MA102: Multivariable Calculus

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Iterated integrals

Let $f : \mathcal{R} \to \mathbb{R}$. Suppose that for each fixed $x \in [a, b]$

$$\phi(x) := \int_c^d f(x, y)dy$$

exists. If ϕ is Riemann integrable on $[a, b]$ then

$$\int_a^b \phi(x)dx = \int_a^b \left(\int_c^d f(x, y)dy \right) dx$$

is called an iterated integral of f over $\mathcal{R}$.

Similarly $\int_c^d \left(\int_a^b f(x, y)dx \right) dy$, when exists, is another iterated integral of f over $\mathcal{R}$.

Fubini's Theorem

Theorem: Let $f : \mathcal{R} \to \mathbb{R}$ be continuous. Then both the iterated limits exist and

$$\begin{aligned} \iint_{\mathcal{R}} f(x,y)dA &= \int_a^b \left(\int_c^d f(x,y)dy \right) dx \\ &= \int_c^d \left(\int_a^b f(x,y)dx \right) dy. \end{aligned}$$

Example: Evaluate $\iint_{\mathcal{R}} xe^{xy} dA$, where $\mathcal{R} = [0,1] \times [0,1]$. Since the function is continuous,

$$\iint_{\mathcal{R}} xe^{xy} dA = \int_0^1 (\int_0^1 xe^{xy} dy) dx = \int_0^1 (e^x - 1) dx = e - 2.$$

Double integrals over general domains

Definition: Let $D \subset \mathbb{R}^2$ be bounded and $f : D \to \mathbb{R}$ be a bounded function. Then f is said to be integrable over D if for some rectangle $\mathcal{R}$ containing D the function

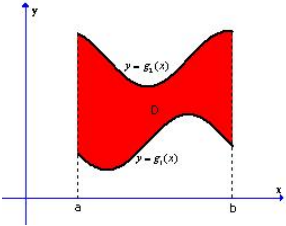
$$F(x,y) := \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

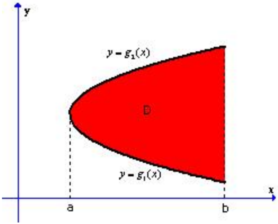
is Riemann integrable over $\mathcal{R}$. The double integral of f over D is then defined by

$$\iint_D f(x,y)dA := \iint_{\mathcal{R}} F(x,y)dA.$$

Remark: Since F is zero outside D the choice of $\mathcal{R}$ is unimportant in defining double integral of f over D.

Special Regions: Type-I Regions



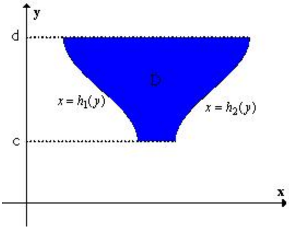


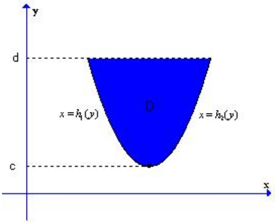
- Type-I Region:

  $$D = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [a, b] \text{ and } g_1(x) \leq y \leq g_2(x)\}$$

  where $g_1(x)$ and $g_2(x)$ are continuous functions on $[a, b]$ and $g_1(x) \leq g_2(x)$ for all $x \in [a, b]$.

Type-II Regions



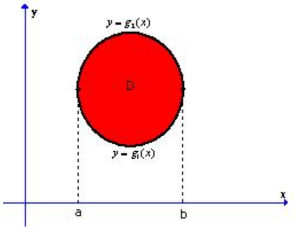


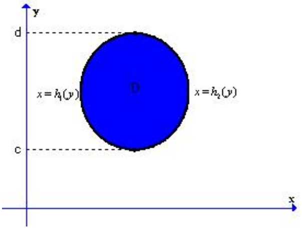
- Type-II Region:

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ y \in [c, d] \text{ and } h_1(y) \leq x \leq h_2(y)\}$$

where $h_1(y)$ and $h_2(y)$ are continuous functions on $[c, d]$ and $h_1(y) \leq h_2(y)$ for all $y \in [c, d]$.

Type-III Regions (Both Type-I and Type-II)





$\mathcal{R}$ is called Type-III region if $\mathcal{R}$ is simultaneously of Type-I and Type-II.

Double integral over special domains

Theorem: Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous. If D is Type-I and $D = \{(x, y) : x \in [a, b] \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$ then f is integrable over D and

$$\iint_D f(x, y)dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy \right) dx.$$

If D is Type-II and
$D = \{(x, y) : \psi_1(y) \leq x \leq \psi_2(y) \text{ and } y \in [c, d]\}$ then f is integrable over D and

$$\iint_D f(x, y)dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx \right) dy.$$

Area and Volume

Let $D \subset \mathbb{R}^2$ be a special (Type-I or Type-II or Type-III) domain and $f : D \to \mathbb{R}$ be continuous. Then

$$\text{Area}(D) = \iint_D dA.$$

If $f(x, y) \geq 0$ then the volume of the solid S bounded by D and the graph of $z = f(x, y)$ is given by

$$\text{Volume}(S) = \iint_D f(x, y) dA.$$

Example

Find the volume of the solid S bounded by elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2,\ y = 2$, and the coordinate planes.

$$\begin{aligned}
\text{Volume}(S) &= \iint_{\mathcal{R}} (16 - x^2 - 2y^2)\,dA \\
&= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,dx\,dy = 48.
\end{aligned}$$

Example

Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

The region D is Type-I and

$$\begin{aligned} \iint_D (x+2y)dA &= \int_{-1}^{1} \left(\int_{2x^2}^{1+x^2} (x+2y)dy \right) dx \\ &= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)dx = \frac{32}{15}. \end{aligned}$$

Riemann sum for Triple integral

Consider the rectangular cube $V := [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ and a bounded function $f : V \to \mathbb{R}$.

Let P be a partition of V into sub-cubes V_{ijk} and $c_{ijk} \in V_{ijk}$ for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $k = 1, \ldots, p$. Also let

$\Delta V_{ijk} := \mathrm{Volume}(V_{ijk}) = \Delta x_i \Delta y_j \Delta z_k$ and $\mu(P) := \max_{ijk} \Delta V_{ijk}$.

Consider the Riemann sum

$$S(P, f) := \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f(c_{ijk}) \Delta V_{ijk}.$$

Triple integral

If $\lim_{\mu(P)\to 0} S(P, f)$ exists then f is said to be Riemann integrable and the (triple) integral of f over V is given by

$$\iiint_V f(x,y,z)dV = \iiint_V f(x,y,z)dxdydz = \lim_{\mu(P)\to 0} S(P,f).$$

Theorem: Let $f : V \to \mathbb{R}$ is continuous. Then

- f is Riemann integrable over V.
- Fubini's theorem holds, i.e, the iterated integrals exist and are equal to $\iiint_V f dV$.

Example

Evaluate $\iiint_V xyz^2 dV$ where $V = [0,1] \times [-1,2] \times [0,3]$.

By Fubini's theorem,

$$\iiint_V f dV = \int_0^3 \left(\int_{-1}^2 \left(\int_0^1 x dx \right) y dy \right) z^2 dz = \frac{27}{4}.$$

Triple integrals over general domains

Let $D \subset \mathbb{R}^3$ be bounded and $f : D \to \mathbb{R}$ be a bounded function. Then f is said to be integrable over D if for some rectangular cube V containing D the function

$$F(x,y,z) := \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in D \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable over V. Then

$$\iiint_D f(x,y,z)dV := \iiint_V F(x,y,z)dV$$

and

$$\mathrm{Volume}(D) := \iiint_D dV.$$

Type-I domain:

A domain $V \subset \mathbb{R}^3$ is **Type-I** if

$$V = \{(x, y, z) : (x, y) \in D \text{ and } u_1(x, y) \leq z \leq u_2(x, y)\}$$

for some $D \subset \mathbb{R}^2$ and continuous functions $u_i : D \to \mathbb{R}$.

If $f : V \to \mathbb{R}$ be continuous and D is a special domain (e.g.,Type-I, Type-II, Type-III) then

$$\iiint_V f(x, y, z)dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz \right) dxdy.$$

Similar results hold for Type-II and Type-III domains.

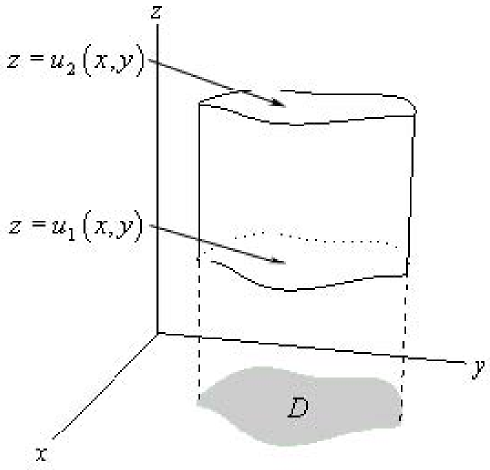


Figure: Type-I domain

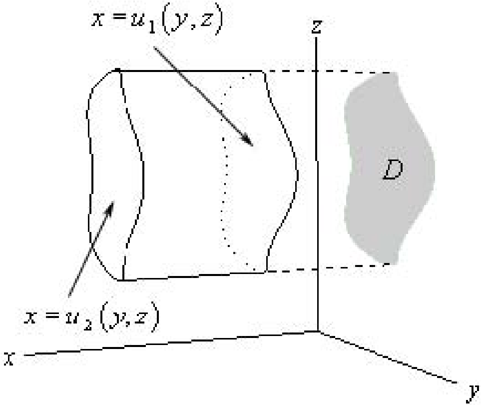


Figure: Type-II domain

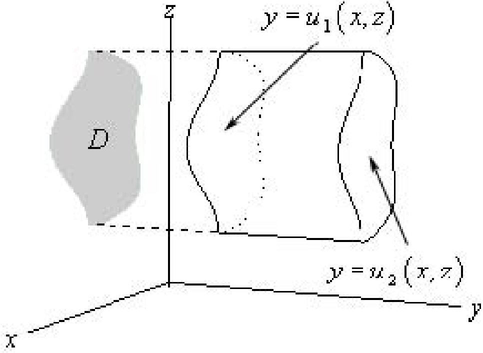


Figure: Type-III domain

Example:

Evaluate $\iiint_V 2x dV$ where V is the region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 3y + z = 6$.

Note that V is Type-I:

$$0 \leq z \leq 6 - 2x - 3y \text{ and } (x, y) \in D,$$

where D is special domain given by

$$0 \leq x \leq 3 \text{ and } 0 \leq y \leq -\frac{2}{3}x + 2.$$

Thus

$$\begin{aligned} \iiint_V 2x dV &= \iint_D \left(\int_0^{6-2x-3y} 2x \, dz \right) dA \\ &= \int_0^3 \int_0^{-\frac{2}{3}x+2} (6 - 2x - 3y) 2x dy dx = 9. \end{aligned}$$

Example:

Find the volume of the region bounded by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

The volume is $V = \iiint_\Omega dzdydx$, where Ω is bounded above by the surface $z = 8 - x^2 - y^2$ and below by the surface $z = x^2 + 3y^2$. Therefore, the limits of z are from $z = x^2 + 3y^2$ to $z = 8 - x^2 - y^2$.

The projection of Ω on xy-plane is the solution of

$$8 - x^2 - y^2 = x^2 + 3y^2 \implies x^2 + 2y^2 = 4.$$

Therefore the limits of x and y are to be determined by $D : x^2 + 2y^2 = 4$.

Example (cont.):

$$
\begin{aligned}
V &= \iint_{\mathcal{R}} \int_{z=x^2+3y^2}^{8-x^2-y^2} dz dA \\
&= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2) dy dx \\
&= \int_{-2}^{2} \left((8 - x^2)y - \frac{4}{3} y^3 \right)_{y=-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \\
&= \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2} dx = 8\pi\sqrt{2}.
\end{aligned}
$$

Example: Change of order of integration

Consider the evaluation of integral $\iint_D \frac{\sin x}{x} dA$ over the triangle formed by $y = 0, x = 1$ and $y = x$.

If we write $D = \{(x, y) : 0 \leq y \leq 1, y \leq x \leq 1\}$, then

$$\iint_D \frac{\sin x}{x} dA = \int_0^1 \left(\int_{x=y}^1 \frac{\sin x}{x} dx \right) dy.$$

The innermost integral is difficult to evaluate.

If we change the order of integration, by taking $D = \{(x, y) : \ 0 \leq x \leq 1, 0 \leq y \leq x\}$, then

$$\iint_D \frac{\sin x}{x} dA = \int_0^1 \int_{y=0}^x \frac{\sin x}{x} dydx = \int_0^1 \sin x dx = 1 - \cos 1.$$

Example: Change of order of integration

Find the volume of the region bounded by $x + z = 1$, $y + 2z = 2$ in the first quadrant.

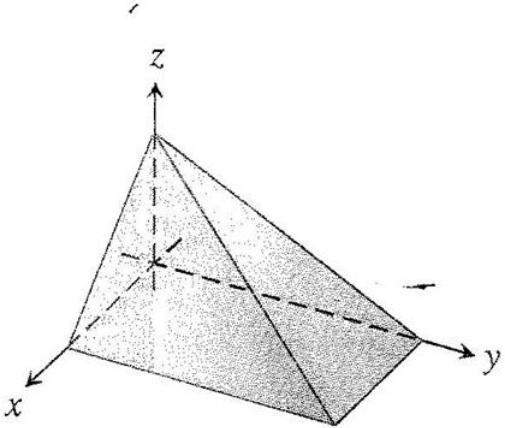


Figure: Region bounded by $x + z = 1$ and $y + 2z = 2, \quad x, y, z \geq 0$

Example: Change of order of integration

Draw line parallel to z-axis and note that the upper surfaces are: $2z + y = 2$ over triangle bounded by $x = 0, y = 2, y = 2x$ and $z = 1 - x$ over the triangle bounded by $y = 0, x = 1, y = 2x$. Therefore,

$$V = \int_{y=0}^{2} \int_{x=0}^{y/2} \int_{z=0}^{\frac{2-y}{2}} dz\, dx\, dy + \int_{x=0}^{1} \int_{y=0}^{2x} \int_{z=0}^{1-x} dz\, dy\, dx = \frac{2}{3}$$

On the other hand, by first drawing the line parallel to x-axis, we get

$$V = \int_{z=0}^{1} \int_{y=0}^{2-2z} \int_{x=0}^{1-z} dx\, dy\, dz = \frac{2}{3}$$

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Example (cont.):

Example: Evaluate $I = \int_{z=0}^{4} \int_{y=0}^{1} \int_{x=2y}^{2} \frac{2\cos(x^2)}{\sqrt{z}} dx\ dy\ dz.$

The integral is difficult to evaluate in the given order of integration. We change the order of integration and evaluate the integral:

$$
\begin{aligned}
I &= \int_{z=0}^{4} \int_{x=0}^{2} \int_{y=0}^{x/2} \frac{2\cos(x^2)}{\sqrt{z}} dy\ dx\ dz. \\
&= \int_{z=0}^{4} \int_{x=0}^{2} \frac{x\cos(x^2)}{\sqrt{z}} dx\ dz = 2\sin 4.
\end{aligned}
$$

Change of variable

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 given by $T(u, v) = (x(u, v), y(u, v))$. Then the Jacobian matrix $J(u, v)$ of T is given by

$$J(u, v) := \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}.$$

Define the Jacobian of T by

$$\frac{\partial(x, y)}{\partial(u, v)} := x_u y_v - x_v y_u = \det J(u, v).$$

Polar coordinates: Define $T(r, \theta) := (r\cos\theta, r\sin\theta)$. Then

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Change of variable for double integrals

Suppose T is injective and $J(u,v)$ is nonsingular. Let $D \subset \mathbb{R}^2$ and $G := T(D)$. Suppose that f is integrable on G. Then

$$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

and

$$\iint_G f(x,y)dxdy = \iint_D f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

Polar coordinates:

$$\iint_G f(x,y)dxdy = \iint_D f(r\cos\theta, r\sin\theta) r dr d\theta.$$

Example

Evaluate the integral $I = \int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dA$.

The given domain is the triangle bounded by $x = 0, y = 0$ and $x + y = 1$. We take the transformation $u = x + y$ and $v = y - 2x$. Under this transformation, the given triangle will be transformed into triangle bounded by
$v = u, v = -2u$ and $u = 1$.
The inverse of this transformation is $x = \frac{u-v}{3}$ and
$y = \frac{2u+v}{3}$. Hence the Jacobian

$$J = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = 1/3.$$

Hence

$$I = \frac{1}{3} \int_0^1 \int_{v=-2u}^{u} \sqrt{u} v^2 dv\, du$$

Example

Using the transformation $u = 2x + 3y$ and $v = x - 3y$, find the value of the integral $I = \iint_{\mathcal{R}} e^{2x+3y} \cos(x - 3y)\, dx\, dy$ where $\mathcal{R}$ is the region bounded by the parallelogram with vertices $(0, 0)$, $(1,\ 1/3)$, $(4/3,\ 1/9)$, $(1/3,\ -2/9)$.

Under the given transformations, $\mathcal{R}$ will be transformed into the rectangle with vertices $(0, 0), (3, 0), (3, 1)$ and $(0, 1)$. Also, $|J| = \frac{1}{9}$. Thus,

$$I = \frac{1}{9} \int_{v=0}^{1} \int_{u=0}^{3} e^{u} \cos v\, du\, dv = \frac{1}{9}(e^{3} - 1) \sin 1.$$

Example

Evaluate $\iiint_G \sqrt{x^2+z^2}\,dV$ where G is the region bounded by the paraboloid $y = x^2 + z^2$ and $y = 4$.

We have

$$\iiint_G f(x,y,z)dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy \right) dxdz,$$

where $D = \{(x,z) : x^2 + z^2 \leq 4\}$.

Setting $x = r\cos\theta$ and $z = r\sin\theta$ for $(r,\theta) \in [0,2] \times [0,2\pi]$,

$$\iiint_G f(x,y,z)dV = \int_0^{2\pi} \int_0^2 r(4-r^2)rdrd\theta = \frac{128\pi}{5}.$$

Change of variable for multiple integrals

Let $D \subset \mathbb{R}^n$ be open and bounded. Let $T : D \to \mathbb{R}^n$ be such that T is C^1, injective and the Jacobian $J(U)$ is nonsingular for $U \in D$.

Let $G := T(D)$ and $f : G \to \mathbb{R}$ be integrable over G. Then

$$dx_1 \cdots dx_n = \left| \frac{\partial(x_1, \cdots, x_n)}{\partial(u_1, \cdots, u_n)} \right| du_1 \cdots du_n$$

and

$$\begin{aligned} \int_G f(X) dx_1 \cdots dx_n &= \int_D f(X(U)) \left| \frac{\partial(x_1, \cdots, x_n)}{\partial(u_1, \cdots, u_n)} \right| du_1 \cdots du_n \\ &= \int_D f(X(U)) \left| \frac{dX}{dU} \right| dU. \end{aligned}$$

Cylindrical coordinates

Consider $T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$. Then

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Thus $dV = rdrd\theta dz$ and

$$\iiint_G f(x, y, z)dV = \iiint_D f(r\cos\theta, r\sin\theta, z)rdrd\theta dz.$$

Example

Evaluate $\iiint_G \sqrt{x^2+y^2}dV$, where G is the region bounded by $x^2+y^2=1, z=4$ and $z=1-x^2-y^2$.

Consider cylindrical coordinates
$D := \{(r,\theta,z) : (r,\theta) \in [0,1]\times[0,2\pi],\ 1-r^2 \le z \le 4\}$.

Then

$$\iiint_G f(x,y,z)dV = \int_0^1 \int_0^{2\pi} \left(\int_{1-r^2}^4 dz \right) r\, rdrd\theta = \frac{12\pi}{5}.$$

Spherical coordinates

Consider $T(\rho, \phi, \theta) = (\rho \sin\phi \ \cos\theta, \rho \sin\phi \ \sin\theta, \rho\cos\phi)$. Then

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\ &= \rho^2\sin\phi.\end{aligned}$$

Thus $dV = \rho^2 \sin\phi d\rho d\phi d\theta$ and

$$\iiint_G f(x,y,z)dV =$$
$$\iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\phi d\theta.$$

Example

Evaluate $\iiint_G e^{(x^2+y^2+z^2)^{3/2}} dV$, where
$G := \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$.

Using spherical coordinates we have

$$\begin{aligned} \iiint_D f(x, y, z) dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^{1} e^{\rho^3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \frac{4}{3}\pi(e-1). \end{aligned}$$

Line Integral: Partition of curves

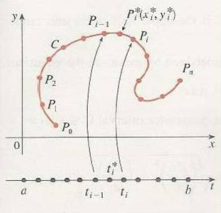

FIGURE 1
A partition of $[a, b]$ determines a partition of C

Let Γ be a curve in $\mathbb{R}^n$ paramatrized by $r : [a, b] \to \mathbb{R}^n$. Then a partition $\mathcal{P} := (a = t_0 < \ldots < t_m = b)$ of $[a, b]$ induces a partition of Γ into m subarcs with arclengths $\Delta s_1, \ldots, \Delta s_m$.

$$\text{Define } \mu(\mathcal{P}) := \max_{1 \le j \le m} \Delta s_j.$$

Riemann sum w.r.t. arclength

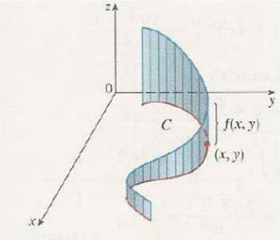

FIGURE 2

Let $f : \Gamma \to \mathbb{R}$. Then for any P_j in the j-th subarc, consider the Riemann sum of f w.r.t. to the partition $\mathcal{P}$

$$S(\mathcal{P}, f) := \sum_{j=1}^{m} f(P_j)\Delta s_j.$$

Line integral

Definition: Suppose that Γ is a piecewise smooth curve in $\mathbb{R}^n$ paramatrized by $r : [a, b] \to \mathbb{R}^n$ and $f : \Gamma \to \mathbb{R}$. Then the line integral of f along Γ is defined by

$$\int_\Gamma f\, ds := \lim_{\mu(\mathcal{P})\to 0} S(\mathcal{P}, f) = \lim_{\mu(\mathcal{P})\to 0} \sum_{j=1}^{m} f(P_j)\Delta s_j$$

if the limit exists (independent of the partitions $\mathcal{P}$ and the chosen points P_j).

Line integral

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if the limit exists (independent of the partitions $\mathcal{P}$ and the chosen points P_j).

Fact: If f is continuous and $r(t)$ is piecewise smooth then we have

$$\int_\Gamma f\, ds = \int_a^b f(r(t))\|r'(t)\|dt.$$

Line integral

For the plane curve $\Gamma : r(t) = (x(t), y(t)), t \in [a, b]$ we have

$$\int_\Gamma f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Line integral

For the plane curve $\Gamma : r(t) = (x(t), y(t)), t \in [a, b]$ we have

$$\int_\Gamma f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Example: Evaluate $\int_\Gamma (2 + x^2 y) ds$, where Γ is the upper half of the circle $x^2 + y^2 = 1$.

Considering $x(t) = \cos t, y(t) = \sin t, 0 \leq t \leq \pi$, we have

$$\int_\Gamma (2 + x^2 y) ds = \int_0^\pi (2 + \cos^2 t \sin t) dt = 2\pi + 2/3.$$

Properties of line integrals

Fact: Let Γ be parametrized by a piecewise smooth curve $r : [a, b] \to \mathbb{R}^n$ and $f, g : \Gamma \to \mathbb{R}$ be continuous. Then the following hold:

- $\int_\Gamma (f + \alpha g) ds = \int_\Gamma f ds + \alpha \int_\Gamma g ds$ for $\alpha \in \mathbb{R}$.

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- $\int_\Gamma (f + \alpha g) ds = \int_\Gamma f ds + \alpha \int_\Gamma g ds$ for $\alpha \in \mathbb{R}$.
- Let $\Gamma = \Gamma_1 + \cdots + \Gamma_m$, where Γ_i is parametrized by smooth curve $r_i : [a_i, b_i] \to \mathbb{R}^n$. Then

$$\int_\Gamma f ds = \int_{\Gamma_1} f ds + \cdots + \int_{\Gamma_m} f ds.$$

Example

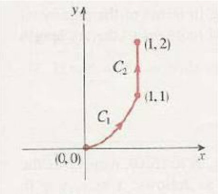

FIGURE 5
$C = C_1 \cup C_2$

Evaluate $\int_\Gamma 2x ds$, where Γ consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the line segment C_2 from $(1, 1)$ to $(1, 2)$. Then

$$\int_\Gamma 2xds = \int_{C_1} 2xds + \int_{C_2} 2xds = \frac{1}{6}(5\sqrt{5} + 11).$$

Application

Suppose a thin wire in the shape of a curve Γ parametrized by a smooth path $r : [a, b] \to \mathbb{R}^2$ has density $\rho(x, y)$. Then the total mass of the wire is given by

$$m = \int_\Gamma \rho(x, y) ds$$

The center of mass $(\overline{x}, \overline{y})$ is given by

$$\overline{x} = \frac{1}{m} \int_\Gamma x\rho(x, y) ds \quad \text{and} \quad \overline{y} = \frac{1}{m} \int_\Gamma y\rho(x, y) ds$$

Application: Example

The wire W has the shape $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 is parametrized by $\gamma_1(t) := (\cos t, \sin t), t \in [0, \pi]$ and Γ_2 is parametrized by $\gamma_2(t) := (t, 0), t \in [-1, 1]$. Let the density function be given by $\rho(x, y) = \sqrt{x^2 + y^2}$.
We have $\|\gamma_i'(t)\| = 1, \rho(\gamma_1(t)) = 1$ and $\rho(\gamma_2(t)) = |t|$.

$$m = \int_\Gamma \rho(x, y) ds = \int_0^\pi dt - \int_{-1}^0 t\ dt + \int_0^1 t\ dt = \pi + 1.$$

The center of mass $(\overline{x}, \overline{y})$ is given by

$$\overline{x} = 0 \quad \text{and} \quad \overline{y} = \frac{2}{1+\pi}.$$

MA102: Multivariable Calculus

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Vector line integral

Definition: Let Γ be a curve in $\mathbb{R}^n$ parametrized by a piecewise smooth path $r : [a, b] \to \mathbb{R}^n$ and let F be a continuous function on an open set containing Γ to $\mathbb{R}^n$. Then the line integral of F over Γ is defined by

$$\int_\Gamma F \bullet dr \quad := \quad \int_a^b F(r(t)) \bullet r'(t)dt = \int_a^b \langle F(r(t)),\, r'(t)\rangle dt.$$

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Note that $[a, b] \longrightarrow \mathbb{R}, t \longmapsto F(r(t)) \bullet r'(t)$ is piecewise continuous and hence Riemann integrable.

Vector line integrals and scalar line integrals

Suppose that Γ is (piecewise) smooth parametrized by r. Then $\|r'(t)\| \neq 0$. Define the tangent vector $T(r(t)) := \frac{r'(t)}{\|r'(t)\|}$ to Γ at $r(t)$.

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Then $F \bullet T$ is the tangential component of F and

$$
\begin{aligned}
\int_\Gamma F \bullet dr &= \int_a^b F(r(t)) \bullet r'(t) dt \\
&= \int_a^b F(r(t)) \bullet T(r(t)) \|r'(t)\| dt \\
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\end{aligned}$$

Vector line integral of F = line integral of $F \bullet T$.

Notations for vector line integrals

- When Γ is closed, that is, $r(a) = r(b)$, the line integral

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- For $n = 3$ and $F = (P, Q, R)$ the line integral is written as

$$\int_\Gamma F \bullet dr = \int_\Gamma (P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz).$$

Examples

- Evaluate $\int_\Gamma F \bullet dr$, where $F(x,y,z) := (xy, yz, zx)$ and $r(t) := (t, t^2, t^2), t \in [0,1]$. We have

$$\int_\Gamma F \bullet dr = \int_0^1 F(r(t)) \bullet r'(t) dt = \int_0^1 (t^3 + 2t^5 + 2t^4) dt = \frac{59}{60}.$$

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$$\int_\Gamma F \bullet dr = \int_0^1 F(r(t)) \bullet r'(t) dt = \int_0^1 (t^3 + 2t^5 + 2t^4) dt = \frac{59}{60}.$$

- Evaluate $\int_\Gamma (yx^2 dx + \sin(\pi y) dy)$, where Γ is the line segment from $(0, 2)$ to $(1, 4)$.

  We have $r(t) = (t,\ 2 + 2t), t \in [0, 1]$. Thus
$$\int_\Gamma (yx^2 dx + \sin(\pi y) dy) =$$
$$= \int_0^1 2\sin(\pi(2 + 2t)) dt + \int_0^1 (2 + 2t) t^2 dt = \frac{7}{6}$$

Oriented path

• A parametrization $r : [a, b] \to \mathbb{R}^n$ determines an orientation or a direction of the curve $\Gamma = r([a, b])$. Indeed, as t varies from a to b, $r(t)$ traverses the path from $r(a)$ to $r(b)$.

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• Let Γ be an oriented path. Denote the reverse orientation of Γ by $-\Gamma$. If $r : [a, b] \to \mathbb{R}^n$ is a parametrization of the oriented path Γ then $\rho : [a, b] \to \mathbb{R}^n$ given by $\rho(t) := r(a + b - t)$ is a parametrization of $-\Gamma$.

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• Let Γ be an oriented path. Then

$$\int_{-\Gamma} F \bullet dr = -\int_{\Gamma} F \bullet dr.$$

Work done

Definition: The work done by a force F on a particle traversing an oriented path Γ is the line integral

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Remark: The total work done by F on a particle traversing the path Γ and then reversing back to the initial point is

$$W = \int_{\Gamma} F \bullet dr + \int_{-\Gamma} F \bullet dr = \int_{\Gamma} F \bullet dr - \int_{\Gamma} F \bullet dr = 0.$$

Green's Theorem

Let $D \subset \mathbb{R}^2$ be a simply connected (no holes) region with positively oriented (counter clockwise direction) boundary ∂D. Let $F = (P, Q)$ be C^1 on D. Then

$$\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \oint_{\partial D} (P(x,y)dx + Q(x,y)dy) \\ &= \oint_{\partial D} F \bullet dr. \end{aligned}$$

Let C be a circle of radius a centered at the origin. Find $\oint_C F \bullet dr$ for $F = (-y, x)$ using Green's theorem.

$$\oint_C F \bullet dr = \iint_D 2dA = 2 \iint_D dA = 2\pi a^2.$$

Applications of Green's Theorem

- Evaluation of area

$$\text{Area}(D) = \iint_D dA = \frac{1}{2}\oint_{\partial D}(xdy - ydx).$$

Taking $F = (0, x)$ and $G = (y, 0)$, we have

$$\begin{aligned} \iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \oint_{\partial D} xdy \\ -\iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \oint_{\partial D} ydx \end{aligned}$$

Vector fields

A vector field in $\mathbb{R}^n$ is a function $F : \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each $X \in \mathbb{R}^n$ a vector $F(X)$. A vector field in $\mathbb{R}^n$ with domain $U \subset \mathbb{R}^n$ is called a vector field on U.

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Geometrically, a vector field F on U is interpreted as attaching a vector to each point of U. Thus, there is a subtle difference between a vector field in $\mathbb{R}^n$ and a function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

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When a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is viewed as a vector field, for each X the vector $F(X)$ is identified with the vector that starts at the point X with the magnitude and direction of $F(X)$.

Thus every vector field on $U \subset \mathbb{R}^n$ is uniquely determined by a function from $U \to \mathbb{R}^n$.

Examples of vector fields

- The gravitational force field describes the force of attraction of the earth on a mass m and is given by

$$F(X) = -\frac{mMG}{r^3}X,$$

where $X = (x, y, z)$, $r := \|X\|$. The vector field F points to the centre of the earth.

- The vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) := (-y, x)$ is a rotational vector field in $\mathbb{R}^2$ which rotates a vector in the anti-clockwise direction by an angle $\pi/2$.
- Let $r : [0, 1] \to \mathbb{R}^n$ be C^1 and $\Gamma := r([0, 1])$. Then $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}^n$ given by $F(r(t)) = r'(t)$ is a tangent vector field on Γ.

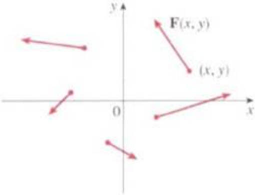

FIGURE 2
Vector field on $\mathbb{R}^2$

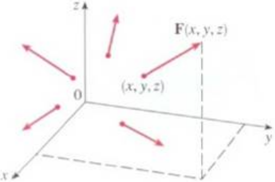

FIGURE 3
Vector field on $\mathbb{R}^3$

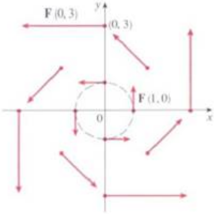

FIGURE 4
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

Figure: Examples of vector fields

Path Independence

Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field on D. We say that the vector field F has independence of path on D if for every pair of piecewise smooth, oriented curves C_1 and C_2 in D with a common initial point and a common final point, we have $\displaystyle\int_{C_1} F \bullet dr = \int_{C_2} F \bullet dr$.

Fact: Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field on D. The vector field F has independence of path on D if and only if the vector line integral $\displaystyle\int_{C} F \bullet dr = 0$ for every piecewise smooth, oriented, closed curve C in D.

Important Example

Let $F(x,y) = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2+y^2}$ for $(x,y) \in D^* = \mathbb{R}^2 \setminus \{(0,0)\}$.
Let $C : r(t) = (\cos t,\ \sin t)$ for $t \in [0,\ 2\pi]$.
Then $r'(t) = (-\sin t,\ \cos t)$ for $t \in [0,\ 2\pi]$.

$$\begin{aligned}
\int_C F \bullet dr &= \int_{t=0}^{2\pi} F(r(t)) \bullet r'(t)\, dt \\
&= \int_{t=0}^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\
&= \int_{t=0}^{2\pi} dt = 2\pi \neq 0
\end{aligned}$$

So, F is NOT independent of path in D^*.

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Gradient vector fields

If $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 scalar field then $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field in $\mathbb{R}^n$.

Gradient vector fields

If $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 scalar field then $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field in $\mathbb{R}^n$.

- A vector field F in $\mathbb{R}^n$ is said to be a gradient vector field or a conservative vector field if there is a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla f$. In such a case, f is called a scalar potential of the vector field F.

Path independence and gradient vector field

Let F be a continuous vector field on an open set $U \subset \mathbb{R}^n$. Consider the following statements:

1. F is a gradient vector field on U.
2. $\int_\Gamma F \bullet dr$ is path independent in U.
3. $\int_\Gamma F \bullet dr = 0$ for any closed path in U.

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3. $\int_\Gamma F \bullet dr = 0$ for any closed path in U.

We also know that (2) $\Leftrightarrow$ (3). The implication (3) $\Rightarrow$ (1) holds under a suitable assumption on U. This is called the first fundamental theorem for line integrals.

Path independence implies gradient field

Definition: A subset $U \subset \mathbb{R}^n$ is said to be path connected if for any two points X and Y in U there is a path $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma(a) = X, \gamma(b) = Y$ and $\gamma([a, b]) \subset U$.

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Theorem (1st Fundamental Thm for line integral):
Let $U \subset \mathbb{R}^n$ be open and path connected and F be a continuous vector field on U. Suppose $\int_\Gamma F \bullet dr$ is independent of Γ for any PC^1 path Γ in U. Then there exists a C^1 function $f : U \to \mathbb{R}$ such that $F = \nabla f$.

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Further, for $X_0 \in U$, define $f : U \to \mathbb{R}$ by

$$f(X) := \int_{X_0}^{X} F \bullet dr$$

where the integral is taken over any PC^1 path joining X_0 to X. Then f is well defined, f is C^1 and $F = \nabla f$.

2nd Fundamental Theorem for line integrals

If $f : [a, b] \to \mathbb{R}$ is C^1 then by FTI $\int_a^b f'(x)dx = f(b) - f(a)$.

Theorem: Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be C^1. Let $r : [a, b] \to \mathbb{R}^n$ be PC^1 such that $r([a, b]) \subset U$. Then

$$\int_\Gamma \nabla f \bullet dr = f(r(b)) - f(r(a)).$$

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$$\int_\Gamma \nabla f \bullet dr = f(r(b)) - f(r(a)).$$

Proof: We have

$$\begin{aligned}\int_\Gamma \nabla f \bullet dr &= \int_a^b \nabla f(r(t)) \bullet r'(t)dt \\ &= \int_a^b \frac{d}{dt} f(r(t))dt = f(r(b)) - f(r(a)).\end{aligned}$$

Thus, gradient field implies path independence.

Gradient vector fields and path independence

In summary, we have the following necessary and sufficient condition for a continuous vector field to be gradient vector field.

Let $U \subset \mathbb{R}^n$ be open and path connected and F be a continuous vector field on U. Then, F is a gradient vector field if and only if F has the path independence property in U.

We now find a necessary and sufficient condition for continuously differentiable vector field to be gradient vector field.

Necessary condition

Let F be a vector field on U with a scalar potential f, that is, $F = \nabla f$. Suppose $F = (F_1, \ldots, F_n)$.

Fact: If a C^1 vector field $F = (F_1, \ldots, F_n)$ on U is conservative then for all i and j

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

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$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

Proof: We have $F_i = \partial_i f \Rightarrow \partial_j F_i = \partial_j \partial_i f = \partial_i \partial_j f = \partial_i F_j$.

Example

Consider $F(x, y) := (3 + 2xy,\ x^2 - 3y^2) =: (P, Q)$. Then $Q_x = 2x = P_y$ so the necessary condition is satisfied.

We wish to find f such that $F = \nabla f$. If f exists then $f_x(x, y) = 3 + 2xy \Rightarrow f(x, y) = 3x + x^2y + h(y)$.

Thus $f_y(x, y) = x^2 + h'(y) = x^2 - 3y^2 \Rightarrow h'(y) = -3y^2$. Hence $h(y) = -y^3 + c$ for some constant c. Consequently,

$$f(x, y) = 3x + x^2y - y^3 + c \text{ and } F = \nabla f.$$

Example

Consider $F(x,y) := (\frac{-y}{x^2+y^2}, \ \frac{x}{x^2+y^2}) = (P,Q)$ for $(x,y) \neq (0,0)$. Then we have $Q_x = P_y$ so the necessary condition is satisfied.

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For the path $\Gamma : r(t) = (\cos t, \sin t), \ t \in [0, 2\pi]$, we have

$$\int_\Gamma F \bullet dr = \int_0^{2\pi} dt = 2\pi.$$

This shows that F is not conservative.

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This shows that F is not conservative.

Remark: The necessary condition $\partial_i F_j = \partial_j F_i$ is also sufficient for conservativeness of F when the domain of F is simply connected. This is a consequence of Green's theorem.

Necessary and sufficient condition for C^1 vector field

Let $F(x, y) := (P, Q)$ be C^1 defined in a simply connected domain U. Then F is a gradient field (conservative field) if and only if

$$Q_x = P_y.$$

Proof: We have alrady proved that $Q_x = P_y$ is a necessary condition. To prove that this condition is sufficient, we apply Green's theorem. Let C be a closed path (positively oriented) in U. Let D be the region bounded by C. Since U is simply connected, so $D \subseteq U$. By Green's theorem,

$$\oint_C F \bullet dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0.$$

Thus, F is conservative.

Surfaces

1. Locus of a point moving in space with 2 degrees of freedom.
2. Level curve of a scalar field $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$. For example, $x^2 + y^2 + z^2 = c, \quad z = x^2 + y^2$, etc.
3. Sometimes surfaces can be described by

   $$\{(x, y, z) : z = f(x, y),\ (x, y) \in D\}.$$

   This is called explicit representation.
4. The unit sphere is a union of two such explicit representations:

   $$\{(x, y, z = \sqrt{1 - x^2 - y^2}) : x^2 + y^2 \leq 1\}$$
   $$\cup\{(x, y, z = -\sqrt{1 - x^2 - y^2}) : x^2 + y^2 \leq 1\}$$

Parametric representation of a surface

A surface may also be described by

$$x = X(u, v), \quad y = Y(u, v), \quad z = Z(u, v),$$

where $u, v \in D$ and D is a connected subset of the uv-plane, for example, plane region like circle, rectangle, etc.

Definition: A continuous function $R : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is called a parametric surface in $\mathbb{R}^3$. The image $S := R(D)$ is called a geometric surface in $\mathbb{R}^3$.

If the surface has an explicit representation given by a continuous function $z = f(x, y), \ (x, y) \in D$, then

$$R(x, y) = x\,\hat{i} + y\,\hat{j} + f(x, y)\,\hat{k}$$

is a parametric representation.

Parametric representation of a sphere of radius a

If we take spherical coordinates, then

$$\begin{aligned} x &= X(\theta,\phi) = a\sin\phi\cos\theta, \\ y &= Y(\theta,\phi) = a\sin\phi\sin\theta, \\ z &= Z(\theta,\phi) = a\cos\phi, \end{aligned}$$

where $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$. This gives a parametric representation of the sphere:

$$R(\phi,\theta) = a\sin\phi\cos\theta\,\hat{i} + a\sin\phi\sin\theta\,\hat{j} + a\cos\phi\,\hat{k},$$

where $0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$.

Parametric representation of a cone

We find a parametrization of the cone

$$z = \sqrt{x^2 + y^2}, \ \ 0 \le z \le 1$$

Here cylindrical coordinates provide everything we need.

$$x(r, \theta) = r\cos\theta, \ y(r, \theta) = r\sin\theta, \ z = \sqrt{x^2 + y^2} = r.$$

Also $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.
So the required parametrization is

$$R(r, \theta) = r\cos\theta\ \hat{i} + r\sin\theta\ \hat{j} + r\ \hat{k}, \ 0 \le r \le 1, 0 \le \theta \le 2\pi.$$

Smooth parametric surface

Let $R : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametric surface and let $R(u, v) = (x(u, v), y(u, v), z(u, v))$. Then the partial derivatives of R, when exist, are given by

$$R_u = (x_u, y_u, z_u) \text{ and } R_v = (x_v, y_v, z_v).$$

The surface $S = R(D)$ is said to be smooth if R is C^1 and $R_u \times R_v \neq 0$ for $(u, v) \in D$.

Assumptions:

- D is connected
- R is injective except possibly on the boundary of D
- R is C^1 and $R_u \times R_v \neq 0$ for $(u, v) \in D$.

MA102: Multivariable Calculus

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Surface Integration: Scalar surface integrals

Let S be a surface parametrized by $R : D \to \mathbb{R}^3$ such that R is C^1. Let $f : S \to \mathbb{R}$ be bounded on S. Then the surface integral of f over S is given by

$$\iint_S f(x, y, z) d\sigma \quad := \quad \iint_D f(R(u, v)) \|R_u \times R_v\| \, dudv$$

whenever the double integral on the right exists.

Example: Evaluate the surface integral $\iint_S (x + y + z) d\sigma$ over the surface of the cylinder $x^2 + y^2 = 9, 0 \leq z \leq 4$.
Solution: Using the cylindrical coordinates, the surface can be represented as

$$R(\theta, z) = 3\cos\theta \ \hat{i} + 3\sin\theta \ \hat{j} + z \ \hat{k}$$

over the parameter domain $\{(\theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq z \leq 4\}$.

Surface Integration: Scalar surface integrals

Then $\|R_\theta \times R_z\| = \sqrt{9\cos^2\theta + 9\sin^2\theta} = 3$. The given integral is equal to

$$
\begin{aligned}
& \iint_S (x+y+z)d\sigma \\
= \;& \iint_S (3\cos\theta + 3\sin\theta + z)\|R_\theta \times R_z\| d\theta dz \\
= \;& 3\int_{z=0}^{4}\int_{\theta=0}^{2\pi} (3\cos\theta + 3\sin\theta + z)d\theta dz \\
= \;& 6\pi \int_0^4 zdz \\
= \;& 48\pi.
\end{aligned}
$$

Oriented surface

Let S be a surface parametrized by a C^1 function $R : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. Let $\hat{n}$ denote the unit normal to S. Then,

$$\hat{n} = \frac{R_u \times R_v}{\|R_u \times R_v\|}.$$

If $\hat{n}$ is a continuous function on D, then S together with $\hat{n}$ is called an oriented surface, that is, the pair $(S, \hat{n})$ is called an oriented surface.

Example: $R_1(\phi, \theta) = a \sin\phi \cos\theta \ \hat{i} + a \sin\phi \sin\theta \ \hat{j} + a \cos\phi \ \hat{k}$, where $0 \leq \phi \leq \pi/2, 0 \leq \theta \leq 2\pi$.

Then $\hat{n}_1 = \frac{\frac{\partial R_1}{\partial \phi} \times \frac{\partial R_1}{\partial \theta}}{\|\frac{\partial R_1}{\partial \phi} \times \frac{\partial R_1}{\partial \theta}\|} = \frac{1}{a} R_1(\phi, \theta)$ is the normal.

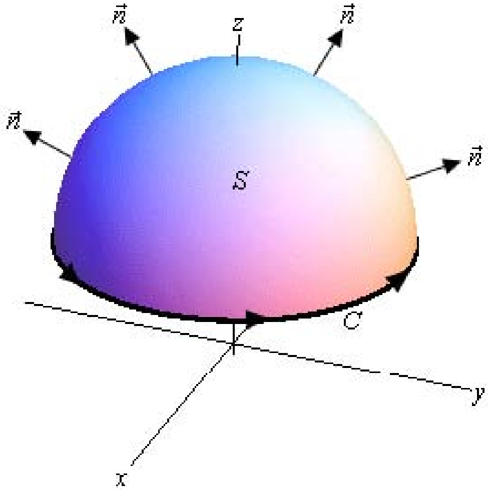

Figure : Orientation induced by $R_1(\phi, \theta)$

Surface integrals of vector fields

Let $(S, \hat{n})$ be an oriented surface in R^3 and let $F : S \to \mathbb{R}^3$ be a continuous vector field. Then $F \bullet \hat{n}$ is the normal component of F.

The surface integral of F (also called the flux integral) over the oriented surface $(S, \hat{n})$ is defined as

$$\iint_S F \bullet \hat{n} d\sigma.$$

If $S = R(D)$, where R is a smooth parametrization of S over the parameter domain D, then

$$\begin{aligned} \iint_S F \bullet \hat{n} d\sigma &= \iint_D F(R(u,v)) \bullet \frac{R_u \times R_v}{\|R_u \times R_v\|} \|R_u \times R_v\| dudv \\ &= \iint_D F(R(u,v)) \bullet (R_u \times R_v) dudv. \end{aligned}$$

Example

Let $F(x, y, z) := (z, y, x)$. Evaluate the flux integral $\iint_S F \bullet \hat{n} d\sigma$ over the unit sphere $S : x^2 + y^2 + z^2 = 1$.

We have
$R(u, v) = (\sin u \cos v, \sin u \sin v, \cos u), (u, v) \in [0, \pi] \times [0, 2\pi]$,

$$R_u \times R_v = (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u).$$

Thus

$$\begin{aligned} \iint_S F \bullet \hat{n} d\sigma &= \int_0^{2\pi} \int_0^{\pi} (2 \sin^2 u \cos u \cos v + \sin^3 u \sin^2 v) du dv \\ &= \frac{4\pi}{3}. \end{aligned}$$

One more example: Surface over the xz-plane

Find the outward flux of $F = yz\,\hat{\imath} + x\,\hat{\jmath} - z^2\,\hat{k}$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$.

Step 1: Writing the Parametric Equation of S

We parameterize S by the equation

$$\phi(x,\, z) = \left(x,\; x^2,\; z\right) \qquad \text{for } (x,\, z) \in D$$

where $D = \{(x, z) \in \mathbb{R}^2 \;:\; 0 \le x \le 1,\; 0 \le z \le 4\}$ in the xz-plane.

Step 2: Computing $\phi_x \times \phi_z$

$$\phi_x = (1,\, 2x,\, 0) \qquad \text{and} \qquad \phi_z = (0,\, 0,\, 1)\,.$$

$$\phi_x \times \phi_z = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\,\hat{\imath} - \hat{\jmath}$$

Example (cont.)

Step 3: Evaluation of the Vector Surface Integral

$$
\begin{aligned}
& \iint_S F \bullet \hat{n}\, d\sigma \\
= & \iint_D F(\phi(x,z)) \bullet (\phi_x \times \phi_z)\ dx\, dz \\
= & \int_{x=0}^{1} \int_{z=0}^{4} (2xyz - x)\ dz\, dx \\
= & \int_{x=0}^{1} \int_{z=0}^{4} (2x^3 z - x)\ dz\, dx \qquad (\text{By putting } \ y = x^2\ ) \\
= & \int_{x=0}^{1} (16x^3 - 4x)\ dx \\
= & \ 2.
\end{aligned}
$$

Curl of a vector field

$F(x, y, z) = M(x, y, z)\ \hat{\imath}\ +\ N(x, y, z)\ \hat{\jmath}\ +\ P(x, y, z)\ \hat{k}$ for $(x, y, z) \in D \subset \mathbb{R}^3$.

The curl of F is denoted by $\mathrm{curl}(F)$ and is defined by

$$\begin{aligned}
\mathrm{curl} F &= \nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x, y, z) & N(x, y, z) & P(x, y, z) \end{vmatrix} \\
&= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) \hat{\imath} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) \hat{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \hat{k}
\end{aligned}$$

Stoke's Theorem (3-D version of Green's Theorem)

Stoke's Theorem: Assume that S is a smooth parametric surface, say $S = R(D)$, where D is a region in the uv-plane bounded by a closed, simple, piecewise smooth curve Γ. Assume that R is C^2 and one-to-one on some open set containing $D \cup \Gamma$. Let C denote the image of Γ, that is, $C = R(\Gamma)$. Let $F(x, y, z) = M(x, y, z)\,\hat{\imath} + N(x, y, z)\,\hat{\jmath} + P(x, y, z)\,\hat{k}$ be a continuously differentiable vector field on S. Then

$$\oint_C F \bullet dr \;=\; \oint_C M\,dx + N\,dy + P\,dz \;=\; \iint_S (\mathrm{curl}(F) \bullet \hat{n})\;d\sigma,$$

where $\hat{n} = (R_u \times R_v)/\|R_u \times R_v\|$. The curve Γ is traversed in the positive (counterclockwise) direction and the curve C is traversed in the direction inherited from Γ through R.

Example

Verify Stoke's Theorem for the vector field $F(x, y, z) = 2z\,\hat{\imath} + 3x\,\hat{\jmath} + 5y\,\hat{k}$ taking S to be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$.

The boundary curve of S is the circle $C : x^2 + y^2 = 4$ in the xy-plane with counterclockwise direction.
$C : r(t) = 2\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath} + 0\,\hat{k}$ for $t \in [0,\ 2\pi]$.

$$\begin{aligned}
\oint_C F \bullet dr &= \int_C 2z\,dx + 3x\,dy + 5y\,dz \\
&= \int_{t=0}^{2\pi} (0 + (6\cos t)(2\cos t) + 0)\ dt = \int_{t=0}^{2\pi} 12\cos^2 t\ dt \\
&= 12\left[\frac{t}{2} + \frac{\sin(2t)}{4}\right]_{t=0}^{2\pi} = 12\pi\ .
\end{aligned}$$

Example (cont.)

$$\mathrm{curl}(F) = 5\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k} \ .$$

$$\hat{n} = \frac{2x\,\hat{\imath} + 2y\,\hat{\jmath} + \hat{k}}{\sqrt{1+4x^2+4y^2}}.$$

$$\begin{aligned}
\iint_S (\mathrm{curl}(F)\bullet\hat{n})\ d\sigma &= \iint_S \frac{(5\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k})\bullet(2x\,\hat{\imath} + 2y\,\hat{\jmath} + \hat{k})}{\sqrt{1+4x^2+4y^2}}\ d\sigma \\
&= \iint_S \frac{10x+4y+3}{\sqrt{1+4x^2+4y^2}}\ d\sigma \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (10r\cos\theta + 4r\sin\theta + 3)\ r\ dr\ d\theta \\
&= \int_{\theta=0}^{2\pi} \left(\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right)\ d\theta \ = \ 12\pi \ .
\end{aligned}$$

Thus,

$$\oint_C F\bullet dr = 12\pi = \iint_S (\mathrm{curl}(F)\bullet\hat{n})\ d\sigma \ .$$

Example

Find $\iint_S \mathrm{curl}(F) \bullet \hat{n} d\sigma$, where $F(x, y, x) = (y^2, xy, xz)$ and S is the upper hemisphere of the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: Using Stoke's theorem, we have

$$\begin{aligned}
& \iint_S (\mathrm{curl}(F) \bullet \hat{n}) \;\; d\sigma \\
= \;& \oint_C [y^2 dx + xydy + xzdz] \;\; (C := \{(x, y, 0) : x^2 + y^2 = 1\}) \\
= \;& \int_{\theta=0}^{2\pi} [\sin^2\theta(-\sin\theta) + \cos^2\theta \sin\theta] \; d\theta \\
= \;& 0.
\end{aligned}$$

Gauss's Divergence Theorem

Divergence Theorem: Let $V \subset \mathbb{R}^3$ be a solid region bounded by an oriented closed surface S, and let $\hat{n}$ be the unit outward normal to S. Let $F = (P, Q, R)$ be a C^1 vector field on any open subset of $\mathbb{R}^3$ containing $V \cup S$. Then

$$\iint_S F \bullet \hat{n} d\sigma = \iiint_V \operatorname{div}(F)\, dV,$$

where $\operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Gauss's Divergence Theorem

Example: Evaluate $\iint_S F \bullet \hat{n} d\sigma$ using Divergence Thm, where $F(x,y,z) = (x+y)\,\hat{\imath} + z^2\,\hat{\jmath} + x^2\,\hat{k}$ and S is the surface of $x^2+y^2+z^2=1,\ z \geq 0$, $\hat{n}$ being the outer normal.

Solution: By Divergence Theorem,

$$\iiint_D \operatorname{div} F \ dV = \iint_S F \bullet \hat{n} d\sigma + \iint_{S_1} F \bullet \hat{n}_1 d\sigma,$$

where $\hat{n}$ is the outer normal to S and $\hat{n}_1$ is the outer normal to $S_1 = \{(x,y,0) : x^2+y^2 \leq 1\}$ and $D = \{(x,y,z) : x^2+y^2+z^2 \leq 1, z \geq 0\}$.
We have $\operatorname{div} F = 1$ and hence $\iiint_D \operatorname{div} F \ dV = \frac{2}{3}\pi$. Now,

$$\iint_{S_1} F \bullet \hat{n}_1 d\sigma = \iint_{S_1} [(x+y)\,\hat{\imath} + z^2\,\hat{\jmath} + x^2\,\hat{k}\,] \bullet (-\,\hat{k}\,) d\sigma = -\iint_{S_1} x^2 d\sigma = -\pi/4.$$

Hence,

$$\iint_S F \bullet \hat{n} d\sigma = \iiint_D \operatorname{div} F \ dV - \iint_{S_1} F \bullet \hat{n}_1 d\sigma = \frac{2\pi}{3} + \frac{\pi}{4} = \frac{11}{12}\pi.$$