PH 102: Physics II

Lecture 18 (Spring 2018)

IIT Guwahati
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LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	6-3-2018	Lorentz Force, Biot-Savart law	5.1, 5.2	1	I, II (5-5:55 pm)
Lec 15	7-3-2018	Lorentz Force, Biot-Savart law	5.1, 5.2	1	III, IV (9-9:55 am)
Tut 8	13-3-2018	Lec 15			
Lec 16	13-3-2018	Divergence & Curl of Magnetostatic Fields, Applications of Ampere's law	5.3	1	I, II (5-5:55 pm)
Lec 16	14-3-2018	Divergence & Curl of Magnetostatic Fields, Applications of Ampere's law	5.3	1	III, IV (9-9:55 am)
Lec 17	14-3-2018	Magnetic Vector Potential, Force & torque on a magnetic dipole	5.4	1	I, II (4-4:55 pm)
Lec 17	15-3-2018	Magnetic Vector Potential, Force & torque on a magnetic dipole	5.4	1	III, IV (10-10:55 am)
Lec 18	15-3-2018	Lec 16+Lec 17 Continues		1	I, II (3-3:55 pm)
Lec 18	16-3-2018	Lec 16+Lec 17 Continues		1	III, IV (11-11:55 am)
Tut 9	20-3-2018	Lec 16+Lec 17+Lec 18			
Lec 19	21-3-2018	Magnetic Materials, Magnetization	6.1	1	I, II, III, IV
Lec 20	23-3-2018	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II, III, IV
Tut 10	3-4-2018	Quiz II			
Lec 21	4-4-2018	Ohm's law, motional emf, electromotive force	7.1	1	I, II, III, IV
Lec 22	6-4-2018	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II, III, IV
Tut 11	10-4-2018	Lec 21+Lec 22			
Lec 23	11-4-2018	Maxwell's equations	7.3	1	I, II, III, IV
Lec 24	13-4-2018	Continuity equation, Poynting Theorem	8.1	1	I, II, III, IV
Tut 12	17-4-2018	Lec 23+Lec 24			
Lec 25	18-4-2018	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II, III, IV
Lec 26	20-4-2018	L11+Electromagnetic waves in matter	9.3	1	I, II, III, IV
Tut 13	24-4-2018	Lec 25+ Lec 26			
Lec 27	25-4-2018	Reflection and transmission: Normal & Oblique Incidence	9.3, 9.4	1	I, II, III, IV
Lec 28	27-4-2018	Lec 27+Discussions	9.3, 9.4	1	I, II, III, IV

Magnetic Dipole

- A current carrying loop with area $\vec{a}$ has a magnetic dipole moment given by $\vec{m} = I\vec{a}$.

- Magnetic dipole moment is independent of the choice of origin.

- The dipole term is the leading order term in the multipole expansion of the vector potential.

- The dipole term is identified as the one that is proportional to inverse of distance (r) in the multipole expansion.

Multipole Expansion of Vector Potential

- Multipole expansion is used to write the potential in the form of a power series in 1/r.

- The vector potential of a current loop can be written as

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\mathfrak{r}} d\vec{l'}$$

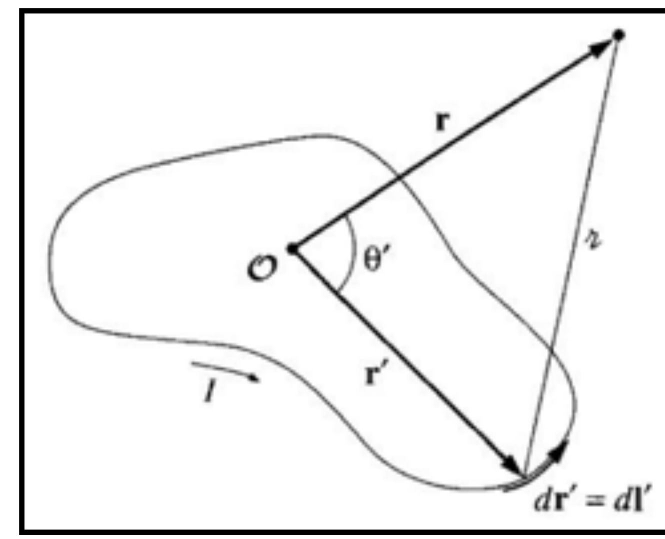


Figure 5.51, Introduction to Electrodynamics, D. J. Griffiths

- Using the standard expansion

$$\frac{1}{\mathfrak{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

and using the standard Legendre polynomials

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l'} + \frac{1}{r^2}\oint r'\cos\theta' d\vec{l'} + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\vec{l'} + \ldots\right]$$

Multipole Expansion of Vector Potential

The power series expansion can also be realised as

$$\mathfrak{r}^2 = r^2 + (r')^2 - 2rr'\cos\theta' = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right]$$

$$\implies \mathfrak{r} = r\sqrt{1+\epsilon}, \epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$$

$$\frac{1}{\mathfrak{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right)$$

$$\implies \frac{1}{\mathfrak{r}} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2 (3\cos^2\theta' - 1)/2 + \left(\frac{r'}{r}\right)^3 (5\cos^3\theta' - 3\cos\theta')/2 + \ldots\right]$$

The first term in the expansion of vector potential is zero due to the vanishing integral of total vector displacement around a closed loop

$$\oint d\vec{l'} = 0$$

Thus, there is no monopole contribution to vector potential. Absence of magnetic monopoles: $\vec{\nabla}\cdot\vec{B} = 0$

Magnetic Dipole

- The dipole contribution to the vector potential is

$$\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' d\vec{l'} = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot \vec{r'}) d\vec{l'}$$

- Using the definition of area of a loop $\vec{a} = \frac{1}{2} \oint \vec{r} \times d\vec{l}$ and the identity $\oint (\vec{c} \cdot \vec{r}) d\vec{l} = \vec{a} \times \vec{c}$

Chapter 1, Introduction to Electrodynamics, D J Griffiths

we can write
$$\oint (\hat{r} \cdot \vec{r'}) d\vec{l} = -\hat{r} \times \int d\vec{a'}$$

- The dipole contribution can now be written as

$$\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}, \quad \vec{m} \equiv I \int d\vec{a} = I\vec{a}$$

Magnetic dipole moment

- In general, $\vec{m} = \frac{1}{2} \oint I(\vec{r} \times d\vec{l}) = \frac{1}{2} \int (\vec{r} \times \vec{J}) d\tau$

Field of a Magnetic Dipole

- Let us consider a magnetic dipole with dipole moment m at the origin, pointing in the z direction.

- The vector potential is given by

$$\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

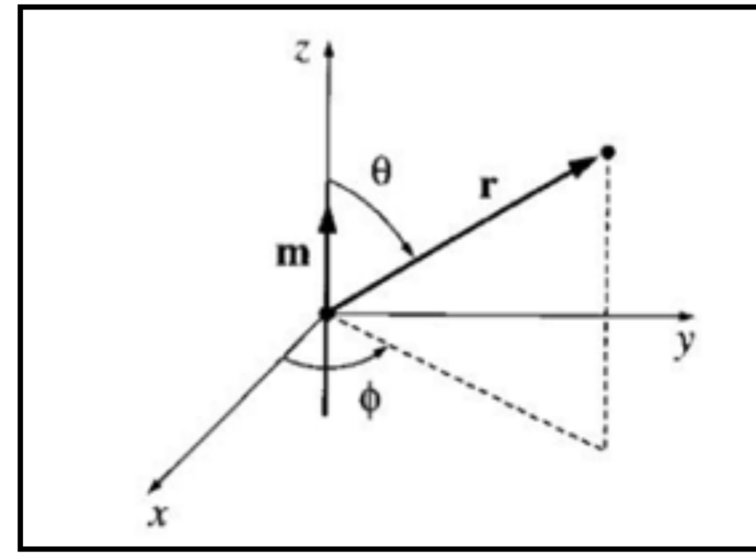


Figure 5.54, Introduction to Electrodynamics, D. J. Griffiths

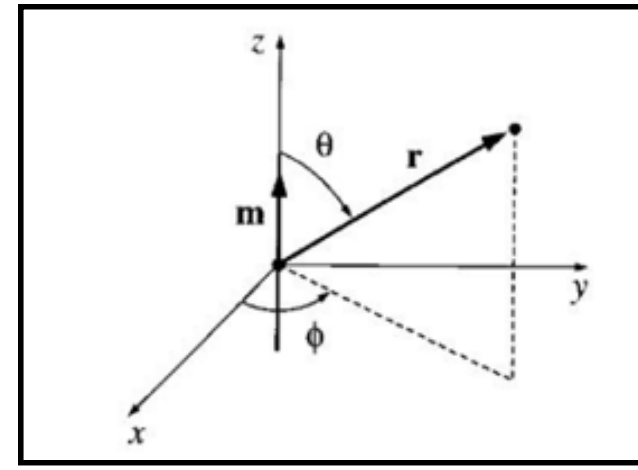
- The magnetic field is given by

$$\vec{B}_{\text{dipole}}(\vec{r}) = \vec{\nabla}\times\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

- The magnetic field of a dipole in coordinate free form can be written as (Problem 5.33, Introduction to Electrodynamics, D. J. Griffiths)

$$\vec{B}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}]$$

The dipole moment can be decomposed as



$$\vec{m} = (\vec{m}\cdot\hat{r})\hat{r} + (\vec{m}\cdot\hat{\theta})\hat{\theta}$$
$$= m\cos\theta\hat{r} - m\sin\theta\hat{\theta}$$

Using this, we get:

$$3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m} = 3m\cos\theta\hat{r} - m\cos\theta\hat{r} + m\sin\theta\hat{\theta}$$
$$= 2m\cos\theta\hat{r} + m\sin\theta\hat{\theta}$$

Therefore,

$$\frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) = \vec{B}_{\text{dipole}}(\vec{r})$$

$$\vec{B}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}]$$

Magnetic Field of Earth

Magnetic dipole axis of earth and its geographic (rotation) axis do not coincide. Also, magnetic south pole is in northern hemisphere so that the north pole of the magnetic compass can point towards geographic north pole. Choosing z axis to be the rotation axis and x axis to pass through prime meridian, the dipole moment is given by

$$\begin{aligned}\vec{m}_E &= m_E(\sin\theta_0\cos\phi_0\hat{i} + \sin\theta_0\sin\phi_0\hat{j} + \cos\theta_0\hat{k}) \\ &= m_E(-0.062\hat{i} + 0.18\hat{j} - 0.98\hat{k})\end{aligned}$$

$$(\theta_0, \phi_0) = (169^\circ, 109^\circ)$$

$$m_E = 7.79 \times 10^{22}\,\mathrm{Am}^2$$

The location of IIT Guwahati is $(\theta_G, \phi_G) = (63.81^\circ, 91.69^\circ) \equiv 26.10^\circ\mathrm{N}, 91.69^\circ\mathrm{E}$

The position vector of IIT Guwahati is

$$\begin{aligned}\vec{r}_G &= r_E(\sin\theta_G\cos\phi_G\hat{i} + \sin\theta_G\sin\phi_G\hat{j} + \cos\theta_G\hat{k}) \\ &= r_E(-0.026\hat{i} + 0.897\hat{j} + 0.44\hat{k})\end{aligned}$$

The angle between $-\vec{m}_E, \vec{r}_G$ is

$$\begin{aligned}\theta_{GE} &= \cos^{-1}\left(\frac{-\vec{r}_G\cdot\vec{m}_E}{|\vec{r}_G||\vec{m}_E|}\right) \\ &= \cos^{-1}(0.268) = 74.4^\circ\end{aligned}$$

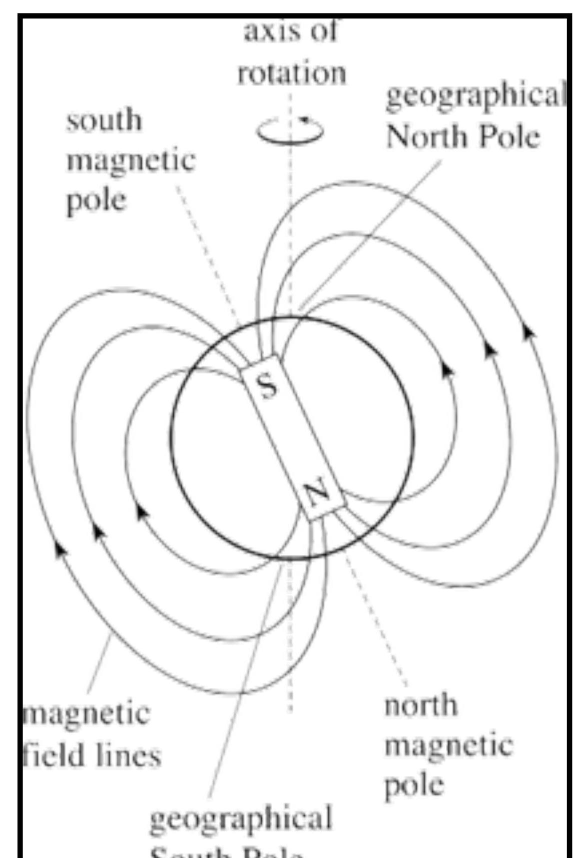


Image credit: http://www.met.reading.ac.uk

Magnetic Field of Earth

Using $\vec{B}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}]$, one can show that

the ratio of the radial and the polar components is

$$\frac{B_r}{B_\theta} = \frac{\frac{\mu_0}{4\pi}\frac{2m}{r^3}\cos\theta}{\frac{\mu_0}{4\pi}\frac{m}{r^3}\sin\theta} = 2\cot\theta$$

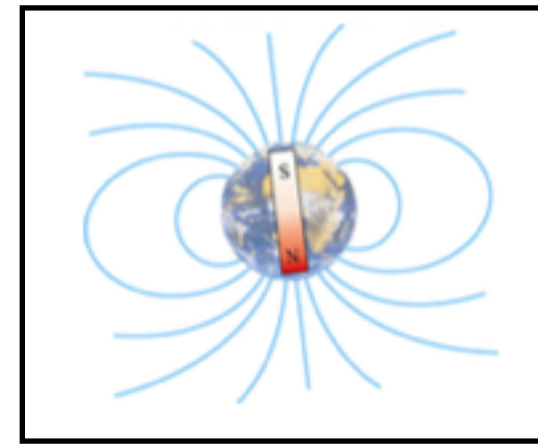


Image credit: MIT

At the location of IIT Guwahati, this value is:

$$\frac{B_r}{B_\theta} = 2\cot\theta_{GE} \approx 0.56$$

Note that here, we are taking θ_{GE} instead of θ keeping in mind the difference between dipole moment axis and the z-axis.

Magnetic Field of Earth

- It is generated in the metallic core of the planet.
- Used for navigation (human, birds….)
- Keeps the earth safe from harmful radiation.
- Lead to the formation of Aurora.

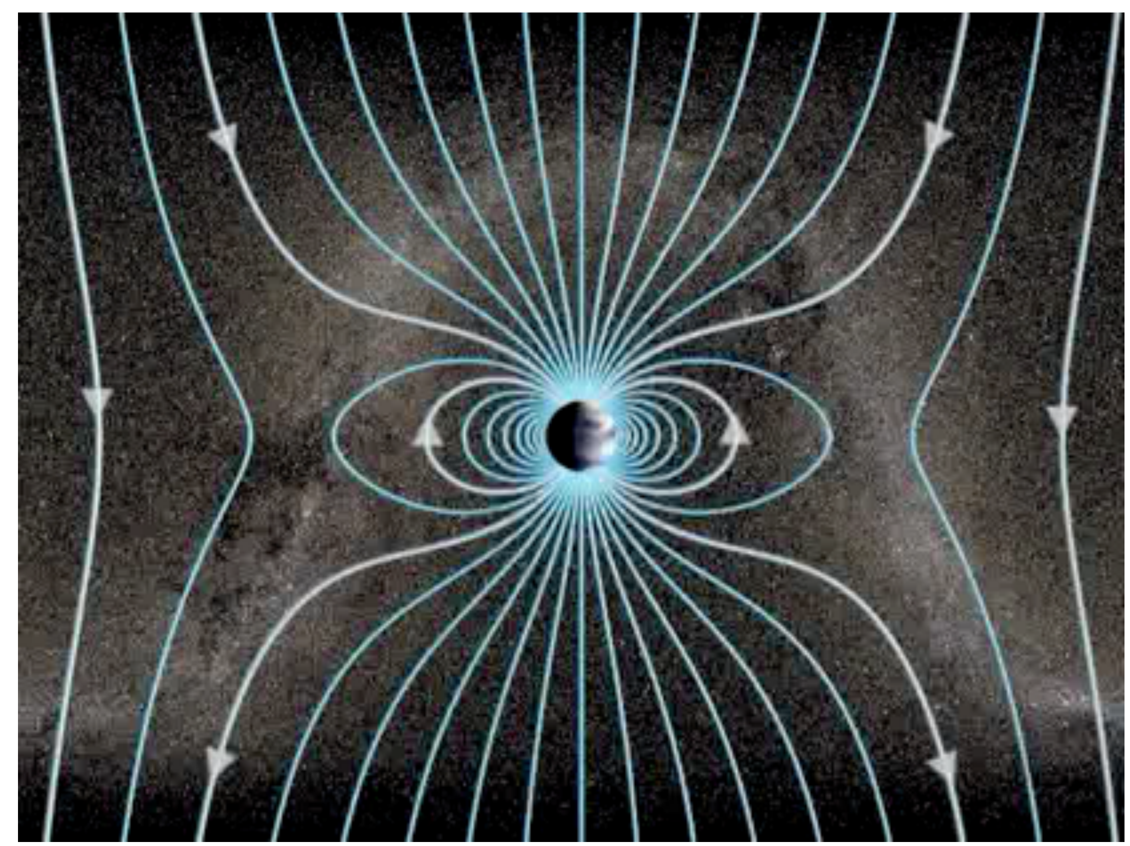


Image credit: NOAA
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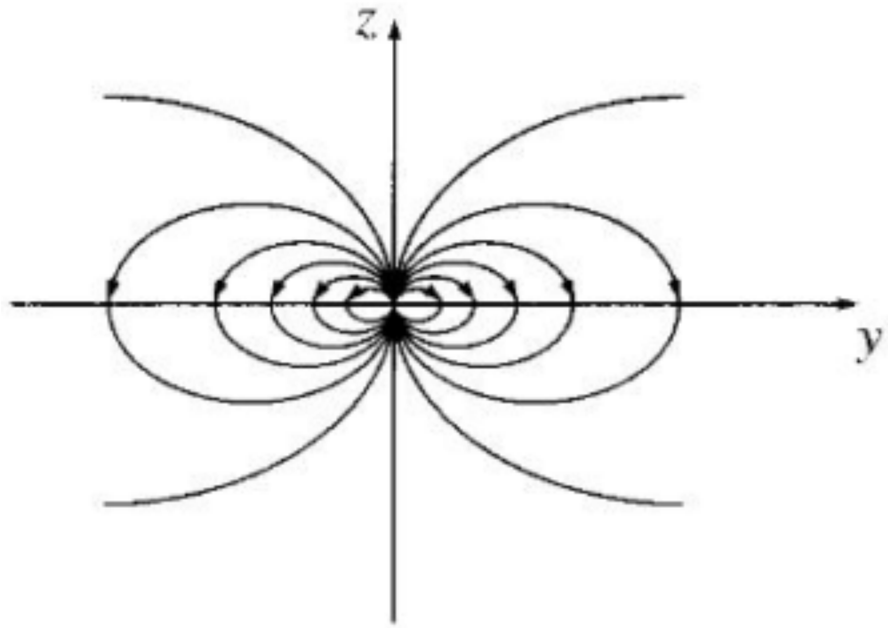
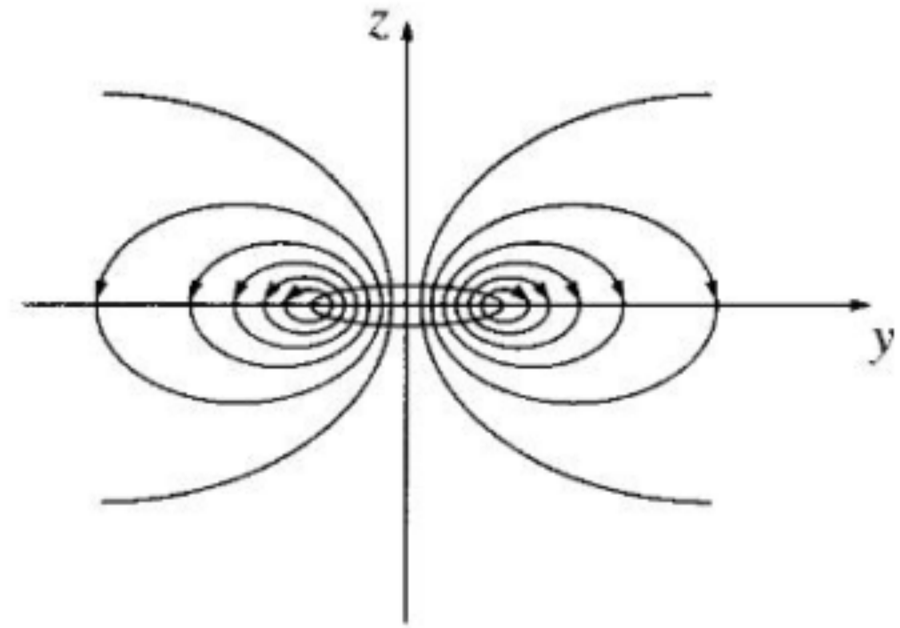
Field of a Magnetic Dipole



(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

Figure 5.55, Introduction to Electrodynamics, D. J. Griffiths

Torques & Forces on Magnetic Dipoles

- A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field.

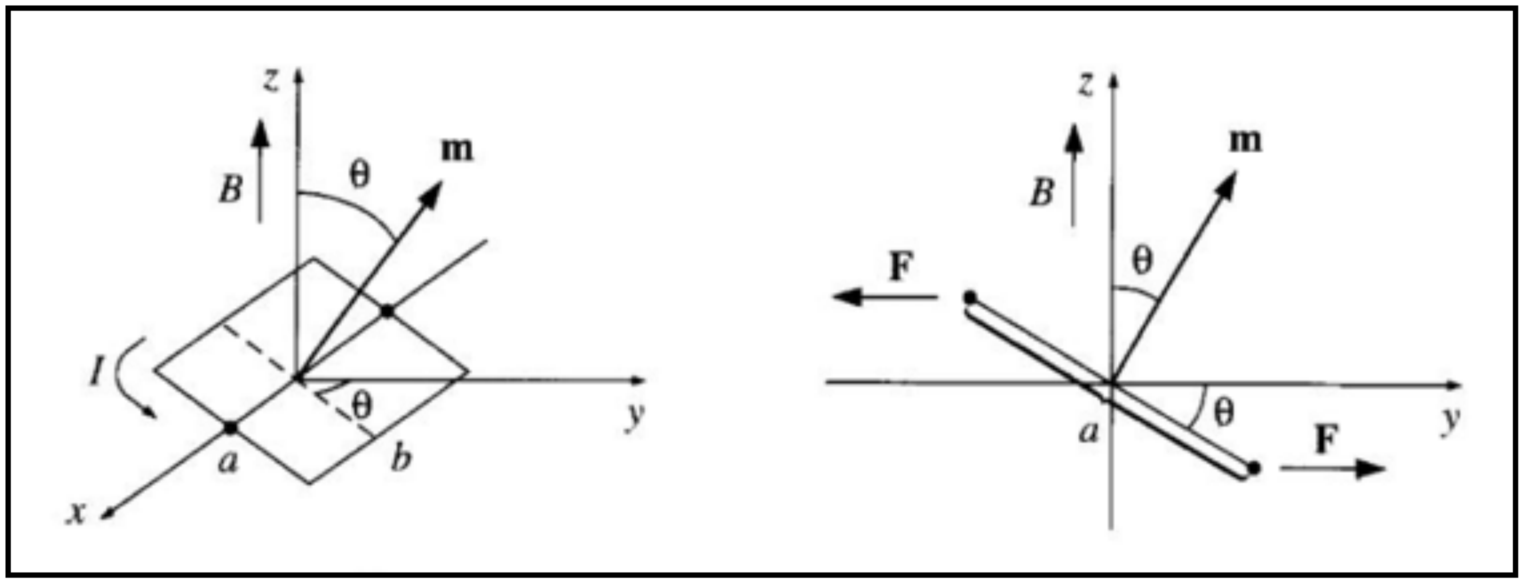


Figure 6.2, Introduction to Electrodynamics, D. J. Griffiths

- Consider a rectangular loop carrying current I, with centre at the origin and magnetic field pointing along the z direction.

- The forces on the two sloping sides (length a) cancel each other.

- The forces on the horizontal sides (length b) cancel each other but they generate a torque

Torques & Forces on Magnetic Dipoles

- The magnitude of the force on each of the horizontal segments is F=IbB.

- The resulting torque is

$$\vec{N} = aF\sin\theta\hat{x} = IabB\sin\theta\hat{x} = mB\sin\theta\hat{x}$$

$$\implies \vec{N} = \vec{m} \times \vec{B}$$

where m=Iab is the magnetic dipole moment of the loop.

- This gives the exact torque on any localised current distribution, in the presence of a uniform field.

- In a nonuniform field, the above formula gives the exact torque about the centre, for a perfect dipole of infinitesimal size.

Torque on a general current loop

Force and torque on an elemental current element of the loop is: $d\vec{F} = I(d\vec{l} \times \vec{B}), d\vec{N} = \vec{r} \times d\vec{F} = I\vec{r} \times (d\vec{l} \times \vec{B})$

Using the identity: $\left[\vec{A} \times (\vec{B} \times \vec{C})\right] + \left[\vec{B} \times (\vec{C} \times \vec{A})\right] + \left[\vec{C} \times (\vec{A} \times \vec{B})\right] = 0$

We have $\left[\vec{r} \times (d\vec{l} \times \vec{B})\right] = -\left[d\vec{l} \times (\vec{B} \times \vec{r})\right] - \left[\vec{B} \times (\vec{r} \times d\vec{l})\right]$ (1)

Also, $d\left[\vec{r} \times (\vec{l} \times \vec{B})\right] = d\vec{r} \times (\vec{l} \times \vec{B}) + \vec{r} \times (d\vec{l} \times \vec{B})$ (Since B is uniform)

$$\implies \vec{r} \times (d\vec{l} \times \vec{B}) = d\left[\vec{r} \times (\vec{l} \times \vec{B})\right] - d\vec{r} \times (\vec{l} \times \vec{B})$$

Using $d\vec{r} \rightarrow d\vec{l}, \vec{l} \rightarrow \vec{r}$

$$\implies \vec{r} \times (d\vec{l} \times \vec{B}) = d\left[\vec{r} \times (\vec{r} \times \vec{B})\right] - d\vec{l} \times (\vec{r} \times \vec{B})$$ (2)

$$(1) + (2) \implies \vec{r} \times (d\vec{l} \times \vec{B}) = \frac{1}{2} d\left[\vec{r} \times (\vec{r} \times \vec{B})\right] - \frac{1}{2}\vec{B} \times (\vec{r} \times d\vec{l})$$

Torque on a general current loop

The net torque on the loop is therefore,

$$N = I \oint \vec{r} \times (d\vec{l} \times \vec{B}) = \frac{1}{2} I \oint d\Big[\underbrace{\vec{r} \times (\vec{r} \times \vec{B})}_{=0}\Big] - \frac{1}{2} I\vec{B} \oint \times (\vec{r} \times d\vec{l})$$

$$-I\vec{B} \times \frac{1}{2} \oint (\vec{r} \times d\vec{l})$$

$$= -\vec{B} \times I\vec{a} = \vec{m} \times \vec{B}$$

Therefore, the torque on a current carrying loop or arbitrary shape is

$\vec{N} = \vec{m} \times \vec{B}$ (Compare with $\vec{N} = \vec{p} \times \vec{E}$ for electric dipole)

Work done in rotating the dipole:

$$W = \int_{\theta_0}^{\theta} N d\theta' = mB(\cos\theta_0 - \cos\theta)$$

$$= U - U_0 = \triangle U$$

Taking reference position as $\theta_0 = \pi/2$, the potential energy of the dipole is:

$$U = -mB\cos\theta = -\vec{m} \cdot \vec{B}$$

similar to $U = -\vec{p} \cdot \vec{E}$ of an electric dipole in an electric field.

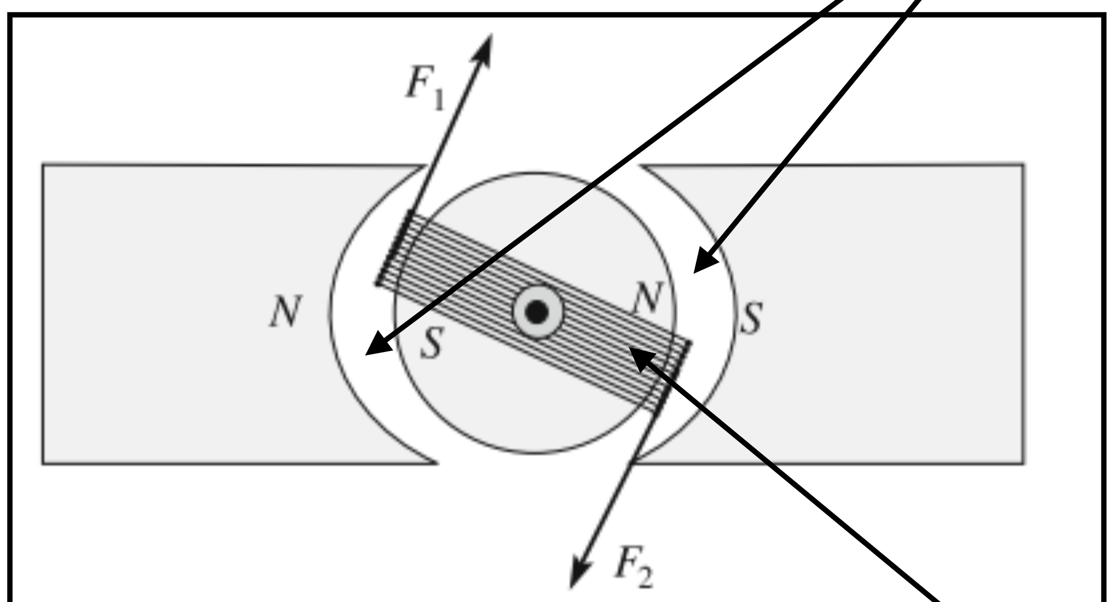
Application: Galvanometer

Radial magnetic field

When current I flows in the coil, there arises a force and a torque.

$$F_1 = F_2 = IaB$$

$$\tau = IBab$$



Equilibrium is reached when torsion moment of the suspension wire balances the torque on the coil.

Coil with N number of turns and sides a, b

Image credit: Springer

Force on Magnetic Dipole

- In a uniform field, the net force on a current loop is zero:

$$\vec{F} = I \oint (d\vec{l} \times \vec{B}) = I \left(\oint d\vec{l} \right) \times \vec{B} = 0$$

as the constant field B can be taken outside the integral.

- In a nonuniform field, the net force is not zero. Consider a circular loop of radius R, current I, suspended above a short solenoid in the fringing region. Here the field has a radial component and a net downward force acts on the loop $F = 2\pi IRB\cos\theta$

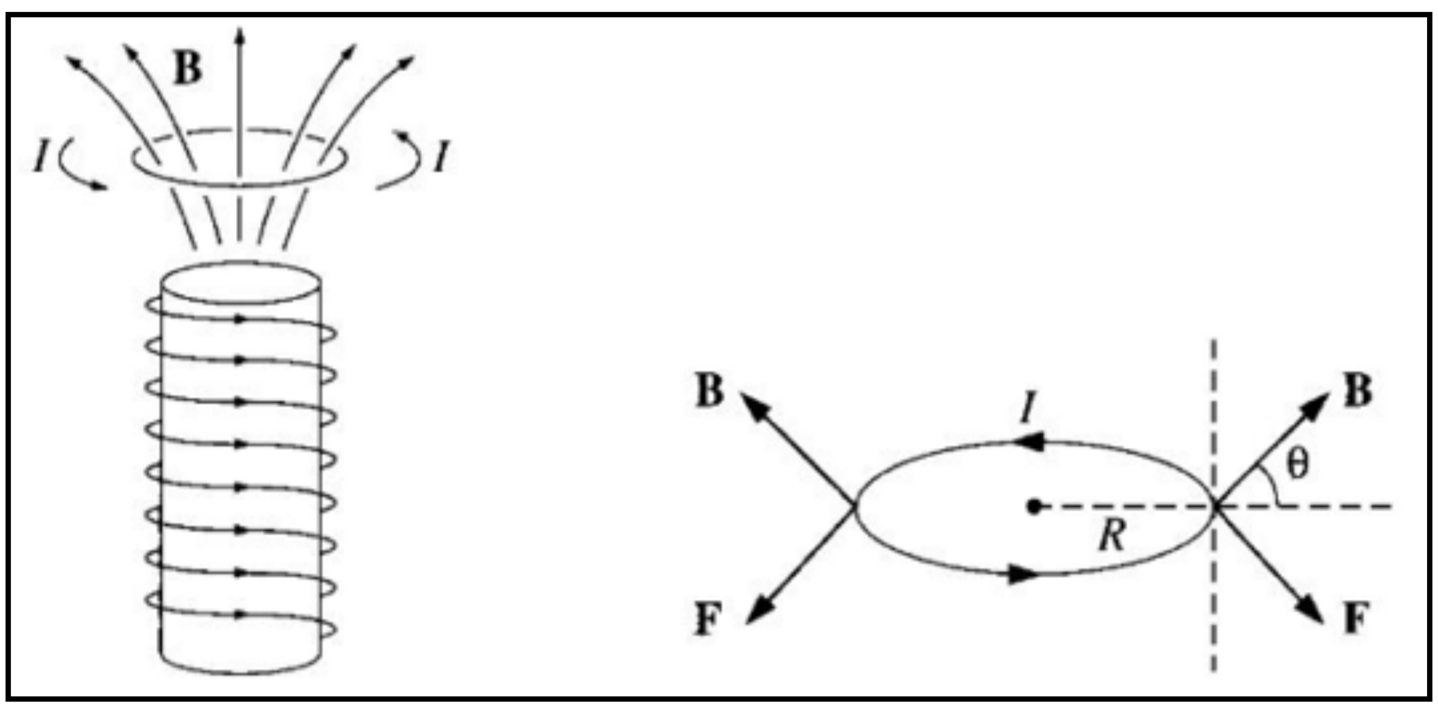


Figure 6.3, Introduction to Electrodynamics, D. J. Griffiths

Force on Magnetic Dipole

For an infinitesimal loop, with dipole moment $\vec{m}$, in a magnetic field $\vec{B}$, the force is $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$

Proof: Assume the dipole to be an infinitesimal square of side e. Choosing the axes as shown in figure, calculate the magnetic force on each of the four sides

$$\vec{F} = I \int (d\vec{l} \times \vec{B})$$

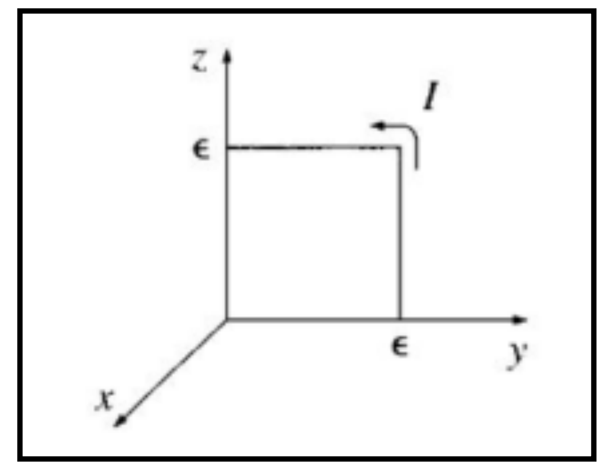


Figure 6.8, Introduction to Electrodynamics, D. J. Griffiths

Force on the elemental square loop is

$$d\vec{F} = I\Big[(dy\hat{y}) \times \vec{B}(0,y,0) + (dz\hat{z}) \times \vec{B}(0,\epsilon,z) - (dy\hat{y}) \times \vec{B}(0,y,\epsilon) - (dz\hat{z}) \times \vec{B}(0,0,z)\Big]$$

$$= I\Big[-(dy\hat{y}) \times \{\vec{B}(0,y,\epsilon) - \vec{B}(0,y,0)\} + (dz\hat{z}) \times \{\vec{B}(0,\epsilon,z) - \vec{B}(0,0,z)\}\Big]$$

Using: $\vec{B}(0,\epsilon,z) \approx \vec{B}(0,0,z) + \epsilon \left.\frac{\partial \vec{B}}{\partial y}\right|_{(0,0,z)}, \vec{B}(0,y,\epsilon) \approx \vec{B}(0,y,0) + \epsilon \left.\frac{\partial \vec{B}}{\partial z}\right|_{(0,y,0)}$

and $$\int dy \frac{\partial \vec{B}}{\partial z}\bigg|_{(0,y,0)} \approx \epsilon \frac{\partial \vec{B}}{\partial z}\bigg|_{(0,0,0)}, \int dz \frac{\partial \vec{B}}{\partial y}\bigg|_{(0,0,z)} \approx \epsilon \frac{\partial \vec{B}}{\partial y}\bigg|_{(0,0,0)}$$ Upto leading order!

we can write the total force as

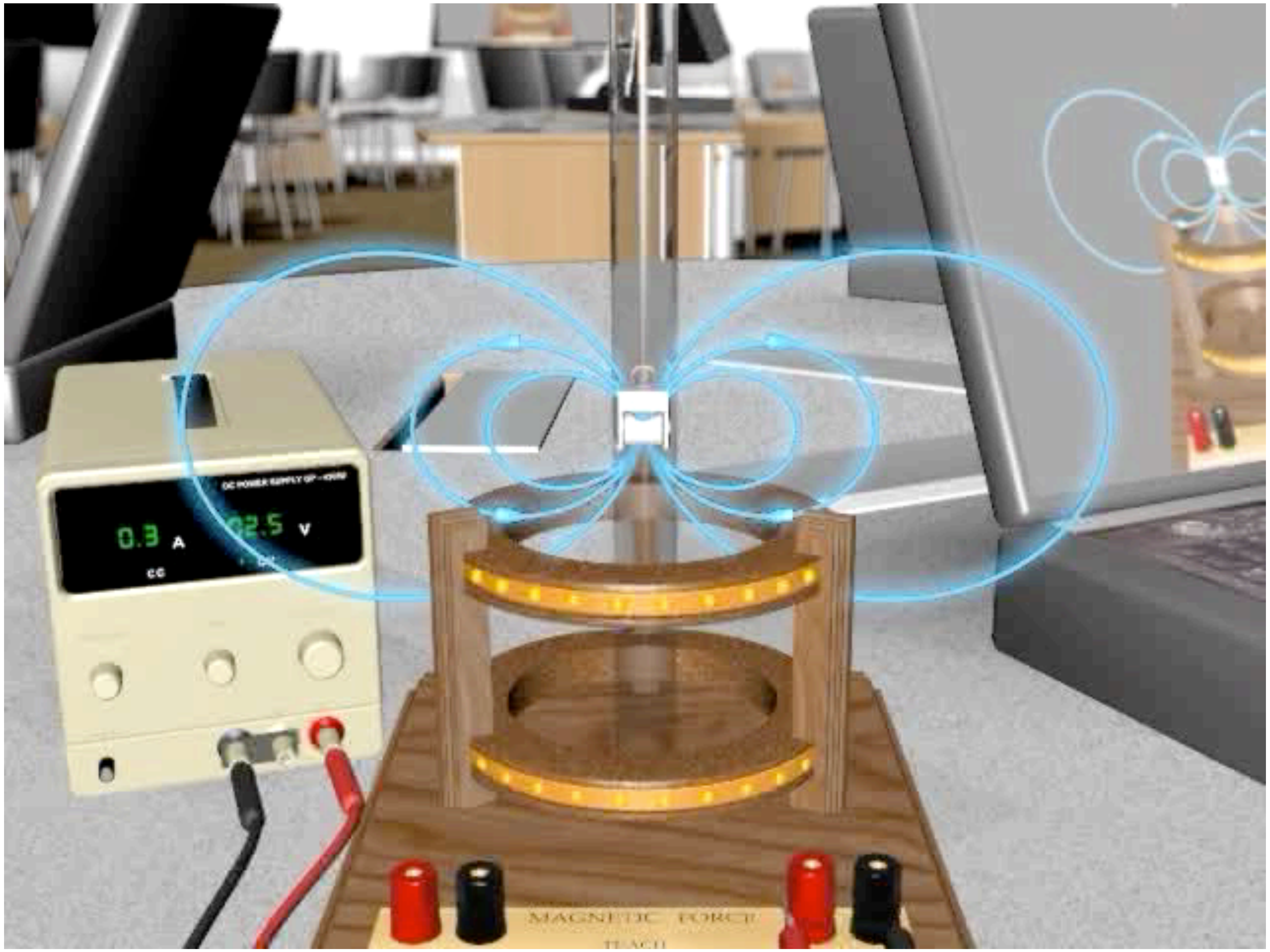
$$\vec{F} = \int d\vec{F} = I\epsilon^2\left[\hat{z} \times \frac{\partial \vec{B}}{\partial y} - \hat{y} \times \frac{\partial \vec{B}}{\partial z}\right]$$
$$= I\epsilon^2\left[\hat{y}\frac{\partial B_x}{\partial y} - \hat{x}\frac{\partial B_y}{\partial y} - \hat{x}\frac{\partial B_z}{\partial z} + \hat{z}\frac{\partial B_x}{\partial z}\right]$$

Using $$\vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \implies \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x}$$

$$\vec{F} = m\left[\hat{y}\frac{\partial B_x}{\partial y} + \hat{x}\frac{\partial B_x}{\partial x} + \hat{z}\frac{\partial B_x}{\partial z}\right] \qquad \vec{m} = m\hat{x} = I\epsilon^2\hat{x}$$

$$\implies \vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

Force on Magnetic Dipole



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