Tutorial – 3 Solution

1 (a)

$$\left|\frac{x^3y}{x^4+y^2}\right| \leqq \left|\frac{\frac{x}{2}(x^4+y^2)}{x^4+y^2}\right| \leq \frac{1}{2}|x|$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0$$

(b)

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| = \left|\frac{(x-y)(x^2+xy+y^2)}{x^2+y^2}\right|$$

$$= |x-y|\left|\frac{x^2+xy+y^2}{x^2+y^2}\right|$$

$$\leq 2|x-y|$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$$

(c)

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$$

choose $y = mx$ then limit is different for different values of m

$\Rightarrow$ limit of $f(x,y)$ does not exists.

(d) $$f(x,y) = \frac{|x|}{y^2} e^{-|x|/y^2}$$

choose $x = my^2$ then limit is different for different values of m hence limit does not exists.

(e) $$f(x,y) = \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$$

changing to polar form we have

$$f(x,y) = \frac{1-\cos r^2}{r^4}$$

~~so this limit does not exists~~

(f) so $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{1-\cos r^2}{r^4}$$

$$= \frac{1}{2}$$

(apply L.H. Rule)

(f) $f(x,y) = \dfrac{\sqrt{x^2y^2+1} - 1}{x^2+y^2} = \dfrac{x^2y^2}{(x^2+y^2)(\sqrt{1+x^2y^2}}$

$\leq \dfrac{1}{2} \dfrac{|xy|}{\sqrt{1+x^2y^2}}$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$

2(a)

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists (choose $y = mx$)

$\lim_{x\to 0} \lim_{y\to 0} f(x,y) = 0 = \lim_{y\to 0} \lim_{x\to 0} f(x,y)$

(b) $|f(x,y)| = |y + x \sin \frac{1}{y}| \leq |x| + |y|$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$

$\lim_{y\to 0} \lim_{x\to 0} f(x,y) = \lim_{y\to 0} y = 0$

$\lim_{x\to 0}$ $\lim_{y\to 0} f(x,y)$ does not exists.

(c) $$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0 \quad f$$

$\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ does not exists.

3(a) $$|f(x,y)| = \left|x \sin\frac{1}{y} + y \sin\frac{1}{x}\right|$$
$$\leq |x| + |y|$$

$$\Rightarrow \lim_{(x,y)\to 0} f(x,y) = 0$$

$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y)$ does not exists

(b) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0 \quad f$$

$\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exists.

(c) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exists.

$\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x,y) = 0$ f

$\lim\limits_{y\to 0} \lim\limits_{(x,y)\to 0,0} f(x,y)$ does not exists.

(4) $\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0} \frac{x^2}{x^2} = 1$

$\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x,y) = \lim\limits_{y\to 0} \frac{-y^2}{y^2} = -1$

(5) 1st part Suppose that f is uniformly continus

$\Rightarrow$ for $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$\|x-y\| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \quad \forall x, y \in A$

$(x_k), (y_k) \subset A$ s.t. $\|x_k - y_k\| \longrightarrow 0$

$\Rightarrow$ for $\delta > 0$ $\exists$ Natural number N s.t.

$\|x_k - y_k\| < \delta \quad \forall \quad n \geq N$

$\Rightarrow \quad \| f(x_k) - f(y_k)| < \epsilon \quad \forall \quad n > N$

$\Rightarrow \quad |f(x_k) - f(y_k)| \longrightarrow 0$

Conversely, suppose that for every $(x_k), (y_k) \subset A$ such that $\|x_k - y_k\| \longrightarrow 0 \Rightarrow \|f(x_k) - f(y_k)\| \longrightarrow 0$

If possible suppose that f is not uniform continuous

Then $\cancel{\exists}\forall \ \delta > 0 \quad \exists \ \epsilon > 0, \ \exists \ x, y \in A$ s.t.

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| > \epsilon$$

choose $\delta = \frac{1}{k}$ then there exist x_k and y_k s.t.

$$|x_k - y_k| < \frac{1}{k} \text{ and } |f(x_k) - f(y_k)| > \epsilon$$

which is a contradiction

and therefore f is uniform cont.

2nd part If possible suppose that f is not uniform continus. then

$\exists \ \epsilon > 0$ and sequence $(x_k), (y_k) \subset A$ such that

$\|x_k - y_k\| \longrightarrow 0$ and $|f(x_k) - f(y_k)| \geq \epsilon$

Since A is compact $\Rightarrow$ (x_k) has convergent subsequence $(x_{k_p}) \longrightarrow x$

$$\| y_{k_p} - x \| = \| y_{k_p} - x_{k_p} + x_{k_p} - x \| \le \| x_{k_p} - y_{k_p} \| + \underbrace{\| x_{k_p} - x \|}_{\longrightarrow 0}$$

$\Rightarrow$ $(y_{k_p}) \longrightarrow x$

so $\| x_{k_p} - y_{k_p} \| \longrightarrow 0$ ~~but~~ $\cancel{| f(x_p) - f(y_p) |}$

|f(x_k_p)-f(y_k_p)| ->|f(x)-f(x)| $= |f(x) - f(x)| = 0$

$\because$ f cont.

which contradicts that $|f(x_{k_p}) - f(y_{k_p})| > \epsilon$

$\Rightarrow$ f uniform continuous

3rd part

choose $(x_k) = (k, 0)$

$(y_k) = (k + \frac{1}{k}, 0)$

Then $\| x_k - y_k \| = \| (\frac{1}{k}, 0) \| \longrightarrow 0$

but $|f(x_k) - f(y_k)| = |k^2 - (k + \frac{1}{k})^2| = 2 + \frac{1}{k^2} > 2$

and therefore it shows that f is not uniform continuous.

(6) $f(x) := \|x\|$

$$|f(x) - f(y)| = |\|x\| - \|y\|| \leq \|x - y\|$$

so $K = 1$

$g(x) := \sqrt{x}$

$$|g(x) - g(y)|^2 = |\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}| (\sqrt{x} + \sqrt{y})$$

$$\leq |x - y|$$

$\Rightarrow$ $|g(x) - g(y)| \leq \sqrt{|x-y|}$

if we choose $\delta = \epsilon^2$ then

$|x-y| < \delta \Rightarrow |g(x) - g(y)| < \epsilon$

which shows that g is uniform contin

If g is L. continuous then $\exists$ $K > 0$ s.t.

$|g(x) - g(y)| < K|x - y|$ $\forall$ x, y

$\Rightarrow$ $K > \dfrac{\sqrt{x} - \sqrt{y}}{x - y}$

$\Rightarrow \quad \frac{\sqrt{x}}{x} < K \quad \forall x \in [0,\infty)$

which is not Possible.

$h(x) := \frac{1}{x}$

choose $x \in (0,1)$ let $(x_n) \subset (0,1)$ s.t. $x_n \to x$

$h(x_n) = \frac{1}{x_n} \longrightarrow \frac{1}{x} = h(x)$

$\Rightarrow$ h is ~~uniform~~ Continuous

take $x_n = \frac{1}{n}$ & $y_n = \frac{1}{n+1}$

Then $|x_n - y_n| = \frac{1}{n(n+1)} \longrightarrow 0$ but

$|h(x_n) - h(y_n)| = 1 \not\longrightarrow 0$

and hence h is not uniform Continuous

7. choose $\epsilon > 0$

Then for $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \quad \forall x, y \in A$

(x_K) is Cauchy $\Rightarrow$ $\delta > 0$ $\exists$ Natural number N

such that $\|x_K - \cancel{x}_\ell\| < \delta$ $\forall$ $K, \ell \geq N$

$$\Rightarrow |f(x_K) - f(x_\ell)| < \epsilon \quad \forall K, \ell \geq N$$

and therefore it shows that $(f(x_K))$ is Cauchy.

$$f(x) := \frac{1}{x}$$

$\left(\frac{1}{n}\right)$ is Cauchy sequence but $f(x_n) = n$ is not Cauchy sequence.