Vector fields, Curl and Divergence

Department of Mathematics
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> When a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is viewed as a vector field, for each $\mathbf{x}$ the vector $F(\mathbf{x})$ is identified with the vector that starts at the point $\mathbf{x}$ and points to $F(\mathbf{x})$, i.e., $F(\mathbf{x})$ is identified with the vector that is obtained by translating $F(\mathbf{x})$ to the point $\mathbf{x}$.

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Thus every vector field on $U \subset \mathbb{R}^n$ is uniquely determined by a function from $U \to \mathbb{R}^n$.

Examples of vector fields

- The gravitational force field describes the force of attraction of the earth on a mass m and is given by

$$\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r},$$

where $\mathbf{r} := (x, y, z)$ and $r := \|\mathbf{r}\|$. The vector field F points to the centre of the earth.

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- The vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) := (-y, x)$ is a rotational vector field in $\mathbb{R}^2$ which rotates a vector in the anti-clockwise direction by an angle $\pi/2$.

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- Let $\mathbf{x} : [0, 1] \to \mathbb{R}^n$ be C^1 and $\Gamma := \mathbf{x}([0, 1])$. Then $F : \Gamma \subset \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\mathbf{x}(t)) = \mathbf{x}'(t)$ is a tangent vector field on Γ.

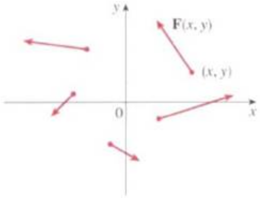

FIGURE 2
Vector field on $\mathbb{R}^2$

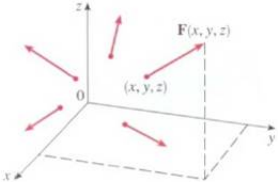

FIGURE 3
Vector field on $\mathbb{R}^3$

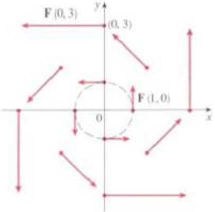

FIGURE 4
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

Figure: Examples of vector fields

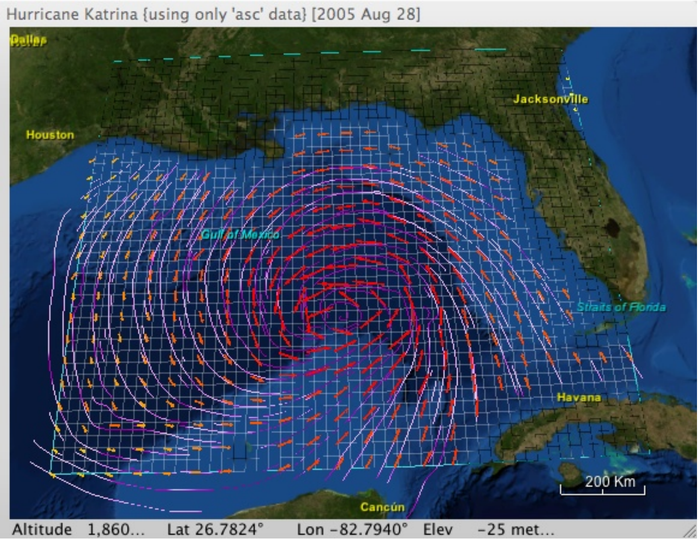

Figure: Vector field representing Hurricane Katrina

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- Let F be a vector field in $\mathbb{R}^n$. Then $F := (f_1, \ldots, f_n)$ for some scalar fields $f_1, \ldots, f_n$ on $\mathbb{R}^n$. We say that F is a C^k vector field if $f_1, \ldots, f_n$ are C^k functions.

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All vector fields are assumed to be C^1 unless otherwise noted.

Gradient vector fields

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Gradient vector fields

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- A vector field F in $\mathbb{R}^n$ is said to be a gradient vector field or a conservative vector field if there is a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla f$. In such a case, f is called a scalar potential of the vector field F.

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Example: The vector field $F(x, y) := (y, -x)$ is not a gradient vector field. Indeed, if $F = \nabla f$ then $f_x = y$ and $f_y = -x$.

Consequently, $f_{xy} = 1$ and $f_{yx} = -1$. Hence F is NOT a C^1 vector field, which is a contradiction.

Integral curves for vector fields

Definition: Let F be a vector field in $\mathbb{R}^n$. Then a C^1 curve $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is said to be an integral curve for the vector field F if $F(\mathbf{x}(t)) = \mathbf{x}'(t)$ for $t \in [a, b]$.

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An integral curve may be viewed as a solution of a system of ODE. Indeed, for $n = 3$ and $F = (P, Q, R)$, we have

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)), \\ y'(t) &= Q(x(t), y(t), z(t)), \\ z'(t) &= R(x(t), y(t), z(t)). \end{aligned}$$

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Note that $\mathbf{r}(t) = (\cos t, \sin t)$ is an integral curve for F. The other integral curves are also circles and are of the form

$$\gamma(t) := (r\cos(t - t_0),\ r\sin(t - t_0)).$$

This follows from the system of ODE $x' = -y$ and $y' = x$ which gives $x'' + x = 0$.

Divergence of vector fields

Definition: Let $F = (f_1, \ldots, f_n)$ be a vector field in $\mathbb{R}^n$. Then the divergence of F is a scalar field on $\mathbb{R}^n$ given by

$$\mathrm{div}F = \partial_1 f_1 + \cdots + \partial_n f_n = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}.$$

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Then applying ∇ to a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ we obtain the gradient vector field

$$\nabla f = (\partial_1 f, \ldots, \partial_n f).$$

Divergence of vector fields

Taking dot product of ∇ with a vector field $F = (f_1, \ldots, f_n)$ we obtain the divergence

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The divergence of $F = (x^2 y, z, xyz)$ is given by

$$\mathrm{div} F = 2xy + 0 + xy = 3xy.$$

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- Next, consider the vector field $F(x, y) := (x, -y)$. Then

  $$\mathrm{div} F = 0$$

  so that neither expansion nor compression takes place.

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- $\det([\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}]) \geq 0$, where $[\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}]$ is the matrix whose columns are $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$.

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It is customary to denote the standard basis in $\mathbb{R}^3$ by

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If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2 v_3 - v_2 u_3)\mathbf{i} + (v_1 u_3 - u_1 v_3)\mathbf{j} + (u_1 v_2 - v_2 u_1)\mathbf{k} \\ &= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}$$

the last equality is only symbolic.

Curl of vector fields in $\mathbb{R}^3$

The curl of a vector field $F = (f_1, f_2, f_3)$ in $\mathbb{R}^3$ is a vector field in $\mathbb{R}^3$ and is given by

$$\mathrm{curl} F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

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Curl of a vector field represents rotational motion when, for example, the vector field represents flow of a fluid.

Irrotational vector field

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Observation: If $\mathrm{curl} F \neq 0$ then F is not a conservative (gradient) vector field.

Examples

- Let $F(x, y, z) := (xy, -\sin z, 1)$. Then
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- Let $F(x, y, z) := (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$. Then $\mathrm{curl} F = 0$ and hence F is irrotational. However, F is NOT a conservative vector field, that is, $F \neq \nabla f$ for some scalar potential f.

Scalar curl

Let $F = (P, Q)$ be a vector field in $\mathbb{R}^2$. Then identifying $\mathbb{R}^2$ with the x-y plane in $\mathbb{R}^3$, F can be identified with the vector field $F = P\mathbf{i} + Q\mathbf{j}$ in $\mathbb{R}^3$. Then we have

$$\mathrm{curl} F = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

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The scalar curl of $F(x, y) := (-y^2, x)$ is given by

$$\partial_x(x) - \partial_y(-y^2) = 1 + 2y.$$

Divergence of curl

Fact: Let F be a C^2 vector field in $\mathbb{R}^3$. Then

$$\mathrm{div}(\mathrm{curl} F) = \nabla \bullet \nabla \times F = 0.$$

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A vector field F in $\mathbb{R}^3$ is said to have vector potential if there exists a vector field G in $\mathbb{R}^3$ such that $F = \mathrm{curl} G$.

Observation: If F has a vector potential then $\mathrm{div}(F) = 0$.

Laplace operator

Example: If $F(x, y, z) := (x, y, z)$ then $\mathrm{div} F = 3 \neq 0$ so F does not have a vector potential.

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For a C^2 function $u : \mathbb{R}^3 \to \mathbb{R}$ the partial differential equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is called Laplace equation.

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*** End ***