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 100\}$, $bd(S) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 100\}$ and $cl(S) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 100\}$.

(h) Clearly S does not contain any open sphere and so $int(S) = \emptyset$. Let $x \in \mathbb{R}^3$. If $x = (\frac{1}{t}, 0, 0) \in S$ choose $\epsilon = \frac{1}{t} - \frac{1}{t+1} > 0$ then $S \cap (B(x, \epsilon) \setminus \{x\}) = \emptyset$. If $x = (x_1, x_2, x_3) \notin S$ then choose $\epsilon = x_1 - \frac{1}{t}$ if $\frac{1}{t} < x_1 < \frac{1}{t+1}$ and $\epsilon = \min\{\frac{|x_1-1|}{2}, \frac{|x_1|}{2}\}$ if $x_1 \neq \frac{1}{t}$. Then $S \cap (B(x, \epsilon) \setminus \{x\}) = \emptyset$.

(i) $int(S) = \emptyset$, limit point $(S) = \emptyset$, $bd(S) = S$ and $cl(S) = S$.

Problem 10. For each of the following sets in their mentioned spaces, find out whether the given set is (i) open, (ii) closed, (iii) bounded.

(a) Space $= \mathbb{R}^2$, $S = \{(x, y) \in \mathbb{R}^2 : xy > 0\}$

(b) Space $= \mathbb{R}^2$, $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } y \geq 0\}$

(c) Space $= \mathbb{R}^2$, $S = \{(x, y) \in \mathbb{R}^2 : y < 1\}$

(d) Space $= \mathbb{R}^3$, $S = \left\{\left(\frac{1}{k}, k, 0\right) \in \mathbb{R}^3 : k \in \mathbb{N}\right\}$

Solution. (a) S is open but not closed and not bounded.

(b) S is not open but closed and bounded.

(c) S is open but not closed and not bounded.

(d) S is not open and not bounded but closed.

Clearly S does not contain any open sphere. Thus $int(S) = \emptyset$ and so S is not open. Suppose S is bounded then there exist $M > 0$ such that $S \subset B(0, M)$, that is, $\|x\| < M$ for all $x \in S$. This gives $1 + k^4 < M^2k^2$ for all $k \in \mathbb{N}$. Taking $k = [M] + 1$ then $1 + k^4 > M$, a contradiction. Thus S is unbounded. There is limit point of this set and so S is closed.

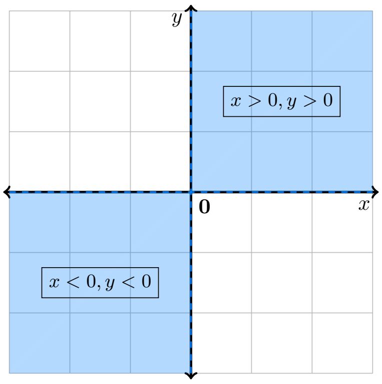

Figure 1: 10(a)

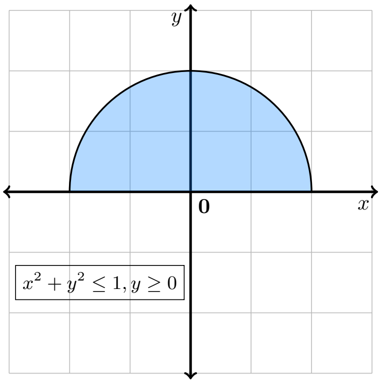

Figure 2: 10(b)

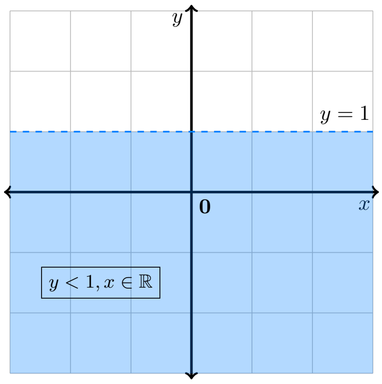

Figure 3: 10(c)

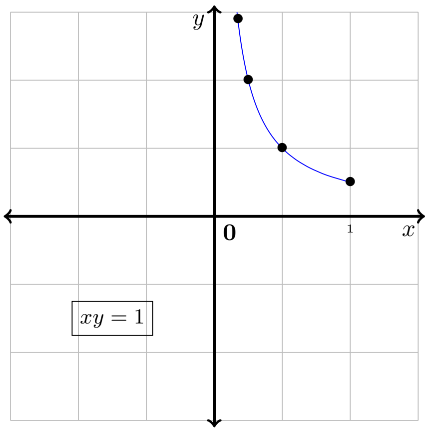

Figure 4: 10(d)