at origin (0,0,0)

1. An electron is injected with a velocity $\vec{u}_0 = \hat{y}\, u_0$ into a region where both an electric field $\vec{E}$ and a magnetic field $\vec{B}$ exist. Describe the motion of the electron if $\vec{E} = \hat{z}\, E_0$, and $\vec{B} = \hat{x}\, B_0$. Discuss the effect of the relative magnitudes of $|\vec{E}|$, and $|\vec{B}|$, on the electron paths.

Sol^n: $q = -|e|$ $\quad \vec{V}\big|_{t=0} = (0, u_0, 0)$ $\quad \vec{E} = (0, 0, E_0)$

$\vec{B} = (B_0, 0, 0)$

Lorentz force equation $F = m\frac{d\bar{V}}{dt} = q\left(\bar{E} + \bar{V}\times\bar{B}\right)$

$m\frac{dV_x}{dt} = 0 \quad \Rightarrow \quad V_x(t) = 0$

$\frac{dV_y}{dt} = -\frac{|e|}{m} B_0 V_z \quad \Rightarrow \quad V_y(t) = (u_0 - E_0/B_0)\cos\omega_c t + E_0/B_0$

$\frac{dV_z}{dt} = -\frac{|e|}{m}(E_0 - B_0 V_y) \quad \Rightarrow \quad V_z(t) = \left(\frac{E_0}{B_0} - u_0\right)\sin\omega_c t$

Cyclotron frequency $= \omega_c = \frac{|e|}{m} B_0$

The electron is injected at the origin ($x(0) = 0$ $y(0) = 0$ $z(0) = 0$)

$x(t) = 0$ since $V_x(t) = 0$

$y(t) = \frac{1}{\omega_c}(u_0 - E_0/B_0)\sin\omega_c t + \frac{E_0 t}{B_0}$

$z(t) = -\frac{1}{\omega_c}(u_0 - E_0/B_0)(1 - \cos\omega_c t)$

Equation of motion

$$\left(y(t) - E_0 t/B_0\right)^2 + \left(z(t) + \frac{(u_0 - E_0/B_0)}{\omega_c}\right)^2 = \frac{(u_0 - E_0/B_0)^2}{\omega_c^2}$$

cycloid motion

If $E_0/B_0 = u_0$

$V_x = 0$; $V_y = u_0$ & $V_z = 0$

$y(t) = u_0 t$ with $x(t) = 0$ & $z(t) = 0$

St line motion

$\Rightarrow a = E_0/u_0 B_0$

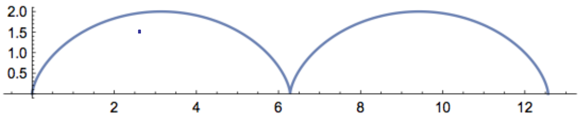
In[1]:=
```
y = (1 - a) Sin[t] + t;
z = - (1 - a) (1 - Cos[t]);
```

In[3]:=
```
a = 0.5;
ParametricPlot[{y, z}, {t, 0, 10 π}]
```

Out[4]=

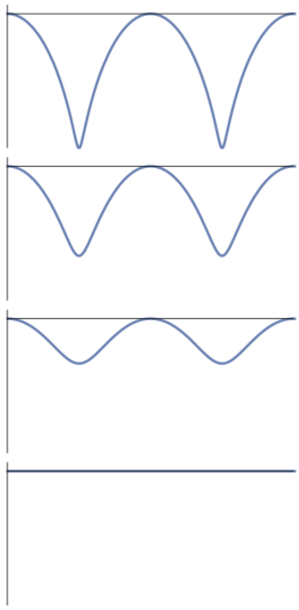
In[5]:=
```
a = 2;
ParametricPlot[{y, z}, {t, 0, 4 π}]
```

Out[6]=



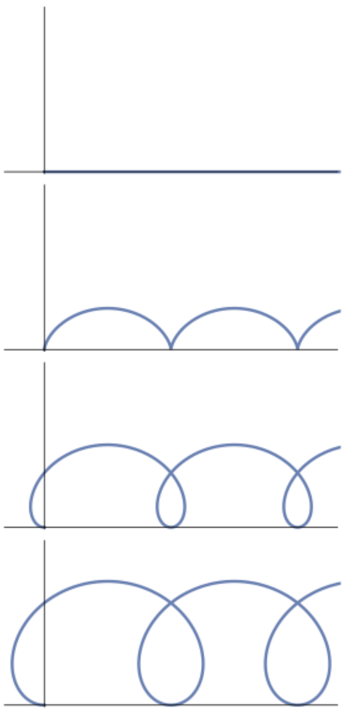
In[7]:=
```
g1 = Grid[Table[
    {ParametricPlot[{y, z}, {t, 0, 10 π}, Ticks -> None, AspectRatio → 1 / 2,
      PlotRange → {{0, 4 π}, {-1.5, 0.1}}]}, {a, 0.25, 1, 0.25}]]
```

Out[7]=



In[8]:=
```
g2 = Grid[Table[
    {ParametricPlot[{y, z}, {t, 0, 10 π}, Ticks -> None,
      AspectRatio → 1 / 2, PlotRange → {{-2, 4 π + 2}, {-0.1, 8}}]}, {a, 1, 4, 1}]]
```

Out[8]=



2. The surface current density in the xoy plane is given to be

$$\vec{K}(x, y) = K_0 \left[\frac{x^2}{a^2}\hat{i} + \frac{y^2}{b^2}\hat{j} \right]$$

Evaluate $\nabla \cdot \vec{K}$ and hence calculate the time rate of change of charge contained in the first quadrant of a circle of unit radius centred on the origin. Show explicitly that for this quadrant, the continuity equation is satisfied in its integral form.

Sol: $\nabla \cdot \bar{K} = -\partial\sigma/\partial t \Rightarrow \dfrac{\partial \sigma}{\partial t} = -2K_0 \left(\dfrac{x}{a^2} + \dfrac{y}{b^2} \right)$

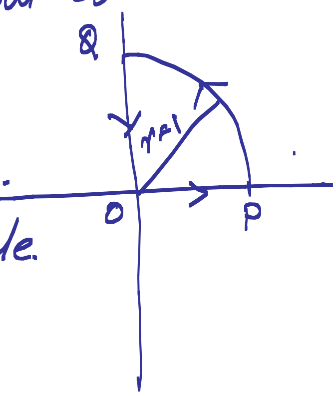
$$= -2K_0 r \left(\frac{\cos\theta}{a^2} + \frac{\sin\theta}{b^2} \right)$$

The charge q is contained in 1^{st} quadrant of a circle of unit radius.

$$\frac{dq}{dt} = \int_{r=0}^{1} \int_{0}^{\pi/2} \frac{\partial \sigma}{\partial t} r \, dr \, d\theta$$

$$= -\frac{2K_0}{3} \left(1/a^2 + 1/b^2 \right)$$

Let us evaluate the total charge q flowing out of the boundaries of the 1^{st} quadrant of unit circle. Along OP, $\vec{K}$ is $\hat{i}$ direction and along OQ, $\vec{K}$ is along $\hat{j}$ direction. No charge is flowing across OP and OQ. $\overset{\frown}{PQ}$ is the part where charge leaves from circle.



$$d\vec{l} = r\hat{\theta}\, d\theta \Big|_{r=1} = \hat{\theta}\, d\theta$$

$$\vec{K} = K_0 \left[\frac{r^2\cos^2\theta}{a^2} (\hat{r}\cos\theta - \hat{\theta}\sin\theta) + \frac{r^2\sin^2\theta}{b^2} (\hat{r}\sin\theta + \hat{\theta}\cos\theta) \right]$$

$$\vec{K} \times d\vec{l} = K_0 \left[\frac{r^2\cos^2\theta}{a^2} (\hat{r}\cos\theta - \hat{\theta}\sin\theta) + \frac{r^2\sin^2\theta}{b^2} (\hat{r}\sin\theta + \hat{\theta}\cos\theta) \right] \times (\hat{r}\, dr + r\hat{\theta}\, d\theta)$$

$$|\vec{K} \times d\vec{l}| = K_0 \left(\frac{\cos^3\theta}{a^2} + \frac{\sin^3\theta}{b^2} \right) d\theta \qquad \int_{P,0}^{Q,\pi/2} |\vec{K} \times d\vec{l}| = \frac{2K_0}{3} \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$$

3. A uniformly charged thin disk of charge density σ radius R and thickness $t < R$ rotates with an angular velocity ω about the z axis of symmetry as shown in Fig. Find out the magnetic field B at distance z in the vertical axis.

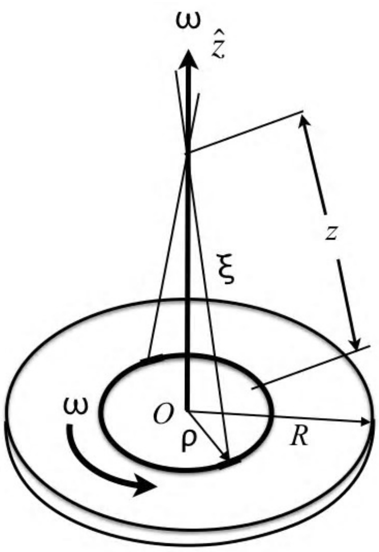


Figure 1: Problem 3

Sol: The current density in the disk is given by $\rho\vec{v} = \sigma\omega\rho\hat{\phi}$ where ρ is the distance from the axis of the disk.

A ring-shaped portion of the disk of radial thickness $d\rho$ thus constitutes a current ring $dI = \sigma\omega\rho\, d\rho$

The $\vec{B}$ field on the z-axis due to this current ring is

$$d\vec{B} = \frac{\mu_0}{2} \frac{dI\, \rho^2}{\xi^3} \hat{z}$$

Note that $\vec{B}$ field on axis of circular loop is

$$\vec{B} = B_z \hat{z} = \frac{\mu_0 I R^2}{2\xi^3} \hat{z} \qquad R = \text{circular loop radius.}$$

$$\Rightarrow \quad d\vec{B} = \frac{\mu_0}{2} \frac{\sigma\omega\rho^3 d\rho}{\xi^3} \hat{z} \qquad \xi^2 = \rho^2 + z^2$$

$$\vec{B} = \int_0^R \left(\frac{\mu_0\omega\sigma}{2}\right) \frac{\rho^3 d\rho}{(\rho^2+z^2)^{3/2}} \hat{z} = \hat{z}\, \frac{\mu_0\omega\sigma}{2} \left[\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - 2|z|\right]$$

4. A thin conducting wire is bent into the shape of a regular polygon of N sides. A current I flows in the wire. Show that the magnetic flux density at the center is

$$\vec{B} = \frac{\mu_o N I}{2\pi b} tan \frac{\pi}{N} \quad \hat{a}_n$$

where b is the radius of the circle circumscribing the polygon and $\hat{a}_n$ is a unit vector normal to the plane of the polygon. Show also that as N becomes very large this result reduces to

$$\vec{B} = \frac{\mu_o I b^2}{2(z^2 + b^2)^{3/2}} \hat{a}_n$$

with $z = 0$.

from Griffiths equation 5·35

$$\vec{B} = \frac{\mu_0 I \hat{\phi}}{4\pi a}(\sin\alpha_1 + \sin\alpha_2)$$

$$= \frac{\mu_0 I}{2\pi a} \frac{l}{\sqrt{l^2+a^2}} \hat{\phi}$$

Sol:

$\tan\alpha = l/a$

polygon with N side $\alpha = \pi/N$

$$\vec{B} = \frac{\mu_0 N I}{2\pi a} \sin\alpha \, \hat{e}_n$$

$$= \frac{\mu_0 N I}{2\pi b} \tan\alpha \, \hat{e}_n$$

$$\vec{B} = \lim_{N\to\infty} \frac{\mu_0 N I}{2\pi b} \tan(\pi/N) \, \hat{e}_n$$

$$= \frac{\mu_0 \cancel{N} I}{2\cancel{\pi} b} \frac{\cancel{\pi}}{\cancel{N}} \hat{e}_n = \frac{\mu_0 I}{2b} \hat{e}_n$$

$$\vec{B} = \left. \frac{\mu_0 I b^2}{2(b^2+z^2)^{3/2}} \right|_{z=0} \hat{e}_n = \frac{\mu_0 I}{2b} \hat{e}_n$$

$\hat{e}_n$ is $\perp$r to the plane of polygon.

5. A curved wire kept in yz plane, carrying a current I_1 subtends an angle $2\theta_0$ at the location of another long straight wire carrying a current I_2 as shown in the figure. Find the force exerted by the straight wire on the curved wire.

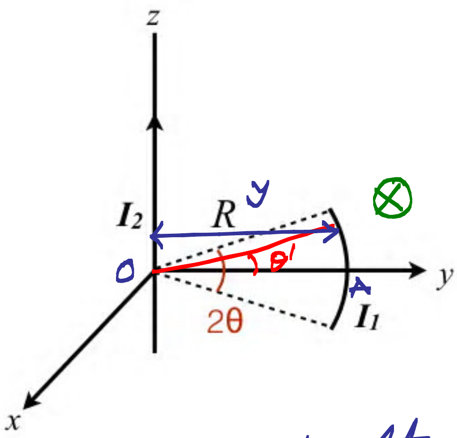


Figure 2:

Sol: The field due to the straight wire at a distance y is $\vec{B} = -\frac{\mu_0 I_2}{2\pi y}\hat{i} = -\frac{\mu_0 I_2}{2\pi R\cos\theta'}\hat{i}$

Force on curved wire

$$\vec{F} = \int I_1\, d\vec{l}\times\vec{B} = -\frac{\mu_0 I_1 I_2}{2\pi R}\int\frac{d\vec{l}\times\hat{i}}{\cos\theta'}$$

Due to symmetry about OA, only the z-component of $d\vec{l}$ will contribute to the force.

$$d\vec{l} = R\,d\theta\,\hat{\theta} = R\cos\theta'\,d\theta'\,\hat{k}$$

$$\vec{F} = -\frac{\mu_0 I_1 I_2}{2\pi}\int_{-\theta}^{\theta} d\theta'\,\hat{j} = -\frac{\mu_0 I_1 I_2}{\cancel{2}\pi}\cancel{2}\theta\,\hat{j}$$

$$= -\frac{\mu_0 I_1 I_2 \theta}{\pi}\hat{j}$$

Note that $y = R\cos\theta'$ $z = R\sin\theta'$

$$d\vec{l} = -R\sin\theta'\,d\theta'\,\hat{y} + R\cos\theta'\,d\theta'\,\hat{z}$$

$$= R\,d\theta'\left(-\sin\theta'\,\hat{y} + \cos\theta'\,\hat{z}\right)$$

$\hat{\theta} = -\sin\theta'\,\hat{r} + \cos\theta'\,\hat{z}$ is not $\hat{\theta}$ in cylinderical co-ordinate system