# Power Series Solutions to the Bessel Equation

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# The Bessel equation

The equation

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0, \tag{1}$$

where $\alpha$ is a nonnegative constant, is called the Bessel equation of order $\alpha$.

The point $x_0 = 0$ is a regular singular point. We shall use the method of Frobenius to solve this equation.

Thus, we seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad x > 0, \tag{2}$$

with $a_0 \neq 0$.

Differentiation of (2) term by term yields

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}.$$

Similarly, we obtain

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Substituting these into (1), we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} \alpha^2 a_n x^{n+r} = 0.$$

This implies

$$x^r \sum_{n=0}^{\infty} [(n+r)^2 - \alpha^2] a_n x^n + x^r \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Now, cancel $x^r$, and try to determine $a_n$'s so that the coefficient of each power of $x$ will vanish.

For the constant term, we require $(r^2 - \alpha^2) a_0 = 0$. Since $a_0 \neq 0$, it follows that

$$r^2 - \alpha^2 = 0,$$

which is the indicial equation. The only possible values of $r$ are $\alpha$ and $-\alpha$.

Case I. For $r = \alpha$, the equations for determining the coefficients are:

$$[(1+\alpha)^2 - \alpha^2]a_1 = 0 \text{ and,}$$
$$[(n+\alpha)^2 - \alpha^2]a_n + a_{n-2} = 0, \quad n \geq 2.$$

Since $\alpha \geq 0$, we have $a_1 = 0$. The second equation yields

$$a_n = -\frac{a_{n-2}}{(n+\alpha)^2 - \alpha^2} = -\frac{a_{n-2}}{n(n+2\alpha)}. \tag{3}$$

Since $a_1 = 0$, we immediately obtain

$$a_3 = a_5 = a_7 = \cdots = 0.$$

For the coefficients with even subscripts, we have

$$a_2 = \frac{-a_0}{2(2+2\alpha)} = \frac{-a_0}{2^2(1+\alpha)},$$
$$a_4 = \frac{-a_2}{4(4+2\alpha)} = \frac{(-1)^2 a_0}{2^4 2!(1+\alpha)(2+\alpha)},$$
$$a_6 = \frac{-a_4}{6(6+2\alpha)} = \frac{(-1)^3 a_0}{2^6 3!(1+\alpha)(2+\alpha)(3+\alpha)},$$

and, in general

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(1+\alpha)(2+\alpha)\cdots(n+\alpha)}.$$

Therefore, the choice $r = \alpha$ yields the solution

$$y(x) = a_0 x^{\alpha}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1+\alpha)(2+\alpha)\cdots(n+\alpha)}\right).$$

Note: The ratio test shows that the power series formula converges for all $x \in \mathbb{R}$.

For $x < 0$, we proceed as above with $x^r$ replaced by $(-x)^r$. Again, in this case, we find that $r$ satisfies

$$r^2 - \alpha^2 = 0.$$

Taking $r = \alpha$, we obtain the same solution, with $x^\alpha$ is replaced by $(-x)^\alpha$. Therefore, the function $y_\alpha(x)$ is given by

$$y_\alpha(x) = a_0|x|^\alpha \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1+\alpha)(2+\alpha)\cdots(n+\alpha)}\right) \tag{4}$$

is a solution of the Bessel equation valid for all real $x \neq 0$.

Case II. For $r = -\alpha$, determine the coefficients from

$$[(1-\alpha)^2 - \alpha^2]a_1 = 0 \text{ and } [(n-\alpha)^2 - \alpha^2]a_n + a_{n-2} = 0.$$

These equations become

$$(1-2\alpha)a_1 = 0 \text{ and } n(n-2\alpha)a_n + a_{n-2} = 0.$$

If $2\alpha$ is not an integer, these equations give us

$$a_1 = 0 \text{ and } a_n = -\frac{a_{n-2}}{n(n-2\alpha)}, \quad n \geq 2.$$

Note that this formula is same as (3), with $\alpha$ replaced by $-\alpha$. Thus, the solution is given by

$$y_{-\alpha}(x) = a_0|x|^{-\alpha}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1-\alpha)(2-\alpha)\cdots(n-\alpha)}\right), \tag{5}$$

which is valid for all real $x \neq 0$.

# Euler's gamma function and its properties

For $s \in \mathbb{R}$ with $s > 0$, we define $\Gamma(s)$ by

$$\Gamma(s) = \int_{0+}^{\infty} t^{s-1} e^{-t} dt.$$

The integral converges if $s > 0$ and diverges if $s \leq 0$.
Integration by parts yields the functional equation

$$\Gamma(s+1) = s\Gamma(s).$$

In general,

$$\Gamma(s+n) = (s+n-1)\cdots(s+1)s\Gamma(s), \text{ for every } n \in \mathbb{Z}^+.$$

Since $\Gamma(1) = 1$, we find that $\Gamma(n+1) = n!$. Thus, the gamma function is an extension of the factorial function from integers to positive real numbers. Therefore, we write

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \quad s \in \mathbb{R}^+.$$

Using this gamma function, we shall simplify the form of the solutions of the Bessel equation. With $s = 1 + \alpha$, we note that

$$(1+\alpha)(2+\alpha)\cdots(n+\alpha) = \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)}.$$

Choose $a_0 = \frac{2^{-\alpha}}{\Gamma(1+\alpha)}$ in (4), the solution for $x > 0$ can be written

$$J_\alpha(x) = \left(\frac{x}{2}\right)^{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1+\alpha)} \left(\frac{x}{2}\right)^{2n}.$$

The function $J_\alpha$ defined above for $x > 0$ and $\alpha \geq 0$ is called the Bessel function of the first kind of order $\alpha$.

When $\alpha$ is a nonnegative integer, say $\alpha = p$, the Bessel function $J_p(x)$ is given by

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p}, \quad (p = 0, 1, 2, \ldots).$$

This is also a solution of the Bessel equation for $x < 0$.

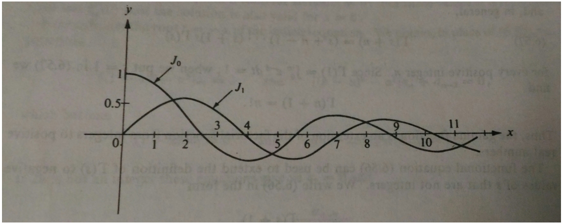


Figure : The Bessel functions $J_0$ and $J_1$.

If $\alpha > 0$, $\alpha \notin \mathbb{Z}^+$, define a new function $J_{-\alpha}(x)$ (replacing $\alpha$ by $-\alpha$)

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1-\alpha)} \left(\frac{x}{2}\right)^{2n}, \; x > 0.$$

With $s = 1 - \alpha$, we note that

$$\Gamma(n+1-\alpha) = (1-\alpha)(2-\alpha)\cdots(n-\alpha)\Gamma(1-\alpha).$$

Thus, the series for $J_{-\alpha}(x)$ is the same as that for $y_{-\alpha}(x)$ in (5) with $a_0 = \frac{2^\alpha}{\Gamma(1-\alpha)}$, $x > 0$. If $\alpha$ is not positive integer, $J_{-\alpha}$ is a solution of the Bessel equation for $x > 0$.

If $\alpha \notin \mathbb{Z}^+$, $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent on $x > 0$. The general solution of the Bessel equation for $x > 0$ is

$$y(x) = c_1 J_\alpha(x) + c_2 J_{-\alpha}(x).$$

# Useful recurrence relations for $J_\alpha$

- $\frac{d}{dx}(x^\alpha J_\alpha(x)) = x^\alpha J_{\alpha-1}(x).$

$$\begin{aligned}
\frac{d}{dx}(x^\alpha J_\alpha(x)) &= \frac{d}{dx}\left\{x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\alpha+n)}\left(\frac{x}{2}\right)^{2n+\alpha}\right\} \\
&= \frac{d}{dx}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2\alpha}}{n!\,\Gamma(1+\alpha+n)2^{2n+\alpha}}\right\} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+2\alpha)x^{2n+2\alpha-1}}{n!\,\Gamma(1+\alpha+n)2^{2n+\alpha}}.
\end{aligned}$$

Since $\Gamma(1+\alpha+n) = (\alpha+n)\Gamma(\alpha+n)$, we have

$$\begin{aligned}
\frac{d}{dx}(x^\alpha J_\alpha(x)) &= \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+2\alpha-1}}{n!\,\Gamma(\alpha+n)2^{2n+\alpha}} \\
&= x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+(\alpha-1)+n)}\left(\frac{x}{2}\right)^{2n+\alpha-1} \\
&= x^\alpha J_{\alpha-1}(x).
\end{aligned}$$

The other relations involving $J_\alpha$ are:

- $\frac{d}{dx}(x^{-\alpha}J_\alpha(x)) = -x^{-\alpha}J_{\alpha+1}(x)$.
- $\frac{\alpha}{x}J_\alpha(x) + J'_\alpha(x) = J_{\alpha-1}(x)$.
- $\frac{\alpha}{x}J_\alpha(x) - J'_\alpha(x) = J_{\alpha+1}(x)$.
- $J_{\alpha-1}(x) + J_{\alpha+1}(x) = \frac{2\alpha}{x}J_\alpha(x)$.
- $J_{\alpha-1}(x) - J_{\alpha+1}(x) = 2J'_\alpha(x)$.

Note: Workout these relations.

*** End ***