PH 102 : Electromagnetism **Department of Physics**
Tutorial 1: Solutions **Indian Institute of Technology Guwahati**

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1. The equation giving a family of ellipsoids is

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

where a, b, c are constants. Find the unit vector normal to each point of the surface of these ellipsoids.

Solution:

Recall that $\vec{\nabla} u$ is perpendicular to $u(x, y, z) =$ const surface as discussed in class. Therefore, the unit normal will be $\hat{n} = \frac{\vec{\nabla} u}{|\vec{\nabla} u|}$. $\vec{\nabla} u = (\hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} + \hat{z}\frac{\partial u}{\partial z}) = \hat{x}\frac{2x}{a^2} + \hat{y}\frac{2y}{b^2} + \hat{z}\frac{2z}{c^2}$ and $|\vec{\nabla} u| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$. Hence, the unit vector $\hat{n}$ is given by

$$\hat{n} = \frac{\hat{x}\frac{x}{a^2} + \hat{y}\frac{y}{b^2} + \hat{z}\frac{z}{c^2}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$$

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2. (a) Find the gradients of the following functions,

$$(i)\ \ln \mathscr{r},$$
$$(ii)\ \mathscr{r}^n,$$

where $\mathscr{r}$ is the magnitude of the vector joining points (x_0, y_0, z_0) and (x, y, z).

(b) Find the divergences of the following vector functions:

$$(i)\ \vec{V}_1 = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z},$$
$$(ii)\ \vec{V}_2 = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}.$$

(c) Find the curl of the vector functions given in (b).

Solution:

(a)(i) $\mathscr{r} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$, hence

$$\begin{aligned}
\vec{\nabla} f &= \frac{1}{2}\vec{\nabla}\ln[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2] \\
&= \frac{1}{2}\left\{\hat{x}\frac{\partial}{\partial x}\ln[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]+\hat{y}\frac{\partial}{\partial y}\ln[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]\right. \\
&\left. +\hat{z}\frac{\partial}{\partial z}\ln[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]\right\} \\
&= \frac{(x-x_0)\hat{x}+(y-y_0)\hat{y}+(z-z_0)\hat{z}}{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = \frac{\vec{\mathscr{r}}}{\mathscr{r}^2}
\end{aligned}$$

(ii) $\mathscr{r} = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. Hence $\mathscr{r}^n = [(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{n/2}$ Now, $\frac{\partial \mathscr{r}^n}{\partial x} = n\mathscr{r}^{n-1}\frac{\partial \mathscr{r}}{\partial x}$. Now, $\frac{\partial \mathscr{r}}{\partial x} = \frac{x-x_0}{\mathscr{r}}$. Hence, $\vec{\nabla}(\mathscr{r})^n = n\mathscr{r}^{n-1}\hat{\mathscr{r}}$;

(b)(i) $\vec{\nabla}.\vec{V}_1 = y+2z+3x$;
(ii)$\vec{\nabla}.\vec{V}_2 = 0+2x+2y$

(c)(i) We will illustrate with levi Civita. One can use traditional methods as well.

$$\begin{aligned}
(\vec{\nabla}\times\vec{V}_1)_x &= \epsilon_{xyz}\partial_y V_{1z}+\epsilon_{xzy}\partial_z V_{1y} = 0-2y \\
(\vec{\nabla}\times\vec{V}_1)_y &= \epsilon_{yzx}\partial_z V_{1x}+\epsilon_{yxz}\partial_x V_{1z} = 0-3z \\
(\vec{\nabla}\times\vec{V}_1)_z &= \epsilon_{zxy}\partial_x V_{1y}+\epsilon_{zyx}\partial_y V_{1x} = 0-x \\
\therefore \vec{\nabla}\times\vec{V}_1 &= -2y\hat{x}-3z\hat{y}-x\hat{z}
\end{aligned}$$

(ii)

$$\begin{aligned}
(\vec{\nabla}\times\vec{V}_2)_x &= \epsilon_{xyz}\partial_y V_{2z}+\epsilon_{xzy}\partial_z V_{2y} = 2z-2z \\
(\vec{\nabla}\times\vec{V}_1)_y &= \epsilon_{yzx}\partial_z V_{2x}+\epsilon_{yxz}\partial_x V_{2z} = 0-0 \\
(\vec{\nabla}\times\vec{V}_1)_z &= \epsilon_{zxy}\partial_x V_{2y}+\epsilon_{zyx}\partial_y V_{2x} = 2y-2y \\
\therefore \vec{\nabla}\times\vec{V}_1 &= 0
\end{aligned}$$

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3. Prove the following product rules.

$$\begin{aligned}
\vec{\nabla}(fg) &= f\vec{\nabla}g+g\vec{\nabla}f \\
\vec{\nabla}.(\vec{A}\times\vec{B}) &= \vec{B}.(\vec{\nabla}\times\vec{A})-\vec{A}.(\vec{\nabla}\times\vec{B}) \\
\vec{\nabla}\times(f\vec{A}) &= f(\vec{\nabla}\times\vec{A})-\vec{A}\times(\vec{\nabla}f)
\end{aligned}$$

Solution:

(i) Use the definition of gradient and product rule of derivatives.

(ii)

$$\begin{aligned}
\vec{\nabla}.(\vec{A}\times\vec{B}) = \partial_i(\vec{A}\times\vec{B})_i &= \partial_i(\epsilon_{ijk}A_jB_k) = \epsilon_{ijk}\partial_i(A_jB_k) \\
&= \epsilon_{ijk}[(\partial_iA_j)B_k + A_j(\partial_iB_k)] \\
&= \epsilon_{kij}(\partial_iA_j)B_k - \epsilon_{jik}(\partial_iB_k)A_j \\
&= B_k(\vec{\nabla}\times\vec{A})_k - (\vec{\nabla}\times\vec{B})_jA_j \\
&= \vec{B}.(\vec{\nabla}\times\vec{A}) - \vec{A}.(\vec{\nabla}\times\vec{B})
\end{aligned}$$

We have used here the fact that repeated indices are summed over and hence i, j, k are dummy indices and used the antisymmetry of levi Civita $\epsilon_{ijk} = -\epsilon_{jik}$.

(iii) Let us focus on the i-th component of the curl:

$$\begin{aligned}
(\vec{\nabla}\times(f\vec{A}))_i &= \epsilon_{ijk}\partial_j(fA_k) = \epsilon_{ijk}(\partial_jf)A_k + \epsilon_{ijk}f(\partial_jA_k) \\
&= ((\vec{\nabla}f)\times\vec{A})_i + f(\vec{\nabla}\times\vec{A})_i = -(\vec{A}\times(\vec{\nabla}f))_i + f(\vec{\nabla}\times\vec{A})_i
\end{aligned}$$

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4. For each of the 2 dimensional vector fields shown in Figures (1) to (6), can you try to comment on their divergence and curl by looking at them?

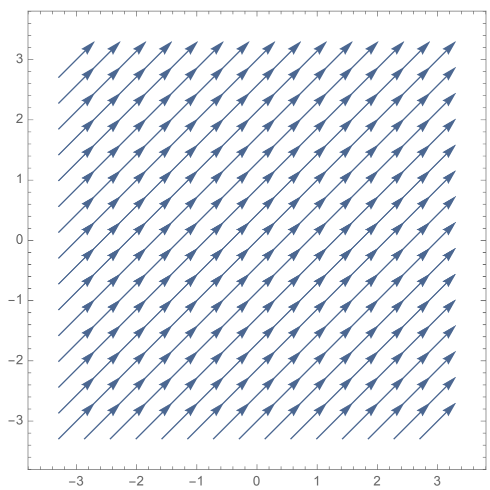

Figure 1

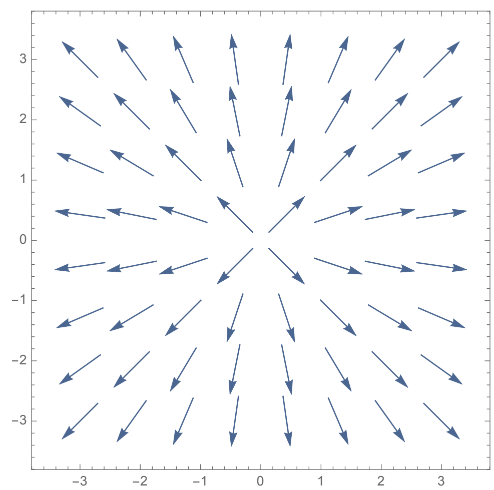

Figure 2

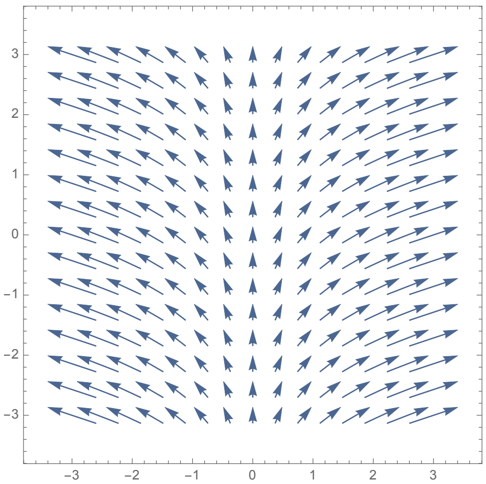

Figure 3

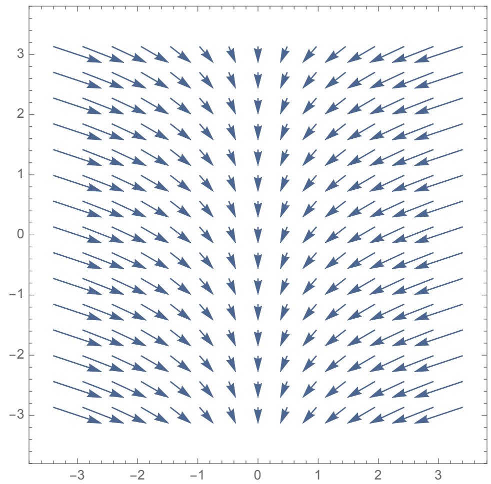

Figure 4

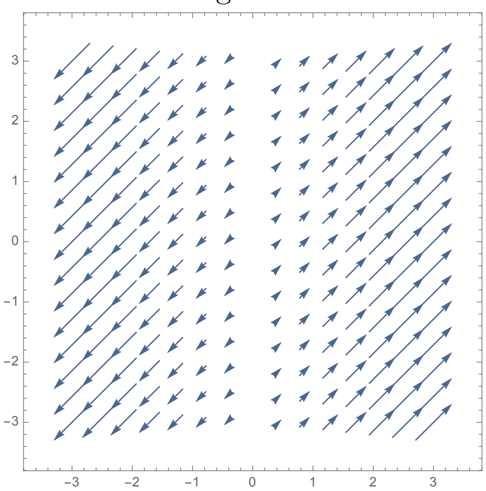

Figure 5

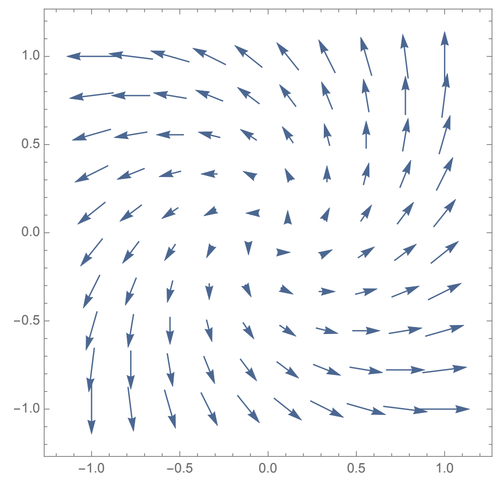

Figure 6

Solution:

The way to see whether a vector field has non-zero divergence and curl are as follows. For divergence, we need to imagine a point in the vector field and see if there are field lines effectively coming out or going into the point. If so, then it will have a non-zero divergence at that point. The way to think about curl is a little more general than just vectors curling around a point. Still, some idea of "circulation" must be involved since that is what a curl does. A good way to think about curl is then to think of a vector field as representing a fluid that actually flows and then to imagine a little paddlewheel placed at a point in the flow. If the vector field somehow represents the rotation of a paddlewheel, then the field has a non-zero curl at that point.

Figure 1:

It has both zero divergence and zero curl, because it does not change in either magnitude or direction in the neighbourhood of any point. We see that all the little arrows, at any point, have the same direction and the same length. Therefore, this is a constant field. But, in general, fields can have different properties at different points. This is a plot of $\vec{v} = \hat{x} + \hat{y}$. $\therefore$ $\vec{\nabla}.\vec{v} = 0, \quad \vec{\nabla} \times \vec{v} = 0$ for the field in Figure 1.

Figure 2:

Note that the vector field is radial and, for given r, its magnitude is constant. First note, that the circulation of the field at any point is zero. This is seen imagining a closed path at any point of the field. Therefore the curl is zero. But the divergence is obviously not zero as the field spreads out from any point that we choose in the field. Note that this is a plot of $\vec{v} = \hat{r}/r$. $\therefore$ $\vec{\nabla}.\vec{v} \neq 0, \quad \vec{\nabla} \times \vec{v} = 0$ for the field in Figure 2. Note added: The exact calculation of the divergence at $r = 0$ may be delayed till we introduce Dirac delta, which will be done in class.

Figure 3 and 4:

Both of them have non-zero divergence and zero curl. If we imagine a wheel placed at any point of the field, the arrows will not be able to create an effective torque to make the wheel rotate. Therefore, the field is irrotational in both cases at all points. Since the length of the arrows in Figure 3 gets larger as we go away from the origin, the vector field in Fig. 3 has positive divergence. On the other hand, we notice that in Figure 4, the arrows are all pointing toward the origin instead of going away from it. If we pick some arbitrary point other than the origin, the lengths of the arrows heading into that point are longer than the lengths of the arrows heading away from the point. Therefore the field in Figure 4 has negative divergence. $\therefore$ $\vec{\nabla}.\vec{v} \neq 0, \quad \vec{\nabla} \times \vec{v} = 0$ for the fields in Figure 3 and 4. The field in Fig.3 is $\vec{v} = x\hat{x} + \hat{y}$, while on Fig.4 we have $\vec{v} = -x\hat{x} - \hat{y}$.

Figure 5:

From Figure 5 it is evident that the vector field shown has non-zero divergence as the field lines are emanating from any chosen point. The field in Figure 5 is an example where it looks like the vectors all fall on diagonal lines, with no perceptible curving, yet there is a non-zero curl. If we put a little paddlewheel at the origin, then the arrows on the left will push down the paddles on the left and the arrows on the right will push up the paddles on the right, which will produce a net imbalance of forces and hence a torque, and thus the paddle will spin. Hence, the field has a nonzero curl at the origin. The vector field is of the form $\vec{v} = x\hat{x} + x\hat{y}$ $\therefore$ $\vec{\nabla}.\vec{v} \neq 0, \quad \vec{\nabla} \times \vec{v} \neq 0$ for the field in Figure 5.

Figure 6: It also has $\vec{\nabla} \times \vec{v} \neq 0, \ \vec{\nabla}.\vec{v} \neq 0$. Notice that the vector field has both "counterclockwise curling" and "outward streaming" behaviour and hence non-zero (positive) curl and non-zero (positive) divergence.

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5. Find the gradient of the scalar potential $\phi(x, y) = \alpha xy$. Provide a clear sketch of the $\phi =$ constant lines in the $x - y$ plane and a representation of its gradient field. Such a field is known as a radial quadrupole field and are used in focussing charged particles.

Solution:

The gradient of ϕ is given by $\vec{\nabla}\phi(x,y) = \alpha(\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(xy)) = \alpha(y\hat{x} + x\hat{y})$. The $\phi =$ constant lines are shown in Figure 7, they are the rectangular hyperbolae as seen from the equation of ϕ.

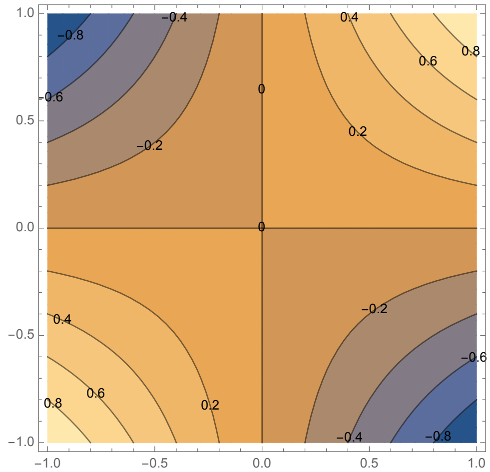

Figure 7

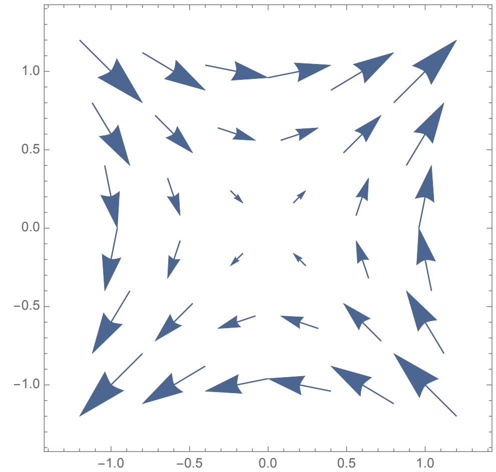

Figure 8

The vector field $\vec{\nabla}\phi$ is plotted in Figure 8 which clearly shows that $\vec{\nabla}\phi$ is perpendicular to the $\phi =$ constant lines as was proved in the class.

6. Find the total work done in moving a particle under the force field given by $\vec{F} = z\hat{x} + z\hat{y} + x\hat{z}$ along the helix C given by $x = \cos t$, $y = \sin t$, $z = t$ from $t = 0$ to $t = \frac{\pi}{2}$.

Solution:

Work done is given by the line integral as $\int_C \vec{F}.\vec{dr}$ where C represents the helical path provided by the parametric equations $x = \cos t$, $y = \sin t$, $z = t$. It is simple to realise that $d\vec{r} = \frac{d\vec{r}}{dt}dt = (\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z})dt = (-\sin t\hat{x} + \cos t\hat{y} + \hat{z})dt$. Hence, work done

$$\begin{aligned}
\int_c \vec{F}.d\vec{r} &= \int_{t=0}^{\pi/2} (z\hat{x} + z\hat{y} + x\hat{z}).(\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z})dt \\
&= \int_{t=0}^{\pi/2} (z\frac{dx}{dt} + z\frac{dy}{dt} + x\frac{dz}{dt})dt \\
&= \int_{t=0}^{\pi/2} (-t\sin t + t\cos t + \cos t)dt \\
&= \frac{1}{2}(\pi - 2)
\end{aligned}$$

7. Show that the work done for a particle moving under the force field $\vec{F} = (2xy + z^3)\hat{x} + x^2\hat{y} + 3xz^2\hat{z}$ from point $a = (1, 1, 0)$ to $b = (2, 2, 0)$ as shown in Figure 1 following path (1) and path (2) is equal. Show that the curl of the force field $\vec{F}$ is zero. As you know that such a force field is called conservative, hence evaluate the scalar potential.

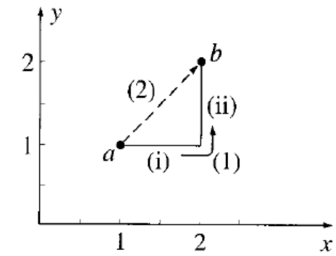

Figure 9: Problem 6

Solution:

Along path 1(i): $\vec{dr} = dx\hat{x}$ and $y = 1, z = 0$. Hence, work done $\int_{1(i)} \vec{F}.\vec{dr} = \int_{x=1}^{2} F_x dx = \int_{x=1}^{2} 2xdx = 3$. Along path 1(ii): $\vec{dr} = dy\hat{y}$ and $x = 2, z = 0$. Hence, work done $\int_{1(ii)} \vec{F}.\vec{dr} = \int_{y=1}^{2} F_y dy = \int_{y=1}^{2} x^2 dy = 4$. Hence, work done along path 1: (i)+(ii)=7.

Along path 2: $y = x, dy = dx, z = 0$. Hence, work done $\int_2 \vec{F}.\vec{dr} = \int_2 (F_x dx + F_y dy) = \int_{x=1}^{2} (2x^2 + x^2)dx = 7$ which is equal to the work done along path (1).

Recall from class, that for conservative forces where $\vec{\nabla} \times \vec{F} = 0$ and hence $\vec{F} = \vec{\nabla}\phi$, work done $\int_a^b \vec{F}.\vec{dr} = \int_a^b (\vec{\nabla}\phi).\vec{dr} = \phi(b) - \phi(a)$ depends only on the end points, independent of chosen path. However, we also note that accidentally one may choose two pathological paths along which the work done is same. Hence, we need to calculate the curl of the force field explicitly to confirm that it is conservative.

$$\begin{aligned}
(\vec{\nabla} \times \vec{F})_x &= \epsilon_{xyz}\partial_y F_z + \epsilon_{xzy}\partial_z F_y = 0 - 0 = 0 \\
(\vec{\nabla} \times \vec{F})_y &= \epsilon_{yzx}\partial_z F_x + \epsilon_{yxz}\partial_x F_z = 3z^2 - 3z^2 = 0 \\
(\vec{\nabla} \times \vec{F})_z &= \epsilon_{zxy}\partial_x F_y + \epsilon_{zyx}\partial_y F_x = 2x - 2x = 0 \\
\therefore \vec{\nabla} \times \vec{F} &= 0
\end{aligned}$$

Now, given that $\vec{\nabla} \times \vec{F} = 0$, we can write the force as a gradient of a scalar potential: $\vec{F} = \vec{\nabla}\phi$.

$$\therefore \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z} = (2xy + z^3)\hat{x} + x^2\hat{y} + 3xz^2\hat{z}$$

which yields

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= (2xy + z^3); \\
\frac{\partial \phi}{\partial y} &= x^2; \\
\frac{\partial \phi}{\partial z} &= 3xz^2
\end{aligned}$$

Integrating the equations respectively we get:

$$\begin{aligned}\phi &= x^2y + xz^3 + f(y,z); \\ \phi &= x^2y + g(x,z); \\ \phi &= xz^3 + h(x,y)\end{aligned}$$

where $f(y,z),\ g(x,z),\ h(x,y)$ are some unknown constants of integration arising out of x, y, z integrations respectively. However, the three solutions are consistent if we choose $f(y,z) = 0,\ g(x,z) = xz^3,\ h(x,y) = x^2y$. Therefore, $\phi(x,y,z) = x^2y + xz^3 + \text{const.}$