MA 102 (Mathematics II)
Department of Mathematics, IIT Guwahati

Tutorial Sheet No. 6 February 22, 2016

Constrained extrema, implicit derivative, vector fields, arclength

(1) Find the extrema of the function $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

(2) Find a point on the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ that is closest to $(0,0,0)$.

(3) Find the maximum and minimum of $f(x,y) = 5x - 3y$ subject to the constraint

$$x^2 + y^2 = 136.$$

(4) Find the global maximum (also called absolute maximum) of $f(x,y) := xy$ on the unit disk $x^2 + y^2 \leq 1$.

(5) Assume that among all rectangular boxes with fixed surface area of 10 square meters there is a box of largest possible volume. Find the dimensions of the optimum box.

(6) Consider the equation $e^{2x-y} + \cos(x^2+xy) - 2 - 2y = 0$ for $(x,y) \in \mathbb{R}^2$. Can the solutions be written as $y = \phi(x)$ and $x = \psi(y)$ in a neighbourhood of 0? If so, compute the derivatives $\phi'(0)$ and $\psi'(0)$.

(7) Show that around the point $(0,1,1)$, the equation $xy - z\log y + e^{xz} = 1$ can be solved locally as $y = f(x,z)$ but cannot be solved locally as $z = g(x,y)$. Find $f_x(0,1)$ and $f_z(0,1)$.

(8) Let S be a surface given by $x^3 + 3y^2 + 8xz^2 - 3z^3y - 1 = 0$. Find all points $(x_0, y_0, z_0) \in \mathbb{R}^3$ such that S is represented as a graph of a differentiable function $z = f(x,y)$ in a neighbourhood of (x_0, y_0, z_0).

(9) Show that each of the following functions $F : \mathbb{R}^2 \to \mathbb{R}^2$ is locally invertible at $(0,0)$ and find the derivative of the inverse function at $(u_0, v_0) := F(0,0)$.
  (a) $F(x,y) := (x + x^2 + e^{x^2y^2},\ -x + y + \sin(xy))$, (b) $F(x,y) := (e^{x+y},\ e^{x-y})$.

(10) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x,y) := (e^x\cos(y),\ e^x\sin(y))$.
  (a) Show that the Inverse Function Theorem is applicable to F at each point in $\mathbb{R}^2$.
  (b) Show that F is not injective on $\mathbb{R}^2$. Does this contradict (a)? Justify your answer.

(11) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be C^1 scalar fields. Show that (a) $\nabla(fg) = f\nabla g + g\nabla f$, (b) $\nabla f^m = mf^{m-1}\nabla f$ and (c) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.

(12) Let F and G be vector fields in $\mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 scalar field. Then show that:

  (a) $\operatorname{div}(F+G) = \operatorname{div} F + \operatorname{div} G$ and $\operatorname{curl}(F+G) = \operatorname{curl} F + \operatorname{curl} G$,
  (b) $\operatorname{div}(fG) = f\operatorname{div} G + G \bullet \nabla f$ and $\operatorname{curl}(fG) = f\operatorname{curl} G + \nabla f \times G$,
  (c) $\operatorname{div}(F \times G) = G \bullet \operatorname{curl} F - F \bullet \operatorname{curl} G$ and $\operatorname{curl}\operatorname{curl} F = \nabla\operatorname{div} F - \nabla^2 F$.

(13) Let $\mathbf{r} = (x, y, z)$ and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$. Then show that

(a) $\nabla r = \frac{\mathbf{r}}{r}$ and $\nabla(\frac{1}{r}) = \frac{-\mathbf{r}}{r^3}$ for $r \neq 0$.
(b) $\operatorname{div}(r^m \mathbf{r}) = (m+3)r^m$
(c) $\operatorname{curl}(r^m \mathbf{r}) = 0$ and $\operatorname{div}\left(\nabla \frac{1}{r}\right) = 0$ for $r \neq 0$.

(14) Find the arclength of parabolic arc $\gamma(t) := (t, t^2)$ for $t \in [0, 4]$.

(15) Find the velocity, the speed and the arclength of the cycloid $\gamma(t) := (t - \sin t,\ 1 - \cos t)$ for $t \in [0, 2\pi]$.

(16) A billiard ball on a square table follows the path $\gamma : [-1, 1] \to \mathbb{R}^3$ given by $\gamma(t) := (|t|,\ |t - 1/2|,\ 0)$. Find the distance travelled by the ball.

(17) Find the arclength of the path $\gamma(t) := (t,\ t \sin t, t \cos t)$ between $(0, 0, 0)$ and $(\pi, 0, -\pi)$.

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