# Physics II: Electromagnetism (PH102)

## Lecture 10

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# Potential in integral form

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's law in the form of the equation

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\vec{r}\,')d\tau'$$

Unfortunately, integrals of this type may be difficult to calculate for many charge configurations except a few simple ones.

Occasionally we can get around this by exploiting symmetry and using Gauss's law, but ordinarily the best strategy is first to calculate the potential, V, given by a somewhat more tractable equation:

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\vec{r}\,')d\tau'}$$

However, this is also cumbersome in many cases and also subject to the knowledge of charge density, and its form.

# Potential: Differential Equation

The way out is to recast the problem in differential form, using Poisson's eqn:

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

which together with appropriate boundary conditions is equivalent to

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\scriptstyle{\mathcal{R}}}\rho(\vec{r}\,')d\tau'.$$

Very often, we are interested in finding the potential in a region where $\rho = 0$. In this case Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0$$

In Cartesian coordinate

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Note: If $\rho = 0$ everywhere then of course $V = 0$, but that's not what we meant. There may be plenty of charges everywhere, but we are confining ourselves to places where there is no charge.

# Ordinary Differential equations and boundary conditions

A typical differential equation in single variable is given with an interval and boundary conditions

$$y''(x) = \frac{d^2y}{dx^2} = 0; \quad a \leq x \leq b$$

$$\text{Boundary conditions :} \quad \begin{aligned} y(a) &= \alpha \\ y(b) &= \beta \end{aligned}$$

Before applying the boundary conditions, the general solution of a differential equation is expressed in terms of arbitrary constants.

Particular solution of a differential equation is obtained from the given boundary conditions.

An ordinary differential equation (ODE) is an equation containing a function of one independent variable and its derivatives. Ex: The equation given above.

# Quick discussion on ODE

Let us start with $$y''(x) = \frac{d^2y}{dx^2} = 0; \quad a \leq x \leq b$$

| Boundary Condition: | Solution | Nature of Solution |
|---|---|---|
| No condition (General Solution) | $y = mx + c$ | Solution exist; Not unique |
| $y(a) = \alpha$ | $y = m(x-a) + \alpha$ | Solution exist; Not unique |
| $y(a) = \alpha\,,\ y(b) = \beta$ | $y = \frac{\beta - \alpha}{b - a}x + \frac{\alpha b - \beta a}{b - a}$ | Solution exist; Unique |
| $y'(a) = \alpha, y(b) = \beta$ | $y = \alpha(x - b) + \beta$ | Solution exist; Unique |
| $y'(a) = \alpha,\ y'(b) = \beta$ | No solution | Solution does not exist |

The goal is to know the

Existence of solution

Uniqueness of solution

for a differential equation

# Quick discussion on differential equations

Now, in general if the function y depends on more than one variable, then the differential equation becomes partial differential equation.

Wave Equation $$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0$$

Schrodinger Equation $$i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi(\vec{r},t) + V(\vec{r})\psi(\vec{r},t)$$

Poisson's Equation $$\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} +\right) = -\frac{\rho}{\epsilon_0}$$

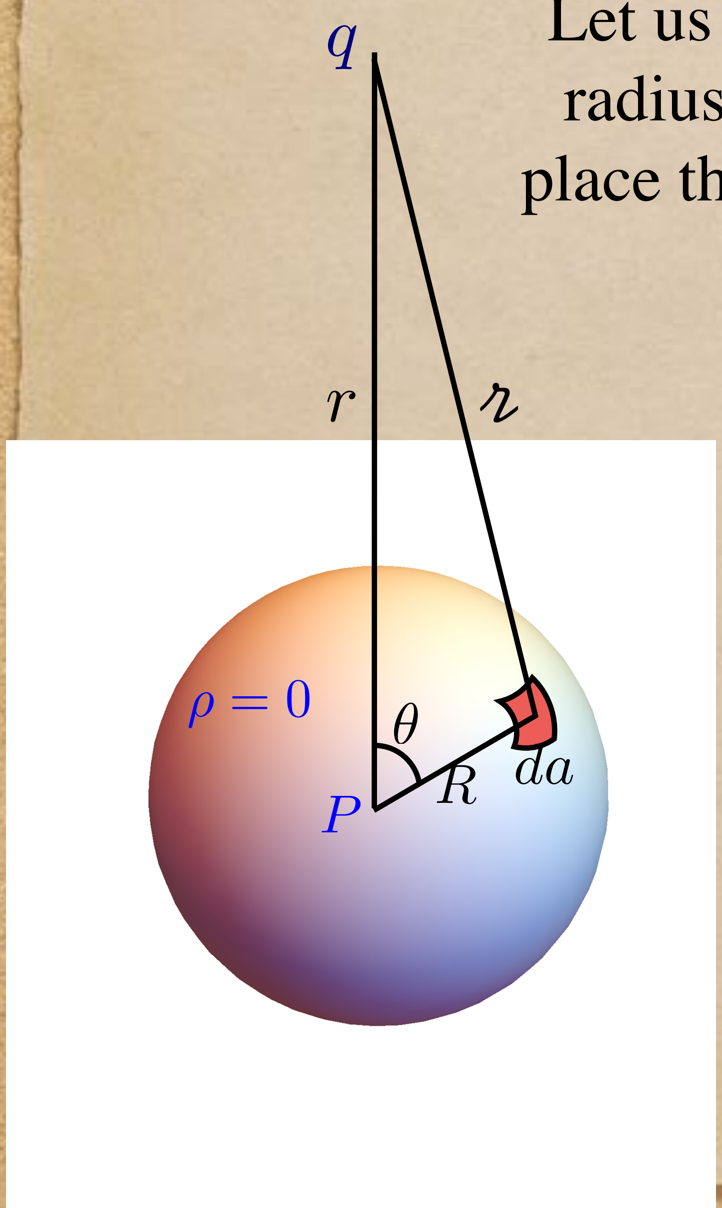
The general solution to PDEs does not contain just two arbitrary constants, neither it contains any finite number despite the fact that it is a second-order equation.

Since, now the boundaries contain infinitely many points and boundary conditions may become more complicated.

We will focus only on those types of PDEs which will be of interest to us in the discussion of electrostatics, mainly the Laplace's equation: $\nabla^2 V = 0$

# Properties of the electrostatic potential: A Theorem:

Consider the electrostatic potential *V(r)* in a region free of charge. (There can of course be charges outside this region producing the $V$. Show that $\bar{V}$, its average value over a sphere of any radius ***R*** equals *V(0)*, its value at the centre. (Note that this sphere is a mathematical surface and lies within the charge free region)




Let us calculate the average of the potential over a spherical surface of radius R due to a single point charge q located outside the sphere. We place the centre of the sphere such that q lies on the z axis. The potential on the surface of the sphere is:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}} \quad \text{where,} \quad \mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta}$$

Therefore, the average of $V$ over the spherical surface

$$\begin{aligned}
\bar{V} &= \frac{1}{4\pi R^2}\frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{q}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}}R^2\sin\theta d\theta d\phi \\
&= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int_0^{\pi}\frac{q}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}}\sin\theta d\theta \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2rR}[(r+R) - (r-R)] = \frac{1}{4\pi\epsilon_0}\frac{q}{r} = V(0) = V(P)
\end{aligned}$$

# Properties of electrostatic potential

Consider a charge free region $\mathcal{V}$. Let point $P$ is inside $\mathcal{V}$. Generalisation of the previous theorem: Consider a collection of point charges $q_1, q_2, \cdots, q_n$ all placed outside the region $\mathcal{V}$. Let $V_1, V_2, \cdots, V_n$ be the potentials due to these charges respectively. Then for each charge $q_i$

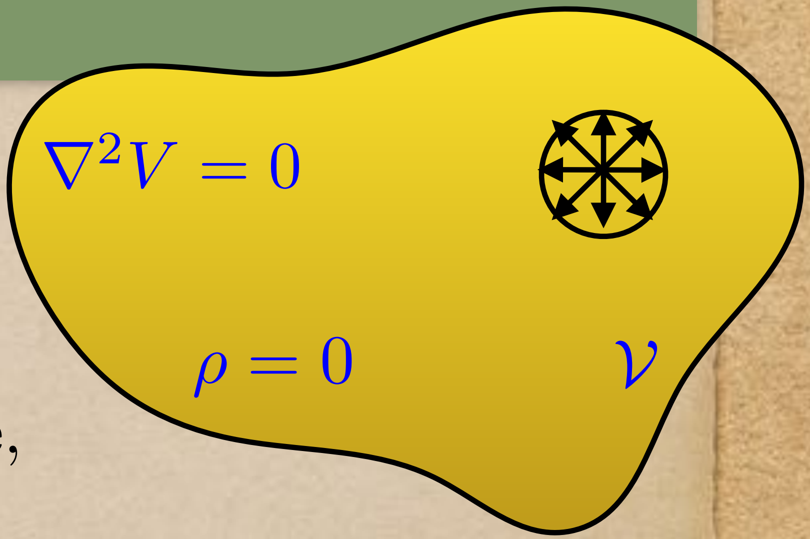
$$\boxed{V_i(0) = V_i(P) = \bar{V}_i = \frac{1}{A}\int_{\mathcal{V}} V_i(\vec{r}_i)\,da}$$

and due to superposition principle

$$\begin{aligned} V(P) &= \sum_{i=1}^{n} V_i(P) \\ &= \sum_{i=1}^{n} \bar{V}_i \\ &= \frac{1}{A}\int_{\mathcal{V}} \left[\sum_{i=1}^{n} V_i(\vec{r}_i)\right] da = \bar{V} \end{aligned}$$

# Properties of electrostatic potential

Let $\mathcal{V}$ be a charge free region. There can be no maximum or minimum in electrostatic potential in $\mathcal{V}$.




Since there is no charge inside $\mathcal{V}$: $\nabla^2 V = 0$

But note that the charges can be present outside as well as on the surface of $\mathcal{V}$.

We know that potentials will be there inside, on the surface, as well as outside $\mathcal{V}$.

Let us choose a point $P$ inside and say that the potential at $P$ has a maxima.

If potential $V$ has a maxima at $P$, then $V$ must be decreasing everywhere surrounding the point $P$.

Now, we know that in whichever directions potential decreases, the electric field is directed in those directions. $\vec{E} = -\vec{\nabla} V$.

So, if potential is decreasing in the surroundings of $P$, then electric field lines will always come out of a Gaussian surface chosen around $P$!

That means there is a positive flux coming out of $P$, but that requires the presence of positive charge at $P$. But $\rho = 0$ inside $\mathcal{V}$. There is no charge.

Our assumption was wrong. There can be no maximum or minimum in electrostatic potential$\mathcal{V}$

# Boundary conditions and Uniqueness theorem:

Laplace's equation does not by itself determine $V$. Suitable boundary conditions must be supplied.

Q. What are appropriate boundary conditions?

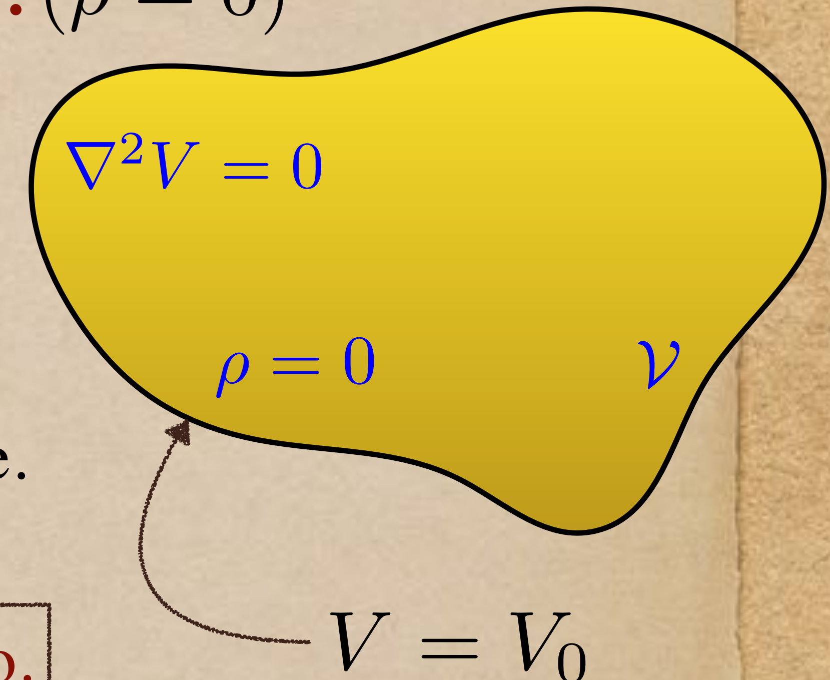
We have seen that this is easy in one dimension. The general solution is of the form $V=mx+c$. This contains two arbitrary constants $m$ and $c$ and we therefore require two boundary conditions.

Two and three dimensional cases are not easy, we confronted PDEs. The boundaries are not two points (like in one dimension) any more. They are surfaces and a surface contains infinitely many points and hence the boundary conditions become more complicated.

But, by now your intuition may tell you that $V$ is uniquely determined by its value at the boundary.

# First Uniqueness Theorem

**The solution to Laplace's Equation in some volume $\mathcal{V}$ is uniquely determined if $V$ is specified on the boundary surface $\mathcal{S}$ . $(\rho = 0)$**




Suppose the solution is not unique in the volume.

Say, there are two solutions to the Laplace's equation: $V_1, \ V_2$

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0 \quad (\text{in } \mathcal{V})$$

Both of which assumes specific values on the boundary surface.

i.e. $V_1$, $V_2$ takes value $V_0$ on the surface.

We want to prove that they must be equal inside the volume too.

The trick is to look at their difference: $V_3 \equiv V_1 - V_2$

Since $V_3$ is a linear superposition of $V_1$ and $V_2$, it must obey Laplace's equation:

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0 \quad \text{(inside the volume)}$$

What is the status of $V_3$ on the surface?

$V_3 = 0$ on the surface because $V_1 = V_2 = V_0$ on the surface and $V_3 \equiv V_1 - V_2$

So, we have $V_3 = 0$ on $\mathcal{S}$ and also $\nabla^2 V_3 = 0$. But, Laplace's equation allows no local maxima or minima. So maxima and minima of $V_3$ are both zero. Therefore $V_3$ must be zero everywhere inside $\mathcal{V}$. $\implies V_1 = V_2$.

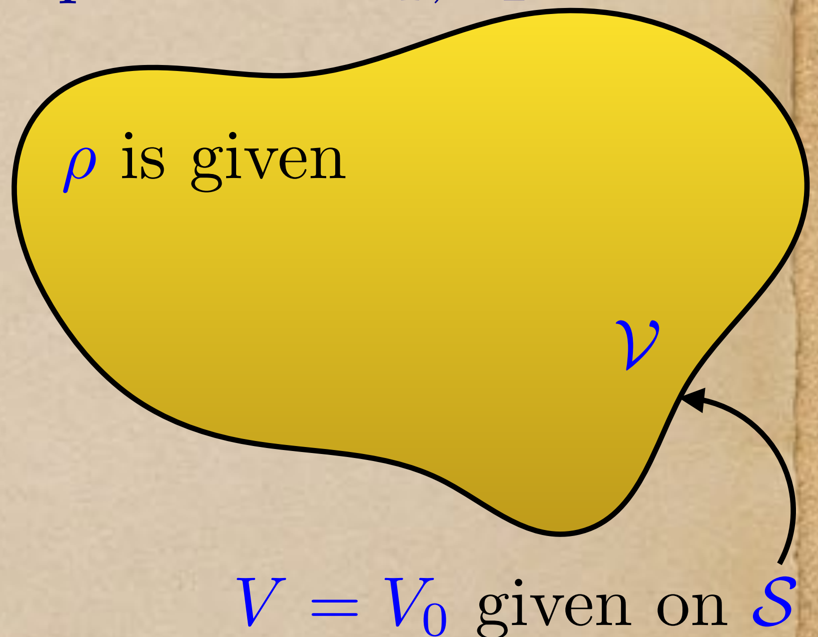
# First Uniqueness Theorem: Improvement

The potential in a volume $\mathcal{V}$ is uniquely determined if (a) the charge density $\rho$ throughout the region, and (b) the value of $V$ on the boundaries, are specified.

Suppose the solution is not unique in the volume $\mathcal{V}$.

Say, there are two solutions in $\mathcal{V}$ satisfying Poisson's equations: $V_1, V_2$

Now, we have $\nabla^2 V_1 = -\frac{\rho}{\epsilon_0}$ and $\nabla^2 V_2 = -\frac{\rho}{\epsilon_0}$




The trick is to look at their difference: $V_3 \equiv V_1 - V_2$

so $$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0$$

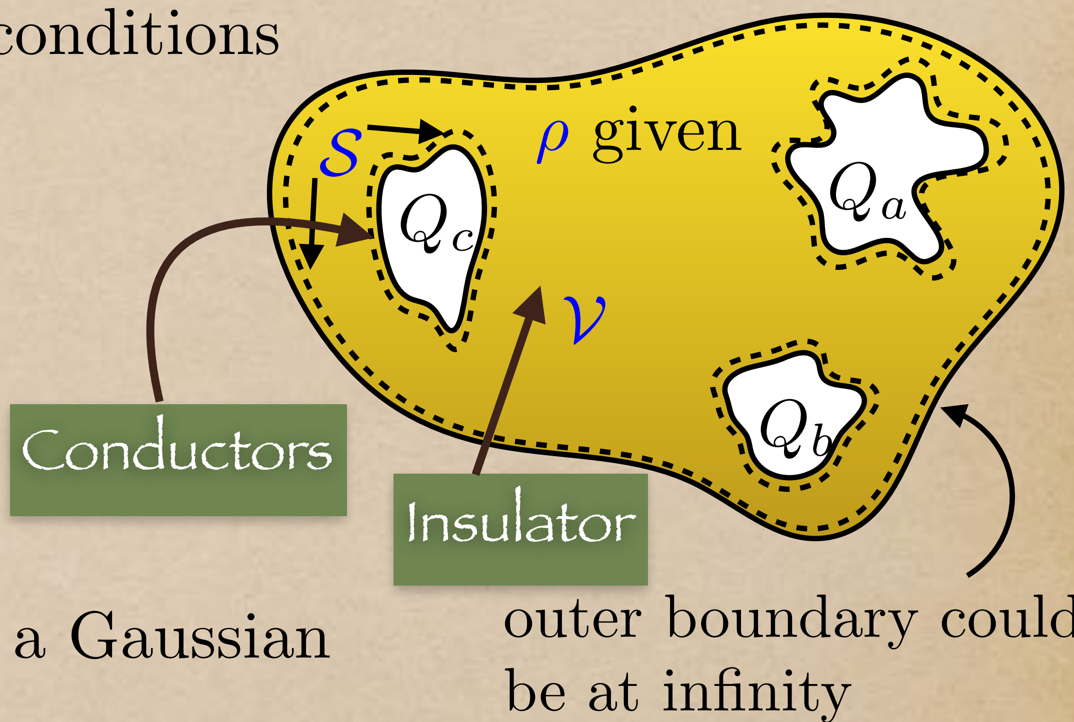
Once again the difference $V_3 = V_1 - V_2$ satisfies Laplace's equation and has the value zero on all boundaries, so $V_3 = 0$ and hence what we had assumed is incorrect.

$V_1 = V_2$ in the volume.

# Conductors and Second Uniqueness Theorem

In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density $\rho$, the electric field is uniquely determined if the total charge on each conductor is given (The region as a whole can be bounded by another conductor, or else unbounded).

Suppose there are two fields satisfying the conditions of the problem. $\vec{E}_1, \vec{E}_2$




In $\mathcal{V}$: region between conductors

$$\vec{\nabla}.\vec{E}_1 = \frac{\rho}{\epsilon_0} + \frac{1}{\epsilon_0}\sum_i Q_i \delta^3(\vec{r} - \vec{r_i})$$

$$\vec{\nabla}.\vec{E}_2 = \frac{\rho}{\epsilon_0} + \frac{1}{\epsilon_0}\sum_i Q_i \delta^3(\vec{r} - \vec{r_i})$$

Both obeys Gauss's law in integral form for a Gaussian surface enclosing each conductor

$$\oint_{\substack{\text{i-th conducting}\\ \text{surface}}} \vec{E}_1.d\vec{a} = \frac{Q_i}{\epsilon_0}, \qquad \oint_{\substack{\text{i-th conducting}\\ \text{surface}}} \vec{E}_2.d\vec{a} = \frac{Q_i}{\epsilon_0}$$

Similarly, for the outer boundary

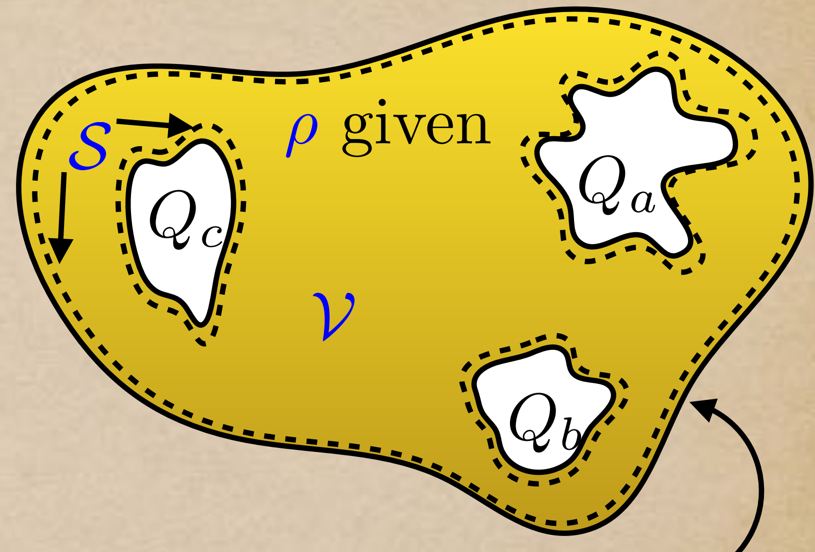
$$\oint_{\substack{\text{outer}\\ \text{boundary}}} \vec{E}_1.d\vec{a} = \frac{Q_{\text{tot}}}{\epsilon_0}, \qquad \oint_{\substack{\text{outer}\\ \text{boundary}}} \vec{E}_2.d\vec{a} = \frac{Q_{\text{tot}}}{\epsilon_0}$$

$$Q_{\text{tot}} = \int \rho \, d\tau + \sum_i Q_i$$

# Conductors and Second Uniqueness Theorem

As before, we examine the difference $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$; which obeys $\vec{\nabla}.\vec{E}_3 = 0$ in the region between the conductors and

$\oint \vec{E}_3.d\vec{a} = 0$ over each boundary surface.




All though we don't know how the charge $Q_i$ is distributed over the $i$-th conductor, we do know that each conductor is an equipotential surface and hence $V_3$ is constant (not necessarily same constant) over each conducting surface.

outer boundary could be at infinity

Use $\vec{\nabla}.(V_3\vec{E}_3) = V_3(\vec{\nabla}.\vec{E}_3) + \vec{E}_3.(\vec{\nabla}V_3) = -(E_3^2)$ since $\vec{\nabla}.\vec{E}_3 = 0$, $\vec{E}_3 = -\vec{\nabla}V_3$

Using Divergence theorem: $\displaystyle\int_{\mathcal{V}} \vec{\nabla}.(V_3\vec{E}_3)d\tau = \oint_{\mathcal{S}} V_3\vec{E}_3.d\vec{a} = -\int_{\mathcal{V}} (E_3)^2 d\tau$

Covers all boundaries of the region

$V_3$ is a constant over each surface and comes out of integral. But $\int \vec{E}_3.d\vec{a} = 0 \implies \int_{\mathcal{V}} (E_3)^2 d\tau = 0 \implies \vec{E}_3 = 0 \implies \vec{E}_1 = \vec{E}_2$.

# In summary...

- Potential can be found out by solving the partial differential equation in the form of Poisson's equation or Laplace's equation
- In a charge free region, the average of potential on spherical surface is equal to the potential at the centre of the sphere.
- When potential solves Laplace's equation in a certain region, it can not have any local maxima or minima.
- A region, where potential solves Laplace's or Poisson's equation and is specified at the boundary, the potential will have an unique solution inside: First Uniqueness theorem.
- In a region surrounded by conductors with specific charges and the region in between with a specific charge density, will have unique electric field : Second Uniqueness theorem