Constrained extrema and Lagrange multipliers
Inverse and implicit function theorems

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Constrained extrema of $f : \mathbb{R}^n \to \mathbb{R}$

Let $U \subset \mathbb{R}^n$ be open and $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Then

Maximize or Minimize $f(\mathbf{x})$
Subject to the constraint $g(\mathbf{x}) = \alpha$.

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Example: Find the extreme values of $f(x, y) = x^2 - y^2$ on the circle $x^2 + y^2 = 1$.

It turns out that f attains minimum at $(0, \pm 1)$ and maximum at $(\pm 1, 0)$ although $\nabla f(0, \pm 1) \neq 0$ and $\nabla f(\pm 1, 0) \neq 0$.

Test for constrained extrema of $f : \mathbb{R}^2 \to \mathbb{R}$

Theorem: Let $f, g : U \subset \mathbb{R}^2 \to \mathbb{R}$ be C^1. Suppose that f has an extremum at $(a, b) \in U$ such that $g(a, b) = \alpha$ and that $\nabla g(a, b) \neq (0, 0)$. Then there is a $\lambda \in \mathbb{R}$, called Lagrange multiplier, such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

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Proof: Let $\mathbf{r}(t)$ be a local parametrization of the curve $g(x, y) = \alpha$ such that $\mathbf{r}(0) = (a, b)$. Then $f(\mathbf{r}(t))$ has an extremum at $t = 0$. Therefore

$$\frac{\mathrm{d}f(\mathbf{r}(t))}{\mathrm{d}t}\Big|_{t=0} = \nabla f(a, b) \bullet \mathbf{r}'(0) = 0.$$

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Now $g(\mathbf{r}(t)) = \alpha \Rightarrow \nabla g(a, b) \bullet \mathbf{r}'(0) = 0$. This shows that $\mathbf{r}'(0) \perp \nabla g(a, b)$ and $\mathbf{r}'(0) \perp \nabla f(a, b)$. Hence $\nabla f(a, b) = \lambda \nabla g(a, b)$ for some $\lambda \in \mathbb{R}$. ■

Method of Lagrange multipliers for $f : \mathbb{R}^2 \to \mathbb{R}$

To find extremum of f subject to the constraint $g(x, y) = \alpha$, define $L(x, y, \lambda) := f(x, y) - \lambda(g(x, y) - \alpha)$ and solve the equations

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\frac{\partial L}{\partial x} &= 0 \Rightarrow \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \\
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• Critical points of L are eligible solutions for constrained extrema.

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Find the extreme values of $f(x, y) = x^2 - y^2$ on the circle $x^2 + y^2 = 1$.

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Now $f(0, 1) = f(0, -1) = -1$ and $f(1, 0) = f(-1, 0) = 1$ so that minimum and maximum values are -1 and 1.

Finding global extrema of $f : \mathbb{R}^2 \to \mathbb{R}$

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- Choose points among eligible solutions in C and the critical points at which f attains extreme values. These extreme values are global extremum.

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Find global maximum and global minimum of the function $f(x, y) := (x^2 + y^2)/2$ such that $x^2/2 + y^2 \leq 1$.

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Next, consider $L(x, y, \lambda) := (x^2 + y^2)/2 - \lambda(x^2/2 + y^2 - 1)$. Then Lagrange multiplier equations are

$$x = \lambda x, \quad y = 2\lambda y, \quad x^2/2 + y^2 = 1.$$

If $x = 0$ then $y = \pm 1$ and $\lambda = 1/2$. If $y = 0$ then $x = \pm\sqrt{2}$ and $\lambda = 1$. If $xy \neq 0$ then $\lambda = 1$ and $\lambda = 1/2$ -which is not possible.

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Thus $(0, \pm 1)$ and $(\pm\sqrt{2}, 0)$ are eligible solutions for the boundary curve. We have $f(0, \pm 1) = 1/2$, $f(\pm\sqrt{2}, 0) = 1$ and $f(0, 0) = 0$.

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Proof:

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Proof:

If $g(\mathbf{x}) = \alpha$ is a closed surface then global extremum is obtained by finding all points where $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$ and choosing those where f is largest or smallest.

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The Lagrangian is given by $L(\mathbf{x}, \lambda) := f(\mathbf{x}) - \lambda(g(\mathbf{x}) - \alpha)$. So, the multiplier equations are $\nabla L(\mathbf{x}, \lambda) = \mathbf{0}$.

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Maximize the function $f(x, y, z) := x + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

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Hence $\mathbf{p} := (1/\sqrt{2}, 0, 1/\sqrt{2})$ and $\mathbf{q} := (-1/\sqrt{2}, 0, -1/\sqrt{2})$ are eligible solutions. This shows that $f(\mathbf{p}) = \sqrt{2}$ and $f(\mathbf{q}) = -\sqrt{2}$.

What does the Implicit function theorem say?

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^1. Consider the curve

$$\mathrm{V}(F) := \{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\}.$$

Does there exist $f : \mathbb{R} \to \mathbb{R}$ such that $V(F) = \mathrm{Graph}(f)$?
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Considering $F(x, y) := x^2 + y^2 - 1$, it follows that $F(x, y) = 0$ cannot be solved for y or x.

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The implicit function theorem says that if $F(a, b) = 0$ and $\nabla F(a, b) \neq (0, 0)$ then in a neighbourhood of (a, b), we have

$$\mathrm{V}(F) = \mathrm{Graph}(f)$$

for some function f.

Implicit function theorem

Theorem: Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}$ be C^1, where U is open. Consider the curve $\mathrm{V}(F) := \{(x, y) \in U : F(x, y) = 0\}$. Let $(a, b) \in \mathrm{V}(F)$. Suppose that $\partial_y F(a, b) \neq 0$.

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- Then there exists $r > 0$ and a C^1 function $g : (a - r, a + r) \to \mathbb{R}$ such that $F(x, g(x)) = 0$ for $x \in (a - r, a + r)$.

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- Further, $\partial_x F(a, b) + \partial_y F(a, b) g'(a) = 0$.

Implicit derivative

Thus if $\partial_y f(a, b) \neq 0$ then in some disk about (a, b) the set of points (x, y) satisfying $F(x, y) = 0$ is the graph of a function $y = g(x)$ with

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Example: Consider $F(x,y) := e^{x-2+(y-1)^2} - 1$ and the equation $F(x,y)=0$. Then $F(2,1)=0$, $\partial_x F(2,1)=1$, and $\partial_y F(2,1)=0$.

Hence $x=g(y)$ for some C^1 function $g:(1-r,1+r)\to\mathbb{R}$. Moreover, $g'(1)=0$.

Implicit function theorem for $F : \mathbb{R}^{n+1} \to \mathbb{R}$

Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be C^1, where U is open. Consider the level set $\mathrm{V}(F) := \{(\mathbf{x} \in U : F(\mathbf{x}, y) = 0\}$. Let $(\mathbf{a}, b) \in \mathrm{V}(F)$. Suppose that $\partial_y F(\mathbf{a}, b) \neq 0$.

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- Further, $\partial_i F(\mathbf{a}, b) + \partial_y F(\mathbf{a}, b)\partial_i g(\mathbf{a}) = 0, i = 1, 2, \ldots, n$

Inverse function theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be linear and represented (standard basis) by an $n \times n$ matrix A. Then f is invertible on $\mathbb{R}^n \Leftrightarrow \det(A) \neq 0$.

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$$J_f(x, y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \Rightarrow \det(J_f(x, y)) = e^{2x} \neq 0.$$

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But $f(x, y) = f(x, y + 2\pi) \Rightarrow f$ is not invertible on $\mathbb{R}^2$.

Inverse function theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be linear and represented (standard basis) by an $n \times n$ matrix A. Then f is invertible on $\mathbb{R}^n \Leftrightarrow \det(A) \neq 0$.

Note that $J_f(\mathbf{x}) = A$. Thus $\det(J_f(\mathbf{x})) \neq 0 \implies f$ is invertible on $\mathbb{R}^n$.

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \longmapsto (e^x \cos y,\ e^x \sin y)$. Then

$$J_f(x, y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \Rightarrow \det(J_f(x, y)) = e^{2x} \neq 0.$$

But $f(x, y) = f(x, y + 2\pi) \Rightarrow f$ is not invertible on $\mathbb{R}^2$.

Moral: Nonsingularity of $J_f(\mathbf{x})$ does not guarantee invertibility of $f : \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$.

Fact: Let $f : \mathbb{R} \to \mathbb{R}$ be C^1 and $f'(x_0) \neq 0$. Then

- f is invertible in a neighborhood of x_0,
- the inverse is continuously differentiable, and
- $(f^{-1})'(f(x_0)) = \dfrac{1}{f'(x_0)}$.

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Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and $\det(J_f(\mathbf{a})) \neq 0$. Then there are open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ such that

- $\mathbf{a} \in U$ and $f(\mathbf{a}) \in V$,
- $f : U \longrightarrow V$ is bijective,
- $f^{-1} : V \to U$ is C^1 and $J_{f^{-1}}(f(\mathbf{a})) = (J_f(\mathbf{a}))^{-1}$.

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- Stronger version: $J_{f^{-1}}(\mathbf{y}) = (J_f(f^{-1}(\mathbf{y})))^{-1}$ for $\mathbf{y} \in V$.

Example

Consider the system $u = x\cos y$, $v = x\sin y$. Then x and y can be expressed as C^1 functions of (u, v) in a neighbourhood of (a, b) when $a \neq 0$.

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Let $f(x, y) := (x\cos y, x\sin y)$. Then

$$J_f(a, b) = \begin{bmatrix} \cos b & -a\sin b \\ \sin b & a\cos b \end{bmatrix} \Rightarrow \det(J_f(a, b)) = a.$$

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Set $(x, y) := f^{-1}(u, v)$. Then

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = J_{f^{-1}}(f(a, b)) = \begin{bmatrix} \cos b & -a\sin b \\ \sin b & a\cos b \end{bmatrix}^{-1}.$$

*** End ***