Series Solution of Linear Ordinary Differential Equations

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Aim: To study methods for determining series expansions for solutions to linear ODE with variable coefficients.

In particular, we shall obtain

- the form of the series expansion,
- a recurrence relation for determining the coefficients, and
- the interval of convergence of the expansion.

Review of power series

A series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots, \quad (1)$$

is called a power series about the point x_0. Here, x is a variable and a_n's are constants.

The series (1) converges at $x = c$ if $\sum_{n=0}^{\infty} a_n(c-x_0)^n$ converges. That is, the limit of partial sums

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(c-x_0)^n < \infty.$$

If this limit does not exist, the power series is said to diverge at $x = c$.

Note that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at $x = x_0$ as

$$\sum_{n=0}^{\infty} a_n(x_0-x_0)^n = a_0.$$

Q. What about convergence for other values of x?

Theorem: **(Radius of convergence)**
For each power series of the form (1), there is a number R $(0 \leq R \leq \infty)$, called the radius of convergence of the power series, such that the series converges absolutely for $|x-x_0| < R$ and diverge for $|x-x_0| > R$.

If the series (1) converges for all values of x, then $R = \infty$. When the series (1) converges only at x_0, then $R = 0$.

Theorem: (**Ratio test**) If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L,$$

where $0 \le L \le \infty$, then the radius of convergence (R) of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is

$$R = \begin{cases} \frac{1}{L} & \text{if } 0 < L < \infty, \\ \infty & \text{if } L = 0, \\ 0 & \text{if } L = \infty. \end{cases}$$

Remark. If the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ does not have a limit, then methods other than the ratio test (e.g. root test) must be used to determine R.

Example: Find R for the series $\sum_{n=0}^{\infty} \frac{(-2)^n}{n+1}(x-3)^n$.
Note that $a_n = \frac{(-2)^n}{n+1}$. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1}(n+1)}{(-2)^n(n+2)}\right| = \lim_{n\to\infty} \frac{2(n+1)}{(n+2)} = 2 = L.$$

Thus, $R = 1/2$. The series converges absolutely for $|x-3| < \frac{1}{2}$ and diverge for $|x-3| > \frac{1}{2}$.

Next, what happens when $|x-3| = 1/2$?

At $x = 5/2$, the series becomes the harmonic series $\sum_{n=0}^{\infty} \frac{1}{n+1}$, and hence diverges. When $x = 7/2$, the series becomes an alternating harmonic series, which converges.

Thus, the power series converges for each $x \in (5/2, 7/2]$.

Given two power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

with nonzero radii of convergence. Then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$

has common interval of convergence.
The formula for the product is

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad \text{where } c_n := \sum_{k=0}^{n} a_k b_{n-k}. \tag{2}$$

This power series in (2) is called the Cauchy product and will converge for all x in the common interval of convergence for the power series of f and g.

Differentiation and integration of power series

Theorem: If $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ has a positive radius of convergence R, then f is differentiable in the interval $|x - x_0| < R$ and termwise differentiation gives the power series for the derivative:

$$f'(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1} \text{ for } |x - x_0| < R.$$

Furthermore, termwise integration gives the power series for the integral of f:

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1} + C \text{ for } |x - x_0| < R.$$

Example: A power series for

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots .$$

Since $\frac{d}{dx}\left\{1/(1-x)\right\} = \frac{1}{(1-x)^2}$, we obtain a power series for

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots .$$

Since $\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt$, integrate the series for $\frac{1}{1+x^2}$ termwise to obtain

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots .$$

Shifting the summation index

The index of a summation in a power series is a dummy index and hence

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{k=0}^{\infty} a_k (x - x_0)^k = \sum_{i=0}^{\infty} a_i (x - x_0)^i.$$

Shifting the index of summation is particularly important when one has to combine two different power series.

Example:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

$$x^3 \sum_{n=0}^{\infty} n^2 (n-2) a_n x^n = \sum_{n=3}^{\infty} (n-3)^2 (n-5) a_{n-3} x^n.$$

Definition: **(Analytic function)**
A function f is said to be analytic at x_0 if it has a power series representation $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ in an neighborhood about x_0, and has a positive radius of convergence.

Example: Some analytic functions and their representations:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n, \quad x > 0.$$

Power series solutions to linear ODEs

Consider linear ODE of the form:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \quad a_2(x) \neq 0. \qquad (*)$$

Writing in the standard form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where $p(x) := a_1(x)/a_2(x)$ and $q(x) := a_0(x)/a_2(x)$.

Definition: A point x_0 is called an ordinary point of $(*)$ if both $p(x) = a_1(x)/a_2(x)$ and $q(x) = a_0(x)/a_2(x)$ are analytic at x_0. If x_0 is not an ordinary point, it is called a singular point of $(*)$.

Example: Find all the singular point points of

$$xy''(x) + x(x-1)^{-1}y'(x) + (\sin x)y = 0, \quad x > 0$$

Here,

$$p(x) = \frac{1}{(1-x)}, \quad q(x) = \frac{\sin x}{x}.$$

Note that $p(x)$ is analytic except at $x = 1$. $q(x)$ is analytic everywhere as it has power series representation

$$q(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots .$$

Hence, $x = 1$ is the only singular point of the given ODE.

Power series method about an ordinary point

Consider the equation

$$2y'' + xy' + y = 0. \tag{$**$}$$

Let's find a power series solution about $x = 0$. Seek a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and then attempt to determine the coefficients a_n's. Differentiate termwise to obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting these power series in $(**)$, we find that

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

By shifting the indices, we rewrite the above equation as

$$\sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

Combining the like powers of x in the three summation to obtain

$$4a_2 + a_0 + \sum_{k=1}^{\infty} [2(k+2)(k+1)a_{k+2} + k a_k + a_k] x^k = 0.$$

Equating the coefficients of this power series equal to zero yields

$$
\begin{aligned}
&4a_2 + a_0 = 0 \\
&2(k+2)(k+1)a_{k+2} + (k+1)a_k = 0, \quad k \geq 1.
\end{aligned}
$$

This leads to the recurrence relation

$$a_{k+2} = \frac{-1}{2(k+2)}a_k, \quad k \geq 1.$$

Thus,

$$
\begin{aligned}
a_2 &= \frac{-1}{2^2}a_0, & a_3 &= \frac{-1}{2\cdot 3}a_1 \\
a_4 = \frac{-1}{2\cdot 4}a_2 &= \frac{1}{2^2\cdot 2\cdot 4}a_0, & a_5 &= \frac{-1}{2\cdot 5}a_3 = \frac{1}{2^2\cdot 3\cdot 5}a_1 \\
&\cdots & &\cdots
\end{aligned}
$$

With a_0 and a_1 as arbitrary constants, we find that

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!} a_0, \quad n \geq 1,$$

and

$$a_{2n+1} = \frac{(-1)^n}{2^n [1 \cdot 3 \cdot 5 \cdots (2n+1)]} a_1, \quad n \geq 1.$$

From this, we have two linearly independent solutions as

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} x^{2n}, \\ y_2(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n [1 \cdot 3 \cdot 5 \cdots (2n+1)]} x^{2n+1}. \end{aligned}$$

Hence the general solution is

$$y(x) = a_0 y_1(x) + a_1 y_2(x).$$

Remark. Suppose we are given the value of $y(0)$ and $y'(0)$, then $a_0 = y(0)$ and $a_1 = y'(0)$. These two coefficients leads to a unique power series solution for the IVP.

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