Indian Institute of Technology Guwahati

Quiz-1, Even Semester, 2014-15

Time: 50 Minutes | *MA 102 Mathematics-II* | Marks: 10

Roll No.: ____________ Name: ________________ Signature: ________________

1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by $f(x,y) = \begin{cases} \dfrac{x}{y} - \dfrac{y}{x} & \text{if } xy \neq 0; \\ 0 & \text{otherwise.} \end{cases}$

   (a) Prove or disprove: f is continuous at $(0,0)$. [1]

   Answer: Put $x = r\cos\theta$ and $y = r\sin\theta$ in $|f(x,y) - f(0,0)|$ then we get $|\cot\theta - \tan\theta|$ which (for certain θ) does not tend to 0 as $r \to 0$.

   1 mark for full correct answer. Otherwise, zero.

   (b) Find all possible directions along which directional derivatives of f at $(0,0)$ exist. [1]

   Answer: Let $V = (u,v)$ with $\|V\| = 1$. Then

$$D_V f(0,0) = \lim_{t\to 0} \frac{f(0+tu, 0+tv) - f(0,0)}{t}$$

   (i) If $uv = 0$, then $D_V f(0,0) = 0$.

   (Till this, one will get 0.5 marks)

   (ii) If $uv \neq 0$, then $D_V f(0,0) = \lim_{t\to 0} \frac{1}{t}\left(\frac{u}{v} - \frac{v}{u}\right)$. Hence, limit exists iff $\frac{u}{v} - \frac{v}{u} = 0$.

   (If one answers only second part, s/he will get only 0.5 marks)

   So all possible directions $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}), (\pm 1, 0), (0, \pm 1)$. Total eight directions.

2. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by $f(x,y) = \begin{cases} xy\log(x^2+y^2) & \text{if } x^2+y^2 \neq 0; \\ 0 & \text{otherwise.} \end{cases}$

(Here, log is the natural logarithm to the base e.)

(a) Show that $f_x(0,0) = f_y(0,0) = 0$. [1]

Answer: $f_x(0,0) = \lim\limits_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \frac{0}{h} = 0$ (0.5 marks)

$f_y(0,0) = \lim\limits_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim\limits_{k\to 0} \frac{0}{k} = 0$ (0.5 marks)

(b) Check the differentiability of f at $(0,0)$. [1]

Answer: Note that $e(h,k) = \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \frac{hk\log(h^2+k^2)}{\sqrt{h^2+k^2}}$ (0.5 marks)

Put $h = r\cos\theta,\ k = r\sin\theta$. Then $|e(h,k)| = |\sin 2\theta\ r\log r| \leq r|\log r| \to 0$ as $r \to 0$. (0.5 marks)

(c) Prove that $(0,0)$ is a saddle point of f. (**Hint:** It is recommended to avoid second derivative test for Part (c)) [1]

Answer: In any neighborhood of $(0,0)$ of radius r, consider $0 < a < \min\{0.5, r\}$. Note that $f(a,a) < 0 = f(0,0) < f(-a,a)$. Hence, $(0,0)$ is a saddle point for f.

1 mark for full correct answer. Otherwise, zero. However, strictly following the definition, a meritorious argument will be given 0.5 marks.

3. Consider the surface $z = 3x^2 - y^2$. Write the equation of the tangent plane, if exists, to the surface at $(0, 0, 0)$. (Do not write your calculations here.) [1]

Answer: $z = 0$.

As such calculations are not expected. However, in case someone shows calculations, marks are awarded as follows. Correct calculations with correct answer gets 1 mark. Zero marks will be awarded in each of the cases: (i) wrong calculations with correct answer (ii) correct calculation with wrong answer (iii) multiple answers (iv) anything equivalent to these.

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4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function. Suppose $f \circ g$ is differentiable at $t = 0$ for all functions $g : \mathbb{R} \longrightarrow \mathbb{R}^2$ with $g(0) = (0, 0)$. Show that directional derivatives of f at $(0, 0)$ exist in all directions. [2]

Answer:

Let $V = (u, v)$ with $\|V\| = 1$. Consider

$$D_V f(0,0) = \lim_{t \to 0} \frac{f(tu, tv) - f(0,0)}{t}$$

Take $g(t) = (tu, tv)$, then

$$D_V f(0,0) = \lim_{t \to 0} \frac{f \circ g(t) - f \circ g(0)}{t}$$

exists as $f \circ g$ is differentiable at $t = 0$. Hence, $D_V f(0,0)$ exists for all V.

1 mark for considering the function g and 1 mark for rest of the correct answer.

5. Let D be a closed subset of $\mathbb{R}^2$. Show that the complement of D in $\mathbb{R}^2$, i.e. $\mathbb{R}^2 \setminus D$, is an open subset of $\mathbb{R}^2$. [2]

Answer:

Write $O = \mathbb{R}^2 \setminus D$ and assume O is not open. Then there exists $X_0 \in O$ which is not an interior point of O. That is, for each $n \in \mathbb{N}$, there exists $X_n \in \mathbb{R}^2$ such that

$$\|X_n - X_0\| < \frac{1}{n}$$

but $X_n \notin O$. It is clear that the sequence (X_n) converges to X_0.

Since $X_n \notin O$, we have $X_n \in D$, for all n. But since D is closed, the limit of (X_n) is in D. That is, $X_0 \in D$; a contradiction. Hence, O is open.

1 mark for assuming O is not open and constructing a sequence.

1 mark for rest of the correct answer.