Tutorial sheet - 1

(1)

$$\|x\| = \|x-y+y\| \le \|x-y\| + \|y\|$$

$$\Rightarrow \quad \|x\| - \|y\| \le \|x-y\| \quad \text{———} \quad (1)$$

again changing x & y in (1), we have

$$\|y\| - \|x\| \le \|y-x\| = \|x-y\| \quad \text{———} \quad (2)$$

Therefore from (1) & (2) we have

$$\big|\|x\| - \|y\|\big| \le \|x-y\|$$

2nd part.

$$\|x+y\|^2 = \langle x+y, \ x+y \rangle$$

$$= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle$$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$$

$\Rightarrow \quad \|x+y\|^2 = \|x\|^2 + \|y\|^2$ iff $\langle x, y\rangle = 0$

OR $\quad \|x+y\|^2 = \|x\|^2 + \|y\|^2$ iff x and y are orthogonal

(2) $p(t) := \|x+ty\|^2$

Then $p(t) \geq 0$ from definition

$$p(t) = \langle x+ty, x+ty\rangle = \|x\|^2 + 2t\langle x,y\rangle + t^2\|y\|^2 \geq 0$$

Recall that a Quadratic ax^2+bx+c, where $a>0$, is always ≥ 0 if and only if $b^2-4ac \leq 0$

$$\Rightarrow \quad 4\,(\langle x,y\rangle)^2 - 4\|x\|^2\|y\|^2 \leq 0$$

$$\Rightarrow \quad |\langle x,y\rangle \leq \|x\|\,\|y\|$$

again $\cos\theta = \dfrac{\langle x,y\rangle}{\|x\|\,\|y\|}$

so $\cos\theta = 1$ iff $x = \alpha y$

OR $\|x\|\,\|y\| = \langle x,y\rangle$ iff $x = \alpha y$

(3) $(x_k) \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$

Suppose $x_k \longrightarrow x$

Then $|\langle x_k, y\rangle - \langle x,y\rangle| = |\langle x_k - x, y\rangle|$

$$\leq \|x_k - x\|\,\|y\| \qquad (\text{C.S. inequality})$$

$$\downarrow 0$$

$$\Rightarrow \quad \langle x_k, y\rangle \longrightarrow \langle x,y\rangle$$

Now take $y = e_i$

$$(x_k) = (x_{k_1}, x_{k_2}, \dots x_{k_i}, \dots x_{k_n})$$

$$x = (x_1, x_2, \dots\dots x_n)$$

Then $\langle x_k, y \rangle = x_{k_i}$

$\langle x, y \rangle = x_i$

So if $\langle x_k, y \rangle \longrightarrow \langle x, y \rangle$ Then $x_{k_i} \longrightarrow x_i$

So each component of x_k convergent

$\Rightarrow (x_k) = (x_{k_1}, \dots x_{k_n}) \longrightarrow (x_1, \dots x_n)$

$\Rightarrow (x_k) \longrightarrow x$

(4)

$$x_k \longrightarrow x$$

$$\Rightarrow \big|\,\|x_k\| - \|x\|\,\big| \leq \|x_k - x\| \longrightarrow 0$$

$$\Rightarrow \|x_k\| \longrightarrow \|x\|$$

Suppose $x_k = (x_{k_1}, x_{k_2} \dots x_{k_n})$ & $x = (x_1 \dots x_n)$

$$\Rightarrow \frac{x_{ki}}{\|x_k\|} \longrightarrow \frac{x_i}{\|x\|}$$

$$\Rightarrow \frac{x_k}{\|x_k\|} \longrightarrow \frac{x}{\|x\|}$$

(5)

Given that $x_k \longrightarrow x$ & $\langle x_k, y\rangle = 0 \ \forall k$

$$\langle x_k, y\rangle = 0 \ \forall k \Rightarrow x_k \perp y$$

$$|\langle x, y\rangle| = |\langle x - x_k, y\rangle| \leq \underbrace{\|x_k - x\|}_{\to 0} \|y\|$$

$$\Rightarrow |\langle x, y\rangle| = 0$$

$$\langle x, y\rangle = 0$$

(6)

$$x_n = \left(n^3 \alpha^n , \frac{[n\alpha]}{n} \right)$$

$$\because \lim_{n\to\infty} \frac{(n+1)^3 \alpha^{n+1}}{n^3 \alpha^n} = \alpha < 1$$

$$\Rightarrow \lim_{n\to\infty} n^3 \alpha^n = 0$$

again $0 \leq n\alpha - [n\alpha] < 1$

$$\Rightarrow 0 \leq \frac{n\alpha - [n\alpha]}{n} < \frac{1}{n}$$

$$\because \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{n\alpha - [n\alpha]}{n} = 0$$ (from Sandwitch Thm

i.e. $\lim_{n\to\infty} \frac{[n\alpha]}{n} = \alpha$.

So $\lim_{n\to\infty} x_n = (0, \alpha)$