Physics II: Electromagnetism (PH102)

Lecture 7

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Electric field and Potential

Electric Field

Electric field is the force per unit charge that would be exerted on a test charge Q $\vec{F} = Q\vec{E}$ where $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$

for a point charge

For continuous charge distribution

$$\begin{aligned}\vec{E}(P) &= \frac{1}{4\pi\epsilon_0}\int_{line}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\lambda dl \\ &= \frac{1}{4\pi\epsilon_0}\int_{sur}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\sigma da \\ &= \frac{1}{4\pi\epsilon_0}\int_{vol}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho d\tau\end{aligned}$$

$$\boxed{\oint_{sur}\vec{E}.\vec{da} = \frac{Q_{\text{enc}}}{\epsilon_0}}$$

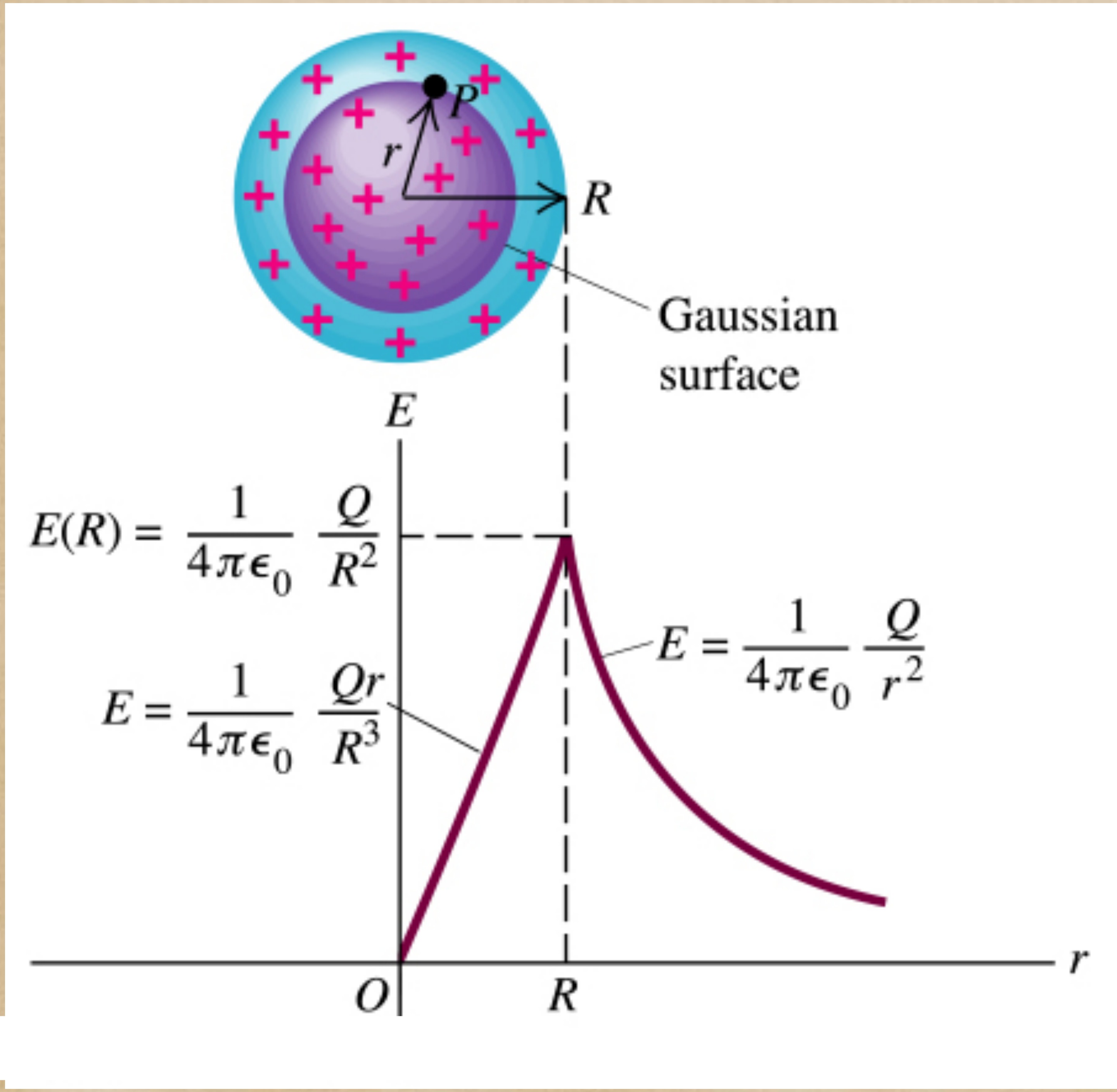
$$\boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}}$$

Gauss's Law: Flux through an enclosed surface is proportional to the charge enclosed by the surface

Most effective way of calculating electric field

Electric field of uniformly charged sphere

$$\oint_{sur} \vec{E}.d\vec{a} = \oint_{sur} |\vec{E}|da = |\vec{E}| \oint_{sur} da = |\vec{E}|4\pi r^2 = \frac{Q_{enc}}{\epsilon_0}$$



Inside:

$$Q_{enc} = (4/3)\pi r^3 \rho$$

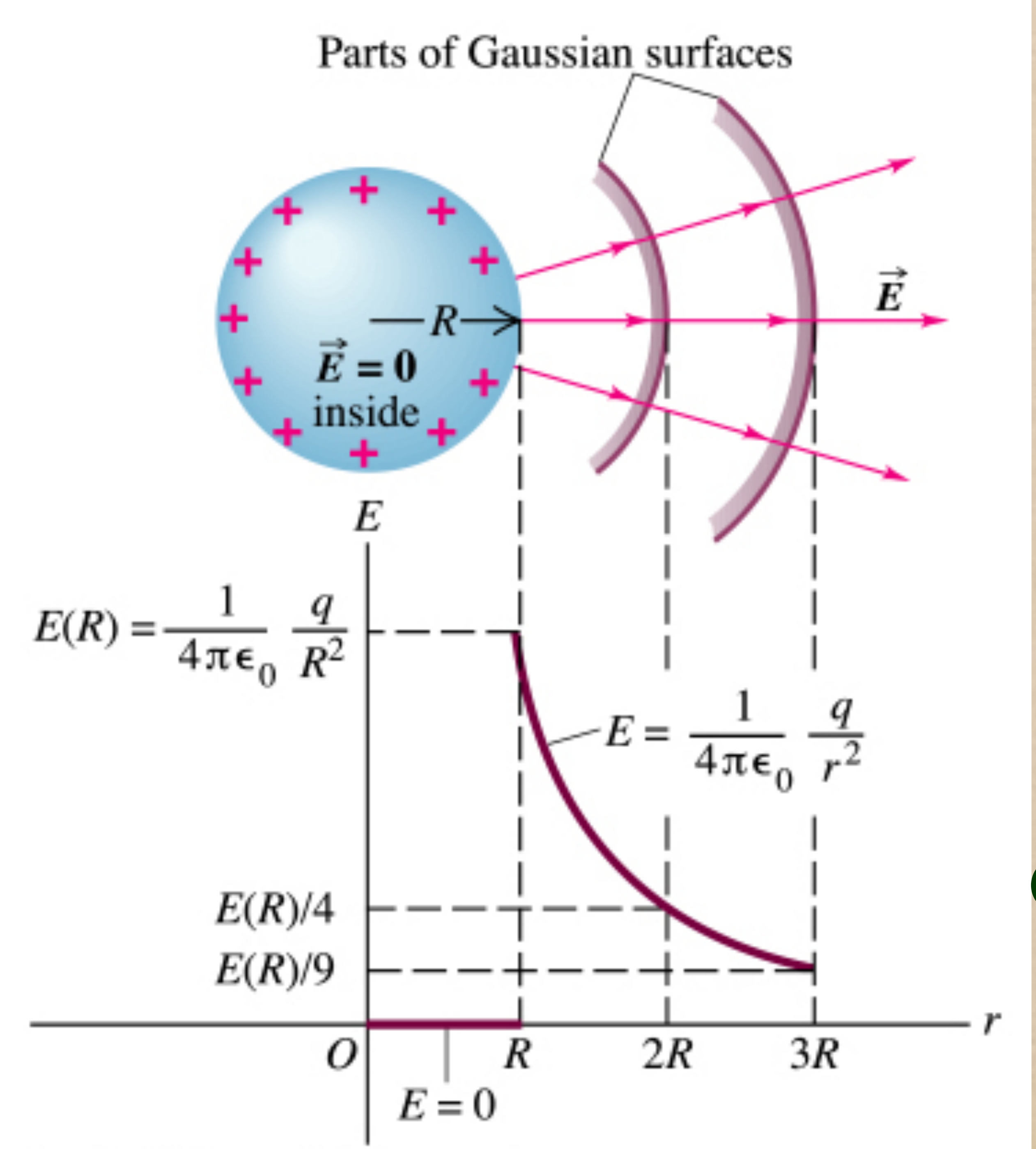
$$\vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r}$$

Outside:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

total charge of the sphere

Fields of conducting sphere



Charges accumulate in the surface of the sphere

Inside the sphere

$$Q_{enc} = 0$$

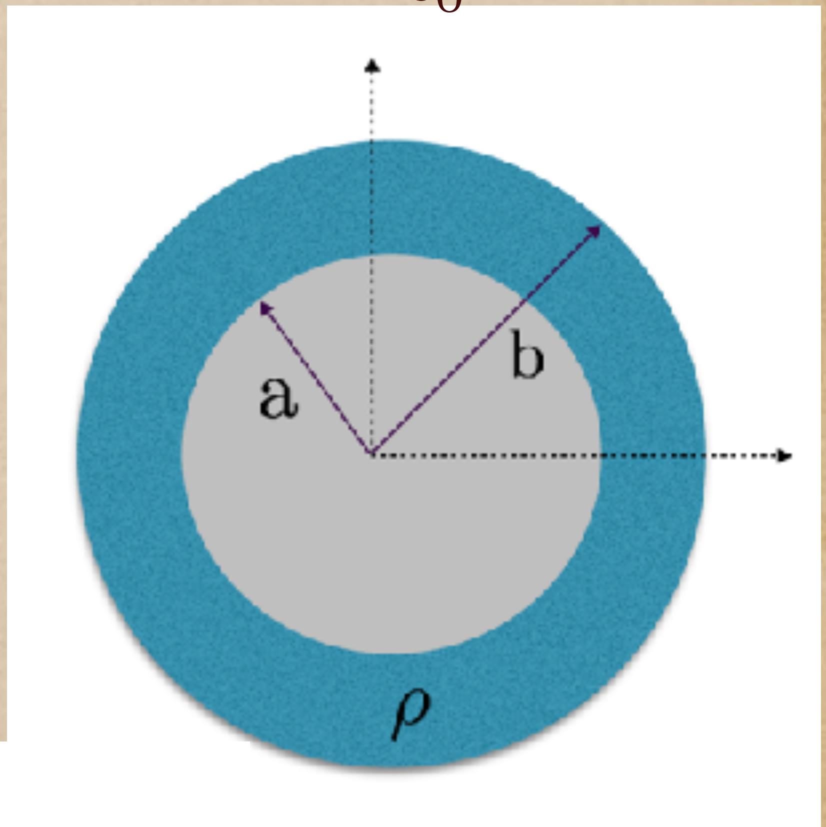
$$\text{Hence, } \vec{E} = 0$$

$$\text{Outside } r > R: \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

Spherical shell....

A hollow spherical shell carries charge density $\rho = \frac{k}{r^2}$ in the region $a \leq r \leq b$. Find electric field in regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$.

$$\oint_{sur} \vec{E}.d\vec{a} = \oint_{sur} |\vec{E}|da = |\vec{E}| \oint_{sur} da = |\vec{E}|4\pi r^2 = \frac{Q_{enc}}{\epsilon_0}$$



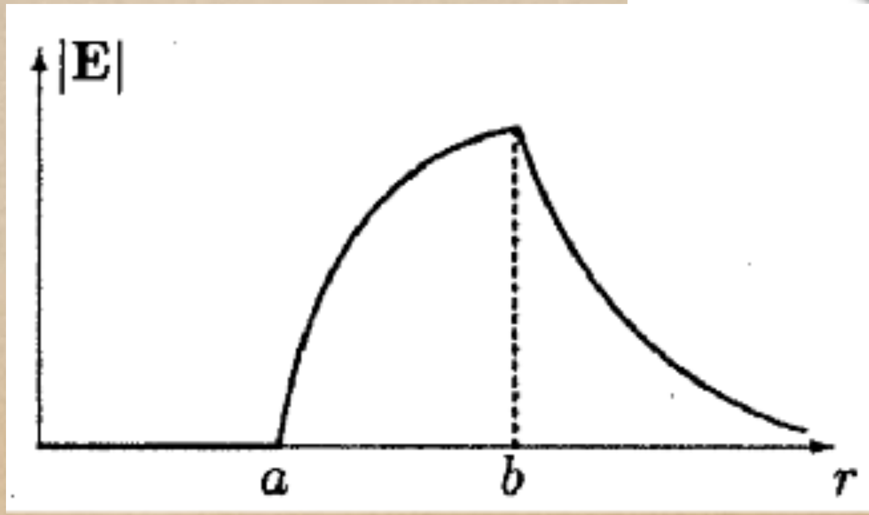
For $r < a : Q_{enc} = 0.$ Hence, $E = 0$

For $a < r < b:$

$$Q_{enc} = \int \rho d\tau = \int \frac{k}{r^2} r^2 dr \sin\theta d\theta d\phi$$
$$= 4\pi k \int_a^r dr = 4\pi k(r-a)$$

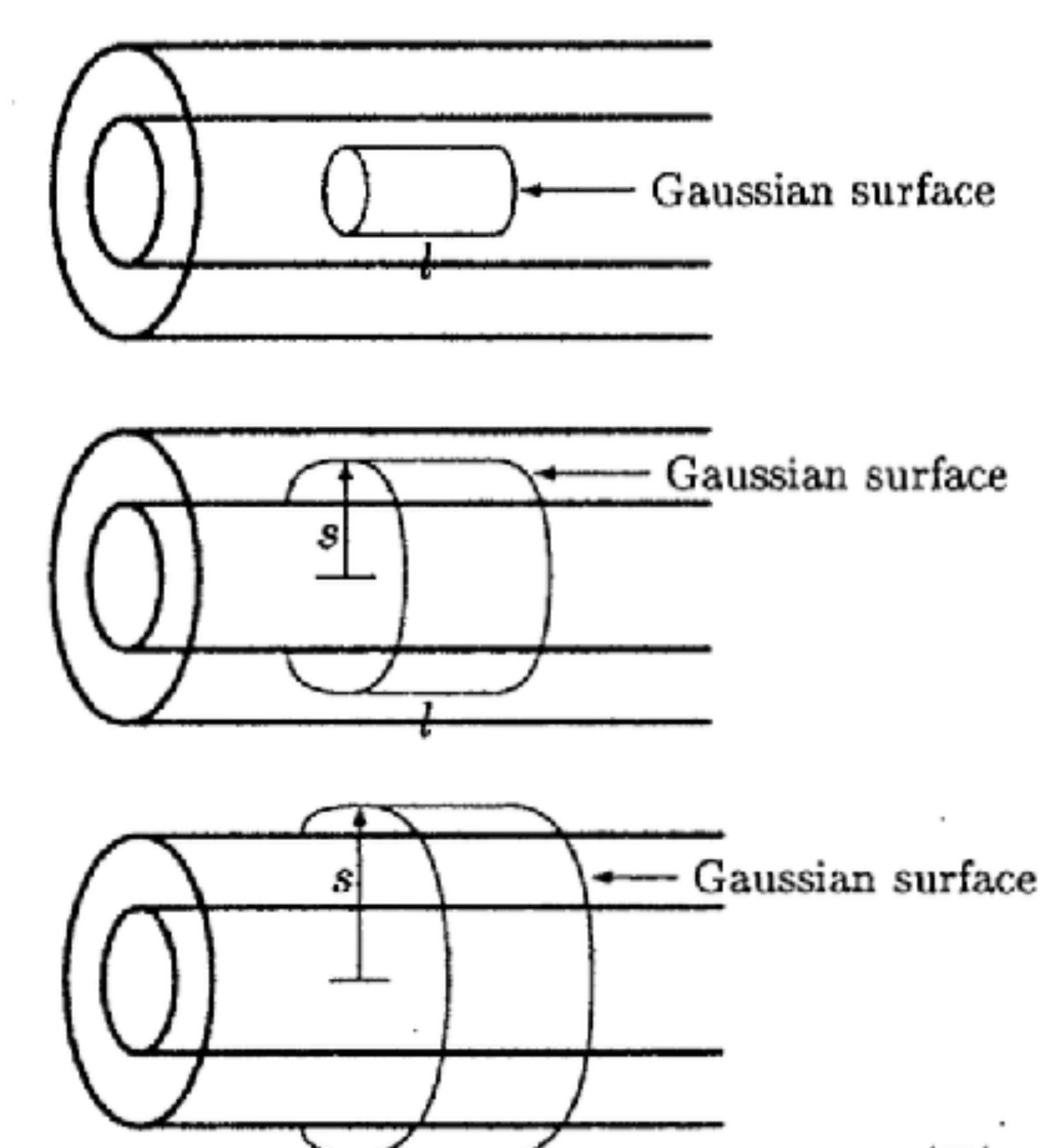
$$\vec{E} = \frac{k}{\epsilon_0}\frac{(r-a)}{r^2}\hat{r}$$

For $r > b$: $\vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\hat{r}$



Neutral Coaxial cable

A long co-axial cable carries uniform volume charge density ρ on the inner cylinder of radius a and uniform surface charge density on the outer cylindrical shell of radius b so that the cable is electrically neutral. Find electric field in regions: $r < a$, $a < r < b$, $r > b$.

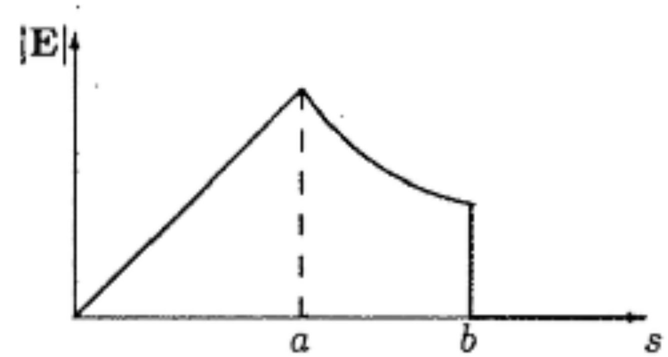


$$\oint \mathbf{E} \cdot d\mathbf{a} = E \cdot 2\pi s \cdot l = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \rho \pi s^2 l;$$

$$\boxed{\mathbf{E} = \frac{\rho s}{2\epsilon_0} \hat{\mathbf{s}}.}$$

$$\oint \mathbf{E} \cdot d\mathbf{a} = E \cdot 2\pi s \cdot l = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} \rho \pi a^2 l;$$

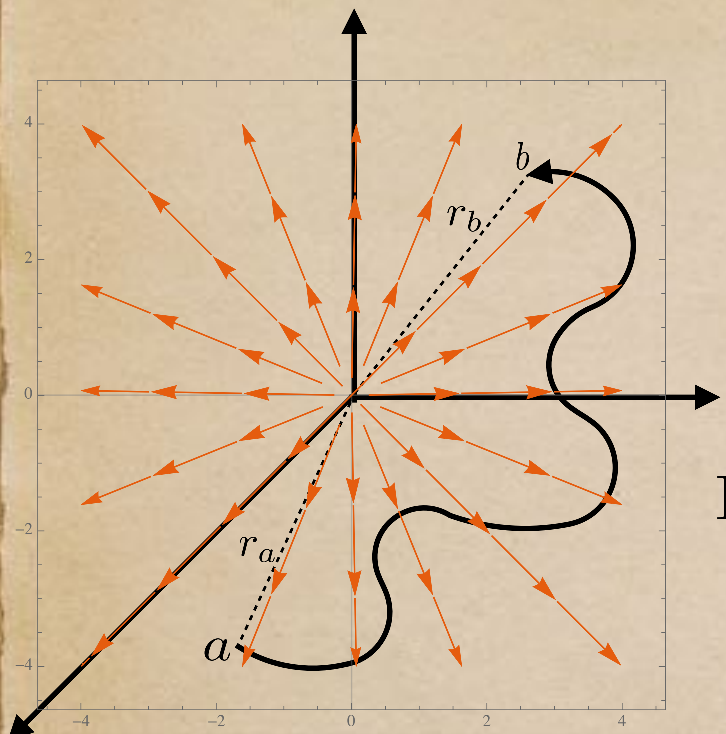
$$\boxed{\mathbf{E} = \frac{\rho a^2}{2\epsilon_0 s} \hat{\mathbf{s}}.}$$



$$\oint \mathbf{E} \cdot d\mathbf{a} = E \cdot 2\pi s \cdot l = \frac{1}{\epsilon_0} Q_{\text{enc}} = 0;$$

$$\boxed{\mathbf{E} = 0.}$$

The curl of Electric field



Field due to a point charge

Curl is zero !

$$\vec{\nabla} \times \vec{E} = 0$$

We will calculate the line integral of Electric field

In spherical coordinates $d\vec{\ell} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$, so that

$$\vec{E}.d\vec{\ell} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

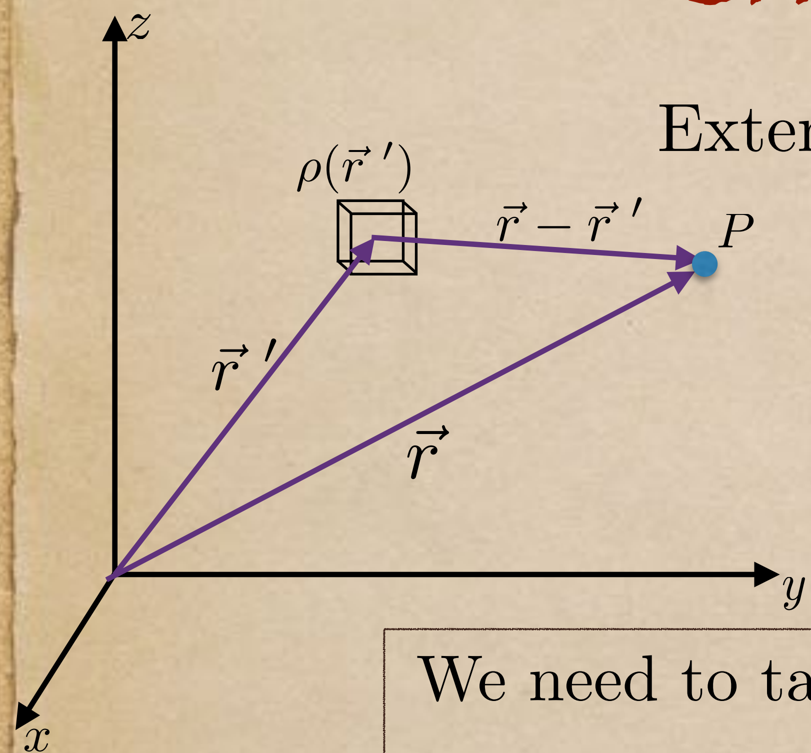
$$\therefore \int_a^b \vec{E}.d\vec{\ell} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}dr$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{q}{r}\Big|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_b} - \frac{q}{r_a}\right)$$

This integral around a closed path is evidently zero, since $r_a = r_b$ for a closed path $\implies \oint \vec{E}.d\vec{\ell} = 0$

Applying Stoke's theorem $\vec{\nabla} \times \vec{E} = 0$

Curl of electric field due to volume charge distribution



Extending the result to arbitrary charge distribution $\rho(\vec{r}\,')$:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}d\tau'$$

We need to take the curl with respect to the variable $\vec{r}$:

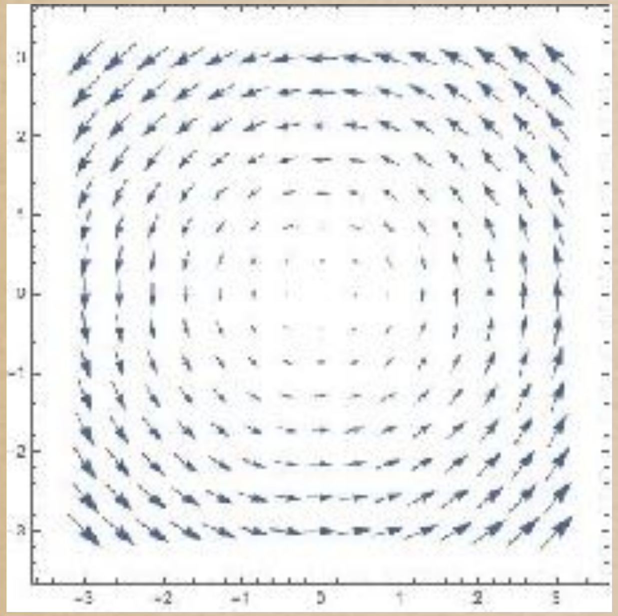
$$\begin{aligned}\vec{\nabla}\times\vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\vec{\nabla}\times\int \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}d\tau' \\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}\,')\Big(\vec{\nabla}\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}d\tau'\Big) = 0\end{aligned}$$

One can easily check

$$\left(\vec{\nabla}\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right)_x = -3(z-z')\frac{(y-y')}{|\vec{r}-\vec{r}\,'|^3} + 3(y-y')\frac{(z-z')}{|\vec{r}-\vec{r}\,'|^3}) = 0$$

Consequences of irrotational electric field

Electric field is not just any kind of vector field, but one with zero curl.



Ex: $\vec{E} = -y\hat{x} + x\hat{y}$ can not be an electric field since its curl is given by $2\hat{z}$.

Recall,

A vector field $\vec{F}(x, y, z)$ is <u>conservative</u>, if $\vec{\nabla} \times \vec{F} = 0$.

$$\boxed{\vec{F} = \vec{\nabla}\phi}$$

Thanks to : $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$

$$\vec{\nabla} \times \vec{E} = 0 \implies \vec{E} = -\vec{\nabla}V$$

V is called electric potential

Electric potential

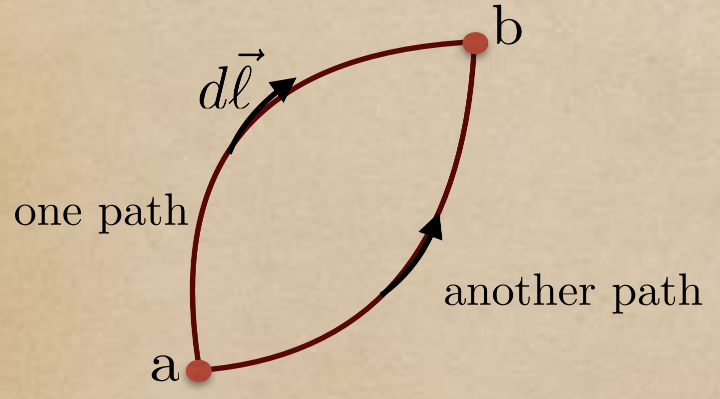
Using fundamental theorem for gradients : $V(b) - V(a) = \int_a^b (\vec{\nabla} V).d\vec{\ell}$, we get

$$\int_a^b (\vec{\nabla} V).d\vec{\ell} = -\int_a^b \vec{E}.d\vec{\ell}.$$

Potential difference between a and b

$$V(b) - V(a) = -\int_{\mathcal{O}}^b \vec{E}.d\vec{\ell} + \int_{\mathcal{O}}^a \vec{E}.d\vec{\ell} = -\int_{\mathcal{O}}^b \vec{E}.d\vec{\ell} - \int_a^{\mathcal{O}} \vec{E}.d\vec{\ell} = -\int_a^b \vec{E}.d\vec{\ell}$$

Because of the fact the line integral is independent of the path, we define a function



$$V(r) = -\int_{\mathcal{O}}^r \vec{E}(\vec{r}\,').d\vec{\ell'}$$

Electric potential at a point r is defined with respect to a reference point

Electric potential: comments

We are dumping the informations of Electric field into a scalar quantity potential. That eases calculations !

- How is it possible ? A vector has three components !

The three components of E are not independent as $\vec{\nabla} \times \vec{E} = 0$

The components are related as:

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

The reference point $\mathcal{O}$ is completely arbitrary. Changing reference point is same as adding a constant C to the potential.

$$\bar{V}(r) = -\int_{\mathcal{O}'}^{r} \vec{E}.d\vec{\ell} = -\int_{\mathcal{O}'}^{\mathcal{O}} \vec{E}.d\vec{\ell} - \int_{\mathcal{O}}^{r} \vec{E}.d\vec{\ell} = C + V(r)$$

where C is the line integral of $\vec{E}$ from the old reference point $\mathcal{O}$ to the new one

Like electric field, potential also obeys superposition principle: At any point the net potential is the sum of all potentials due to different source charges

$$V = V_1 + V_2 + V_3 + \ldots$$

Solving for potential

So far, we have obtained two important laws regarding the electric field

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \times \vec{E} = 0$$

$\vec{E} = -\vec{\nabla} V$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$

Poisson's equation

In source free region (no charge) : $\rho = 0$

$$\boxed{\nabla^2 V = 0}$$

Laplace's equation

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times (-\vec{\nabla} V) = 0$$

No condition on V since curl of gradient is always zero. $\vec{\nabla} \times \vec{E} = 0$ permits $\vec{E} = -\vec{\nabla} V$.

We require only one differential equation to determine scalar potential

Potential for a point charge

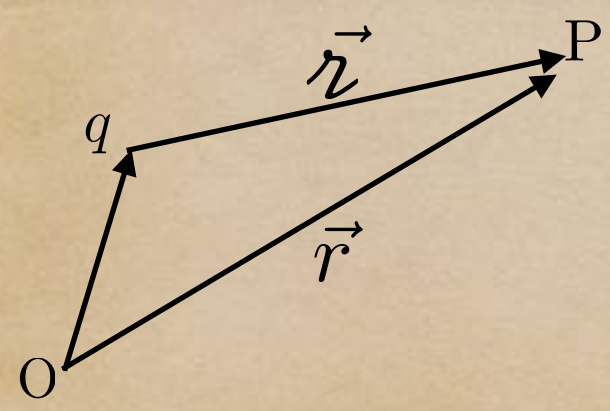
Point charge: The electric field is $\vec{E} = (1/4\pi\epsilon_0)(q/r^2)\hat{r}$ and $d\vec{\ell} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$ so that

$$\vec{E}.d\vec{\ell} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

Setting the reference point $\mathcal{O}$ at infinity, the potential of a point charge q at origin is

$$V(r) = -\frac{1}{4\pi\epsilon_0}\int_\infty^r \frac{q}{r'^2}dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\Big|_\infty^r = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

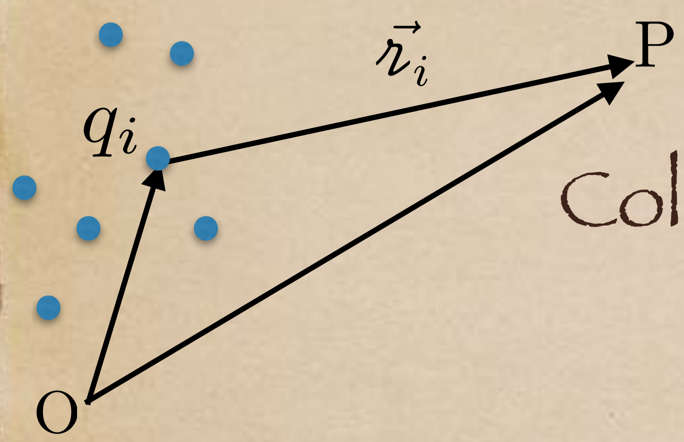
- The conventional negative sign in front of $V = -\int \vec{E}.d\vec{\ell}$ was chosen in order to make the potential of a positive charge come out positive.



In general the potential for a point charge q is

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}}$$

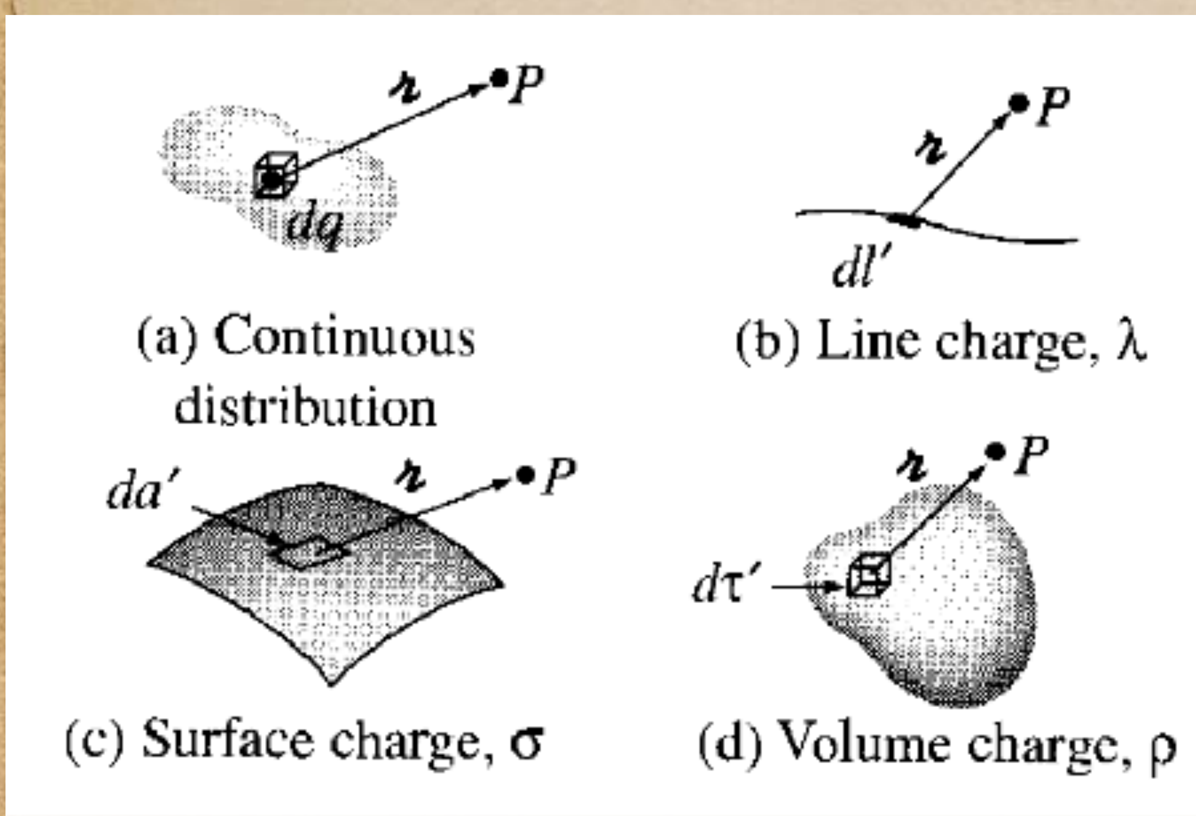
Potential due to charge distributions



Collection of point charges :

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$



(a) Continuous distribution

(b) Line charge, λ

(c) Surface charge, σ

(d) Volume charge, ρ

$$V(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\vec{r}\,')}{\mathscr{r}}d\ell'$$

Line charge

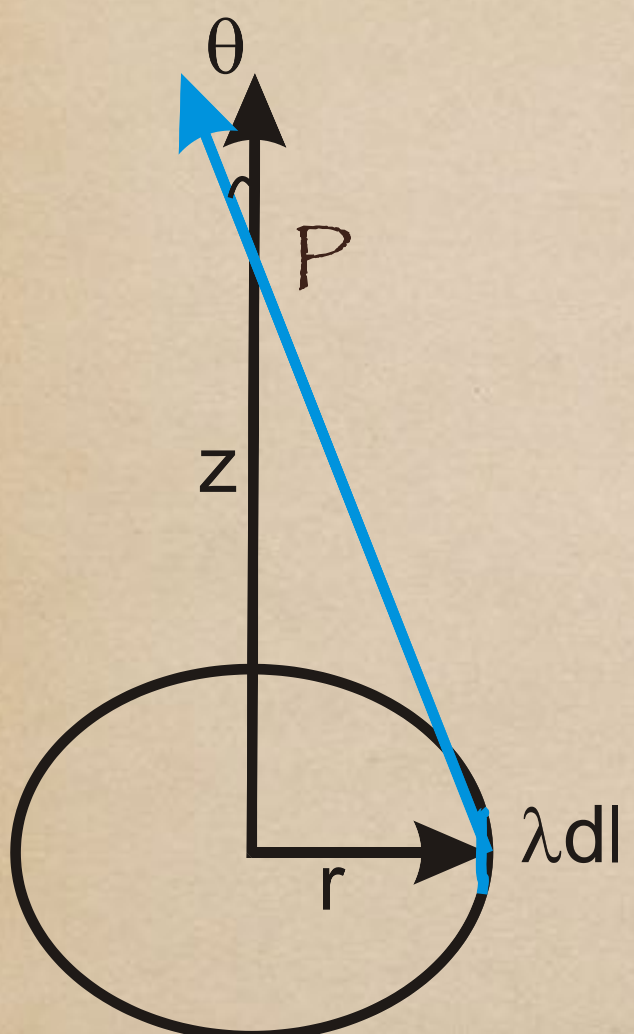
$$V(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\vec{r}\,')}{\mathscr{r}}da'$$

Surface charge

$$V(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}\,')}{\mathscr{r}}d\tau'$$

Volume charge

Example: Potential due to a ring



$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$

$$\boxed{dq = \lambda dl'}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda}{\sqrt{r^2+z^2}}dl'$$

$$dl' = r d\phi$$

(r is constant)

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{r}{\sqrt{r^2+z^2}}\int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi r}{\sqrt{r^2+z^2}}$$

For $z >> r$:

$$V(z) \simeq \frac{\lambda}{4\pi\epsilon_0}\frac{2\pi r}{z} = \frac{1}{4\pi\epsilon_0}\frac{q}{z}$$

$$q = \lambda(2\pi r)$$

behaves as a point charge !

Potential inside and outside of solid charged sphere of radius R

I will use: $$\boxed{V(r) = -\int_{\mathcal{O}}^{r} \vec{E}(\vec{r}\,').d\vec{\ell}'}$$

Therefore, I need electric field inside and outside

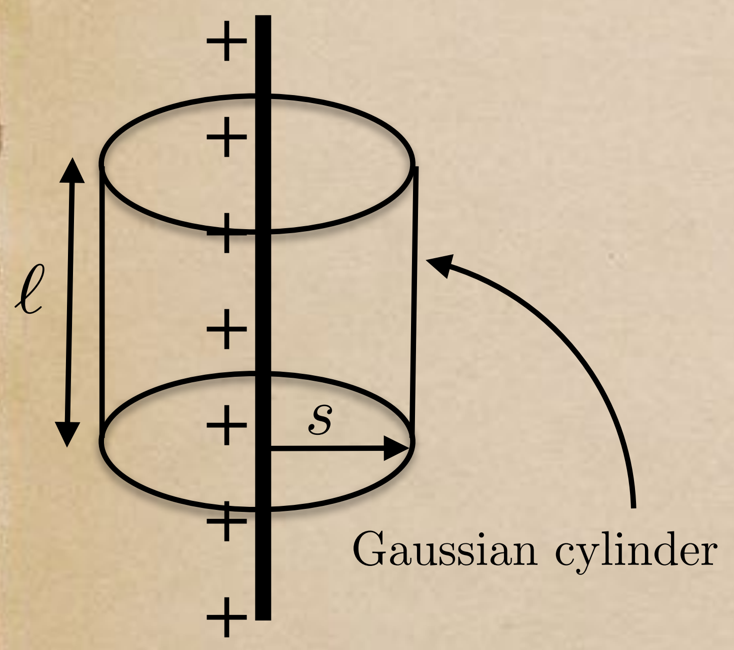
$$\begin{cases} \text{Outside the sphere } (r > R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \\ \text{Inside the sphere } (r < R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}. \end{cases}$$

- $r > R: \quad V(r) = -\int_{\infty}^{r} (\frac{1}{4\pi\epsilon_0}\frac{q}{r^2})dr = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$
- $r < R: \quad V(r) = -\int_{\infty}^{R} (\frac{1}{4\pi\epsilon_0}\frac{q}{r^2})dr - \int_{R}^{r} (\frac{1}{4\pi\epsilon_0}\frac{q}{R^3})rdr$

$$= \frac{q}{4\pi\epsilon_0}\Big[\frac{1}{R} - \frac{1}{R^3}\Big(\frac{r^2 - R^2}{2}\Big)\Big]$$

Potential due to line charge

Field can be evaluated by enclosing the line charge with a Gaussian cylinder of length ℓ and radius s.



The field only depends on distance from the line charge and directed away from it. Contribution to the flux from the top and bottom caps of the cylinder are zero.

$$\int \vec{E}.d\vec{a} = |E| 2\pi s \ell = \frac{\lambda \ell}{\epsilon_0}$$

Gauss's law

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

using

$$V(r) = -\int_{\mathcal{O}}^{r} \vec{E}(\vec{r}\,').d\vec{\ell}'$$

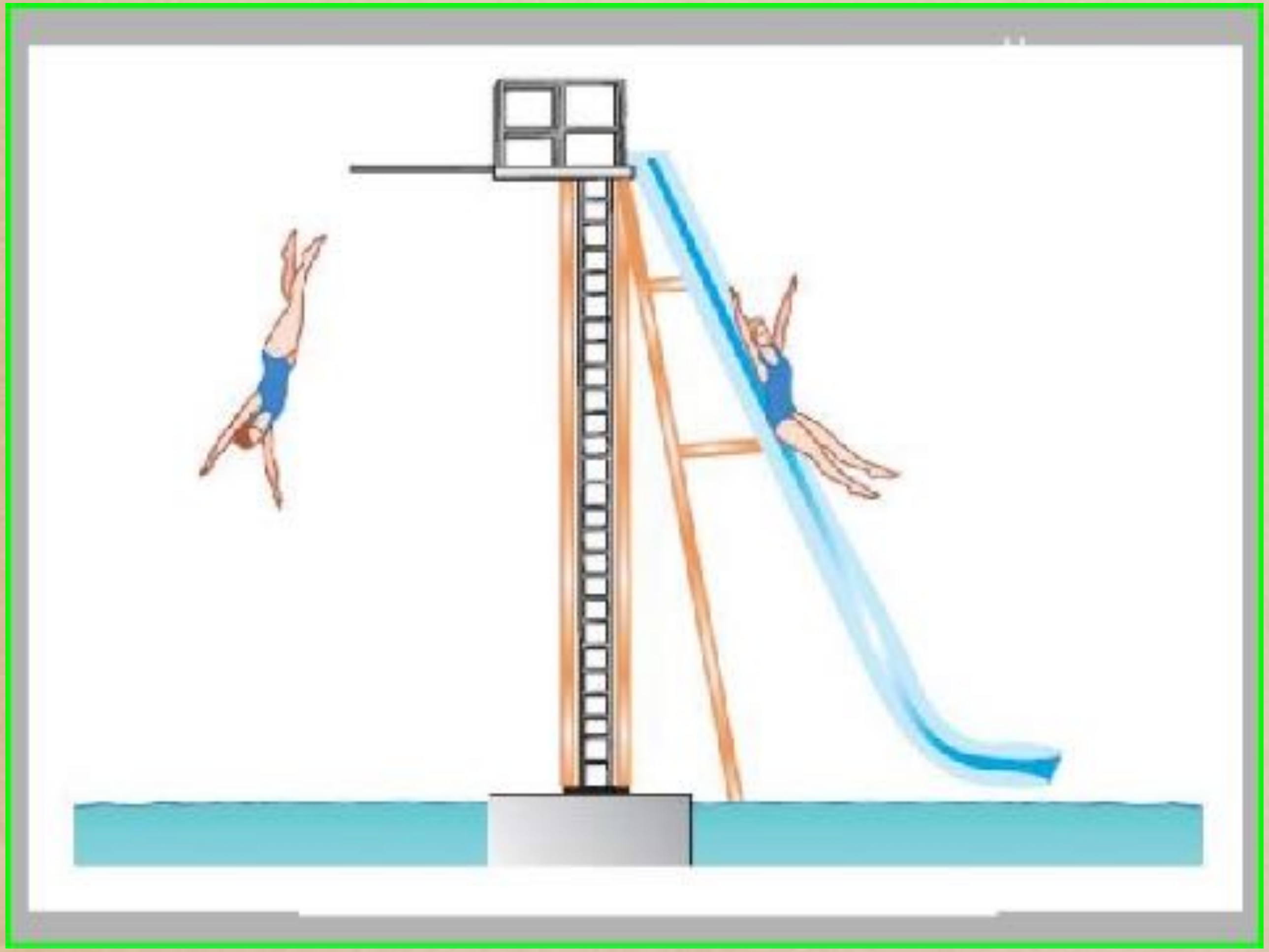
- $$\vec{E} = -\vec{\nabla}V \implies V = -\frac{\lambda}{2\pi\epsilon_0}\ln s + \text{Constant}$$

Warning: Unlike the case of point charge, the reference point can not be taken at infinity! Why?

logarithm gets undefined! Choose the reference point at $r = 1$ where the potential becomes zero.

$$\therefore \; V = -\frac{\lambda}{2\pi\epsilon_0}\ln s$$ with respect to reference point at $r = 1$

How exactly the potential defined here is related to the concept of potential energy ?



Gravity: $G\frac{Mm}{r^2} = mg$

Acceleration due to gravity

$$g = \frac{GM}{r^2}$$

In a similar way, gravitational potential due to earth

$$V_G = \frac{GM}{r} = gr$$

Potential energy for mass m = mgr

The reason that a gravitational potential energy could be defined was due to an Irrotational or conservative gravitational force field

In summary...

- Irrotational electric field allows one to have a scalar potential to do the job for the electric field.
- Mathematically, it is simpler to deal with a scalar potential than a vector field.
- Potential can be evaluated from integrating charge distributions, or line integrating electric field or by solving Poisson's /Laplace's equations.
- Care should be taken in choosing the reference point for finding potential while integrating electric field, so that we do not end up with ill-defined function.