1. Currents I and $-I$ flow along lines at $y = d/2$ and $y = -d/2$, respectively, on the yz plane, as shown in Figure. Determine the vector potential at point P sufficiently far from the z-axis.

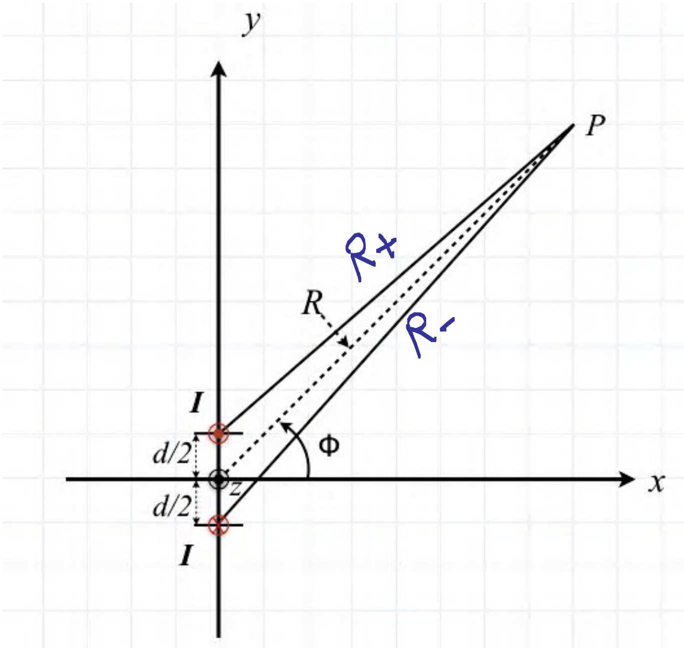


Figure 1: Problem 1

Sol^n:

The vector potential due to current I flow along line at $y = d/2$ is given as

$$A_{z+} = \frac{\mu_0 I}{2\pi} \ln \frac{R_0}{R_+}$$

$R_0 \rightarrow$ the distance to the reference point at which A_{z+} is zero

||ly $$A_{z-} = -\frac{\mu_0 I}{2\pi} \ln \frac{R_0}{R_-}$$

The vector potential

$$A_z = A_{z+} + A_{z-} = \frac{\mu_0 I}{2\pi} \ln \frac{R_-}{R_+}$$

$$R_+ = \left(R^2 + \frac{d^2}{4} - Rd\sin\varphi\right)^{1/2} \qquad R \gg d$$

$$\simeq R\left(1 - \frac{d}{2R}\sin\varphi\right)$$

$$R_- \simeq R\left(1 + \frac{d}{2R}\sin\varphi\right)$$

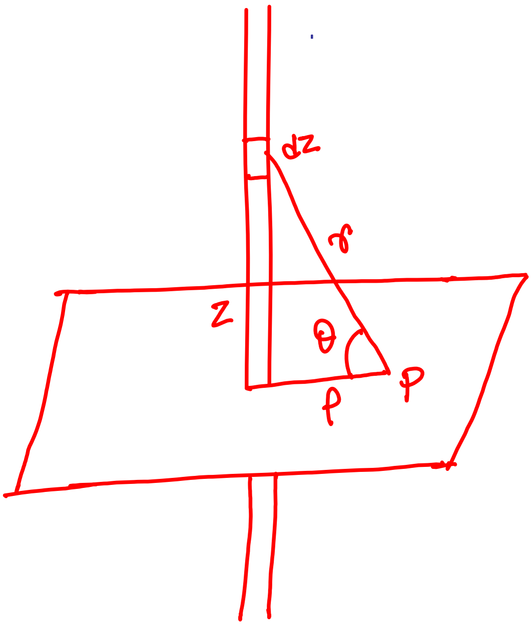
$$A_z(R,\phi) \simeq \frac{\mu_0 I}{2\pi}\ln\frac{1 + d/2R\,\sin\varphi}{1 - d/2R\,\sin\varphi}$$

$$\simeq \frac{\mu_0 I d}{2\pi R}\sin\phi.$$

$$|\vec{m}| = Id$$

$$A_z \cong \frac{\mu_0 |\vec{m}|}{2\pi R}\sin\varphi$$

at sufficient distance $R \gg d$ the equality holds. In this case the total vector sum of the current is zero, and hence $\vec{A} \to 0$ at infinity.



$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{k}\int_{-L}^{L}\frac{dz}{(\rho^2+z^2)^{1/2}}$$

$$z = \rho\tan\theta$$

$$dz = \rho\sec^2\theta\, d\theta$$

$$\vec{A} = \frac{\mu_0 I}{2\pi}\hat{k}\ \ln(\sec\alpha + \tan\alpha)$$

$$\tan\alpha = L/\rho$$

$$\sec\alpha = \frac{L}{\rho}\left(1+\frac{\rho^2}{L^2}\right)^{1/2} \approx \frac{L}{\rho}\left(1-\frac{\rho^2}{2L^2}\right)$$

$$\vec{A} = \frac{\mu_0 I}{2\pi}\hat{k}\left[\ln(2L) - \ln(\rho)\right]$$

as expected $L\to\infty$, the above expression diverges. $\vec{A}$ itself is not physical while $\nabla\times A$ is. constant term (which diverges in the limit of $L\to\infty$) is of no consequence and $\vec{A}$ is given by

$$\vec{A} = -\frac{\mu_0 I}{2\pi}\hat{k}\ \ln\rho$$

$$= \frac{\mu_0 I}{2\pi}\hat{k}\ \ln\left(\frac{\rho_{ref}}{\rho}\right)$$

2. If $\mathbf{B}$ is uniform, show that $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r} \times \mathbf{B})$. That is, check that $\nabla \cdot \mathbf{A} = \mathbf{0}$ and $\nabla \times \mathbf{A} = \mathbf{B}$. Is this result unique or are there other function with same divergence and curl ?

Sol^n:

$$\nabla . A = -\frac{1}{2} \nabla . (\bar{r} \times \bar{B})$$

$$= -\frac{1}{2} \left[\bar{B} . (\nabla \times \bar{r}) - \bar{r} . (\nabla \times \bar{B}) \right]$$

$$= -\frac{1}{2} \left[\bar{B} . 0 - \bar{r} . 0 \right] = 0$$

Since B = uniform

$$\nabla \times A = -\frac{1}{2} \nabla \times (\bar{r} \times \bar{B})$$

$$= -\frac{1}{2} \left[(\bar{B} . \nabla) \bar{r} - \overbrace{(\bar{r} . \nabla) \bar{B}}^{0} + \underbrace{\bar{r} (\nabla . \bar{B})}_{0} - \bar{B} \underbrace{(\nabla . \bar{r})}_{3} \right]$$

$$= -\frac{1}{2} \left[\bar{B} - 3\bar{B} \right] = \bar{B}$$

3. A current distribution produces the vector potential

$$\mathbf{A}(\mathbf{r},\theta,\phi) = \widehat{\phi}\frac{\mu_0}{4\pi}\frac{A_0 \sin\theta}{r}e^{-\lambda r}$$

What is the magnetic dipole moment associated with this current distribution ?

Solⁿ:

$$\bar{B} = \bar{\nabla}\times\bar{A} = \frac{\mu_0 A_0}{4\pi}\left[\hat{r}\,\frac{2\cos\theta}{r^2} + \hat{\theta}\,\frac{\lambda\sin\theta}{r}\right] e^{-\lambda r}$$

$$\bar{\nabla}\times\bar{B} = \mu_0 \bar{J}$$

$$\vec{J} = \frac{1}{\mu_0}\left\{\nabla\times\bar{B}\right\}$$

$$= \frac{A_0}{4\pi}\sin\theta\left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\}e^{-\lambda r}$$

$$\vec{m} = \frac{1}{2}\int(\bar{r}\times\bar{J})\,d^3r$$

$$= -\frac{A_0}{8\pi}\int d^3r\,\hat{\theta}\;r\sin\theta\left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\}e^{-\lambda r}$$

$$\hat{\theta} = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta$$

$$\vec{m} = +\frac{A_0}{8\pi}\hat{k}\int r^3\sin^3\theta\left(\frac{2}{r^3} - \frac{\lambda^2}{r}\right)e^{-\lambda r}\,dr\,d\theta$$

$$= \frac{A_0}{8\pi}\hat{k}\,\frac{4}{3}\left(\frac{2}{\lambda} - \frac{2}{\lambda}\right) = 0$$

4. A filamentary current loop traverses eight edges of a cube with side length $2b$ as shown in Figure 2. Find the magnetic dipole moment $\mathbf{m}$ of this structure.

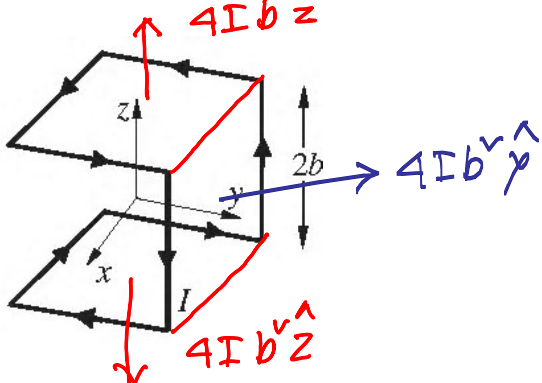


Figure 2: Problem3

$$\bar{m} = 4Ib^2\hat{y}$$

5. Another way to fill in the "missing link" in Figure 3 is to look for a magnetostatic analog to

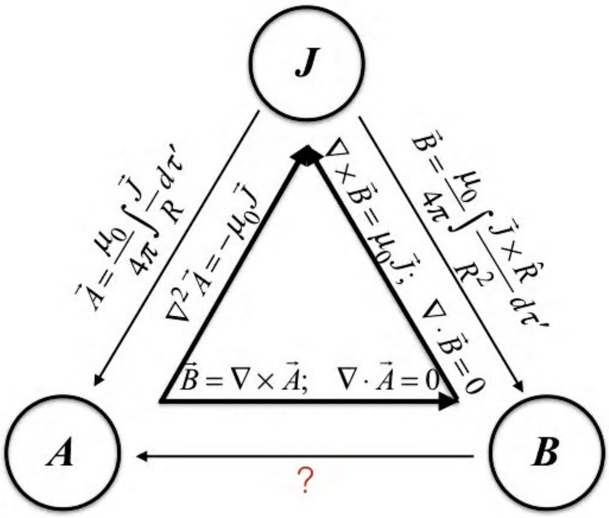


Figure 3: Problem 5

$$\Phi(\mathbf{r}) = -\int_{O}^{r} \mathbf{E} \cdot d\mathbf{l}.$$

The obvious candidate would be

$$\mathbf{A}(\mathbf{r}) = -\int_{O}^{r} \mathbf{B} \times d\mathbf{l}.$$

(a) Test this formula for the simplest possible case-uniform $\mathbf{B}$ (use the origin as your reference point). Is the result consistent with Prob. 2? You could cure this problem by throwing in a factor of 1/2, but the flaw in this equation runs deeper.

(b) Show that $\int \mathbf{B} \times d\mathbf{l}$ is not independent of path, by calculating $\oint (\mathbf{B} \times d\mathbf{l})$ around the rectangular loop shown in Figure.

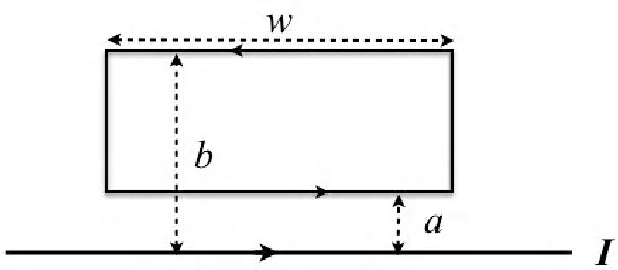


Figure 4: Problem 5(b)

(c) Use $\mathbf{A}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda r)\, d\lambda$ to find the vector potential for uniform B.

(d) Use $\mathbf{A}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda r)\, d\lambda$ to find the vector potential of an infinite straight wire carrying a steady current I. Does $\mathbf{A}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda r)\, d\lambda$ automatically satisfy $\nabla \cdot \mathbf{A} = 0$?

Sol: (a) For a uniform field $\bar{B}$

$$\int_0^r (\bar{B} \times d\bar{l}) = \bar{B} \times \int_0^r d\bar{l} = \bar{B} \times \bar{r}$$

$$A = \frac{1}{2}(\bar{r} \times \bar{B}) = -\frac{1}{2}(\bar{B} \times \bar{r})$$

(b) $$\bar{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$\oint \bar{B} \times d\bar{l} = \oint \left\{ \frac{\mu_0 I}{2\pi s}\hat{\phi} \right\} \times d\bar{l}$$

$$= \frac{\mu_0 I}{2\pi a}\hat{s} - \frac{\mu_0 I}{2\pi b}\hat{s} = \frac{\mu_0 I}{2\pi}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{s}$$

(c) $$\bar{A} = -\bar{r} \times \bar{B}\int_0^1 \lambda \, d\lambda = -\frac{\bar{r} \times \bar{B}}{2}$$

(d) $$\bar{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \qquad \bar{B}(\lambda s) = \frac{\mu_0 I}{2\pi \lambda s}\hat{\phi}$$

$$\bar{A} = -\frac{\mu_0 I}{2\pi s}(\bar{r} \times \hat{\phi})\int_0^1 \lambda \frac{1}{\lambda} d\lambda$$

$$= -\frac{\mu_0 I}{2\pi s}(\bar{r} \times \hat{\phi})$$

But $\vec{r}$ here is the vector from origin in cylinderical co-ordinates $\vec{r} = s\hat{s} + z\hat{z}$

$$\vec{A} = -\frac{\mu_0 I}{2\pi s}\left[s(\hat{s}\times\hat{\phi}) + z(\hat{z}\times\hat{\phi})\right]$$

$$= -\frac{\mu_0 I}{2\pi s}\left(z\hat{s} - s\hat{z}\right)$$

Let $\bar{L} \equiv \int_0^1 \lambda \bar{B}(\lambda r)\, d\lambda$

$$\nabla\cdot A = -\nabla\cdot(\bar{r}\times\bar{L})$$

$$= -\left[\bar{L}\cdot \nabla\times\bar{r} - \bar{r}\cdot(\nabla\times\bar{L})\right]$$

$$= r\cdot \nabla\times\bar{L}$$

$$\bar{\nabla}\times\bar{L} = \int_0^1 \lambda\,(\bar{\nabla}\times\bar{B})\,d\lambda = \mu_0\int_0^1 \lambda^2 J(\lambda r)\,d\lambda$$

$$\nabla\times B = \mu_0 J$$

$$\Rightarrow \quad \nabla\times B(\lambda r) = \mu_0 \lambda J(\lambda r)$$

$$\bar{\nabla}\cdot\bar{A} = \mu_0 \bar{r}\cdot\int_0^1 \lambda^2 J(\lambda r)\,d\lambda = 0$$

in the region where $J = 0$

To construct an explicit counter example, we need the field at a point where $J \neq 0$ — say inside a wire with current

$$B\, 2\pi s = \mu_0 I_{enc} \qquad B = \frac{\mu_0 J}{2} s\, \hat{\phi}$$

$$A = -\, r \times \int_0^1 \lambda \left(\frac{\mu_0 J}{2}\right) \lambda s \hat{\phi}\, d\lambda$$

$$= -\frac{\mu_0 J}{6} s\, (\bar{r} \times \hat{\phi})$$

$$= \frac{\mu_0 J s}{6} (z\hat{s} - s\hat{z})$$

$$\bar{\nabla} \cdot \bar{A} = \frac{\mu_0 J}{6}\left[\frac{1}{s}\frac{\partial}{\partial s}(s^2 z) + \frac{\partial}{\partial z}(-s^2)\right]$$

$$= \frac{\mu_0 J z}{3} \neq 0$$

Conclusion $\bar{A}(\bar{r}) = -\, r \times \int_0^1 \lambda B(\lambda r)\, d\lambda$ does not automatically yield $\nabla \cdot A = 0$