Variation of Parameters, Use of a Known Solution to Find Another and Cauchy-Euler Equation

Department of Mathematics
IIT Guwahati

Variation of Parameters

The variation of parameter is a more general method for finding a particular solution (y_p). The method applies even when the coefficients of the differential equation are functions of x.

Consider $L(y) = g(x)$, where

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y,$$

where $p_{n-1}(x), \ldots, p_0(x) \in C(I)$. We know the general solution to $L(y) = g$ is given by

$$y(x) = y_h(x) + y_p(x),$$

where y_h is the general solution to $Ly = 0$ and $y_p(x)$ is a particular solution to $L(y) = g$.

Suppose we know a fundamental solution set $\{y_1, \ldots, y_n\}$ for $L(y) = 0$. Then

$$y_h(x) = C_1 y_1(x) + \cdots + C_n y_n(x).$$

In this method, seek a particular solution y_p of the form

$$y_p(x) = v_1(x) y_1(x) + \cdots + v_n(x) y_n(x),$$

and try to determine the functions $v_1, \ldots, v_n$.

Differentiating y_p,

$$y_p' = \sum_{i=1}^{n} v_i y_i' + \sum_{i=1}^{n} v_i' y_i.$$

To avoid second and higher-order derivatives of v_i's, we impose the condition

$$\sum_{i=1}^{n} v_i' y_i = 0. \tag{1}$$

Therefore,

$$y_p' = \sum_{i=1}^{n} v_i y_i', \qquad \text{if } \sum_{i=1}^{n} v_i' y_i = 0$$

Again, differentiating y_p', we obatin

$$y_p'' = \sum_{i=1}^{n} v_i y_i'' + \sum_{i=1}^{n} v_i' y_i' = \sum_{i=1}^{n} v_i y_i'', \qquad \text{if } \sum_{i=1}^{n} v_i' y_i' = 0$$

Therefore,

$$y_p' = \sum_{i=1}^{n} v_i y_i', \qquad \text{if } \sum_{i=1}^{n} v_i' y_i = 0$$

erentiating y_p', we obatin

Therefore,

$$y_p' = \sum_{i=1}^{n} v_i y_i', \qquad \text{if } \sum_{i=1}^{n} v_i' y_i = 0$$

erentiating y_p', we obatin

Therefore,

$$y_p' = \sum_{i=1}^{n} v_i y_i', \qquad \text{if } \sum_{i=1}^{n} v_i' y_i = 0$$

erentiating y_p', we obatin

Therefore,

$$y_p' = \sum_{i=1}^{n} v_i y_i', \qquad \text{if } \sum_{i=1}^{n} v_i' y_i = 0$$

erentiating y_p', we obatin

$$L(y_p) = v_1\left(y_1^{(n)} + p_{n-1}y_1^{(n-1)} + p_{(n-2)}y_1^{(n-3)} + \cdots + p_0 y_1\right) +$$
$$v_2\left(y_2^{(n)} + p_{n-1}y_2^{(n-1)} + p_{(n-2)}y_2^{(n-3)} + \cdots + p_0 y_2\right) + \cdots +$$
$$v_n\left(y_n^{(n)} + p_{n-1}y_n^{(n-1)} + p_{(n-2)}y_n^{(n-3)} + \cdots + p_0 y_n\right) + \sum_{i=1}^{n} v_i' y_i^{(n-1)}.$$

Therefore, if we seek $v_1', \ldots, v_n'$ that satisfy the system

$$\begin{aligned} y_1 v_1' + \cdots + y_n v_n' &= 0, \\ y_1' v_1' + \cdots + y_n' v_n' &= 0, \\ \vdots \quad + \quad \vdots \quad + \vdots &= \vdots \\ y_1^{(n-1)} v_1' + \cdots + y_n^{(n-1)} v_1' &= g. \end{aligned}$$

then

$$L(y_p) = v_1 \times 0 + v_2 \times 0 + \cdots + v_n \times 0 + g = g$$

$\Longrightarrow y_p$ is a particular solution of $L(y) = g$.

$$L(y_p) = v_1\left(y_1^{(n)} + p_{n-1}y_1^{(n-1)} + p_{(n-2)}y_1^{(n-3)} + \cdots + p_0 y_1\right) +$$
$$v_2\left(y_2^{(n)} + p_{n-1}y_2^{(n-1)} + p_{(n-2)}y_2^{(n-3)} + \cdots + p_0 y_2\right) + \cdots +$$
$$v_n\left(y_n^{(n)} + p_{n-1}y_n^{(n-1)} + p_{(n-2)}y_n^{(n-3)} + \cdots + p_0 y_n\right) + \sum_{i=1}^{n} v_i' y_i^{(n-1)}.$$

Therefore we can solve the matrix equation to obtain $v_1', \ldots, v_n'$,

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-2)(x)} & y_2^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} v_1'(x) \\ v_2'(x) \\ \vdots \\ \vdots \\ v_n'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ g(x) \end{bmatrix}.$$

Because

$$\begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W(y_1, \ldots, y_n)(x) \neq 0$$

on I, which is true as $\{y_1, \ldots, y_n\}$ is a fundamental solution set.

Therefore we can solve the matrix equation to obtain $v_1', \ldots, v_n'$,

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-2)(x)} & y_2^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} v_1'(x) \\ v_2'(x) \\ \vdots \\ \vdots \\ v_n'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ g(x) \end{bmatrix}.$$

$$v_k'(x) = \frac{\begin{vmatrix} y_1(x) & \cdots & 0 & \cdots & y_n(x) \\ \vdots & & \vdots & & \\ y_1^{(n-2)(x)} & \cdots & 0 & \cdots & y_n^{(n-2)(x)} \\ y_1^{(n-1)(x)} & \cdots & g(x) & \cdots & y_n^{(n-1)(x)} \end{vmatrix}}{W(y_1, y_2, \cdots, y_n)(x)}$$

i.e, $$v_k'(x) = \frac{g(x)W_k(x)}{W(y_1, \ldots, y_n)(x)}, \quad k = 1, \ldots, n,$$

where $W_k(x)$ is obtained from $W(y_1, \ldots, y_n)(x)$ by replacing kth column by $[0, \ldots, 0, 1]^T$.
We can express $W_k(x)$ as

$$W_k(x) = (-1)^{(n-k)} W(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)(x)$$

for $k = 1, \ldots, n$.

Integrating $v_k'(x)$ yields

$$v_k(x) = \int \frac{g(x)W_k(x)}{W(y_1, \ldots, y_n)(x)} dx, \quad k = 1, \ldots, n.$$

Finally, substituting the v_k's back into y_p, we obtain

$$y_p(x) = v_1(x)y_1(x) + \cdots + v_n(x)y_n(x)$$

we obtain

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{g(x)W_k(x)}{W(y_1, \ldots, y_n)(x)} dx.$$

For $n = 2$, v_1' and v_2' are given by

$$v_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ g(x) & y_2'(x) \end{vmatrix}}{W(y_1, y_2)(x)} = \frac{-g(x)y_2(x)}{W(y_1, y_2)(x)}, \quad v_2'(x) = \frac{g(x)y_1(x)}{W(y_1, y_2)(x)},$$

where $W(y_1, y_2)(x) \neq 0$. Integrating these equations, we obtain

$$v_1(x) = \int \frac{-g(x)y_2(x)}{W(y_1, y_2)(x)} dx, \quad v_2(x) = \int \frac{g(x)y_1(x)}{W(y_1, y_2)(x)} dx.$$

Thus, the particular solution is given by

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x).$$

Example: Consider $y'' + y = \ cosec\, x$.

$$y_h(x) = c_1 \sin x + c_2 \cos x.$$

The two linearly independent solutions are $y_1(x) = \sin x$ and $y_2(x) = \cos x$ and $W(y_1, y_2) = -1 \neq 0$.

$$v_1(x) = \int \frac{-g(x)y_2(x)}{W(y_1,y_2)(x)}dx = \int \frac{-\cos x\, cosec\, x}{-1}dx = \log(\sin x).$$

$$v_2(x) = \int \frac{g(x)y_1(x)}{W(y_1,y_2)(x)}dx = \int \frac{\sin x\, cosec\, x}{-1}dx = -x.$$

$$y_p = \sin x \log(\sin x) - x \cos x.$$

The general solution is
$y(x) = c_1 \sin x + c_2 \cos x + \sin x \log(\sin x) - x \cos x.$

Use of a known solution to find another

Assume that $y_1(x) \neq 0$ is a known solution of $L(y) = 0$, where

$$L(y) = y'' + p(x)y' + q(x)y.$$

We know $L(cy_1) = 0$, where c is any arbitrary constant. Replace c by an unknown function $v(x)$ so that $L(y_2) = 0$, where $y_2 = v(x)y_1(x)$.

Suppose $L(y_2) = L(vy_1) = 0$. Then, we have

$$v(y_1'' + py_1' + qy_1) + v''y_1 + v'(2y_1' + py_1) = 0.$$

Since $L(y_1) = 0$, we have

$$v''y_1 + v'(2y_1' + py_1) = 0 \Rightarrow \frac{v''}{v'} = -2\frac{y_1'}{y_1} - p.$$

$$\frac{v''}{v'} = -2\frac{y_1'}{y_1} - p \Longrightarrow \frac{z'}{z} = -2\frac{y_1'}{y_1} - p, \quad z = v'.$$

Integrating

$$z(x) = \frac{1}{y_1^2}e^{-\int pdx} \Longrightarrow v(x) = \int \frac{1}{y_1^2}e^{-\int pdx}dx.$$

Thus, the second solution is $y_2(x) = v(x)y_1(x)$.

Example: Given that $y_1 = e^x$ is a solution to $y'' - 2y' + y = 0$. Determine the second linear independent solution y_2.

Note that $v(x) = x$. The second linearly dependent solution is

$$y_2(x) = vy_1 = xe^x.$$

Cauchy-Euler Equation

An equation of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x),$$

where a_i's are constants is called Cauchy-Euler equation.

The substitution $x = e^t$ transform the above equation into an equation with constant coefficients. For simplicity, take $n = 2$.

Assume that $x > 0$ and let $x = e^t$. By the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t = x\frac{dy}{dx},$$

hence

$$x\frac{dy}{dx} = \frac{dy}{dt}.$$

Differentiating $x\frac{dy}{dx} = \frac{dy}{dt}$ with respect to t, we find that

$$\begin{aligned}
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{dx}{dt}\frac{dy}{dx} + x\frac{d}{dt}\left(\frac{dy}{dx}\right) \\
&= \frac{dy}{dt} + x\frac{d^2y}{dx^2}\frac{dx}{dt} = \frac{dy}{dt} + x\frac{d^2y}{dx^2}e^t \\
&= \frac{dy}{dt} + x^2\frac{d^2y}{dx^2}.
\end{aligned}$$

Thus

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

Substituting into the equation we obtain the constant coefficient ODE

$$a_2\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+a_1\frac{dy}{dt}+a_0y=g(e^t),$$

which may be written as

$$a_2\frac{d^2y}{dt^2}+(a_1-a_2)\frac{dy}{dt}+a_0y=g(e^t).$$

Note: Observe that in the proof it is assumed that $x>0$. If $x<0$, the substitution $x=-e^t$ will reduced the Cauchy-Euler equation to constant coefficients ODE. The method can be applied to higher-order Cauchy-Euler equation.

Example: Consider $x^2y'' - 2xy' + 2y = x^3, \quad x > 0$.
Setting $x = e^t$, we obtain

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = e^{3t},$$

or

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}.$$

The GS to the homogeneous equation is

$$y_h(x) = c_1e^t + c_2e^{2t} = c_1x + c_2x^2.$$

To find a particular solution, let $y_p = Ae^{3t}$. Then, $A = \frac{1}{2}$ hence, $y_p = \frac{1}{2}e^{3t} = \frac{1}{2}x^3$. The GS is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1x + c_2x^2 + \frac{1}{2}x^3, \quad x > 0. \end{aligned}$$

*** End ***