Basic Definitions, Existence and Uniqueness Results for First-Order IVP

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Texts/References:

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Dennis G. Zill and Michael R. Cullen, Differential equations with boundary value problems

Definition: An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is said to be a differential equation(DE).

Definition: A DE involving ordinary derivatives of one or more dependent variables w.r.t a single independent variable is called an ordinary differential equation(ODE).

A general form of the nth order ODE:

$$F(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x)) = 0, \tag{1}$$

where $y'(x) = \frac{dy}{dx},\ y''(x) = \frac{d^2y}{dx^2}, \cdots, y^{(n)}(x) = \frac{d^ny}{dx^n}$.

- The order of a DE is the order of the highest derivative that occurs in the equation.
- The degree of a DE is the power of the highest order derivative occurring in the differential equation.
- Eq. (1) is linear if F is linear in $y, y', y'', \ldots, y^{(n)}$, with coefficients depending on the independent variable x. Eq. (1) is called nonlinear if it is not linear.

Examples:

- $y''(x) + 3y'(x) + xy(x) = 0$
  (second-order, first-degree, linear)
- $y''(x) + 3y(x)y'(x) + xy(x) = 0$
  (second-order, first-degree, nonlinear)
- $(y''(x))^2 + 3y'(x) + xy^2(x) = 0$
  (second-order, second-degree, nonlinear)

Definition: A DE involving partial derivatives of one or more dependent variables w.r.t more than one independent variable is called a partial differential equation(PDE).
A PDE for a function $u(x_1, x_2, \ldots, x_n)$ $(n \geq 2)$ is a relation of the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_1 x_1}, u_{x_1 x_2}, \ldots,) = 0, \quad (2)$$

where F is a given function of the independent variables $x_1, x_2, \ldots, x_n$, and of the unknown function u and of a finite number of its partial derivatives.

Examples:

- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ (first-order equation)
- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (second-order equation)

We shall consider only ODE.

Definition: A function $\phi(x) \in C^n((a,b))$ that satisfies

$$F(x, \phi(x), \phi'(x), \phi''(x), \cdots, \phi^n(x)) = 0, \quad x \in (a,b)$$

is called an explicit solution to the equation on (a,b).

Example: $\phi(x) = x^2 - x^{-1}$ is an explicit solution to

$$y''(x) - 2\frac{y}{x^2} = 0.$$

Note that $\phi(x)$ is an explicit solution on $(-\infty, 0)$ and also on $(0, \infty)$.

Definition: (Initial Value Problem)
Find a solution $y(x) \in C^n((a,b))$ that satisfies

$$F(x, y, y'(x), \cdots, y^{(n)}(x)) = 0, \quad x \in (a,b)$$

and the n initial conditions(IC)

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1},$$

where $x_0 \in (a,b)$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.

First-order IVP: $F(x, y, y'(x)) = 0, \quad y(x_0) = y_0$.

Second-order IVP: $F(x, y, y'(x), y''(x)) = 0$,
$y(x_0) = y_0,\ y'(x_0) = y_1$.

Example: The function $\phi(x) = \sin x - \cos x$ is a solution to IVP: $y''(x) + y(x) = 0, \ \ y(0) = -1,\ y'(0) = 1$. on $\mathbb{R}$.

Consider the following IVPs:

$$|y'| + 2|y| = 0, \quad y(0) = 1 \text{ (no solution)}.$$

$$y'(x) = x, \quad y(0) = 1 \text{ (a unique solution } y = \frac{1}{2}x^2 + 1).$$

$$xy'(x) = y - 1, \quad y(0) = 1 \text{ (many solutions } y = 1 + cx).$$

Observation:

Thus, an IVP

$$F(x, y, y') = 0, \quad y(x_0) = y_0$$

may have none, precisely one, or more than one solution.

Well-posed IVP

An IVP is said to be well-posed if

- it has a solution,
- the solution is unique and,
- the solution is continuously depends on the initial data y_0 and f.

Theorem **(Peano's Theorem)**:

Let $\mathscr{R} \;:\; |x - x_0| \le a,\ |y - y_0| \le b$ be a rectangle. If $f \in C(\mathscr{R})$ then the IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

has at least one solution $y(x)$. This solution is defined for all x in the interval $|x - x_0| \le h$, where

$$h = \min\{a,\ \frac{b}{K}\}, \quad K = \max_{(x,y)\in\mathscr{R}} |f(x, y)|.$$

Example: Let $\mathscr{R}\ :\ |x-0| \leq 3,\ |y-0| \leq 3$ be a rectangle. Let $f(x,y) = xy$. Then $f \in C(\mathscr{R})$. Then the IVP

$$y'(x) = f(x,y), \quad y(0) = 0$$

has at least one solution $y(x)$. This solution is defined for all x in the interval $|x-0| \leq h$, where

$$h = \min\{3,\ \frac{3}{K}\}, \quad K = \max_{(x,y)\in\mathscr{R}} |xy|.$$

Theorem **(Picard's Theorem):**

Let $f \in C(\mathscr{R})$ and satisfy the Lipschitz condition with respect to y in $\mathscr{R}$, i.e., there exists a number L such that

$$|f(x,y_2) - f(x,y_1)| \leq L|y_2 - y_1| \quad \forall (x,y_1),\, (x,y_2) \in \mathscr{R}.$$

Then, the IVP

$$y'(x) = f(x,y), \quad y(x_0) = y_0$$

has a unique solution $y(x)$. This solution is defined for all x in the interval $|x - x_0| \leq h$, where

$$h = \min\{a,\ \frac{b}{K}\}, \quad K = \max_{(x,y)\in\mathscr{R}} |f(x,y)|$$

Example: Consider the IVP:

$$y'(x) = |y|, \quad y(1) = 1.$$

$f(x, y) = |y|$ is continuous and satisfies Lipschitz condition w.r.t y in every domain $\mathscr{R}$ of the xy-plane. The point $(1, 1)$ certainly lies in some such domain $\mathscr{R}$. The IVP has a unique solution ϕ defined on some $|x - 1| \leq h$ about $x_0 = 1$.

Corollary to Picard's Theorem:

Let f, $\frac{\partial f}{\partial y} \in C(\mathscr{R})$. Then the IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

has a unique solution $y(x)$. This solution is defined for all x in the interval $|x - x_0| \leq h$, where

$$h = \min\{a, \ \frac{b}{K}\}, \quad K = \max_{(x,y)\in\mathscr{R}} |f(x, y)|.$$