Physics II: Electromagnetism (PH102)

Lecture 3

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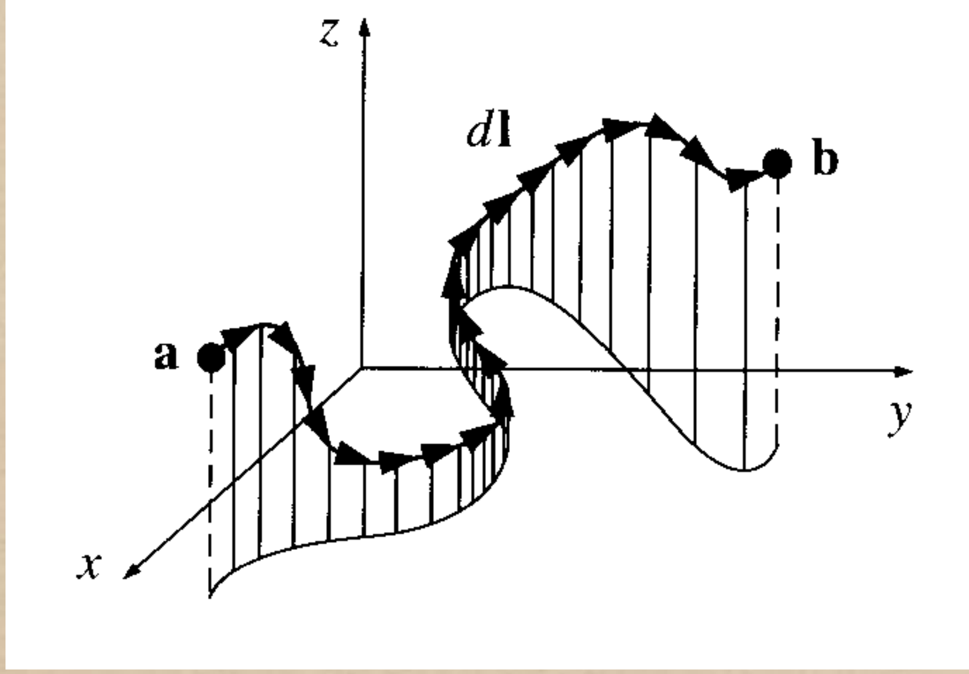
Theorems relating gradient, divergence and curl

Gradient theorem

$$\int_a^b (\vec{\nabla}T).\vec{dl} = T(b) - T(a)$$



Integral of a derivative (here gradient) of a function is given by the value of the function at boundaries

$\int_a^b (\vec{\nabla}T).\vec{dl}$ is independent of the path taken from a to b

$$\oint (\vec{\nabla}T).\vec{dl} = 0$$

Line integral of conservative vector field

Recall, for $\vec{F} = \vec{\nabla}\phi$

Then, $$\int_a^b \vec{F}.d\vec{r} = \int_a^b \vec{\nabla}\phi.d\vec{r} = \int_a^b \left(\hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}\right).(dx\hat{x} + dy\hat{y} + dz\hat{z})$$

$$= \int_a^b \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

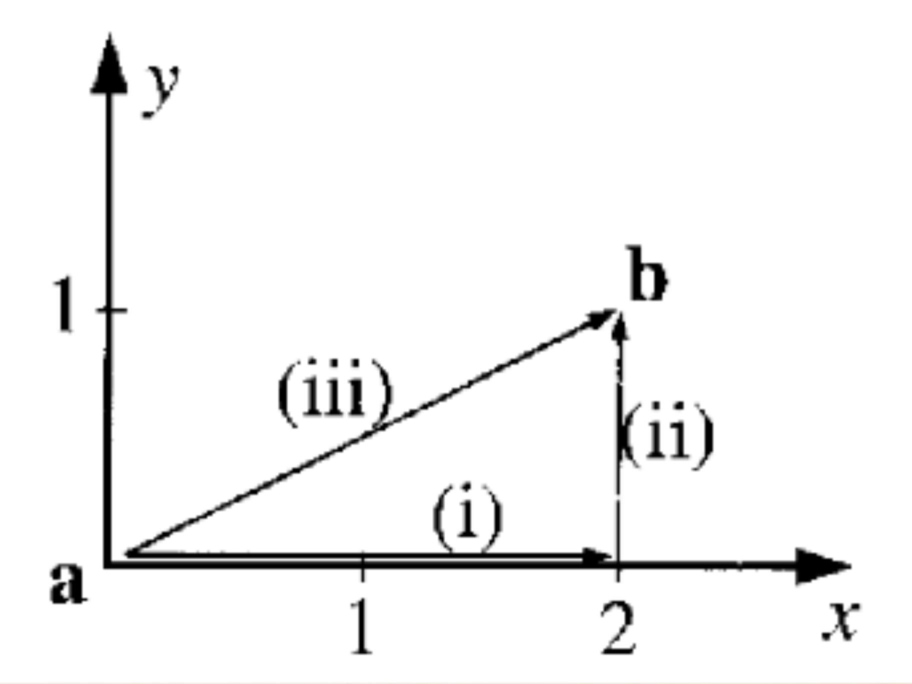
$$= \int_a^b d\phi = \phi(b) - \phi(a) \longrightarrow \oint \vec{F}.d\vec{r} = 0$$

Gradient theorem has already been proved for conservative vector fields with $\vec{F} = \vec{\nabla}\phi$

Example on gradient theorem...

- Check the fundamental theorem for gradients for $T = xy^2$ using two end points a:(0,0,0) and b:(2,1,0)

choose two paths: (i)+(ii) and (iii)



$\longrightarrow \quad \vec{\nabla}T = y^2\hat{x} + 2xy\hat{y}$

Along (i): $x : 0 \rightarrow 2;\ y = 0,\ dy = dz = 0.$

Hence, $\vec{\nabla}T.\vec{dl} = 0$

Along (ii): $x = 2;\ y : 0 \rightarrow 1,\ dx = dz = 0.$

Hence, $\vec{\nabla}T.\vec{dl} = 2xydy = 4ydy$

$$\int_{(ii)} \vec{\nabla}T.\vec{dl} = \int_0^1 4ydy = 2 \longrightarrow \int_{(i)+(ii)} \vec{\nabla}T.\vec{dl} = 2$$

Along (iii): $x : 0 \rightarrow 2;\ y = \frac{1}{2}x,\ dy = \frac{1}{2}dx.$

Hence, $\vec{\nabla}T.\vec{dl} = y^2dx + 2xydy = \frac{3}{4}x^2dx$

$$\int_{(iii)} \vec{\nabla}T.\vec{dl} = \int_0^2 \frac{3}{4}x^2dx = 2$$

Hence, proved..

Same as the other path.

Points to note…

Given a vector field if we calculate the line integral along two different paths to be the same the vector field may Not be conservative

It is very much possible, that accidentally the chosen paths yielded same answer, but if you check with another path, it will end up giving a separate answer.

Hence, to check whether a vector field is conservative or not, one better calculate the curl of it and if it is zero, then we know for sure that the line integral of that will be independent of the chosen path.

Fundamental theorem for divergence

$$\int_{\text{volume}} \vec{\nabla}.\vec{V}\, d\tau = \oint_{\text{surface}} \vec{V}.\vec{da}$$

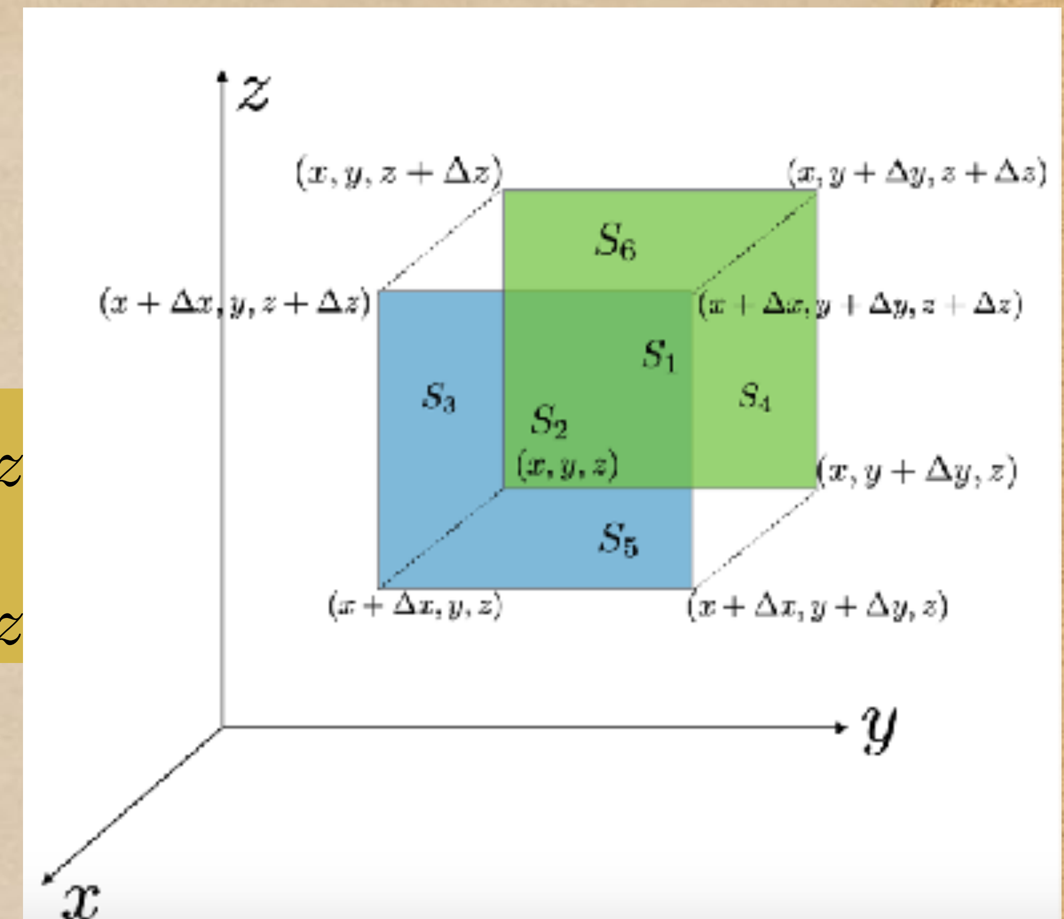
Gauss's Theorem or
Green's Theorem
Divergence Theorem

Integral (over volume) of a derivative (divergence) of a function is equal to the value of the function at boundary (surface that bounds the volume)

If $\vec{V}$ represents velocity of an incompressible fluid then the integral on RHS represents `flux' of the fluid

$$\int \text{all faucets in the volume} = \int \text{flow out through the surface}$$

Illustration: The flux of a vector 'field' through a cube



Lets choose $\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$

Outward flux through surface 1:

$$\phi_1 = \int_{\mathcal{S}_1} \vec{V}.\hat{n}da = \int_{\mathcal{S}_1} (V_x\hat{x} + V_y\hat{y} + V_z\hat{z}).(-\hat{x})dydz$$

$$\sim -V_x(\vec{r_1})\Delta y\Delta z$$

where $V_x(\vec{r_1})$ is the value of V_x at the centre of surface 1

Similarly, for flux through surface 2:

$$\phi_2 = \int_{\mathcal{S}_2} \vec{V}.\hat{n}da = \int_{\mathcal{S}_2} (V_x\hat{x} + V_y\hat{y} + V_z\hat{z}).(\hat{x})dydz$$

$$\sim V_x(\vec{r_2})\Delta y\Delta z$$

Now, $\vec{r_2} = \vec{r_1} + \hat{x}\Delta x$

One can do Taylor Series expansion :

$$V_x(\vec{r_2}) = V_x(\vec{r_1}) + \frac{\partial V_x}{\partial x}\Delta x + \mathcal{O}\{(\Delta x)^2\}$$

Example on divergence continued……

The sum of outward fluxes through surface 1 and 2 :

$$\phi_1 + \phi_2 = \frac{\partial V_x}{\partial x}\Delta x \Delta y \Delta z$$

Similarly, $$\phi_3 + \phi_4 = \frac{\partial V_y}{\partial y}\Delta x \Delta y \Delta z$$

$$\phi_5 + \phi_6 = \frac{\partial V_z}{\partial z}\Delta x \Delta y \Delta z$$

Total outward flux of V :

$$\oint_{\mathrm{Surface}} \vec{V}.\vec{da} = (\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z})\Delta x \Delta y \Delta z$$

$$= \int_{\mathrm{Volume}} \vec{\nabla}.\vec{V} dxdydz$$

Example: Divergence theorem

Evaluate $\oint_S \vec{V}.\vec{da}$ where $\vec{V} = 4xz\ \hat{x} - y^2\ \hat{y} + yz\ \hat{z}$ and S is the surface of a cube bounded by x=0, x=1, y=0, y=1, z=0 and z=1

Using divergence theorem : $\oint_S \vec{V}.\vec{da} = \int_v \vec{\nabla}.\vec{V} d\tau$

$$\text{RHS} = \int_v (4z-y)d\tau = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z-y)dzdydx = 3/2$$

- LHS: Six surface integrals

Along (i): $\vec{da} = dydz\ \hat{x}, x = 1.$

Hence, $\int \vec{V}.\vec{da} = \int V_x dydz = \int_0^1\int_0^1 4xzdydz = 2$

Along (ii): $\vec{da} = -dydz\ \hat{x}, x = 0.$

Hence, $\int \vec{V}.\vec{da} = \int V_x dydz = 0$

Along (iii): $\vec{da} = dxdz\ \hat{y}, y = 1.$

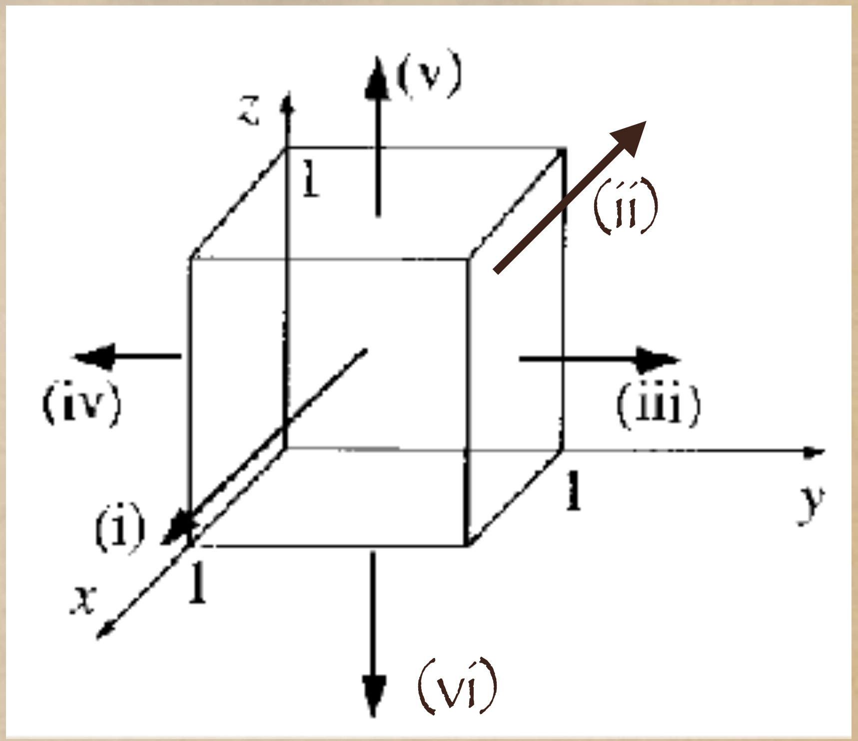
Hence, $\int \vec{V}.\vec{da} = \int V_y dxdz = \int y^2 dxdz = -1$

Along (iv): $\vec{da} = -dxdz\ \hat{y}, y = 0.$

Hence, $\int \vec{V}.\vec{da} = \int V_y dxdz = \int y^2 dxdz = 0$

Along (v): $\vec{da} = dxdy\ \hat{z}, z = 1.$

Hence, $\int \vec{V}.\vec{da} = \int V_z dxdy = \int yzdxdy = \frac{1}{2}$



Along (vi): $\vec{da} = -dxdy\ \hat{z}, z = 0.$

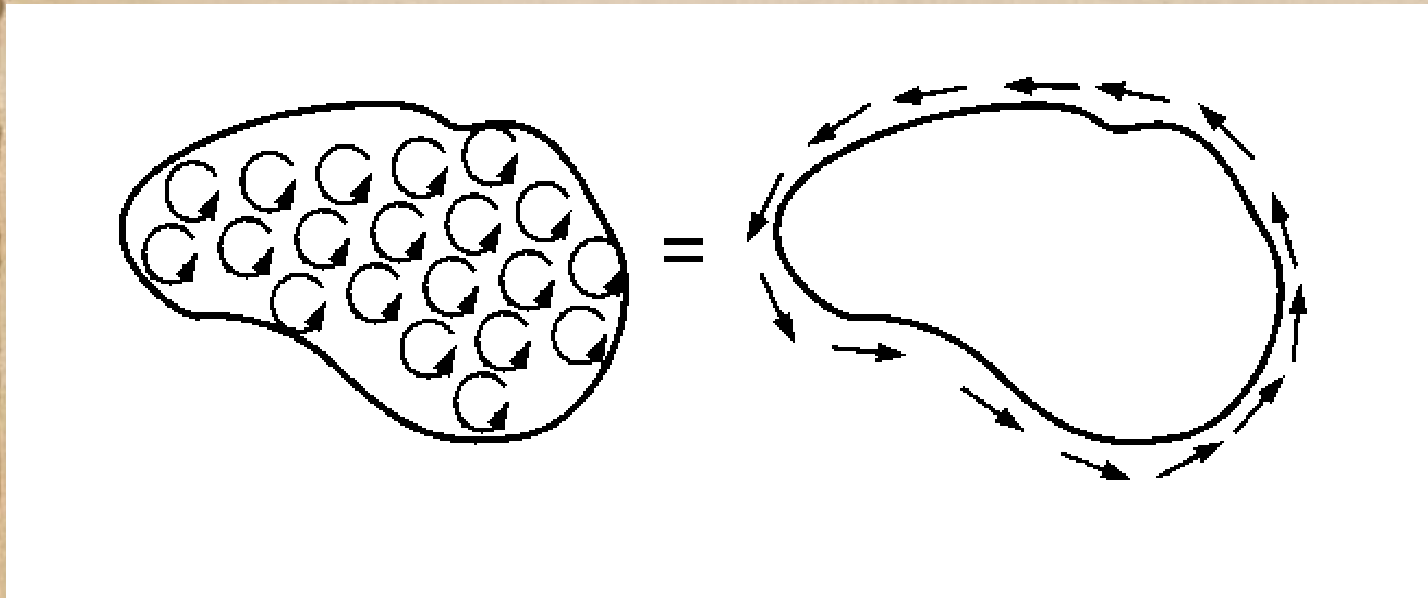
Hence, $\int \vec{V}.\vec{da} = \int yz\ dxdy = 0$

(i)+(ii)+..(vi)=3/2

Fundamental Theorem for Curls

$$\int_S (\vec{\nabla} \times \vec{V}).\vec{da} = \oint_{\text{line}} \vec{V}.\vec{dl}$$

Stoke's Theorem



Integral of a derivative (curl) over a region (surface) is equal to the value of the function at boundary (the perimeter)

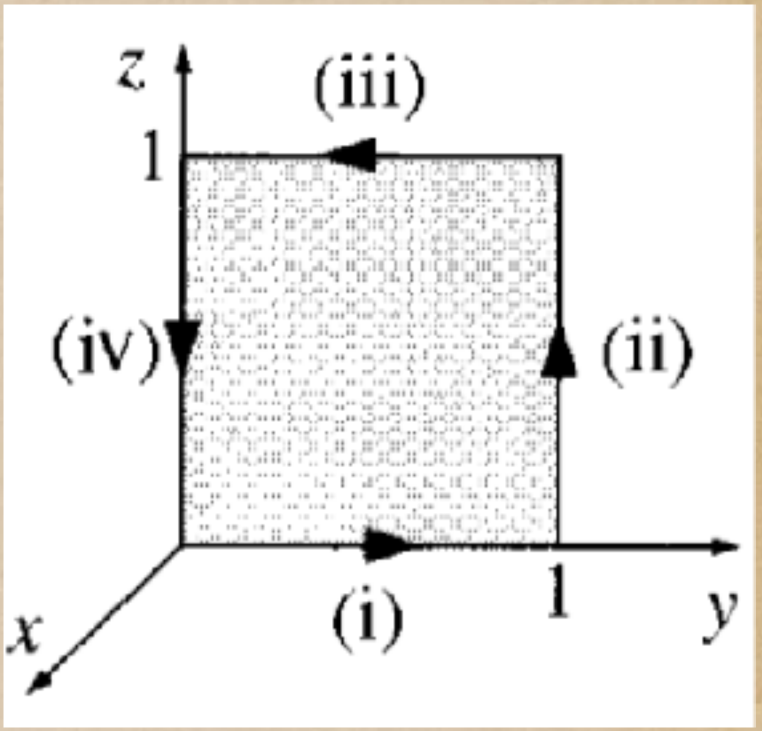
Right hand rule: if the fingers point in the direction of line integral, the thumb fixes direction of $\vec{da}$

$\int_S (\vec{\nabla} \times \vec{V}).\vec{da}$ depends only on the boundary line, not on the particular surface used as there can be infinitely many surfaces one can choose

$\oint_S (\vec{\nabla} \times \vec{V}).\vec{da} = 0$ Since the boundary shrinks to a point like balloon and RHS vanishes

Example on Stoke's theorem.........

Check stoke's theorem for $\vec{V} = (2xz + 3y^2)\hat{y} + 4yz^2\hat{z}$ for the surface shown in Fig.



$\vec{\nabla} \times \vec{V} = (4z^2 - 2x)\hat{x} + 2z\hat{z}$ and $\vec{da} = dydz\ \hat{x}$

$$\int_s (\vec{\nabla} \times \vec{V}).\vec{da} = \int_{y=0}^{1} \int_{z=0}^{1} 4z^2 dydz = 4/3$$

$(i)\ x = 0;\ y : 0 \to 1;\ z = 0;\quad \int_0^1 \vec{V}.\vec{dl} = \int_0^1 3y^2 dy = 1$

$(ii)\ x = 0;\ y = 1;\ z : 0 \to 1;\quad \int_0^1 \vec{V}.\vec{dl} = \int_0^1 4z^2 dz = 4/3$

$(iii)\ x = 0;\ y : 1 \to 0;\ z = 1;\quad \int_1^0 \vec{V}.\vec{dl} = \int_1^0 3y^2 dy = -1$

$(iv)\ x = 0;\ y = 0;\ z : 1 \to 0;\quad \int_1^0 \vec{V}.\vec{dl} = 0$

$$\oint \vec{V}.\vec{dl} = 4/3$$

Therefore $\int_S (\vec{\nabla} \times \vec{V}).\vec{da} = \oint_{\text{line}} \vec{V}.\vec{dl}$

Checked

Relations among fundamental theorems

Gradient theorem: $$\oint (\vec{\nabla} T).\vec{dl} = 0$$

From theorem on curl: $$\oint \vec{\nabla} T.\vec{dl} = \int_{sur} [\vec{\nabla} \times \vec{\nabla} T].\vec{da} = 0$$

Moreover it holds for any surface. Hence the integrand must vanish everywhere. Hence, $\vec{\nabla} \times \vec{\nabla} T = 0$ ← nothing new !!

From theorem on curl: $$\oint_S (\vec{\nabla} \times \vec{V}).\vec{da} = 0$$

Using divergence theorem:

$$\oint_{\text{sur}} (\vec{\nabla} \times \vec{V}).\vec{da} = \int_{\text{vol}} [\vec{\nabla}.(\vec{\nabla} \times \vec{V})] d\tau = 0$$

$$\vec{\nabla}.(\vec{\nabla} \times \vec{V}) = 0$$

Thus, three fundamental theorems are mutually consistent !

In summary...

- The integral of a derivative of a function equals to the value of the function at boundary
- Gradient theorem: $\int_a^b (\vec{\nabla} T).\vec{dl} = T(b) - T(a)$
- Divergence theorem: $\int_{\text{volume}} \vec{\nabla}.\vec{V} d\tau = \oint_{\text{surface}} \vec{V}.\vec{da}$
- Fundamental theorem of curls: $\int_S (\vec{\nabla} \times \vec{V}).\vec{da} = \oint_{\text{line}} \vec{V}.\vec{dl}$
- They are mutually consistent !