Surface Integrals
Stokes' Theorem, Divergence Theorem

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Parametric surface

Definition: A continuous function $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is called a parametric surface in $\mathbb{R}^3$. The image $S := \Phi(D)$ is called a geometric surface surface in $\mathbb{R}^3$.

• Let $f : D \to \mathbb{R}$ be continuous. Then $\mathrm{Graph}(f) \subset \mathbb{R}^3$ is a surface parametrized by $\Phi : D \to \mathbb{R}^3$ given by

$$\Phi(u, v) := (u, v, f(u, v)).$$

• The sphere $S : x^2 + y^2 + z^2 = r^2$ is parametrized by $\Phi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ given by

$$\Phi(u, v) := (r \sin u \cos v, r \sin u \sin v, r \cos u).$$

Smooth parametric surface

Let $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametric surface and let $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$. Then the partial derivatives of Φ, when exist, are given by

$$\Phi_u = (x_u, y_u, z_u) \text{ and } \Phi_v = (x_v, y_v, z_v).$$

The parametric surface $S = \Phi(D)$ is said to be smooth if Φ is C^1 and $\Phi_u \times \Phi_v \neq 0$ for $(u, v) \in D$.

Assumptions:

- D is connected
- Φ is injective except possibly on the boundary ∂D
- Φ is C^1 and $\Phi_u \times \Phi_v \neq 0$ for $(u, v) \in D$.

Surface area and surface area differential

The surface area differential is given by

$$dS = \|\Phi_u \times \Phi_v\| dudv.$$

Set $E := \|\Phi_u\|^2$, $G := \|\Phi_v\|^2$ and $F := \Phi_u \bullet \Phi_v$. Then the surface area of S is given by

$$\begin{aligned} \mathrm{Area}(S) &= \mathrm{Area}(\Phi(D)) = \iint_D \|\Phi_u \times \Phi_v\| dudv \\ &= \iint_D \sqrt{EG - F^2}\, dudv. \end{aligned}$$

- $\mathrm{Area}(S) = \iint_D \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2}\ dudv$

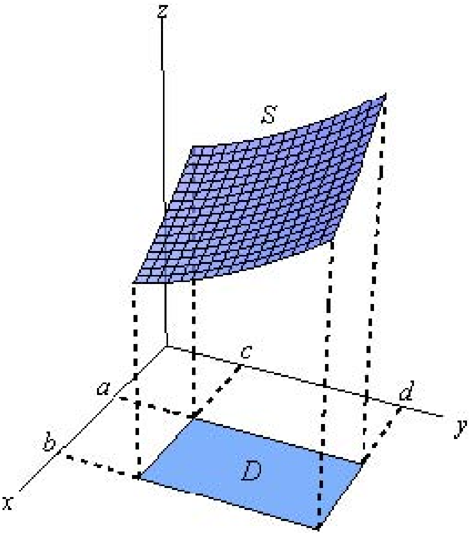

Figure: Surface element

Examples

- Consider the cylinder S parametrized by

$$\Phi(u, v) := (r\cos u, r\sin u, v), (u, v) \in [0, 2\pi] \times [0, h].$$

$$\mathrm{Area}(S) = \int_0^h \int_0^{2\pi} \|\Phi_u \times \Phi_v\| dudv = \int_0^h \int_0^{2\pi} rdudv = 2\pi rh.$$

- For the sphere S given by

$$\Phi(u, v) := (r\sin u\cos v, r\sin u\sin v, r\cos u)$$

$$\begin{aligned}\mathrm{Area}(S) &= \int_0^{\pi} \int_0^{2\pi} \|\Phi_u \times \Phi_v\| dudv \\ &= \int_0^{\pi} \int_0^{2\pi} r^2 \sin u \, dudv = 4\pi r^2.\end{aligned}$$

Surface integrals of scalar fields

Let S be a surface paramatrized by $\Phi : D \to \mathbb{R}^3$ and let $f : S \to \mathbb{R}$ be continuous. Then the surface integral of f over S is given by

$$\begin{aligned}\iint_S f(x,y,z)dS &:= \iint_D f(\Phi(u,v))\|\Phi_u \times \Phi_v\|\, dudv \\ &= \lim_{\mu(P)\to 0} \sum_{i,j} f(c_{ij})\Delta S_{ij}.\end{aligned}$$

Also

$$\iint_S fdS = \iint_D f(\phi(u,v))\sqrt{EG - F^2}\, dudv.$$

Example

Evaluate $\iint_S x^2 dS$ over the sphere $S : x^2 + y^2 + z^2 = 1$.

We have $\|\Phi_u \times \Phi_v\| = \sin u$ and

$$\begin{aligned}\iint_S x^2 dS &= \int_0^{2\pi}\int_0^{\pi} \sin^3 u \cos^2 v \, dudv \\ &= \int_0^{\pi} \sin^3 u du \int_0^{2\pi} \cos^2 v dv = 4\pi/3.\end{aligned}$$

Oriented surface

Informal: A surface $S \subset \mathbb{R}^3$ is orientable if it has two sides.

Formal: A surface $S \subset \mathbb{R}^3$ is orientable if there exists a continuous vector field $\mathbf{n} : S \to \mathbb{R}^3$ such that $\mathbf{n}(x, y, z)$ is a unit normal to S at (x, y, z).

A surface $S \subset \mathbb{R}^3$ together with a continuous normal vector field $\mathbf{n}$ on it is called an oriented surface, that is, the pair $(S, \mathbf{n})$ is called an oriented surface.

Let $\Phi : D \to \mathbb{R}^3$ be a paramertization of an oriented surface $(S, \mathbf{n})$. Then Φ is called a consistent parametrization if

$$\mathbf{n}(x, y, z) = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|}.$$

Positively oriented surface

A closed oriented surface $(S, \mathbf{n})$ is called positively oriented if the unit normal vector field $\mathbf{n}$ on S points outward.

For the unit sphere $\Phi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$,

$$\mathbf{n} = \frac{\Phi_u \times \Phi_v}{\|\Phi_u \times \Phi_v\|} = \Phi(u, v)$$

which points outward. Thus $(S, \mathbf{n})$ is positively oriented.

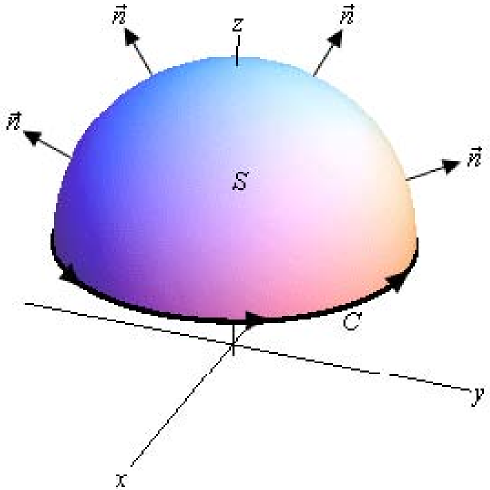

Figure: Oriented surface

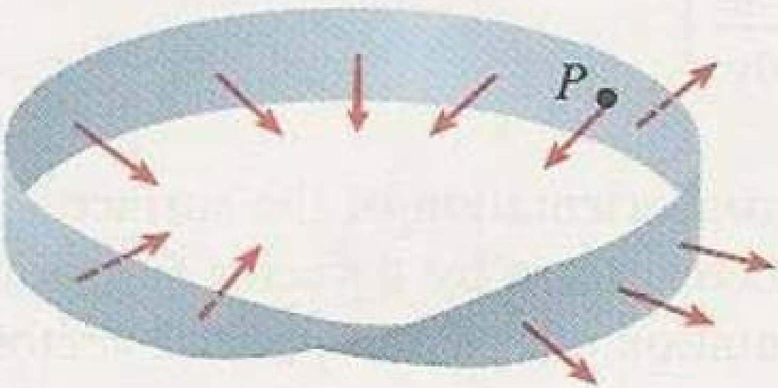

FIGURE 5
A Möbius strip

Surface integrals of vector fields

Let $(S, \mathbf{n})$ be an oriented surface in R^3 and let $F : S \to \mathbb{R}^3$ be a continuous vector field. Then $F \bullet \mathbf{n}$ is the normal component of F.

Interpretation: Flux of F through the oriented surface S per unit surface area $= F \bullet \mathbf{n}$.

The surface integral of F over the oriented surface $(S, \mathbf{n})$ is defined by

$$\iint_S F \bullet d\mathbf{S} := \iint_S (F \bullet \mathbf{n}) dS.$$

- $\iint_S F \bullet d\mathbf{S}$ is the flux of F through the oriented surface S.

Surface integrals of vector fields

Let $\Phi : D \to \mathbb{R}^3$ be a consistent parametrization of the oriented surface $(S, \mathbf{n})$. Then

$$\iint_S F \bullet d\mathbf{S} := \iint_S F \bullet \mathbf{n} dS = \iint_D F \bullet (\Phi_u \times \Phi_v) dudv.$$

Example

Let $F(x, y, z) := (z, y, x)$. Evaluate $\iint_S F \bullet d\mathbf{S}$ over the unit sphere $S : x^2 + y^2 + z^2 = 1$.

We have

$\Phi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u), (u, v) \in [0, \pi] \times [0, 2\pi]$,

$$\Phi_u \times \Phi_v = (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u).$$

Thus

$$\begin{aligned} \iint_S F \bullet d\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi} (2 \sin^2 u \cos u \cos v + \sin^3 u \sin^2 v) du dv \\ &= \frac{4\pi}{3}. \end{aligned}$$

Stokes' Theorem

Stokes' Theorem: Let $(S, \mathbf{n})$ be a (piecewise) smooth oriented surface with (piecewise) smooth positively oriented boundary ∂S. Let $F : S \to \mathbb{R}^3$ be C^1 vector field. Then

$$\iint_S \mathrm{curl}(F) \bullet \mathbf{n} dS = \int_{\partial S} F \bullet T ds = \int_{\partial S} F \bullet d\mathbf{r},$$

where T is the tangent field on ∂S.

Divergence Theorem: Let $V \subset \mathbb{R}^3$ be a solid region with positively oriented boundary surface $(S, \mathbf{n})$. Let $F : V \to \mathbb{R}^3$ be C^1. Then

$$\iint_S F \bullet \mathbf{n} dS = \iiint_V \mathrm{div}(F)\, dV.$$

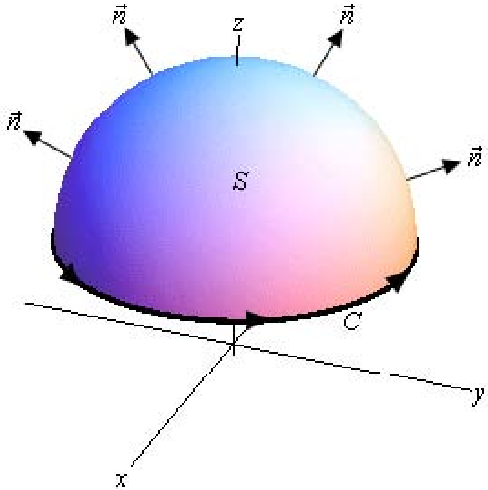

Figure: Oriented surface

Green's theorem in vector form

Let $D \subset \mathbb{R}^2$ be a simply connected (no holes) region with positively oriented boundary ∂D. Let $F = (P, Q)$ be C^1 vector field on D. Identifying $F = (P, Q)$ with $F = (P, Q, 0)$ we have

$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint_D \mathrm{curl}(F) \bullet \mathbf{k}\, dA \\ &= \oint_{\partial D} F \bullet \mathbf{dr}.\end{aligned}$$

Examples:

• Let $F = (ye^z, xe^z, xye^z)$. Then for any oriented surface S with positively oriented boundary ∂S

$$\int_{\partial S} F \bullet d\mathbf{r} = \iint_S \mathrm{curl}(F) \bullet d\mathbf{S} = 0$$

because $\mathrm{curl}(F) = 0$.

• If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is such that $\mathrm{curl} F = 0$ then F is a gradient vector field. Indeed, by Stoke's theorem

$$\oint_{\partial S} F \bullet d\mathbf{r} = \iint_S \mathrm{curl}(F) \bullet d\mathbf{S} = 0.$$

Thus the line integral is path independent and hence F is conservative.

Example:

Let $F = (2x, y^2, z^2)$ and $S : x^2 + y^2 + z^2 = 1$. Evaluate the surface integral

$$\iint_S F \bullet d\mathbf{S}.$$

By Divergence theorem

$$\begin{aligned}
\iint_S F \bullet d\mathbf{S} &= \iiint_V \mathrm{div}(F) dV \\
&= 2 \iiint_V (1 + y + z) dV \\
&= 2 \iiint_V dV = \frac{8\pi}{3}.
\end{aligned}$$

*** End ***