MA 102 (Ordinary Differential Equations)

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Tutorial Sheet No. 2 **Date:** March 15, 2018

Exact differential equations; Integrating Factors; Higher-order linear IVPs; Wronskian.

(1) Under what conditions, the following differential equations are exact?
(a) $(ax + by)dx + (kx + ly)dy = 0$; (b) $[f(x) + g(y)]dx + [h(x) + l(y)]dy = 0$;
(c) $(x^3 + xy^2)dx + (ax^2y + bxy^2)dy = 0$.

Solution: (a) $b = k$; (b) $g'(y) = h'(x)$; (c) $a = 1,\ b = 0$;

(2) Are the following equations exact? If exact, obtain the general solution.
(a) $(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0$. (b) $(x - 2xy + e^y)dx + (y - x^2 + xe^y)dy = 0$.

Solution: (a) Since $M_y(x, y) = 2x = N_x(x, y)$, the equation is exact. Then, there exists $f(x, y)$ such that $f_x = M$ and $f_y = N$. Integrate $f_x = M$ w.r.t. x to have

$$f(x, y) = \int (2xy - \sec^2 x)dx + g(y) = x^2y - \tan x + g(y).$$

Now, $x^2 + 2y = f_y(x, y) = N(x, y) = x^2 + 2y \implies g'(y) = 2y$. Thus, $g(y) = y^2$. Hence, the solution to the equation is given implicitly by $f(x, y) = c$ i.e., $x^2y - \tan x + y^2 = c$.

(b) In this case, $M_y = -2x + e^y = N_x$. Arguing as in (a), we obtain

$$f(x, y) = \int (x - 2xy + e^y)dx + g(y) = \frac{x^2}{2} - x^2y + xe^y + g(y).$$

and $g(y) = y^2/2$.
The general solution is thus given by $f(x, y) = c$ i.e., $\frac{x^2}{2} - x^2y + xe^y + y^2/2 = c$.

(3) In each case find an integrating factor and solve:
(a) $y' - (2/x)y = x^2 \cos x$, (b) $ydx + (x^2y - x)dy = 0$, (c) $y(2x^2y^3 + 3)dx + x(x^2y^3 - 1)dy = 0$

Solution: (a) I.F. $\mu(x) = e^{\int p(x)dx} = e^{\int -(2/x)dx} = x^{-2}$. Multiplying by $\mu(x) = x^{-2}$ yields $\frac{d}{dx}(x^{-2}y) = \cos x$. Integrating and solving for y, we obtain $y(x) = x^2 \sin x + Cx^2$.

(b) Note that $(M_y - N_x)/N = -(2/x)$. I.F. $\mu(x) = e^{\int -(2/x)dx} = x^{-2}$. The general solution is $\frac{1}{2}y^2 - \frac{y}{x} = C$ where $x \neq 0$.

(c) $M_y = 8x^2y^3 + 3 \neq N_x = 3x^2y^3 - 1$. Multiplying the equation throughout by x^ay^b, we have

$$(2x^{a+2}y^{b+4} + 3x^ay^{b+1})dx + (x^{a+3}y^{b+3} - x^{a+1}y^b)dy = 0.$$

The exactness condition leads to

$$2(b + 4)x^{a+2}y^{b+3} + (b + 1)3x^ay^b = (a + 3)x^{a+2}y^{b+3} - (a + 1)x^ay^b.$$

Divide both sides by x^ay^b, we obtain $2b + 8 = a + 3,\ \ 3b + 3 = -a - 1 \implies a = 7/5,\ \ b = -9/5$.
Thus, I.F. is $x^{7/5}y^{-9/5}$.

(4) Show that if $(N_x - M_y)/(xM - yM) = g(xy)$ then the equation $M(x,y)dx + N(x,y)dy = 0$ has an integrating factor of the form $\mu(xy)$, where $\mu(u) = \exp\left(\int g(u)du\right)$.

Solution: Note that

$$\mu'(u) = \exp\left(\int g(u)du\right)\left(\int g(u)du\right)' = \mu(u)g(u).$$

Since $\mu(xy)Mdx + \mu(xy)Ndy = 0$ is exact, we have

$$\begin{aligned}\frac{\partial}{\partial y}\{\mu(xy)M\} &= \mu'(xy)\frac{\partial(xy)}{\partial y}M + \mu(xy)M_y \\ &= \mu(xy)\left[g(xy)xM + M_y\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}\{\mu(xy)N\} &= \mu'(xy)\frac{\partial(xy)}{\partial x}N + \mu(xy)N_x \\ &= \mu(xy)\left[g(xy)yN + N_x\right].\end{aligned}$$

Now, $(N_x - M_y) = (xM - yN)g(xy) \Longrightarrow yNg(xy) + N_x = xMg(xy) + M_y$, hence

$$\frac{\partial}{\partial y}\{\mu(xy)M\} = \frac{\partial}{\partial x}\{\mu(xy)N\}.$$

(5) Find the particular solution of
(a) $xy' + 3y = \frac{\sin x}{x^2}$, $x \neq 0$, $y(\pi/2) = 1$.
(b) $y' + y = f(x)$, $y(0) = 0$, where $f(x) = \begin{cases} 2, & 0 \leq x < 1, \\ 0, & x \geq 1. \end{cases}$
(c) $x^2y' + xy = \frac{y^3}{x}$, $y(1) = 1$, $x \neq 0$.

Solution: (a) I.F. $\mu(x) = x^3$. The general solution is $x^3y(x) = -\cos x + C$. The initial condition $y(\pi/2) = 1 \implies C = \frac{\pi^3}{8}$. The particular solution is $yx^3 + \cos x = \frac{\pi^3}{8}$.

(b) Split the problem into two IVPs: I. For $0 \leq x < 1$, $y' + y = 2$, $y(0) = 0$;
II. For $x \geq 1$, $y' + y = 0$, $y(1) = k$, where $\lim_{x\to 1-}\phi_1(x) = k$, and $\phi_1(x)$ is the solution of I. The solution of IVP I is $\phi_1(x) = 2 - 2e^{-x}$. Now,

$$\lim_{x\to 1-}\phi_1(x) = \lim_{x\to 1-}(2 - 2e^{-x}) = 2 - 2e^{-1} = k.$$

This value $k = 2 - 2e^{-1}$ would be the initial condition for IVP II. The solution of IVP II is $\phi_2(x) = (2e-2)e^{-x}$ for $x \geq 1$. The solution of the given IVP is

$$y(x) = \begin{cases} \phi_1(x) = 2 - 2e^{-x}, & 0 \leq x < 1, \\ \phi_2(x) = (2e-2)e^{-x}, & x \geq 1. \end{cases}$$

(c) Multiplying both sides of the equation by y^{-3} and substituting $v(x) = y^{-2}$ transform into a linear equation $v' - (2/x)v = -(2/x^3)$. Its general solution is $v(x) = \frac{x^{-2}}{2} + Cx^2$. Replacing $v = y^{-2}$, we obtain $y^{-2} = \frac{x^{-2}}{2} + Cx^2$. Using IC $y(1) = 1$, we obtain $y^2 = \frac{2x^2}{x^4+1}$.

(6) Given that $y_1(x) = x$ is a solution of $\dfrac{dy}{dx} = -y^2 + xy + 1$, obtain the general solution.

Solution: This is a Riccati equation with one solution $f(x) = x$ is known. The substitution $y(x) = f(x) + \frac{1}{v(x)} = x + \frac{1}{v}$ transforms the given equation to a linear equation in $v(x)$, i.e., $\frac{dv}{dx} - xv = 1$. Its solution is $e^{-x^2/2}v(x) = \int e^{-x^2/2}dx + C$. Now, replacing $v(x) = 1/(y(x) - x)$, we obtain

$$\frac{e^{-x^2}/2}{y - x} = \int e^{-x^2/2}dx + C.$$