Physics II: Electromagnetism (PH102)

Lecture 2

Subhaditya Bhattacharya
IITG
Email: subhab@iitg.ernet.in, Office No: EVC 14, Phone: 3558

$\int$ Integral calculus of vectors

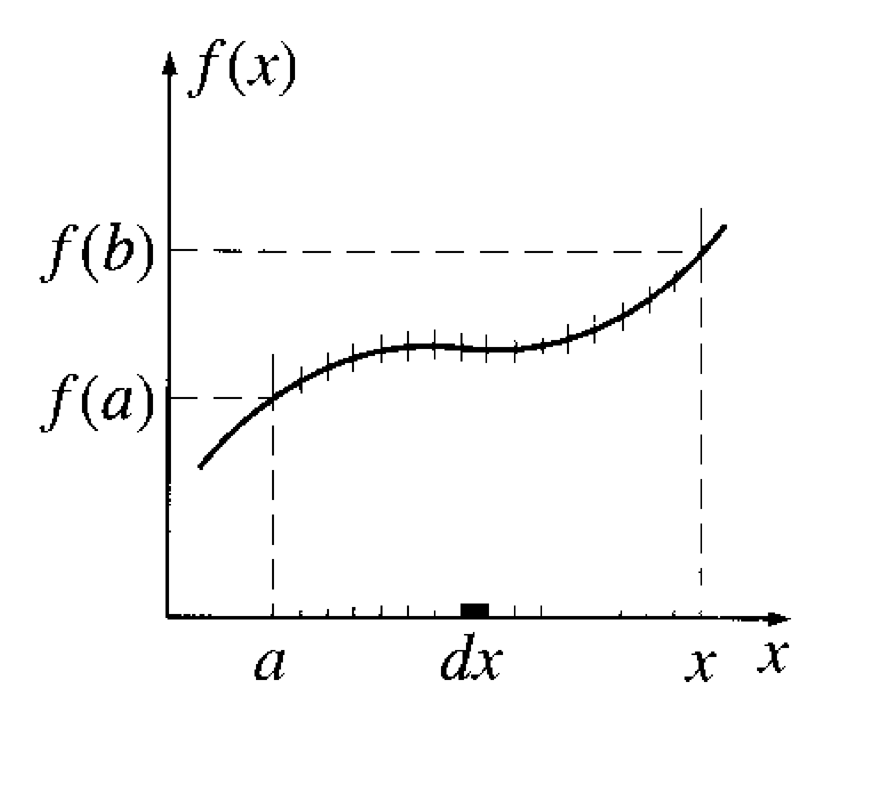
Fundamental Theorem of Integral Calculus

Suppose f(x) is a function of one variable. Then,

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

$$\rightarrow \int_a^b F(x)\, dx = f(b) - f(a)$$

where $F(x) = \dfrac{df}{dx}$



$\int_a^x f(x)dx$ represents the area under the curve $f(x)$ with x axis.

However, for scalar and vector fields, in general the integral involve more than one variable: hence we have more possibilities: Line, surface volume integrals involving vectors

Ordinary integral for vector valued functions

- Simplest possibility : Suppose we have a vector field that depends on a single variable u.

$$\vec{R}(u) = R_1(u)\ \hat{x} + R_2(u)\ \hat{y} + R_3(u)\ \hat{z}$$

We can then integrate over the parameter u:

$$\int \vec{R}(u)du = \hat{x}\int R_1(u)du\ + \hat{y}\int R_2(u)du\ + \hat{z}\int R_3(u)du$$

If $\vec{R}(u) = \frac{d}{du}(\vec{S}(u))$, then

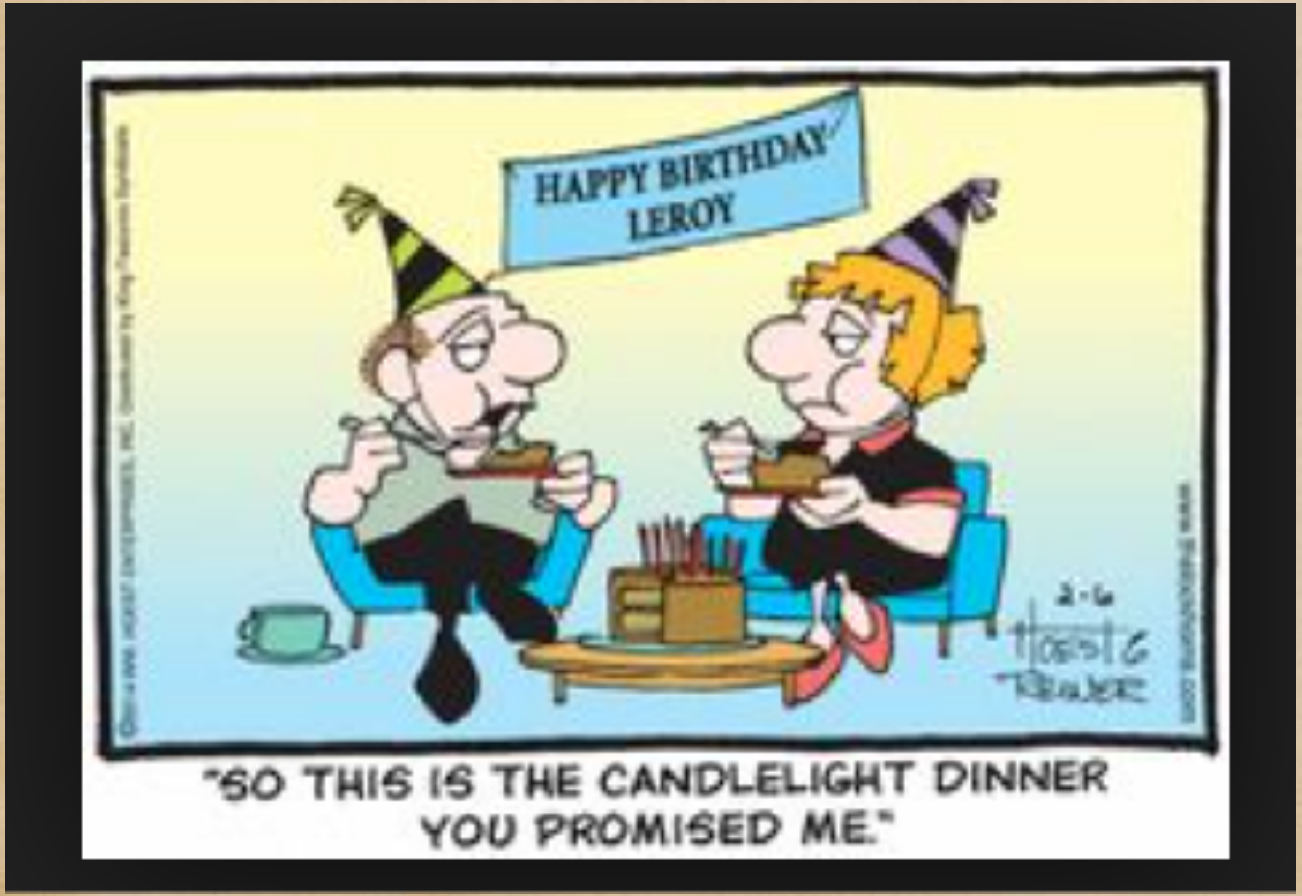
$$\int_a^b \vec{R}(u)du = \int_a^b \frac{d}{du}(\vec{S}(u))du = \vec{S}(b) - \vec{S}(a)$$

A quick example...

Suppose: $\vec{R}(u) = u^2\hat{x} + 2u^3\hat{y} + 5\hat{z}$

$$\int_1^2 \vec{R}(u)du = \hat{x}\int u^2 du + \hat{y}\int 2u^3 du + \hat{z}\int 5du$$

$$= \left[\frac{u^3}{3}\hat{x} + \frac{u^4}{2}\hat{y} + 5u\hat{z}\right]_1^2$$



However line integrals itself may be more complicated if it has to follow a given path

Parametric equations for a path

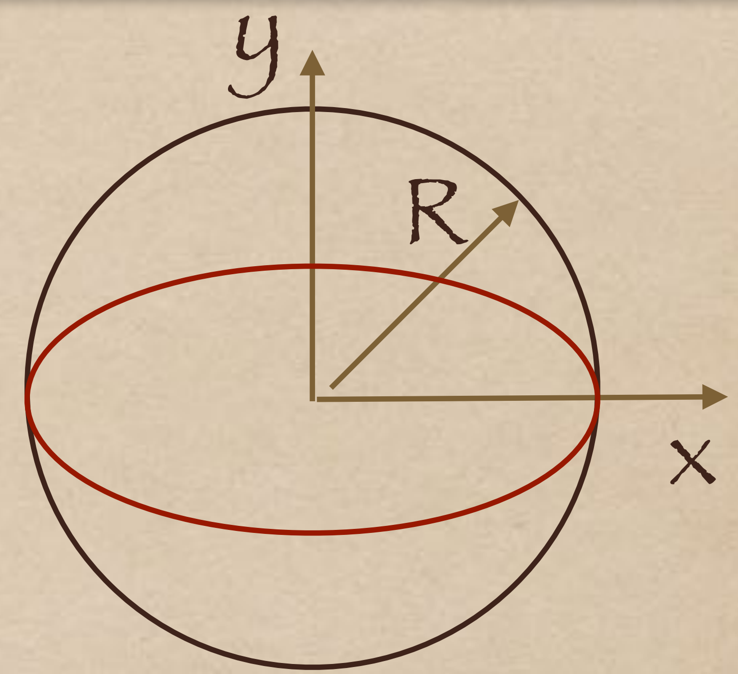
A curve in two dimension plane may be described by one single variable. They are called parametric equations.

Circle : $x^2 + y^2 = R^2$

Parametric equations:

$x = R\cos\theta;$

$y = R\sin\theta \ (0 \leq \theta \leq 2\pi)$



Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Parametric equations:

$x = a\cos t;$

$y = b\sin t \ (0 \leq t \leq 2\pi)$

its extremely hard to parametrise an arbitrary curve

Parametric equations are often helpful to evaluate line, surface or volume integrals along a chosen path or surface or over a volume

Line integral of scalar field

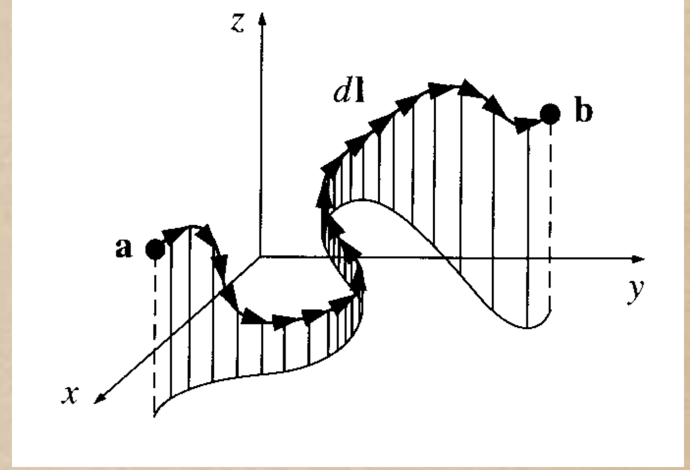
- An integral of the form $\int_C f(x,y,z)dl$

where dl is chosen along a particular curve C parametrised by t

$$\vec{dl}(t) = dx(t)\hat{x} + dy(t)\hat{y} + dz(t)\hat{z}$$

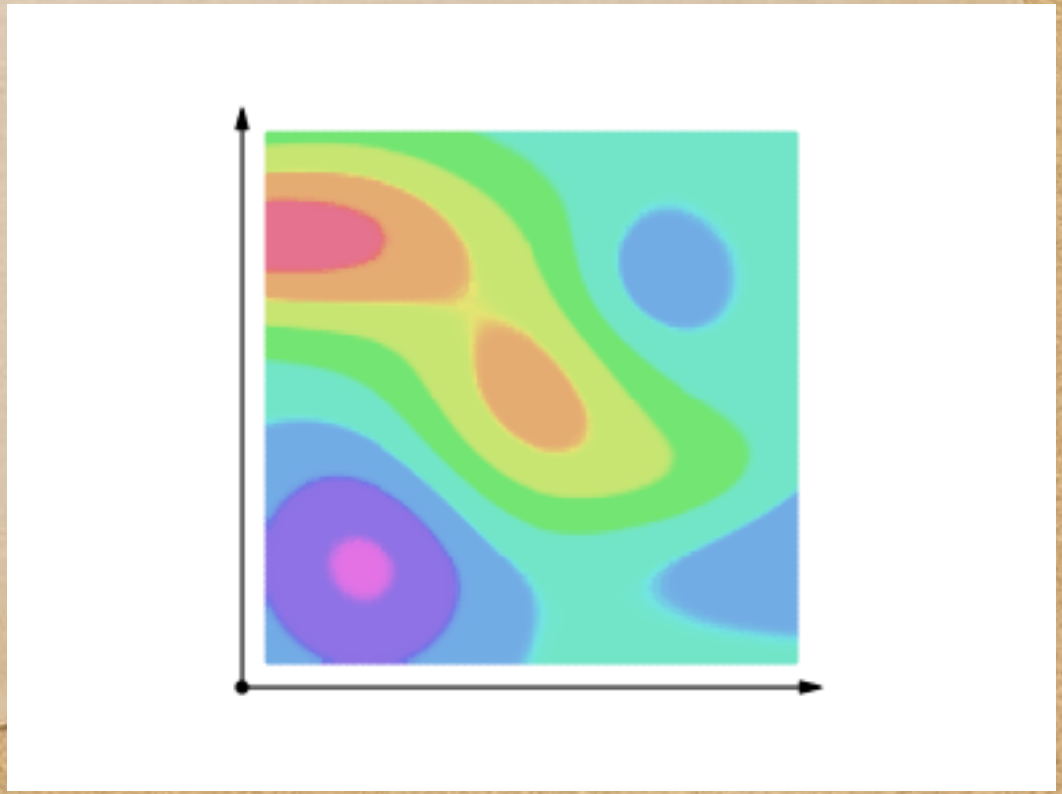
$$= (\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z})dt = \vec{r}'(t)dt$$

$$dl = |\vec{dl}| = |\vec{r}'(t)|dt$$



where $|\vec{r}'(t)| = \left[(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2\right]^{1/2}$

$$\int_C f\,dl = \int_C f(\vec{r}(t))|\vec{r}'(t)|dt$$

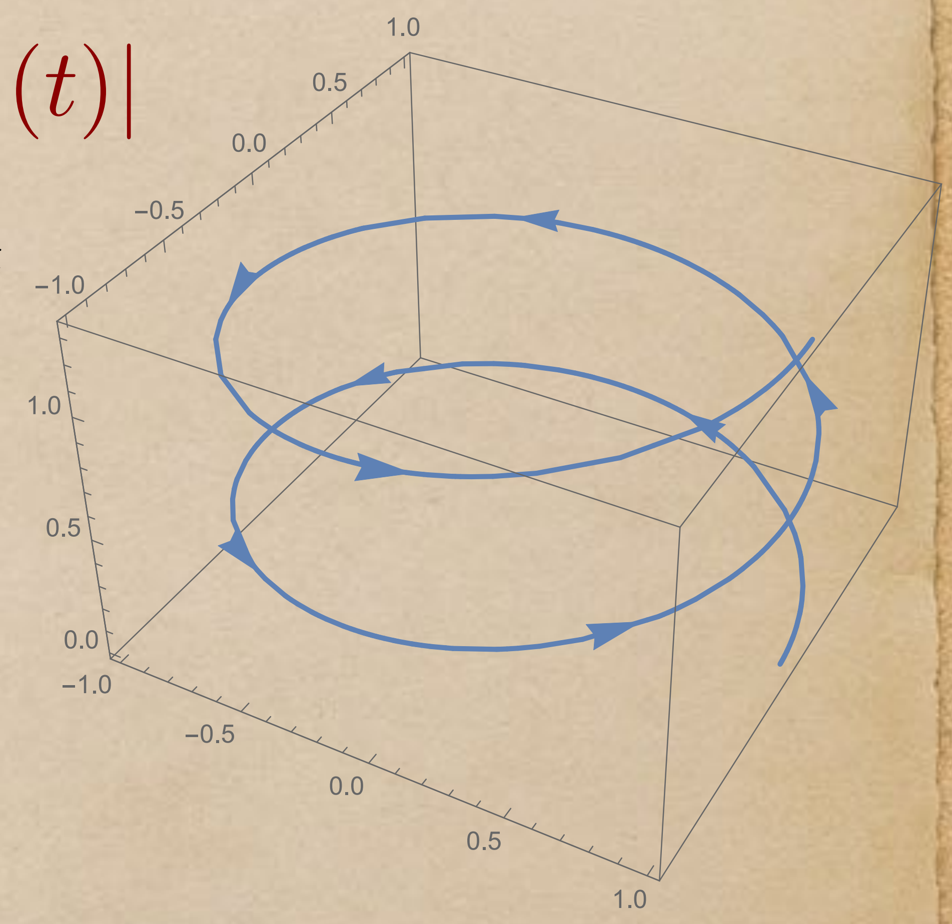


Example.....

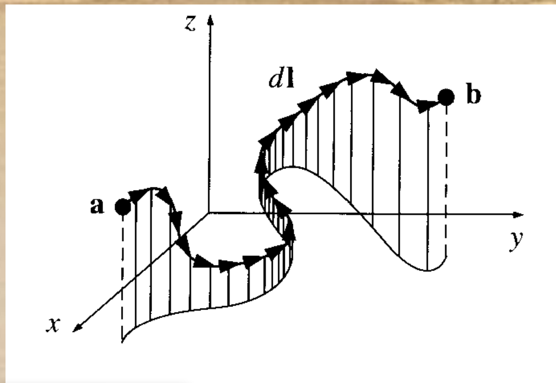
Evaluate $\int_C xyz\ dr$ where C is the helix given by $\vec{r}(t) = (\cos t,\ \sin t,\ 3t)$, $0 \le t \le 4\pi$.

$|\vec{r}'(t)|$

$$\begin{aligned}
\int_C xyz\ dr &= \int_0^{4\pi} 3t\cos t\sin t\sqrt{\sin^2 t + \cos^2 t + 9}\,dt \\
&= \int_0^{4\pi} 3t\left(\frac{1}{2}\sin 2t\right)\sqrt{1+9}dt \\
&= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t\sin 2t dt \\
&= \frac{3\sqrt{10}}{2}\left(\frac{1}{4}\sin 2t - \frac{t}{2}\cos 2t\right)\Bigg|_0^{4\pi} \\
&= -3\sqrt{10}\pi
\end{aligned}$$



Possible Line Integrals



- Four possibilities:

$\int_C \phi \; dl$	Line integral of a scalar field	Scalar
$\int_C \phi \; \vec{dl}$	Line integral of a scalar field	Vector
$\int_C \vec{V} . \; \vec{dl}$	Line integral of vector fields	Scalar
$\int_C \vec{V} \times \; \vec{dl}$	Line integral of vector fields	Vector

C represents a closed or open contour

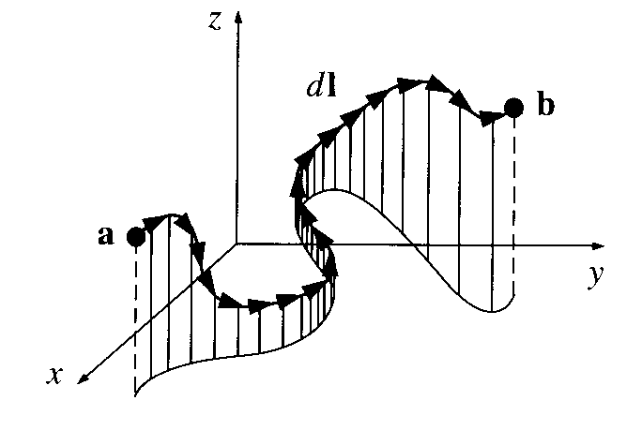
$$\vec{dl} = \hat{x} \; dx + \hat{y} \; dy + \hat{z} \; dz$$

Line Integral of Vector fields

We need to integrate a vector field along path C

Vector field: $\vec{v} = v_x \ \hat{x} + v_y \ \hat{y} + v_z \ \hat{z}$

Curve C: $\vec{r}(u) = x(u) \ \hat{x} + y(u) \ \hat{y} + z(u) \ \hat{z}$



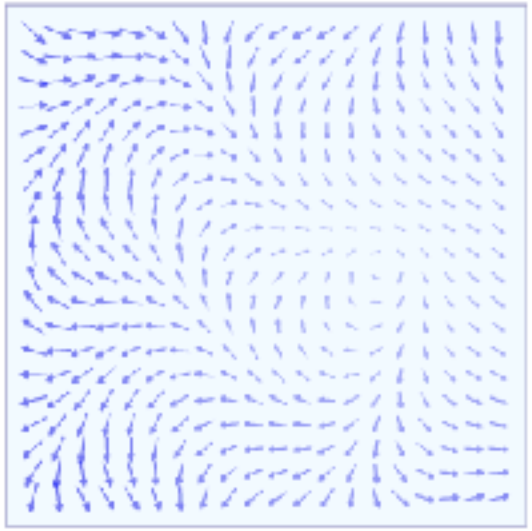
The line integral is of the form

$$\int_a^b \vec{v}.\vec{dl} = \int_a^b v_x dx + \int_a^b v_y dy + \int_a^b v_z dz$$

Line integral may or may not depend on the path taken

Example: Work done

$$W = \int_a^b \vec{F}.\vec{dl}$$



If the path is closed, then line integral is denoted by

$$\oint \vec{v}.\vec{dl}$$

$\int_C \vec{v} \times \vec{dl}$ lacks physical significance

Line integral of vector fields using parametric equation

$$\int_C \vec{F}.d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)).\vec{r}\,'(t)\ dt$$

Ex: A force field is given by $\vec{F}(x,y,z) = 8x^2yz\ \hat{x} + 5z\ \hat{y} - 4xy\ \hat{z}$. Find the work done in moving a particle along a curve parametrised by (t, t^2, t^3); $0 \le t \le 1$.

$$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\hat{x} + 5t^3\hat{y} - 4t(t^2)\hat{z} = 8t^7\hat{x} + 5t^3\hat{y} - 4t^3\,\hat{z}$$

Parametric equations of the path

$$\Longrightarrow \boxed{\begin{array}{rcl} \vec{r}(t) & = & t\hat{x} + t^2\hat{y} + t^3\,\hat{z} \\ \vec{r}\,'(t) & = & \hat{x} + 2t\hat{y} + 3t^2\,\hat{z} \end{array}}$$

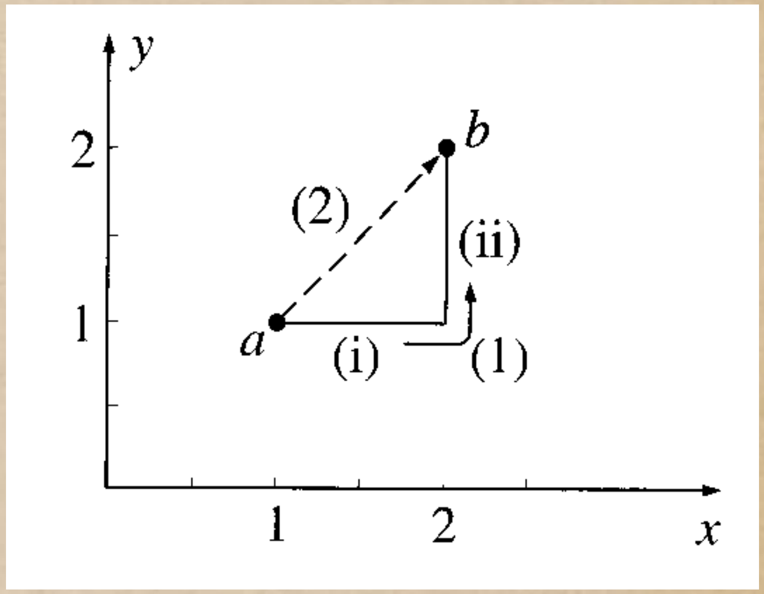
Work done :

$$\begin{aligned} \int_C \vec{F}.d\vec{r} &= \int_0^1 8t^7 + 10t^4 - 12t^5\ dt \\ &= (t^8 + 2t^5 - 2t^6)\Big|_0^1 \\ &= 1 \end{aligned}$$

An illustrious example of line integral of vector

Line integral of $\vec{V} = y^2\hat{x} + 2x(y+1)\hat{y}$ from a to b along (1) and (2)

$$\vec{dl} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$



Along path (1): (i) dy=dz=0, (ii) dx=dz=0

$$(i) \int \vec{V}.\vec{dl} = \int_{(1,1,0)}^{(2,1,0)} V_x dx = 1$$

$$(ii) \int \vec{V}.\vec{dl} = \int_{(2,1,0)}^{(2,2,0)} V_y dy = \int_{(2,1,0)}^{(2,2,0)} 4(y+1)dy = 10$$

Hence, Along path (1): (i)+(ii)=11

The method of line integral depends on the choice of path

Along path (2): y=x, dy=dx and dz=0

$$\int_{x=1}^{x=2} \vec{V}.\vec{dl} = \int_{x=1}^{x=2} (3x^2 + 2x)dx = 10$$

In closed loop, we usually move counterclockwise

$$\oint \vec{V}.\vec{dl} = \int_1 \vec{V}.\vec{dl} - \int_2 \vec{V}.\vec{dl} = 11 - 10 = 1$$

Line integral of conservative vector field

A vector field $\vec{F}(x,y,z)$ is <u>conservative</u>, if $\vec{\nabla}\times\vec{F}=0$.

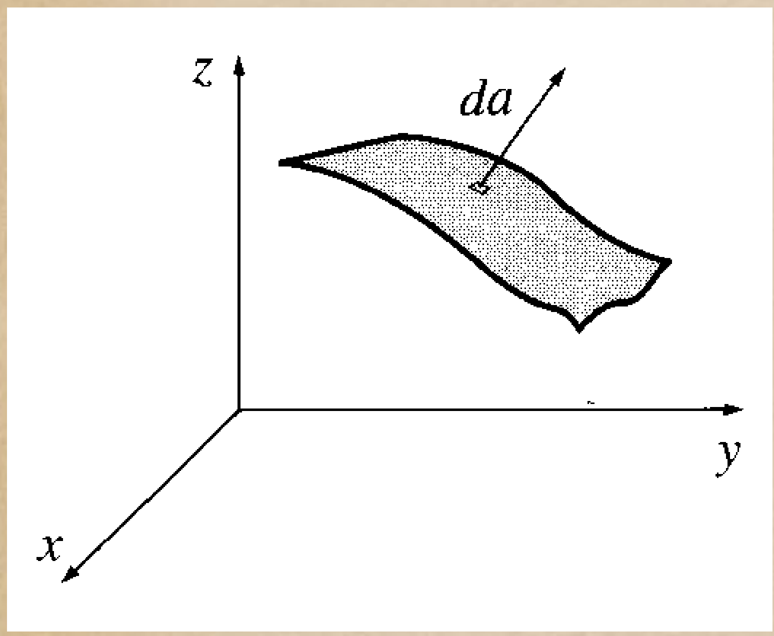
$$\vec{F}=\vec{\nabla}\phi \qquad \text{Thanks to: } \vec{\nabla}\times(\vec{\nabla}\phi)=0$$

Then,
$$\begin{aligned}\int_a^b \vec{F}.d\vec{r} &= \int_a^b \vec{\nabla}\phi.d\vec{r}=\int_a^b\left(\hat{x}\frac{\partial\phi}{\partial x}+\hat{y}\frac{\partial\phi}{\partial y}+\hat{z}\frac{\partial\phi}{\partial z}\right).(dx\hat{x}+dy\hat{y}+dz\hat{z})\\ &= \int_a^b\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right)\\ &= \int_a^b d\phi=\phi(b)-\phi(a)\end{aligned}$$

$$\boxed{\oint \vec{F}.\vec{dr}=0}$$

If the vector field can be expressed as a gradient of a scalar field then the line integral is independent of the path chosen and only depends on end points.

Surface Integral



Infinitesimal patch of area : $\vec{da}$

Direction of area vector is perpendicular to surface.

Closed surface: Outward normal is positive

Open Surface: Any direction perpendicular may be chosen

Form of surface integral: $\int_S \vec{V}.\vec{da}$ $\qquad$ $\vec{da} = dx\ dy\ \hat{n}$

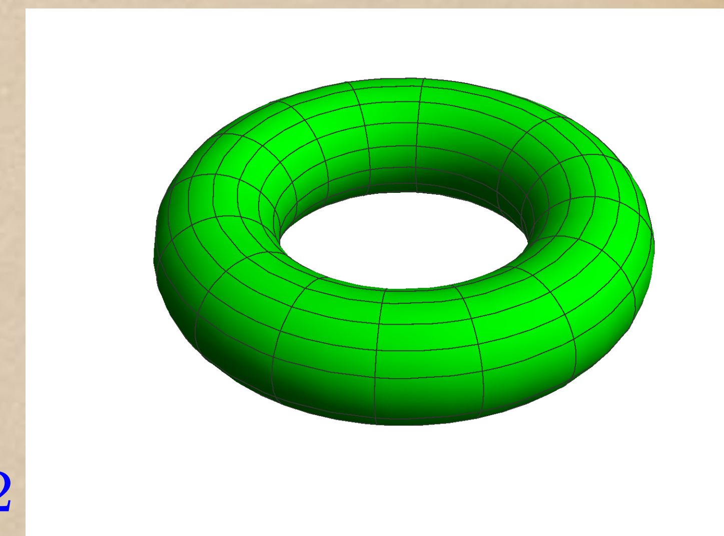
For a closed surface : $\oint \vec{V}.\vec{da}$ $\qquad$ $\hat{n}$ is the unit normal

If vector represents flow of a fluid (mass per unit area per unit time) then surface integral represents total mass per unit time passing through the surface: FLUX

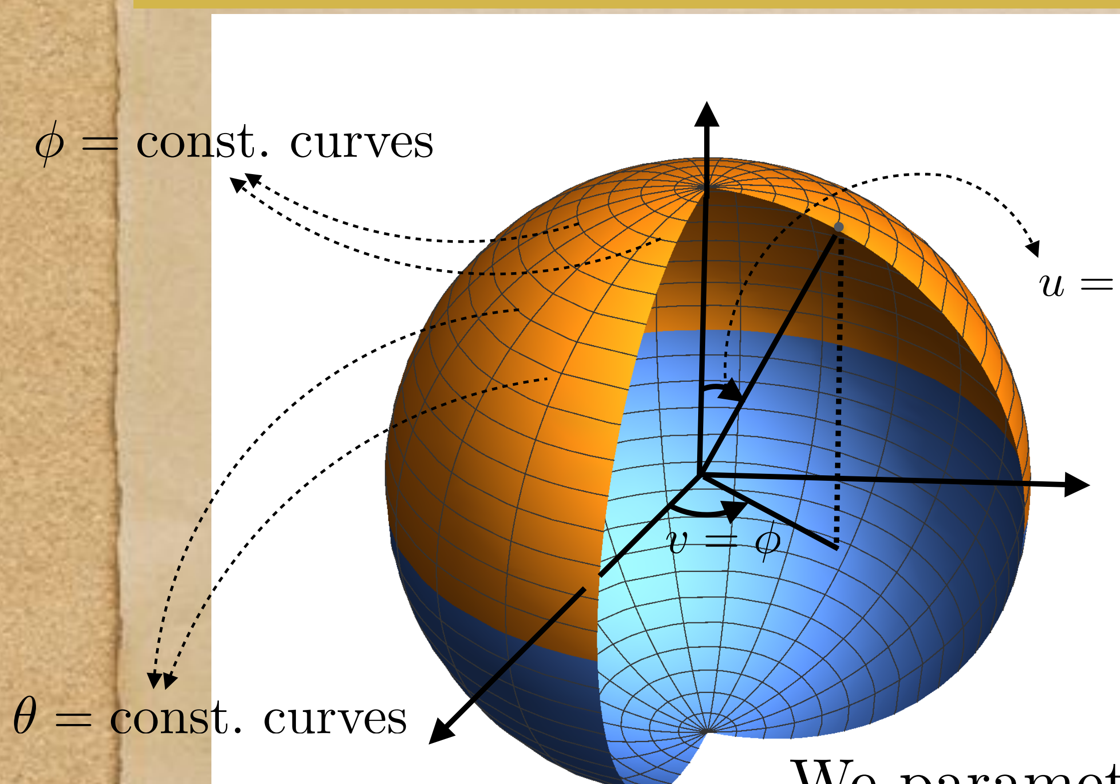
How to represent a surface ?

$z = f(x,y)$ or $x = f(y,z)$ or $u = f(x,z)$ is one of the standard form to represent surfaces.

$$z^2 = a^2 - (c - \sqrt{x^2+y^2})^2 \text{ is a torus}$$



Another way to represent: $f(x,y,z) = \text{constant}$



Sphere :

$$x^2 + y^2 + z^2 = a^2$$

Parametric representation of surface:

$$\vec{S}(u,v) = (x(u,v), y(u,v), z(u,v))$$

two real valued functions

We parametrise the sphere as

$$\vec{S}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta);\ 0 \le \theta \le \pi,$$

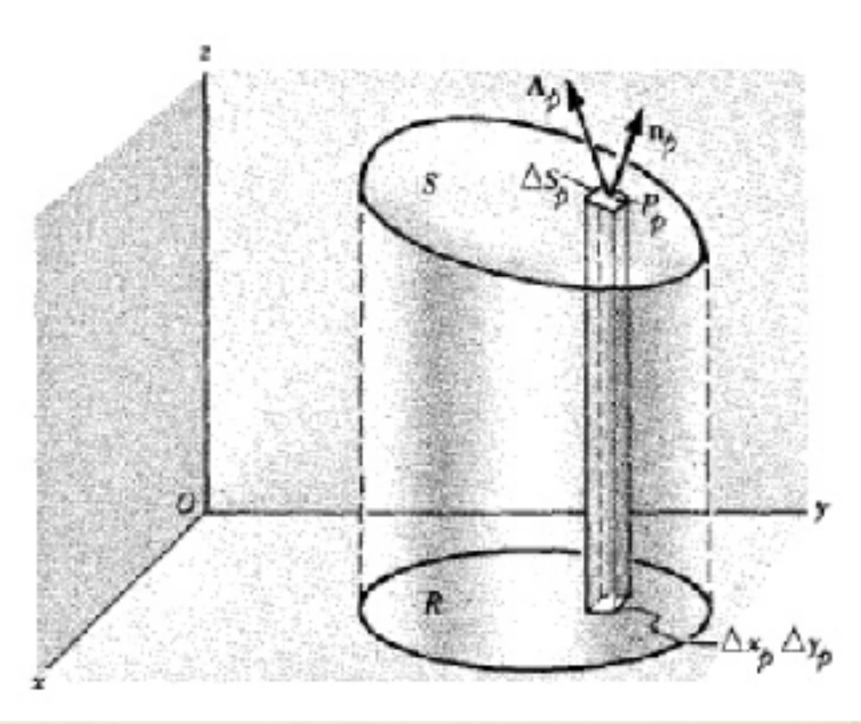
⋆ For $\theta = \frac{\pi}{2}$ (i.e. const.), $\vec{S}(\theta = \frac{\pi}{2}, \phi) = (\cos\phi, \sin\phi, 0) \implies$ Circle (lattitude)

⋆ For $\phi = \frac{\pi}{2}$ (i.e. const.), $\vec{S}(\theta, \phi = \frac{\pi}{2}) = (0, \sin\theta, \cos\theta) \implies$ Circle (longitude)

How to evaluate surface integral ?

- Let S be a smooth surface as z=f(x,y)
- Let $\hat{n}$ be the unit vector perp to an elementary area dS
- Project it on xy plane where the area is dx dy

$$\int_S \vec{A}.\hat{n}\ dS = \sum \vec{A}_p.\hat{n}_p\ \Delta S_p$$



- The projection of ΔS_p on xy plane is

$$|\hat{n}_p \Delta S_p.\hat{z}| = |\hat{n}_p.\hat{z}|\Delta S_p = \Delta x_p \Delta y_p$$

Hence, $dxdy = |\hat{n}.\hat{z}|dS$

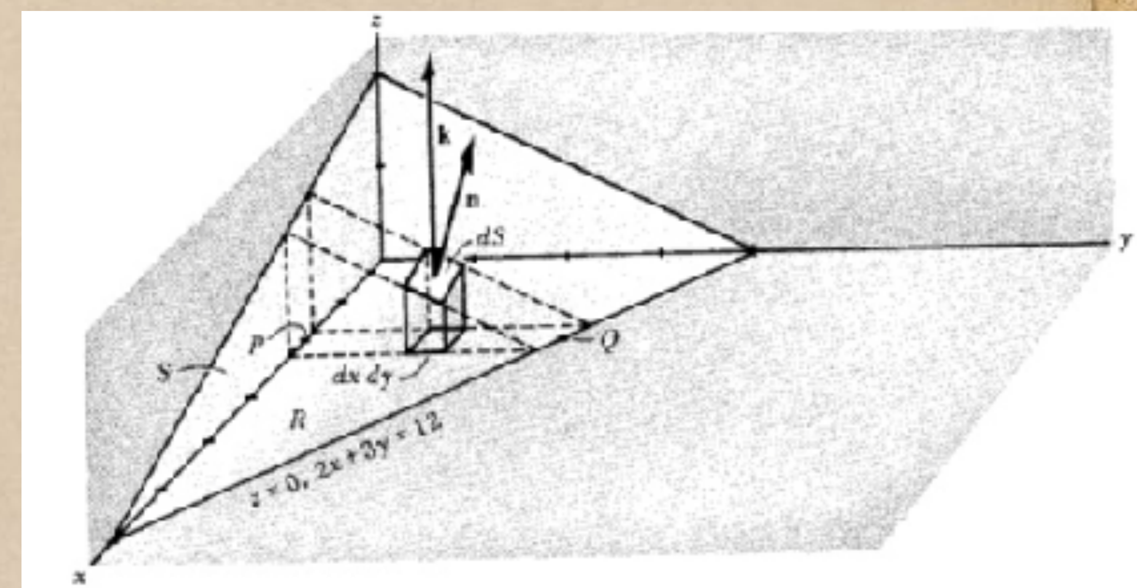
$$\Delta S_p = \frac{\Delta x_p \Delta y_p}{|\hat{n}_p.\hat{z}|} \longrightarrow dS = \frac{dxdy}{|\hat{n}.\hat{z}|}$$

$$\longrightarrow \int_S \vec{A}.\hat{n}\ dS = \int_R \vec{A}.\hat{n}\ \frac{dxdy}{|\hat{n}.\hat{z}|}$$

An example of surface integral

Evaluate $\int_S \vec{V}.\vec{da}$ where $\mathcal{S}$ is that part of the plane 2x+3y+6z=12 which is located in the first octant

$$\vec{V} = 18z\hat{x} - 12\hat{y} + 3y\hat{z}$$



- Recall perpendicular to the surface 2x+3y+6z=12 is obtained by

$$\vec{\nabla}(2x + 3y + 6z) \qquad \hat{n} = \frac{2\hat{x} + 3\hat{y} + 6\hat{z}}{\sqrt{2^2 + 3^2 + 6^2}}$$

$$\longrightarrow \frac{dxdy}{|\hat{n}.\hat{z}|} = \frac{7}{6}dxdy$$

$$\vec{V}.\hat{n} = \frac{36z - 36 + 18y}{7} = \frac{36 - 12x}{7}$$

using $\boxed{z = (12 - 2x - 3y)/6}$

$$\int_S \vec{V}.\hat{n}\; da = \int_R (6 - 2x)\; dxdy = \int_{x=0}^{6} \int_{y=0}^{(12-2x)/3} (6 - 2x)\; dydx = 24$$

$$da = \frac{dxdy}{|\hat{n}.\hat{z}|}$$

keep x fixed and integrate over y

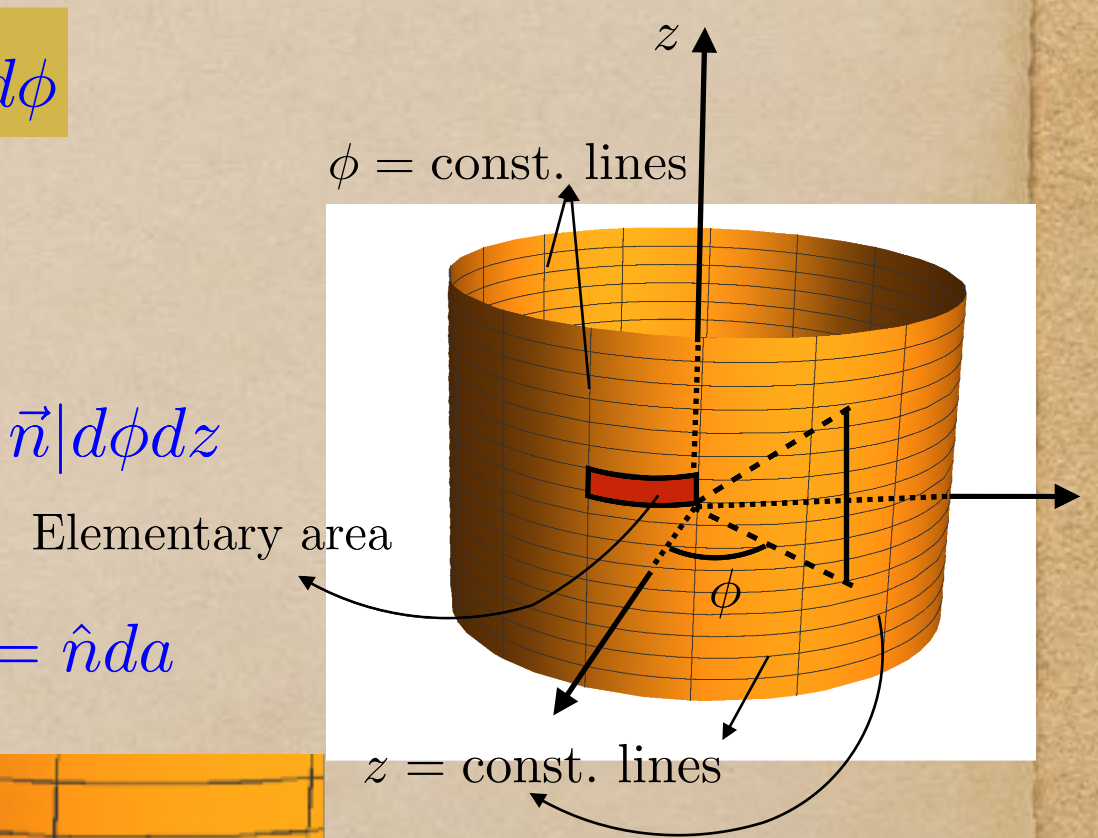
Example: Parametrised Surface

Suppose we have a cylinder of radius $R = 3$ units and parametrised by ϕ, z.

$$\vec{r}(\phi, z) = (3\cos\phi, 3\sin\phi, z);\ 0 \le \phi \le 2\pi, \quad -2 \le z \le 2.$$

Take a point on the cylinder at $\phi = 0, z = 1$ and then $\vec{r}(\phi = 0, z = 1) = (3, 0, 1)$

Then $\overrightarrow{AB} = (\partial\vec{r}/\partial z)dz$ and $\overrightarrow{AC} = (\partial\vec{r}/\partial\phi)d\phi$

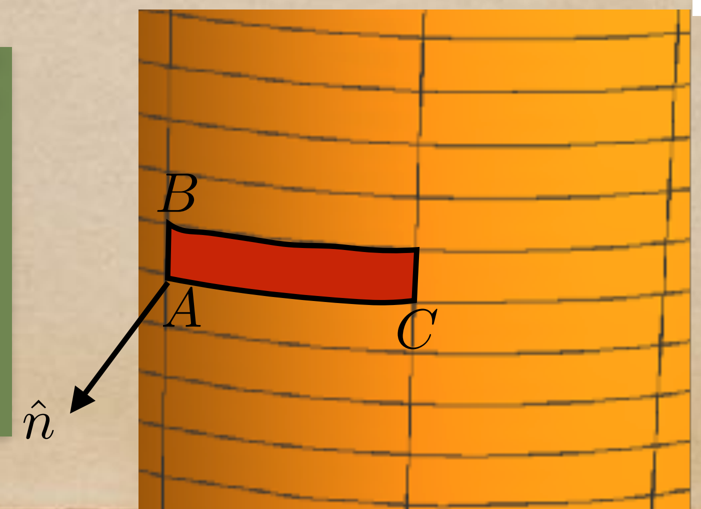


Normal vector at A: $\vec{n} = \left(\frac{\partial\vec{r}}{\partial\phi}\right) \times \left(\frac{\partial\vec{r}}{\partial z}\right)$

Scalar area element $\quad da = |\overrightarrow{AC} \times \overrightarrow{AB}| = |\vec{n}|d\phi dz$

Elementary vector area $\quad d\vec{a} = \left(\frac{\vec{n}}{|\vec{n}|}\right) da = \hat{n}da$

Then I can evaluate integral of a given vector field following previous strategy



Volume Integral of scalar or vector fields

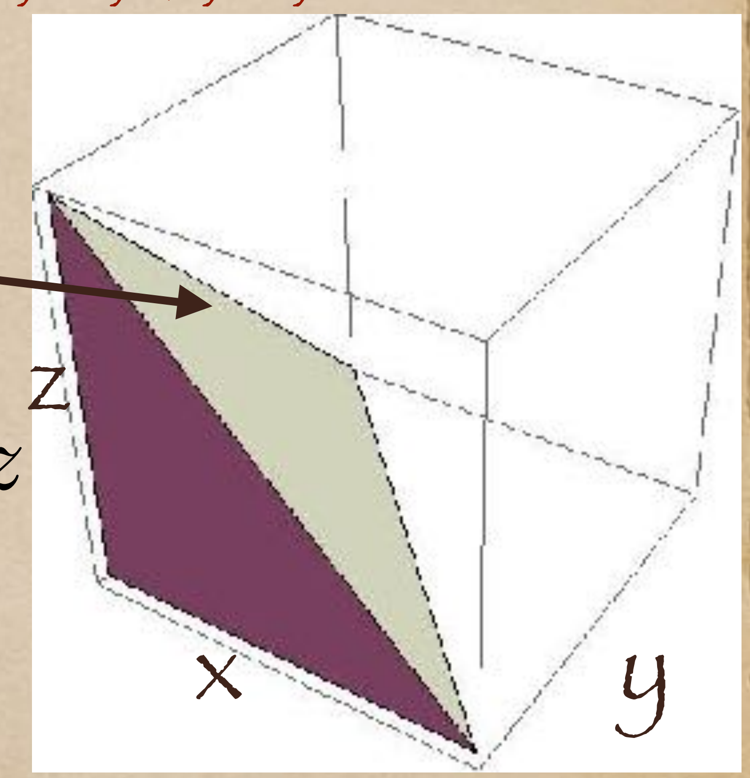
$$\int_{\mathcal{V}} \vec{V} d\tau = \int_{\mathcal{V}} (V_x \hat{x} + V_y \hat{y} + V_z \hat{z}) dxdydz$$

$$\int_{\mathcal{V}} \phi(x, y, z) \ d\tau = \int_{\mathcal{V}} \phi(x, y, z) \ dxdydz$$

Volume integral element is a scalar quantity. Hence, volume integral can be done on both scalar and vector functions.

An example of volume integral

Calculate volume integral of $T = z^2$ over tetrahedron with corners at (0,0,0), (0,1,0), (0,0,1), (1,0,0).



Sloping surface is x+y+z=1

$$\int z^2 \; dxdydz = \int z^2 \Big[\int \Big(\int dx \Big) dy \Big] dz$$

$$\int_0^{1-y-z} dx = 1 - y - z$$

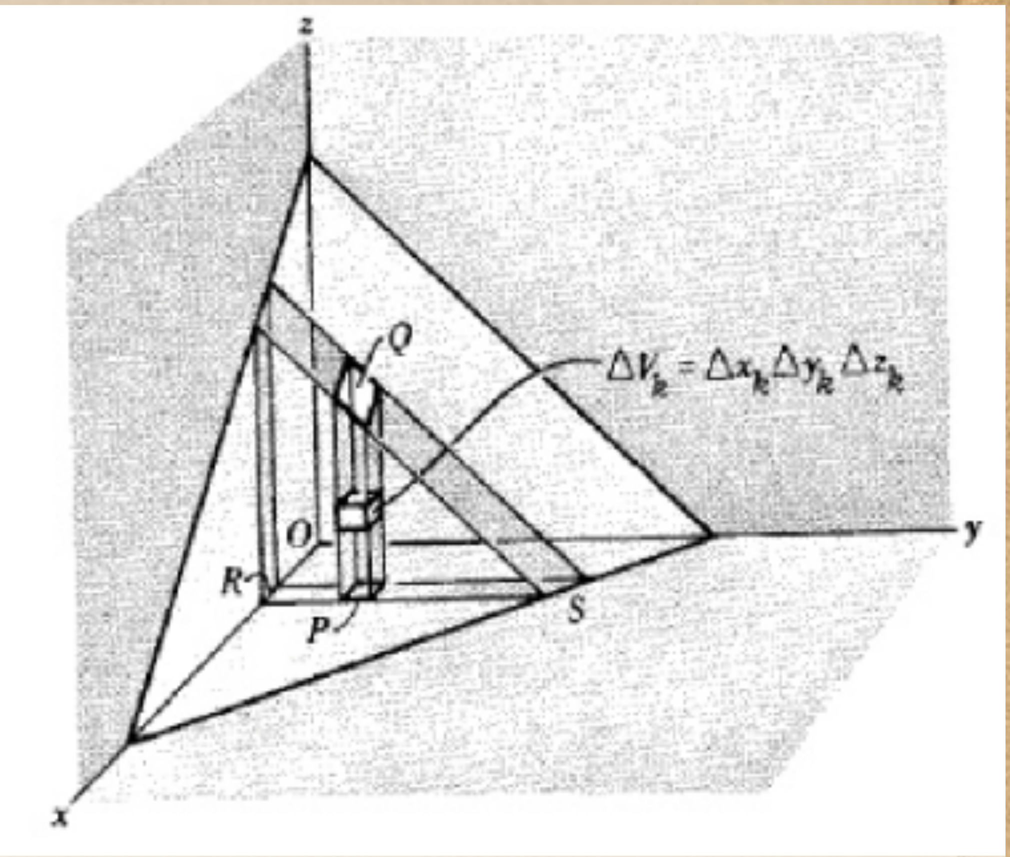
$$\int_0^{1-z} (1-y-z)dy = 1/2 - z + z^2/2$$

$$\int_0^1 (1/2 - z + z^2/2) z^2 dz = 1/60$$

Another example...

Suppose, we need to integrate over a volume bounded by 4x+2y+z=8 plane with x=0, y=0 and z=0 planes for a scalar field $\phi = 45x^2y$

- Choose x, y constant and integrate over z upto Q: z=8-4x-2y
- Next integrate over y upto y=4-2x
- Then integrate over x from 0 to 2



$$\begin{aligned}
\int_{\mathcal{V}} \phi d\tau &= \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2 y dz dy dx \\
&= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2 y(8-4x-2y) dy dx \\
&= 45\int_{x=0}^{2} \frac{8}{3} x^2 (2-x)^3 dx \\
&= 120\int_{x=0}^{2} x^2(2-x)^3 dx \\
&= 120 \times \frac{16}{15} = 128
\end{aligned}$$

In summary...

- Concept of line integral: $\int_a^b \vec{v}.\vec{dl} = \int_a^b v_x dx + \int_a^b v_y dy + \int_a^b v_z dz$
- One can also use parametric form $\int_C \vec{F}.d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)).\vec{r}\,'(t)\ dt$
- One can also evaluate $\int \phi \vec{dr}$ $\int \vec{F} \times \vec{dr}$
- concept of representing a surface z=f(x,y)
- Surface integral of vector $\int_S \vec{A}.\hat{n}\ dS = \int_R \vec{A}.\hat{n}\ \frac{dxdy}{|\hat{n}.\hat{z}|}$
- Surface integral (parametrized by u, v) of scalar field

$$\int_S f da = \int_R f(\vec{r}(u,v)) |\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}| dudv$$

- Surface integral (parametrised by u,v) of vector fields

$$\int_S \vec{A}(\vec{r}(u,v)).\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| \hat{n} dudv$$

- Volume integral is scalar and hence can be done for scalar or vector fields $\int_\mathcal{V} \phi(x,y,z)\ d\tau$ $\int_\mathcal{V} \vec{V}(x,y,z)\ d\tau$