MA 102 (Ordinary Differential Equations)

IIT Guwahati

Tutorial Sheet No. 13 **Homework**

Power Series Solutions to Differential Equations.

(1) Classify the singular points of the following differential equations:
(a) $e^x y'' - (x^2-1)y' + 2xy = 0$; (b) $(x^2+x)y'' + 3y' - 6xy = 0$;
(c) $(\sin x)y'' + (\cos x)y = 0$; (d) $\ln(x-1)y'' + (\sin 2x)y' - e^x y = 0$.

(2) Determine the convergence set of the given power series:
(a) $\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x+2)^n$; (b) $\sum_{n=1}^{\infty} \frac{3}{n^3}(x-2)^n$; (c) $\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1}(x-1)^n$.

(3) Compute the indicial equation and their roots of the given differential equations:
(a) $(x^2-x-2)y'' + (x^2-4)y' - 6xy = 0$; (b) $x^2y'' + 2(x-3x^2)y' + e^x y = 0$;
(c) $x^2y'' + (\sin x)y' + (\cos x)y = 0$.

(4) Find a series solution about the regular singular point of the following equations:
(a) $xy'' + 4y' - xy = 0$; (b) $4x^2y'' + 2x^2y' - (x+3)y = 0$;

(5) Using Rodrigues' formula prove the following properties of the Legendre polynomials.
(a) For each $n \geq 0$, $P_n(1) = 1$. Moreover, $P_n(x)$ is the only polynomial which satisfies the Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ and $P_n(1) = 1$.
(b) $\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases}$
(c) If $f(x)$ is a polynomial of degree n, we have $f(x) = \sum_{k=0}^{n} c_k P_k(x)$, where $c_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)dx$.
(d) Use orthogonality relation to show that $\int_{-1}^{1} g(x)P_n(x)dx = 0$ for every polynomial $g(x)$ with $\deg(g(x)) < n$.

(6) Show that the value of the integral $\int_{-1}^{1} P_n(x)P'_{n+1}(x)dx$ is independent of n.

(7) Find a general solution to the following equations using Bessel functions of the first kind.
(a) $4x^2y'' + 4xy' + (4x^2-1)y = 0$; (b) $x^2y'' + xy' + x^2y = 0$.

(8) Prove the following identities:
(a) $\frac{d}{dx}(x^\alpha J_\alpha(x)) = x^\alpha J_{\alpha-1}(x)$; (b) $\frac{d}{dx}(x^{-\alpha} J_\alpha(x)) = -x^{-\alpha} J_{\alpha+1}(x)$.

(9) From the relation in Problem 8, deduce the recurrence relations.
(c) $\frac{\alpha}{x}J_\alpha(x) + J'_\alpha(x) = J_{\alpha-1}(x)$; (d) $\frac{\alpha}{x}J_\alpha(x) - J'_\alpha(x) = J_{\alpha+1}(x)$.

(10) Use the relation in Problem 9 to deduce the formulas:
(a) $J_{\alpha-1}(x) + J_{\alpha+1}(x) = \frac{2\alpha}{x}J_\alpha(x)$; (b) $J_{\alpha-1}(x) - J_{\alpha+1}(x) = 2J'_\alpha(x)$.

(11) Show that between two consecutive positive roots of $J_1(x)$, there is a root of $J_0(x)$.