# MA 102 (Mathematics II)

## Department of Mathematics, IIT Guwahati

Tutorial Sheet No. 5 February 08, 2016

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### Chain rule, tangent and normal, Jacobian matrix

(1) Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where $m$ is a nonnegative integer. If $f$ is differentiable then show that $\langle x, \nabla f(x) \rangle = mf(x)$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define $F, G : \mathbb{R}^2 \to \mathbb{R}$ by $F(x,y) := \int_0^{x+y} f(t)dt$ and $G(x,y) := \int_0^{xy} f(t)dt$. Show that $F$ and $G$ are differentiable and determine $DF(x,y)$ and $DG(x,y)$.

(3) Let $f(x,y,z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of $f$ at $(1,-1,3)$ and the equations of the tangent plane and the normal line to the surface $f(x,y,z) = 7$ at $(1,-1,3)$.

(4) Find $D_u f(2,2,1)$, where $f(x,y,z) = 3x - 5y + 2z$ and $u$ is the unit vector in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2,2,1)$.

(5) Find the equation of the tangent plane to the graphs of the following functions at the given point:
(a) $f(x,y) := x^2 - y^4 + e^{xy}$ at the point $(1,0,2)$
(b) $f(x,y) = \tan^{-1}\dfrac{y}{x}$ at the point $(1, \sqrt{3}, \dfrac{\pi}{3})$.

(6) Check the following functions for differentiability and determine the Jacobian Matrix.
(a) $f(x,y) = (e^{x+y}+y, xy^2)$ (b) $f(x,y) = (x^2+\cos y, e^x y)$ (c) $f(x,y,z) = (ze^x, -ye^z)$.

(7) Let $z = x^2 + y^2$, and $x = 1/t, y = t^2$. Compute $\dfrac{dz}{dt}$ by (a) expressing $z$ explicitly in terms of $t$ and (b) chain rule.

(8) Let $w = 4x + y^2 + z^3$ and $x = e^{rs^2}, y = \log\dfrac{r+s}{t}, z = rst^2$. Find $\dfrac{\partial w}{\partial s}$.

(9) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0,0) := 0$ and, for $(x,y) \neq (0,0)$, $f(x,y) := xy\dfrac{x^2 - y^2}{x^2 + y^2}$.
(a) Show that $\frac{\partial f}{\partial y}(x,0) = x$ for $x \in \mathbb{R}$ and $\frac{\partial f}{\partial x}(0,y) = -y$ for $y \in \mathbb{R}$.
(b) Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

(10) Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Show that

$$\lim_{h \to 0} \frac{f(x+h) - [f(x) + \langle \nabla f(x), h \rangle + \frac{1}{2}\langle H_f(x)h, h \rangle]}{\|h\|^2} = 0,$$

where $H_f(x)$ is the Hessian of $f$ at $x$.

(11) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable and $x = r\cos\theta, y = r\sin\theta$. Show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

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