PH 102: Physics II

Lecture 16 (Spring 2018)

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LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	6-3-2018	Lorentz Force, Biot-Savart law	5.1, 5.2	1	I, II (5-5:55 pm)
Lec 15	7-3-2018	Lorentz Force, Biot-Savart law	5.1, 5.2	1	III, IV (9-9:55 am)
Tut 8	13-3-2018	Lec 15			
Lec 16	13-3-2018	Divergence & Curl of Magnetostatic Fields, Applications of Ampere's law	5.3	1	I, II (5-5:55 pm)
Lec 16	14-3-2018	Divergence & Curl of Magnetostatic Fields, Applications of Ampere's law	5.3	1	III, IV (9-9:55 am)
Lec 17	14-3-2018	Magnetic Vector Potential, Force & torque on a magnetic dipole	5.4	1	I, II (4-4:55 pm)
Lec 17	15-3-2018	Magnetic Vector Potential, Force & torque on a magnetic dipole	5.4	1	III, IV (10-10:55 am)
Lec 18	15-3-2018	Lec 16+Lec 17 Continues		1	I, II (3-3:55 pm)
Lec 18	16-3-2018	Lec 16+Lec 17 Continues		1	III, IV (11-11:55 am)
Tut 9	20-3-2018	Lec 16+Lec 17+Lec 18			
Lec 19	21-3-2018	Magnetic Materials, Magnetization	6.1	1	I, II, III, IV
Lec 20	23-3-2018	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II, III, IV
Tut 10	3-4-2018	Quiz II			
Lec 21	4-4-2018	Ohm's law, motional emf, electromotive force	7.1	1	I, II, III, IV
Lec 22	6-4-2018	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II, III, IV
Tut 11	10-4-2018	Lec 21+Lec 22			
Lec 23	11-4-2018	Maxwell's equations	7.3	1	I, II, III, IV
Lec 24	13-4-2018	Continuity equation, Poynting Theorem	8.1	1	I, II, III, IV
Tut 12	17-4-2018	Lec 23+Lec 24			
Lec 25	18-4-2018	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II, III, IV
Lec 26	20-4-2018	L11+Electromagnetic waves in matter	9.3	1	I, II, III, IV
Tut 13	24-4-2018	Lec 25+ Lec 26			
Lec 27	25-4-2018	Reflection and transmission: Normal & Oblique Incidence	9.3, 9.4	1	I, II, III, IV
Lec 28	27-4-2018	Lec 27+Discussions	9.3, 9.4	1	I, II, III, IV

Ampere's Law

- Magnetic field due to an infinitely long wire carrying current I is

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

- The field "circles" around the wire as shown in the figure.

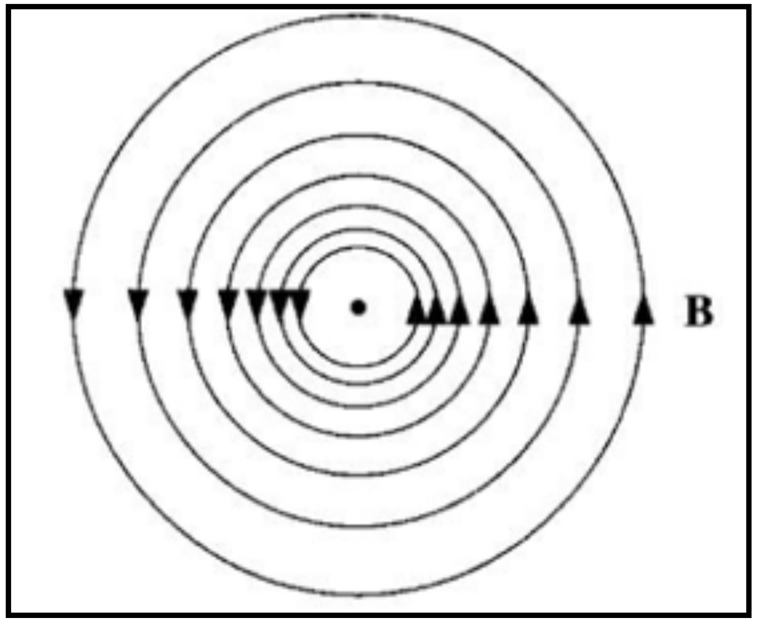


Fig. 5.27 (Introduction to Electrodynamics, D. J. Griffiths)

- The line integral of magnetic field around a circular path of radius r, entered at the wire is

$$\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi r} dl = \frac{\mu_0 I}{2\pi r} \oint dl = \mu_0 I$$

Ampere's Law

- The line integral of B around a closed path does not depend upon the distance r of the path from the wire: B decreases at the same rate as the circumference increases.

Ampere's Law (loop of arbitrary shape)

- In cylindrical polar coordinates (r, ϕ, z) the magnetic field due to an infinite wire carrying current along the z axis is $$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$
- Using the cylindrical line element $$d\vec{l} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}$$
- The line integral of B around an arbitrary closed loop is

$$\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\oint \frac{rd\phi}{r} = \frac{\mu_0 I}{2\pi}\oint d\phi = \mu_0 I$$

- If the loop does not enclose the wire at all, then ϕ will go from ϕ_1 to ϕ_2 and back again resulting in $\int d\phi = 0$

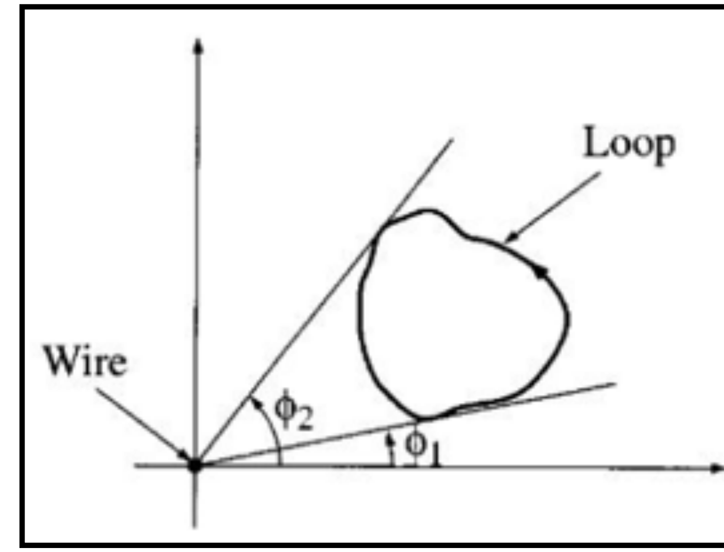


Fig. 5.28 (Introduction to Electrodynamics, D. J. Griffiths)

- If there is a bundle of straight wires, the ones passing through the loop contributes $\mu_0 I$ to the magnetic field

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

Ampere's Law

For a loop of the specific shape (shown in figure)

$$\oint \vec{B}\cdot d\vec{l} = \int_{ab}\vec{B}\cdot d\vec{l} + \int_{bc}\vec{B}\cdot d\vec{l} + \int_{cd}\vec{B}\cdot d\vec{l} + \int_{da}\vec{B}\cdot d\vec{l}$$

$$= 0 + B_2(r_2\theta) + 0 + B_1 r_1(2\pi - \theta)$$

$$= \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + \frac{\mu_0 I}{2\pi r_1} r_1(2\pi - \theta)$$

$$\implies \oint \vec{B}\cdot d\vec{l} = \mu_0 I$$

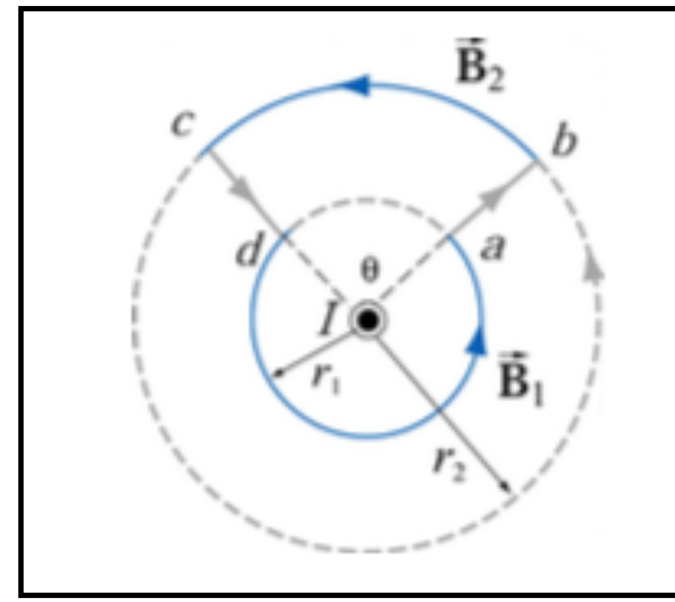


Image credit: MIT

Ampere's law is applicable to symmetric configurations such as:

Infinite long straight wire, infinite large 2D sheet, infinite solenoid, toroid carrying steady currents.

Divergence & Curl of Magnetic Field

- If J is the volume current density then the current enclosed by the integration path is $I_{\text{enc}} = \int \vec{J} \cdot d\vec{a}$ where the integral is taken over the surface bounded by the loop.

- Using Stoke's theorem

$$\oint \vec{B} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

- Since this should be true for any area, we have

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$ Ampere's Law

Divergence & Curl of Magnetic Field From Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau'$$ Biot-Savart Law

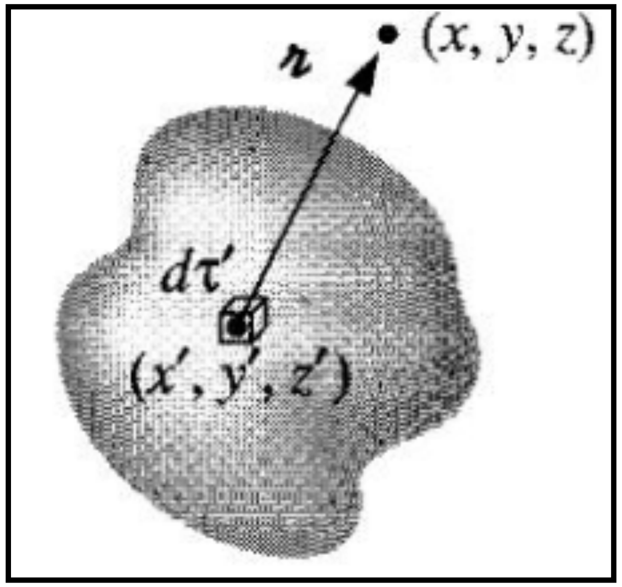


Fig. 5.30 (Introduction to Electrodynamics, D. J. Griffiths)

$$\vec{\nabla}\cdot\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}\right) d\tau'$$

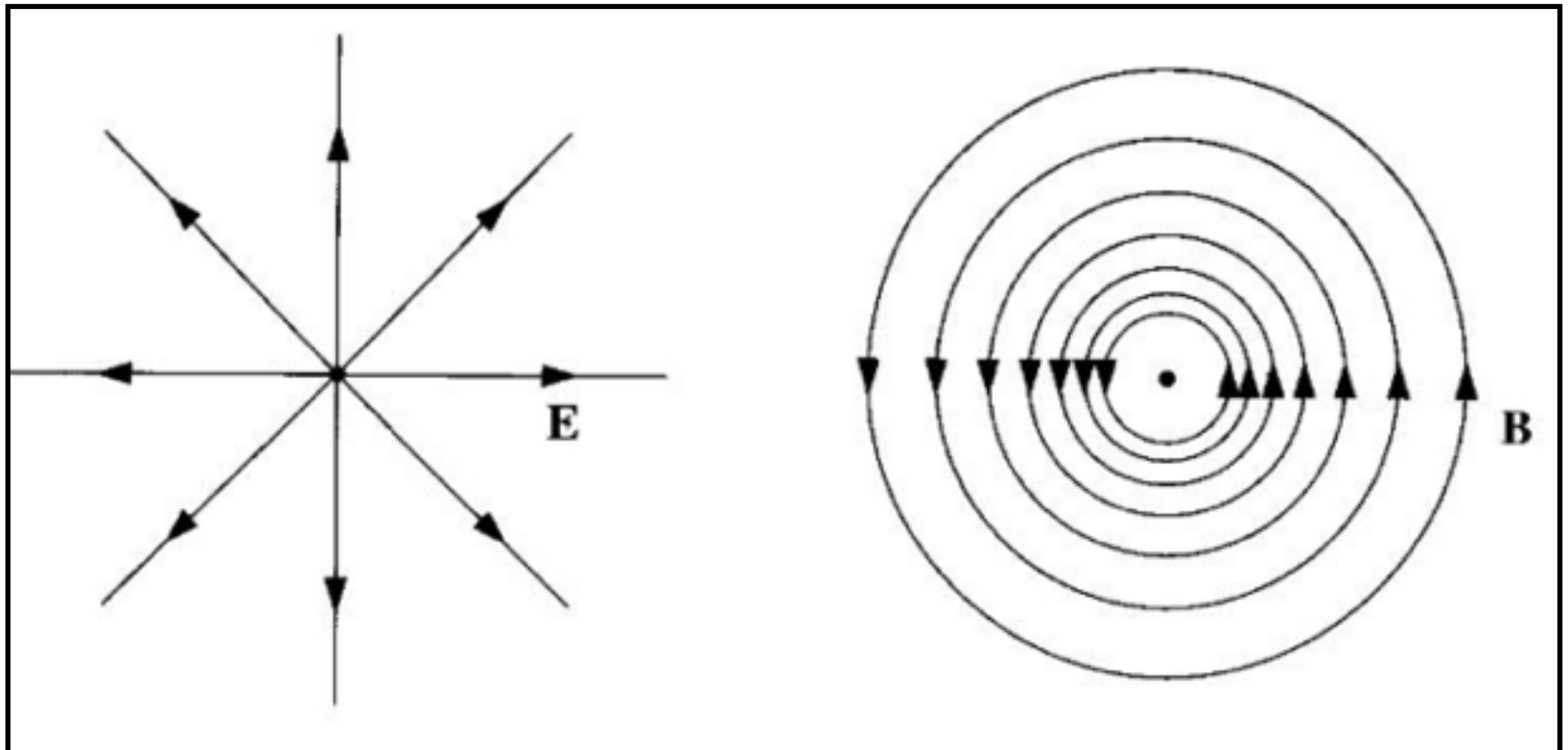
$$\vec{\mathfrak{r}} = \vec{r} - \vec{r'}$$

$$\vec{\nabla}\cdot\left(\vec{J}(\vec{r'}) \times \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right) = \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\cdot(\vec{\nabla}\times\vec{J}(\vec{r'})) - \vec{J}(\vec{r'})\cdot\left(\vec{\nabla}\times\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)$$

Using $\quad \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

Divergence & Curl of Magnetic Field

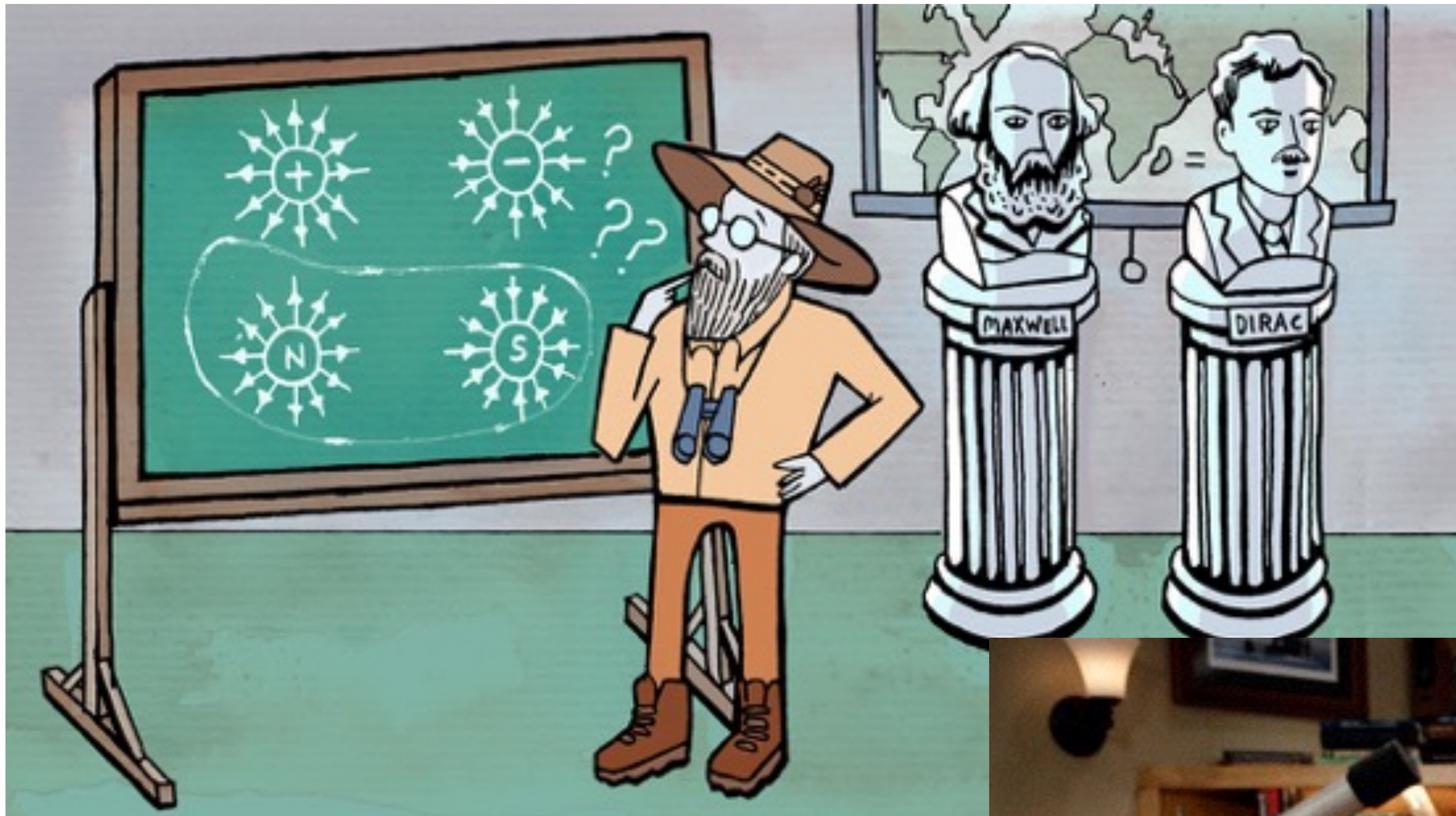
- We have $\vec{\nabla} \times \vec{J}(\vec{r'}) = 0, \vec{\nabla} \times \left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right) = 0$ which implies $\vec{\nabla} \cdot \vec{B} = 0$
- The magnetic field is divergence-less: no net outflow of magnetic lines of force through a closed surface. No magnetic analog to electric charge.



(a) Electrostatic field of a point charge
(b) Magnetostatic field of a long wire

Fig. 5.44 (Introduction to Electrodynamics, D. J. Griffiths)

No Magnetic Monopoles



$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\int_S \vec{B} \cdot d\vec{a} = 0$$

Monopole Expedition
(The Big Bang Theory)

Image Credit:
Symmetrymagazine, CBS

Curl of Magnetic Field

- Similarly, $\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) d\tau'$

- Using the identity

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$$

we have

$$\vec{\nabla} \times \left(\vec{J}(\vec{r'}) \times \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) = \vec{J}(\vec{r'}) \left(\vec{\nabla} \cdot \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) - (\vec{J}(\vec{r'}) \cdot \vec{\nabla}) \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

The other 2 terms identically vanish $\left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \cdot \vec{\nabla} \right) \vec{J}(\vec{r'}) = 0, \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} (\vec{\nabla} \cdot \vec{J}(\vec{r'})) = 0$

Curl of Magnetic Field

- We know the identity: $\vec{\nabla} \cdot \left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) = 4\pi\delta^3(\vec{\mathfrak{r}})$
- Since $\vec{\mathfrak{r}} = \vec{r} - \vec{r'}$ and derivatives act only on $\hat{\mathfrak{r}}/\mathfrak{r}^2$ one can have the interchange $\vec{\nabla} \leftrightarrow \vec{\nabla}'$ just by incorporating an extra minus sign. Therefore,

$$-(\vec{J}(\vec{r'}) \cdot \vec{\nabla}) \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} = (\vec{J}(\vec{r'}) \cdot \vec{\nabla}') \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

- Taking the x-component of the RHS of above expression

$$(\vec{J}(\vec{r'}) \cdot \vec{\nabla}') \left(\frac{x - x'}{\mathfrak{r}^3} \right) = \vec{\nabla}' \cdot \left(\frac{x - x'}{\mathfrak{r}^3} \vec{J}(\vec{r'}) \right) - \left(\frac{x - x'}{\mathfrak{r}^3} \right) (\vec{\nabla}' \cdot \vec{J}(\vec{r'}))$$

Using $\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} f)$

Curl of Magnetic Field

- For steady current $\vec{\nabla}' \cdot \vec{J}(\vec{r'}) = 0$ and hence

$$\left(-(\vec{J}(\vec{r'}) \cdot \vec{\nabla})\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)_x = \vec{\nabla}' \cdot \left(\frac{x - x'}{\mathfrak{r}^3}\vec{J}(\vec{r'})\right)$$

- Therefore, the integral becomes

$$\int_V \vec{\nabla}' \cdot \left(\frac{x - x'}{\mathfrak{r}^3}\vec{J}(\vec{r'})\right) d\tau' = \oint_S \frac{x - x'}{\mathfrak{r}^3}\vec{J}(\vec{r'}) \cdot d\vec{a'}$$

which vanishes for J=0 at the surface S (Current is zero on the boundary, all current is inside).

- The curl of magnetic field is therefore, $\left(\vec{\nabla} \cdot \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} = 4\pi\delta^3(\vec{\mathfrak{r}})\right)$

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r'})4\pi\delta^3(\vec{r} - \vec{r'})d\tau' = \mu_0\vec{J}(\vec{r})$$

Ampere's Law

Application of Ampere's Law

- The role of Ampere's law in the context of Biot-Savart law of magnetostatics is equivalent to that of Gauss's law in the context of Coulomb's law in electrostatics.

- For currents with appropriate symmetry (infinite straight lines, infinite planes, infinite solenoids, toroids), Ampere's law in integral form can be applied to simplify the calculation of magnetic field.

- See solved examples 5.7-5.10 (Introduction to Electrodynamics, D. J. Griffiths)

Example 5.7 (Introduction to Electrodynamics, D J Griffiths): Find the magnetic field a distance s from a long straight wire, carrying a steady current I.

Since the problem has a symmetry, it is obvious that the magnitude of magnetic field is constant around an amperian loop of radius s, centred on the wire. Using Ampere's law:

$$\oint \vec{B} \cdot d\vec{l} = B \oint dl = (2\pi s)B = \mu_0 I$$

$$\implies B = \frac{\mu_0 I}{2\pi s}$$

Same answer as Ex 5.5 (Lecture 1), but obtained in a much simpler way

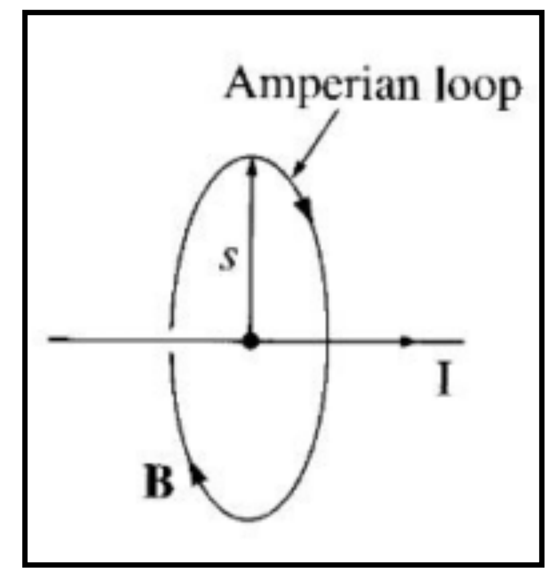


Fig. 5.32 (Introduction to Electrodynamics, D. J. Griffiths)

Magnetic field of an infinite surface current:

Surface current is given by: $\vec{K} = K\hat{x}$

By symmetry, $\vec{B}$ should be along $\pm\hat{y}$

For the Amperian loop:

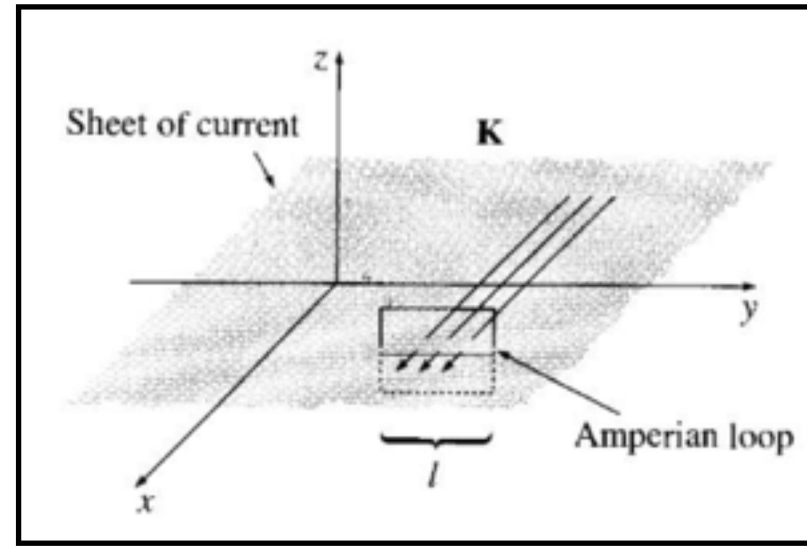


Fig. 5.33 (Introduction to Electrodynamics, D. J. Griffiths)

$$\oint \vec{B}\cdot d\vec{l} = 2Bl = \mu_0 K l$$

Why not along x, z?

$$\vec{B} = \begin{cases} +\frac{\mu_0}{2}K\hat{y} & \text{for } z < 0, \\ -\frac{\mu_0}{2}K\hat{y} & \text{for } z > 0. \end{cases}$$

Linear solenoid:

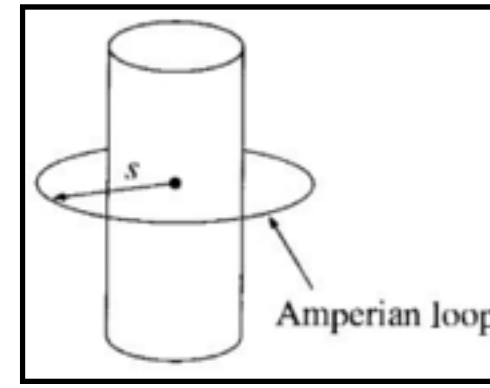
For the circular loop, enclosed current = 0

$$\oint \vec{B}\cdot d\vec{l} = 0 = B_\phi(2\pi s) \quad \Longrightarrow \quad B_\phi = 0$$

Hence, for rectangular loop 1:

$$\oint \vec{B}\cdot d\vec{l} = [B(a) - B(b)]L = \mu_0 I = 0 \quad \Longrightarrow \quad B(a) = B(b)$$

Thus, the field outside is same everywhere.



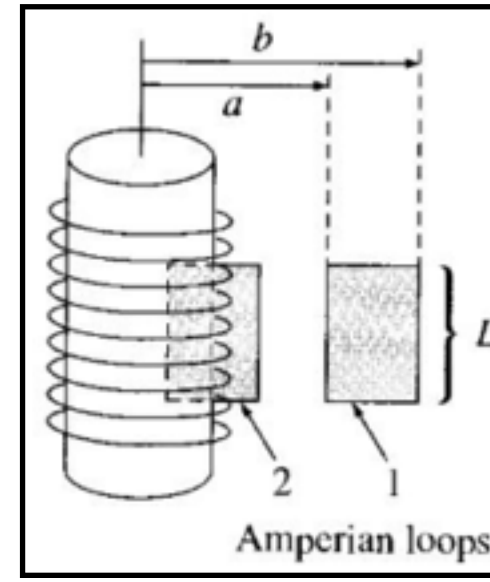


Fig. 5.36, 5.37 (Introduction to Electrodynamics, D. J. Griffiths)

Since field has to vanish at infinity it means $B_{\text{outside}} = 0$ (everywhere)

For rectangular loop 2: $\oint \vec{B}\cdot d\vec{l} = BL = \mu_0(nL)I \Longrightarrow \vec{B} = \mu_0 n I\hat{z}$ (inside)

(Much easier than using Biot-Savart law, Problem 5.44, Lecture 1)

Linear Solenoid (Finite)

Find the magnetic field at the axis of a finite solenoid of length L, Radius R, n number of turns per unit length with its centre coinciding with the origin.

Consider an elemental length of the solenoid and use the expression for magnetic field on the axis of a current carrying loop. Integrate over the length of the solenoid and show that the field at the axis is

$$B_z = \frac{\mu_0 n I}{2}\left[\frac{L/2 - z}{\sqrt{(z-L/2)^2 + R^2}} + \frac{L/2 + z}{\sqrt{(z+L/2)^2 + R^2}}\right]$$

Toroidal Solenoid:

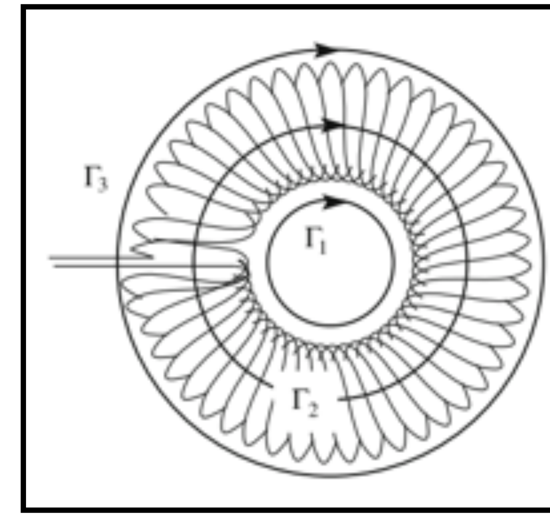


Image credit: Springer

Applying Ampere's law for circle Γ_1

$$\oint \vec{B} \cdot d\vec{l} = B(2\pi s) = 0 \implies B = 0 \qquad s < R_1$$

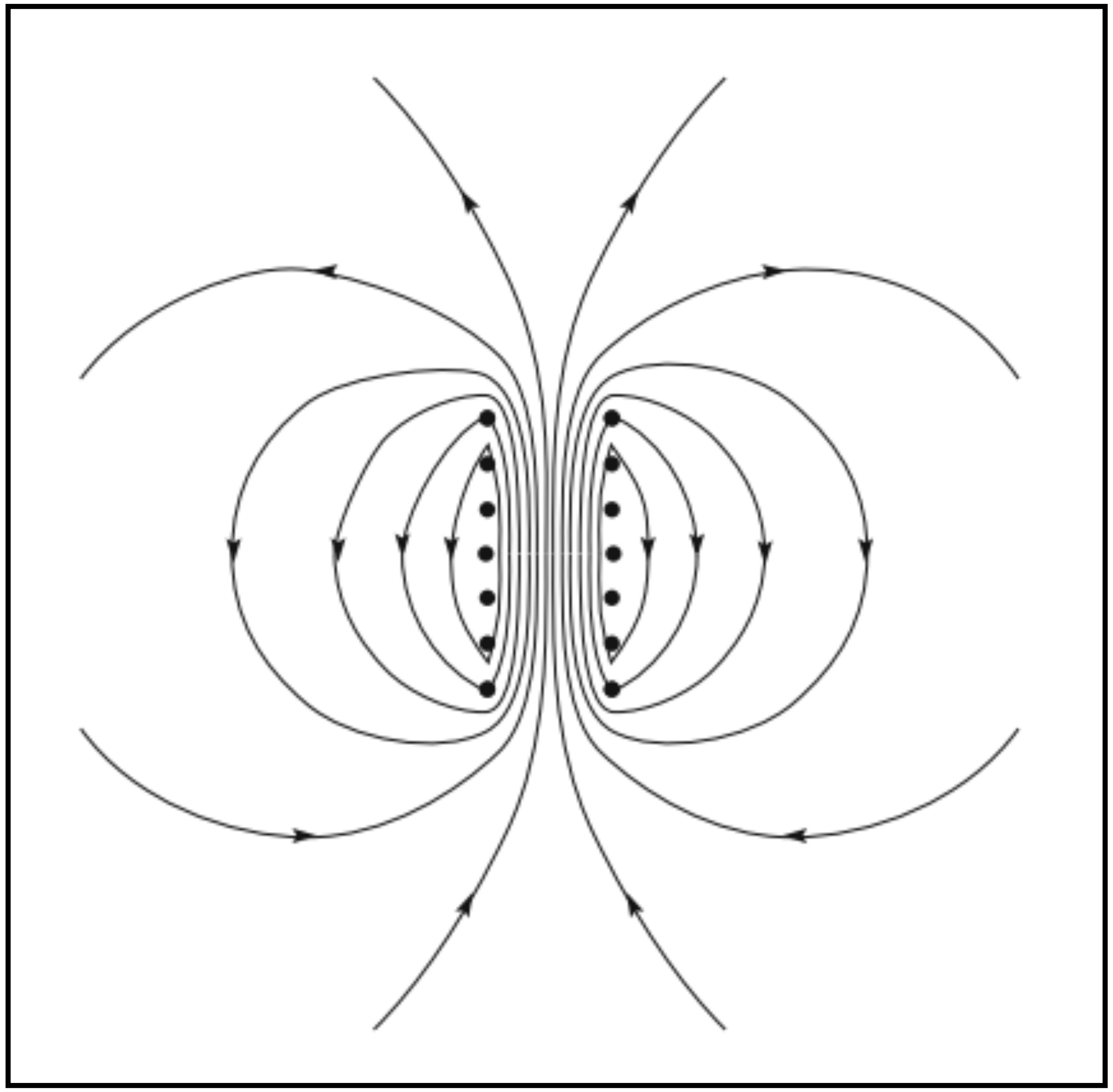
For circle Γ_3 also, the enclosed current is zero due to equal ad opposite currents and hence B=0 (outside).

For circle Γ_2: $\oint \vec{B} \cdot d\vec{l} = B(2\pi s) = \mu_0 NI \implies B = \frac{\mu_0 NI}{2\pi s}, \quad \vec{B}(\vec{r}) = \frac{\mu_0 NI}{2\pi r}\hat{\phi}, \quad R_1 < r < R_2$

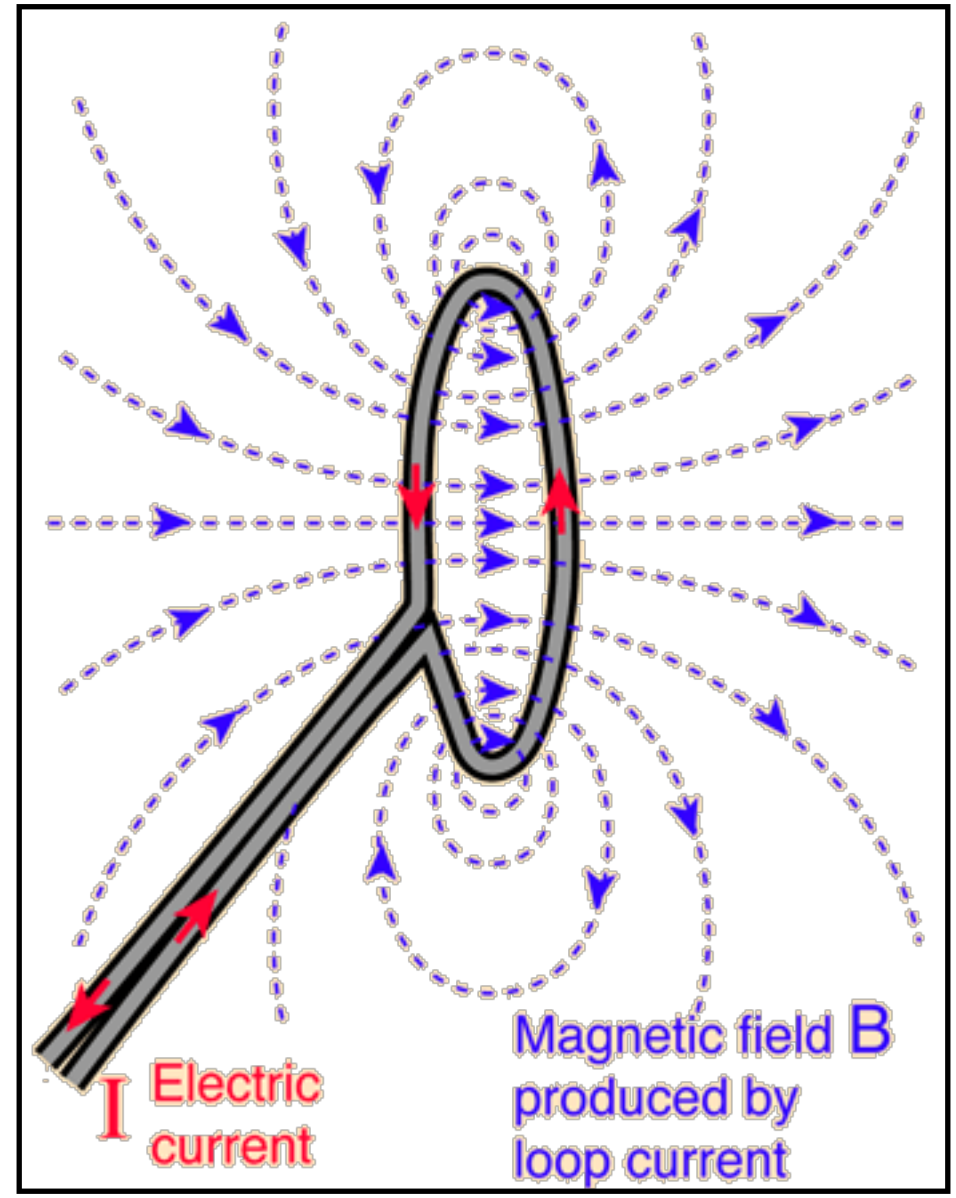
For $R_2 - R_1 \ll R_1$ one can assume the length of the solenoid to be $2\pi r$ so that $n = N/(2\pi r)$ is the no. of turns per unit length.

$$B = \mu_0 n I \qquad R_1 < r < R_2$$

See example 5.10 (Introduction to Electrodynamics, D J Griffiths): Show that the magnetic field of the toroid is circumferential!



Field lines for solenoid



Field lines for a
current carrying loop

Image credit: Springer, hyperphysics

Exercise: Calculate the magnetic field inside an outside of an infinitely long hollow cylinder (of radius R) carrying uniform surface current $\vec{K} = K\hat{z}$ along the axis of the cylinder, compare it with the result for a linear solenoid having axis along the same direction.

Use Ampere's law and show that the field (which is in the circumferential direction) is given by:

$$B_\phi = \begin{cases} \frac{\mu_0 K R}{r}, & r > R \\ 0, & r < R \end{cases}$$

Exercise: Calculate magnetic field inside and outside of an infinitely long solid cylinder (of radius R) having uniform volume current $\vec{J} = J\hat{z}$

Use Ampere's law and show that the field (which is in the circumferential direction) is given by:

$$B_\phi = \begin{cases} \frac{\mu_0 J R^2}{2r}, & r > R \\ \frac{\mu_0 J r}{2}, & r < R \end{cases}$$