Physics II: Electromagnetism (PH102)

Lecture 5

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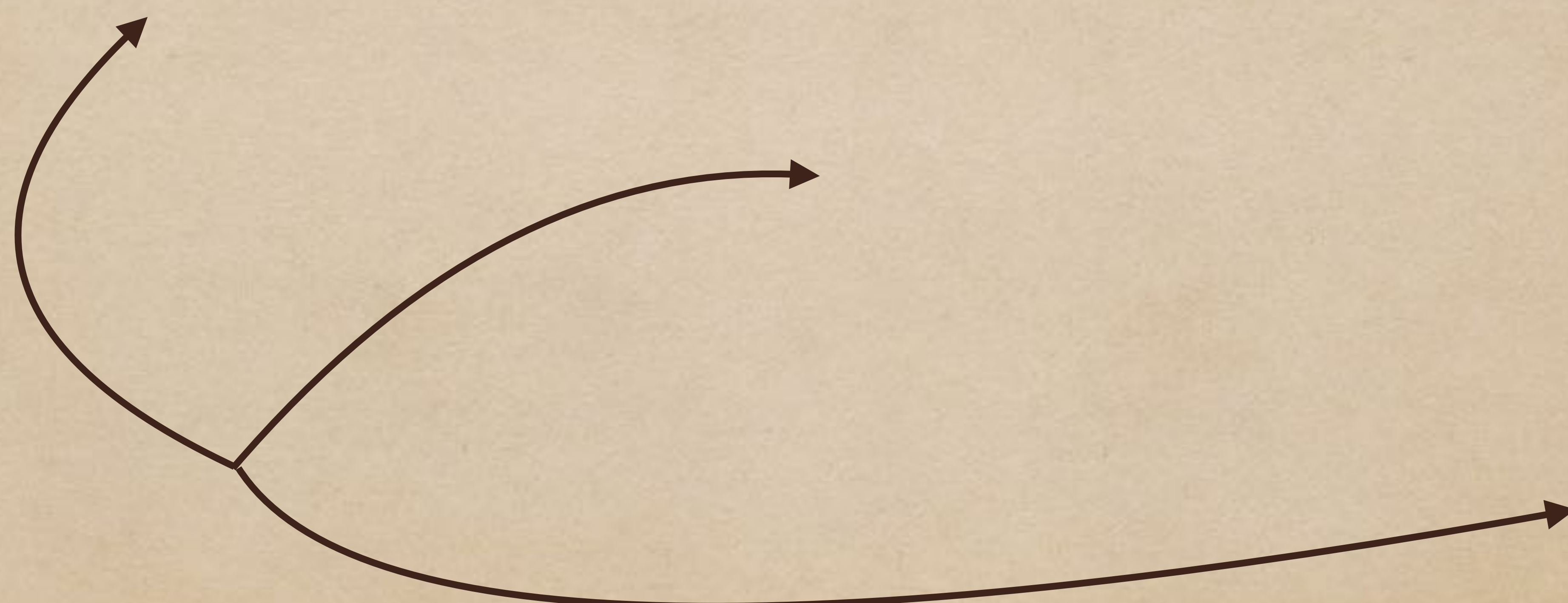
Calculus in Curvilinear coordinates



Curvilinear coordinates

u_1, u_2, u_3 describe curvilinear coordinates

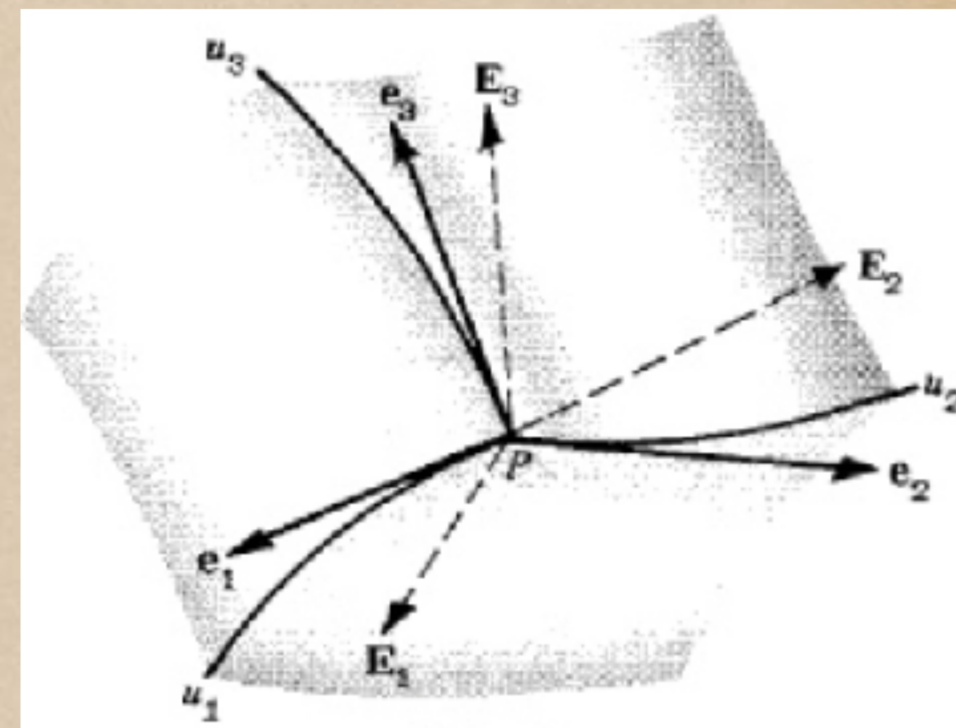
$\hat{e_1}$: unit vector along curve u_1 at P.
$\hat{e_1} = (\partial\vec{r}/\partial u_1)/|\partial\vec{r}/\partial u_1|$

$$\partial\vec{r}/\partial u_1 = h_1\hat{e_1};\ \ h_1 = |\partial\vec{r}/\partial u_1|$$

Similarly unit vectors along curves u_2, u_3 are respectively $\hat{e}_2$ and $\hat{e}_3$

$$\partial\vec{r}/\partial u_2 = h_2\hat{e}_2;\ \ h_2 = |\partial\vec{r}/\partial u_2|$$

$$\partial\vec{r}/\partial u_3 = h_3\hat{e}_3;\ \ h_3 = |\partial\vec{r}/\partial u_3|$$



Line Element : $\vec{dr} = \frac{\partial\vec{r}}{\partial u_1}du_1 + \frac{\partial\vec{r}}{\partial u_2}du_2 + \frac{\partial\vec{r}}{\partial u_3}du_3$
$= h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$

Line element is identified given the transformation

$x = x(u_1, u_2, u_3),\ \ y = y(u_1, u_2, u_3),$
$z = z(u_1, u_2, u_3)$

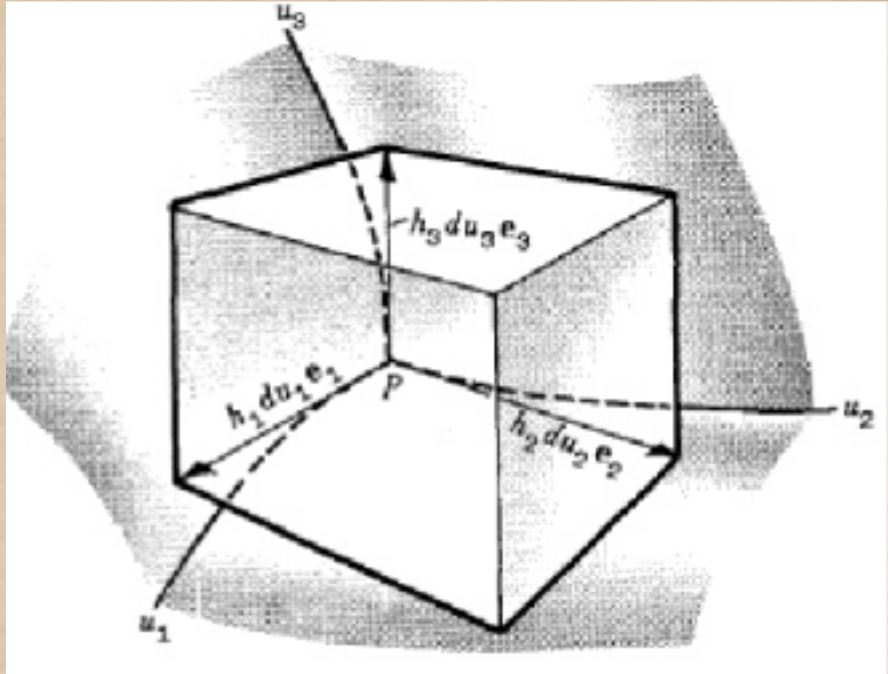
Surface and volume elements in Curvilinear coordinates

$u_1 - u_2$ plane $\longrightarrow$ u_3 is constant

$\vec{da} = h_1 h_2 du_1 du_2 (\hat{e}_1 \times \hat{e}_2)$

For orthogonal coordinates: $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$

Hence, $\vec{da} = h_1 h_2 du_1 du_2 \hat{e}_3$



$u_2 - u_3$ plane $\longrightarrow$ u_1 is constant

$\vec{da} = h_2 h_3 du_2 du_3 (\hat{e}_2 \times \hat{e}_3)$

For orthogonal coordinates: $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$

Hence, $\vec{da} = h_2 h_3 du_2 du_3 \hat{e}_1$

$u_3 - u_1$ plane $\rightarrow$ u_2 is constant

Similarly, $\vec{da} = h_1 h_3 du_1 du_3 \hat{e}_2$

One needs to find h_1, h_2, h_3 given the transformations.

$$\boxed{\text{Volume element: } d\tau = h_1 h_2 h_3 du_1 du_2 du_3}$$

Gradient, Divergence, Curl in Curvilinear coordinates

$$\vec{\nabla}\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\hat{e}_3$$

$$\vec{\nabla}.\vec{V} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 V_1)}{\partial u_1} + \frac{\partial(h_3 h_1 V_2)}{\partial u_2} + \frac{\partial(h_1 h_2 V_3)}{\partial u_3}\right)$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

For orthogonal coordinates

$$\nabla^2\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]$$

Spherical polar coordinates

Transformation rules :

$$x = r\sin\theta\cos\phi,\;\; y = r\sin\theta\sin\phi,\;\; z = r\cos\theta,$$

$$r = \sqrt{(x^2+y^2+z^2)}$$

$$u_1 \to r;\;\; u_2 \to \theta; u_3 \to \phi \qquad \hat{e}_1 \to \hat{r};\;\; \hat{e}_2 \to \hat{\theta};\;\; \hat{e}_3 \to \hat{\phi}$$

$$h_1 = |\partial\vec{r}/\partial r| = 1;\;\; h_2 = |\partial\vec{r}/\partial\theta| = r;\;\; h_3 = |\partial\vec{r}/\partial\phi| = r\sin\theta$$

$$\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\phi}$$

$$\vec{\nabla}.\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial\theta}(\sin\theta V_\phi) - \frac{\partial V_\theta}{\partial\phi}\Big]\hat{r} + \frac{1}{r}\Big[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial}{\partial r}(rV_\phi)\Big]\hat{\theta} + \frac{1}{r}\Big[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\Big]\hat{\phi}$$

$$\nabla^2T = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial T}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial T}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

Examples on spherical polar coordinates

- Evaluate $\vec{\nabla} f(r)$ $\longrightarrow$ $\hat{r}\dfrac{df}{dr}$

using, $\vec{\nabla} T = \dfrac{\partial T}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial T}{\partial \theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial T}{\partial \phi}\hat{\phi}$

- Evaluate $\vec{\nabla} r^n = \hat{r} n r^{n-1}$

Think of your efforts to calculate the same using Cartesian coordinates !

- Evaluate $\vec{\nabla}.(\hat{r} r^n)$

using, $\vec{\nabla}.\vec{V} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 V_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta V_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial V_\phi}{\partial \phi}$

$$\vec{\nabla}.(\hat{r} r^n) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) = (n+2) r^{n-1}$$

Verify that: $\vec{\nabla}.(\hat{r} f(r)) = \frac{2}{r} f(r) + \frac{df}{dr}$ and $\vec{\nabla} \times (\hat{r} f(r)) = 0$

Gradient, divergence and curl of any functions with spherical symmetry is best evaluated using spherical polar coordinates

Verifying theorems on grad, div or curl in spherical polar coordinates

(a) Check the divergence theorem for the function $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.

(a) $\nabla\cdot\mathbf{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2) = \frac{1}{r^2}4r^3 = 4r$

$\int(\nabla\cdot\mathbf{v}_1)d\tau = \int(4r)(r^2\sin\theta\,dr\,d\theta\,d\phi) = (4)\int_0^R r^3 dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi = (4)\left(\frac{R^4}{4}\right)(2)(2\pi) = \boxed{4\pi R^4}$

$\int \mathbf{v}_1\cdot d\mathbf{a} = \int(r^2\hat{\mathbf{r}})\cdot(r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}) = r^4\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi = 4\pi R^4$ ✓ (*Note:* at surface of sphere $r = R$.)

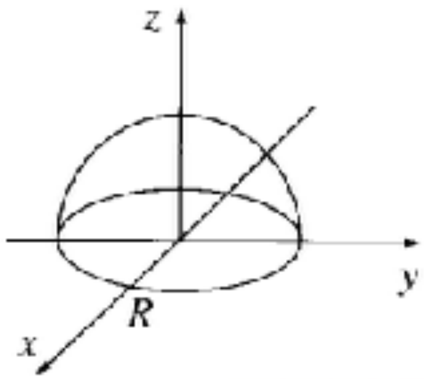
Note here that surface integral can easily be performed without taking the projection on x-y plane and so on...

A vector field to be integrated over a geometry with spherical symmetry is better dealt with spherical polar coordinates

Another example...

(b) Assume : $\mathbf{v} = (r\cos\theta)\,\hat{\mathbf{r}} + (r\sin\theta)\,\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\,\hat{\boldsymbol{\phi}}.$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin



$$\begin{aligned}
\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\\
&= \frac{1}{r^2}\,3r^2\cos\theta + \frac{1}{r\sin\theta}\,r\,2\sin\theta\cos\theta + \frac{1}{r\sin\theta}\,r\sin\theta(-\sin\phi)\\
&= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi
\end{aligned}$$

$$\int(\nabla\cdot\mathbf{v})d\tau = \int(5\cos\theta - \sin\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi = \int_0^R r^2\,dr \int_0^{\frac{\theta}{2}} \Big[\underbrace{\int_0^{2\pi}(5\cos\theta - \sin\phi)\,d\phi}_{\hookrightarrow 2\pi(5\cos\theta)}\Big]\, d\theta\sin\theta$$

$$= \left(\frac{R^3}{3}\right)(10\pi)\underbrace{\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\ ,d\theta}_{\hookrightarrow \left.\frac{\sin^2\theta}{2}\right|_0^{\frac{\pi}{2}} = \frac{1}{2}}$$

$$= \boxed{\frac{5\pi}{3}R^3.}$$

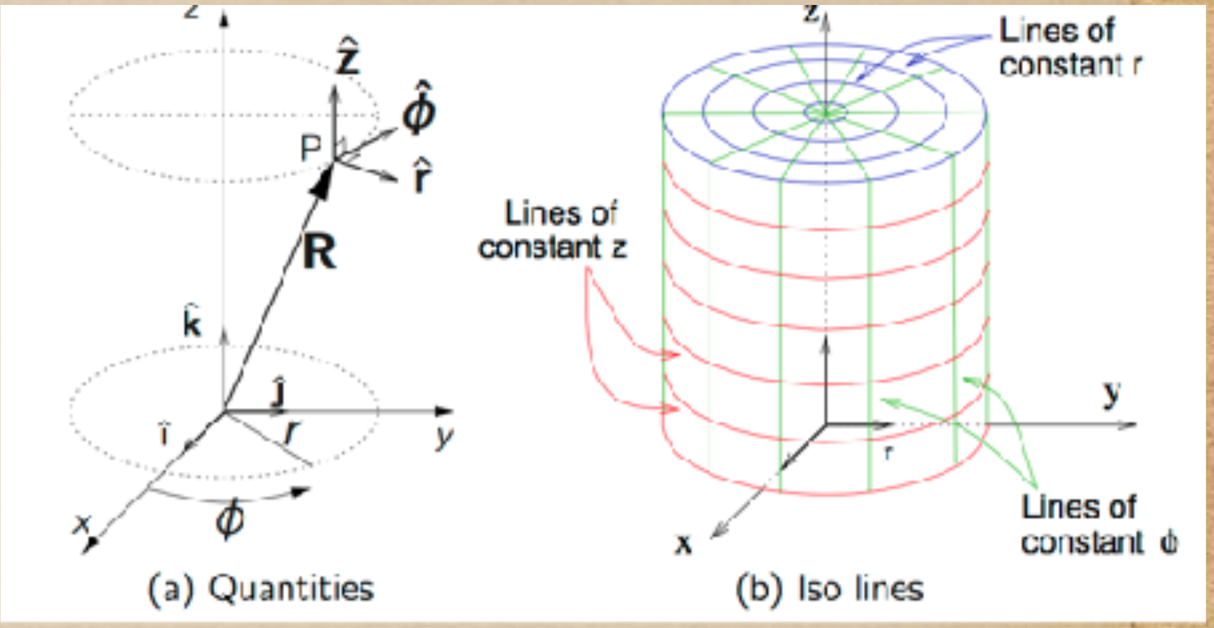
- Two surfaces—one the hemisphere: $da = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$; $r = R$; $\phi: 0 \to 2\pi$, $\theta: 0 \to \frac{\pi}{2}$.
  $\int\mathbf{v}\cdot d\mathbf{a} = \int(r\cos\theta)R^2\sin\theta\,d\theta\,d\phi = R^3\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta\int_0^{2\pi}d\phi = R^3\left(\frac{1}{2}\right)(2\pi) = \pi R^3.$
- other the flat bottom: $d\mathbf{a} = (dr)(r\sin\theta\,d\phi)(+\hat{\boldsymbol{\theta}}) = r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$ (here $\theta = \frac{\pi}{2}$). $r: 0 \to R$, $\phi: 0 \to 2\pi$.
  $\int\mathbf{v}\cdot d\mathbf{a} = \int(r\sin\theta)(r\,dr\,d\phi) = \int_0^R r^2\,dr\int_0^{2\pi}d\phi = 2\pi\frac{R^3}{3}.$

Total: $\int\mathbf{v}\cdot d\mathbf{a} = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3.$ ✓

Cylindrical polar coordinates

$$x = s\cos\phi, \;\; y = s\sin\phi, \;\; z = z$$

$$s \geq 0, \;\; 0 \leq \phi \leq 2\pi, \;\; -\infty \leq z \leq \infty$$



Unit vectors

$$\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y}$$
$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$
$$\hat{z} = \hat{z}$$

$$h_1 \rightarrow 1, \;\; h_2 \rightarrow s, \;\; h_3 \rightarrow 1; \quad \hat{e}_1 \rightarrow \hat{s}, \;\; \hat{e}_2 \rightarrow \hat{\phi}, \;\; \hat{e}_3 \rightarrow \hat{z}$$

Line element: $\vec{dl} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}$

Surface elements:

$$z = \text{constant:} \;\; \vec{da}_1 = sdsd\phi(\pm\hat{z})$$
$$s = \text{constant:} \;\; \vec{da}_2 = sd\phi dz\hat{s}$$

Volume element $d\tau = sdsd\phi dz$

Gradient, Divergence and Curl in Cylindrical polar coordinates

$$\vec{\nabla}T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

$$\vec{\nabla}.\vec{V} = \frac{1}{s}\frac{\partial}{\partial s}(sV_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

$$\vec{\nabla}\times\vec{V} = \left(\frac{1}{s}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial V_s}{\partial z} - \frac{\partial V_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(sV_\phi) - \frac{\partial V_s}{\partial \phi}\right)\hat{z}$$

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

Example....

(a) Find the divergence of the function

$$\mathbf{v} = s(2 + \sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(a)
$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\, s(2+\sin^2\phi)\right) + \frac{1}{s}\frac{\partial}{\partial\phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)\\ &= \frac{1}{s}2s(2+\sin^2\phi) + \frac{1}{s}s(\cos^2\phi - \sin^2\phi) + 3\\ &= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3\\ &= 4 + \sin^2\phi + \cos^2\phi + 3 = \boxed{8.}\end{aligned}$$

(b) $\int(\nabla\cdot\mathbf{v})d\tau = \int(8)s\,ds\,d\phi\,dz = 8\int_0^2 s\,ds\int_0^{\frac{\pi}{2}} d\phi\int_0^5 dz = 8(2)\left(\frac{\pi}{2}\right)(5) = \boxed{40\pi.}$

Meanwhile, the surface integral has five parts:

- top: $z = 5$, $d\mathbf{a} = s\,ds\,d\phi\,\hat{\mathbf{z}}$; $\mathbf{v}\cdot d\mathbf{a} = 3z\,s\,ds\,d\phi = 15s\,ds\,d\phi$. $\int\mathbf{v}\cdot d\mathbf{a} = 15\int_0^2 s\,ds\int_0^{\frac{\pi}{2}} d\phi = 15\pi$.
- bottom: $z = 0$, $d\mathbf{a} = -s\,ds\,d\phi\,\hat{\mathbf{z}}$; $\mathbf{v}\cdot d\mathbf{a} = -3z\,s\,ds\,d\phi = 0$. $\int\mathbf{v}\cdot d\mathbf{a} = 0$.
- back: $\phi = \frac{\pi}{2}$, $d\mathbf{a} = ds\,dz\,\hat{\boldsymbol{\phi}}$; $\mathbf{v}\cdot d\mathbf{a} = s\sin\phi\cos\phi\,ds\,dz = 0$. $\int\mathbf{v}\cdot d\mathbf{a} = 0$.
- left: $\phi = 0$, $d\mathbf{a} = -ds\,dz\,\hat{\boldsymbol{\phi}}$; $\mathbf{v}\cdot d\mathbf{a} = -s\sin\phi\cos\phi\,ds\,dz = 0$. $\int\mathbf{v}\cdot d\mathbf{a} = 0$.
- front: $s = 2$, $d\mathbf{a} = s\,d\phi\,dz\,\hat{\mathbf{s}}$; $\mathbf{v}\cdot d\mathbf{a} = s(2+\sin^2\phi)s\,d\phi\,dz = 4(2+\sin^2\phi)d\phi\,dz$.
  $\int\mathbf{v}\cdot d\mathbf{a} = 4\int_0^{\frac{\pi}{2}}(2+\sin^2\phi)d\phi\int_0^5 dz = (4)(\pi + \frac{\pi}{4})(5) = 25\pi$.

So $\oint\mathbf{v}\cdot d\mathbf{a} = 15\pi + 25\pi = 40\pi$. ✓

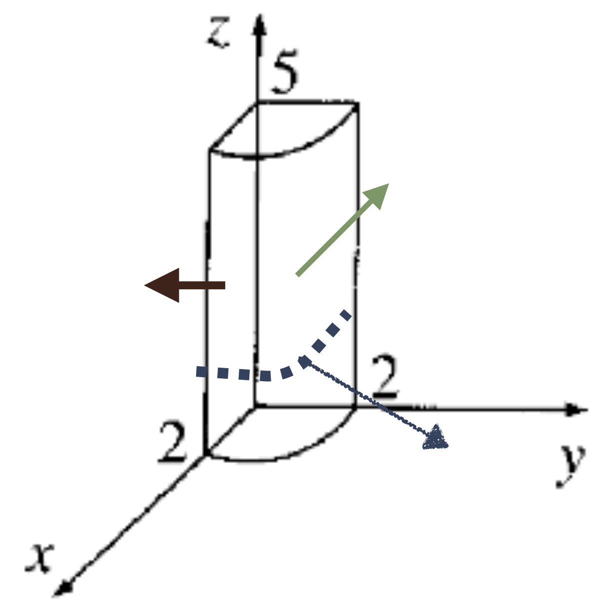


Figure 1.43

Again, no projection required to evaluate the surface integral of the cylindrical surface

Divergence of $\vec{V} = \frac{\hat{r}}{r^2}$

$$\vec{\nabla}.\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{1}{r^2}) = 0 \quad \longrightarrow \int_{vol} \vec{\nabla}.\vec{V}\,d\tau = 0$$

Suppose we calculate divergence of $\vec{V}$ over a sphere of radius R centred at origin using surface integral of $\vec{V}$ using divergence theorem

$$\int_{vol} \underbrace{\vec{\nabla}.\vec{V}}_{0\ !!!}\,d\tau = \oint \vec{V}.\vec{da} = \int \left(\frac{1}{R^2}\hat{r}\right).(R^2 \sin\theta d\theta d\phi \hat{r})$$

$$= \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi$$

Is divergence theorem wrong ?

Actually, we neglected contributions from the origin which is solely contributing to non-zero surface integral.

$\vec{\nabla}.\vec{V}$ is zero everywhere excepting at $r = 0$, origin

Dirac Delta Function

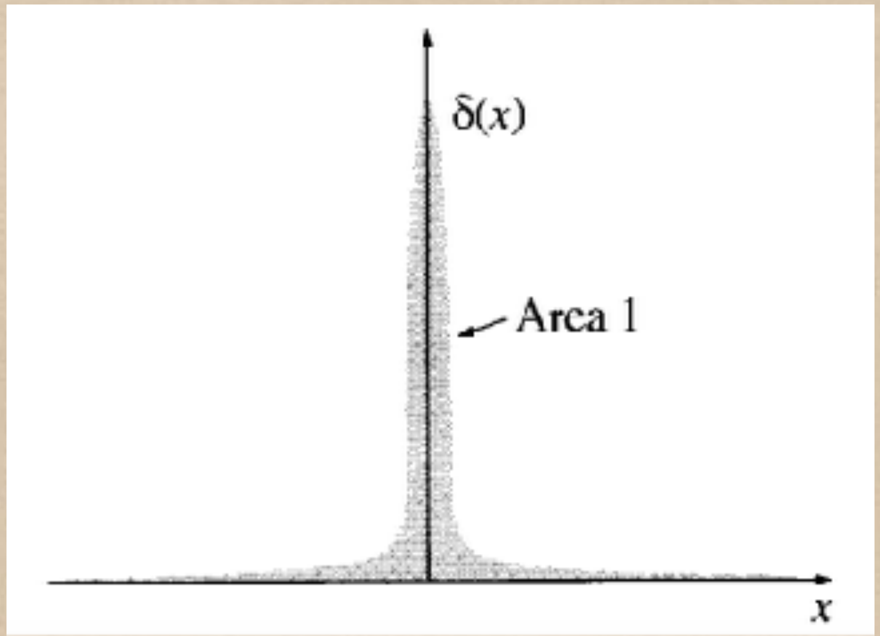
We need a mathematical function which is zero elsewhere excepting at one particular point

Dirac delta in one dimension

$$\delta(x) = \begin{cases} 0 & \text{for} \quad x \neq 0 \\ \infty & \text{for} \quad x = 0 \end{cases}$$

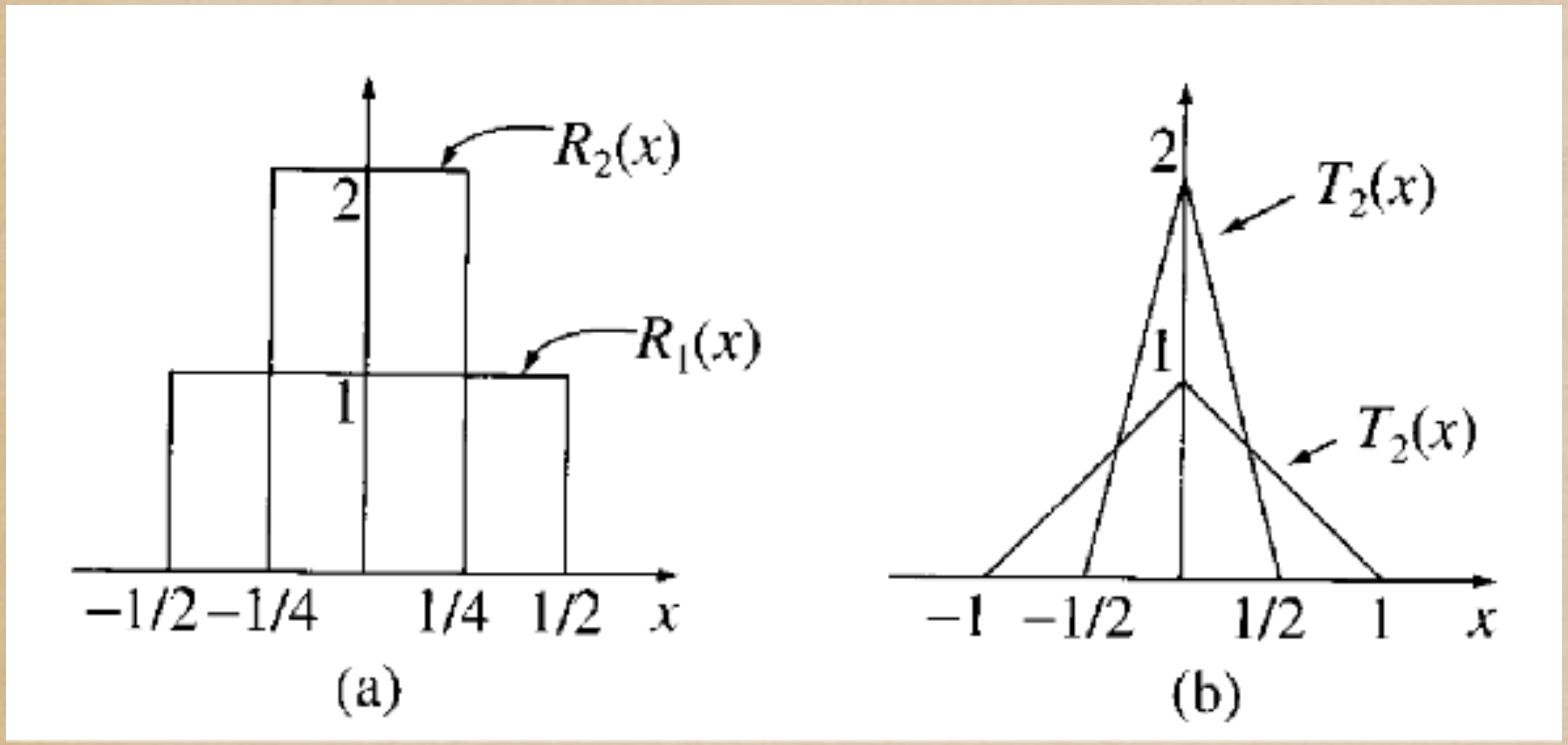
Dirac delta function requires to satisfy

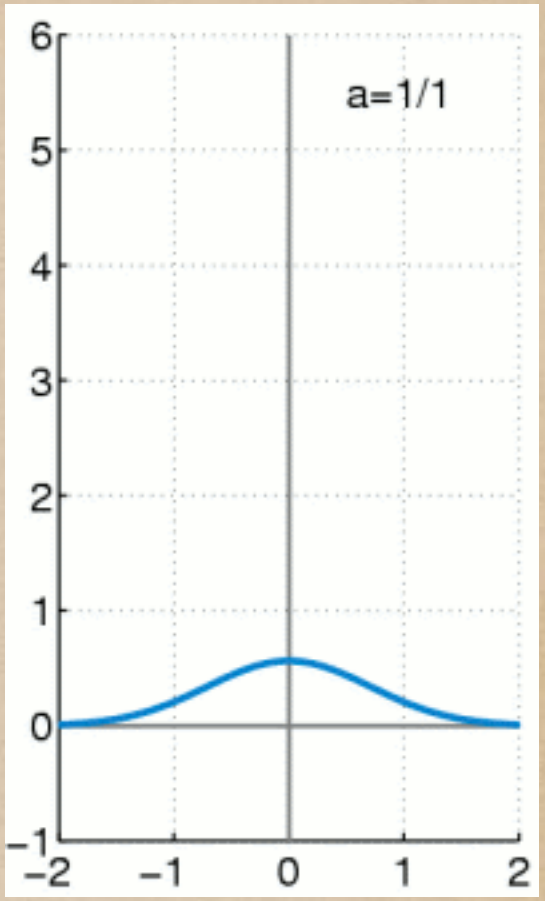
$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$



Technically, Dirac delta is not a function as the value is not finite at one particular point

How a Dirac delta function is realised ?





Dirac delta is the limit of a sequence of functions such as
Rectangles $R_n(x)$, of height n and width $\frac{1}{n}$
Isosceles triangles $T_n(x)$, of height n and base $\frac{2}{n}$

$$\boxed{\delta(x) = \lim_{n\to\infty} \eta_n(x)} \qquad \eta_n \to \{R_n, T_n\}$$

Properties of Dirac Delta

$\boxed{f(x)\delta(x) = f(0)\delta(x)} \longrightarrow$ is zero everywhere excepting $x = 0$

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0)$$

Note that the limit of the integral need not run to infinity; it is sufficient to extend across the delta function

We can of course shift the spike from x=0 to x=a

$$\boxed{\delta(x-a) = \begin{cases} 0 & \text{for} \quad x \neq a \\ \infty & \text{for} \quad x = a \end{cases}}$$

Similarly $\displaystyle\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)dx = f(a)$

Example: $\displaystyle\int_0^3 x^3\delta(x-2)\,dx. = 2^3 = 8$

However if the upper limit is 1, then the integral is 0

Scaling of Dirac Delta

Show that

$$\delta(kx) = \frac{1}{|k|}\delta(x),$$

where k is any (nonzero) constant. (In particular, $\delta(-x) = \delta(x)$.)

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \pm\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{dy}{k}$$

$$= \pm\frac{1}{k}f(0) = \frac{1}{|k|}f(0)$$

for k positive or negative

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \int_{-\infty}^{\infty} f(x)\Big[\frac{1}{|k|}\delta(x)\Big]dx$$

$$\boxed{\delta(kx) = \frac{1}{|k|}\delta(x)}$$

Dirac Delta in three dimension

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

is zero everywhere except at (0,0,0) where it blows up !

$$\int_{\text{all space}} \delta^3(\vec{r})d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1$$

$$\boxed{\int_{\text{all space}} f(\vec{r})\delta^3(\vec{r}-\vec{r_0})d\tau = f(\vec{r_0})}$$

Shifting the peak of delta function

Let us get back to the divergence paradox !

Let us assume: $\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$ Why ?

Then, $\boxed{\int_{\text{vol}} \vec{\nabla}.\vec{V}d\tau = \int_{\text{vol}} 4\pi\delta^3(\vec{r})d\tau = 4\pi}$

Satisfies divergence theorem !

Examples on 3 dim Dirac delta

Evaluate the integral

$$J = \int_{\mathcal{V}} (r^2 + 2)\, \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) d\tau,$$

where $\mathcal{V}$ is a sphere of radius R centered at the origin.

Using

$$\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

$$J = \int_{\mathcal{V}} (r^2 + 2) 4\pi\delta^3(\mathbf{r})\, d\tau = 4\pi(0+2) = 8\pi.$$

(a) Write an expression for the electric charge density $\rho(\mathbf{r})$ of a point charge q at $\mathbf{r}'$. Make sure that the volume integral of ρ equals q.

(a) $\boxed{\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}').}$ Check: $\int \rho(\mathbf{r}) d\tau = q \int \delta^3(\mathbf{r} - \mathbf{r}')\, d\tau = q.$

(b) $\int_{\mathcal{V}} |\mathbf{r} - \mathbf{b}|^2 \delta^3(5\mathbf{r})\, d\tau$, where $\mathcal{V}$ is a cube of side 2, centered on the origin, and $\mathbf{b} = 4\,\hat{\mathbf{y}} + 3\,\hat{\mathbf{z}}$.

(b) $\int (\mathbf{r} - \mathbf{b})^2 \frac{1}{5^3} \delta^3(\mathbf{r})\, d\tau = \frac{1}{125} b^2 = \frac{1}{125}(4^2 + 3^2) = \boxed{\frac{1}{5}.}$

Summary...

- Curvilinear coordinates are often more useful concerning the symmetries of a particular problem.
- Spherical polar and cylindrical polar coordinates are widely used due to the abundance of such geometry.
- Line element in curvilinear coordinates can be known from the transformation properties.
- Gradient, divergence, curl and surface and volume elements can be found therefore.
- Dirac delta function is required to represent a point source of charge, mass and non-zero divergences at a particular point.