MA 102 (Mathematics II)

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(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(0,0)$. Suppose that for $U := (3/5, 4/5)$ and $V := (1/\sqrt{2}, 1/\sqrt{2})$, we have $D_U f(0,0) = 12$ and $D_V f(0,0) = -4\sqrt{2}$. Then determine $f_x(0,0)$ and $f_y(0,0)$.

Solution. Set $(\alpha, \beta) := \nabla f(0,0)$. Then $(\alpha,\beta) \bullet (3/5, 4/5) = 12 \Rightarrow 3\alpha + 4\beta = 60$ and $(\alpha,\beta) \bullet (1/\sqrt{2}, 1/\sqrt{2}) = -4\sqrt{2} \Rightarrow \alpha + \beta = -8$.
Hence, $f_x(0,0) = \alpha = -92$ and $f_y(0,0) = \beta = 84$. $\square$

(2) Find the direction where the directional derivative is greatest for the function $f(x,y) = 3x^2y^2 - x^4 - y^4$ at the point $(1,2)$.

Solution. $f_x(1,2) = 20$, $f_y(1,2) = -20$. Directional derivative is greatest when pointing in the direction of the gradient $(20, -20)$. Hence, the direction is $\frac{1}{\sqrt{2}}(\hat{\imath} - \hat{\jmath})$. $\square$

(3) Let $f(x,y) = \dfrac{1}{2}\ln(x^2+y^2) + tan^{-1}(\dfrac{y}{x})$, $P = (1,3)$. Find the direction in which $f(x,y)$ is increasing the fastest at P. Find the derivative of $f(x,y)$ in this direction.

Solution. We have $f_x(1,3) = -1/5$ and $f_y(1,3) = 2/5$. Directional derivative is greatest when pointing in the direction of the gradient $(-1/5, 2/5)$. Hence, the direction is
$U = -1/\sqrt{5}\hat{\imath} + 2/\sqrt{5}\hat{\jmath}$. The derivative in the direction of U is $f_U(1,3) = f_x(1,3)u_1 + f_y(1,3)u_2 = 1/\sqrt{5}$ $\square$

(4) A heat-seeking bug is a bug that always moves in the direction of the greatest increase in heat. Find the direction along which the heat-seeking bug will move when it is placed at the point $(2,1)$ on a metal plate heated so that the temperature at (x,y) is given by $T(x,y) = 50y^2e^{\frac{-1}{5}(x^2+y^2)}$.

Solution. $T_x(2,1) = -40/e$, $T_y(2,1) = 80/e$. Therefore, the bug will move in the direction $-1/\sqrt{5}\ \hat{\imath} + 2/\sqrt{5}\ \hat{\jmath}$. $\square$

(5) Let $f(x,y,z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of f at $(1,-1,3)$ and the equations of the tangent plane and the normal line to the surface $f(x,y,z) = 7$ at $(1,-1,3)$.

Solution. We have $\nabla f(1,-1,3) = \left(\frac{\partial f}{\partial x}(1,-2,3), \frac{\partial f}{\partial y}(1,-1,3), \frac{\partial f}{\partial z}(1,-1,3)\right) = (0,4,6)$. The tangent plane to the surface $f(x,y,z) = 7$ at the point $(1,-1,3)$ is given by

$$0 \times (x-1) + 4 \times (y+1) + 6 \times (z-3) = 0, \quad \text{i.e., } 2y + 3z = 7.$$

The Normal Line to the surface $f(x,y,z) = 7$ at the point $(1,-1,3)$ is given by $(x,y,z) = (1,-1,3) + t(0,4,6)$ for $t \in \mathbb{R}$. That is, $x = 1, y = -1+4t, z = 3+6t$, $t \in \mathbb{R}$. $\square$

(6) Find $D_U f(2,2,1)$, where $f(x,y,z) = 3x - 5y + 2z$ and U is the unit vector in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2,2,1)$.

Solution. We have $U = \frac{(2,2,1)}{\sqrt{2^2+2^2+1^2}} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ and $\nabla f(2,2,1) = (3,-5,2)$. Therefore, $D_U f(2,2,1) = \nabla f(2,2,1) \bullet U = \frac{6}{3} - \frac{10}{3} + \frac{2}{3} = -\frac{2}{3}$. $\square$

(7) Find equations for the tangent plane and the normal line to the level surface $x^2+y^2+z^2 = 4$ at the point $P_0 = (-1,\ 1,\ \sqrt{2})$

Solution. Equation of the tangent plane is $x - y - \sqrt{2}z + 4 = 0$. Equation of the normal line is $(x,y,z) = (-1,1,\sqrt{2}) + t(-2,2,2\sqrt{2}), t \in \mathbb{R}$. $\square$

(8) Find equations for the tangent plane and normal line to the surface $z = 6 - 3x^2 - y^2$ at the point $P_0 = (1,\ 2,\ -1)$.

Solution. Equation of the tangent plane is $6x + 4y + z - 13 = 0$. Equation of the normal line is $(x,y,z) = (1,2,-1) + t(6,4,1), t \in \mathbb{R}$. $\square$

(9) Find the equation of the tangent plane to the graphs of the following functions at the given point:
(a) $f(x,y) = x^2 - y^4 + e^{xy}$ at the point $(1,0,2)$
(b) $f(x,y) = \tan^{-1}\dfrac{y}{x}$ at the point $(1, \sqrt{3}, \dfrac{\pi}{3})$.

Proof. The equation of tangent plane to the surface $z = f(x,y)$ at the point (x_0,y_0) is

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

(a) We have $f_x = 2x + ye^{xy}$ and $f_y = -4y^3 + xe^{xy}$. The equation of the tangent plane at $(1,0,2)$ is given by $z = 2(x-1) + 1(y-0) + 2 \Rightarrow z = 2x + y$.
(b) The equation of the tangent plane is given by

$$z = \frac{\pi}{3} - \frac{\sqrt{3}}{4}(x-1) + \frac{1}{4}(y - \sqrt{3}) \Rightarrow 3\sqrt{3}x - 3y + 12z - 4\pi = 0.$$

$\square$

(10) Check the following functions for differentiability, and then find the Jacobian Matrix.
(a) $f(x,y) = (e^{x+y} + y, xy^2)$ (b) $f(x,y) = (x^2 + \cos y, e^x y)$ (c) $f(x,y,z) = (ze^x, -ye^z)$.

Solution. Let $f(x,y) = (e^{x+y} + y, xy^2) = (f_1(x,y), f_2(x,y))$. The first order partial derivatives of the component functions $f_1(x,y) = e^{x+y} + y$, $f_2(x,y) = xy^2$ exist and continuous everywhere in $\mathbb{R}^2$. Hence, f_1, f_2 are differentiable by the sufficient condition. This proves that f is differentiable in $\mathbb{R}^2$. Using similar argument, we find that the remaining two functions are also differentiable. The Jacobian matrices are

(a) $J_f(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y}+1 \\ y^2 & 2xy \end{bmatrix}$.

(b) $J_f(x,y) = \begin{bmatrix} 2x & -\sin y \\ ye^x & e^x \end{bmatrix}$.

(c) $J_f(x,y,z) = \begin{bmatrix} ze^x & 0 & e^x \\ 0 & -e^z & -ye^z \end{bmatrix}$.

$\square$

(11) Let $z = x^2 + y^2$, and $x = 1/t, y = t^2$. Compute $\dfrac{dz}{dt}$ by (a) expressing z explicitly in terms of t and (b) chain rule.

Solution. (a) By direct substitution we have $z = x^2 + y^2 = t^{-2} + t^4$ for $t \neq 0$. Therefore $\dfrac{dz}{dt} = -2t^{-3} + 4t^3$.

(b) Note that $\dfrac{\partial z}{\partial x} = 2x$, $\dfrac{\partial z}{\partial y} = 2y$, $\dfrac{dx}{dt} = -t^{-2}$, $\dfrac{dy}{dt} = 2t$. Therefore by chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x)(-t^{-2}) + (2y)(2t) = -2t^{-3} + 4t^3.$$

$\square$

(12) Let $w = 4x + y^2 + z^3$ and $x = e^{rs^2}, y = \log\dfrac{r+s}{t}, z = rst^2$. Find $\dfrac{\partial w}{\partial s}$.

Solution. By chain rule,

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= \left(\frac{\partial}{\partial x}(4x + y^2 + z^3)\right)\left(\frac{\partial}{\partial s}(e^{rs^2})\right) + \left(\frac{\partial}{\partial y}(4x + y^2 + z^3)\right)\left(\frac{\partial}{\partial s}\left(\log\frac{r+s}{t}\right)\right) \\
&\quad + \left(\frac{\partial}{\partial z}(4x + y^2 + z^3)\right)\left(\frac{\partial}{\partial s}(rst^2)\right) \\
&= 8rse^{rs^2} + 2y\left(\frac{t}{r+s}\right)\left(\frac{1}{t}\right) + 3rt^2z^2 = 8rse^{rs^2} + \frac{2}{r+s}\log\frac{r+s}{t} + 3r^3s^2t^6.
\end{aligned}$$

$\square$

(13) If $w = \sqrt{x} + yz^3$, $x(r,\ s) = 1 + r^2 + s^2$, $y(r,\ s) = rs$, $z(r,\ s) = 3r$, then find $\partial w/\partial r$ and $\partial w/\partial s$ using the chain rule.

Solution.

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{1}{2\sqrt{x}}2r + z^3 s + 3yz^3 \cdot 3 \\
&= \frac{r}{\sqrt{1+r^2+s^2}} + 108r^3 s. \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{1}{2\sqrt{x}}2s + z^3 r + 3yz^3 \cdot 0 \\
&= \frac{s}{\sqrt{1+r^2+s^2}} + 27r^4.
\end{aligned}$$

□

(14) For the following functions, compute the mixed partial derivatives at all points in $\mathbb{R}^2$. Further find out at each point, whether the mixed derivatives are equal or not?

(a) $f(x,\ y) = x\sin y + y\sin x + xy$

(b) $f(x,\ y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ for $(x,\ y) \neq (0,\ 0)$ and $f(0,\ 0) = 0$

Solution. (a)

$$\begin{aligned}
f_x(x,y) &= \sin y + y\cos x + y \\
f_y(x,y) &= x\cos y + \sin x + x \\
f_{xy}(x,y) &= 1 + \cos y + \cos x \\
f_{yx} &= 1 + \cos y + \cos x.
\end{aligned}$$

Mixed derivatives are equal everywhere.

(b) We have $f_x(0,0) = 0 = f_y(0,0)$.

For $(x,y) \neq (0,0)$, we have

$$\begin{aligned}
f_x(x,y) &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \\
f_y(x,y) &= \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \\
f_{xy}(x,y) &= \frac{x^8 + 10x^6y^2 - 10x^2y^6 - y^8}{(x^2+y^2)^4} \\
f_{yx}(x,y) &= \frac{x^8 + 10x^6y^2 - 10x^2y^6 - y^8}{(x^2+y^2)^4}.
\end{aligned}$$

Again,

$$\begin{aligned}
f_{xy}(0,0) &= \lim_{t\to 0}\frac{f_x(0,t) - f_x(0,0)}{t} = -1 \\
f_{yx}(0,0) &= \lim_{t\to 0}\frac{f_y(t,0) - f_y(0,0)}{t} = 1.
\end{aligned}$$

Hence, mixed partial derivatives are equal at every $(x,y) \neq (0,0)$ only.

□

(15) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $F(x,\ y) = (\sin x \cos y,\ \sin x \sin y,\ \cos x \cos y)$. Show that F is differentiable in $\mathbb{R}^2$ and find its Jacobian matrix.

Solution. Let $F(x,y) = (f_1(x,y), f_2(x,y), f_3(x,y))$. The first order partial derivatives of the component functions $f_1(x,y) = \sin x \cos y$, $f_2(x,y) = \sin x \sin y$, $f_3(x,y) = \cos x \cos y$ exist and continuous everywhere in $\mathbb{R}^2$. Hence, f_1, f_2, f_3 are differentiable by the sufficient condition. This proves that F is differentiable in $\mathbb{R}^2$.
The Jacobian matrix is given by

$$J_F(x,y) = \begin{bmatrix} \cos x \cos y & -\sin x \sin y \\ \cos x \sin y & \sin x \cos y \\ -\sin x \cos y & -\cos x \sin y \end{bmatrix}.$$

□

(16) Using Taylor's formula find the quadratic and cubic approximations of the function $f(x,\ y) = e^x \cos(y)$ near the origin.

Solution. Quadratic approximation: $1 + x + x^2/2 - y^2/2$.
Cubic approximation: $1 + x + x^2/2 - y^2/2 + x^3/6 - (xy^2)/2$.

□

(17) Find the first three terms in the Taylor's formula for the function $f(x,\ y) = \cos x \cos y$ at origin. Find a quadratic approximation of f near the origin. How accurate is the approximation if $|x| \le 0.1$ and $|y| \le 0.1$?

Solution. Quadratic approximation: $1 - x^2/2 - y^2/2$.
The remainder term is: $[x^3 \sin(\theta x)\cos(\theta y)]/6 + [x^2 y \cos(\theta x)\sin(\theta y)]/2 + [xy^2 \sin(\theta x)\cos(\theta y)]/2 + [y^3 \cos(\theta x)\sin(\theta y)]/6$, $0 < \theta < 1$. Thus the absolute error is bounded by $2(0.1)^3/6 + (0.1)^3 = \frac{4}{3}(0.1)^3$ if $|x| \le 0.1$ and $|y| \le 0.1$.

□