Solution of Constant Coefficients ODE

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Homogeneous linear equations with constant coefficients

Aim: To find a basis for $Ker(L)$. That is, to find a set of fundamental solution to the homogeneous equation $L(y) = 0$, where

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

and $a_n \neq 0$, $a_{n-1}, \ldots, a_0$ are real constants.
For $y = e^{rx}$, we find

$$\begin{aligned} L(e^{rx}) &= a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_0 e^{rx} \\ &= e^{rx}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0) = e^{rx} P(r), \end{aligned}$$

where $P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$.
Thus $L(e^{rx}) = 0$ provided r is a root of the auxiliary equation

$$P(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0.$$

Case I (Distinct real roots): Let $r_1, \ldots, r_n$ be real and distinct roots. The n solutions are given by

$$y_1(x) = e^{r_1 x},\ y_2(x) = e^{r_2 x}, \ldots, y_n(x) = e^{r_n x}.$$

We need to show

$$c_1 e^{r_1 x} + \cdots + c_n e^{r_n x} = 0 \Longrightarrow c_1 = c_2 = \cdots = c_n = 0.$$

$P(r)$ can be factored as

$$P(r) = a_n(r - r_1)(r - r_2)\cdots(r - r_n).$$

Writing the operator L as

$$L = P(D) = a_n(D - r_1)\cdots(D - r_n).$$

Now, construct the polynomial $P_k(r)$ by deleting the factor $(r - r_k)$ from $P(r)$. Then

$$L_k := P_k(D) = a_n(D - r_1)\cdots(D - r_{k-1})(D - r_{k+1})\cdots(D - r_n).$$

By linearity

$$L_k(\sum_{i=1}^{n} c_i e^{r_i x}) = L_k(0) \Rightarrow c_1 L_k(e^{r_1 x}) + \cdots + c_n L_k(e^{r_n x}) = 0.$$

Since $L_k = P_k(D)$, we find that $L_k(e^{rx}) = e^{rx} P_k(r)$ for all r. Thus

$$\sum_{i=1}^{n} c_i e^{r_i x} P_k(r_i) = 0 \Longrightarrow c_k e^{r_k x} P_k(r_k) = 0,$$

as $P_k(r_i) = 0$ for $i \neq k$. Since r_k is not a root of $P_k(r)$, then $P_k(r_k) \neq 0$. This yields $c_k = 0$. As k is arbitrary, we have

$$c_1 = c_2 = \cdots = c_n = 0.$$

Theorem: If $P(r) = 0$ has n distinct roots $r_1, r_2, \ldots, r_n$. Then the general solution of $L(y) = 0$ is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x},$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Example: Consider $y'' - 3y' + 2y = 0$. The auxiliary equation $P(r) = r^2 - 3r + 2 = 0$ has two roots $r_1 = 1,\ \ r_2 = 2$. The general solution is $y(x) = C_1 e^x + C_2 e^{2x}$..

Case II (Repeated roots): If r_1 is a root of multiplicity m. Then

$$P(r) = (r - r_1)^m \tilde{P}(r),$$

where $\tilde{P}(r) = a_n(r - r_{m+1}) \cdots (r - r_n)$ and $\tilde{P}(r_1) \neq 0$. Now

$$L(e^{rx}) = e^{rx}(r - r_1)^m \tilde{P}(r)$$

Setting $r = r_1$, we see that $e^{r_1 x}$ is a solution. To find other solutions, we note that $\frac{\partial^k}{\partial r^k} L(e^{rx}) = \frac{\partial^k}{\partial r^k}[e^{rx}(r - r_1)^m \tilde{P}(r)]$. Now,

$$\frac{\partial^k}{\partial r^k} L(e^{rx})|_{r=r_1} = 0 \quad \text{if } k \leq m - 1.$$

$$\implies \ L\left[\frac{\partial^k}{\partial r^k}(e^{rx})|_{r=r_1}\right] = 0.$$

Thus,

$$\frac{\partial^k}{\partial r^k}(e^{rx})|_{r=r_1} = x^k e^{r_1 x}$$

will be a solution to $L(y) = 0$ for $k = 0, 1, \ldots, m-1$.
So, m distinct solutions are

$$e^{r_1 x},\ xe^{r_1 x},\ \cdots,\ x^{m-1}e^{r_1 x}.$$

Theorem: If $P(r) = 0$ has the real root r_1 occurring m times and the remaining roots $r_{m+1}, r_{m+2}, \ldots, r_n$ are distinct, then the general solution of $L(y) = 0$ is

$$\begin{aligned} y(x) &= (C_1 + C_2 x + C_3 x^2 + \cdots + C_m x^{m-1})e^{r_1 x} \\ &\quad + C_{m+1}e^{r_{m+1}x} + \cdots + C_n e^{r_n x}, \end{aligned}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Example: Consider $y^{(4)} - 8y'' + 16y = 0$. In this case, $r_1 = r_2 = 2$ and $r_3 = r_4 = -2$. The general solution is

$$y = (C_1 + C_2 x)e^{2x} + (C_3 + C_4 x)e^{-2x}.$$

Case III (Complex roots): If $\alpha + i\beta$ is a non-repeated complex root of $P(r) = 0$ so is its complex conjugate. Then, both

$$e^{(\alpha+i\beta)x} \quad \text{and} \quad e^{(\alpha-i\beta)x}$$

are solution to $L(y) = 0$. Then, the corresponding part of the general solution is of the form

$$e^{\alpha x}(C_1\cos(\beta x) + C_2\sin(\beta x)).$$

Theorem: If $P(r) = 0$ has non-repeated complex roots $\alpha + i\beta$ and $\alpha - i\beta$, the corresponding part of the general solution is

$$e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right).$$

If $\alpha + i\beta$ and $\alpha - i\beta$ are each repeated roots of multiplicity m, then the corresponding part of the general solution is

$$e^{\alpha x}\Big[\left(C_1 + C_2 x + C_3 x^2 + \cdots + C_m x^{m-1}\right)\cos(\beta x) \\ +\left(C_{m+1} + C_{m+2} x + \cdots + C_{2m} x^{m-1}\right)\sin(\beta x)\Big],$$

where $C_1, C_2, \ldots, C_{2m}$ are arbitrary constants.

Example: Consider $y^{(4)} - 2y''' + 2y'' - 2y' + y = 0$. Here, $r_1 = r_2 = 1$, $r_3 = i$ and $r_4 = -i$. The general solution is

$$y = (C_1 + C_2 x)e^x + (C_3 \cos x + C_4 \sin x).$$

Particular solution of constant coefficients ODE

Method of undetermined coefficients: A simple procedure for finding **a particular solution**(y_p) to a non-homogeneous equation $L(y) = g$, when L **is a linear differential operator with constant coefficients** and when $g(x)$ is of special type:

That is, when $g(x)$ is either

- a polynomial in x,
- an exponential function $e^{\alpha x}$,
- trigonometric functions $\sin(\beta x), \cos(\beta x)$

or finite sums and products of these functions.

Case I. For finding y_p to the equation $L(y) = p_n(x)$, where $p_n(x)$ is a polynomial of degree n. Try a solution of the form

$$y_p(x) = A_n x^n + \cdots + A_1 x + A_0$$

and match the coefficients of $L(y_p)$ with those of $p_n(x)$:

$$L(y_p) = p_n(x).$$

Remark: This procedure yields $n+1$ linear equations in $n+1$ unknowns $A_0, \ldots, A_n$.

Example: Find y_p to $L(y)(x) := y'' + 3y' + 2y = 3x + 1$.

Try the form $y_p(x) = Ax + B$ and attempt to match up $L(y_p)$ with $3x + 1$. Since

$$L(y_p) = 2Ax + (3A + 2B),$$

equating

$$2Ax + (3A + 2B) = 3x + 1 \implies A = 3/2 \text{ and } B = -7/4.$$

Thus, $y_p(x) = \frac{3}{2}x - \frac{7}{4}$.

Case II: The method of undetermined coefficients will also work for equations of the form

$$L(y) = ae^{\alpha x},$$

where a and α are given constants. Try y_p of the form

$$y_p(x) = Ae^{\alpha x}$$

and solve $L(y_p)(x) = ae^{\alpha x}$ for the unknown coefficients A.

Example: Find y_p to $L(y)(x) := y'' + 3y' + 2y = e^{3x}$.
Seek $y_p(x) = Ae^{3x}$. Then

$$L(y_p) = 9Ae^{3x} + 3(3Ae^{3x}) + 2(Ae^{3x}) = 20Ae^{3x}.$$

Now, $L(y_p) = e^{3x} \implies 20Ae^{3x} = e^{3x} \implies A = 1/20$.
Thus, $y_p(x) = (1/20)e^{3x}$.

Case III: For an equation of the form

$$L(y) = a\cos\beta x + b\sin\beta x,$$

try y_p of the form

$$y_p(x) = A\cos\beta x + B\sin\beta x$$

and solve $L(y_p) = a\cos\beta x + b\sin\beta x$ for the unknowns A and B.

Example: Find y_p to $L(y) := y'' - y' - y = \sin x$.
Seek $y_p(x)$ of the form $y_p(x) = A\cos x + B\sin x$. Then

$$L(y_p) = \sin x \implies A = 1/5, \quad B = -2/5.$$

Thus, $y_p(x) = \frac{1}{5}\cos x - \frac{2}{5}\sin x$.

Example: Find y_p to $L(y) := y'' - y' - 12y = e^{4x}$.

Note that $y_h(x) = c_1 e^{4x} + c_2 e^{-3x}$. Try finding y_p with the guess $y_p(x) = Ae^{4x}$ as before. Since e^{4x} is a solution to the corresponding homogeneous equation $L(y) = 0$, we replace this choice of y_p by $y_p(x) = Axe^{4x}$. Since $L(xe^{4x}) \neq 0$, there exists a particular solution of the form

$$y_p(x) = Axe^{4x}.$$

Remark: If $L(y_p) = 0$ then replace $y_p(x)$ by $xy_p(x)$. If $L(xy_p) = 0$ the replace xy_p by $x^2 y_p$ and so on. Thus, employing $x^s y_p$, where s is the smallest nonnegative integer such that $L(x^s y_p) \neq 0$.

Form of y_p:

- $g(x) = p_n(x) = a_n x^n + \cdots + a_1 x + a_0$,
  $y_p(x) = x^s P_n(x) = x^s\{A_n x^n + \ldots + A_1 x + A_0\}$

- $g(x) = ae^{\alpha x}$, $y_p(x) = x^s A e^{\alpha x}$

- $g(x) = a\cos\beta x + b\sin\beta x$,
  $y_p(x) = x^s\{A\cos\beta x + B\sin\beta x\}$

- $g(x) = p_n(x)e^{\alpha x}$, $y_p(x) = x^s P_n(x) e^{\alpha x}$

- $g(x) = p_n(x)\cos\beta x + q_m(x)\sin\beta x$,
  where $q_m(x) = b_m x^m + \cdots + b_1 x + b_0$.
  $y_p(x) = x^s\{P_N(x)\cos\beta x + Q_N(x)\sin\beta x\}$,
  where $Q_N(x) = B_N x^N + \cdots + B_1 x + B_0$ and
  $N = \max(n, m)$

- $g(x) = ae^{\alpha x}\cos\beta x + be^{\alpha x}\sin\beta x$,
  $y_p(x) = x^s\{Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x\}$
- $g(x) = p_n(x)e^{\alpha x}\cos + q_m(x)e^{\alpha x}\sin\beta x$,
  $y_p(x) = x^s e^{\alpha x}\{P_N(x)\cos\beta x + Q_N(x)\sin\beta x\}$, where
  $N = \max(n, m)$.

Note:

1. The nonnegative integer s is chosen to be the smallest integer so that no term in y_p is a solution to $L(y) = 0$.

2. $P_n(x)$ must include all its terms even if $p_n(x)$ has some terms that are zero.

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