Solution of Constant Coefficients ODE

Department of Mathematics
IIT Guwahati

Thus,

$$\frac{\partial^k}{\partial r^k}(e^{rx})|_{r=r_1} = x^k e^{r_1 x}$$

will be a solution to $L(y) = 0$ for $k = 0, 1, \ldots, m-1$.
So, m distinct solutions are

$$e^{r_1 x},\ xe^{r_1 x},\ \cdots,\ x^{m-1}e^{r_1 x}.$$

Theorem: If $P(r) = 0$ has the real root r_1 occurring m times and the remaining roots $r_{m+1}, r_{m+2}, \ldots, r_n$ are distinct, then the general solution of $L(y) = 0$ is

$$\begin{aligned} y(x) \;&=\; (C_1 + C_2 x + C_3 x^2 + \cdots + C_m x^{m-1})e^{r_1 x} \\ &\quad + C_{m+1}e^{r_{m+1}x} + \cdots + C_n e^{r_n x}, \end{aligned}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Example: Consider $y^{(4)} - 8y'' + 16y = 0$. In this case, $r_1 = r_2 = 2$ and $r_3 = r_4 = -2$. The general solution is

$$y = (C_1 + C_2 x)e^{2x} + (C_3 + C_4 x)e^{-2x}.$$

Case III (Complex roots): If $\alpha + i\beta$ is a non-repeated complex root of $P(r) = 0$ so is its complex conjugate. Then, both

$$e^{(\alpha+i\beta)x} \quad \text{and} \quad e^{(\alpha-i\beta)x}$$

are solution to $L(y) = 0$. Then, the corresponding part of the general solution is of the form

$$e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)).$$

Theorem: If $P(r) = 0$ has non-repeated complex roots $\alpha + i\beta$ and $\alpha - i\beta$, the corresponding part of the general solution is

$$e^{\alpha x}\left(C_1\cos(\beta x) + C_2\sin(\beta x)\right).$$

If $\alpha + i\beta$ and $\alpha - i\beta$ are each repeated roots of multiplicity m, then the corresponding part of the general solution is

$$e^{\alpha x}\Big[\left(C_1 + C_2x + C_3x^2 + \cdots + C_mx^{m-1}\right)\cos(\beta x) \\ +\left(C_{m+1} + C_{m+2}x + \cdots + C_{2m}x^{m-1}\right)\sin(\beta x)\Big],$$

where $C_1, C_2, \ldots, C_{2m}$ are arbitrary constants.

Example: Consider $y^{(4)} - 2y''' + 2y'' - 2y' + y = 0$. Here, $r_1 = r_2 = 1$, $r_3 = i$ and $r_4 = -i$. The general solution is

$$y = (C_1 + C_2x)e^x + (C_3\cos x + C_4\sin x).$$

Particular solution of constant coefficients ODE

Method of undetermined coefficients: A simple procedure for finding **a particular solution**(y_p) to a non-homogeneous equation $L(y) = g$, when L **is a linear differential operator with constant coefficients** and when $g(x)$ is of special type:

That is, when $g(x)$ is either

- a polynomial in x,
- an exponential function $e^{\alpha x}$,
- trigonometric functions $\sin(\beta x), \cos(\beta x)$

or finite sums and products of these functions.

Case I. For finding y_p to the equation $L(y) = p_n(x)$, where $p_n(x)$ is a polynomial of degree n. Try a solution of the form

$$y_p(x) = A_n x^n + \cdots + A_1 x + A_0$$

and match the coefficients of $L(y_p)$ with those of $p_n(x)$:

$$L(y_p) = p_n(x).$$

Remark: This procedure yields $n + 1$ linear equations in $n + 1$ unknowns $A_0, \ldots, A_n$.

Example: Find y_p to $L(y)(x) := y'' + 3y' + 2y = 3x + 1$.

Try the form $y_p(x) = Ax + B$ and attempt to match up $L(y_p)$ with $3x + 1$. Since

$$L(y_p) = 2Ax + (3A + 2B),$$

equating

$$2Ax + (3A + 2B) = 3x + 1 \Longrightarrow A = 3/2 \text{ and } B = -7/4.$$

Thus, $y_p(x) = \frac{3}{2}x - \frac{7}{4}$.

Case II: The method of undetermined coefficients will also work for equations of the form

$$L(y) = ae^{\alpha x},$$

where a and α are given constants. Try y_p of the form

$$y_p(x) = Ae^{\alpha x}$$

and solve $L(y_p)(x) = ae^{\alpha x}$ for the unknown coefficients A.

Example: Find y_p to $L(y)(x) := y'' + 3y' + 2y = e^{3x}$.
Seek $y_p(x) = Ae^{3x}$. Then

$$L(y_p) = 9Ae^{3x} + 3(3Ae^{3x}) + 2(Ae^{3x}) = 20Ae^{3x}.$$

Now, $L(y_p) = e^{3x} \implies 20Ae^{3x} = e^{3x} \implies A = 1/20.$
Thus, $y_p(x) = (1/20)e^{3x}$.

Case III: For an equation of the form

$$L(y) = a\cos\beta x + b\sin\beta x,$$

try y_p of the form

$$y_p(x) = A\cos\beta x + B\sin\beta x$$

and solve $L(y_p) = a\cos\beta x + b\sin\beta x$ for the unknowns A and B.

Example: Find y_p to $L(y) := y'' - y' - y = \sin x$.
Seek $y_p(x)$ of the form $y_p(x) = A\cos x + B\sin x$. Then

$$L(y_p) = \sin x \implies A = 1/5, \quad B = -2/5.$$

Thus, $y_p(x) = \frac{1}{5}\cos x - \frac{2}{5}\sin x$.

Example: Find y_p to $L(y) := y'' - y' - 12y = e^{4x}$.

Note that $y_h(x) = c_1 e^{4x} + c_2 e^{-3x}$. Try finding y_p with the guess $y_p(x) = Ae^{4x}$ as before. Since e^{4x} is a solution to the corresponding homogeneous equation $L(y) = 0$, we replace this choice of y_p by $y_p(x) = Axe^{4x}$. Since $L(xe^{4x}) \neq 0$, there exists a particular solution of the form

$$y_p(x) = Axe^{4x}.$$

Remark: If $L(y_p) = 0$ then replace $y_p(x)$ by $xy_p(x)$. If $L(xy_p) = 0$ the replace xy_p by $x^2 y_p$ and so on. Thus, employing $x^s y_p$, where s is the smallest nonnegative integer such that $L(x^s y_p) \neq 0$.

Form of y_p:

- $g(x) = p_n(x) = a_n x^n + \cdots + a_1 x + a_0,$
  $y_p(x) = x^s P_n(x) = x^s\{A_n x^n + \ldots + A_1 x + A_0\}$

- $g(x) = ae^{\alpha x}, \;\; y_p(x) = x^s A e^{\alpha x}$

- $g(x) = a\cos\beta x + b\sin\beta x,$
  $y_p(x) = x^s\{A\cos\beta x + B\sin\beta x\}$

- $g(x) = p_n(x)e^{\alpha x}, \;\; y_p(x) = x^s P_n(x) e^{\alpha x}$

- $g(x) = p_n(x)\cos\beta x + q_m(x)\sin\beta x,$
  where $q_m(x) = b_m x^m + \cdots + b_1 x + b_0$ and $p_n(x)$ as above.
  $y_p(x) = x^s\{P_N(x)\cos\beta x + Q_N(x)\sin\beta x\},$
  where $Q_N(x) = B_N x^N + \cdots + B_1 x + B_0,$
  $P_N(x) = A_N x^N + \cdots + A_1 x + A_0$ and $N = \max(n, m)$.

- $g(x) = ae^{\alpha x}\cos\beta x + be^{\alpha x}\sin\beta x$,
  $y_p(x) = x^s\{Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x\}$
- $g(x) = p_n(x)e^{\alpha x}\cos + q_m(x)e^{\alpha x}\sin\beta x$,
  $y_p(x) = x^s e^{\alpha x}\{P_N(x)\cos\beta x + Q_N(x)\sin\beta x\}$, where
  $N = \max(n, m)$.

Note:

1. The nonnegative integer s is chosen to be the smallest integer so that no term in y_p is a solution to $L(y) = 0$.

2. $P_n(x)$ or $P_N(x)$ must include all its terms even if $p_n(x)$ has some terms that are zero. Similarly for $Q_N(x)$.

*** End ***