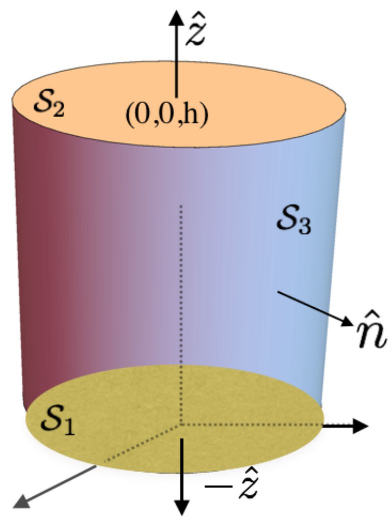
PH 102 : Electromagnetism **Department of Physics**
Tutorial 2 **Indian Institute of Technology Guwahati**

Please solve the star ($\star$) marked problems first and discuss the rest if time permits.

1. $\star$ Calculate the flux of the vector $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$ over the surface of a right circular cylinder of radius R bounded by the surfaces $z = 0$ and $z = h$ with the centre of the base of the cylinder situated at origin. Calculate it directly as well as by use of the divergence theorem.

Solution:

Let the base of the cylinder be at $z = 0$ and the top at $z = h$. The origin is at the centre of the base $(x, y, z) = (0, 0, 0)$. The cylinder has three surfaces. For the bottom surface ($\mathcal{S}_1$), located at $z = 0$, the direction of the normal is along $-\hat{z}$. The surface integral for this surface is $\int_{\mathcal{S}_1} \vec{F}.d\vec{a} = \int_{\mathcal{S}_1} F_z\ (-\hat{z})dxdy = \int_{\mathcal{S}_1}(-z)dxdy = 0$.
For the top surface ($\mathcal{S}_2$), located at $z = h$, the normal is along $\hat{z}$. The surface integral is $\int_{\mathcal{S}_2} \vec{F}.d\vec{a} = \int_{\mathcal{S}_2} F_z\ (\hat{z})dxdy = \int_{\mathcal{S}_2}(z)dxdy = h\pi R^2$. For the curved surface $\mathcal{S}_3$, the direction of the normal is radially outward. Hence the unit normal to $\mathcal{S}_3$ is given by $\hat{n} = \frac{\vec{\nabla}(x^2+y^2)}{|\vec{\nabla}(x^2+y^2)|} = \frac{x\hat{x}+y\hat{y}}{R}$.



Now for the cylindrical surface ($\mathcal{S}_3$), we can project the elementary area either on $y - z$ plane or $x - z$ plane. Choosing the projection on the $x - z$ plane, the surface integral is (please discuss all the steps in details in the tutorial, you may take a look at the lecture slides in moodle)

$$\begin{aligned}
\int_{\mathcal{S}_3} \vec{F}.\hat{n} da &= \int_{\mathcal{S}_3} \vec{F}.\hat{n} \frac{dxdz}{|\hat{n}.\hat{y}|} \\
&= \int_{z=0}^{h} \int_{x=-R}^{R} (x\hat{x} + y\hat{y} + z\hat{z}).\frac{x\hat{x} + y\hat{y}}{R} \frac{dxdz}{|\frac{x\hat{x}+y\hat{y}}{R}.\hat{y}|} \\
&= \frac{1}{R} \int_{z=0}^{h} \int_{x=-R}^{R} (x^2 + y^2) \frac{dxdz}{|\frac{y}{R}|} = \int_{z=0}^{h} \int_{x=-R}^{R} \frac{(x^2 + y^2)}{|y|} dxdz \\
&= h \int_{x=-R,\ y>0}^{R} \frac{(x^2 + y^2)}{y} dx + h \int_{x=-R,\ y<0}^{R} \frac{(x^2 + y^2)}{-y} dx
\end{aligned}$$

Now, $y = \pm\sqrt{R^2 - x^2}$ for $y > 0$ and $y < 0$ respectively. Therefore,

$$\int_{\mathcal{S}_3} \vec{F}.\hat{n} da = 2h \int_{x=-R}^{R} \left(\frac{x^2}{\sqrt{R^2 - x^2}} + \sqrt{R^2 - x^2} \right) dx = 2\pi R^2 h$$

Thus the total flux (contribution from all the three surfaces) is $3\pi R^2 h$. This can also be seen by the divergence theorem: $\vec{\nabla}.\vec{F} = 3$. The volume integral is: $\int_{\mathcal{V}} (\vec{\nabla}.\vec{F}) dxdydz = \int_{\mathcal{V}} 3dxdydz = \int_{z=0}^{h} \int_{x=-R}^{R} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 3dxdydz = 3\pi R^2 h$.

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2. $\star$ Let $\vec{F} = 2xz\hat{x} - xy\hat{y} + y^2\hat{z}$. Evaluate $\int_{\mathcal{V}} \vec{F} d\tau$ where $\mathcal{V}$ is the region bounded by the surfaces $x = 0$, $y = 0$, $y = 6$, $z = x^2$, $z = 4$, as pictured in Figure 1.

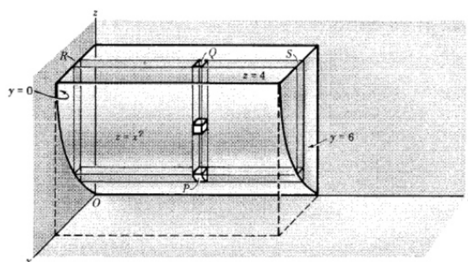

Figure 1: Volume element

Solution:

The region $\mathcal{V}$ is obtained by (i) keeping x, y fixed, and integrating from $z = x^2 \rightarrow z = 4$ (base to top of column PQ), (ii) then by keeping x fixed and then integrating between $y = 0 \rightarrow y = 6$ (R to S in the slab), (iii) finally integrating from $x = 0 \rightarrow x = 2$ (where $z = x^2$ meets $z = 4$). Then the required integral is

$$\begin{aligned}\int_{\mathcal{V}} \vec{F} d\tau &= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (2xz\hat{x} - x\hat{y} + y^2\hat{z}) dzdydx \\ &= \hat{x}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz dzdydx - \hat{y}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x dzdydx \\ &\quad +\hat{z}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2 dzdydx \\ &= 128\hat{x} - 24\hat{y} + 384\hat{z}\end{aligned}$$

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3. Check the fundamental theorem for gradients using $T = x^2 + 4xy + 2yz^3$ joining the points $a = (0; 0; 0)$, $b = (1; 1; 1)$ following three different paths as shown in Figure 2.

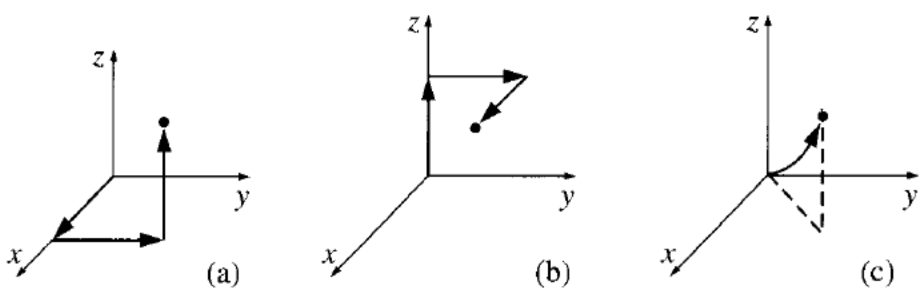

Figure 2: Three different paths

(a) $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$
(b) $(0, 0, 0) \to (0, 0, 1) \to (0, 1, 1) \to (1, 1, 1)$
(c) The parabolic path $z = x^2, x = y$.

Solution:

Fundamental theorem of gradients: $\int_a^b \vec{\nabla} T.d\vec{r} = T(b) - T(a)$. Note, $T(b) = 1 + 4 + 2 = 7$ and $T(a) = 0$. Hence, $T(b) - T(a) = 7$. Now the gradient: $\vec{\nabla} T = (2x + 4y)\hat{x} + (4x + 2z^3)\hat{y} + 6yz^2\hat{z}$. Hence, $\vec{\nabla} T.d\vec{r} = (2x + 4y)dx + (4x + 2z^3)dy + 6yz^2dz$.

(a) <u>On $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$:</u>

<u>Along segment $(0, 0, 0) \to (1, 0, 0)$:</u> $x : 0 \to 1$, $y = z = 0, dy = dz = 0$. Hence, $\int \vec{\nabla} T.d\vec{r} = \int_{x=0}^{1} 2xdx = 1$.

<u>Along segment $(1, 0, 0) \to (1, 1, 0)$:</u> $y : 0 \to 1$, $x = 1, z = 0$, $dx = dz = 0$. Hence, $\int \vec{\nabla} T.d\vec{r} = \int_{y=0}^{1} 4dy = 4$.

<u>Along segment $(1, 1, 0) \to (1, 1, 1)$:</u> $z : 0 \to 1$, $x = y = 1, dx = dy = 0$. Hence, $\int \vec{\nabla} T.d\vec{r} = \int_{z=0}^{1} 6z^2dz = 2$.

Combining all the contributions, we get: $\int \vec{\nabla} T.d\vec{r} = 7$.

(b) $\underline{\text{On } (0,0,0) \rightarrow (0,0,1) \rightarrow (0,1,1) \rightarrow (1,1,1) :}$

$\underline{\text{Along segment } (0,0,0) \rightarrow (0,0,1)}$: $z : 0 \rightarrow 1,\ y = x = 0, dy = dx = 0$. Hence, $\int \vec{\nabla} T.d\vec{r} = \int_{z=0}^{1} 0dz = 0$.

$\underline{\text{Along segment } (0,0,1) \rightarrow (0,1,1)}$: $y : 0 \rightarrow 1,\ x = 0, z = 1,\ dx = dz = 0$. Hence, $\int \vec{\nabla} T.d\vec{r} = \int_{y=0}^{1} 2dy = 2$.

$\underline{\text{Along segment } (0,1,1) \rightarrow (1,1,1)}$: $x : 0 \rightarrow 1,\ z = y = 1, dz = dy = 0$. Hence, $\int \vec{\nabla} T.d\vec{r} = \int_{x=0}^{1} (2x+4)dx = 5$.

Combining all the contributions, we get: $\int \vec{\nabla} T.d\vec{r} = 7$.

(c) $\underline{\text{The parabolic path: } z = x^2; x = y}$: $x : 0 \rightarrow 1, y = x, z = x^2$. Hence, $dy = dx,\ dz = 2xdx$. Therefore, $\int \vec{\nabla} T.d\vec{r} = \int_{x=0}^{1}[(2x+4x)dx + (4x+2x^6)dx + (6xx^4)2xdx] = 7$. Hence checked.

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4. $\star$ (a) Using Stoke's theorem calculate the line integral of $\vec{F} = 2z\hat{x} + x\hat{y} + y\hat{z}$ over a circle of radius R in the xy plane centered at the origin. Take the open surface to be a hemisphere in $z > 0$ (Fig. 3).
(b) Calculate the same using Divergence theorem imagining the hemispherical surface as well as the disc on the $x - y$ plane to form a closed surface.

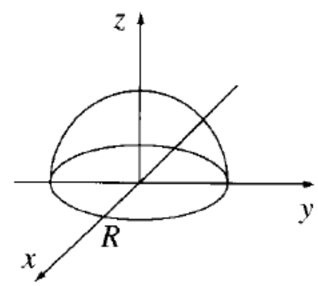

Figure 3: Hemisphere

Solution:

(a) Stoke's theorem states: $\int_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).d\vec{a} = \oint_C \vec{F}.d\vec{r}$. The curl is given by: $\vec{\nabla} \times \vec{F} = \hat{x} + 2\hat{y} + \hat{z}$. The unit normal on the surface of the hemisphere in the radially outward direction is given by $\hat{n} = \frac{\vec{\nabla}(x^2+y^2+z^2)}{|\vec{\nabla}(x^2+y^2+z^2)|} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{R}$. Now for the hemispherical surface ($\mathcal{S}$) we can project it on $x - y$ plane. Thus we need to calculate the surface integral

$$\begin{aligned}
\int_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).\hat{n}da &= \int_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).\hat{n}\frac{dxdy}{|\hat{n}.\hat{z}|} \\
&= \int_{x=-R}^{R} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} (\hat{x} + 2\hat{y} + \hat{z}).\frac{x\hat{x} + y\hat{y} + z\hat{z}}{R} \frac{dxdy}{|\frac{x\hat{x}+y\hat{y}+z\hat{z}}{R}.\hat{z}|} \\
&= \frac{1}{R}\int_{x=-R}^{R} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} (x + 2y + z)\frac{dxdy}{|\frac{z}{R}|} \\
&= \int_{x=-R}^{R} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} (x + 2y + \sqrt{R^2 - x^2 - y^2})\frac{dxdy}{\sqrt{R^2 - x^2 - y^2}} \\
&= I_1 + I_2 + I_3
\end{aligned}$$

Note here, that as the hemisphere is already in $z > 0$ region, $|z| = z$, with z > 0, unlike problem 1. The hemisphere being symmetrical with respect to x and y coordinates, the first two integrals, I_1, I_2 vanish and we are left with $I_3 = \int_{x=-R}^{R} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dxdy = \int_{x=-R}^{R} 2\sqrt{R^2 - x^2}dx = \pi R^2$. The line integral can be calculated as follows:

$$\oint_C \vec{F}.d\vec{r} = \oint_C (2zdx + xdy + ydz)$$

Since the boundary of the open surface (hemisphere) is the circle on the $x - y$ plane, the first and the third terms in the integral give zero as $z = 0$ on C. We are left with $\oint_C xdy$ which can be parameterized in polar coordinates with $x = R\cos\theta, y = R\sin\theta$ so that we get: $\oint_C xdy = \int_{\theta=0}^{2\pi} R^2 \cos^2\theta d\theta = \pi R^2$. Thus, Stokes theorem is verified.

(b) Using divergence theorem,

$$\int_{\mathcal{V}} (\vec{\nabla}.(\vec{\nabla} \times \vec{F}))d\tau = \oint_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).\hat{n}da = 0,$$

as, $(\vec{\nabla}.(\vec{\nabla} \times \vec{F})) = 0$. Therefore,

$$\begin{aligned} \oint_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).\hat{n}da &= \int_{\mathcal{S}_1} (\vec{\nabla} \times \vec{F}).\hat{n}da + \int_{\mathcal{S}_2} (\vec{\nabla} \times \vec{F}).\hat{n}da = 0. \\ \therefore \int_{\mathcal{S}_1} (\vec{\nabla} \times \vec{F}).\hat{n}da &= -\int_{\mathcal{S}_2} (\vec{\nabla} \times \vec{F}).\hat{n}da \end{aligned}$$

In the above expressions $\mathcal{S}_1$ is the hemispherical surface and $\mathcal{S}_2$ is the surface of the disc situated on the $x - y$ plane. Hence

$$\begin{aligned} \int_{\mathcal{S}_1} (\vec{\nabla} \times \vec{F}).\hat{n}da &= -\int_{\mathcal{S}_2} (\vec{\nabla} \times \vec{F}).\hat{n}da \\ &= -\int_{\mathcal{S}_2} (\hat{x} + 2\hat{y} + \hat{z}).(-\hat{z})dxdy \\ &= \int_{\mathcal{S}_2} dxdy = \pi R^2 \end{aligned}$$

Hence verified.

5. $\star$ Prove that the cylindrical coordinate system is orthogonal and express velocity and acceleration of a particle in cylindrical polar coordinates.

Solution:

The position vector in cylindrical coordinate is $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = s\cos\phi\hat{x} + s\sin\phi\hat{y} + z\hat{z}$. The tangent vectors to s, ϕ, z curves are given by $\frac{\partial\vec{r}}{\partial s}$, $\frac{\partial\vec{r}}{\partial\phi}$, $\frac{\partial\vec{r}}{\partial z}$ respectively. Recall s curve is the curve on which ϕ, z are constants.

$$\frac{\partial \vec{r}}{\partial s} = \cos\phi\hat{x} + \sin\phi\hat{y},\ \frac{\partial \vec{r}}{\partial \phi} = -s\sin\phi\hat{x} + s\cos\phi\hat{y},\ \frac{\partial \vec{r}}{\partial z} = \hat{z}.$$

Hence the unit vectors in these directions are

$$\begin{aligned}
\hat{s} &= \frac{\frac{\partial \vec{r}}{\partial s}}{\left|\frac{\partial \vec{r}}{\partial s}\right|} = \cos\phi\hat{x} + \sin\phi\hat{y}, \\
\hat{\phi} &= \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = -\sin\phi\hat{x} + \cos\phi\hat{y}, \\
\hat{z} &= \frac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|} = \hat{z}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\hat{s}.\hat{\phi} &= (\cos\phi\hat{x} + \sin\phi\hat{y}).(-\sin\phi\hat{x} + \cos\phi\hat{y}) = 0, \\
\hat{s}.\hat{z} &= (\cos\phi\hat{x} + \sin\phi\hat{y}).\hat{z} = 0, \\
\hat{z}.\hat{\phi} &= \hat{z}.(-\sin\phi\hat{x} + \cos\phi\hat{y}) = 0.
\end{aligned}$$

Therefore the unit vectors are mutually perpendicular and the coordinate system is orthogonal.

In Cartesian coordinates position vector is $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. Hence velocity and acceleration vectors are $\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$, $\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$. Using above results, we can recast the unit vectors in Cartesian coordinates in terms of unit vectors of Cylindrical Polar coordinates as follows

$$\begin{aligned}
\hat{x} &= \cos\phi\hat{s} - \sin\phi\hat{\phi}, \\
\hat{y} &= \sin\phi\hat{s} + \cos\phi\hat{\phi}, \\
\hat{z} &= \hat{z}.
\end{aligned}$$

Hence, the position vector in Cylindrical Polar coordinates is given by

$$\begin{aligned}
\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} &= s\cos\phi(\cos\phi\hat{s} - \sin\phi\hat{\phi}) + s\sin\phi(\sin\phi\hat{s} + \cos\phi\hat{\phi}) + z\hat{z} \\
&= s\hat{s} + z\hat{z}.
\end{aligned}$$

Therefore, the velocity vector in Cylindrical Polar coordinates,

$$\begin{aligned}
\vec{v} = \frac{d\vec{r}}{dt} &= \frac{ds}{dt}\hat{s} + s\frac{d\hat{s}}{dt} + \frac{dz}{dt}\hat{z} \\
&= \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z},
\end{aligned}$$

where, we have used $\frac{d\hat{s}}{dt} = -\sin\phi\dot{\phi}\hat{x} + \cos\phi\dot{\phi}\hat{y} = \dot{\phi}\hat{\phi}$. Proceeding in the similar manner, the acceleration can be found to be

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}) = (\ddot{s} - s\dot{\phi}^2)\hat{s} + (s\ddot{\phi} + 2\dot{s}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}.$$

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6. In PH 101 you encountered the momentum operator in quantum mechanics. Recall that the momentum operator had the form $p = \frac{\hbar}{i}\frac{d}{dx}$ in one dimension. Now that we have discussed everything in general in three dimensions, $\vec{p} = \frac{\hbar}{i}\vec{\nabla}$. Hence the angular momentum operator $\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i}(\vec{r} \times \vec{\nabla})$. Show that the angular momentum operator in spherical polar coordinate is of the form

$$\vec{L} = \frac{\hbar}{i}\left(-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

Solution:

The gradient operator in the spherical polar coordinates as discussed in the class

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

Now, $\vec{r} = r\hat{r}$. Hence,

$$\vec{L} = \frac{\hbar}{i}\left[r(\hat{r} \times \hat{r})\frac{\partial}{\partial r} + (\hat{r} \times \hat{\theta})\frac{\partial}{\partial\theta} + (\hat{r} \times \hat{\phi})\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

Now, $\hat{r} \times \hat{r} = 0,\ \hat{r} \times \hat{\theta} = \hat{\phi},\ \hat{r} \times \hat{\phi} = -\hat{\theta}$. Hence,

$$\vec{L} = \frac{\hbar}{i}\left(-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \hat{\phi}\frac{\partial}{\partial\theta}\right)$$
