Solutions for Tutorial 3

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1. Suppose f is not uniformly continuous. Then $\exists\ \epsilon > 0$ such that for any $\delta > 0$, there will exists points X and Y in D s.t. $||X-Y|| < \delta$ but, $|f(X) - f(Y)| \geq \epsilon$.
Taking $\delta = \frac{1}{k}$, we get that for each $k \in \mathbb{N}$ and X_k, $Y_k \in D$ such that $||X_k - Y_k|| < \frac{1}{k}$ but,

$$|f(X_k) - f(Y_k)| \geq \epsilon. \tag{1}$$

Now since D is bounded the sequence $\{X_k\}$ is bounded and so by Bolzano Weierstrass Theorem it has a convergent subsequence. So $\exists X \in \mathbb{R}^n$ and a subsequence $< X_{k_l} >$ s.t. $X_{k_l} \to X$ as $l \to \infty$.
Now since $||Y_{k_l} - X|| \leq ||Y_{k_l} - X_{k_l}|| + ||X_{k_l} - X|| < \frac{1}{k_l} + ||X_{k_l} - X||$, we have $Y_{k_l} \to X$ as $l \to \infty$. Since D is closed $X \in D$. Now f is continuous at X so $f(X_{k_l}) \to f(X)$ and $f(Y_{k_l}) \to f(X)$ as $l \to \infty$. Thus $f(X_{k_l}) - f(Y_{k_l}) \to 0$ as $l \to \infty$.
But this is a contradicts (1). Hence we are done.

2. Let $X_k = (\sqrt{k\pi}, 0, 0, \cdots, 0)$ and $Y_k = (\sqrt{k\pi + \frac{\pi}{2}}, 0, 0, \cdots, 0)$.Then

$$||X_k - Y_k|| = \sqrt{k\pi + \frac{\pi}{2}} - \sqrt{k\pi} = \frac{\frac{\pi}{2}}{\sqrt{k\pi + \frac{\pi}{2}} + \sqrt{k\pi}} < \frac{\frac{\pi}{2}}{\sqrt{k\pi}} \to 0$$

as $k \to \infty$.
But

$$|f(X_k) - f(Y_k)| = |sin||X_k||^2 - sin||Y_k||^2| = 1 \nrightarrow 0.$$

Thus f is not uniformly continuous.

3. By Lagrange's MVT, $|f(x) - f(y)| = |f'(z)||x-y|$ for some z between x and y.
Thus

$$\begin{aligned} |f(x) - f(y)| &= \frac{1}{2\sqrt{z}}|x-y| \\ &\leq \frac{1}{2}|x-y| \quad \text{[since } x, y \geq 1] \end{aligned}$$

Thus $f(x) = \sqrt{x}$ is Lipschitz on $[1, \infty)$.

4.

$$F \times G(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_1(t) & F_2(t) & F_3(t) \\ G_1(t) & G_2(t) & G_3(t) \end{vmatrix} = (F_2(t)G_3(t) - F_3(t)G_2(t))\hat{i} + (F_3(t)G_1(t) - F_1(t)G_3(t))\hat{j} + (F_1(t)G_2(t) - F_2(t)G_1(t))\hat{k}$$

Thus

$$\begin{aligned} (F \times G)'(t) &= [F_2'(t)G_3(t) + F_2(t)G_3'(t) - (F_3'(t)G_2(t) + F_3(t)G_2'(t))]\hat{i} \\ &+ [F_3'(t)G_1(t) + F_3(t)G_1'(t) - (F_1'(t)G_3(t) + F_1(t)G_3'(t))]\hat{j} \\ &+ [F_1'(t)G_2(t) + F_1(t)G_2'(t) - (F_2'(t)G_1(t) + F_2(t)G_1'(t))]\hat{k} \\ &= F'(t) \times G(t) + F(t) \times G'(t) \end{aligned}$$

5.a.

$$\begin{aligned} r(\theta) &= (2cos^2\theta, 2cos\theta sin\theta), 0 \leq \theta \leq \pi \\ r'(\theta) &= (-4cos\theta sin\theta, -2sin^2\theta + 2cos^2\theta) \end{aligned}$$

Thus Arc length
$= \int_0^{\pi} \sqrt{16cos^2\theta sin^2\theta + 4sin^4\theta + 4cos^4\theta - 8cos^2\theta sin^2\theta} d\theta$
$= \int_0^{\pi} \sqrt{4sin^4\theta + 4cos^4\theta + 8cos^2\theta sin^2\theta} d\theta$
$= 2\int_0^{\pi} (sin^2\theta + cos^2\theta)$
$= 2\pi$

b.

$$r(t) = (t^2, t^3), 1 \leq t \leq 2$$
$$r'(t) = (2t, 3t^2)$$

Arc length
$= \int_1^2 \sqrt{4t^2 + 9t^4} dt$
$= \int_1^2 t\sqrt{4 + 9t^2} dt$
$= \frac{1}{18} \int_{13}^{40} \sqrt{z} dz,$ put $4 + 9t^2 = z$
$= \frac{1}{27}(40^{\frac{3}{2}} - 13^{\frac{3}{2}})$

6.a.

$$r(t) = (\frac{t^2}{2}\hat{i}, \frac{t^3}{3}\hat{k}), 1 \leq t \leq 2$$
$$r'(t) = (t\hat{i}, t^2\hat{k})$$

Thus
$s(t)$
$= \int_0^t \sqrt{u^2 + u^4} du$
$= \int_0^t u\sqrt{1 + u^2} du$
$= \frac{1}{2} \int_0^t 2u\sqrt{1 + u^2} du$
$= \frac{1}{2} \int_1^{1+t^2} \sqrt{z} dz$
$= \frac{1}{3}[(1 + t^2)^{\frac{3}{2}} - 1]$
or, $(1 + t^2)^{\frac{3}{2}} = 3s + 1$
or, $1 + t^2 = (3s + 1)^{\frac{2}{3}}$
or, $t = \sqrt{(3s + 1)^{\frac{2}{3}} - 1}$

Thus the arc length parametrization is given by,
$r(s) = \frac{((3s+1)^{\frac{2}{3}} - 1)}{2}\hat{i} + \frac{((3s+1)^{\frac{2}{3}} - 1)^{\frac{3}{2}}}{3}\hat{k}$

b.

$$r(t) = 3cost^2\hat{i} + 3sint^2\hat{j}$$
$$r'(t) = 6t(-sint^2)\hat{i} + 6t(cost^2)\hat{j}$$

Thus, $s(t) = \int_0^t \sqrt{36u^2 sinu^2 + 36u^2 cosu^2} du$
$= \int_0^t 6u du$
$= 3t^2$
Thus, $s = 3t^2$ or, $t = \sqrt{\frac{s}{3}}$
Thus $r(s) = 3cos(\frac{s}{3})\hat{i} + 3sin(\frac{s}{3})\hat{j}$

7.
$T(t) = \frac{-asint}{\sqrt{a^2+b^2}}\hat{i} + \frac{acost}{\sqrt{a^2+b^2}}\hat{j} + \frac{b}{\sqrt{a^2+b^2}}\hat{k}$
$N(t) = -cost\hat{i} - sint\hat{j}$

$\kappa = \frac{a}{\sqrt{a^2+b^2}}$

8.

$(1, 1, \frac{2}{3})$ corresponds to $t = 1$

Equation of tangent:
$t \longrightarrow (1, 1, \frac{2}{3}) + t(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

Equation of the normal:
$t \longrightarrow (1, 1, \frac{2}{3}) + t(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$

Equation of Binormal:
$t \longrightarrow (1, 1, \frac{2}{3}) + t(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$

9.
$s(t) = v_0 t$
Thus,
$s(t) = r_0 \theta(t)$
$v_0 t = r_0 \theta(t)$
$\theta(t) = \frac{v_0 t}{r_0}$

Thus position vector
$R(t) = r_0(cos\theta(t), sin\theta(t)) = r_0(cos\frac{v_0 t}{r_0}, sin\frac{v_0 t}{r_0})$,
$v(t) = \frac{d}{dt}(R(t)) = v_0(-sin\frac{v_0 t}{r_0}, cos\frac{v_0 t}{r_0})$,
$a(t) = \frac{d}{dt}(v(t)) = \frac{v_0^2}{r_0}(-cos\frac{v_0 t}{r_0}, -sin\frac{v_0 t}{r_0})$.