Multivariable Calculus

Note Title 2/22/2013

Lectures 1 and 2:

- The space $\mathbb{R}^n$
- Convergence in $\mathbb{R}^n$

1. Why do Analysis?

(a) Sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}$$

What about

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} \; ?$$

Consider the "matrix"

$$(a_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

row-sum $= 1 + 0 + 0 + \cdots = 1$

Column-sum $= 0 + 0 + 0 + 0 + \cdots = 0$

(b) Integral:

$$V = \iint f(x,y)\,dx\,dy$$

Is $\int \left(\int f(x,y)\,dx \right) dy =$

$$\int \left(\int f(x,y)\,dy \right) dx \;?$$

Consider

$$f(x,y) = e^{-xy} - xy\,e^{-xy}$$

and $\displaystyle\int_{x=0}^{\infty}\int_0^1 f(x,y)\,dx\,dy$

Then

$$\int_0^{\infty}\left(\int_0^1 f(x,y)\,dy\right)dx$$

$$= \int_0^{\infty} y\,e^{-xy}\Big|_0^1\,dx = \int_0^{\infty} e^{-x}\,dx = \underline{1}$$

and

$$\int_0^1\left(\int_0^{\infty} f(x,y)\,dx\right)dy$$

$$= \int_0^1 x\,e^{-xy}\Big|_0^{\infty}\,dy = 0$$

(c) limit of a function.

Is $\lim_{x \to a} \lim_{y \to b} f(x,y) =$

$$\lim_{y \to b} \lim_{x \to a} f(x,y) \; ?$$

Consider
$$f(x,y) = \frac{x^2}{x^2+y^2}$$

Then $\lim_{x \to 0} \lim_{y \to 0} f(x,y) = 1$

and $\lim_{y \to} \lim_{x \to 0} f(x,y) = 0$

(d) Partial Derivatives

is

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \; ?$$

Consider

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & \text{else} \end{cases}$$

Then

$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = 1$$

and

$$\frac{\partial^2 f(0,0)}{\partial y \partial x} = 0$$

(e) Differentiability:

$$f(x,y) = \left(e^{xy},\ \sin(x^2+y^2),\ xy\right)$$

Is f differentiable?

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

What does it mean to say that f is differentiable?

Analysis is the Foundation of Calculus.

2. Review of analysis in $\mathbb{R}$.

- $(\mathbb{R}, +, \cdot, |\cdot|)$ is an ordered field.

- Lub property holds in $\mathbb{R}$ (also known as completeness property)

$\Updownarrow$ — Convergence of sequence.

- Monotone convergence property

$(x_n)\uparrow$: $x_1 \le x_2 \le \cdots \le x_n \le \cdots$

$(x_n)\downarrow$: $x_1 \geq x_2 \geq \cdots \geq x_n \cdots$

— Bounded + Monotone = Convergence

Cauchy criterion holds in $\mathbb{R}$ ($\mathbb{R}$ is complete)

(x_n) is Cauchy $\Rightarrow$ (x_n) is convergent

Cauchy: $|x_n - x_m| \to 0$ as $m, n \to \infty$

Formally: For $\varepsilon > 0$, $\exists$ $p \in \mathbb{N}$ s.t.

$m, n \geq p \Longrightarrow |x_n - x_m| < \varepsilon$

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- <u>Bolzano–Weierstrass theorem</u>

  - A bounded sequence has a convergent subsequence.

    $\Updownarrow$

  - A bounded infinite subset of $\mathbb{R}$ has a limit point.

What is a limit point?

Limit Point: $a \in \mathbb{R}$ and $A \subset \mathbb{R}$

a is a limit point if for any $\varepsilon > 0$

$(a-\varepsilon, a+\varepsilon) \cap A \setminus \{a\} \neq \phi$

$$\Updownarrow$$

$(a-\varepsilon, a+\varepsilon) \cap A$ contains infinitely many elements of A.

$$\Updownarrow$$

$\exists (x_n) \subset A \setminus \{a\}$ s.t $x_n \longrightarrow a$

- Heine-Borel theorem

Closed + Bounded = Compact

- Compact: $A \subset \mathbb{R}$ is compact if $(x_n) \subset A$ then $\exists (x_{n_k}) \subset (x_n)$ s.t $x_{n_k} \longrightarrow x$ in $\mathbb{R}$ and $x \in A$

Q. Can these results be generalised to $\mathbb{R}^n$?

3. The Euclidean Space $\mathbb{R}^n$

$$\mathbb{R}^n := \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$$

$$= \{ (x_1, x_2, \ldots x_n) : x_i \in \mathbb{R} \}$$

- $x = (x_1, x_2 \ldots x_n)$, $y = (y_1, y_2 \ldots y_n) \in \mathbb{R}^n$

$x + y := (x_1 + y_1, \ldots, x_n + y_n)$

- $\alpha x := (\alpha x_1, \ldots, \alpha x_n)$, $\alpha \in \mathbb{R}$

— $(\mathbb{R}^n, +, \cdot)$ is a vector space over $\mathbb{R}$.

- Absolute value: $(\mathbb{R}, |\cdot|)$

$$|x| := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

- Fundamental properties:

(i) $|x| \geq 0$ and $|x| = 0 \iff x = 0$

(ii) $|\alpha x| = |\alpha|\,|x|$, $\forall \alpha \in \mathbb{R}, x \in \mathbb{R}$

iii) $|x+y| \leq |x| + |y|$, $\forall x, y \in \mathbb{R}$

- Norm $(\mathbb{R}^n, \|\cdot\|)$

$$\|x\| := \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

Fundamental properties.

(i) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$

(ii) $\|\alpha x\| = |\alpha|\,\|x\|$, $\forall \alpha \in \mathbb{R}, x \in \mathbb{R}^n$

iii) $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{R}^n$

$\|x\| \longrightarrow$ Euclidean norm

Distance:

$$d(x,y) = \|x-y\|$$
$$= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

is the Euclidean distance between x and y.

- <u>Inner product / dot product.</u>

$$\langle \, , \, \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\langle x, y \rangle := x_1 y_1 + \cdots + x_n y_n$$
$$= x \cdot y$$

Then $\|x\| = \sqrt{\langle x, x \rangle}$

- Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|$$

Angle between vectors:

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \leq 1$$

$\exists$ unique $\theta \in [0, \pi]$ s.t.

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \, \|y\|}, \quad x \neq 0, y \neq 0.$$

$$\Rightarrow \boxed{\langle x, y \rangle = \|x\| \, \|y\| \cos\theta}$$

Orthogonality:

$\langle x, y \rangle = 0$ then $x \perp y$.

4. Convergence in $\mathbb{R}^n$.

A function $\mathbb{N} \longrightarrow \mathbb{R}^n, k \longmapsto x_k$

is called a sequence, written as

(x_k), $(x_k)_{k=1}^{\infty}$, $\{x_k\}$, $\{x_k\}_{k=1}^{\infty}$

Remark:

Each term x_K of (x_K) is a vector in $\mathbb{R}^n$ ie

$$x_K = (x_{1,K}\ x_{2,K}, \ldots x_{n,K}) \in \mathbb{R}^n$$

- **Convergence:** Let $(x_K) \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then $x_K \longrightarrow x$ if:

Informal: $\|x_K - x\| \longrightarrow 0$ as $K \to \infty$

Formal: For $\epsilon > 0$, $\exists\ p \in \mathbb{N}$ s.t

$$\boxed{K \geq p \implies \|x_K - x\| < \epsilon}$$

Same old defn !!!

$$\begin{array}{ccccccccc} x_K & = & (x_{1,K}, & x_{2,K}, & \ldots, & x_{n,K}) \\ \downarrow & \Longleftrightarrow & \downarrow & \downarrow & & \downarrow \\ x & = & (x_1, & x_2, & \ldots, & x_n) \end{array}$$

Indeed, $|x_j - x_{j,K}| \leq \|x - x_K\| \longrightarrow 0$

Theorem: Let $(x_k) \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

Then $x_k \longrightarrow x \iff x_{j,k} \longrightarrow x_j$ as $k \longrightarrow \infty$ for $j = 1, 2, \cdots n$.

Spl. Case: $X_k = (x_k, y_k) \in \mathbb{R}^2$.

Then $X_k \longrightarrow X = (x, y) \in \mathbb{R}^2$

$\iff$ $x_k \longrightarrow x$ and $y_k \longrightarrow y$.

Proof $|x_{j,k} - x_j| \leq \|x_k - x\| \longrightarrow 0$

$$\|x_k - x\| \leq \underbrace{|x_{1,k} - x_1|}_{\downarrow\, 0} + \cdots + \underbrace{|x_{n,k} - x_n|}_{\downarrow\, 0}$$

$\Longrightarrow \|x_k - x\| \longrightarrow 0$ as $k \to \infty$ $\boxtimes$

Moral: Convergence of sequences in $\mathbb{R}^n$ is essentially the same as that in $\mathbb{R}$.

Example: $x_n = ((-1)^n, 1/n)$, $x_n = (n \sin 1/n, e^{-n})$

- Completeness of $\mathbb{R}^n$.

Obviously, L.U.B. Prop, MC Prop have no analogues in $\mathbb{R}^n$.

Cauchy's criterion is amenable to generalization in $\mathbb{R}^n$.

- Cauchy sequence: $(x_K) \subset \mathbb{R}^n$

informal: $\|x_K - x_p\| \longrightarrow 0$ as, $K, p \to \infty$

Formal: For $\varepsilon > 0$, $\exists$ $m \in \mathbb{N}$ s.t.

$$\boxed{K, p \geq m \Longrightarrow \|x_K - x_p\| < \varepsilon}$$

Same old defn. !!!

Fact: $(X_K) = ((x_K, y_K)) \in \mathbb{R}^2$ is Cauchy

$\Longleftrightarrow$ (x_K) and (y_K) are Cauchy.

<u>Theorem:</u> $\mathbb{R}^n$ is Complete ie

$(x_K) \subset \mathbb{R}^n$ converges in $\mathbb{R}^n \iff$

(x_K) is a Cauchy sequence.

<u>Proof:</u> (x_K) converges $\Rightarrow (x_K)$ Cauchy.

Suppose (x_K) is Cauchy. $\Rightarrow (x_{j,K})_{K=1}^{\infty}$ is

Cauchy for each $j = 1, 2, \cdots n$.

$\Rightarrow \quad x_{j,K} \longrightarrow x_j$ for some $x_j \in \mathbb{R}$

because $\mathbb{R}$ is Complete.

$\Rightarrow \quad x_K \longrightarrow x = (x_1, x_2, \cdots x_n)$. $\square$

• <u>Theorem (Bolzano-Weierstrass)</u>

If $(x_K) \subset \mathbb{R}^n$ is bdd (ie $\|x_K\| \le$ Constant)

then (x_K) has a Convergent

Subsequence.

<u>Proof</u>. Spl. case: $X_K = (x_K, y_K) \in \mathbb{R}^2$

$(\cancel{X}_K)$ bdd $\Rightarrow (x_K)$ and (y_K) are bdd.

By B-W in $\mathbb{R}$, $\exists \left(x_{K_{p_m}}\right) \subset (x_K)$

s.t. $x_{K_{p_m}} \longrightarrow x$ as $p \to \infty$.

Again by B-W, $\exists \left(y_{K_{p_m}}\right) \subset (y_K)$

s.t. $y_{K_{p_m}} \longrightarrow y$ as $m \to \infty$

$$\Rightarrow \left(x_{K_{p_m}}, y_{K_{p_m}}\right) \longrightarrow (x, y)$$

$\qquad \parallel$

$X_{K_{p_m}}$ gives a subsequence of (X_K). $\boxed{\text{///}}$

- A bounded infinite subset of $\mathbb{R}^n$ has a limit point.

<u>Remark</u>: Bolzano-Weierstrass does not hold in infinite dimension.

Def: $A \subset \mathbb{R}^n$ is said to be bounded if $\exists\ M > 0$ such that $\|x\| \leq M\ \forall\ x \in A$.

Theorem (Heine-Borel):

Closed + bounded = Compact

Proof: $A \subset \mathbb{R}^n$ Compact $\Rightarrow$ A is closed and bounded (why?)

Suppose A is closed and bdd.

Let $(x_K) \subset A \xrightarrow{B-W} \exists (x_{K_p}) \subset (x_K)$

s.t $x_{K_p} \longrightarrow x \in \mathbb{R}^n$.

Since A is closed, $x \in A$

$\Longrightarrow$ A is compact. $\boxtimes$

— End —