MA 102 (Mathematics II)

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Partial and directional derivatives, differentiability

(1) The *kinetic energy* of an object with a constant mass m and position $\mathbf{r}(t) \in \mathbb{R}^n$ at time $t \in \mathbb{R}$ is defined to be $K(t) := \frac{1}{2}mv^2(t)$, where $v(t) := \|\mathbf{r}'(t)\|$. Determine $K'(t)$.

Solution: We have $K(t) = \frac{1}{2}m\mathbf{r}'(t) \bullet \mathbf{r}'(t) \Rightarrow K'(t) = m(\mathbf{r}'(t) \bullet \mathbf{r}''(t))$. ■

(2) Find the unit tangent vector to $\mathbf{r}(t) = (e^t, 2t, 2e^{-t})$. Also find the speed of a moving object with position $\mathbf{r}(t) = (3\sin(2t), 5\cos(2t), 4\sin(2t))$ in feet at time $t \in \mathbb{R}$ in seconds.

Solution: Speed is the magnitude of the velocity $\mathbf{r}'(t)$. Hence

$$\begin{aligned}
\|\mathbf{r}'(t)\| &= \|(6\cos(2t), -10\sin(2t), 8\cos(2t))\| \\
&= \sqrt{36\cos^2(2t) + 100\sin^2(2t) + 64\cos^2(2t)} \\
&= \sqrt{100\cos^2(2t) + 100\sin^2(2t)} = 10 \text{ feet per second.} ■
\end{aligned}$$

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0,0) = 0$ and $f(x,y) = \dfrac{xy}{x^2+y^2}$. Show that f is not continuous at $(0,0)$ but the partial derivatives of f exist on $\mathbb{R}^2$. Show that the partial derivatives are not continuous at $(0,0)$.

Solution: Obviously f is not continuous and $f_x(0,0) = 0 = f_y(0,0)$. Now

$$f_x(x,y) = \frac{y(y^2 - x^2)}{(x^2+y^2)^2}$$

shows that f_x is not continuous at $(0,0)$. Indeed, $f_x(0, 1/n) = n \to \infty$. Similarly, f_y is not continuous at $(0,0)$. ■

(4) Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, where U is open. If the first order partial derivatives of f exist on U and are bounded then show that f is continuous on U.

Solution: We have $f(a+x, b+y) - f(a,b) = [f(a+x, b+y) - f(a, b+y)] + [f(a, b+y) - f(a,b)]$. By MVT, there exists $0 < \theta_i < 1$ for $i = 1, 2$ such that

$$\begin{aligned}
[f(a+x, b+y) - f(a, b+y)] + [f(a, b+y) - f(a,b)] &= f_x(a + \theta_1 x, b+y)x + \\
&\quad f_y(a, b + \theta_2 y)y.
\end{aligned}$$

This shows $|f(a+x, b+y) - f(a,b)| \le \text{const.}(|x| + |y|) \to 0$ as $(x,y) \to (0,0)$. Hence f is continuous at (a,b). ■

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \quad \text{for } (x,y) \neq (0,0).$$

Show that f is continuous at $(0,0)$ and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0,0)$.

Solution: Since $|f(x,y)| \le x^2+y^2$, f is continuous at $(0,0)$ and $f_x(0,0) = f_y(0,0) = 0$. We have

$$f_x(x,y) = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$

The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at (0,0), while $\frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)$ is unbounded in any such disc. (Consider$(x,y) = \left(\frac{1}{\sqrt{n\pi}}, 0\right)$ for n a large positive integer.) Thus f_x is unbounded in any disc around (0,0). ■

(6) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $f_x(x,y) = 0 = f_y(x,y)$ for all $(x,y) \in \mathbb{R}^2$ then show that f is a constant function.

Solution: We have $f(x,y) - f(0,0) = [f(x,y) - f(0,y)] + [f(0,y) - f(0,0)]$. By MVT, there exists $0 < \theta_i < 1$ for $i = 1,2$ such that

$$[f(x,y) - f(0,y)] + [f(0,y) - f(0,0)] = f_x(\theta_1 x, y)x + f_y(0, \theta_2 y)y = 0.$$

This shows $f(x,y) - f(0,0) = 0$ for all $(x,y) \in \mathbb{R}^2$. Hence f is constant. ■.

(7) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0,0) = 0 = g(0,0)$ and, for $(x,y) \neq (0,0)$,

$$f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}, \qquad g(x,y) = \frac{\sin^2(x+y)}{|x|+|y|}.$$

Examine differentiability and the existence of partial and directional derivatives of f and g at $(0,0)$.

Solution: (i) We have $\nabla f(0,0) = (0,0)$ and f is differentiable. Solved in the class.
(ii) We have $g_x(0,0) = \lim\limits_{h\to 0}\dfrac{\sin^2(h)/|h|}{h} = \lim\limits_{h\to 0}\dfrac{\sin^2(h)}{h|h|}$ which shows that the limit does not exist. Similarly, $g_y(0,0)$ does not exist. Hence g is not differentiable. ■

(8) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = 0$ if $y = 0$ and and $f(x,y) = \dfrac{y}{|y|}\sqrt{x^2+y^2}$, if $y \neq 0$. Show that f is continuous at $(0,0)$, $D_u f(0,0)$ exists for all unit vector u but f is not differentiable at $(0,0)$.

Solution: Since $f(0,0) = 0$ and $|f(x,y)| \le \sqrt{x^2+y^2}$, f is continuous at (0,0). Let $u = (u_1, u_2)$ be a unit vector in $\mathbb{R}^2$ with $u_2 \neq 0$. Then

$$D_u f(0,0) = \lim_{t\to 0}\frac{f(tu) - f(0)}{t} = \frac{u_2}{|u_2|}.$$

On the other hand, if $u = (1,0)$ then $D_u f(0,0) = f_x(0,0) = 0$. Hence $D_u f(0,0)$ exist for every unit vector $u \in \mathbb{R}^2$. Next, we have $\nabla f(0,0) = (0,1)$ and

$$\lim_{(h,k)\to(0,0)}\frac{|f(h,k) - \nabla f(0,0)\bullet(h,k)|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)}\frac{\left|\frac{k}{|k|}\sqrt{h^2+k^2} - k\right|}{\sqrt{h^2+k^2}}$$
$$= \lim_{(h,k)\to(0,0)}\left|\frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}\right|$$

which shows that the limit does not exist. Hence f is not differentiable at $(0,0)$. ■

(9) Find the directional derivative of $f(x,y) = y^3 - 2x^2 + 3$ at the point $(1,2)$ in the direction of $u := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Also, find the directional derivative of $f(x,y) = \log(x^2+y^2)$ at $(1,-3)$

in the direction of $u := (2, -3)$.

Solution: (i) We have $f_x(x,y) = -4x$, $f_y(x,y) = 3y^2$ which are continuous. Therefore

$$D_u f(1,2) = \nabla f(1,2) \bullet u = f_x(1,2)\frac{1}{2} + f_y(1,2)\frac{\sqrt{3}}{2} = -2 + 6\sqrt{3}.$$

(ii) Next, we have $f_x(x,y) = \frac{2x}{x^2+y^2}$ and $f_y(x,y) = \frac{2y}{x^2+y^2}$ which are continuous at $(1,-3)$. Hence for $u = (\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}})$, we have $D_u f(1,-3) = \nabla f(1,-3) \bullet u = \dfrac{11}{5\sqrt{3}}$. ■

(10) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(0,0)$. Suppose that for $u := (3/5, 4/5)$ and $v := (1/\sqrt{2}, 1/\sqrt{2})$, we have $D_u f(0,0) = 12$ and $D_v f(0,0) = -4\sqrt{2}$. Then determine $f_x(0,0)$ and $f_y(0,0)$.

Solution: Set $(\alpha, \beta) := \nabla f(0,0)$. Then $(\alpha, \beta) \bullet (3/5, 4/5) = 12 \Rightarrow 3\alpha + 4\beta = 60$ and $(\alpha, \beta) \bullet (1/\sqrt{2}, 1/\sqrt{2}) = -4\sqrt{2} \Rightarrow \alpha + \beta = -8$. Hence $f_x(0,0) = \alpha = -92$ and $f_y(0,0) = \beta = 84$. ■

(11) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose that $\partial_i f(x,y)$ exists and g is differentiable at $f(x,y)$. Show that $\partial_i (g \circ f)(x,y)$ exists and $\partial_i (g \circ f)(x,y) = g'(f(a))\partial_i f(x,y)$.

Solution: Define $\psi(x) := f(x,y)$ and $\phi(x) = g(\psi(x)) = g(f(x,y))$. Then $\psi'(x) = f_x(x,y)$. Hence $\phi'(x)$ exists and by chain rule $\phi'(x) = g'(\psi(x))\psi'(x) = g'(f(x,y))f_x(x,y)$. ■

(12) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable. Using chain rule determine the partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$(i) f(x,y) := g(xy^2 + 1), \quad (ii) f(x,y) := g(4x + 7y), \quad (iii) f(x,y) := g(x - y).$$

Also, examine differentiability of f and determine the derivative, if it exists.

Solution: Easy. Apply chain rule. Continuous partial derivatives $\Rightarrow$ differentiability and in such a case the derivative is given by $Df(a,b)(h,k) = \nabla f(a,b) \bullet (h,k)$ for all $(h,k) \in \mathbb{R}^2$. ■

(13) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $a \in \mathbb{R}^2$ and suppose that $\nabla f(a) \neq 0$. Show that the maximum value of the directional derivative $D_u f(a)$ is $\|\nabla f(a)\|$ and is attained in the direction of $\nabla f(a)$ with $u = \nabla f(a)/\|\nabla f(a)\|$. Also show that the minimum value of $D_u f(a)$ is $-\|\nabla f(a)\|$ and is attained in the direction of $-\nabla f(a)$.

Solution: We have $D_u f(a) = \nabla f(a) \bullet u$. By Cauchy-Schwarz inequality $|D_u f(a)| \leq \|\nabla f(a)\|$. Hence the result follows. ■

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