Systems of First Order Differential Equations

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A first order system of n (not necessarily linear) equations in n unknown functions $x_1(t)$, $x_2(t)$, $\ldots$, $x_n(t)$ in normal form is given by

$$\begin{aligned} x_1'(t) &= f_1(t, x_1, x_2, \ldots, x_n), \\ x_2'(t) &= f_2(t, x_1, x_2, \ldots, x_n), \\ &\vdots \\ x_n'(t) &= f_n(t, x_1, x_2, \ldots, x_n). \end{aligned}$$

Higher-order differential equations often can be rewritten as first-order system. We can convert the nth order ODE

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}) \tag{1}$$

into a first-order system as follows.

Setting

$$x_1(t) := y(t),\ x_2(t) := y'(t),\ \ldots,\ x_n(t) := y^{(n-1)}(t).$$

we obtain n first-order equations:

$$\begin{aligned}
x_1'(t) &= y'(t) = x_2(t),\\
x_2'(t) &= y''(t) = x_3(t),\\
&\vdots\\
x_{n-1}'(t) &= y^{(n-1)}(t) = x_n(t),\\
x_n'(t) &= y^{(n)}(t) = f(t, x_1, x_2, \ldots, x_n).
\end{aligned} \tag{2}$$

If (1) has n initial conditions:

$$y(t_0) = \alpha_1,\ \ y'(t_0) = \alpha_2,\ \ \ldots,\ y^{(n-1)}(t_0) = \alpha_n,$$

then the system (2) has initial conditions:

$$x_1(t_0) = \alpha_1,\ \ x_2(t_0) = \alpha_2,\ \ \ldots,\ x_{(n)}(t_0) = \alpha_n.$$

Example: $y''(t) + 3y'(t) + 2y(t) = 0;\ \ y(0) = 1,\ \ y'(0) = 3.$

Setting

$$x_1(t) := y(t) \quad \text{and} \quad x_2(t) := y'(t)$$

we obtain

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -3x_2(t) - 2x_1(t). \end{aligned}$$

The ICs transform to $x_1(0) = 1,\ x_2(0) = 3.$

We shall consider only linear systems of first-order ODEs.

Consider the linear system in the normal form:

$$
\begin{aligned}
x_1'(t) &= a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t),\\
x_2'(t) &= a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t),\\
&\vdots\\
x_n'(t) &= a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t).
\end{aligned}
$$

In matrix and vector notations, we write it as

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t), \tag{3}$$

where $\mathbf{x}(t) = [x_1(t), \ldots, x_n(t)]^T$, $\mathbf{f}(t) = [f_1(t), \ldots, f_n(t)]^T$, and $A(t) = [a_{ij}(t)]$ is a $n \times n$ matrix.

When $\mathbf{f} = 0$ the linear system (3) is said to be homogeneous.

Definition: The IVP for the system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{4}$$

is to find a vector function $\mathbf{x}(t) \in C^1$ that satisfies the system (4) on an interval I and the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0 = (x_{1,0}, \ldots, x_{n,0})^T$, where $t_0 \in I$ and $\mathbf{x}_0 \in \mathbb{R}^n$.

Theorem: (Existence and Uniqueness)

Let $A(t)$ and $\mathbf{f}(t)$ are continuous on I and $t_0 \in I$. Then, for any choice of $\mathbf{x}_0 = (x_{1,0}, \ldots, x_{n,0})^T \in \mathbb{R}^n$, there exists a unique solution $\mathbf{x}(t)$ to the IVP

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

on the whole interval I.

Example: Consider the IVP:

$$\mathbf{x}'(t) = \begin{bmatrix} t^3 & \tan t \\ t & \sin t \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \sqrt{1-t} \\ 0 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This IVP has a unique solution on the interval $(-\pi/2, 1)$.

Definition: The Wronskian of n vector functions $\mathbf{x}_1(t) = (x_{1,1}, \ldots, x_{n,1})^T, \ldots, \mathbf{x}_n(t) = (x_{1,n}, \ldots, x_{n,n})^T$ is defined as

$$\begin{aligned} W(\mathbf{x}_1, \ldots, \mathbf{x}_n)(t) \quad &:= \quad \begin{vmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{vmatrix} \\ &= \quad \det[\mathbf{x}_1 \ \mathbf{x}_2 \ \ldots \ \mathbf{x}_n]. \end{aligned}$$

Theorem: Let $A(t)$ is an $n \times n$ matrix of continuous functions. If $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$ are linearly independent solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}$ on I, then $W(t) := \det[\mathbf{x}_1 \ \mathbf{x}_2 \ \ldots \ \mathbf{x}_n] \neq 0$ on I.

Proof. Suppose $W(t_0) = 0$ at some point $t_0 \in I$. Now, $W(t_0) = 0 \implies \mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, ..., $\mathbf{x}_n(t_0)$ are L.D. . Then, $\exists$ scalars $c_1, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) + \ldots + c_n\mathbf{x}_n(t_0) = \mathbf{0}.$$

Note that $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t)$ and $\mathbf{z}(t) = \mathbf{0}$ are both solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$ on I and $\sum_{i=1}^{n} c_i\mathbf{x}_i(t_0) = \mathbf{z}(t_0) = 0$. By the existence and uniqueness theorem

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_n\mathbf{x}_n(t) = \mathbf{0}, \ \ \forall t \in I$$

which contradicts to the fact that $\mathbf{x}_1$, ..., $\mathbf{x}_n$ are L.I. . Hence, $W(t_0) \neq 0$. Since $t_0 \in I$ is arbitrary, the result follows.

Theorem:(**Abel's formula**)

If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are n solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, then

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \left\{ \sum_{i=1}^{n} a_{ii}(s) \right\} ds \right),$$

where a_{ii}'s are the main diagonal elements of A.

Proof: Prove for $n = 3$.

Fact:

- The Wronskian of solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ is either zero or never zero on I.
- A set of n solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ on I is linearly independent on I if and only if $W(\mathbf{x}_1, \ldots, \mathbf{x}_n)(t) \neq 0$ on I.

Representation of Solutions

Theorem:(Homogeneous case)
Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n linearly independent solutions to

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t), \quad t \in I,$$

where $A(t)$ is continuous on I. Then, every solution to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ can be expressed in the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t),$$

where c_i's are constants.

Definition: A set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of n linearly independent solutions to

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t), \quad t \in I \qquad (*)$$

is called a fundamental solution set for $(*)$ on I.

The matrix $\Phi(t)$ defined by

$$\begin{aligned} \Phi(t) &:= [\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ \ldots\ \mathbf{x}_n(t)] \\ &= \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{bmatrix} \end{aligned}$$

is called a fundamental matrix for $(*)$.

Note: 1. We can use $\Phi(t)$ to express the general solution

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \Phi(t)\mathbf{c}, \text{ where } \mathbf{c} = (c_1, \ldots, c_n)^T.$$

2. Since $\det \Phi(t) = W(\mathbf{x}_1, \ldots, \mathbf{x}_n) \neq 0$ on $I \implies \Phi(t)$ is invertible for every $t \in I$.

Example: The set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix},$$

is a fundamental solution set for the system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ on $\mathbb{R}$, where $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Note that $A\mathbf{x}_i(t) = \mathbf{x}_i'(t)$, $i = 1, 2, 3$. Further,

$$W(t) = \begin{vmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{vmatrix} = -3 \neq 0.$$

The fundamental matrix $\Phi(t) = \begin{bmatrix} e^{2t} & -e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & e^{-t} & -e^{-t} \end{bmatrix}$.

Thus, the GS is

$$\mathbf{x}(t) = \Phi(t)\mathbf{c} = c_1 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}.$$

Theorem:(Non-homogeneous case)
let $\mathbf{x}_p$ be a particular solution to

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t), \quad t \in I, \qquad (**)$$

and let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a fundamental solution set on I for the corresponding homogeneous system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$. Then every solution to $(**)$ can be expressed in the form

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t) \\ &= \Phi(t)\mathbf{c} + \mathbf{x}_p(t). \end{aligned}$$

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