MA 102 (Multivariable Calculus)

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Lipschitz continuity: $f : \mathbb{R}^n \to \mathbb{R}^m$

Definition: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$. Then f is Lipschitz continuous on A if there is $M > 0$ such that

$$X, Y \in A \implies \|f(X) - f(Y)\| \leq M \|X - Y\|.$$

Lipschitz continuity $\implies$ Uniform Continuity $\implies$ Continuity

Examples:

1. $f : \mathbb{R}^n \to \mathbb{R}, X \mapsto \|X\|$ is Lipschitz continuous.
2. $f : [0, \infty) \to \mathbb{R}, x \mapsto \sqrt{x}$ is uniformly continuous but NOT Lipschitz.
3. $f : (0, 1) \to \mathbb{R}, x \mapsto 1/x$ is continuous but NOT uniformly continuous.

Diffential calculus

Task: Extend differential calculus to the functions:

Case I: $f : A \subset \mathbb{R}^n \to \mathbb{R}$
Case II: $f : A \subset \mathbb{R} \to \mathbb{R}^n$
Case III: $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$

Question: What does it mean to say that f is differentiable?

Parametric curve $\Gamma : \mathbb{R} \to \mathbb{R}^n$

A continuous function $\Gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ where $I = (a,\ b)$ or $I = [a,\ b]$ or $I = \mathbb{R}$ is called a parametric curve in $\mathbb{R}^n$. The curve $C := \Gamma([a, b])$ is parameterized by $\Gamma(t)$.

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Examples:

- A line is determined by a point and a direction.

  To find an equation of a line in $\mathbb{R}^3$, choose a point P_0 on the line and a non-zero vector V parallel to the line.

  If X_0 is the position vector of the point P_0, then the equation of the line is given by $\Gamma(t) = X_0 + \ t\ V$ for $t \in \mathbb{R}$.

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- $\Gamma : [0, 2\pi] \to \mathbb{R}^3$ given by $\Gamma(t) := (\cos t, \sin t, t)$ parameterizes a circular helix.

$\Gamma : [0, 2\pi] \to \mathbb{R}^2$ given by $\Gamma(t) := (\cos t, \sin t)$ parameterizes the circle $x^2 + y^2 = 1$.

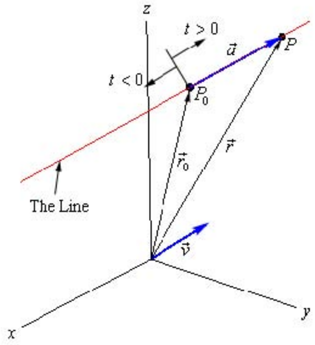


Figure : Line $\Gamma(t) = r_0 + tV$

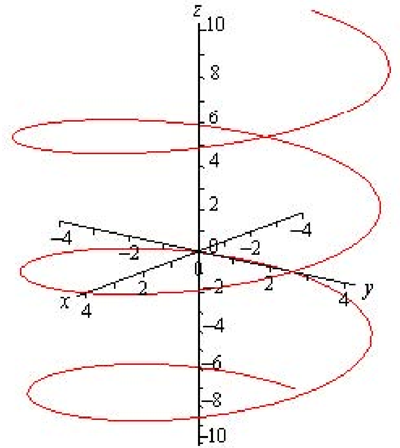


Figure : Helix $\Gamma(t) = (4\cos t, 4\sin t, t)$

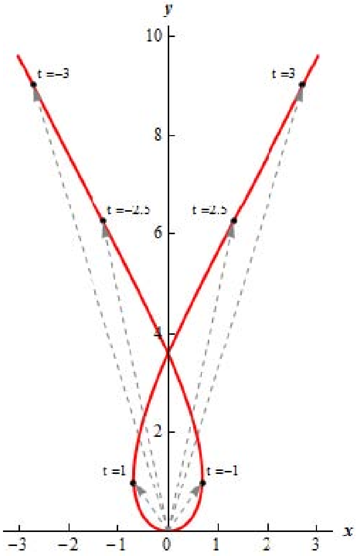


Figure : Plane curve $\Gamma(t) = (t - 2\sin t, t^2)$

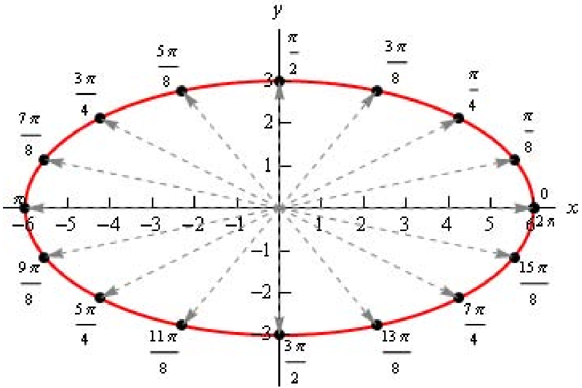
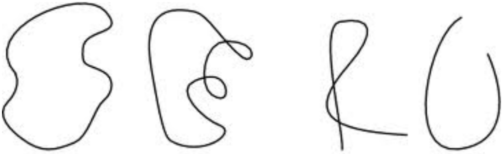


Figure : Ellipse $\Gamma(t) = (6\cos t, 3\sin t)$

Closed Curves

A curve C in $\mathbb{R}^n$ is closed if C has a parametrization $\Gamma : [a,\ b] \subset \mathbb{R} \to \mathbb{R}^n$ such that $\Gamma(a) = \Gamma(b)$.



closed curves *open curves*

Examples:

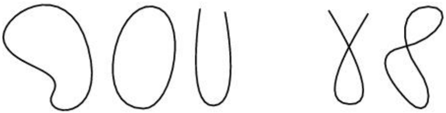
$C : \Gamma(t) = (3\cos t,\ 3\sin t)$ for $t \in [0,\ 2\pi]$ is a closed curve.
C: $\Gamma(t) = (t,\ 1 - t)$ for $t \in [0,\ 1]$ is not closed.

Simple Curves

A curve C in $\mathbb{R}^n$ is simple if C has a parametrization $\Gamma : I \subseteq \mathbb{R} \to \mathbb{R}^n$ where I is an interval in $\mathbb{R}$ such that for any two interior points t_1 and t_2 of I

$$t_1 \neq t_2 \qquad \Longrightarrow \qquad \Gamma(t_1) \neq \Gamma(t_2)\ .$$

That is, Γ does not have self-intersecting points except for the endpoints (that is, the terminal point and the initial point can be same).



simple curves *nonsimple curves*

Examples: $C : \Gamma(t) = (3\cos t,\ 3\sin t)$ for $t \in [0,\ 2\pi]$ is a simple curve.
$C : \Gamma(t) = (\cos(t),\ \sin(2t))$ for $t \in [0,\ 2\pi]$ is not simple.

Differentiability of $r : \mathbb{R} \to \mathbb{R}^n$

Definition: Let $r : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ and $t_0 \in (a, b)$. If

$$r'(t_0) = \frac{\mathrm{d}r}{\mathrm{d}t}(t_0) := \lim_{t \to t_0} \frac{r(t) - r(t_0)}{t - t_0}$$

exists then r is differentiable at t_0. The derivative $r'(t_0)$ is called the velocity vector.

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Fact:

- $r(t) = (r_1(t), \ldots, r_n(t))$, where $r_i : (a, b) \to \mathbb{R}$.
- r is differentiable at $t_0 \iff$ each r_i is differentiable at t_0, $i = 1, 2, \ldots, n$. Further, $r'(t_0) = (r_1'(t_0), \ldots, r_n'(t_0))$.
- r differentiable at $t_0 \Rightarrow r$ continuous at t_0.

Sum and product rules

Fact: Let $f, g : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ be differentiable at $t_0 \in (a, b)$. Then for $\alpha \in \mathbb{R}$

1. $f + g$ and αf are differentiable at t_0. Further, $(f+g)'(t_0) = f'(t_0) + g'(t_0)$ and $(\alpha f)'(t_0) = \alpha f'(t_0)$.

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1. $f + g$ and αf are differentiable at t_0. Further, $(f+g)'(t_0) = f'(t_0) + g'(t_0)$ and $(\alpha f)'(t_0) = \alpha f'(t_0)$.
2. $f \bullet g$ defined by $(f \bullet g)(t) := \langle f(t), g(t) \rangle$ is differentiable at t_0 and
$$(f \bullet g)'(t_0) = f'(t_0) \bullet g(t_0) + f(t_0) \bullet g'(t_0).$$
3. Chain rule: $\dfrac{\mathrm{d} f(h(t_0))}{\mathrm{d} t} = h'(t_0) f'(h(t_0))$.

Velocity and tangent vectors

Let $r : (a, b) \to \mathbb{R}^n$ be differentiable. Then treating $r(t)$ as the position of a moving object at time t, we have

$$\text{scaled secant } = \frac{r(t+\Delta t) - r(t)}{\Delta t} \to r'(t) \text{ as } \Delta t \to 0.$$

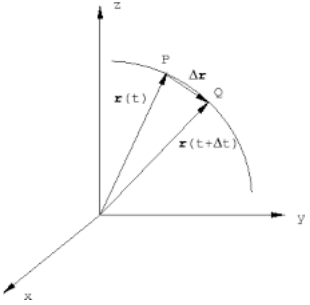

Figure : As $\Delta t \to 0$, Q approaches P

Velocity and tangent vectors

Scaled secant $\rightarrow$ tangent vector to the curve at $r(t)$ as $\Delta t \rightarrow 0$.

Velocity and tangent vectors

Scaled secant $\to$ tangent vector to the curve at $r(t)$ as $\Delta t \to 0$.

Thus velocity vector $v(t) := r'(t)$ is tangent to the curve at $r(t)$.

If $r(t) := (\cos t, \sin t)$ then $v(t) = r'(t) = (-\sin t, \cos t)$.

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Smooth curve

Smooth curve: A curve C in $\mathbb{R}^n$ is smooth if C has a parametrization $\Gamma : I \subseteq \mathbb{R} \to \mathbb{R}^n$ where $I = (a,\ b)$ or $I = [a,\ b]$ or $I = \mathbb{R}$ such that

- $\Gamma'(t)$ exists and continuous at each interior point of I,
- $\Gamma'(t) \neq 0$ at each interior point of I.

Examples:

$C : \Gamma(t) = (a\cos t,\ a\sin t)$ for $t \in [0,\ 2\pi]$ is a smooth curve.
$C : \Gamma(t) = (t,\ t^2)$ for $t \in \mathbb{R}$ is a smooth curve.

Piecewise smooth curve

A curve C in $\mathbb{R}^n$ is piecewise smooth if C is composed of finite number of smooth curves $C_1, \cdots, C_n$ (say) such that the end of point C_k is the initial point of C_{k+1} for each k.

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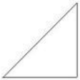

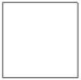

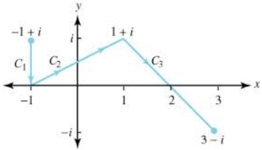

Examples:
Boundary of Triangular region (Triangle).
Rectangular path
Polygonal path formed by a finite number of line segments

Length of a Piecewise Smooth / Smooth Curve

Polygonal approximations of curves:

Let $\Gamma : [a, b] \to \mathbb{R}^n$ be a parametric curve. For a partition $P := \{t_0, \ldots, t_m\}$ of $[a, b]$, define

$$\ell(P, \Gamma) := \sum_{j=1}^{m} \|\Gamma(t_j) - \Gamma(t_{j-1})\|$$

and $\mu(P) := \max\{t_j - t_{j-1} : j = 1, \ldots, m\}$.

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and $\mu(P) := \max\{t_j - t_{j-1} : j = 1, \ldots, m\}$.

Note that $\ell(P, \Gamma) \leq \ell(Q, \Gamma)$ if Q is a refinement of P. Hence

$$\lim_{\mu(P)\to 0} \ell(P, \Gamma) = \sup_{P} \ell(P, \Gamma).$$

Length of a Piecewise Smooth / Smooth Curve

Arclength of a curve

Let $\ell(\Gamma) := \sup\limits_{P} \ell(P, \Gamma)$. If $\ell(\Gamma)$ is finite then Γ is said to be rectifiable (finite length) and $\ell(\Gamma)$ is said to be the arclength of Γ.

Theorem: Let $\Gamma : [a, b] \to \mathbb{R}^n$ be a smooth (or piecewise smooth) path. Then Γ is rectifiable and

$$\ell(\Gamma) = \lim_{\mu(P)\to 0} \ell(P, \Gamma) = \int_a^b \|\Gamma'(t)\| dt.$$

Case $n = 2, 3$

If a smooth curve C in $\mathbb{R}^2$ has a parametrization $C : \Gamma(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$ for $t \in [a,\ b]$ then

$$\Gamma'(t) = x'(t)\,\hat{\imath} + y'(t)\,\hat{\jmath} \qquad \text{for } t \in [a,\ b]\ ,$$

$$\|\Gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2} \qquad \text{for } t \in [a,\ b]\ ,$$

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt\ .$$

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If $C : \Gamma(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$ for $t \in [a,\ b]$ then

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Length of a Piecewise Smooth / Smooth Curve

Example 1: Find the length of the circle
$C : \Gamma(t) = (a\cos t,\ a\sin t)$ for $t \in [0,\ 2\pi]$.

$$\Gamma'(t) = (-a\sin t,\ a\cos t) \qquad \text{for } t \in [0,\ 2\pi]\ .$$

$$\|\Gamma'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a \qquad \text{for } t \in [0,\ 2\pi]\ .$$

$$L = \int_0^{2\pi} a\ dt\ =\ 2\pi\ a\ .$$

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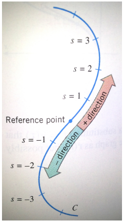
Example 2: Find the length of the helix
$C : \Gamma(t) = (\cos t,\ \sin t,\ t)$ for $t \in [0,\ 2\pi]$.

$$\Gamma'(t) = (-\sin t,\ \cos t,\ 1) \qquad \text{for } t \in [0,\ 2\pi]\ .$$

$$\|\Gamma'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = \sqrt{2} \qquad \text{for } t \in [0,\ 2\pi]\ .$$

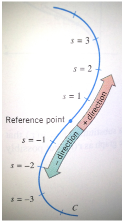
$$L = \int_0^{2\pi} \sqrt{2}\ dt\ =\ \sqrt{2} \times 2\pi = 2\sqrt{2}\ \pi\ .$$

Arc Length Parametrization of a smooth curve



- Select an arbitrary point $P(t_0)$ (say) on the curve C to serve as a base point (or reference point).

Arc Length Parametrization of a smooth curve



- Select an arbitrary point $P(t_0)$ (say) on the curve C to serve as a base point (or reference point).
- Starting from the point, choose one direction along the curve to be the positive direction (direction we get for $t > t_0$) and the other to be the negative direction ($t < t_0$).

Arc Length Parametrization of a smooth curve

- If $P(t)$ is a point on the curve, let $s(t)$ be the distance from $P(t_0)$ to $P(t)$ along the curve. Each value of s determines a point on C and this parametrizes C with respect to s. We call s is an arc length parameter for the curve. If $t > t_0$, $s(t) > 0$. If $t < t_0$, $s(t) < 0$.

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Arclength differential: Let $\Gamma : [a, b] \to \mathbb{R}^n$ be a smooth curve. Define $s : [a, b] \to [0, \ell]$ by

$$s(t) := \int_a^t \|\Gamma'(\tau)\| d\tau,$$

where $\ell := \ell(\Gamma)$. Thus,

$$\frac{ds}{dt} = \|\Gamma'(t)\|.$$

Example

Find the arc length parametrization of the circular helix $\Gamma(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$ for $t \in \mathbb{R}$ that has base point $\Gamma(0) = (1,\ 0,\ 0)$ in the positive direction.

$\Gamma(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$, $\Gamma'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}$.

$$s = s(t) = \int_0^t \|\Gamma'(\tau)\|\,d\tau = \int_0^t \sqrt{2}\,d\tau = \sqrt{2}t \quad \text{for } t \in \mathbb{R}.$$

$$\Longrightarrow \qquad t = s/\sqrt{2}$$

Arc Length Parametrization of C is

$$F(s) = \cos(s/\sqrt{2})\,\hat{\imath} + \sin(s/\sqrt{2})\,\hat{\jmath} + (s/\sqrt{2})\,\hat{k} \qquad \text{for } s \in \mathbb{R}.$$

Example

A bug, starting at the reference point (1, 0, 0) of the helix $\Gamma(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$ for $t \in \mathbb{R}$, walks up the helix for a distance of 10 units. What are the bug's final coordinates?

The arc length parametrization of the helix with the base point $(1, 0, 0)$ is

$$F(s) \; = \; \cos(s/\sqrt{2})\,\hat{\imath} + \sin(s/\sqrt{2})\,\hat{\jmath} + (s/\sqrt{2})\,\hat{k} \qquad \text{for } s \in \mathbb{R}\;.$$

If $s = 10$ then

$$F(10) \; = \; \cos(10/\sqrt{2})\,\hat{\imath} + \sin(10/\sqrt{2})\,\hat{\jmath} + (10/\sqrt{2})\,\hat{k}\;.$$

The bug's final coordinates are

$$(\cos(10/\sqrt{2}),\; \sin(10/\sqrt{2}),\; 10/\sqrt{2})\;.$$

Tangent Vector, Normal Vector, Binormal Vector

Let C be a smooth curve in $\mathbb{R}^n$ with a parametrization $\Gamma(t)$ for $t \in I$ where I is an interval in $\mathbb{R}$.

- The unit tangent vector to the curve C at the point $\Gamma(t)$ is

$$T(t) := \frac{\Gamma'(t)}{\|\Gamma'(t)\|} \qquad \text{for} \quad t \in \ \text{int}(I)\ .$$

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Theorem: Let I be an interval and F be a vector valued function on I such that $\|F(t)\|$ is constant for all $t \in I$. Then $F'(t)$ is perpendicular to $F(t)$ for each $t \in I$.

Proof: Use dot product rule for derivative.

Tangent Vector, Normal Vector, Binormal Vector

- If $T'(t) \neq 0$ for all $t \in \operatorname{int}(I)$, then the unit normal vector or principal normal vector to the curve C at the point $\Gamma(t)$ is

$$N(t) := \frac{T'(t)}{\|T'(t)\|} \qquad \text{for} \quad t \in \operatorname{int}(I)\ .$$

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The unit normal always points in the direction in which the tangent is turning.

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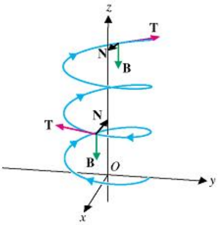
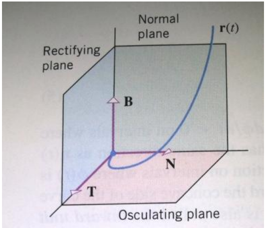
$$N(t) := \frac{T'(t)}{\|T'(t)\|} \qquad \text{for} \quad t \in \text{ int}(I)\ .$$
The unit normal always points in the direction in which the tangent is turning.
- The binormal vector to the curve C at the point $\Gamma(t)$ is
$$B(t) := T(t) \times N(t) \qquad \text{for} \quad t \in \text{ int}(I)\ .$$

T, N and B are mutually perpendicular



The vectors T, N, B are mutually perpendicular unit vectors.

T, N and B are mutually perpendicular



If the fingers on the right hand curl around from T toward N, then the thumb will point in the direction of B.

Trajectories of Objects in the Space: T, N and B System (or Frenet Frame)

- The binormal vector $B(t)$ can be expressed as

$$B(t) = \frac{\Gamma'(t) \times \Gamma''(t)}{\|\Gamma'(t) \times \Gamma''(t)\|} .$$

- The binormal vector $B = B(t)$ is perpendicular to the unit tangent vector $T = T(t)$ and the unit normal vector $N = N(t)$.
- The vectors T, N, B are mutually perpendicular unit vectors. The T, N and B system is used in the analysis of trajectories of spacecrafts.
- $B = T \times N$, $N = B \times T$ and $T = N \times B$.

Example

Let $\Gamma(t) = 3\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath} + 4t\,\hat{k}$.

$$
\begin{aligned}
\Gamma'(t) &= 3\cos t\,\hat{\imath} - 3\sin t\,\hat{\jmath} + 4\,\hat{k}\\
\|\Gamma'(t)\| &= 5\\
T(t) &= \frac{\Gamma'(t)}{\|\Gamma'(t)\|} = \frac{1}{5}\left(3\cos t\,\hat{\imath} - 3\sin t\,\hat{\jmath} + 4\,\hat{k}\right)\\
T'(t) &= \frac{1}{5}(-3\sin t\,\hat{\imath} - 3\cos t\,\hat{\jmath})\\
\|T'(t)\| &= \frac{3}{5}\\
N(t) &= \frac{T'(t)}{\|T'(t)\|} = (-1)(\sin t\,\hat{\imath} + \cos t\,\hat{\jmath})\\
B(t) &= T(t)\times N(t) = \frac{1}{5}\left(4\cos t\,\hat{\imath} - 4\sin t\,\hat{\jmath} - 3\,\hat{k}\right)
\end{aligned}
$$

Curve is parameterized in terms of arc length

We have $\frac{ds}{dt} = \|\Gamma'(t)\|$

If Γ is a smooth curve, then $\frac{ds}{dt} = \|\Gamma'(t)\| > 0$.

Thus, as a function of t, s is invertible and

$$\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{\|\Gamma'(t)\|}.$$

Using chain rule, we have

$$\frac{d\Gamma}{ds} = \frac{d\Gamma}{dt}\frac{dt}{ds} = \frac{\Gamma'(t)}{\|\Gamma'(t)\|} = T(t).$$

Curve is parameterized in terms of arc length

If C is a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ that is parametrized by arc length $\Gamma(s)$ for $s \in I \subseteq \mathbb{R}$ then

$$\text{Unit Tangent Vector:} \qquad T(s) = \Gamma'(s)$$

$$\text{Unit Normal Vector:} \qquad N(s) = \frac{\Gamma''(s)}{\|\Gamma''(s)\|}$$

$$\text{Binormal Vector:} \qquad B(s) = \frac{\Gamma'(s) \times \Gamma''(s)}{\|\Gamma''(s)\|}.$$

MA 102 (Multivariable Calculus)

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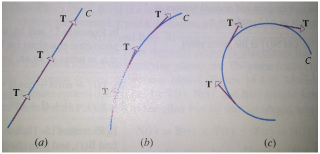
Curvature: A measure of how sharply the curve bends?

Suppose that a curve C is parametrized in terms of arc length. Let T denote the unit tangent vector to C.
The sharpness of the bend in C is closely related to $\frac{dT}{ds}$, which is the rate of change of T with respect to s.



If C is a straight line (no bend), then the direction of T remains constant and hence the curvature becomes zero.
If C bends slightly, then T undergoes a gradual change of direction.
If C bends sharply, then T undergoes a rapid change of direction.

Curvature of a Smooth Curve

If C is a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ that is parametrized by arc length $\Gamma(s)$ for $s \in I \subseteq \mathbb{R}$ then the curvature of C at the point $\Gamma(s)$, denoted by $\kappa = \kappa(s)$ (Greek letter kappa), is defined by

$$\kappa(s) \;=\; \left\| \frac{dT}{ds} \right\| \;=\; \|\Gamma''(s)\| \;.$$

Theorem: If C is a smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ with the parametric equation $\Gamma(t)$ for $t \in I \subseteq \mathbb{R}$ such that $\Gamma''(t)$ exists at all interior points of I then the curvature $\kappa = \kappa(t)$ of C at the point $\Gamma(t)$ is given by

$$\kappa(t) = \frac{\|T'(t)\|}{\|\Gamma'(t)\|} = \frac{\|\Gamma'(t) \times \Gamma''(t)\|}{\|\Gamma'(t)\|^3}$$

Example

$\Gamma(s) = (a\cos(s/a),\ a\sin(s/a))$ for $s \in [0,\ 2\pi a]$ is the arclength parametrization of $x^2 + y^2 = a^2$.

Unit Tangent Vector: $T(s) = \Gamma'(s) = (-\sin(s/a),\ \cos(s/a))$

$$\Gamma''(s) = \frac{-1}{a}(\cos(s/a),\ \sin(s/a))$$

Curvature: $\kappa(s) = \|\Gamma''(s)\| = \dfrac{1}{a}$

Unit Normal Vector: $N(s) = \dfrac{\Gamma''(s)}{\|\Gamma''(s)\|} = (-\cos(s/a),\ -\sin(s/a))$

Binormal Vector: $B(s) = \dfrac{\Gamma'(s) \times \Gamma''(s)}{\|\Gamma''(s)\|} = \hat{k}$

Partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. Then

$$\frac{\partial f}{\partial x}(a, b) := \lim_{t \to 0} \frac{f(a + t, b) - f(a, b)}{t},$$

when exists, is called partial derivative of f at (a, b) w.r.t to the first variable.

Partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$

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when exists, is called partial derivative of f at (a, b) w.r.t to the first variable.

Other notations for $\dfrac{\partial f}{\partial x}(a, b)$:

$$f_x(a, b),\ \partial_x f(a, b),\ \partial_1 f(a, b).$$

Partial derivative $\dfrac{\partial f}{\partial y}(a, b)$ w.r.t. the second variable is defined similarly.

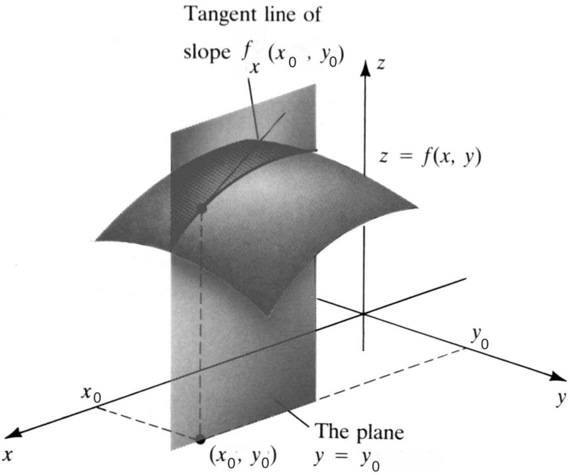

Figure : Graph of $z = f(x, y)$ and geometric interpretation of $\partial_x f(x_0, y_0)$.

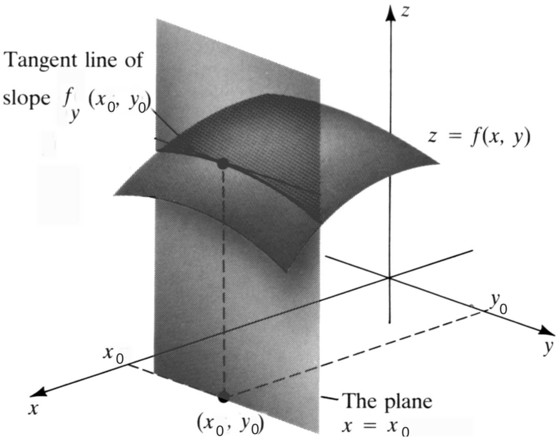

Figure : Graph of $z = f(x, y)$ and geometric interpretation of $\partial_y f(x_0, y_0)$.

Examples

- Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) := 0$ and $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Then

$$\partial_1 f(0,0) = \partial_2 f(0,0) = 0$$

even though f is NOT continuous at $(0,0)$.

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$$\partial_1 f(0,0) = \partial_2 f(0,0) = 0$$

even though f is NOT continuous at $(0,0)$.

- Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and

$$f(x,y) := \begin{cases} x\sin(1/y) + y\sin(1/x) & \text{if } x \neq 0, y \neq 0, \\ x\sin(1/x) & \text{if } x \neq 0, y = 0, \\ y\sin(1/y) & \text{if } x = 0, y \neq 0. \end{cases}$$

Then f is continuous at $(0,0)$ but neither $\partial_1 f(0,0)$ nor $\partial_2 f(0,0)$ exists.

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Then f is continuous at $(0,0)$ but neither $\partial_1 f(0,0)$ nor $\partial_2 f(0,0)$ exists.

Moral: Partial derivatives $\not\Rightarrow$ continuity $\not\Rightarrow$ Partial derivatives

Partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $X_0 \in \mathbb{R}^n$. Then

$$\frac{\partial f}{\partial x_i}(X_0) := \lim_{t \to 0} \frac{f(X_0 + te_i) - f(X_0)}{t},$$

when exists, is called partial derivative of f at X_0 w.r.t to the i-th variable.

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when exists, is called partial derivative of f at X_0 w.r.t to the i-th variable.

Other notations for $\dfrac{\partial f}{\partial x_i}(X_0)$:

$$f_{x_i}(X_0),\ \partial_{x_i} f(X_0),\ \partial_i f(X_0).$$

If $\partial_i f(X_0)$ exists for $i = 1, 2, \ldots, n$, then f is said to have first order partial derivatives at X_0.

Gradient

Gradient: If all the first order partial derivatives of f exist at X_0, then the vector $(\frac{\partial f}{\partial x_1}(X_0), \frac{\partial f}{\partial x_2}(X_0), \ldots, \frac{\partial f}{\partial x_n}(X_0))$ is called the gradient of f at X_0, and is denoted by $\nabla f(X_0)$.

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$$\nabla f(X_0) := \left(\frac{\partial f}{\partial x_1}(X_0), \frac{\partial f}{\partial x_2}(X_0), \ldots, \frac{\partial f}{\partial x_n}(X_0)\right).$$

Sum, product and chain rule

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ and $X_0 \in \mathbb{R}^n$. Suppose $\partial_i f(X_0)$ and $\partial_i g(X_0)$ exist. Then

- $\partial_i(\alpha f)(X_0) = \alpha \partial_i f(X_0)$ for $\alpha \in \mathbb{R}$,
- $\partial_i(f + g)(X_0) = \partial_i f(X_0) + \partial_i g(X_0)$,
- $\partial_i(fg)(X_0) = \partial_i f(X_0) g(X_0) + f(X_0) \partial_i g(X_0)$.
- If $h : \mathbb{R} \to \mathbb{R}$ is differentiable at $f(X_0)$ then $\partial_i(h \circ f)(X_0)$ exists and $\partial_i(h \circ f)(X_0) = h'(f(X_0)) \partial_i f(X_0)$.

Sufficient condition for continuity

Theorem

Suppose one of the partial derivatives exist at (a, b) and the other partial derivative is bounded in a neighborhood of (a, b). Then $f(x, y)$ is continuous at (a, b).

Proof: Board

Remark: The above conditions are not necessary.

Example: Consider the function $f(x, y) = |x| + |y|$. This is a continuous function at $(0, 0)$. But both the partial derivatives do not exist at $(0, 0)$.

Directional derivatives of $f : \mathbb{R}^n \to \mathbb{R}$

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $X_0 \in \mathbb{R}^n$. Also let $U \in \mathbb{R}^n$ with $\|U\| = 1$. Then the limit, when exists,

$$\begin{aligned} \mathrm{D}_U f(X_0) &:= \lim_{t\to 0} \frac{f(X_0 + tU) - f(X_0)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} f(X_0 + tU)_{|t=0}, \\ &= \text{rate of change of } f \text{ at } X_0 \text{ in the direction of } U, \end{aligned}$$

is called directional derivative of f at X_0 in the direction of U.

- $\mathrm{D}_U f(X_0)$, also denoted by $\frac{\partial f}{\partial U}(X_0)$, is the rate of change of f at X_0 in the direction U.

MA102: Multivariable Calculus

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Examples

1. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := \sqrt{|xy|}$. Then $\partial_1 f(0,0) = 0 = \partial_2 f(0,0)$ and f is continuous at $(0,0)$. However, $\mathrm{D}_U f(0,0)$ does NOT exist for $u_1 u_2 \neq 0$.

Examples

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2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x, y) := \dfrac{x^2 y}{x^4 + y^2}$ if $(x, y) \neq (0,0)$. Then f is NOT continuous at $(0,0)$, $\partial_1 f(0,0) = 0 = \partial_2 f(0,0)$ and $\mathrm{D}_U f(0,0)$ exits for all U. Further, $\mathrm{D}_U f(0,0) = u_1^2 / u_2$ for $u_1 u_2 \neq 0$.

Examples

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Moral: Partial derivatives $\not\Rightarrow$ Directional derivative $\not\Rightarrow$ Continuity $\not\Rightarrow$ Directional derivative.

Properties of directional derivatives

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $X_0 \in \mathbb{R}^n$. Also let $U \in \mathbb{R}^n$ with $\|U\| = 1$.

Then

- Sum, product and chain rule similar to those of $\partial_i f(X_0)$ hold for $\mathrm{D}_U f(X_0)$.
- If $\mathrm{D}_U f(X_0)$ exists for all nonzero $U \in \mathbb{R}^n$ then f is said to have directional derivatives in all directions.
- Obviously $\partial_i f(X_0) = \mathrm{D}_{e_i} f(X_0)$. Hence $\mathrm{D}_U f(X_0)$ exists in all directions $U \Rightarrow \partial_i f(X_0)$ exist for $i = 1, 2, \ldots, n$.

Differential Calculus for $f : \mathbb{R}^n \to \mathbb{R}$

Question: Let $f : \mathbb{R}^n \to \mathbb{R}$. What does it mean to say that f is differentiable?

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Task: Define differentiability of f at $X_0 \in \mathbb{R}^n$ and determine the derivative $\mathrm{D}f(X_0)$.

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Wish List:

- f is differentiable at $X_0 \Rightarrow f$ is continuous at X_0.

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Wish List:

- f is differentiable at $X_0 \Rightarrow f$ is continuous at X_0.
- Sum, product and chain rules hold for $\mathrm{D}f(X_0)$.
- Mean Value Theorem and Taylor's Theorem hold for f.

Differentiability of $f : (c, d) \subset \mathbb{R} \to \mathbb{R}$

1. f is differentiable at $a \in (c, d)$ if there exists $\alpha \in \mathbb{R}$ such that
$$\alpha = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

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1. f is differentiable at $a \in (c, d)$ if there exists $\alpha \in \mathbb{R}$ such that
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In other words, f is differentiable at a if there exists $\varepsilon = \varepsilon(h)$ and a constant α satisfying $f(a+h) - f(a) = h \cdot \alpha + h \cdot \varepsilon$ such that $\varepsilon \to 0$ as $h \to 0$.

Differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$

Differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$:
Let D be an open subset of $\mathbb{R}^2$.

Definition 1: A function $f : D \to \mathbb{R}$ is differentiable at a point $(a, b) \in D$ if there exist $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\varepsilon_1 = \varepsilon_1(h, k), \varepsilon_2 = \varepsilon_2(h, k)$ such that

$$f(a + h, b + k) - f(a, b) = h \cdot \alpha_1 + k \cdot \alpha_2 + h\varepsilon_1 + k\varepsilon_2,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(h, k) \to (0, 0)$.

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where $\varepsilon_1, \varepsilon_2 \to 0$ as $(h, k) \to (0, 0)$.

We call the pair (α_1, α_2) the total derivative of f at (a, b).

Differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$

Fact: If (α_1, α_2) is the total derivative of f at (a, b), then letting (h, k) approach $(0, 0)$ along the x-axis and y-axis, we have $\alpha_1 = f_x(a, b)$ and $\alpha_2 = f_y(a, b)$, respectively.

Differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$

Fact: If (α_1, α_2) is the total derivative of f at (a, b), then letting (h, k) approach $(0, 0)$ along the x-axis and y-axis, we have $\alpha_1 = f_x(a, b)$ and $\alpha_2 = f_y(a, b)$, respectively.

Example 1: The following function is NOT differentiable at $(0, 0)$.

$$f(x, y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y}, & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

Solution: $|f(x, y)| \leq |x| + |y| \leq 2\sqrt{x^2 + y^2}$ implies that f is continuous at $(0, 0)$.

Differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$

We have

- $$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$
- $$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

If f is differentiable at $(0,0)$, then we can deduce that $\sin \frac{1}{h} \to 0$ as $h \to 0$, which is a contradiction.

Differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$

Example 2: The function f defined by $f(x, y) = \sqrt{|xy|}$ is NOT differentiable at the origin.

Solution: If f is differentiable at $(0, 0)$, then there exist $\varepsilon_1, \varepsilon_2$ such that

$$f(h, k) = \varepsilon_1 h + \varepsilon_2 k.$$

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Taking $h = k$, we get

$$\frac{|h|}{h} = \varepsilon_1 + \varepsilon_2.$$

This implies that $(\varepsilon_1 + \varepsilon_2) \not\to 0$ as $h \to 0$ along the line $h = k$.

Another definition of differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$

Recall that $f : (c, d) \to \mathbb{R}$ is differentiable at $a \in (c, d)$ if there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|f(a + h) - f(a) - \alpha h|}{|h|} = 0.$$

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Recall that $f : (c, d) \to \mathbb{R}$ is differentiable at $a \in (c, d)$ if there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \alpha h|}{|h|} = 0.$$

Definition 2: Let D be open in $\mathbb{R}^2$. Then $f : D \to \mathbb{R}$ is differentiable at a point $(a, b) \in D$ if

$$\lim_{(h,k) \to (0,0)} \frac{|f(a+h, b+k) - f(a, b) - f_x(a, b)h - f_y(a, b)k|}{\sqrt{h^2 + k^2}} = 0.$$

Another definition of differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$

We use the following notations

1. $\Delta f = f(a+h, b+k) - f(a,b)$, the total variation of f
2. $df = hf_x(a,b) + kf_y(a,b)$, the total differential of f.
3. $\rho = \sqrt{h^2 + k^2}$

Then Definition 2 takes the form:

f is differentiable at (a,b) if $\lim_{\rho \to 0} \frac{\Delta f - df}{\rho} = 0.$

Theorem

Definition 1 and Definition 2 of differentiability of $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ are equivalent.

Differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$

Example 3: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & x = y = 0. \end{cases}$$

Then, f is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = 0, f_y(0,0) = 0$. By taking $h = \rho\cos\theta, k = \rho\sin\theta$, we obtain

$$\frac{\Delta f - df}{\rho} = \frac{h^2k^2}{\rho^3} = \frac{\rho^4\cos^2\theta\sin^2\theta}{\rho^3} = \rho\cos^2\theta\sin^2\theta.$$

This implies that $\left|\frac{\Delta f - df}{\rho}\right| \leq \rho \to 0$ as $\rho \to 0$. Hence, f is differentiable at $(0,0)$.

Differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$

Example 4: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq 0 \\ 0, & x = y = 0 \end{cases}.$$

Then, f is NOT differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = 0, f_y(0,0) = 0$.
By taking $h = \rho\cos\theta, k = \rho\sin\theta$, we obtain

$$\frac{\Delta f - df}{\rho} = \frac{h^2 k}{\rho^3} = \frac{\rho^3 \cos^2\theta \sin\theta}{\rho^3} = \cos^2\theta\sin\theta.$$

The limit does not exist as $\rho \to 0$. Therefore, f is NOT differentiable at $(0,0)$.

Sufficient condition for differentiability

We now prove a sufficient condition for differentiability of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

Theorem

Suppose that one of the partial derivatives f_x and f_y exists at (a, b) and the other is continuous at (a, b). Then, f is differentiable at (a, b).

Remark: Continuity of f_x or f_y at (a, b) is not necessary for differentiability of f at (a, b).

Sufficient condition for differentiability

Example 5: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x^3 \sin \frac{1}{x^2} + y^3 \sin \frac{1}{y^2} & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

Then f is differentiable at $(0,0)$, but f_x and f_y are not continuous at $(0,0)$.

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Then f is differentiable at $(0,0)$, but f_x and f_y are not continuous at $(0,0)$.

Solution: We have

$$f_x(x,y) = \begin{cases} 3x^2 \sin \frac{1}{x^2} - 2\cos \frac{1}{x^2} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$$

and

$$f_y(x,y) = \begin{cases} 3y^2 \sin \frac{1}{y^2} - 2\cos \frac{1}{y^2} & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

Sufficient condition for differentiability

Clearly, partial derivatives f_x and f_y are not continuous at $(0, 0)$.

However,

$$f(0 + h, 0 + k) - f(0, 0) = f(h, k) = h^3 \sin \frac{1}{h^2} + k^3 \sin \frac{1}{k^2}$$

Then,

$$f(h, k) = 0 + 0 + \varepsilon_1 h + \varepsilon_2 k,$$

where $\varepsilon_1 = h^2 \sin \frac{1}{h^2}$ and $\varepsilon_2 = k^2 \sin \frac{1}{k^2}$.

It is easy to check that $\varepsilon_1, \varepsilon_2 \to 0$ as $(h, k) \to (0, 0)$. Hence, f is differentiable at $(0, 0)$.

Sufficient condition for differentiability

Remark: There are functions for which directional derivatives exist in any direction, but the function is not differentiable.

Example 6: The function

$$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

is not differentiable, but all the directional derivatives exist.

MA102: Multivariable Calculus

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Differentiability of $f : A \subset \mathbb{R}^n \to \mathbb{R}$

A function $f : A \to \mathbb{R}$ is said to be differentiable at $X_0 \in A$ if there exists $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$\lim_{H \to 0} \frac{|f(X_0 + H) - f(X_0) - \alpha \bullet H|}{\|H\|} = 0.$$

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $H = (h_1, h_2, \ldots, h_n)$, then

$$\alpha \bullet H = \langle \alpha, H \rangle = \sum_{i=1}^{n} \alpha_i \cdot h_i.$$

Fact: If f is differentiable at X_0, then $\alpha_1 = \partial_1 f(X_0)$, $\ldots$, $\alpha_n = \partial_n f(X_0)$.

Differentiability of $f : A \subset \mathbb{R}^n \to \mathbb{R}$

Differentiability and linear maps

f is differentiable at $a \in (c, d)$ if there exists a linear map $L : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - L(h)|}{|h|} = 0.$$

Let $A \subset \mathbb{R}^n$ be open. Then $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $X_0 \in A$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{H \to 0} \frac{|f(X_0+H) - f(X_0) - L(H)|}{\|H\|} = 0. \qquad (*)$$

The linear map L is called the derivative of f at X_0 and is denoted by $\mathrm{D}f(X_0)$, that is, $L = \mathrm{D}f(X_0)$.

Differentiability of $f : A \subset \mathbb{R}^n \to \mathbb{R}$

Fact: If $L : \mathbb{R}^n \to \mathbb{R}$ is linear, then $L(X) = P \bullet X = \langle X, P \rangle$ for some $P \in \mathbb{R}^n$.

Theorem: If $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $X_0 \in A$ then partial derivatives $\partial_1 f(X_0), \ldots, \partial_n f(X_0)$ exist and the derivative $\mathrm{D}f(X_0) : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\mathrm{D}f(X_0)(X) = \nabla f(X_0) \bullet X = \langle X, \nabla f(X_0) \rangle,$$

where $\nabla f(X_0) := (\partial_1 f(X_0), \ldots, \partial_n f(X_0))$ is the gradient of f at X_0.

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Proof.
Considering $H := te_i$ for $t \in \mathbb{R}$ in $(*)$ and letting $t \to 0$, it ready follows that $\partial_i f(X_0)$ exists for all $i = 1, 2, \ldots, n$. $\square$

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Example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x,y) := xy\dfrac{x^2 - y^2}{x^2 + y^2}$ if $(x,y) \neq (0,0)$. Then

- f is continuous at $(0,0)$ and $\nabla f(0,0) = (0,0)$.
- $$\frac{|f(h,k) - f(0,0) - \nabla f(0,0) \bullet (h,k)|}{\|(h,k)\|} \leq \frac{|hk|}{\|(h,k)\|} \to 0.$$

  Hence, f is differentiable at $(0,0)$.

Affine approximation

Let $X_0 \in \mathbb{R}^n$. Define the error function $E : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$E(H) := \frac{f(X_0 + H) - f(X_0) - \nabla f(X_0) \bullet H}{\|H\|}.$$

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- Then f is differentiable at X_0 if and only if

$$f(X_0 + H) = f(X_0) + \nabla f(X_0) \bullet H + E(H)\|H\|$$

and $E(H) \to 0$ as $\|H\| \to 0$.

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- The function $T(H) = f(X_0) + \nabla f(X_0) \bullet H$ approximates $f(X_0 + H)$ for small $\|H\| \iff f$ is differentiable at X_0.

Implications of differentiability

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $X_0 \in \mathbb{R}^n$.

- If f is differentiable at X_0 then f is continuous at X_0.
- If f is differentiable at X_0 then directional derivatives exist for all $U \in \mathbb{R}^n$ and

$$\mathrm{D}_U f(X_0) = \nabla f(X_0) \bullet U.$$

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Proof: Continuity follows from

$$f(X_0 + H) = f(X_0) + \nabla f(X_0) \bullet H + E(H)\|H\|$$

and the fact that $E(H) \to 0$ as $\|H\| \to 0$.

Second part: Put $H = tU$. Note that $\|H\| = \|tU\| = |t|$.

Example

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x,y) := \dfrac{x^2 y}{x^4 + y^2}$ if $(x,y) \neq (0,0)$. Then

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- For $U = (u_1, u_2)$ such that $u_1 u_2 \neq 0$, we have

$$\mathrm{D}_U f(0,0) = u_1^2/u_2 \neq \nabla f(0,0) \bullet U.$$

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Moral: The equality $\mathrm{D}_U f(X_0) = \nabla f(X_0) \bullet U$ may not hold if f is NOT differentiable at X_0.

A geometric interpretation of gradient

Let $D \subseteq \mathbb{R}^2$ and let (a, b) be an interior point of D. Let $f : D \to \mathbb{R}$ be differentiable at (a, b) and suppose $\nabla f(a, b) \neq (0, 0)$.

$$\mathrm{D}_U f(a, b) = \nabla f(a, b) \bullet U = \|\nabla f(a, b)\| \cos\theta,$$

where θ is the angle between $\nabla f(a, b)$ and U.

1. $\mathrm{D}_U f(a, b)$ is maximum when $\cos\theta = 1$. Thus, near (a, b), $U = \nabla f(a, b)/\|\nabla f(a, b)\|$ is the direction in which f increases most rapidly.
2. $\mathrm{D}_U f(a, b)$ is minimum when $\cos\theta = -1$. Thus, near (a, b), $U = -\nabla f(a, b)/\|\nabla f(a, b)\|$ is the direction in which f decreases most rapidly.
3. $\mathrm{D}_U f(a, b) = 0$ when $\cos\theta = 0$. Thus, near (a, b), $U = \pm(f_y(a, b), -f_x(a, b))/\|\nabla f(a, b)\|$ are the directions of no change in f.

Properties of derivative

Fact: Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $X_0 \in \mathbb{R}^n$. Then

- $\mathrm{D}(f + \alpha g)(X_0) = \mathrm{D}f(X_0) + \alpha \mathrm{D}g(X_0)$.
- $\mathrm{D}(fg)(X_0) = \mathrm{D}f(X_0)g(X_0) + f(X_0)\mathrm{D}g(X_0)$.

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- $\mathrm{D}(fg)(X_0) = \mathrm{D}f(X_0)g(X_0) + f(X_0)\mathrm{D}g(X_0)$.

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $X_0 \in \mathbb{R}^n$. If $\partial_i f(X_0)$ exists for $i = 1, 2, \ldots, n$, and are continuous on $B(X_0, \varepsilon)$ for some $\varepsilon > 0$, then f is differentiable at X_0.