Homogeneous Linear Systems With Constant Coefficients Contd...

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We shall extend techniques for scalar differential equations to systems.

For example, a GS to $x'(t) = ax(t)$, where a is a constant, is $x(t) = ce^{at}$. Analogously, we shall show that a GS to the system

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where A is a constant matrix, is

$$\mathbf{x}(t) = e^{At}\mathbf{c}.$$

Task: To define the matrix exponential e^{At}.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. The norm on T is given by

$$\|T\| = \max_{|\mathbf{x}|\leq 1} |T(\mathbf{x})|, \quad |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Theorem: Given a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ and $t_0 > 0$, the series

$$\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$$

is absolutely and uniformly convergent for all $|t| \leq t_0$.

Proof. Let $\|T\| = a$. Then for $|t| \leq t_0$,

$$\left\| \frac{T^k t^k}{k!} \right\| \leq \frac{\|T\|^k |t|^k}{k!} \leq \frac{a^k t_0^k}{k!}.$$

But, $\sum_{k=0}^{\infty} \frac{a^k t_0^k}{k!} = e^{a t_0}$. By the Weierstrass M-test, the series $\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$ is absolutely and uniformly convergent for all $|t| \leq t_0$.

Definition: The exponential of the linear operator T is then defined by

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}.$$

Assume that T is represented by the $n \times n$ matrix A w.r.t. the standard basis for $\mathbb{R}^n$ and define the exponential e^{At} as follows:

Definition: Let A be an $n \times n$ matrix. Then for $t \in \mathbb{R}$,

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

e^{At} is an $n \times n$ matrix which can be computed in terms of the eigenvalues and eigenvectors of A.

If A is a diagonal matrix, then the computation of e^{At} is simple.

Example: Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}, \cdots, \quad A^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}.$$

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} A^k \frac{t^k}{k!} \\ &= \begin{bmatrix} \sum_{k=0}^{\infty}(-1)^k \frac{t^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty}(2)^k \frac{t^k}{k!} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}. \end{aligned}$$

Theorem: Let A and B be $n \times n$ constant matrices and $r, s, t \in \mathbb{R}$ (or $\in \mathbb{C}$). Then

- $e^{A0} = e^0 = I$.
- $e^{A(t+s)} = e^{At}e^{As}$.
- $(e^{At})^{-1} = e^{-At}$.
- $e^{(A+B)t} = e^{At}e^{Bt}$, provided that $AB = BA$.
- $e^{rIt} = e^{rt}I$.

Theorem: If P and A are $n \times n$ matrices and $PAP^{-1} = B$, then

$$e^{Bt} = Pe^{At}P^{-1}.$$

Proof. Using the definition of e^{At},

$$\begin{aligned} e^{Bt} &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(PAP^{-1})^k t^k}{k!} \\ &= P \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(At)^k}{k!} P^{-1} = Pe^{At}P^{-1}. \end{aligned}$$

Corollary: If $P^{-1}AP = \mathrm{diag}[\lambda_j]$ then $e^{At} = P\,\mathrm{diag}[e^{\lambda_j t}]\,P^{-1}$.

Corollary: If $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ then $e^{A} = e^{a}\begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$.

Proof. If $\lambda = a + ib$, it follows by induction that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{k} = \begin{bmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{bmatrix}.$$

Thus,

$$\begin{aligned} e^{A} &= \sum_{k=0}^{\infty} \begin{bmatrix} Re(\frac{\lambda^k}{k!}) & -Im(\frac{\lambda^k}{k!}) \\ Im(\frac{\lambda^k}{k!}) & Re(\frac{\lambda^k}{k!}) \end{bmatrix} \\ &= \begin{bmatrix} Re(e^{\lambda}) & -Im(e^{\lambda}) \\ Im(e^{\lambda}) & Re(e^{\lambda}) \end{bmatrix} = e^{a}\begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}. \end{aligned}$$

If $a = 0$, then e^{A} is simply a rotation through b radians.

Lemma: Let A be a square matrix, then

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Proof. We have

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \lim_{h\to 0} \frac{e^{A(t+h)} - e^{At}}{h} \\
&= \lim_{h\to 0} e^{At}\frac{(e^{Ah} - I)}{h} \\
&= e^{At} \lim_{h\to 0} \lim_{k\to\infty} \left(A + \frac{A^2 h}{2!} + \cdots + \frac{A^k h^{k-1}}{k!}\right) = Ae^{At}.
\end{aligned}$$

In the above, we have used the fact that the series defining e^{Ah} converges uniformly for $|h| \leq 1$.

Note that

$$\frac{d}{dt}(e^{At}) = Ae^{At}$$

$\implies$ e^{At} is a solution to the matrix differential equation $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Since e^{At} is invertible it follows that the columns of e^{At} form a fundamental solution set for $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Theorem: If A is an $n \times n$ constant matrix, then the columns of e^{At} form a fundamental solution set for

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

Therefore, e^{At} is a fundamental matrix for the system, and a GS is

$$\mathbf{x}(t) = e^{At}\mathbf{c} = \Phi(t)\mathbf{c}.$$

Theorem: (**The fundamental theorem for linear systems**)
Let A be an $n \times n$ matrix. Then for a given $\mathbf{x}_0 \in \mathbb{R}^n$, the IVP $\mathbf{x}' = A\mathbf{x}; \;\; \mathbf{x}(0) = \mathbf{x}_0$ has a unique solution given by $\mathbf{x}(t) = e^{At}\mathbf{x}_0$.

Proof. If $\mathbf{x}(t) = e^{At}\mathbf{x}_0$, then $\mathbf{x}'(t) = \frac{d}{dt}e^{At}\mathbf{x}_0 = A\mathbf{x}(t), \;\; t \in \mathbb{R}$. Also, $\mathbf{x}(0) = I\mathbf{x}_0 = \mathbf{x}_0$.

Uniqueness: Let $\mathbf{x}(t)$ be any solution of the IVP. Set $\mathbf{y}(t) = e^{-At}\mathbf{x}(t)$. Then

$$\begin{aligned} \mathbf{y}'(t) &= -Ae^{-At}\mathbf{x}(t) + e^{-At}\mathbf{x}'(t) \\ &= -Ae^{-At}\mathbf{x}(t) + e^{-At}A\mathbf{x}(t) = \mathbf{0}, \;\; \text{for all } t \in \mathbb{R}. \\ &\Rightarrow \;\; \mathbf{y}(t) \text{ is a constant.} \end{aligned}$$

Further, $\mathbf{y}(0) = \mathbf{x}_0$. Thus, any solution of the IVP is given by

$$\mathbf{x}(t) = e^{At}\mathbf{y}(t) = e^{At}\mathbf{x}_0.$$

When A has complex eigenvalues

Theorem: Let A be a real matrix of size $2n \times 2n$. If A has $2n$ **distinct complex eigenvalues** $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ and corresponding complex eigenvectors $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\bar{w}_j = \mathbf{u}_j - i\mathbf{v}_j$, $j = 1, \ldots, n$, then $\{\mathbf{u}_1,\ \mathbf{v}_1,\ \ldots,\ \mathbf{u}_n,\ \mathbf{v}_n\}$ is a basis for $\mathbb{R}^{2n}$, the matrix

$$P = [\mathbf{v}_1 \ \ \mathbf{u}_1 \ \ \mathbf{v}_2 \ \ \mathbf{u}_2 \ \ \cdots \ \ \mathbf{v}_n \ \ \mathbf{u}_n]$$

is invertible and

$$P^{-1}AP = \mathrm{diag}\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

a real $2n \times 2n$ matrix with 2×2 blocks along the diagonal.

Remark: Instead of P if we use

$$Q = [\mathbf{u}_1 \ \ \mathbf{v}_1 \ \ \mathbf{u}_2 \ \ \mathbf{v}_2 \ \ \cdots \ \ \mathbf{u}_n \ \ \mathbf{v}_n]$$

then

$$Q^{-1}AQ = \text{diag}\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}.$$

Using the above result, the solution of the IVP $\mathbf{x}' = Ax$, $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = P \ \text{diag}\, e^{a_j t} \begin{bmatrix} \cos(b_j t) & -\sin(b_j t) \\ \sin(b_j t) & \cos(b_j t) \end{bmatrix} P^{-1}\mathbf{x}_0$$

Example: Solve the IVP $\mathbf{x}' = A\mathbf{x}, \;\; \mathbf{x}(0) = \mathbf{x}_0$ with

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

A has complex eigenvalues $\lambda_1 = 1 + i \;\; \lambda_2 = 2 + i$ (as well as $\bar{\lambda}_1 = 1 - i \;\; \bar{\lambda}_2 = 2 - i$). A corresponding pair of complex eigenvectors is

$$\mathbf{w}_1 = \mathbf{u}_1 + i\mathbf{v}_1 = [i\ 1\ 0\ 0]^T \quad \text{and} \quad \mathbf{w}_2 = \mathbf{u}_2 + i\mathbf{v}_2 = [0\ 0\ 1+i\ 1]^T.$$

The matrix

$$P = [\mathbf{v}_1\ \mathbf{u}_1\ \mathbf{v}_2\ \mathbf{u}_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution to IVP is

$$\mathbf{x}(t) = P \begin{bmatrix} e^t \cos t & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^{2t} \cos t & -e^{2t} \sin t \\ 0 & 0 & e^{2t} \sin t & e^{2t} \cos t \end{bmatrix} P^{-1}\mathbf{x}_0$$

$$= \begin{bmatrix} e^t \cos t & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^{2t}(\cos t + \sin t) & -e^{2t} \sin t \\ 0 & 0 & e^{2t} \sin t & e^{2t}(\cos t - \sin t) \end{bmatrix} \mathbf{x}_0$$

When A has both real and complex eigenvalues.

Theorem: If A has distinct real eigenvalues λ_j and corresponding eigenvectors $\mathbf{v}_j$, $j = 1, \ldots, k$ and distinct complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ and corresponding eigenvectors $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\bar{\mathbf{w}}_j = \mathbf{u}_j - i\mathbf{v}_j$, $j = k+1, \ldots, n$, then the matrix

$$P = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_k \ \mathbf{v}_{k+1} \ \mathbf{u}_{k+1} \ \cdots \ \mathbf{v}_n \ \mathbf{u}_n]$$

is invertible and

$$P^{-1}AP = \mathrm{diag}[\lambda_1, \ldots, \lambda_k, B_{k+1}, \ldots, B_n],$$

where $B_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$ for $j = k+1, \ldots, n$.

Example: Solve the IVP $\mathbf{x}' = A\mathbf{x}, \ \mathbf{x}(0) = \mathbf{x}_0$ with

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = -3, \ \lambda_2 = 2 + i(\bar{\lambda}_2 = 2 - i)$.
The corresponding eigenvectors

$$\mathbf{v}_1 = [1 \ \ 0 \ \ 0]^T \ \text{ and } \ \mathbf{w}_2 = \mathbf{u}_2 + i\mathbf{v}_2 = [0 \ \ 1 + i \ \ 1]^T$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ \ P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

The solution of IVP is

$$\mathbf{x}(t) = P \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}\cos t & -e^{2t}\sin t \\ 0 & e^{2t}\sin t & e^{2t}\cos t \end{bmatrix} P^{-1}\mathbf{x}_0$$

$$= \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}(\cos t + \sin t) & -2e^{2t}\sin t \\ 0 & e^{2t}\sin t & e^{2t}(\cos t - \sin t) \end{bmatrix} \mathbf{x}_0.$$

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