

MA 102 (Mathematics II)
Department of Mathematics, IIT Guwahati

Tutorial Sheet No. 1

January 11, 2016

The Euclidean space \mathbb{R}^n , Norms, convergence of sequences in \mathbb{R}^n ,

- (1) Consider the Euclidean norm $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and show that $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that the vectors \mathbf{x} and \mathbf{y} are orthogonal if and only if
- $$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$
- (2) Let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be such that (a) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = 0$, (b) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, (c) $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, and (d) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. The function $\langle \cdot, \cdot \rangle$ is called an innerproduct on \mathbb{R}^n . Define $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for $\mathbf{x} \in \mathbb{R}^n$. Show that $p(t) := \|\mathbf{x} + t\mathbf{y}\|^2 \geq 0$ for all $t \in \mathbb{R}$ and hence prove the Cauchy-Schwarz inequality $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that the equality holds if and only if $\mathbf{x} = \alpha \mathbf{y}$ for some $\alpha \in \mathbb{R}$.
- (3) Let $(\mathbf{x}_k) \subset \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Show that $\mathbf{x}_k \rightarrow \mathbf{x}$ in \mathbb{R}^n if and only if for every $\mathbf{y} \in \mathbb{R}^n$ the sequence $(\langle \mathbf{x}_k, \mathbf{y} \rangle) \subset \mathbb{R}$ converges to $\langle \mathbf{x}, \mathbf{y} \rangle$, that is, $\langle \mathbf{x}_k, \mathbf{y} \rangle \rightarrow \langle \mathbf{x}, \mathbf{y} \rangle$ in \mathbb{R} .
- (4) Let $(\mathbf{x}_k) \subset \mathbb{R}^n$ be such that $\mathbf{x}_k \rightarrow \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Show that the sequence $(\|\mathbf{x}_k\|) \subset \mathbb{R}$ converges to $\|\mathbf{x}\|$. Additionally suppose that $\mathbf{x} \neq 0$ and $\mathbf{x}_k \neq 0$ for all k , and define $\mathbf{y}_k := \mathbf{x}_k / \|\mathbf{x}_k\|$ and $\mathbf{y} := \mathbf{x} / \|\mathbf{x}\|$. Show that $\mathbf{y}_k \rightarrow \mathbf{y}$.
- (5) Let $(\mathbf{x}_k) \subset \mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Suppose that $\mathbf{x}_k \rightarrow \mathbf{x}$ and that $\langle \mathbf{x}_k, \mathbf{y} \rangle = 0$ for all k . Show that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
- (6) Let $\alpha \in (0, 1)$ and let $\mathbf{x}_n = (n^3 \alpha^n, \frac{1}{n} [n\alpha])$ for all $n \in \mathbb{N}$. (For each $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x .) Examine whether the sequence (\mathbf{x}_n) converges in \mathbb{R}^2 . Also, find $\lim_{n \rightarrow \infty} \mathbf{x}_n$ if it exists.
- (7) (**Bolzano-Weierstrass Theorem**) Let $(\mathbf{x}_n) \subset \mathbb{R}^2$ be bounded. Show that (\mathbf{x}_n) has a subsequence that converges in \mathbb{R}^2 .
- (8) (**Heine-Borel Theorem**) Let $S \subset \mathbb{R}^n$. Show that S is compact if and only if S is bounded and closed in \mathbb{R}^n .

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