MA 102 (Ordinary Differential Equations)

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Tutorial Sheet No. 12 **Date:** April 25, 2016

Systems of Linear Differential Equations.

(1) Rewrite the given scalar equation as a first-order system in normal form. Express the system in the matrix form $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$.
(a) $y''(t) - 3y'(t) - 11y(t) = \sin t$; (b) $y^{(4)}(t) + y(t) = t^2$.

(2) Determine over what interval we are assured that there is a unique solution to the following initial value problem.
(a) $\mathbf{x}'(t) = \begin{bmatrix} \cos t & \sqrt{t} \\ t^3 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \tan t \\ e^t \end{bmatrix}, \ \mathbf{x}(2) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.
(b) $\mathbf{x}'(t) = \begin{bmatrix} t^2 & 1+3t \\ 1 & \sin t \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \ \mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(3) The vector functions $\mathbf{x}_1 = [e^{-t}, 2e^{-t}, e^{-t}]^T$, $\mathbf{x}_2 = [e^t, 0, e^t]^T$, $\mathbf{x}_3 = [e^{3t}, -e^{3t}, 2e^{3t}]^T$ are solutions to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Determine whether they form a fundamental solution set. If they do, find a fundamental matrix for the system and give a general solution.

(4) Let $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ be two fundamental matrices for the same system $\mathbf{x}'(t) = A\mathbf{x}$. Then, there exists a nonsingular constant matrix $\mathbf{C}$ such that $\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{C}$.

(5) Find a fundamental matrix of the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ by computing e^{At}.
(a) $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, (b) $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, (c) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (d) $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$.

(6) Solve the initial value problems $\mathbf{x}'(t) = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ for the matrix
(a) $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$, (b) $A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, (c) $A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$,
(d) $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(7) Let $\mathbf{x}(t)$ be a nontrivial solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $A + A^T$ is positive definite. Prove that $\|\mathbf{x}(t)\|$ is an increasing function of t. (Here, $\|\cdot\|$ denotes the Euclidean norm.)

(8) Let A be a real 3×3 matrix such that $A^T = -A$. Let $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$ be a real solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Prove that
(a) $\|\mathbf{x}(t)\|$ is independent of t.
(b) If $\mathbf{v} \in Ker(A)$ then $\mathbf{x}(t) \cdot \mathbf{v}$ is independent of t.

(9) Solve the nonhomogeneous linear system $\mathbf{x}(t) = A\mathbf{x}(t) + \mathbf{f}(t)$ with initial condition $\mathbf{x}(0) = [1, 0]^T$, where (a) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix}$; (b) $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} t \\ 1+2t \end{bmatrix}$.

(10) Show that $\Phi(t) = \begin{bmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{bmatrix}$ is a fundamental matrix solution of the nonautonomous linear system $\mathbf{x}'(t) = A(t)\mathbf{x}$ with $A(t) = \begin{bmatrix} -2\cos^2 t & -1-\sin 2t \\ 1-\sin 2t & -2\sin^2 t \end{bmatrix}$. Find the inverse of $\Phi(t)$ and

solve $\mathbf{x}'(t) = A(t)\mathbf{x} + \mathbf{f}(t),\ \mathbf{x}(0) = \mathbf{x}_0$ with $A(t)$ as given above and $\mathbf{f}(t) = [1,\ \ e^{-2t}]^T$.

(11) Find all critical points of each of the following plane autonomous systems:
$(a)\ x_1'(t) = -x_1 + x_2,\ \ x_2'(t) = x_1 - x_2;\ (b)\ x_1'(t) = x_1^2 + x_2^2 - 6,\ \ x_2'(t) = x_1^2 - x_2.$
$(c)\ x_1'(t) = x_1^2 e^{x_2},\ \ x_2'(t) = x_2(e^{x_1} - 1).$

(12) Determine the nature of critical point $(0,0)$ of each of the linear autonomous systems $\mathbf{x}'(t) = A\mathbf{x}(t)$, and determine whether or not the critical point is stable.
$(a)\ A = \begin{bmatrix} 5 & -3 \\ 4 & -3 \end{bmatrix},\ (b)\ A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix},\ \ (c)\ \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}.$