♣ Tutorial, Solution ♣

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Problem 1. Examine if the limits as $(x, y) \to (0, 0)$ exist?

(a) $f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2-y^2} & \text{if } x \neq \pm y, \\ 0 & \text{if } x = \pm y. \end{cases}$

(b) $f(x,y) = xy\dfrac{x^2-y^2}{x^2+y^2}$

(c) $f(x,y) = \dfrac{\sin(xy)}{x^2+y^2}$

(d) $f(x,y) = \dfrac{|x|}{y^2}e^{-\frac{|x|}{y^2}}$

(e) $f(x,y) = \dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$

Solution. (a) If we approach the origin along any line $y = mx$, then $f(x,y) \to 0$. But let us approach the origin along a curve $y = +\sqrt{x^2 - mx^3}$; here $y \to 0$ as $x \to 0$.

$$f(x,y) = \frac{x^3 + (x^2 - mx^3)^{3/2}}{mx^3} = \frac{1 + (1-mx)^{3/2}}{m},$$

which has different values for different m. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Alternative Method: Consider the sequence (x_n, y_n) where $x_n = 1/n$, $y_n = 1/n + 1/n^2$. Then,

$$f(x_n, y_n) = \frac{1 + (1+\frac{1}{n})^3}{-(2+\frac{1}{n})} \to -1 \neq 0 = f(0), \quad \text{as } n \to \infty$$

Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ is not unique and the limit, in fact, does not exist.

(b) We are trying to establish $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$. Let $\epsilon > 0$ be given. Consider $\delta = \sqrt{\epsilon} > 0$, then $\forall (x,y) \in B(0,\delta)$, that is, $\sqrt{x^2+y^2} < \delta$,

$$\left| xy\frac{x^2-y^2}{x^2+y^2} \right| \leq |x||y| \leq x^2 + y^2 < \delta^2 = \epsilon.$$

Above inequality follows from the fact that, $\forall (x,y) \in \mathbb{R}^2$

$$|x| < \sqrt{x^2+y^2}, \quad |y| < \sqrt{x^2+y^2}, \quad |x^2-y^2| < \sqrt{x^2+y^2}$$

Thus $\delta = \sqrt{\epsilon}$ satisfy the requirement of the definition of limit. Therefore

$$\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$$

Alternative Method: We use polar coordinates to find the indicated limit, if it exists. Let $x = r\cos\theta, y = r\sin\theta$. Note that $(x,y) \to (0,0)$ is equivalent to $r \to 0$.

$$\left| xy\frac{x^2-y^2}{x^2+y^2} \right| = \left| r^2 \sin\theta\cos\theta \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2} \right| = \left| \frac{r^2}{4} \right| |\sin(4\theta)| \leq \frac{r^2}{4} \to 0, \quad \text{as } r \to 0$$

Solution (Cont.)

Therefore $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$.

(c) If we approach the origin along the line $y = 0$, then $f(x, y) \to 0$. Again if we approach the origin along the line $y = x$; here $y \to 0$ as $x \to 0$.

$$f(x, y) = \frac{\sin x^2}{2x^2} \to \frac{1}{2}, \qquad \text{Using L'Hopital's Rule}$$

Clearly $\lim_{(x,y)\to(0,0)} f(x, y)$ is not unique and the limit, in fact, does not exist.

Alternative Method: If we approach the origin along the line $y = mx$ $(m \neq 0)$; here $y \to 0$ as $x \to 0$,

$$f(x, mx) = \frac{\sin(mx^2)}{x^2(1 + m^2)} = \frac{\sin(mx^2)}{mx^2}\frac{m}{(1 + m^2)} \to \frac{m}{(1 + m^2)} \qquad \text{as } x \to 0,$$

which has different values for different m. Hence $\lim_{(x,y)\to(0,0)} f(x, y)$ is not unique and the limit, in fact, does not exist

(d) If we approach the origin along the line $x = 0$, then $f(x, y) \to 0$. Again if we approach the origin along the line $y = mx$ $(m \neq 0)$; here $y \to 0$ as $x \to 0$.

$$f(x, y) = \frac{|x|}{m^2x^2}e^{-\frac{|x|}{m^2x^2}} = \frac{1}{m^2|x|}e^{-\frac{1}{m^2|x|}} \to 0, \qquad \text{Using L'Hopital's Rule}$$

Thus $(x, y) \to (0, 0)$ along any straight line passing through the origin, we have $f(x, y)$ tending to zero.

Let us approach the origin origin along the parabola $x = y^2$, we have $f(y^2, y) = e^{-1} \to 0$, as $(x, y) \to (0, 0)$. Hence $\lim_{(x,y)\to(0,0)} f(x, y)$ is not unique and the limit, in fact, does not exist.

(e) We use polar coordinates to find the indicated limit, if it exists. Let $x = r\cos\theta, y = r\sin\theta$. Note that $(x, y) \to (0, 0)$ is equivalent to $r \to 0$. Now we have $f(r, \theta) = \frac{1-\cos r}{r^4}$. Now repeated application of L'Hopital's Rule gives us

$$\lim_{r\to 0}\frac{1 - \cos r^2}{r^4} = \lim_{r\to 0}\frac{1 - \cos r^2}{r^4} = \lim_{r\to 0}\frac{2r\sin r^2}{4r^3} = \lim_{r\to 0}\frac{4r\cos r^2}{12r} = 1$$

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{1 - \cos(x^2 + y^2)}{(x^2 + y^2)^2} = 1.$

Problem 2. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at (0, 0), where for all $(x, y) \in \mathbb{R}^2$,

(a) $f(x, y) = \begin{cases} xy\cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

(b) $f(x, y) = \begin{cases} 1 & \text{if } x > 0 \,\&\, 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$

(c) $f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

(d) $f(x, y) = \begin{cases} \frac{x^3y}{x^4+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

(e) $f(x, y) = \begin{cases} \frac{\sin(x+y)}{|x|+|y|} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

(f) $f(x, y) = \begin{cases} xy\ln(x^2 + y^2) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

Solution. (a) Let $\epsilon > 0$. Take $\delta = \sqrt{\epsilon} > 0$ and $(x, y) \in B(0, \delta)$, that is, $\sqrt{x^2 + y^2} < \delta$. Then we have following

$$\|f(x, y) - f(0, 0)\| = |xy \cos(1/x) - 0| \leq |x||y| \leq (x^2 + y^2) < \delta^2 = \epsilon.$$

Thus f is continuous at 0.
Alternative Method: Let $(x_n, y_n) \subset \mathbb{R}^2$ such that $(x_n, y_n) \to (0, 0)$. Then we have $|f(x_n, y_n)| = |x_n y_n \cos(1/x_n)| \leq |x_n||y_n| \to 0$ and so $f(x_n, y_n) \to 0$ as $(x_n, y_n) \to (0, 0)$. Hence f is continuous at 0.

(b) Take $(x_n, y_n) = (\frac{1}{\sqrt{n}}, \frac{1}{2n}) \in \mathbb{R}^2$ for $n \in \mathbb{N}$. Then $(x_n, y_n) \to 0$ as $n \to \infty$ and $f(x_n, y_n) = 1$ for all n, as $0 < y_n = \frac{1}{2n} < x_n^2 = \frac{1}{n^2}$. Thus $f(x_n, y_n) \to 1 \neq 0 = f(0, 0)$. Hence f is not continuous at $(0, 0)$.

Alternative Method: If we approach the origin along the curve $y = x^3$ $(x < 1)$; $f(x, x^3) = 1 \to 1 \neq 0 = f(0, 0)$. Hence $\lim_{(x,y)\to(0,0)} f(x, y)$ is not unique and so f is not continuous at $(0, 0)$.

(c) Let $\epsilon > 0$. Take $\delta = \epsilon > 0$ and $(x, y) \in B(0, \delta)$, that is, $\sqrt{x^2 + y^2} < \delta$. Then we have following

$$\|f(x, y) - f(0, 0)\| = \left|\frac{x^3}{x^2 + y^2}\right| \leq |x| \leq \sqrt{x^2 + y^2} < \delta^2 = \epsilon,$$

Second last inequality hold because $x^2 \leq x^2 + y^2$, for all $(x, y) \in \mathbb{R}^2$. Thus f is continuous at 0.

Alternative Method: Let $(x_n, y_n) \subset \mathbb{R}^2$ such that $(x_n, y_n) \to (0, 0)$. Then we have $|f(x_n, y_n)| = |x_n^3/(x_n^2 + y_n^2)| \leq |x_n| \to 0$ and so $f(x_n, y_n) \to 0$ as $(x_n, y_n) \to (0, 0)$. Hence f is continuous at 0.

(d) Let $\epsilon > 0$. Take $\delta = 2\epsilon > 0$ and $(x, y) \in B(0, \delta)$, that is, $\sqrt{x^2 + y^2} < \delta$. Then we have following

$$\|f(x, y) - f(0, 0)\| = \left|\frac{x^3 y}{x^4 + y^2}\right| = \left|\frac{x(2x^2 y)}{2(x^4 + y^2)}\right| \leq \frac{|x|}{2} \leq \frac{\sqrt{x^2 + y^2}}{2} = \epsilon.$$

Second last inequality hold because $2ab \leq a^2 + b^2$, for all $(a, b) \in \mathbb{R}^2$. Thus f is continuous at 0.
Alternative Method: Let $(x_n, y_n) \subset \mathbb{R}^2$ such that $(x_n, y_n) \to (0, 0)$. Then we have $|f(x_n, y_n)| = \left|\frac{x_n(2x_n^2 y_n)}{2(x_n^4 + y_n^2)}\right| \leq \frac{|x_n|}{2} \to 0$ and so $f(x_n, y_n) \to 0$ as $(x_n, y_n) \to (0, 0)$. Hence f is continuous at 0.

(e) Let us approach $(0, 0)$ via positive x-axis, that means, $y = 0$ and $x > 0$ such that $x \to 0$. Then $f(x, y) = \frac{\sin x}{x} \to 1$ as $x \to 0$. Thus $\lim_{(x,y)\to(0,0)} f(x, y) \neq 0 = f(0, 0)$ and so f is not continuous at $(0, 0)$.

Note: In fact one can show that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist by approach the origin via negative x-axis.

(f) Let $x = r \cos\theta$ and $y = r \sin\theta$. Then we have $|f(x, y)| = |f(r, \theta)| = |r^2 \sin(2\theta) \ln r| \to 0$ as $r \to 0$, using L'Höspital rule. Thus $f(x, y) \to f(0, 0)$ as $(x, y) \to (0, 0)$ and so f is continuous at 0.

Exercise. *Examine the continuity of* $f : \mathbb{R}^2 \to \mathbb{R}$ *at (0, 0), where for all* $(x, y) \in \mathbb{R}^2$*,* *(a)* $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$ *(b)* $f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$

Solution (Hints). (a) If you apply same technique as apply in solution of (2)(c) then you will get $|f(x,y)| \leq 1$, which will not gives you any thing. Take $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n})$ and show that f is not continuous at $(0,0)$

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Let us approach $(0,0)$ via $y = mx$ line. Then we have following

$$f(x,y) = \frac{xy}{x^2+y^2} = \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}.$$

Thus $\lim f(x,y)$ as $(x,y) \to (0,0)$ depends on m, that means, depend on line $y = mx$. The value of $\lim_{(x,y)\to(0,0)} f(x,y)$ is different for different values of m and hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Thus f is not continuous at $(0,0)$.

(b) Take $(x_n, y_n) = (\frac{1}{\sqrt{n}}, \frac{1}{n})$ and shows that $f(x_n, y_n) \to \frac{1}{2}$ as $n \to \infty$ and so f is not continuous at $(0,0)$.
Alternative Method: Let us consider $y = mx^2$ curve to approach $(0,0)$ and show that The value of $f(x,y) = \frac{m}{1+m^2}$. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ is different for different values of m and hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Thus f is not continuous at $(0,0)$.

Problem 3. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function at $X_0 \in \mathbb{R}^2$ and that $|f(X_0)| > 2$. Show that there is a $\delta > 0$ such that $|f(X)| > 2$ whenever $\|X - X_0\| < \delta$.

Solution. Let $\epsilon = |f(X_0)| - 2 > 0$. Since f is continuous at X_0, there exist $\delta > 0$ such that $|f(X) - f(X_0)| < \epsilon$ for all X satisfying $\|X - X_0\| < \delta$, that means,

$$f(X_0) - |f(X_0)| + 2 < f(X) < f(X_0) + |f(X_0)| - 2, \tag{3.1}$$

whenever $\|X - X_0\| < \delta$. If $f(X_0) > 0$ then left side of Equation (3.1) gives $f(X) > 2$ and if $f(X_0) < 0$ then right side of Equation (3.1) gives $f(X) < -2$. Hence $|f(X)| > 2$ whenever $\|X - X_0\| < \delta$.

Problem 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 0$ if $x \in \mathbb{Q}$, $y \in \mathbb{Q}$ and $f(x,y) = xy$ otherwise. Find all the points in $\mathbb{R}^2$ where f is continuous.

Solution. Given $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \mathbb{Q} \times \mathbb{Q} \\ xy & \text{otherwise .} \end{cases}$$

Check continuity at $(x,0)$: Let $(x_n, y_n) \subset \mathbb{R}^2$ such that $(x_n, y_n) \to (x,0)$. Then $|f(x_n, y_n)| \leq |x_n||y_n| \to 0$, as $|x_n| \to |x|$ and $|y_n| \to 0$. Thus $f(x_n, y_n) \to 0$. Thus f is continuous at $(x,0)$.

Solution (Cont.)

Similarly f is continuous at $(0, y)$.

Check continuity at nonzero points: Let $(x, y) \in \mathbb{R}^2$ such that $x \neq 0$ and $y \neq 0$.
Case 1: Let $(x, y) \in \mathbb{Q} \times \mathbb{Q}$. Then there exist a sequence $x_n \in \mathbb{R} \setminus \mathbb{Q}$ and $y_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to x$ and $y_n \to y$. Then $f(x_n, y_n) = x_n y_n \to xy \neq 0 = f(x, y)$. Thus f is not continuous at $(x, y) \in \mathbb{Q} \times \mathbb{Q}$.
Case 2: Let $(x, y) \in \mathbb{Q}^c \times \mathbb{Q}^c$. Then there exist a sequence $x_n \in \mathbb{Q}$ and $y_n \in \mathbb{Q}$ such that $x_n \to x$ and $y_n \to y$. Then $f(x_n, y_n) = 0 \to 0 \neq xy = f(x, y)$. Thus f is not continuous at $(x, y) \in \mathbb{Q}^c \times \mathbb{Q}^c$.
Case 3: Let $(x, y) \in \mathbb{Q} \times \mathbb{Q}^c$ or $(x, y) \in \mathbb{Q}^c \times \mathbb{Q}$. Then there exist a sequence $x_n \in \mathbb{Q}$ and $y_n \in \mathbb{Q}$ such that $x_n \to x$ and $y_n \to y$. Then $f(x_n, y_n) = 0 \to 0 \neq xy = f(x, y)$. Thus f is not continuous at $(x, y) \in (\mathbb{Q} \times \mathbb{Q}^c) \cup (\mathbb{Q}^c \times \mathbb{Q})$.

Hence f is continuous only on x-axis and y-axis.