Physics II: Electromagnetism (PH102)

Lecture 1

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Syllabus

- Vector Calculus: Gradient, Divergence and Curl, Line, Surface, and Volume integrals, Gauss's divergence theorem and Stokes' theorem in Cartesian, Spherical polar and cylindrical polar coordinates, Dirac Delta function.
- Electrostatics: Gauss's law and its applications, Divergence and Curl of Electrostatic fields, Electrostatic Potential, Boundary conditions, Work and Energy, Conductors, Capacitors, Laplace's equation, Method of images, Boundary value problems in Cartesian Coordinate Systems, Dielectrics, Polarization, Bound Charges, Electric displacement, Boundary conditions in dielectrics, Energy in dielectrics, Forces on dielectrics. (upto mid sem)

- Magnetostatics: Lorentz force, Biot-Savart and Ampere's laws and their applications, Divergence and Curl of Magnetostatic fields, Magnetic vector Potential, Force and torque on a magnetic dipole, Magnetic materials, Magnetization, Bound currents, Boundary conditions.
- Electrodynamics: Ohm's law, Motional EMF, Faraday's law, Lenz's law, Self and Mutual inductance, Energy stored in magnetic field, Maxwell's equations, Continuity Equation, Poynting Theorem, Wave solution of Maxwell Equations.
- Electromagnetic waves: Polarization, reflection & transmission at oblique incidences.Electrostatics: Field, Potentials

References

Texts:

1. D. J. Griffiths, Introduction to Electrodynamics, 3rd Ed., Prentice-Hall of India, 2005.

References:

1. N. Ida, Engineering Electromagnetics, Springer, 2005.
2. M. N. O. Sadiku, Elements of Electromagnetics, Oxford, 2006.
3. R. P. Feynman, R. B. Leighton and M. Sands, The Feynman Lectures on Physics, Vol.II,Norosa Publishing House, 1998.
4. I. S. Grant and W. R. Phillips, Electromagnetism, John Wiley, 1990.

Lecture notes will be uploaded in moodle for your reference.

Schedule and grading

Two classes a week ->

Wednesday: 9:00 am (div III and iv) & 4:00 pm (div I and II),

Friday: 11:00 am (div III and iv) and 2:00 pm (div I and II),

The class on Thursday will be kept as backup.

One tutorial-> Tuesday: 8:00 am

For queries and doubts: `Office hours' Tuesday, 5:00-6:00 pm

Examination	Date	Marks
Quiz 1	30th Jan, 2018	10%
Mid-sem	3rd March, 2018	30%
Quiz 2	To be decided	10%
End-sem	4th May, 2018	50%

- 75% attendance is a must for appearing in end sem exam
- Attendance in tutorials is a must.

Why study Electromagnetism ?

However, before we start electromagnetism, let us go through some mathematical prerequisites that we will need in three dimensional universe !

Vectors and their algebra : addition, dot and cross products

- A quantity with magnitude and direction: $\vec{A}$

- Addition: method of parallelogram

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

Ex: Force, displacement, velocity ...

- Dot Product: scalar quantity $\vec{A}.\vec{B} = AB\cos\theta$

- Cross product: vector quantity:

$$\vec{A} \times \vec{B} = AB\sin\theta\hat{n}$$

Recall: scalars are only numbers

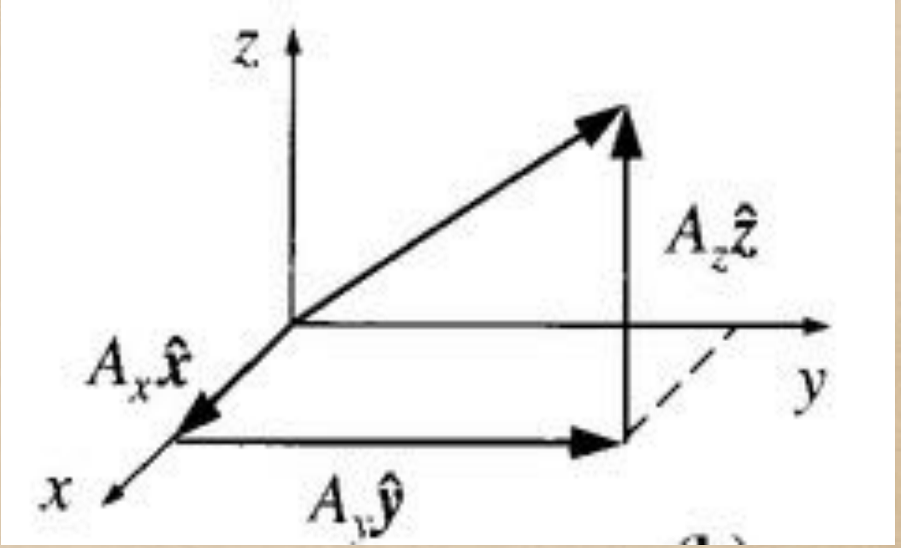
Component Notation of vectors

- Three components of a vector in 3 dimension

  Cartesian coordinates: $\hat{x}, \ \hat{y}, \ \hat{z}$ unit vectors.

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\vec{A} = |\vec{A}|\hat{a} \quad A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$



- Dot product :

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z \quad \hat{i}.\hat{i} = 1, \ \ \hat{i}.\hat{j} = 0, \ (i, j = x, y, z)$$

- Cross product:

$$\vec{A} \times \vec{B} = (A_xB_y - B_xA_y)\hat{z} + (A_zB_x - B_zA_x)\hat{y} + (A_yB_z - A_zB_y)\hat{x}$$

$$\hat{i} \times \hat{i} = 0, \ (i = x, y, z)$$

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector Identities

distributive laws: $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

$$\vec{A} \times (\vec{B}+\vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

Some more

$$\begin{aligned}
\vec{A} \times \vec{A} &= 0 \\
\vec{A}.(\vec{A} \times \vec{B}) &= 0 \\
\vec{A}.(\vec{B} \times \vec{C}) &= (\vec{A} \times \vec{B}).\vec{C} \\
\vec{A} \times (\vec{B} \times \vec{C}) &= \vec{B}\,(\vec{A}.\vec{C}) - \vec{C}\,(\vec{A}.\vec{B})
\end{aligned}$$

One can prove all the identities traditionally, however life becomes much easier with Levi-Ci-Vita symbol

Levi ci vita Symbol

Dot product in component notation

$$\vec{A}.\vec{B} = \sum_i A_i B_i$$

$$= A_x B_x + A_y B_y + A_z B_z$$

Cross-product in component notation

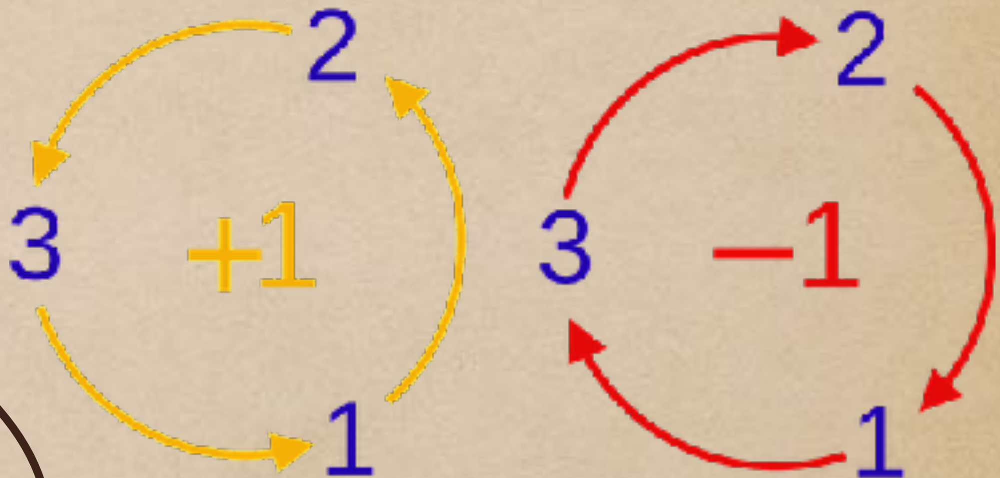
$$(\vec{A} \times \vec{B})_i = \sum_{j,k} \epsilon_{ijk} A_j B_k$$

$$(\vec{A} \times \vec{B})_x = \epsilon_{xyz} A_y B_z + \epsilon_{xzy} A_z B_y$$

$$(\vec{A} \times \vec{B})_y = \epsilon_{yxz} A_x B_z + \epsilon_{yzx} A_z B_x$$

$$(\vec{A} \times \vec{B})_z = \epsilon_{zxy} A_x B_y + \epsilon_{zyx} A_y B_x$$

Levi-ci-vita symbol



$$\epsilon_{xyz} = \epsilon_{yzx} = \epsilon_{zxy} = 1$$

$$\epsilon_{xzy} = \epsilon_{zyx} = \epsilon_{yxz} = -1$$

$$\epsilon_{iij} = \epsilon_{iii} = 0$$

Using Levi ci vita: An example

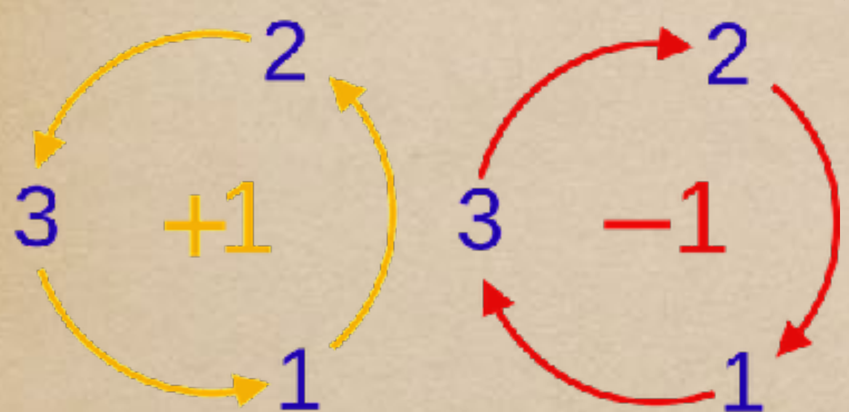
Let us show that : $\vec{A}.(\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}).\vec{C}$

$$LHS = A_i \epsilon_{ijk} B_j C_k = \epsilon_{ijk} A_i B_j C_k$$

$$RHS = \epsilon_{ijk} A_j B_k C_i = \epsilon_{kij} A_i B_j C_k$$
$$= \epsilon_{ijk} A_i B_j C_k$$

(i,j,k are dummy)

Using cyclic permutation



Note: We drop the summation notation over repeated indices to avoid clutter

Useful identity:

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\delta_{ij} = 0 \text{ for } i \neq j$$
$$\delta_{ij} = 1 \text{ for } i = j$$

Kronecker delta

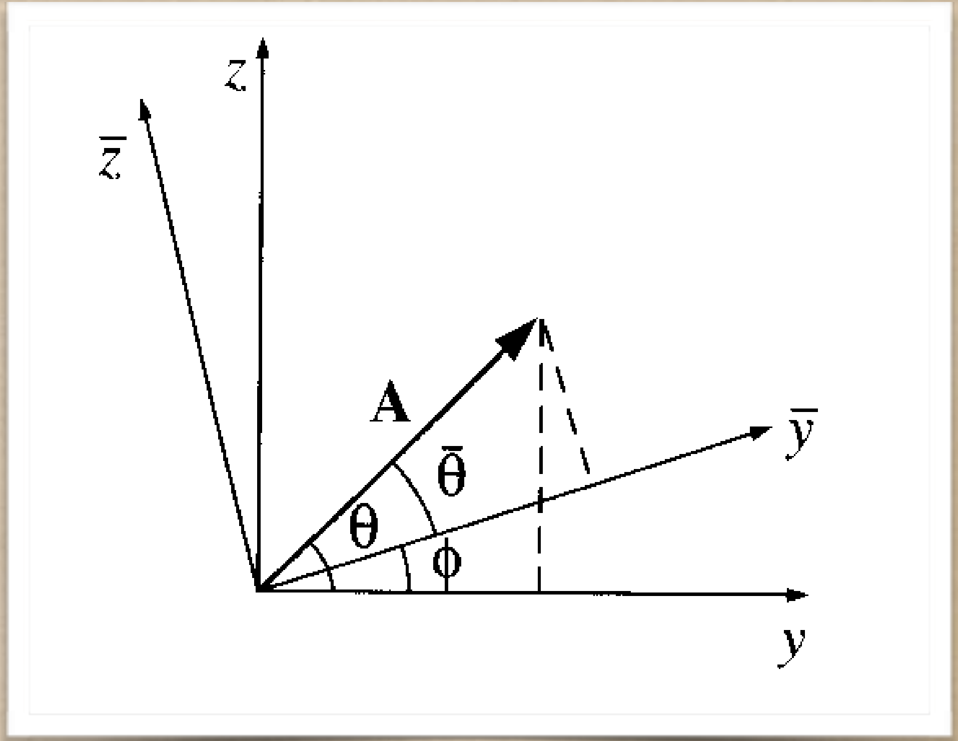
How vectors transform ?

As vectors have components corresponding to each unit vectors/choice of basis, the components transform under coordinate transformation

- Rotation about x-axis:

$$
\begin{aligned}
A_y &= A\cos\theta, \quad A_z = A\sin\theta \\
\bar{A}_y &= A\cos\bar{\theta} = A\cos(\theta-\phi) \\
&= \cos\phi A_y + \sin\phi A_z \\
\bar{A}_z &= -\sin\phi A_y + \cos\phi A_z
\end{aligned}
$$

$$\begin{pmatrix} \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$



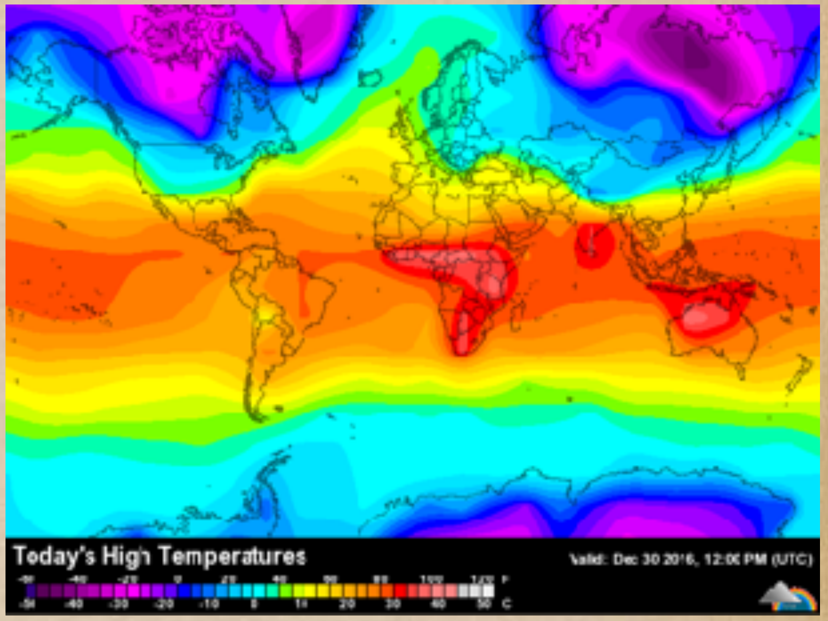
- For rotation about arbitrary axis:

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \implies \bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j$$

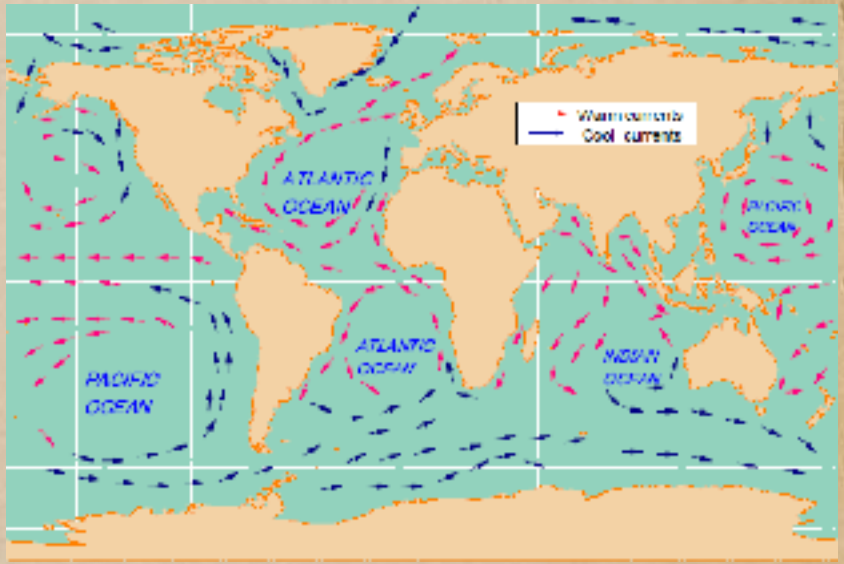
However the vector remains unchanged while the components change due to coordinate transformation

Concepts of field

- Scalar Field: Associate a number at every point in space -> T(x,y,z); Example: Temperature



- Vector Field: Associate a vector at each point in space -> $\vec{A}(x, y, z)$ Example: Gravitational field, electromagnetic field, ocean current etc



Scalar and vector fields are functions of more than one variable

Differential calculus for scalar field: Gradient

Infinitesimal change in a scalar field :

$$dT(x,y,z) = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$= \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right).(dx\hat{x} + dy\hat{y} + dz\hat{z})$$

involves partial derivatives

$$= \vec{\nabla}T.\vec{dl} = |\vec{\nabla}T||\vec{dl}|\cos\theta$$

for fixed $|\vec{dl}|$, dT is max when $\theta = 0$

$$\boxed{\vec{\nabla}T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}}$$

Gradient of a scalar field

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Vector differential operator or nabla

Gradient of a scalar: Vector which points in the direction of maximum increase of a function T(x,y,z) and the magnitude gives the slope !

An immediate corollary on gradient

Show that $\vec{\nabla}\phi(x,y,z)$ is a vector perpendicular to the surface $\phi(x,y,z)=C$ where C is a constant

$$d\phi = \vec{\nabla}\phi.\vec{dl} = 0$$

Hence $\vec{\nabla}\phi$ is perpendicular to infinitesimal displacement vector $\vec{dl}$ in the constant ϕ plane.

$\vec{\nabla}\phi(x,y,z) = 0$ in the maximum, minimum or in the saddle point !

An example on gradient...

Suppose the height of the hill (in feet) behind the CC is $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$, where y is the distance (in miles) north, x the distance west of the IITG admin. building Where is the top of the hill and how high is the hill? How steep is the slope 1 mile north and 1 mile west of admin building?

Let's calculate the gradient !

$$\begin{aligned} \vec{\nabla} h(x,y) &= \frac{\partial h}{\partial x}\hat{x} + \frac{\partial h}{\partial y}\hat{y} \\ &= 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y} \end{aligned}$$

If the hilltop is located at $(\bar{x}, \bar{y})$, then $\vec{\nabla} h|_{(\bar{x},\bar{y})} = 0$

(condition for maxima)

This gives
$$\begin{aligned} 2\bar{y} - 6\bar{x} - 18 &= 0 \\ 2\bar{x} - 8\bar{y} + 28 &= 0. \end{aligned} \quad \Longrightarrow \quad \bar{x} = -2 \text{ and } \bar{y} = 3$$

The hill top is 3 miles north and 2 miles east of the admin building.

Height $h(-2,3) = 720$ feet.

Differential calculus for vector: Divergence of a vector

$$\vec{\nabla}.\vec{V} = (\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}).(V_x\ \hat{x} + V_y\ \hat{y} + V_z\ \hat{z})$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

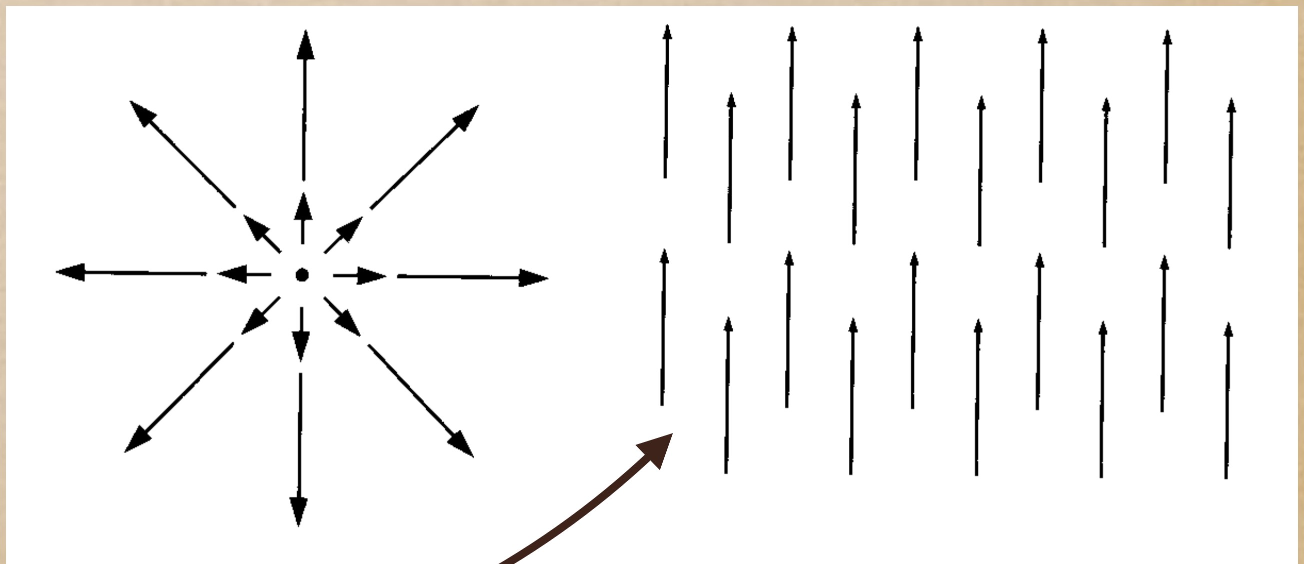
take a scalar product of a vector with nabla operator

$$\boxed{(\vec{\nabla}.\vec{V}) = \sum_i \nabla_i V_i}$$

component notation

Divergence measures how much a vector is spread out

A vector field is called solenoidal if the divergence is zero.



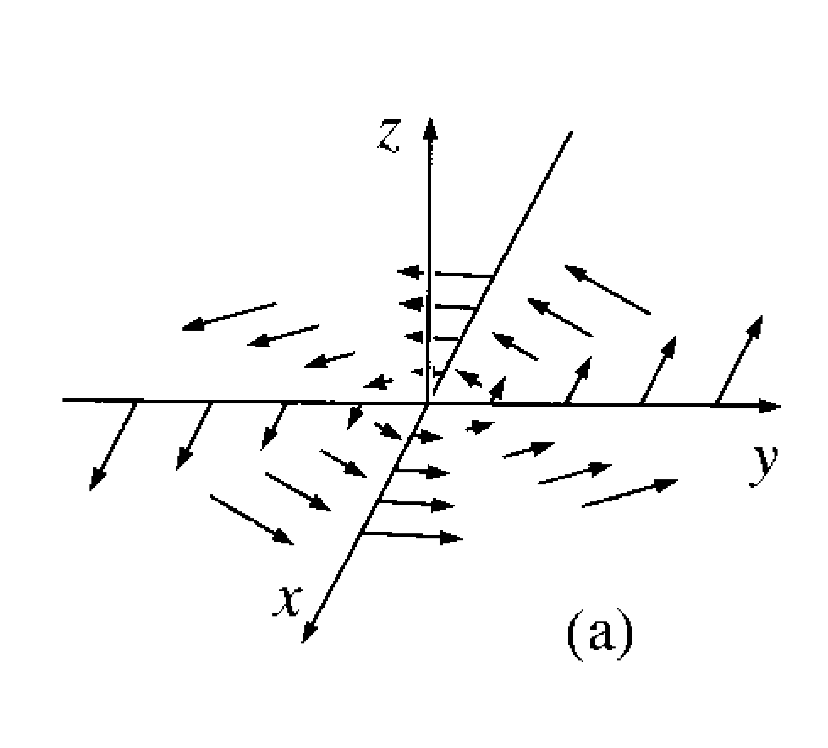
(a) (b)

Curl of a vector

Take a cross product of a vector with nabla operator

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= (\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z})\hat{x} + (\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x})\hat{y} + (\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})\hat{z}$$



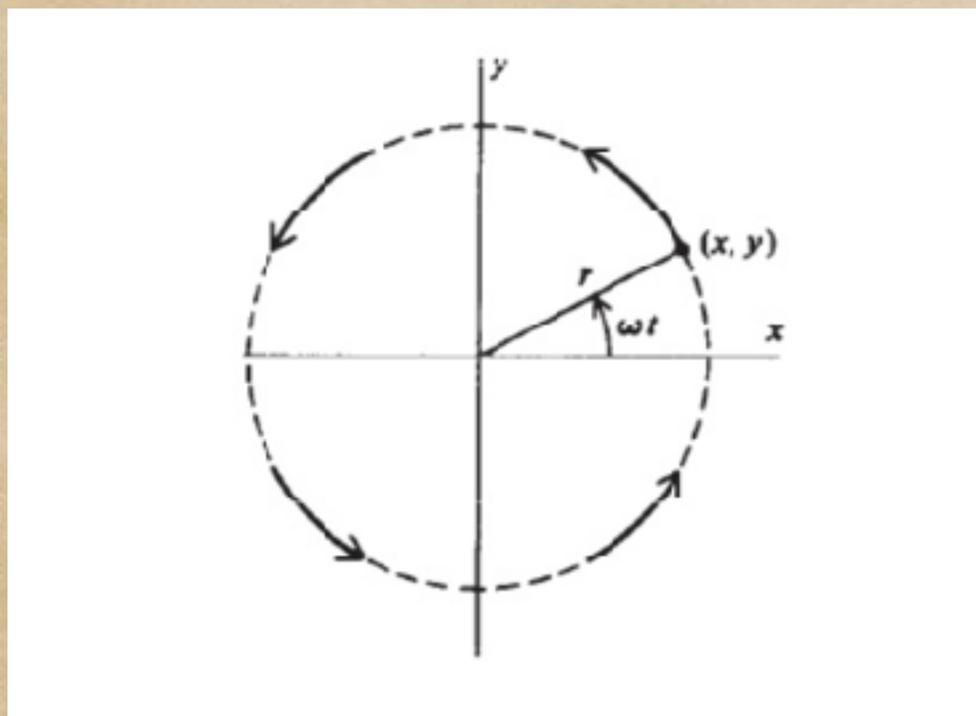
A vector field with non-zero curl

Curl of a vector measures how much the vector swirls around the point at which the curl is measured

$$\boxed{(\vec{\nabla} \times \vec{V})_i = \epsilon_{ijk} \nabla_j V_k}$$

using levi ci vita symbol as before

How does curl represent swirling of a vector field ?



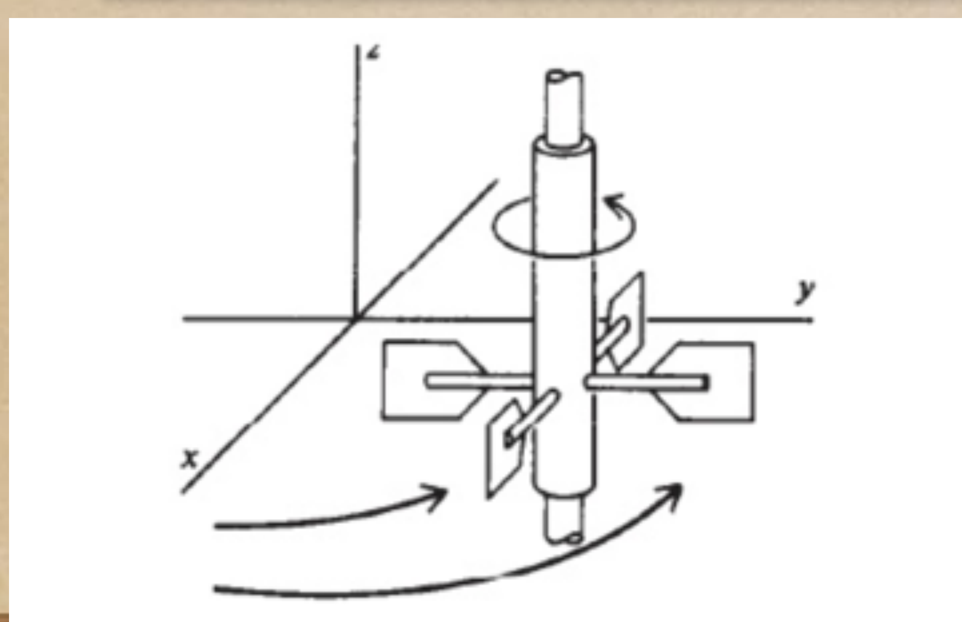
- Lets take an example of rotation

$$x = r\cos\omega t, \;\; y = r\sin\omega t$$

- velocity vector field:

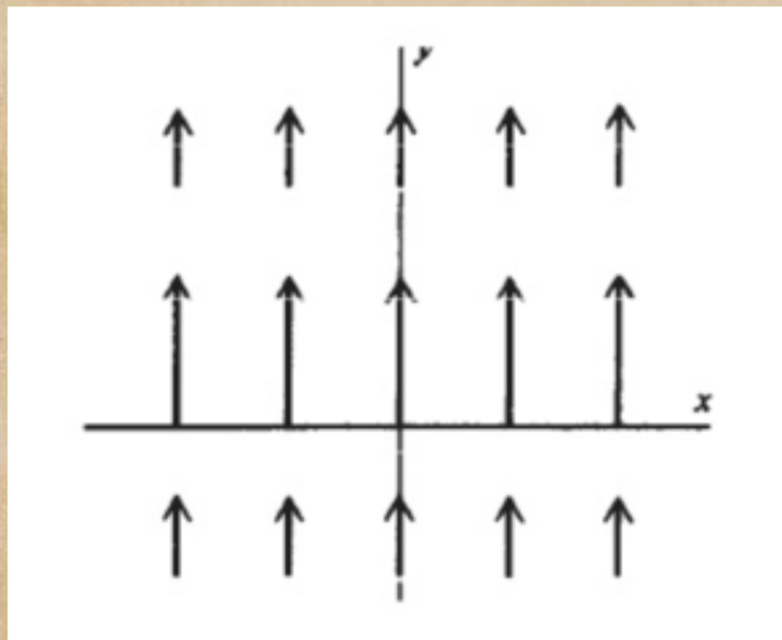
$$\begin{aligned}\vec{v} &= \hat{x}(\frac{dx}{dt}) + \hat{y}(\frac{dy}{dt}) \\ &= r\omega[-\hat{x}\sin\omega t + \hat{y}\cos\omega t] \\ &= \omega(-\hat{x}y + \hat{y}x)\end{aligned}$$

Curl of velocity vector field: $\vec{\nabla}\times\vec{v} = 2\hat{z}\omega$ Non-zero !!



Curl of a vector field can be ideated by imagining a paddlewheel in the field. If it starts spinning, then curl is non-zero. The direction of rotation will yield the direction of curl.

A more interesting example on curl !!



- Imagine a vector field : $\vec{v} = \hat{y}\ v_0\ e^{-\frac{y^2}{\lambda^2}}$

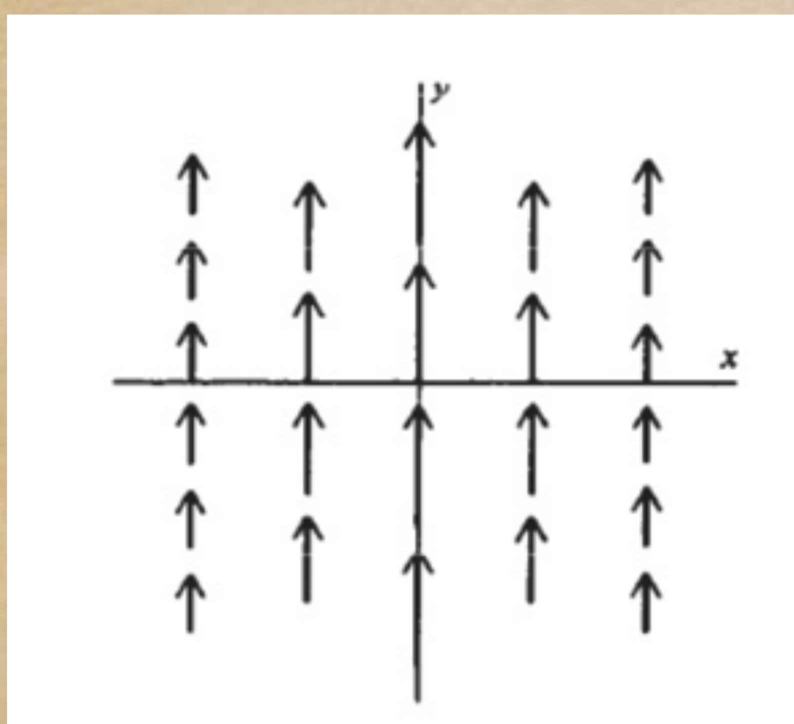
  The field lines point along y although the magnitude varies with y !

$$\boxed{\vec{\nabla} \times \vec{v} = 0}$$

as per expectation

- However, if we consider, $\vec{v} = \hat{y}\ v_0\ e^{-\frac{x^2}{\lambda^2}}$

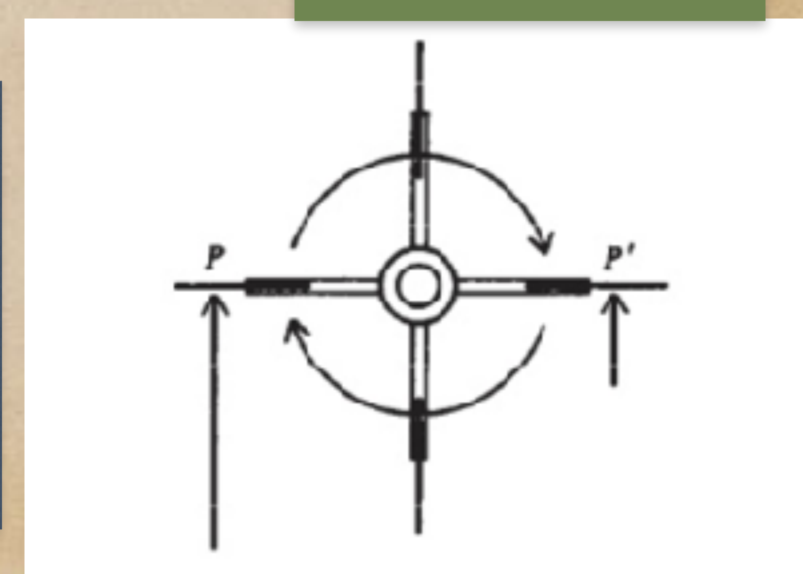
field lines still pointing along y, but magnitude varies with x



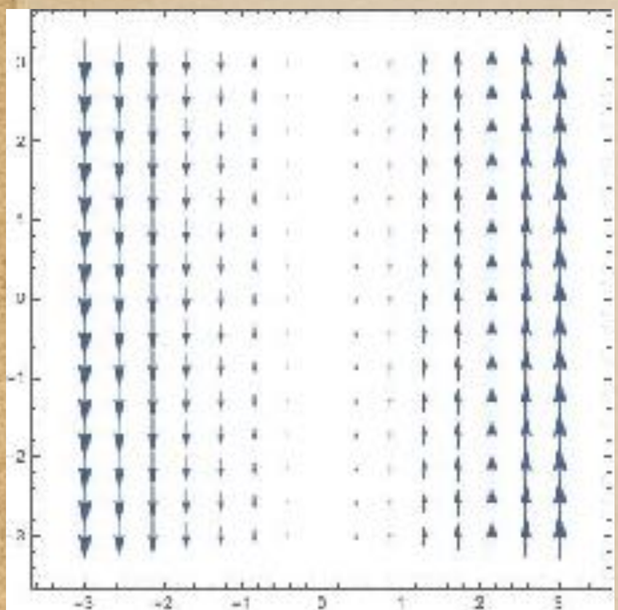
$$\boxed{\vec{\nabla} \times \vec{v} = -\hat{z}\ v_0\ \frac{2x}{\lambda^2}\ e^{-\frac{x^2}{\lambda^2}}}$$

Then curl is non-zero ! and is counter intuitive

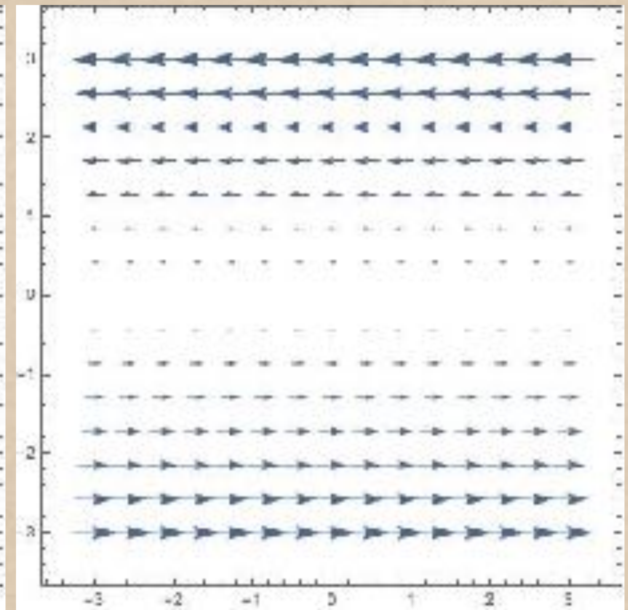
This can be understood by the non-zero torque on the paddlewheel in the field !



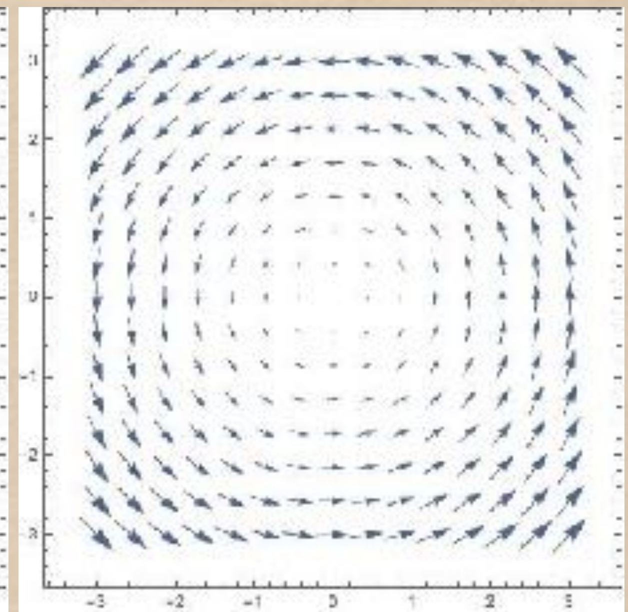
A few more examples !



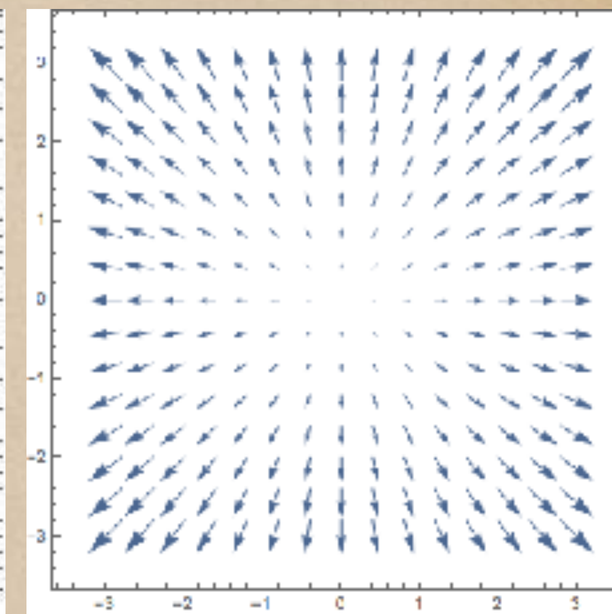
$\vec{V}_1 = x\ \hat{y}$



$\vec{V}_2 = -y\ \hat{x}$



$\vec{V}_3 = -y\ \hat{x} + x\ \hat{y}$



$\vec{V}_4 = x\ \hat{x} + y\ \hat{y}$

$\vec{\nabla} \times \vec{V}_1 = 1\hat{z}$ $\quad$ $\vec{\nabla} \times \vec{V}_2 = 1\hat{z}$ $\quad$ $\vec{\nabla} \times \vec{V}_3 = 2\hat{z}$ $\quad$ $\vec{\nabla} \times \vec{V}_4 = 0$

$$\vec{\nabla}.\vec{V}_1 = 0$$
$$\vec{\nabla}.\vec{V}_2 = 0$$
$$\vec{\nabla}.\vec{V}_3 = 0$$

$$\vec{\nabla}.\vec{V}_4 \neq 0$$

solenoidal fields

A vector field is irrotational if the curl is zero

Some identities involving Gradient , Divergence and Curl

$$\vec{\nabla}.(\vec{A}+\vec{B}) = \vec{\nabla}.\vec{A}+\vec{\nabla}.\vec{B}$$ (This also applies to curl)

$$\vec{\nabla}(f+g) = \vec{\nabla}f+\vec{\nabla}g$$

$$\vec{\nabla}(fg) = f\vec{\nabla}g+g\vec{\nabla}f$$

$$\vec{\nabla}.(f\vec{A}) = f\vec{\nabla}.\vec{A}+\vec{A}.\vec{\nabla}f$$

$$\vec{\nabla}.(\vec{A}\times\vec{B}) = \vec{B}.(\vec{\nabla}\times\vec{A})-\vec{A}.(\vec{\nabla}\times\vec{B})$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}.\vec{\nabla})\vec{A}-(\vec{A}.\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}.\vec{B})-\vec{B}(\vec{\nabla}.\vec{A})$$

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A})-\vec{A}\times\vec{\nabla}f$$

Second Derivatives

(i) Divergence of gradient of a scalar

$$\vec{\nabla}.(\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

Laplacian operator

(ii) Curl of gradient of a scalar

$$\vec{\nabla} \times (\vec{\nabla} f) = 0$$

$$= \epsilon_{ijk}\partial_j\partial_k f = -\epsilon_{ikj}\partial_j\partial_k f$$ (antisymmetry of levi ci vita)

$$= -\epsilon_{ijk}\partial_k\partial_j f = -\epsilon_{ijk}\partial_j\partial_k f$$ (using dummy)

(iii) Divergence of curl of a vector

$$\vec{\nabla}.(\vec{\nabla} \times \vec{V}) = 0$$

(iv) Gradient of divergence $\vec{\nabla}(\vec{\nabla}.\vec{V})$

(v) Curl of curl of a vector

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla}.\vec{V}) - \nabla^2\vec{V}$$

To summarise…

- Theory of Electromagnetism involves the idea of scalar and vector fields
- Vector analysis is simpler using component notation in particular with levi ci vita symbols for cross products
- Differential calculus for fields relies on three important quantities: Gradient, Divergence and Curl involving vector differential operators nabla $\vec{\nabla}$
- Scalar double derivative: Laplacian; Curl of gradient of a scalar and divergence of curl of a vector are identically zero

Note: Although we may evaluate the dot product $\vec{V}.\vec{\nabla}$ and cross product $\vec{V} \times \vec{\nabla}$ involving nabla, they are neither divergence nor curl of those vectors!