Note:

- The main purpose of the tutorial is to provide you with an opportunity to interact with a teacher.
- The teacher will assist you in clearing your doubts and answer your queries regarding the topics covered in lectures.
- This problem sheet is given to you for your practice. You are expected to attempt these problems before you come to the tutorial class.
- In case you find a problem very difficult, do ask your teacher to help you. The teacher will not solve all problems in the class but just provide you with enough hints. Solutions will be uploaded on Moodle course.
- These problems also indicate the difficulty level of the exams.

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1. [**G1.11**] Find the gradients of the following functions:

   (a) $f(x, y, z) = x^2 + y^3 + z^4$.
   (b) $f(x, y, z) = x^2y^2z^4$.
   (c) $f(x, y, z) = e^x \sin(y) \ln(z)$.
   (d) $f(x, y, z) = r^n$ where $r = \sqrt{x^2 + y^2 + z^2}$.

   Solution:

   (a) $\nabla f = 2x\hat{\mathbf{x}} + 3y^2\hat{\mathbf{y}} + 4z^3\hat{\mathbf{z}}$
   (b) $\nabla f = 2xy^2z^4\hat{\mathbf{x}} + 2x^2yz^4\hat{\mathbf{y}} + 4x^2y^2z^3\hat{\mathbf{z}}$
   (c) $\nabla f = e^x \sin(y) \ln(z)\hat{\mathbf{x}} + e^x \cos(y) \ln(z)\hat{\mathbf{y}} + \frac{e^x \cos(y)}{z}\hat{\mathbf{z}}$
   (d) Since $\frac{\partial}{\partial x} r^n = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r}$. Thus $\nabla f = nr^{n-2}\mathbf{r}$

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2. [**G1.12**] The height of a certain hill (in feet) is given by

$$h(x, y) = 10\left(2xy - 3x^2 - 4y^2 - 18x + 28y + 12\right)$$

   where y is the distance (in miles) north, x the distance east of South Hadley.

   (a) Where is the top of the hill located?
   (b) How high is the hill?
   (c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In which direction is the slope steepest at that point?

   Solution:

   Gradient of h

$$\nabla h(x, y) = 10(2y - 6x - 18)\hat{\mathbf{x}} + 10\,(2x - 8y + 28)\,\hat{\mathbf{y}}$$

   (a) Say, hill top is located at (x^*, y^*). Then $\nabla h(x^*, y^*) = 0$. This gives

$$\left.\begin{aligned} 2y^* - 6x^* - 18 = 0 \\ 2x^* - 8y^* + 28 = 0 \end{aligned}\right\} \begin{aligned} x^* = -2 \\ y^* = 3 \end{aligned}$$

   The hilltop is at 3 miles north and 2 miles west of South Hadley.

(b) $h(-2,3) = 720\text{ft}$.

(c) Putting in $x = 1, y = 1$, we get $\nabla h(1,1) = 220(-\hat{\mathbf{x}} + \hat{\mathbf{y}})$. Thus the slope is $220\sqrt{2}$ ft/mile and direction is northwest.

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3. **[G1.13]** Let (x_0, y_0, z_0) be a fixed point. Let $\mathbf{R} = \hat{\mathbf{x}}(x - x_0) + \hat{\mathbf{y}}(y - y_0) + \hat{\mathbf{z}}(z - z_0)$ and $R = |\mathbf{R}|$

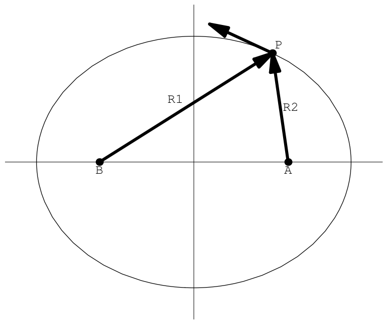

Figure 1: Problem 6

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(a) Show that $\nabla R = \mathbf{R}/R$.

(b) The figure shows an ellipse with foci at points A and B. Let P be a point on the ellipse. Show that lines AP and BP make equal angles with the tangent to the ellipse at P. [Hint: Use the fact that $R_1 + R_2 = \text{Constant}$.]

Solutions:

(a) $R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$, then

$$\frac{\partial R}{\partial x} = \frac{1}{2} \frac{2(x - x_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = \frac{\mathbf{R} \cdot \hat{\mathbf{x}}}{R}$$

Then $\nabla R = \mathbf{R}/R$. Notice that ∇R is a unit vector in the direction of $\mathbf{R}$.

(b) The ellipse is a level curve of function $f(x,y) = R_1 + R_2$, thus the gradient must be $\perp$ to the unit tangent vector $\mathbf{t}$ to the ellipse at P. So, $\mathbf{t} \cdot \nabla(R_1 + R_2) = 0 \implies \mathbf{t} \cdot \hat{\mathbf{R}}_1 = -\mathbf{t} \cdot \hat{\mathbf{R}}_2$. Hence lines AP and BP make equal angles with the tangent to the ellipse at P. This means that if ellipse has reflecting surface, all rays emitted from one focus would all converge to the other focus.

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4. **[G1.15]** Calculate the divergence of the following vector functions:

(a) $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$

(b) $\mathbf{v} = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}$

(c) $\mathbf{v} = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$

Solutions:

(a) $\nabla \cdot \left(x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}\right) = 0$

(b) $\nabla \cdot (xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}) = y + 2z + 3x$

(c) $\nabla \cdot \left(y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}\right) = 2x + 2y$

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5. **[G1.18]** Calculate curls of the vector functions in Prob 4.

Solution:

(a) $\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3xz^2 & -2xz \end{vmatrix} = -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}$

(b) $\nabla \times \mathbf{v} = -2y\hat{\mathbf{x}} + -3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}$

(c) $\nabla \times \mathbf{v} = 0$

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6. **[G1.20]** Prove the following product rules:

$$\begin{aligned} \nabla (fg) &= f\nabla g + g\nabla f \\ \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\ \nabla \times (f\mathbf{A}) &= f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f) \end{aligned}$$

Solutions:

No tricks here! Use the brute force. For example, z component of the third identity is

$$\begin{aligned} [\nabla \times (f\mathbf{A})]_z &= \frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x) \\ &= f\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) + \left(\frac{\partial}{\partial x}f\right)A_y - \left(\frac{\partial}{\partial y}f\right)A_x \\ &= f[\nabla \times A]_z + [(\nabla f) \times A]_z \end{aligned}$$

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7. **[G 1.27]** Prove that the curl of a gradient of a smooth function is always zero. Check it for the function $f(x, y, z) = x^2y^3z^4$.

Solution:

Curl of a gradient is

$$\begin{aligned} \nabla \times (\nabla t) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \end{vmatrix} \\ &= \hat{\mathbf{x}}\left(\frac{\partial^2 t}{\partial y\partial z} - \frac{\partial^2 t}{\partial z\partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 t}{\partial z\partial x} - \frac{\partial^2 t}{\partial x\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial^2 t}{\partial x\partial y} - \frac{\partial^2 t}{\partial y\partial x}\right) \\ &= 0 \quad \text{by equality of cross-derivatives .} \end{aligned}$$

Now,$\nabla f = 2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}$, so

$$\begin{aligned} \nabla \times (\nabla f) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix} \\ &= \hat{\mathbf{x}}\left(3 \times 4x^2y^2z^3 - 4 \times 3x^2y^2z^3\right) + \hat{\mathbf{y}}\left(4 \times 2xy^3z^3 - 2 \times 4xy^3z^3\right) \\ &+ \hat{\mathbf{z}}\left(2 \times 3xy^2z^4 - 3 \times 2xy^2z^4\right) \\ &= 0. \end{aligned}$$

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8. **[G1.25]** Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4$. (b) $T_b = \sin x \sin y \sin z$. (c) $T_c = e^{-5x}\sin 4y \cos 3z$. (d) $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$.

Solutions:

(a) $\nabla^2 T_a = 2$. (b) $\nabla^2 T_b = -3\sin x\,\sin y\,\sin z$. (c) $\nabla^2 T_c = 0$. (d) Laplacian of a vector field is defined as

$$\nabla^2 \mathbf{v} = \hat{\mathbf{x}}\nabla^2 v_x + \hat{\mathbf{y}}\nabla^2 v_y + \hat{\mathbf{z}}\nabla^2 v_z.$$

[Note that the cartesian unit vectors are constants.] Then $\nabla^2 \mathbf{v} = 2\hat{\mathbf{x}} + 6x\hat{\mathbf{y}}$.

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9. Find the length of one turn of a helical wire(with radius R and pitch p).

Solution:

The parametrization for given helix $C:\ \mathbf{r}(\phi) = \left(R\cos\phi, R\sin\phi,\ p\frac{\phi}{2\pi}\right)$ where $\phi : 0 \to 2\pi$.

$$\left|\mathbf{r}'(\phi)\right| = |(-R\sin\phi, R\cos\phi, p/2\pi)| = \frac{1}{2\pi}\left(4\pi^2 R^2 + p^2\right)^{1/2}$$

The length of helix is

$$\int_C dl = \int_0^{2\pi} \left|\mathbf{r}'(\phi)\right| d\phi = \left(4\pi^2 R^2 + p^2\right)^{1/2}.$$

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10. Find the work done by the force field $\mathbf{F}(x, y) = x\hat{\mathbf{x}} + (y+2)\hat{\mathbf{y}}$ in moving an object along an arch of the cycloid $\mathbf{r}(t) = (t - \sin t)\,\hat{\mathbf{x}} + (1 - \cos t)\,\hat{\mathbf{y}}$, $0 \le t \le 2\pi$.

Solution:

Since $\mathbf{r}(t) = (t - \sin t)\,\hat{\mathbf{x}} + (1 - \cos t)\,\hat{\mathbf{y}}$, $d\mathbf{r} = ((1 - \cos t)\,\hat{\mathbf{x}} + \sin t\hat{\mathbf{y}})\,dt$. Then the work

$$\begin{aligned}\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^{2\pi} [(t - \sin t)\,\hat{\mathbf{x}} + (3 - \cos t)\,\hat{\mathbf{y}}] \cdot [(1 - \cos t)\,\hat{\mathbf{x}} + \sin t\hat{\mathbf{y}}]\,dt \\ &= 2\pi^2\end{aligned}$$
