Physics II: Electromagnetism (PH102)

Lecture 11

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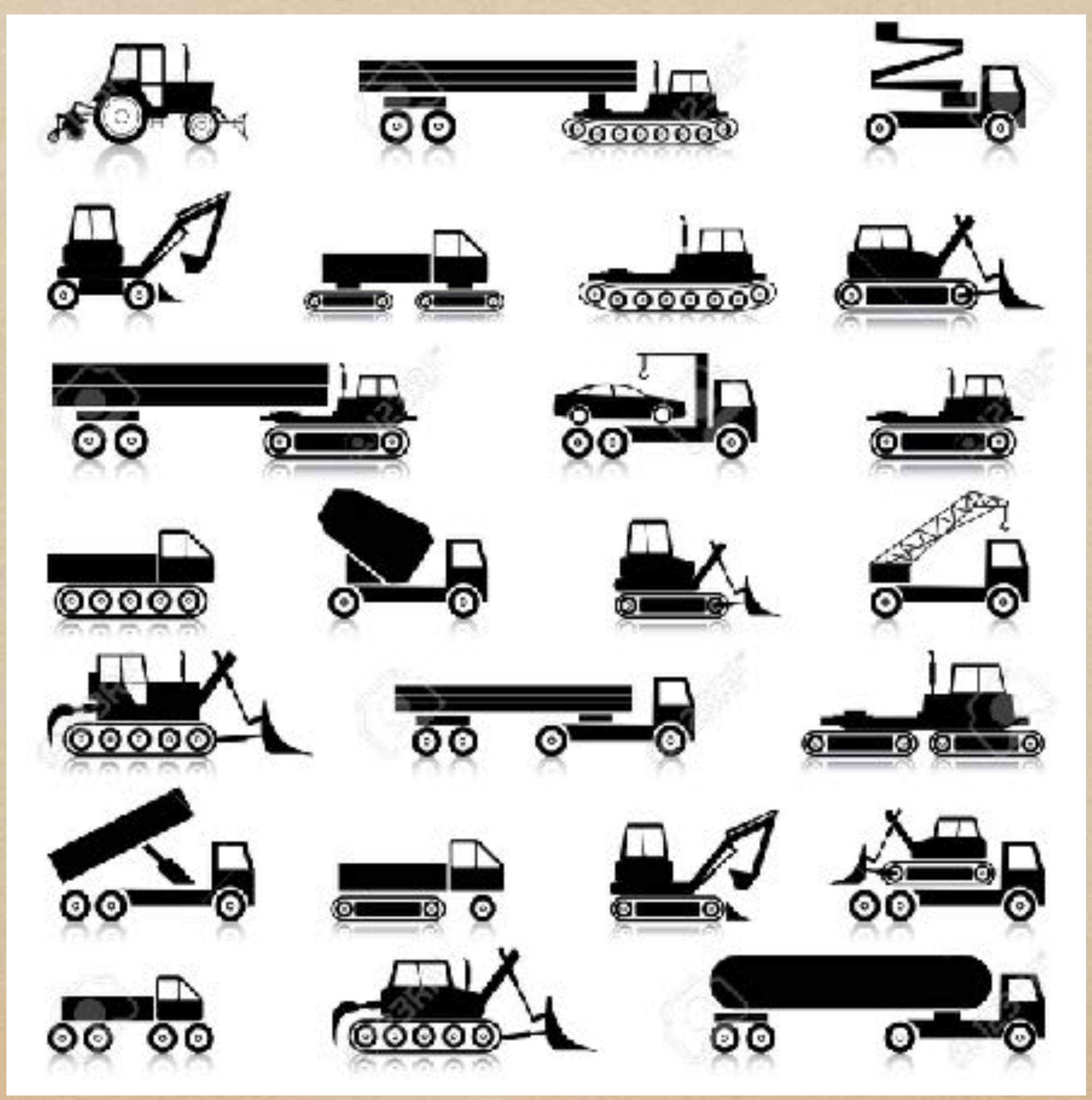
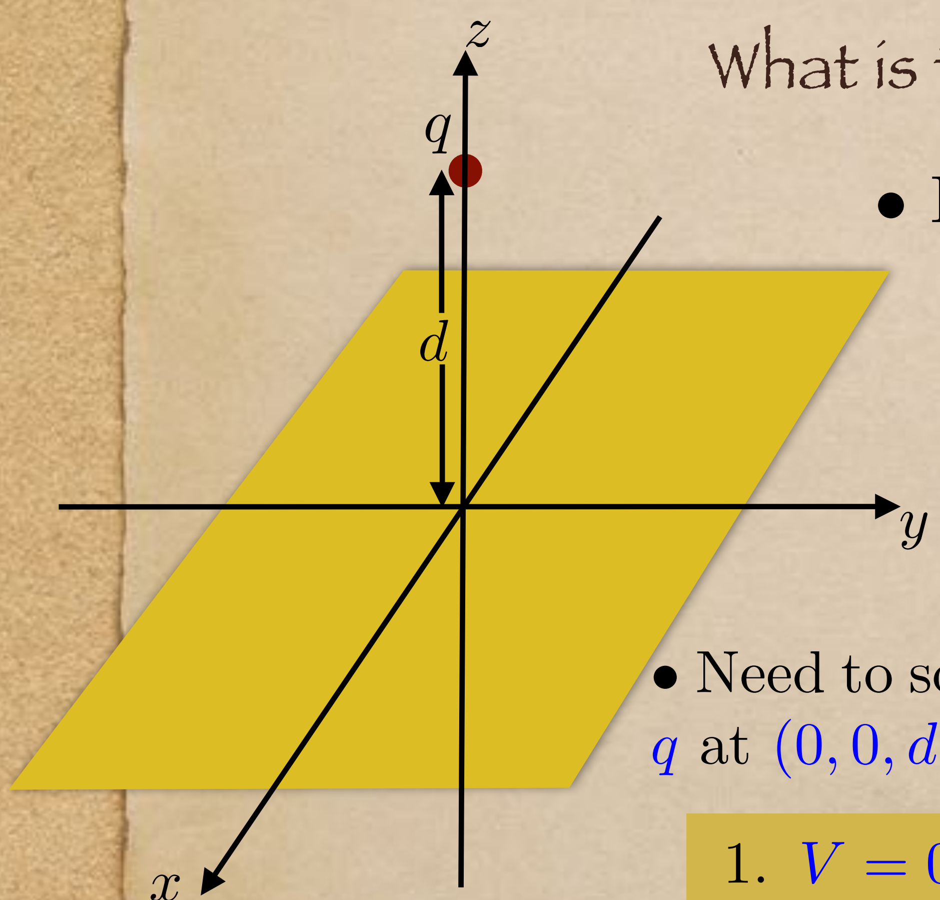
Special techniques to solve for potential



A point charge on top of a conductor

Suppose, we need to solve for the case where we have placed a point charge at a certain point above a conductor plane (infinitely spread).



What is the potential above ? • It is not $\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$.

• Because q will induce negative charges on the conductor.

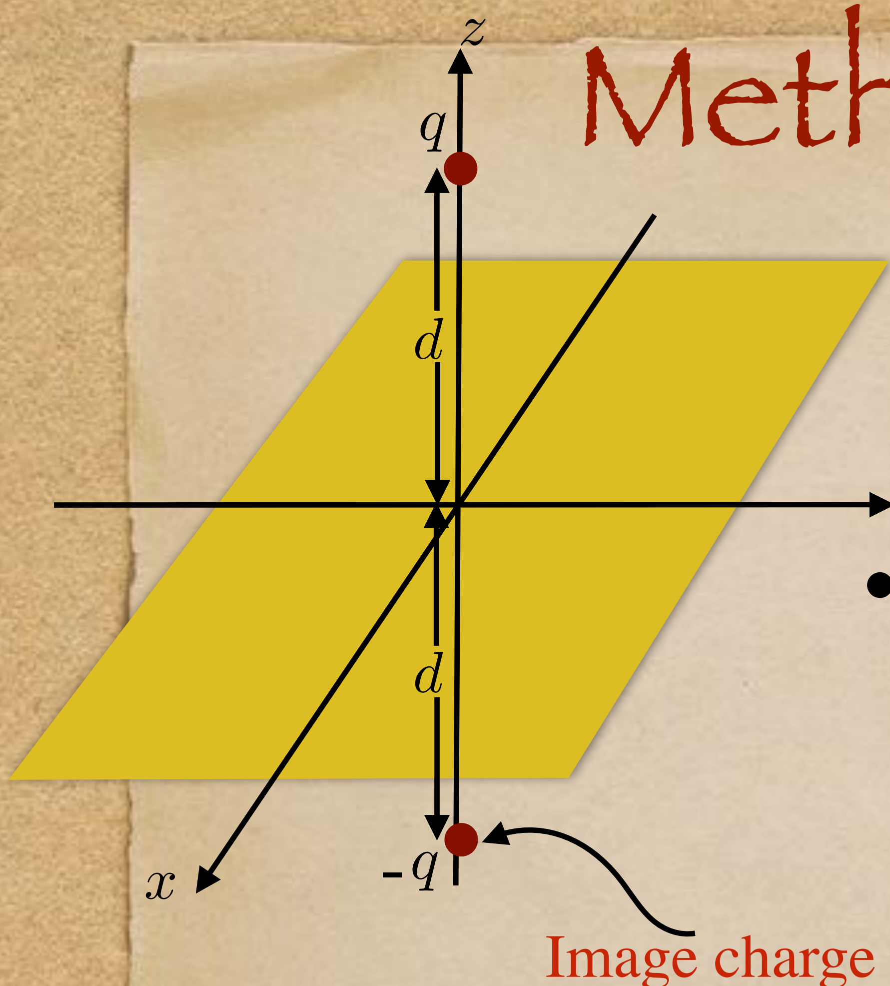
The potential will be due to the charge and the induced charge together.

• Need to solve Poisson's equation in the region $z > 0$, with a single point charge q at $(0, 0, d)$, subject to the boundary condition:

1. $V = 0$ at $z = 0$ (since conducting plane is grounded),
2. $V \to 0$ far from the charge (i.e. for $x^2 + y^2 + z^2 >> d$).

• First uniqueness theorem tells us that there is only one function that meets these requirements. If by trick or clever guess we can find the function, it is going to be the answer.

Method of Image charges



We will forget the actual problem and imagine two point charges +q (0,0,d) and -q (0,0,-d)

- Potential for such a configuration

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

- It follows that (1) $V = 0$ at $z = 0$; (2) $V \to 0$ for $x^2+y^2+z^2 >> d^2$.
- And the only charge in $z > 0$ is $+q$ at $(0,0,d)$.

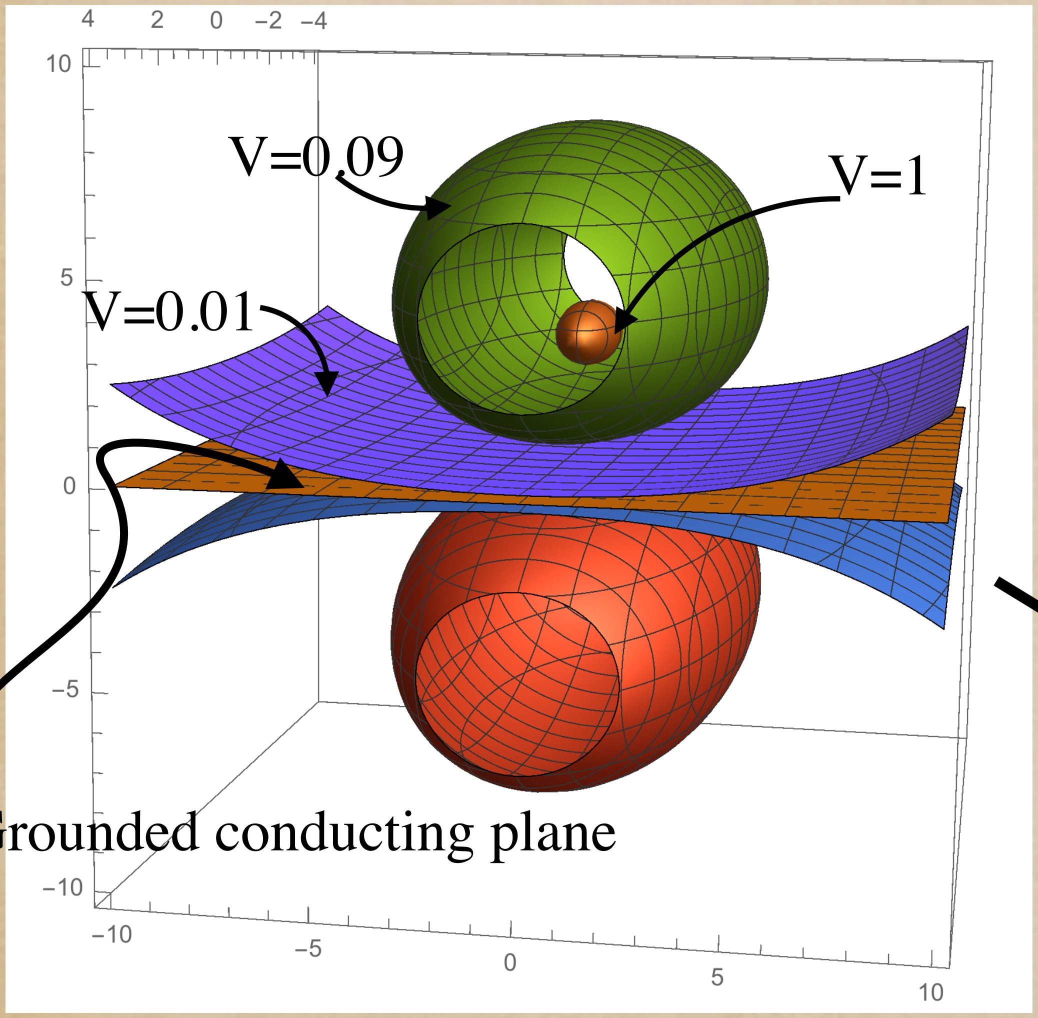
This is precisely the boundary conditions provided to solve the problem.

Hence, this is the solution ! The potential for a charged particle in front of an infinitely grounded conductor can be obtained by above formula and assuming an 'image charge' with conductor behaving as mirror

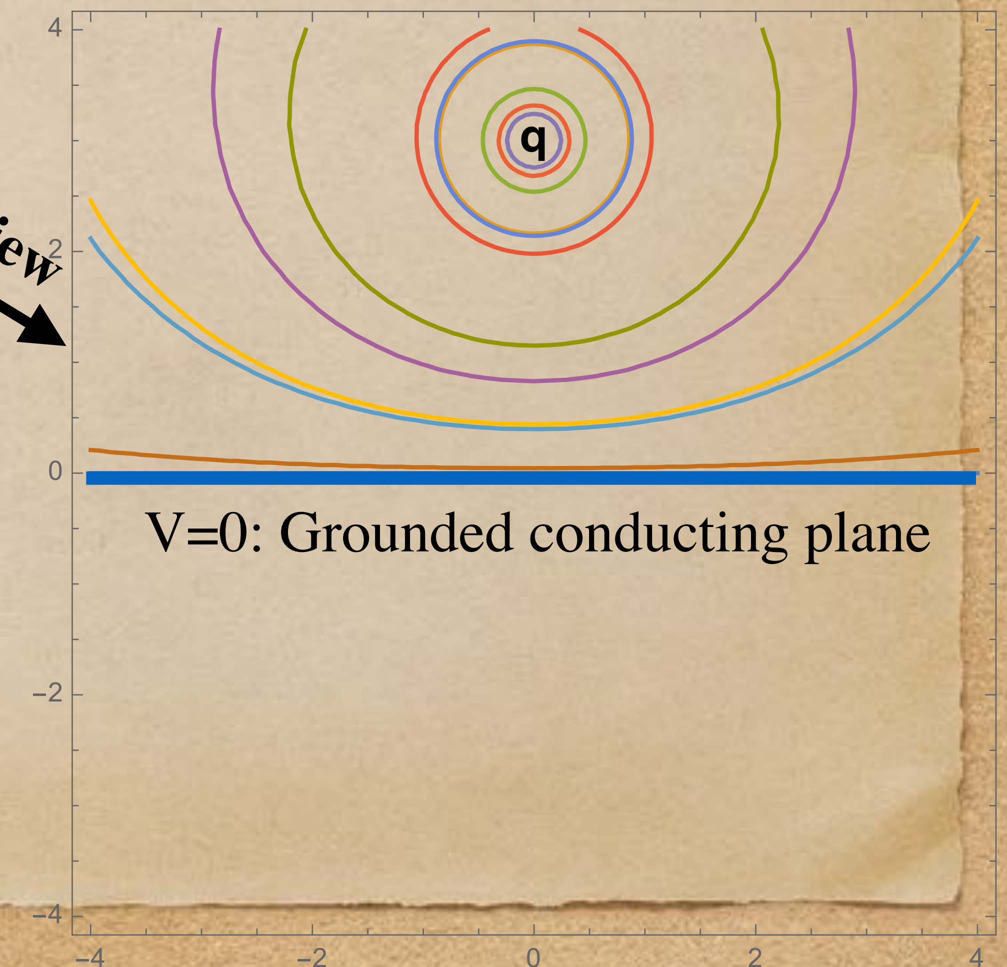
Equipotential surfaces for solutions obtained by method of images

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Equipotential surface: $V(x,y,z) = V_0$



Cross sectional view

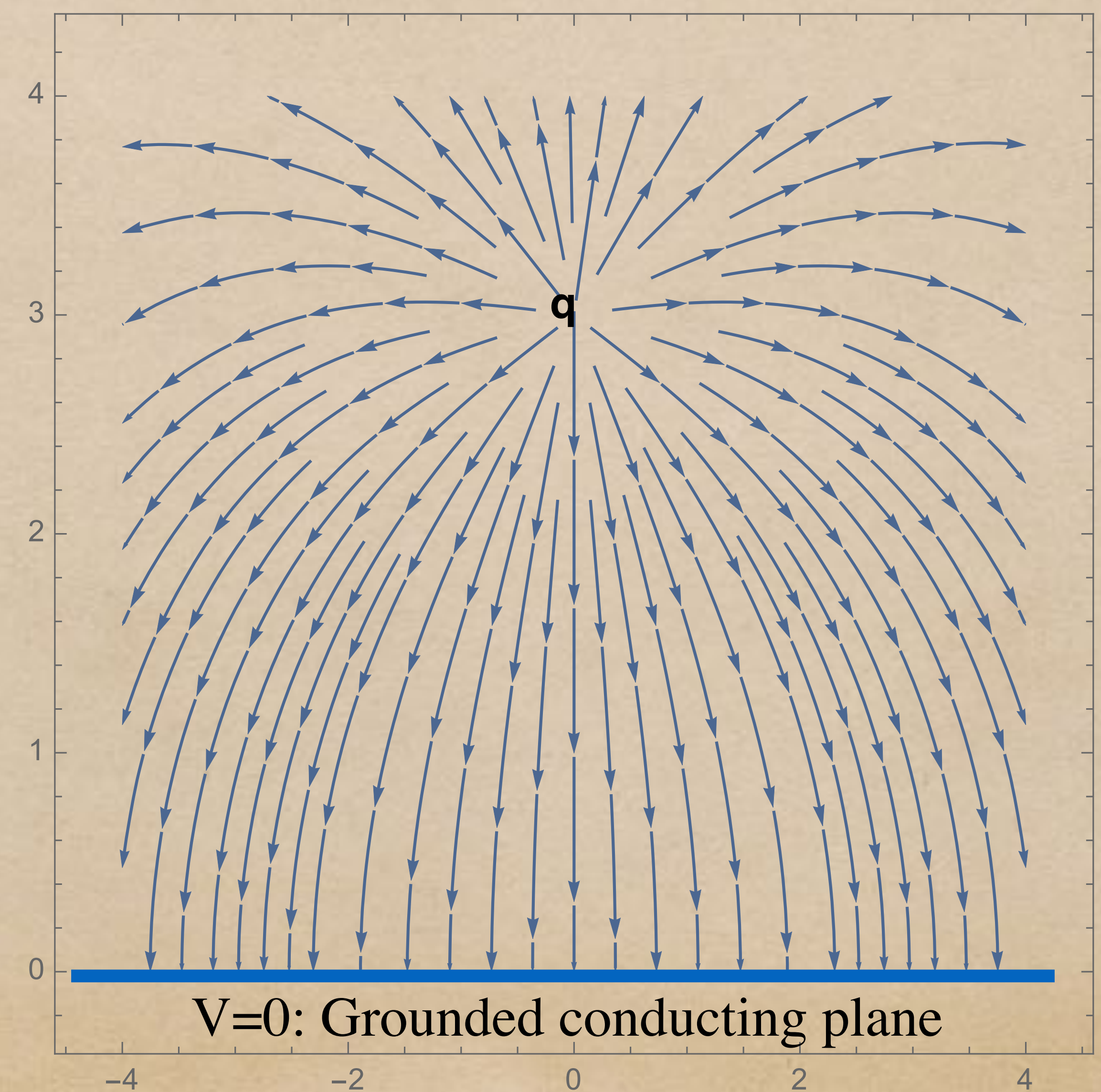


We are not interested in solutions for z<0. Actually, the region where image charge is present, the solution is unphysical.

Electrostatic field in image problem

We will simply calculate the gradient of the potential to find electric field.

$$\vec{E}(x,y,z) = -\vec{\nabla}V = \frac{q}{4\pi\epsilon_0}\left[\frac{x\hat{x}+y\hat{y}+(z-d)\hat{z}}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{x\hat{x}+y\hat{y}+(z+d)\hat{z}}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$



This is how the electric field spreads out for a point charge in front of a conductor !

Can I calculate induced surface charge ?

Of course yes, as we already know the potential at z>0

Recall, that the electric field immediately above a charged surface :

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n} \implies \sigma = -\epsilon_0 \frac{\partial V}{\partial n} \implies \boxed{\sigma = -\epsilon_0 \frac{\partial V}{\partial z}}$$

Electric field below is zero !

where $\frac{\partial V}{\partial n}$ is the normal derivative at the surface, and for us, the direction of the normal is in z direction:

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left[\frac{-q(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$

$$\boxed{\sigma(x,y) = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = -\frac{\epsilon_0}{4\pi\epsilon_0}\frac{2qd}{(x^2+y^2+d^2)^{3/2}} = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}}$$

Well, the negative sign upfront is obvious !

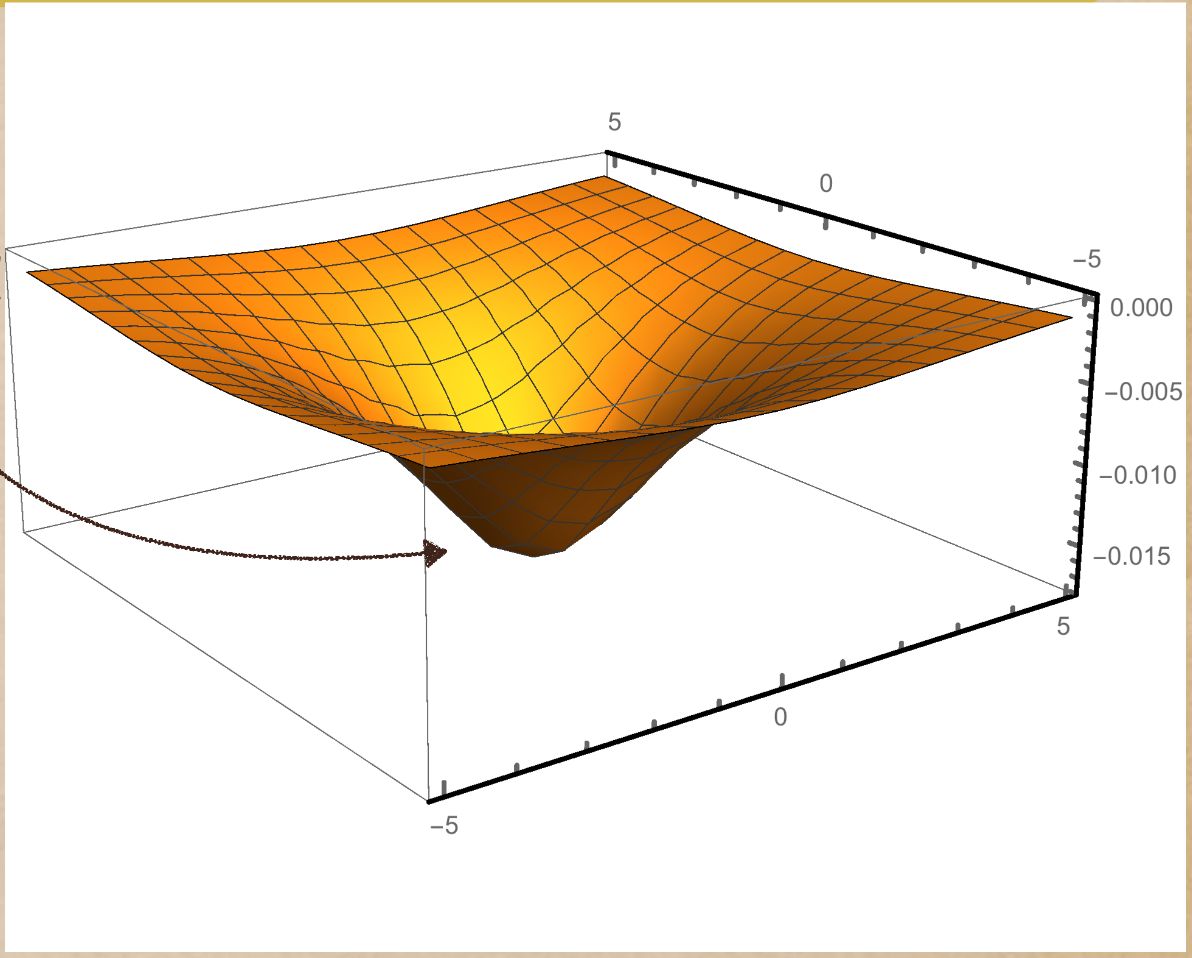
The charge induced will have opposite sign to the test charge

Surface charge density on the conductor

Induced surface charge density:

$$\sigma(x,y) = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = -\frac{\epsilon_0}{4\pi\epsilon_0}\frac{2qd}{(x^2+y^2+d^2)^{3/2}} = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

Let us plot the density !



Induced charge is maximum at $x=0, y=0$!

Total charge induced on the plate ?

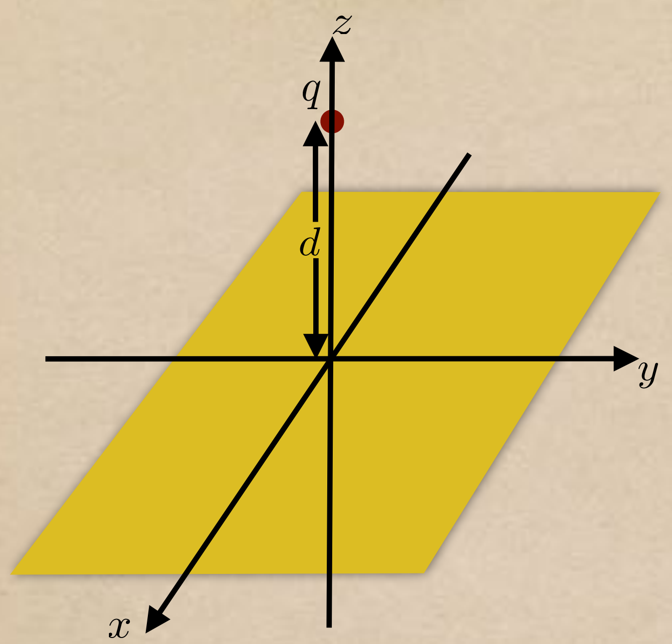
So, the total charge: $Q = \int \sigma da$

It is easier to perform the integral in polar coordinate: $r^2 = x^2 + y^2$ and $da = r dr d\phi$

$$\begin{aligned} Q &= \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r dr d\phi \\ &= \frac{qd}{\sqrt{r^2+d^2}}\bigg|_0^{\infty} = -q \end{aligned}$$

As expected, the total charge induced is -q

Summarise: Method of images



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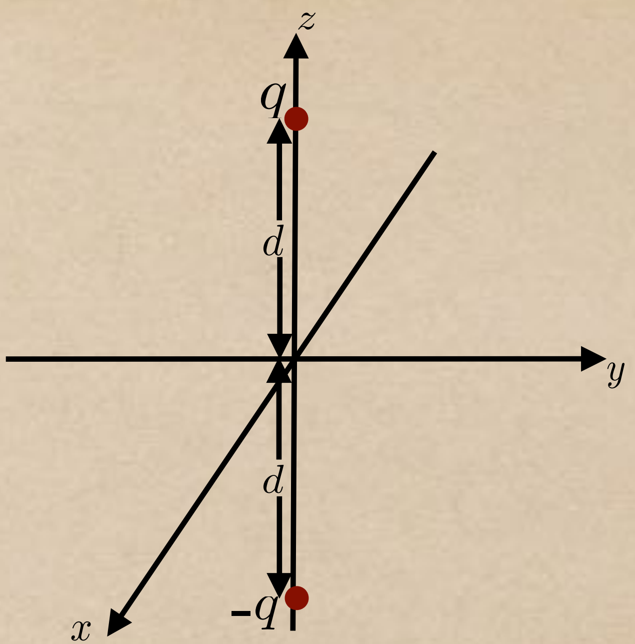


Is everything same about the two situations ?

Energy is not the same!

Where is the discrepancy ?

For point charge and the plane

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$

Energy is half than that when there are two point charges

For two point charges

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}$$

Only the upper (z>0) region contributed to the energy.

Recall

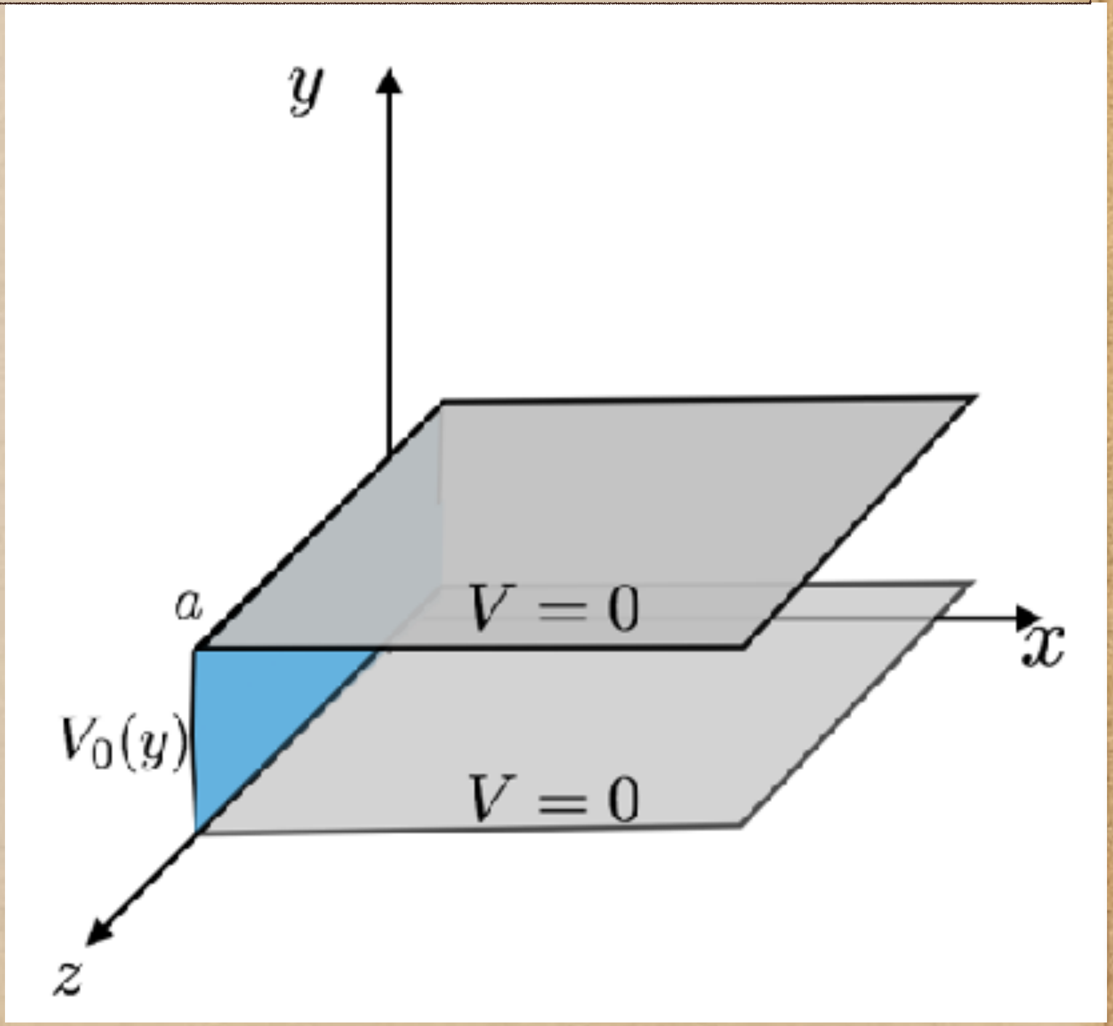
$$W = \frac{\epsilon_0}{2}\int E^2 d\tau$$

We have considered both upper (z>0) and lower (z<0) regions and by symmetry both contributed equally.

Solving Laplace's equation directly

Method of separation of variables

- Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside this infinite "slot". (Griffiths, Example 3.3)



- There is a translational symmetry along z direction, therefore the potential must be independent of z.
- Our region of interest is $x > 0, 0 < y < a$.
- We need to solve Laplace's eqn: in 2 dim

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Boundary conditions on surface

$$\begin{aligned} V(x,y,z) &= V_0(y) \quad \text{for } x = 0 \\ V(x,y,z) &= 0 \quad \text{for } y = 0,\ y = a \\ V(x,y,z) &= 0 \quad \text{for } x \to \infty \end{aligned}$$

Separation of variables: Strategy

• Look for solutions of the form $\boxed{V(x,y) = X(x)Y(y).}$ There is no guarantee that such solutions exists. But in some cases, such separation may be possible.

But, this sounds funny !

Not all solutions can be written in that product form.

Gives only a tiny subset of all possible solutions

• These kind of product solutions may not fit the boundary conditions. Apply as many conditions as possible to narrow down number of possible solutions.

• Laplace's equation is linear: $\nabla^2 V_1 = 0, \quad \nabla^2 V_2 = 0 \implies \nabla^2(\alpha V_1 + \beta V_2) = 0$

• Hence $\alpha V_1 + \beta V_2$ is also a solution to Laplace's equation. It is therefore possible that some linear combination of such solutions may fit the remaining boundary conditions.

This is how we can reach to a generic solution using separation of variables

Separation of variables: Strategy

Substitute the solution $V(x,y) = X(x)Y(y)$ into Laplace's equation

$$\begin{aligned}\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2}{\partial x^2}(X(x)Y(y)) + \frac{\partial^2}{\partial y^2}(X(x)Y(y)) &= 0 \\ Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} &= 0\end{aligned}$$

Dividing both sides by $X(x)Y(y)$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}$$

LHS is a function of x, while RHS is a function of y only !

This necessarily means that each side must be equal to a constant!

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = k^2 \quad \text{(say)}.$$

We get two separate equations:

$$\begin{aligned}\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} &= k^2 \\ -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} &= k^2\end{aligned}$$

A PDE has been converted to 2 separate ODEs and ODEs are much easier to solve!

Separation of variables

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k^2 \implies X(x) = Ae^{kx} + Be^{-kx}$$

$$-\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = k^2 \implies Y(y) = C\sin ky + D\cos ky$$

$$V(x,y) = X(x)Y(y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

To fix A,B,C, we need to use boundary conditions

• Since as $x \to \infty$, $V \to 0$: $\implies X(x) \to 0$, hence coefficient of e^{kx} must vanish. This means $A = 0$. Absorbing B into C and D, we get

$$V(x,y) = e^{-kx}(C\sin ky + D\cos ky)$$

• Since, at $y = 0$, $V = 0$: $\implies Y(y) = 0$: only possible if coefficient of $\cos ky$ is zero. Hence $D = 0$. $\therefore V(x,y) = Ce^{-kx}\sin ky$.

• Again, at $y = a$, $V = 0$: $\implies Y(y) = 0$, i.e. $Y(a) = C\sin ka = 0$

$$\implies k = k_n = \frac{n\pi}{a}, \quad (n = 1, 2, 3, \cdots)$$

Infinite number of solutions !

$$\boxed{V_n(x,y) = C_n e^{-k_n x}\sin k_n y}$$

Separation of variables

We haven't used one boundary condition yet: $V(x,y,z) \quad = \quad V_0(y) \quad \text{for } x=0$

- We have countably infinite number of solutions: $V_n = C_n e^{-n\pi x/a}\sin(n\pi y/a)$.

- Unless $V_0(y)$ just happens to have the form $\sin(n\pi y/a)$ for some n, we can not fit the final boundary condition at $x=0$.

- Separation of variable has given us an infinite family of solutions (one for each n), but, none of them by itself satisfies the final boundary condition!

n	k_n	$V_n(x, y)$	$V_n(\infty, y)$	$V_n(x, 0)$	$V_n(x, a)$	$V_n(0, y)$
1	$\frac{\pi}{a}$	$e^{-\pi x/a}\sin(\frac{\pi y}{a})$	0	0	0	$\sin(\frac{\pi y}{a})$
2	$\frac{2\pi}{a}$	$e^{-2\pi x/a}\sin(\frac{2\pi y}{a})$	0	0	0	$\sin(\frac{2\pi y}{a})$
3	$\frac{3\pi}{a}$	$e^{-3\pi x/a}\sin(\frac{3\pi y}{a})$	0	0	0	$\sin(\frac{3\pi y}{a})$
..	..	..	..	..	..	..

How to incorporate the final condition $V(x,y,z) \quad = \quad V_0(y) \quad \text{for } x=0$ **?**

Separation of variables: Linear combination

Construct a linear combination out of the solutions V_n of the Laplace's Equation:

$$V(x,y) = \sum_{n=1}^{\infty} \alpha_n V_n(x,y) = \alpha_1 V_1(x,y) + \alpha_2 V_2(x,y) + \cdots$$

Each V_n satisfies Laplace's equation separately. Therefore:

$$\nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \cdots = 0\alpha_1 + 0\alpha_2 + \cdots = 0$$

Exploiting this fact, we can patch together the separable solutions to construct a more general solution:

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right).$$

Still satisfies the three boundary conditions

We must use the fourth boundary condition to find C_n

i.e. we must find $C_1, C_2, \cdots$ etc. such that the final boundary condition:

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$

Fourier's trick

Our problem is essentially solved if we can uniquely find $C_1, C_2, \cdots$ such that

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

Fourier's sine series

The good news is that the existence of such a unique set of numbers C_n is guaranteed by Dirichlet's theorem: virtually any function $V_0(y)$ (even having some finite number of discontinuities) can be expanded in the above Fourier Sine Series.

Let m be a positive integer, then multiply by $\sin(m\pi y/a)$ both sides and integrate

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) = \sin\left(\frac{m\pi y}{a}\right) V_0(y)$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) dy = \int_0^a \sin\left(\frac{m\pi y}{a}\right) V_0(y) dy$$

The integral in LHS is 0 if $m \neq n$ and is $\frac{a}{2}$ if $m = n$.

Essentially all the terms in the series drop out, except the one with $n = m$!

$$C_m = \frac{2}{a} \int_0^a \sin\left(\frac{m\pi y}{a}\right) V_0(y) dy$$

Final solution !

So, we know how to find the unknown constant, using Fourier Integral, but unless $V_0(y)$ is specified, we can't find it.

Let $V_0(y) \equiv V_0 = \text{constant}$ for all y. Then, we have

$$\begin{aligned} \boxed{C_n} &= \frac{2}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) V_0(y)dy \\ &= \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) dy \\ &= \frac{2V_0}{n\pi}(1-\cos n\pi) \\ &= \begin{cases} \frac{4V_0}{n\pi} & \text{if } \boxed{n} \text{ is odd;} \\ 0 & \text{if } \boxed{n} \text{ is even.} \end{cases} \end{aligned}$$

Therefore the final solution: $$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right).$$

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5..}^{\infty} \frac{1}{n} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right).$$

Summary !

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Love Affairs and Differential Equations

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The purpose of this note is to suggest an unusual approach to the teaching of some standard material about systems of coupled ordinary differential equations. The approach relates the mathematics to a topic that is already on the minds of many college students: the time-evolution of a love affair between two people. Students seem to enjoy the material, taking an active role in the construction, solution, and interpretation of the equations.

Method of Images can be helpful for finding potential for point charges in front of a grounded conducting plane.

We need to utilise boundary conditions to evaluate the solution of Laplace's equation through the method of separation of variables.