Homogeneous Linear Systems With Constant Coefficients

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Consider the homogeneous system

$$\mathbf{x}'(t) = A\mathbf{x}(t), \tag{1}$$

where A is a real $n \times n$ matrix.
Goal: To find a fundamental solution set for (1).

We seek solutions of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where λ is a constant and $\mathbf{v}$ is a constant vector such that

$$\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v} \Longrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Thus,

$$\begin{aligned} \mathbf{x}(t) \quad &= \quad e^{\lambda t}\mathbf{v} \text{ is a solution of } \mathbf{x}'(t) = A\mathbf{x}(t) \\ &\Longleftrightarrow \quad \lambda \text{ and } \mathbf{v} \text{ satisfy } (A - \lambda I)\mathbf{v} = \mathbf{0}. \end{aligned}$$

$(A - \lambda I)\mathbf{v} = \mathbf{0}$ has a nontrivial solution $\iff$ $\det(A - \lambda I) = \mathbf{0}$.

Thus

$$\lambda \text{ is an eigenvalue of } A \iff \mathbb{P}(\lambda) = 0,$$

where $\mathbb{P}(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of A.

Finding the eigenvalues of A is equivalent to finding the zeros of $\mathbb{P}(\lambda)$. $\mathbb{P}(\lambda) = 0$ is called the characteristics equation of A.

Note that $e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$ if λ is an eigenvalue and $\mathbf{v}$ is a corresponding eigenvector.

Q. Can we obtain n linear independent solutions to $\mathbf{x}' = A\mathbf{x}$ by finding all the eigenvalues and eigenvectors of A?

Some essential results from linear algebra

Theorem: Let A be an $n \times n$ matrix. The following statements are equivalent:

- A is singular.
- $\det A = 0$.
- $A\mathbf{x} = 0$ has nontrivial solution ($\mathbf{x} \neq \mathbf{0}$).
- The columns of A form a linearly dependent set.

Definition: (Eigenvalues and Eigenvectors)
The numbers λ for which

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

has at least one nontrivial solution $\mathbf{v}$ are called eigenvalues of A. The corresponding nontrivial solutions are called the eigenvectors of A associated with λ.

Example: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = 0.$$

$\lambda_1 = 1, \ \ \lambda_2 = -1.$ To find the eigenvectors corresponding to $\lambda_1 = 1$, we solve

$$(A - \lambda_1 I)\mathbf{v} = \mathbf{0} \Longrightarrow \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The eigenvector associated with $\lambda_1 = 1$ is

$$\mathbf{v}_1 = r \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R}.$$

Similarly, for $\lambda_2 = -1$, we solve

$$(A - \lambda_2 I)\mathbf{v} = \mathbf{0} \Longrightarrow \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The eigenvector associated with $\lambda_2 = -1$ is

$$\mathbf{v}_2 = r \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R}.$$

Theorem: If $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of A and $\mathbf{v}_i$ is an eigenvectors associated with λ_i, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Finding the general solution to $\mathbf{x}' = A\mathbf{x}$

Theorem: Suppose $A = (a_{ij})_{n\times n}$ has n linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$. Let λ_i be the eigenvalue corresponding to $\mathbf{v}_i$. Then

$$\{e^{\lambda_1 t}\mathbf{v}_1, e^{\lambda_2 t}\mathbf{v}_2, \ldots, e^{\lambda_n t}\mathbf{v}_n\}$$

is a fundamental solution set on $\mathbb{R}$ for $\mathbf{x}' = A\mathbf{x}$. Then the general solution (GS) of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n,$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Proof.

$$W(t) = \det[e^{\lambda_1 t}\mathbf{v}_1, \ldots, e^{\lambda_n t}\mathbf{v}_n] = e^{(\lambda_1+\cdots+\lambda_n)t}\det[\mathbf{v}_1, \ldots, \mathbf{v}_n] \neq 0.$$

Thus, $\{e^{\lambda_1 t}\mathbf{v}_1, e^{\lambda_2 t}\mathbf{v}_2, \ldots, e^{\lambda_n t}\mathbf{v}_n\}$ is a fundamental solution set and hence, the GS is given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n.$$

Example: Find the GS of

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \text{where } A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. The corresponding eigenvectors (with $r = 1$) are

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The GS is

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Uncoupling Normal Systems

We know the GS to scalar equation $x'(t) = ax(t)$ is $x(t) = ce^{at}$, where $c = x(0)$.

The easiest normal systems to solve are systems of the form

$$\mathbf{x}'(t) = D\mathbf{x}(t),$$

where D is an $n \times n$ diagonal matrix. Such a system actually consists of n uncoupled equations

$$x_i'(t) = d_{ii}x_i(t), \quad i = 1, \ldots, n,$$

whose solution is

$$x_i(t) = c_i e^{d_{ii}t},$$

where the c_i's are arbitrary constants.

Example: Consider the uncoupled system

$$\begin{aligned} x_1'(t) &= -x_1(t) \\ x_2'(t) &= 2x_2(t). \end{aligned}$$

Writing this system in the matrix form $\mathbf{x}'(t) = A\mathbf{x}(t)$, where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The method of separation of variables yield the GS

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{2t} \end{aligned}$$

In matrix form

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \mathbf{c}, \quad \text{where} \quad \mathbf{c} = \mathbf{x}(0).$$

Diagonalization Technique

The diagonalization technique is used to reduce the linear system

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

to an uncoupled linear system.

Theorem: If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are real distinct, then any set of corresponding eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ form a basis for $\mathbb{R}^n$. The matrix

$$P = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$$

is invertible and $P^{-1}AP = \mathrm{diag}[\lambda_1, \ldots, \lambda_n]$.

Reducing the system $\mathbf{x}' = A\mathbf{x}$ to an uncoupled system:

Define $\mathbf{y} = P^{-1}\mathbf{x}$. Then $\mathbf{x} = P\mathbf{y}$.

Now,

$$\begin{aligned} \mathbf{y}' &= P^{-1}\mathbf{x}' \\ &= P^{-1}AP\mathbf{y} \\ &= \mathrm{diag}[\lambda_1, \ldots, \lambda_n]\mathbf{y}. \end{aligned}$$

The uncoupled linear system has the solution

$$\mathbf{y}(t) = \mathrm{diag}[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}]\mathbf{y}(0)$$

Since $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$ and $\mathbf{x}(t) = P\mathbf{y}(t)$, it follows that

$$\mathbf{x}(t) = P\,\mathrm{diag}[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}]\,P^{-1}\mathbf{x}(0).$$

Example: Consider $x_1' = -x_1 - 3x_2$; $x_2' = 2x_2$. Here

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$. A pair of corresponding eigenvectors is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The matrix

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Note that $P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

We obtain the uncoupled linear system

$$y_1' = -y_1 \quad y_2' = 2y_2.$$

The GS is given by

$$y_1(t) = c_1 e^{-t}, \quad y_2(t) = c_2 e^{2t}.$$

The GS to the original system is

$$\mathbf{x}(t) = P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}\mathbf{c}, \quad \mathbf{c} = \mathbf{x}(0).$$

$$x_1(t) = c_1 e^{-t} + c_2(e^{-t} - e^{2t}), \;\; x_2(t) = c_2 e^{2t}.$$

*** End ***