Physics II: Electromagnetism (PH102)

Lecture 8

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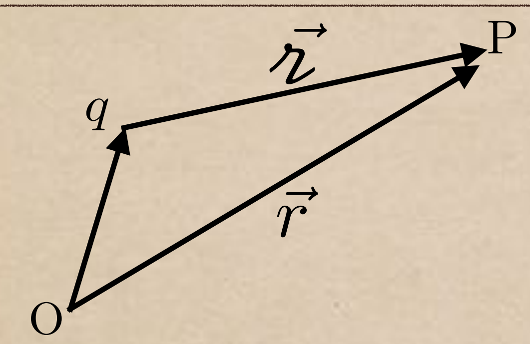
Electric Field to Potential and Energy

Electric potential

$$\vec{\nabla} \times \vec{E} = 0 \implies \vec{E} = -\vec{\nabla} V$$

V is called electric potential



In general the potential for a point charge q is

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}}$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$

Poisson's equation

In source free region (no charge) : $\rho = 0$

$$\nabla^2 V = 0$$

Laplace's equation

$$\boxed{V(r) = -\int_{\mathcal{O}}^{r} \vec{E}(\vec{r}\,').d\vec{\ell}'}$$

Potential for continuous charge distributions

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}\,')}{\mathscr{r}} d\ell'$$

Line charge

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r}\,')}{\mathscr{r}} da'$$

Surface charge

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{\mathscr{r}} d\tau'$$

Volume charge

Potential due to uniformly charged spherical shell

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{r}\, da', \qquad r^2 = R^2 + z^2 - 2Rz\cos\theta'$$

$$\epsilon_0 4\pi V(z) = \sigma \int \frac{R^2 \sin\theta' d\theta' d\varphi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$= 2\pi R^2 \sigma \int_0^{\pi} \frac{\sin\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}\, d\theta'$$

$$= 2\pi R^2 \sigma \left(\frac{1}{Rz} \sqrt{R^2 + z^2 - 2Rz\cos\theta'} \right)\Bigg|_0^{\pi}$$

$$= \frac{2\pi R\sigma}{z} \left(\sqrt{R^2 + z^2 + 2Rz} - \sqrt{R^2 + z^2 - 2Rz} \right)$$

$$= \frac{2\pi R\sigma}{z} \left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2} \right]$$

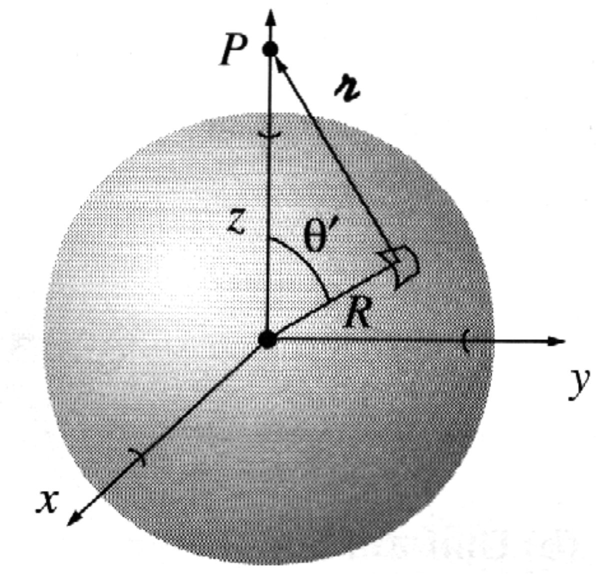


Figure 2.33

$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z) - (z-R)] = \frac{R^2\sigma}{\varepsilon_0 z}, \quad outside$$

$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z) - (R-z)] = \frac{R\sigma}{\varepsilon_0}, \quad inside$$

Note: outside the shell, z>R

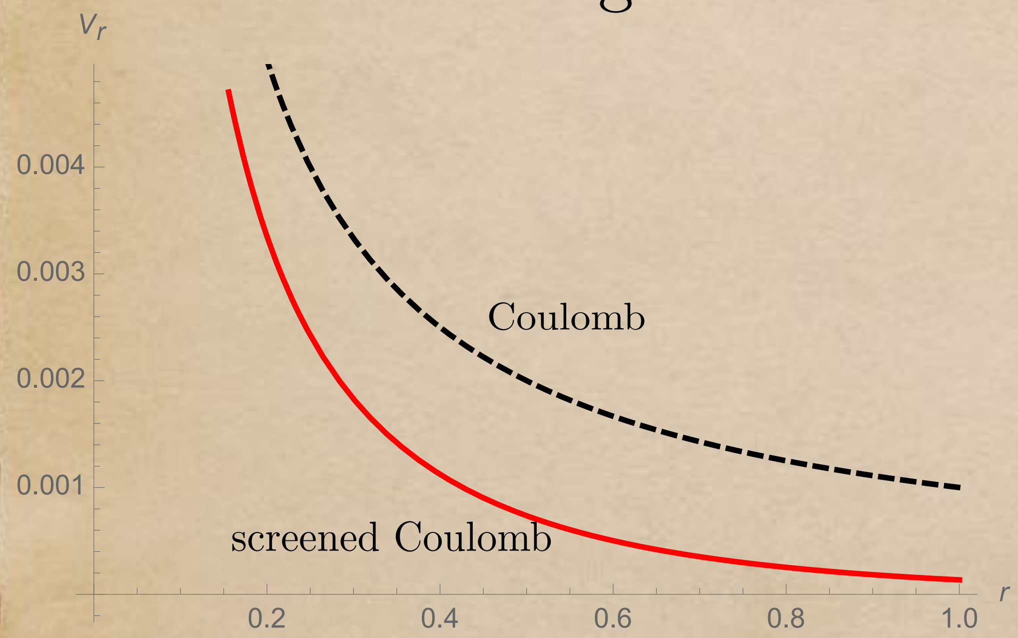
Screened Coulomb Potential

Consider the so-called "screened Coulomb potential" of a point charge q that arises, for example, in plasma physics

$$\boxed{V(r) = \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}}$$

λ is a constant called screening length.

Determine the charge distribution $\rho(r)$ that produces this potential.



We will use $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$

$$\vec{E} = -\vec{\nabla}V = -\frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{e^{-r/\lambda}}{r}\right)\hat{r}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\hat{r}$$

$$\vec{\nabla}.\vec{E} = \frac{q}{4\pi\epsilon_0}\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left[\left\{\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right)\right\}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda} + \frac{\hat{r}}{r^2}.\vec{\nabla}\left\{\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\right\}\right]$$

Screened Coulomb potential

Using the identity $\vec{\nabla}.(f\vec{A}) = f(\vec{\nabla}.\vec{A}) + \vec{A}.(\vec{\nabla}f)$ and $\vec{\nabla}.(\hat{r}/r^2) = 4\pi\delta^3(r)$

$$\begin{aligned}
\vec{\nabla}.\vec{E} &= \frac{q}{4\pi\epsilon_0}\left[e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)4\pi\delta^3(\vec{r}) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)\right)\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)4\pi\delta^3(\vec{r}) - \frac{1}{r\lambda^2}e^{-r/\lambda}\right] \equiv \frac{\rho}{\epsilon_0}
\end{aligned}$$

Thus

$$\boxed{\rho = q\delta^3(\vec{r}) - \frac{q}{4\pi\lambda^2 r}e^{-r/\lambda}}$$

There is a point charge at the origin

In addition, there is an exponential decay charge density

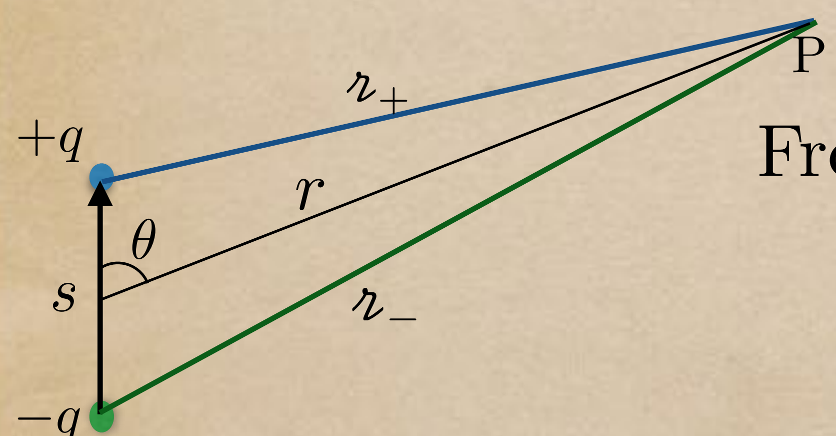
Dipole

An electric dipole consists of two equal and opposite charges $(\pm q)$ separated by a distance `s'

We will calculate potential and electric field due to electric dipole

$$V(P) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

Superposition



From the geometry:

$$\begin{aligned} \mathscr{r}_\pm^2 &= r^2 + \left(\frac{s}{2}\right)^2 \mp rs\cos\theta \\ &= r^2\left(1 \mp \frac{s}{r}\cos\theta + \frac{s^2}{4r^2}\right) \end{aligned}$$

We are interested in the regime $r >> s$

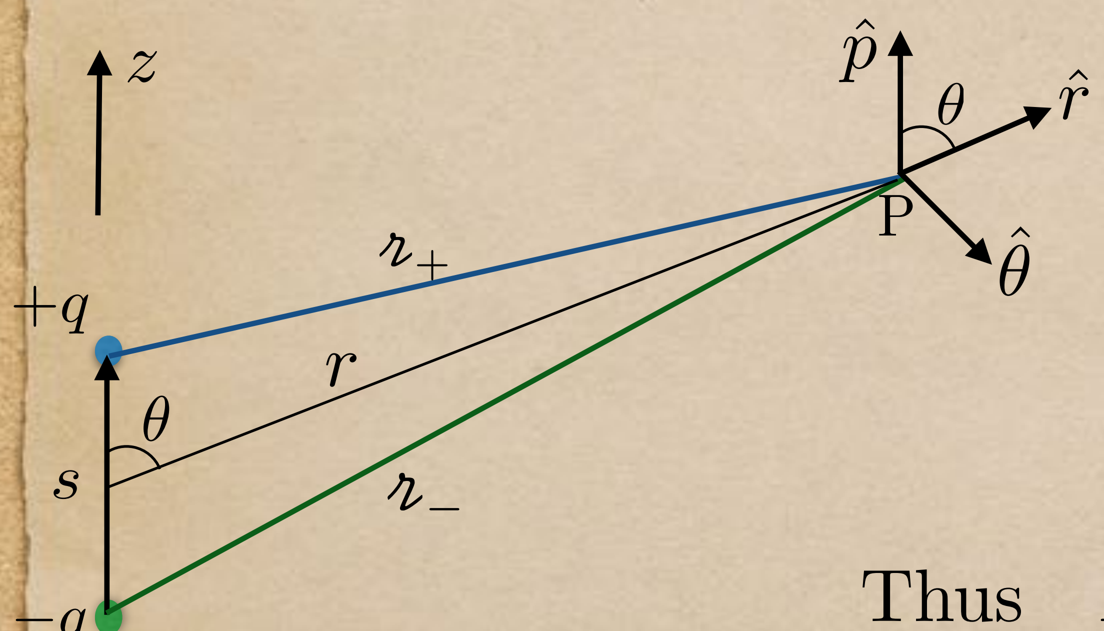
$$\frac{1}{\mathscr{r}_\pm} \simeq \frac{1}{r}\left(1 \mp \frac{s}{r}\cos\theta\right)^{-1/2} \simeq \frac{1}{r}\left(1 \pm \frac{s}{2r}\cos\theta\right)$$

Dipole moment

Thus $\dfrac{1}{\imath_+} - \dfrac{1}{\imath_-} \simeq \dfrac{s}{r^2}\cos\theta$ and hence $\boxed{V(P) \simeq \dfrac{1}{4\pi\epsilon_0}\dfrac{qs\cos\theta}{r^2}}$

$\boxed{\text{Dipole moment: } \vec{p} = q\vec{s}}$ $\longrightarrow$ $qs\cos\theta = p\cos\theta = \vec{p}.\hat{r}$

$$\boxed{V(P) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}.\hat{r}}{r^2}}$$



Now, $\vec{E} = -\vec{\nabla}V$

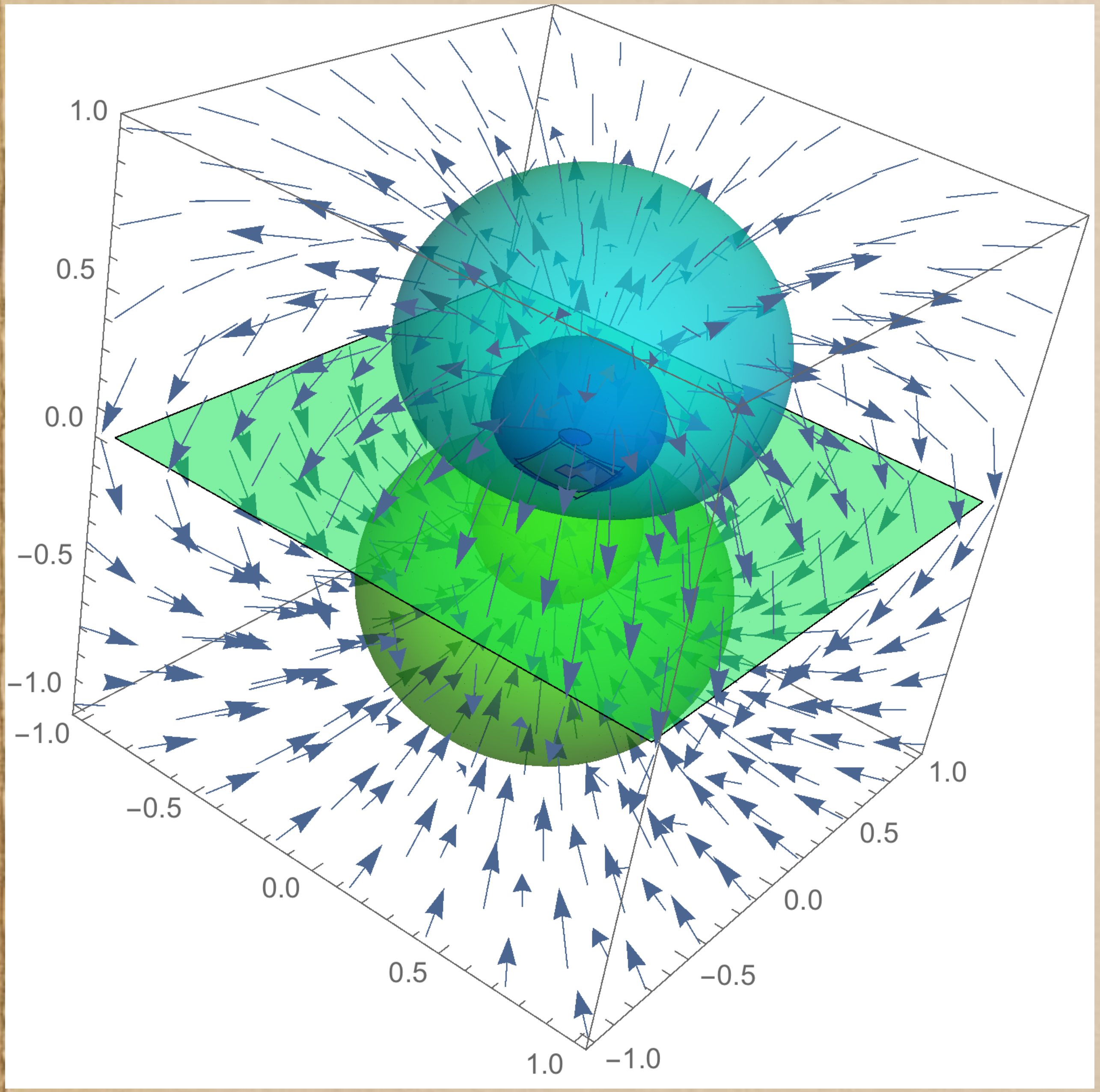
$$\vec{\nabla} \equiv \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

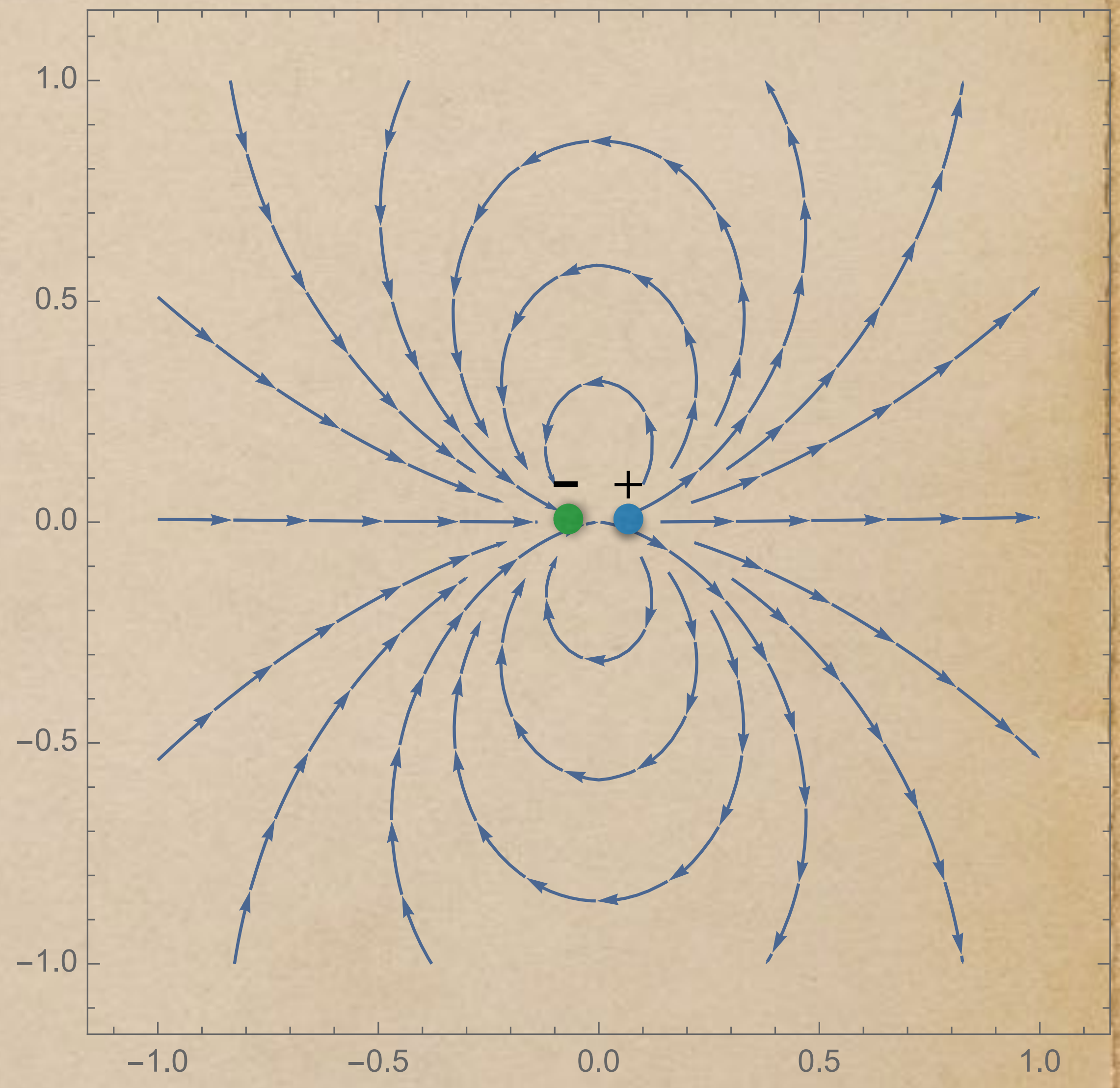
Thus
$$\vec{E} = -\frac{p}{4\pi\epsilon_0}\vec{\nabla}\frac{\cos\theta}{r^2}$$
$$= -\frac{p}{4\pi\epsilon_0}\left(\hat{r}\frac{\partial}{\partial r}\frac{\cos\theta}{r^2} + \hat{\theta}\frac{1}{r^3}\frac{\partial\cos\theta}{\partial\theta}\right)$$
$$= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

Assumes spherical polar and direction of $\vec{p}$ along z

Dipole potential falls of as inverse square and field falls of as inverse cube

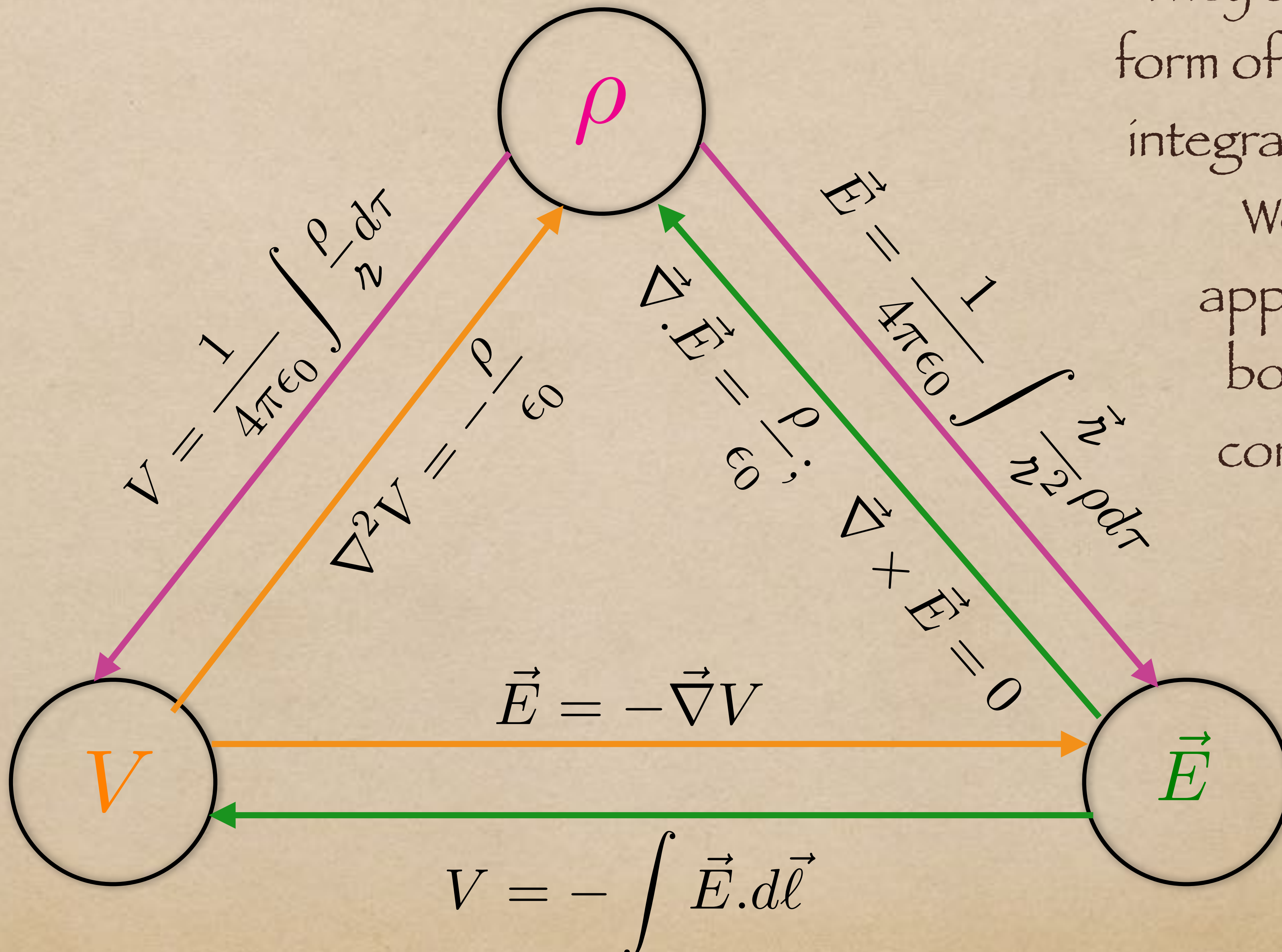
Field of dipole





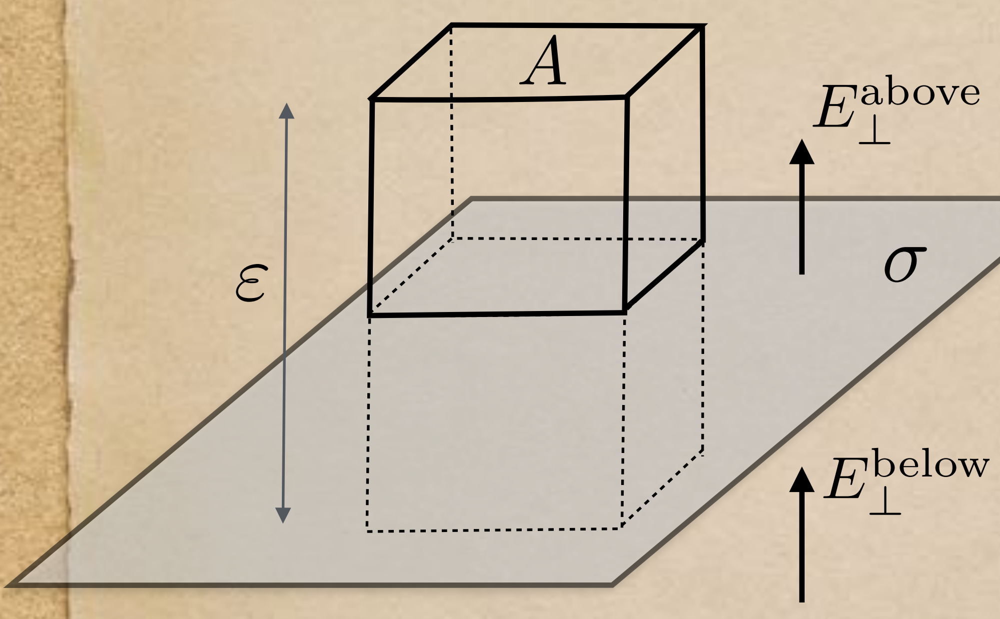
Field, potential and charge density

We had two assumptions: (i) Coulomb's law, (ii) Superposition principle



They are all in the form of differential, integral equations. We need appropriate boundary conditions

Electrostatic boundary conditions

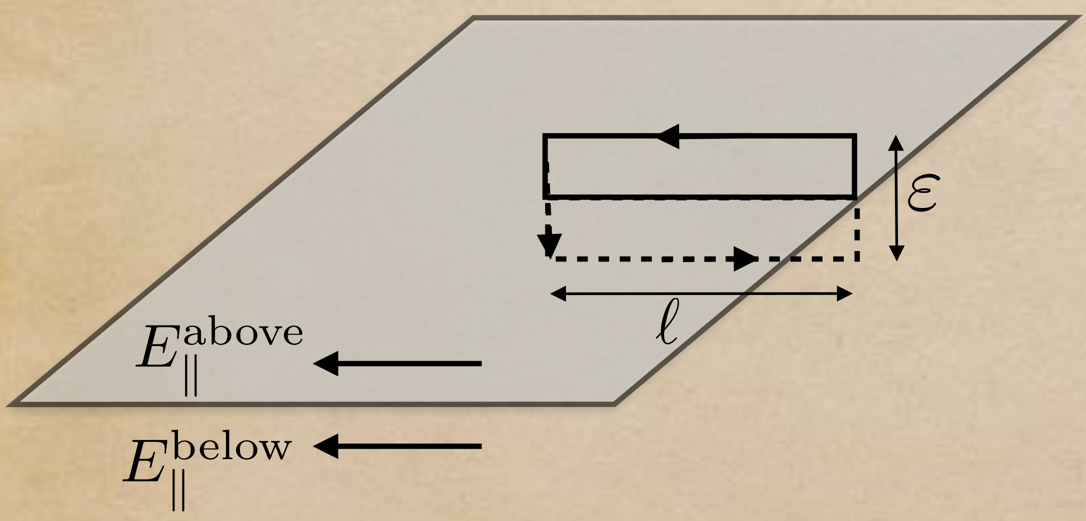


$$\oint_{\mathcal{S}} \vec{E}.d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{1}{\epsilon_0}\sigma A$$

As $\varepsilon \to 0$ $\quad E_\perp^{\text{above}} A - E_\perp^{\text{below}} A = \frac{1}{\epsilon_0}\sigma A$

Normal component of the electric field is discontinuous by an amount

$$\boxed{E_\perp^{\text{above}} - E_\perp^{\text{below}} = \frac{\sigma}{\epsilon_0}}$$



Since $\vec{E}$ is conservative field, we have $\oint \vec{E}.d\vec{\ell} = 0$ always. Applying this to the sides as $\varepsilon \to 0$: $E_\parallel^{\text{above}}\ell - E_\parallel^{\text{below}}\ell = 0$

$\therefore$ for the tangential component

$$\boxed{E_\parallel^{\text{above}} = E_\parallel^{\text{below}}}$$

Electrostatic boundary conditions:

The boundary conditions on $\vec{E}$ can be combined to a single formula

$$\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{n}$$

$\hat{n}$: unit vector $\perp$ to surface

Electric field is discontinuous whenever there is a surface charge density

However, potential is continuous across the boundary

$$V_{\text{above}} - V_{\text{below}} = \int_a^b \vec{E}.d\vec{\ell}$$

The path length shrinks to zero at the boundary and hence,

$$V_{\text{above}} = V_{\text{below}}$$

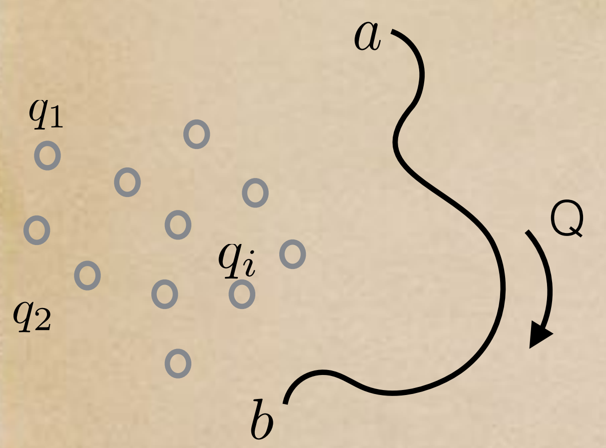
As $\vec{E} = -\vec{\nabla} V$

$$\vec{\nabla} V_{\text{above}} - \vec{\nabla} V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{n}$$

or, $$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

where $\dfrac{\partial V}{\partial n} = \vec{\nabla} V.\hat{n}$ is the normal derivative of of V (rate of change in the direction $\perp$) to surface)

Work and energy in electrostatics



How much work is needed to move a point charge Q from a to b ?

Work done by $\vec{F}$ $$W = \int_a^b \vec{F}.d\vec{\ell} = -Q\int_a^b \vec{E}.d\vec{\ell} = Q[V(b) - V(a)]$$

independent of path, depends only on end points

Potential difference between a and b is the work done per unit charge for moving the particle from a to b

$$V(b) - V(a) = \frac{W}{Q}$$

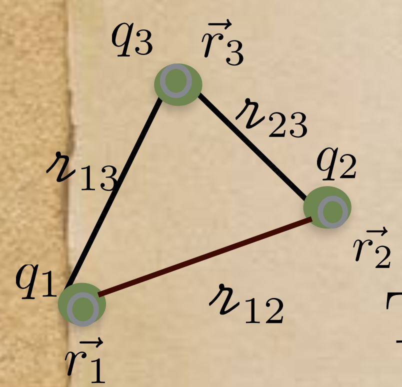
Work done per unit charge : Potential

$$W = Q[V(r) - V(\infty)]$$

$\rightarrow$ $W = QV(r)$ if reference at ∞ with $V(\infty) = 0$

Energy of point charge distribution

How much work is needed to assemble entire collection of charges ?



No work is done to bring the first one.

Total work done in placing the second charge $W = \dfrac{1}{4\pi\epsilon_0}\left(\dfrac{q_1 q_2}{\mathscr{r}_{12}}\right)$

Total work done in placing the third charge q_3 at $\vec{r}_3$

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}}\right)}$$

together with the second one

Generalised formula for n charges

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\boxed{j>i}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

not to calculate each pair twice

$$= \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\boxed{j\neq i}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

here I have calculated each pair twice and then divided by two

Energy of point charge distribution

You can also write the energy as:

$$\boxed{W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}}$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right)$$

$$\boxed{= \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i).}$$

where $V(\vec{r}_i)$ is the scalar potential experienced by the i-th point charge due to other point charges.

W represents the amount of work needed to assemble a configuration of point charges.

Note that this is also the same amount of energy required to dismantle the charge configuration/distribution

Energy of continuous charge distribution

Suppose, we have a volume charge distribution with density ρ

Each volume element $d\tau$ contains charge $\rho d\tau$.

Generalising the previous case:

$$W = \frac{1}{2}\int\int \frac{1}{4\pi\epsilon_0}\frac{\rho(1)\rho(2)}{\mathscr{r}_{12}} d\tau_1 d\tau_2$$

$$= \frac{1}{2}\int \rho(1) \int \frac{1}{4\pi\epsilon_0}\frac{\rho(2)}{\mathscr{r}_{12}} d\tau_2 d\tau_1$$

potential $V(1)$ at (1)

since (2) no longer appears

$$= \frac{1}{2}\int \rho V d\tau$$

The energy of the charge $\rho d\tau$ is the product of this charge and the potential at the same point. The total energy therefore is the integral over $V\rho d\tau$

The factor of 1/2

The $\frac{1}{2}$ is still required because we are counting energies twice. The mutual energies of two charges is the charge of one times the potential at it due to the other. Or, it can be taken as the second charge times the potential at it from the first. Thus for two point charges we could write:

$$\begin{aligned} W &= q_1 V(1) = q_1 \frac{1}{4\pi\epsilon_0}\frac{q_2}{\mathscr{r}_{12}} \\ \text{or} \quad W &= q_2 V(2) = q_2 \frac{1}{4\pi\epsilon_0}\frac{q_1}{\mathscr{r}_{12}} \end{aligned}$$

But, note: we could also write

$$\boxed{W = \frac{1}{2}[q_1 V(1) + q_2 V(2)]}$$

The integral in the previous slide corresponds to the sum of both terms in the brackets of the above equation. That is why we need the factor $\frac{1}{2}$.

Energy of continuous charge distribution

Volume charge distribution ρ

$$W = \frac{1}{2}\int \rho V \, d\tau$$

Corresponding integrals for line and surface charges will be $\int \lambda V \, d\ell$ and $\sigma V \, da$.

Recall that $\rho = \epsilon_0 \vec{\nabla}.\vec{E}$ and $\vec{E} = -\vec{\nabla}V$. Therefore $\rho = -\epsilon_0 \nabla^2 V$. So that we can write

$$W = -\frac{\epsilon_0}{2}\int V \nabla^2 V \, d\tau$$

$$\begin{aligned} V\nabla^2 V &= V\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) \\ &= \frac{\partial}{\partial x}\left(V\frac{\partial V}{\partial x}\right) - \left(\frac{\partial V}{\partial x}\right)^2 + \frac{\partial}{\partial y}\left(V\frac{\partial V}{\partial y}\right) - \left(\frac{\partial V}{\partial y}\right)^2 + \frac{\partial}{\partial z}\left(V\frac{\partial V}{\partial z}\right) - \left(\frac{\partial V}{\partial z}\right)^2 \\ &= \vec{\nabla}.(V\vec{\nabla}V) - (\vec{\nabla}V).(\vec{\nabla}V). \end{aligned}$$

Therefore $$W = \frac{\epsilon_0}{2}\int (\vec{\nabla}V).(\vec{\nabla}V) d\tau - \frac{\epsilon_0}{2}\int \vec{\nabla}.(V\vec{\nabla}V) d\tau$$

But using Gauss's theorem $$\int_{vol.} \vec{\nabla}.(V\vec{\nabla}V) d\tau = \int_{surf.} (V\vec{\nabla}V).\hat{n} da.$$

Energy of continuous charge distribution

Hence $$W = \frac{\epsilon_0}{2}\left(\int_{\text{vol}} (\vec{\nabla}V).(\vec{\nabla}V)d\tau - \int_{\text{surf}} (V\vec{\nabla}V).\hat{n}da\right).$$

Evaluate surface integral in the case when surface goes to infinity.

why ?

So that volume integral becomes integral over all space

How?

Take a spherical surface of enormous radius R with centre at the origin

Far away from charges, V varies as $1/R$ and $\vec{\nabla}V$ as $1/R^2$, but surface area increases as R^2

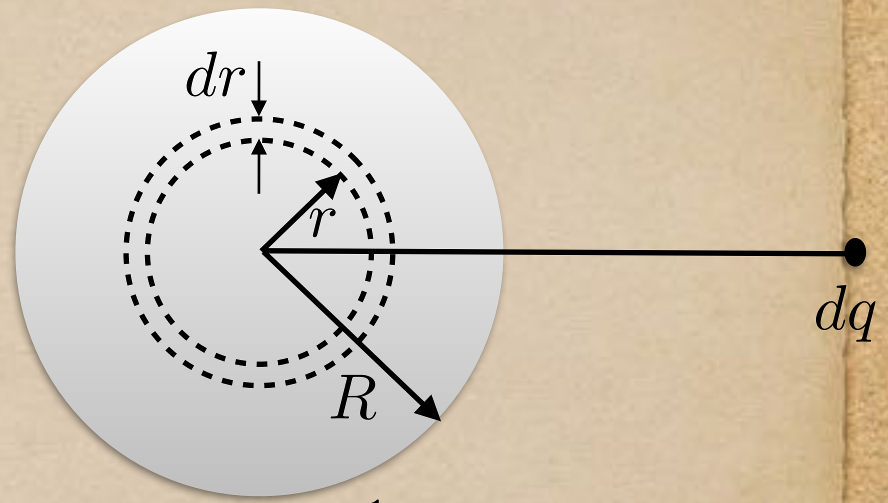
Surface integral falls off as $(1/R)(1/R^2)R^2 \sim (1/R)$ as R increases

Including all space ($R \to \infty$) means the surface integral $\to 0$

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} (\vec{\nabla}V).(\vec{\nabla}V)d\tau = \frac{\epsilon_0}{2}\int_{\text{all space}} \vec{E}.\vec{E}d\tau$$

Energy of uniformly solid sphere

Imagine building up the sphere by accumulating thin layers of infinitesimal thickness between r to r+dr



Work done in bringing a charge dq to it is $dW = dqV = dq\frac{1}{4\pi\epsilon_0}\frac{q_r}{r}$

charge of the sphere upto radius r

$$\text{If } \rho \text{ is the charge density then} \quad q_r = \rho\,\frac{4}{3}\pi r^3$$

$$\text{and} \quad dq = \rho\, 4\pi r^2 dr$$

$$\implies dW = \frac{4\pi\rho^2 r^4 dr}{3\epsilon_0}$$

$$W = \frac{4\pi\rho^2}{3\epsilon_0}\int_0^R r^4 dr = \frac{4\pi\rho^2 R^5}{15\epsilon_0} = \frac{4\pi}{15\epsilon_0}\frac{q^2}{(\frac{4}{3}\pi R^3)^2}R^5 = \frac{3}{5}\frac{q^2}{4\pi\epsilon_0 R}$$

Energy is proportional to the square of total charge and inversely proportional to radius

Energy of uniformly solid sphere

- By using $W = \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{E}.\vec{E}\, d\tau$

We have seen that
$$\vec{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \hat{r} & r < R \\ \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} & r > R \end{cases}$$

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \left(\frac{1}{4\pi\epsilon_0}\right)^2 q^2 \left[\int_0^R \left(\frac{r}{R^3}\right)^2 4\pi r^2 dr + \int_R^\infty \frac{1}{r^4} 4\pi r^2 dr\right] \\ &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \left[\frac{1}{R^6} \left(\frac{r^5}{5}\right)\Bigg|_0^R + \left(-\frac{1}{r}\right)\Bigg|_R^\infty\right] = \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \left(\frac{1}{R} + \frac{1}{5R}\right) = \frac{3}{5} \frac{q^2}{4\pi\epsilon_0 R} \end{aligned}$$

- You can also use $W = \frac{1}{2} \int \rho\, V\, d\tau$ with $V = \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left(3 - \frac{r^2}{R^2}\right)$

- use $W = \frac{\epsilon_0}{2} \left(\int_{\text{vol}} (\vec{\nabla} V).(\vec{\nabla} V) d\tau - \int_{\text{surf}} (V \vec{\nabla} V).\hat{n} da\right)$

and check what happens if you evaluate the surface integral at infinity

Where is the energy located ?

If there is a pair of interacting charges, the combination has certain energy.

Is it then one of the charges store the energy ?

In case of electrostatics, it is really hard to answer. Is it stored in the field as $W = \frac{\epsilon_0}{2}\int E^2 d\tau$ may suggest or is it stored in the charges as $W = \frac{1}{2}\int \rho V d\tau$ implies?

It is best to think that the energy is located in space where the electric field is. Define energy density $w = \frac{\epsilon_0 |\vec{E}|^2}{2}$ such that a small volume $d\tau$ will contain electrostatic energy $w\ d\tau$

Each volume element $d\tau = dxdydz$ in an electric field contains the energy $(\epsilon_0/2)|\vec{E}|^2 d\tau$

Self energy of a point charge

Electric field of point charge placed at origin : $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

Electrostatic energy $W = \frac{\epsilon_0}{2}\int E^2\, d\tau$

$$= \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_{r=0}^{\infty} \frac{1}{r^4}\, 4\pi r^2 dr$$

$$= \frac{q^2}{8\pi\epsilon_0}\int_{r=0}^{\infty}\frac{1}{r^2}dr$$

$$= -\frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{r}\right)\Big|_{r=0}^{r=\infty}$$

diverges for r->0

There is an infinite amount of energy in the field of a point charge

Whats wrong ?

There is something wrong in the theory of electromagnetism at very small distances to describe point charges.

Interaction energy of two point charges

Take two charges q_1, q_2 at $\vec{r}_1$, $\vec{r}_2$ respectively.

Electric field at any point $\vec{r}$: $\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$ due to the two charges respectively

Electrostatic energy :
$$
\begin{aligned}
W_{\text{tot}} &= \frac{\epsilon_0}{2}\int |\vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})|^2 \, d\tau \\
&= \frac{\epsilon_0}{2}\int |\vec{E}_1(\vec{r})|^2 d\tau + \frac{\epsilon_0}{2}\int |\vec{E}_2(\vec{r})|^2 d\tau + \frac{\epsilon_0}{2}\int 2\vec{E}_1(\vec{r}).\vec{E}_2(\vec{r}) d\tau \\
&= W_1 + W_2 + \epsilon_0 \int \vec{E}_1(\vec{r}).\vec{E}_2(\vec{r}) d\tau
\end{aligned}
$$

Cross term

Because electrostatic energy is quadratic in the field, it does not obey the superposition principle!

Interaction energy : $$W_{\text{int}} = \epsilon_0 \int \vec{E}_1(\vec{r}).\vec{E}_2(\vec{r}) \, d\tau = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\mathscr{r}}$$

Therefore, only interaction energy is defined in classical theory

In summary....

- An electric dipole consists of two equal and opposite charges. The potential drops as inverse square of distance while electric field drops as inverse cube of distance, apart from being proportional to dipole moment itself.
- Electric field is discontinuous across a surface carrying surface charge density, while the potential remains the same above and below.
- Potential amounts to work done per unit charge.
- Electrostatic energy contained in a charge configuration can be calculated by the work done of moving the charges in presence of others.
- The easiest way to calculate the energy of electrostatic system for all space, where the surface integral vanishes.