

1. [G 3.1] Find the average potential over a spherical surface of radius R due to a point charge q located inside. Show that in general,
 $V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\epsilon_0 R}$,
 where V_{center} is the potential at the center due to all the external charges, and Q_{enc} is the total enclosed charge.

Solution:

Set up coordinate system such that the origin is at the center of the sphere and the charge q is at distance d on the z -axis. Let S : spherical surface (centered at origin, radius $R > d$). Average potential on S

$$\begin{aligned}\bar{V}_{ave} &= \frac{1}{4\pi R^2} \int_S V(\mathbf{r}) dS \\ &= \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \frac{R^2 \sin\theta d\theta d\phi}{\sqrt{R^2 + d^2 - 2Rd \cos\theta}} \\ &= \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} 2\pi R^2 \int_{-1}^1 \frac{du}{\sqrt{R^2 + d^2 - 2Rdu}} \\ &= \frac{1}{2} \frac{q}{4\pi\epsilon_0} \left[\frac{\sqrt{R^2 + d^2 - 2Rdu}}{(1/2)(-2Rd)} \right]_{-1}^1 = \frac{q}{4\pi\epsilon_0 R}\end{aligned}$$

The argument is exactly the same as in Sect. 3.1.4, except that since $z < R$, $\sqrt{z^2 + R^2 - 2zR} = (R - z)$, instead of $(z - R)$. Hence $V_{ave} = \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} [(z + R) - (R - z)] = \frac{1}{4\pi\epsilon_0} \frac{q}{R}$. If there is more than one charge inside the sphere, the average potential due to interior charges is $\frac{1}{4\pi\epsilon_0} \frac{Q_{enc}}{R}$, and the average due to exterior charges is V_{center} , so $V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\epsilon_0 R}$. ✓

2. (a) Two spherical conducting shells of radii r_a and r_b are arranged concentrically and are charged to the potentials V_a and V_b , respectively. If $r_b > r_a$, find the potential at points between the shells, and at points $r > r_b$. (b) Two long cylindrical shells of radii r_a and r_b arranged coaxially and are charged to the potentials V_a and V_b , respectively. Find the potential at points between the cylindrical shells.

Solution:

- (a) First, notice that the solution is independent of θ and ϕ owing to spherical symmetry of the problem. Then,

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = 0 \implies r^2 \frac{\partial V}{\partial r} = c \implies V = -\frac{c}{r} + k.$$

Applying BC, we get

$$V = -\frac{(V_a - V_b) r_a r_b}{r_a - r_b} \frac{1}{r} + \frac{V_a r_a - V_b r_b}{r_a - r_b}$$

- (b) Again, here the solution must be independent of z and ϕ . Then,

$$\nabla^2 V = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) = 0 \implies s \frac{\partial V}{\partial s} = c \implies V = c \ln s + k.$$

Applying BC we get,

$$V = \frac{V_a - V_b}{\ln(s_a/s_b)} \ln s + \frac{V_a \ln s_b + V_b \ln s_a}{\ln(s_a/s_b)}$$

3. [G 3.7] Consider a grounded conducting sphere of radius R . A charge q is placed at a distance $a(> R)$ on the z axis. The image charge $q' = -Rq/a$ is kept at distance $b = R^2/a$ on the z axis.

- (a) Using the law of cosines, show that $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r'} + \frac{q'}{r''} \right)$ (where r' and r'' are the distances from q and q' respectively) can be written as follows:

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{r^2 + a^2 - 2ra \cos \theta}} - \frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra \cos \theta}} \right]$$

where r and θ are the usual spherical polar coordinates. In this form it is obvious that $V = 0$ on the sphere, $r = R$.

- (b) Find the induced surface charge on the sphere, as a function of θ . Integrate this to get the total induced charge.
(c) Calculate the energy of this configuration.

Solution: (Sorry, I could not get script r in tex. so r' is script r and r'' is primed script r in the solution)

(a) From Fig. 3.13: $r = \sqrt{r^2 + a^2 - 2ra \cos \theta}$; $r' = \sqrt{r^2 + b^2 - 2rb \cos \theta}$. Therefore:

$$\begin{aligned} \frac{q'}{r'} &= -\frac{R}{a} \frac{q}{\sqrt{r^2 + b^2 - 2rb \cos \theta}} \quad (\text{Eq. 3.15}), \text{ while } b = \frac{R^2}{a} \quad (\text{Eq. 3.16}). \\ &= -\frac{q}{\left(\frac{a}{R}\right) \sqrt{r^2 + \frac{R^4}{a^2} - 2r \frac{R^2}{a} \cos \theta}} = -\frac{q}{\sqrt{\left(\frac{ar}{R}\right)^2 + R^2 - 2ra \cos \theta}}. \end{aligned}$$

Therefore:

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r} + \frac{q'}{r'} \right) = \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{\sqrt{r^2 + a^2 - 2ra \cos \theta}} - \frac{1}{\sqrt{R^2 + (ra/R)^2 - 2ra \cos \theta}} \right\}.$$

Clearly, when $r = R$, $V \rightarrow 0$.

(b) $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$ (Eq. 2.49). In this case, $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$ at the point $r = R$. Therefore,

$$\begin{aligned} \sigma(\theta) &= -\epsilon_0 \left(\frac{q}{4\pi\epsilon_0} \right) \left\{ -\frac{1}{2} (r^2 + a^2 - 2ra \cos \theta)^{-3/2} (2r - 2a \cos \theta) \right. \\ &\quad \left. + \frac{1}{2} (R^2 + (ra/R)^2 - 2ra \cos \theta)^{-3/2} \left(\frac{a^2}{R^2} 2r - 2a \cos \theta \right) \right\} \Big|_{r=R} \\ &= -\frac{q}{4\pi} \left\{ -(R^2 + a^2 - 2Ra \cos \theta)^{-3/2} (R - a \cos \theta) + (R^2 + a^2 - 2Ra \cos \theta)^{-3/2} \left(\frac{a^2}{R} - a \cos \theta \right) \right\} \\ &= \frac{q}{4\pi} (R^2 + a^2 - 2Ra \cos \theta)^{-3/2} \left[R - a \cos \theta - \frac{a^2}{R} + a \cos \theta \right] \\ &= \frac{q}{4\pi R} (R^2 - a^2) (R^2 + a^2 - 2Ra \cos \theta)^{-3/2}. \end{aligned}$$

$$\begin{aligned} q_{\text{induced}} &= \int \sigma da = \frac{q}{4\pi R} (R^2 - a^2) \int (R^2 + a^2 - 2Ra \cos \theta)^{-3/2} R^2 \sin \theta d\theta d\phi \\ &= \frac{q}{4\pi R} (R^2 - a^2) 2\pi R^2 \left[-\frac{1}{Ra} (R^2 + a^2 - 2Ra \cos \theta)^{-1/2} \right] \Big|_0^\pi \\ &= \frac{q}{2a} (a^2 - R^2) \left[\frac{1}{\sqrt{R^2 + a^2 + 2Ra}} - \frac{1}{\sqrt{R^2 + a^2 - 2Ra}} \right]. \end{aligned}$$

But $a > R$ (else q would be *inside*), so $\sqrt{R^2 + a^2 - 2Ra} = a - R$.

$$\begin{aligned} &= \frac{q}{2a} (a^2 - R^2) \left[\frac{1}{(a + R)} - \frac{1}{(a - R)} \right] = \frac{q}{2a} [(a - R) - (a + R)] = \frac{q}{2a} (-2R) \\ &= \boxed{-\frac{qR}{a} = q'}. \end{aligned}$$

(c) The force on q , due to the sphere, is the same as the force of the image charge q' , to wit:

$$F = \frac{1}{4\pi\epsilon_0} \frac{qq'}{(a-b)^2} = \frac{1}{4\pi\epsilon_0} \left(-\frac{R}{a}q^2\right) \frac{1}{(a-R^2/a)^2} = -\frac{1}{4\pi\epsilon_0} \frac{q^2 Ra}{(a^2-R^2)^2}.$$

To bring q in from infinity to a , then, we do work

$$W = \frac{q^2 R}{4\pi\epsilon_0} \int_{\infty}^a \frac{\bar{a}}{(\bar{a}^2 - R^2)^2} d\bar{a} = \frac{q^2 R}{4\pi\epsilon_0} \left[-\frac{1}{2} \frac{1}{(\bar{a}^2 - R^2)} \right]_{\infty}^a = \boxed{-\frac{1}{4\pi\epsilon_0} \frac{q^2 R}{2(a^2 - R^2)}}.$$

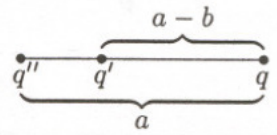
4. [G 3.8] Consider a point charge q situated at a distance a from the center of a grounded conducting sphere of radius R . The same basic model will handle the case of a sphere at any potential V_0 (relative to infinity) with the addition of a second image charge. What charge should you use, and where should you put it? Find the force of attraction between a point charge q and a neutral conducting sphere.

Solution:

Place a second image charge, q'' , at the *center* of the sphere; this will not alter the fact that the sphere is an *equipotential*, but merely *increase* that potential from zero to $V_0 = \frac{1}{4\pi\epsilon_0} \frac{q''}{R}$;

$$\boxed{q'' = 4\pi\epsilon_0 V_0 R \text{ at center of sphere.}}$$

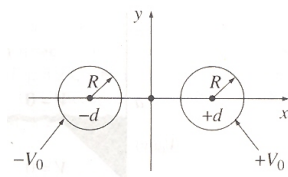
For a *neutral* sphere, $q' + q'' = 0$.



$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0} q \left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2} \right) = \frac{qq'}{4\pi\epsilon_0} \left(-\frac{1}{a^2} + \frac{1}{(a-b)^2} \right) \\ &= \frac{qq'}{4\pi\epsilon_0} \frac{b(2a-b)}{a^2(a-b)^2} = \frac{q(-Rq/a)(R^2/a)(2a-R^2/a)}{4\pi\epsilon_0 a^2(a-R^2/a)^2} \\ &= \boxed{-\frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a}\right)^3 \frac{(2a^2-R^2)}{(a^2-R^2)^2}}. \end{aligned}$$

(Drop the minus sign, because the problem asks for the force of *attraction*.)

5. [G 3.11] Two long, straight copper pipes, each of radius R , are held a distance $2d$ apart. One is at potential V_0 , the other at $-V_0$. Find the potential everywhere.



Solution: The solution depends on problem 2.47, so the solution is given here.

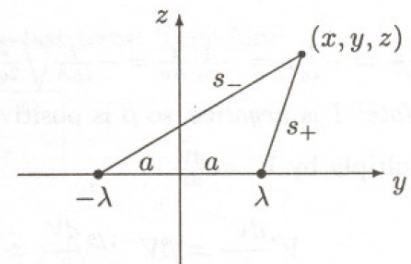
Problem 2.47

- (a) Potential of $+\lambda$ is $V_+ = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_+}{a}\right)$, where s_+ is distance from λ_+ (Prob. 2.22).
Potential of $-\lambda$ is $V_- = +\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_-}{a}\right)$, where s_- is distance from λ_- .

$$\therefore \text{Total } V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_-}{s_+}\right).$$

Now $s_+ = \sqrt{(y-a)^2 + z^2}$, and $s_- = \sqrt{(y+a)^2 + z^2}$, so

$$V(x, y, z) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{(y+a)^2 + z^2}}{\sqrt{(y-a)^2 + z^2}}\right) = \boxed{\frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right]}.$$



(b) Equipotentials are given by $\frac{(y+a)^2+z^2}{(y-a)^2+z^2} = e^{(4\pi\epsilon_0 V_0/\lambda)} = k = \text{constant}$. That is:

$$y^2 + 2ay + a^2 + z^2 = k(y^2 - 2ay + a^2 + z^2) \Rightarrow y^2(k-1) + z^2(k-1) + a^2(k-1) - 2ay(k+1) = 0, \text{ or}$$

$$y^2 + z^2 + a^2 - 2ay \left(\frac{k+1}{k-1} \right) = 0. \text{ The equation for a circle, with center at } (y_0, 0) \text{ and radius } R, \text{ is}$$

$$(y - y_0)^2 + z^2 = R^2, \text{ or } y^2 + z^2 + (y_0^2 - R^2) - 2yy_0 = 0.$$

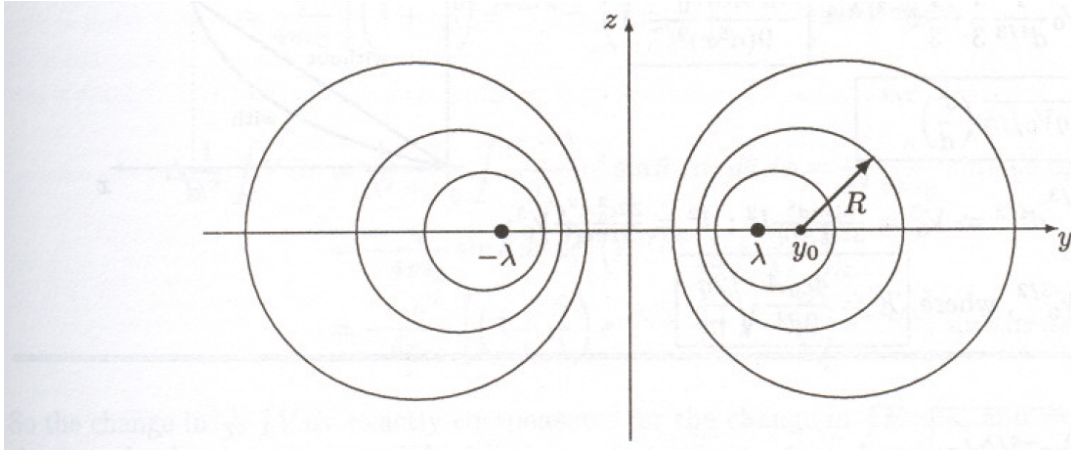
Evidently the equipotentials are circles, with $y_0 = a \left(\frac{k+1}{k-1} \right)$ and

$$a^2 = y_0^2 - R^2 \Rightarrow R^2 = y_0^2 - a^2 = a^2 \left(\frac{k+1}{k-1} \right)^2 - a^2 = a^2 \frac{(k^2+2k+1-k^2-2k-1)}{(k-1)^2} = a^2 \frac{4k}{(k-1)^2}, \text{ or}$$

$$R = \frac{2a\sqrt{k}}{|k-1|}; \text{ or, in terms of } V_0:$$

$$y_0 = a \frac{e^{4\pi\epsilon_0 V_0/\lambda} + 1}{e^{4\pi\epsilon_0 V_0/\lambda} - 1} = a \frac{e^{2\pi\epsilon_0 V_0/\lambda} + e^{-2\pi\epsilon_0 V_0/\lambda}}{e^{2\pi\epsilon_0 V_0/\lambda} - e^{-2\pi\epsilon_0 V_0/\lambda}} = \boxed{a \coth \left(\frac{2\pi\epsilon_0 V_0}{\lambda} \right)}.$$

$$R = 2a \frac{e^{2\pi\epsilon_0 V_0/\lambda}}{e^{4\pi\epsilon_0 V_0/\lambda} - 1} = a \frac{2}{(e^{2\pi\epsilon_0 V_0/\lambda} - e^{-2\pi\epsilon_0 V_0/\lambda})} = \frac{a}{\sinh \left(\frac{2\pi\epsilon_0 V_0}{\lambda} \right)} = \boxed{a \operatorname{csch} \left(\frac{2\pi\epsilon_0 V_0}{\lambda} \right)}.$$



From Prob. 2.47 (with $y_0 \rightarrow d$):

$$V = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \right], \text{ where } a^2 = y_0^2 - R^2 \Rightarrow \boxed{a = \sqrt{d^2 - R^2}},$$

and

$$\left\{ \begin{array}{l} a \coth(2\pi\epsilon_0 V_0/\lambda) = d \\ a \operatorname{csch}(2\pi\epsilon_0 V_0/\lambda) = R \end{array} \right\} \Rightarrow (\text{dividing}) \quad \frac{d}{R} = \cosh \left(\frac{2\pi\epsilon_0 V_0}{\lambda} \right), \text{ or } \boxed{\lambda = \frac{2\pi\epsilon_0 V_0}{\cosh^{-1}(d/R)}}.$$

6. [G 3.12] Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a constant potential V_0 , and the other, from $y = a/2$ to $y = a$, is at potential $-V_0$. Find the potential in the infinite slot.

Solution:

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a) \quad (\text{Eq. 3.30}), \quad \text{where } C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a) dy \quad (\text{Eq. 3.34}).$$

In this case $V_0(y) = \begin{cases} +V_0, & \text{for } 0 < y < a/2 \\ -V_0, & \text{for } a/2 < y < a \end{cases}$. Therefore,

$$C_n = \frac{2}{a} V_0 \left\{ \int_0^{a/2} \sin(n\pi y/a) dy - \int_{a/2}^a \sin(n\pi y/a) dy \right\} = \frac{2V_0}{a} \left\{ -\frac{\cos(n\pi y/a)}{(n\pi/a)} \Big|_0^{a/2} + \frac{\cos(n\pi y/a)}{(n\pi/a)} \Big|_{a/2}^a \right\}$$

$$= \frac{2V_0}{n\pi} \left\{ -\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) \right\} = \frac{2V_0}{n\pi} \left\{ 1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right) \right\}.$$

The term in curly brackets is:

$$\left\{ \begin{array}{ll} n=1 & : 1 - 1 - 2 \cos(\pi/2) = 0, \\ n=2 & : 1 + 1 - 2 \cos(\pi) = 4, \\ n=3 & : 1 - 1 - 2 \cos(3\pi/2) = 0, \\ n=4 & : 1 + 1 - 2 \cos(2\pi) = 0, \end{array} \right\} \text{ etc. (Zero if } n \text{ is odd or divisible by 4, otherwise 4.)}$$

Therefore

$$C_n = \begin{cases} 8V_0/n\pi, & n = 2, 6, 10, 14, \text{ etc. (in general, } 4j + 2, \text{ for } j = 0, 1, 2, \dots), \\ 0, & \text{otherwise.} \end{cases}$$

So

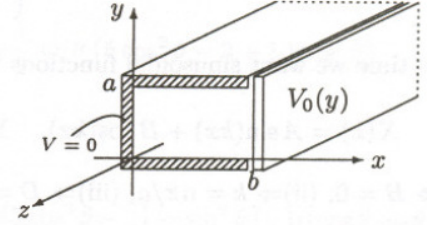
$$V(x, y) = \frac{8V_0}{\pi} \sum_{n=2,6,10,\dots} \frac{e^{-n\pi x/a} \sin(n\pi y/a)}{n} = \frac{8V_0}{\pi} \sum_{j=0}^{\infty} \frac{e^{-(4j+2)\pi x/a} \sin[(4j+2)\pi y/a]}{(4j+2)}.$$

7. [G 3.14] A rectangular pipe, running parallel to the z -axis (from $-\infty$ to ∞), has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specific potential $V_0(y)$.
- (a) Develop a general formula for the potential within the pipe.
- (b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).

Solution:

$$(a) \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \text{ with boundary conditions}$$

$$\left\{ \begin{array}{ll} (i) & V(x, 0) = 0, \\ (ii) & V(x, a) = 0, \\ (iii) & V(0, y) = 0, \\ (iv) & V(b, y) = V_0(y). \end{array} \right\}$$



As in Ex. 3.4, separation of variables yields

$$V(x, y) = (Ae^{kx} + Be^{-kx}) (C \sin ky + D \cos ky).$$

Here (i) $\Rightarrow D = 0$, (iii) $\Rightarrow B = -A$, (ii) $\Rightarrow ka$ is an integer multiple of π :

$$V(x, y) = AC (e^{n\pi x/a} - e^{-n\pi x/a}) \sin(n\pi y/a) = (2AC) \sinh(n\pi x/a) \sin(n\pi y/a).$$

But $(2AC)$ is a constant, and the most general linear combination of separable solutions consistent with (i), (ii), (iii) is

$$V(x, y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi x/a) \sin(n\pi y/a).$$

It remains to determine the coefficients C_n so as to fit boundary condition (iv):

$$\sum C_n \sinh(n\pi b/a) \sin(n\pi y/a) = V_0(y). \text{ Fourier's trick } \Rightarrow C_n \sinh(n\pi b/a) = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a) dy.$$

Therefore

$$C_n = \frac{2}{a \sinh(n\pi b/a)} \int_0^a V_0(y) \sin(n\pi y/a) dy.$$

$$(b) C_n = \frac{2}{a \sinh(n\pi b/a)} V_0 \int_0^a \sin(n\pi y/a) dy = \frac{2V_0}{a \sinh(n\pi b/a)} \times \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{2a}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\dots} \frac{\sinh(n\pi x/a) \sin(n\pi y/a)}{n \sinh(n\pi b/a)}.$$

8. [G 3.15] A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded. The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0 . Find the potential inside the box.

Solution:

Same format as Ex. 3.5, only the boundary conditions are:

$$\left. \begin{array}{lll} \text{(i)} & V = 0 & \text{when } x = 0, \\ \text{(ii)} & V = 0 & \text{when } x = a, \\ \text{(iii)} & V = 0 & \text{when } y = 0, \\ \text{(iv)} & V = 0 & \text{when } y = a, \\ \text{(v)} & V = 0 & \text{when } z = 0, \\ \text{(vi)} & V = V_0 & \text{when } z = a. \end{array} \right\}$$

This time we want sinusoidal functions in x and y , exponential in z :

$$X(x) = A \sin(kx) + B \cos(kx), \quad Y(y) = C \sin(l y) + D \cos(l y), \quad Z(z) = E e^{\sqrt{k^2+l^2}z} + G e^{-\sqrt{k^2+l^2}z}.$$

(i) $\Rightarrow B = 0$; (ii) $\Rightarrow k = n\pi/a$; (iii) $\Rightarrow D = 0$; (iv) $\Rightarrow l = m\pi/a$; (v) $\Rightarrow E + G = 0$. Therefore

$$Z(z) = 2E \sinh(\pi \sqrt{n^2 + m^2} z/a).$$

Putting this all together, and combining the constants, we have:

$$V(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{n,m} \sin(n\pi x/a) \sin(m\pi y/a) \sinh(\pi \sqrt{n^2 + m^2} z/a).$$

It remains to evaluate the constants $C_{n,m}$, by imposing boundary condition (vi):

$$V_0 = \sum \sum [C_{n,m} \sinh(\pi \sqrt{n^2 + m^2})] \sin(n\pi x/a) \sin(m\pi y/a).$$

According to Eqs. 3.50 and 3.51:

$$C_{n,m} \sinh(\pi \sqrt{n^2 + m^2}) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a \int_0^a \sin(n\pi x/a) \sin(m\pi y/a) dx dy = \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \frac{16V_0}{\pi^2 n m}, & \text{if both are odd.} \end{cases}$$

Therefore

$$V(x, y, z) = \frac{16V_0}{\pi^2} \sum_{n=1,3,5,\dots} \sum_{m=1,3,5,\dots} \frac{1}{nm} \sin(n\pi x/a) \sin(m\pi y/a) \frac{\sinh(\pi \sqrt{n^2 + m^2} z/a)}{\sinh(\pi \sqrt{n^2 + m^2})}.$$

Additional Problem:

1. Consider an insulated spherical conductor with radius a and net charge q . Another point charge q is placed outside the conductor at a distance d from the center of the conductor. Show that if $d/a = (1 + \sqrt{5})/2$, the point charge is in equilibrium. [Hint: Show that d/a must be the solution of the quintic equation $x^5 - 2x^3 - 2x^2 + x + 1 = 0$. There are three real and two imaginary solutions. Verify that $x^2 - x - 1$ is a factor of the quintic polynomial.]