Tutorial Sheet No. 1 (SOLUTIONS):

(1) Using triangle inequality we have

$\|X\| = \|X - Y + Y\| \le \|X - Y\| + \|Y\| \Rightarrow \|X\| - \|Y\| \le \|X - Y\|$. Similarly
$\|Y\| = \|Y - X + X\| \le \|Y - X\| + \|X\| \Rightarrow \|Y\| - \|X\| \le \|X - Y\|$ and hence
$\big|\|X\| - \|Y\|\big| \le \|X - Y\|$. For the second part note that

$\|X + Y\|^2 = \langle X + Y, X + Y\rangle = \langle X, X\rangle + \langle Y, Y\rangle + 2\langle X, Y\rangle = \|X\|^2 + \|Y\|^2 + 2\langle X, Y\rangle$.

(2) For forward implication using Cauchy-Schwartz inequality note that

$|\langle X_k, Y\rangle - \langle X, Y\rangle| = |\langle X_k - X, Y\rangle| \le \|X_k - X\|\|Y\|$. For backward implication take $Y = e_i = (0, \dots, \underbrace{1}_{i-th\ place}, \dots, 0)$ for $i = 1, 2, \dots, n$.

(3) First part follows from (1). Combining the hypothesis and the first part we can prove the second part.

(4) Follows from (2).

(5) (a) $x = 1, y = \sqrt{3}, z = -2$. Then $\rho = \sqrt{1 + 3 + 4} = 2\sqrt{2}$ and therefore using the relation between spherical coordinates and Cartesian coordinates we get $-2 = 2\sqrt{2}\cos\phi \Rightarrow \phi = \frac{3\pi}{4}$; $1 = 2\sqrt{2}\sin\frac{3\pi}{4}\cos\theta \Rightarrow \theta = \frac{\pi}{3}$. Thus $(\rho, \phi, \theta) = (2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{3})$.

(b) $x = 1, y = -1, z = \sqrt{2}$. Again using the similar relations we conclude that $\rho = \sqrt{1 + 1 + 2} = 2$; $\sqrt{2} = 2\cos\phi \Rightarrow \phi = \frac{\pi}{4}$; $1 = 2\frac{1}{\sqrt{2}}\cos\theta \Rightarrow \theta = \frac{7\pi}{4}$. Thus $(\rho, \phi, \theta) = (2, \frac{\pi}{4}, \frac{7\pi}{4})$.

(6) (a) $\rho = 5, \phi = \frac{\pi}{6}, \theta = \frac{\pi}{4}$. Using the relation between spherical coordinates and Cartesian coordinates we get $x = 5.\frac{1}{2}.\frac{1}{\sqrt{2}} = \frac{5}{2\sqrt{2}}$; $y = 5.\frac{1}{2}.\frac{1}{\sqrt{2}} = \frac{5}{2\sqrt{2}}$; $z = 5.\frac{\sqrt{3}}{2}$. Thus $(x, y, z) = (\frac{5}{2\sqrt{2}}, \frac{5}{2\sqrt{2}}, \frac{5\sqrt{3}}{2})$.

(b) $\rho = 7, \phi = \frac{\pi}{2}, \theta = \frac{\pi}{2}$. Again using the similar relations we conclude that $x = 7.1.0 = 0$; $y = 7.1.1 = 7$; $z = 0$. Thus $(x, y, z) = (0, 7, 0)$.

(7) (a) $r = \sqrt{3}, \theta = \frac{\pi}{6}, z = 3$. Using the relation between spherical coordinates and cylindrical coordinates we get $\theta = \frac{\pi}{6}$; $\rho = \sqrt{3 + 9} = 2\sqrt{3}$; $3 = 2\sqrt{3}\cos\phi \Rightarrow \phi = \frac{\pi}{6}$. Thus $(\rho, \phi, \theta) = (2\sqrt{3}, \frac{\pi}{6}, \frac{\pi}{6})$.

(b) $r = 1, \theta = \frac{\pi}{4}, z = -1$. Again using the similar relations we conclude that $\theta = \frac{\pi}{4}$; $\rho = \sqrt{1 + 1} = \sqrt{2}$; $-1 = \sqrt{2}\cos\phi \Rightarrow \phi = \frac{3\pi}{4}$. Thus $(\rho, \phi, \theta) = (\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{4})$.

(8) (a) $\rho = 5, \phi = \frac{\pi}{4}, \theta = \frac{2\pi}{3}$. Using the relation between spherical coordinates and cylindrical coordinates we get $\theta = \frac{2\pi}{3}$; $r = 5\sin\frac{\pi}{4} = \frac{5}{\sqrt{2}}$; $z = 5\cos\frac{\pi}{4} = \frac{5}{\sqrt{2}}$. Thus $(r, \theta, z) = (\frac{5}{\sqrt{2}}, \frac{2\pi}{3}, \frac{5}{\sqrt{2}})$.

(b) $\rho = 1, \phi = \frac{\pi}{2}, \theta = \frac{7\pi}{6}$. Again using the similar relations we conclude that $\theta = \frac{7\pi}{6}$; $r = 1\sin\frac{\pi}{2} = 1$; $z = 1\cos\frac{\pi}{2} = 0$. Thus $(r, \theta, z) = (1, \frac{7\pi}{6}, 0)$.

(9) (a) (i) interior points $= \emptyset$; (ii) limit points $= \emptyset$; (iii) boundary points $= \{1, 2, 3, 4\}$; (iv) closure of the set $= \{1, 2, 3, 4\}$.

(b) (i) interior points $= \emptyset$; (ii) limit points $= \mathbb{R}$; (iii) boundary points $= \mathbb{R}$; (iv) closure of the set $= \mathbb{R}$.

(c) (i) interior points $= S$; (ii) limit points $= [0, 1]$; (iii) boundary points $= \{0, 1\}$; (iv) closure of the set $= [0, 1]$.

(d) (i) interior points $= \emptyset$; (ii) limit points $= [0, 1] \times \{0\}$; (iii) boundary points $= [0, 1] \times \{0\}$; (iv) closure of the set $= [0, 1] \times \{0\}$.

(e) (i) interior points $= S$; (ii) limit points $= [0,1] \times \mathbb{R}$; (iii) boundary points $= \{0,1\} \times \mathbb{R}$; (iv) closure of the set $= [0,1] \times \mathbb{R}$.

(f) (i) interior points $= \emptyset$; (ii) limit points $= \mathbb{R}^2$; (iii) boundary points $= \mathbb{R}^2$; (iv) closure of the set $= \mathbb{R}^2$.

(g) (i) interior points $= S$; (ii) limit points $= \{x^2+y^2 \leq 1\}$; (iii) boundary points $= \{x^2+y^2 = 1\}$; (iv) closure of the set $= x^2 + y^2 \leq 1$.

(h) (i) interior points $= \emptyset$; (ii) limit points $= \{(0,0,0)\}$; (iii) boundary points $= S \cup \{(0,0,0)\}$; (iv) closure of the set $= S \cup \{(0,0,0)\}$.

(i) (i) interior points $= \emptyset$; (ii) limit points $= \emptyset$; (iii) boundary points $= S$; (iv) closure of the set $= S$.

(10) (a) (i) open; (ii) not closed; (iii) not bounded.

(b) (i) not open; (ii) closed; (iii) bounded.

(c) (i) open; (ii) not closed; (iii) not bounded.

(d) (i) not open; (ii) closed; (iii) not bounded.