MA 102 (Ordinary Differential Equations)

IIT Guwahati

Tutorial Sheet No. 2,3 **Date:** March 22, 2018

Exact differential equations; Integrating Factors; Higher-order linear IVPs; Wronskian.

(1) Find the value of n such that the curves $x^n + y^n = c_1$ are the orthogonal trajectories of the family $y = \dfrac{x}{1 - c_2 x}$, where c_1 and c_2 are arbitrary constants.

Solution: $y' = 1/(1 - c_2 x)^2$. Since $1 - c_2 x = x/y$, we have $y' = \frac{y^2}{x^2}$, which is the differential equation of the given family of curves. The differential equation of the orthogonal trajectories is $y' = -\frac{x^2}{y^2}$. Separating variables and integrating we obtain the family of orthogonal trajectories $x^3 + y^3 = c_1$. Thus, $n = 3$.

(2) Determine the largest interval (a, b) in which the given IVP is certain to have a unique solution:
(a) $e^x y'' - \frac{y'}{x-3} + 3y = \ln x,\ y(1) = 3,\ y'(1) = 2.$
(b) $(1 - x)y'' - 3xy' + 3y = \sin x,\ y(0) = 1,\ y'(0) = 1.$
(c) $x^2 y'' + 4y = \cos x,\ y(1) = 0,\ y'(1) = -1.$

Solution: (a) $(0, 3)$; (b) $(-\infty, 1)$; (c) $(0, \infty)$.

(3) Let y_1 and y_2 be two solutions of $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$ defined in the interval $[a, b]$. Show that if their Wronskian $W(y_1, y_2) = 0$ at least one point in $[a, b]$ then $W(y_1, y_2) = 0$ for all $x \in [a, b]$.

Solution: Done in the class.

(4) If y_1 and y_2 are linearly independent solutions of $xy'' + 2y' + xe^x y = 0$ and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.

Solution: By Abel's formula, $W(y_1, y_2)(x) = C \exp\left(\int -\frac{2}{x} dx\right) = Cx^{-2}$.

$$W(y_1, y_2)(1) = 2 \implies W(y_1, y_2)(x) = 2x^{-2} \implies W(y_1, y_2)(5) = 2/25.$$

(5) (a) Verify that the functions $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ are linearly independent solutions of the differential equation $x^2 y'' - 4xy' + 6y = 0$ on $(-\infty, \infty)$; (b) Show that y_1 and y_2 are linearly dependent on $(-\infty, 0)$, but are linearly independent on $(-\infty, \infty)$; (c) Although y_1 and y_2 are linearly independent, show that $W(y_1, y_2) = 0$ for all $x \in (-\infty, \infty)$. Does this violate the fact that $W(y_1, y_2) = 0$ for every $x \in (-\infty, \infty)$ implies y_1 and y_2 are linearly dependent?

Solution: y_1 and y_2 satisfy the given differential equation. On $(-\infty, 0)$, $y_2 = (-1)y_1$, hence linearly dependent. On $(-\infty, \infty)$, consider $c_1 x^3 + c_2 x^2 |x| = 0$. If $x = 1$ then $c_1 + c_2 = 0$, and if $x = -1$, $c_1 - c_2 = 0$. This implies $c_1 = c_2 = 0$. Hence, y_1 and y_2 are linearly independent on $(-\infty, \infty)$. Note that on $0 < x < \infty$, $W(y_1, y_2)(x) = 0$. Thus, $W(y_1, y_2) = 0$ on $(-\infty, \infty)$. It doesn't violate the fact. Observe that $p(x) = -4/x$ and $q(x) = 6/x^2$ fail to be continuous at $x = 0$. Thus, the continuity assumption on the coefficients p and q cann't be dropped.

(6) Let $p(x), q(x) \in C(I)$. Assume that the functions $y_1,\ y_2 \in C^2(I)$ are solutions of the differential equations $y'' + p(x)y' + q(x)y = 0$ on an open interval I. Prove that (a) if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval; (b) if

y_1 and y_2 have a common point of inflection x_0 in I, then they cannot be a fundamental set of solutions on that interval.

Solution: (a) Since $y_1(x_0) = y_2(x_0) = 0$ for some $x_0 \in I$, we find that $W(y_1, y_2)(x_0) = 0$ for some $x_0 \in I$. But, $W(y_1, y_2)(x) = W(y_1, y_2)(x_0)exp[-\int_{x_0}^{x} p(t)dt]$ for all $x \in I$. Thus, $W(y_1, y_2)(x_0) = 0 \Rightarrow W(y_1, y_2)(x) = 0\ \forall x \in I$ Hence, $W(y_1, y_2)(x) = 0\ \forall x \in I \Rightarrow y_1$ and y_2 cannot form a fundamental set of solutions on I.

(b) IGNORE THIS PROBLEM. THE QUESTION MAY BE WRONG.

(7) Let $S = \{f : \mathbb{R} \to \mathbb{R} \,|\, L(f) = 0\}$, where $L(f) := f''' + f'' - 2$. Find the $Ker(L)$. Let $S_0 \subset Ker(L)$ be the subspace of solutions g such that $\lim_{x\to\infty} g(x) = 0$. Find $g \in S_0$ such that $g(0) = 0$ and $g'(0) = 2$.

Solution: The auxiliary equation (AE) $r^3 + r^2 - 2 = 0 \Rightarrow (r-1)(r^2 + 2r + 2) = 0$. $Ker(L) =$ span $\{e^x, e^{-x}\cos x, e^{-x}\sin x\}$. $f(x) \in S$ has the form $f(x) = c_1e^x + c_2e^{-x}\cos x + c_3e^{-x}\sin x$. S_0 is obtained by putting $c_1 = 0$. Thus, $g(x) \in S_0$ has the form $g(x) = c_2e^{-x}\cos x + c_3e^{-x}\sin x$. Using the IC $g(0) = 0$ and $g'(0) = 2$, we obtain $c_2 = 0$ and $c_3 = 2$. So, $g(x) = 2e^{-x}\sin x$.

(8) Find the general solution of the following differential equations.

(a) $\dfrac{d^4y}{dx^4} + y(x) = 0.$

(b) $\dfrac{d^5y}{dx^5} - 2\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} = 0.$

(c) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y(x) = 0.$

(d) $\dfrac{d^5y}{dx^5} + 5\dfrac{d^4y}{dx^4} + 10\dfrac{d^3y}{dx^3} + 10\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + y(x) = 0.$

Solution: (a) The AE is $r^4 + 1 = 0$. We know the nth roots of $z = r(\cos\theta + i\sin\theta)$ are given by $z^{1/n} = r^{1/n}\left[\cos(\frac{\theta+2k\pi}{n}) + i\sin(\frac{\theta+2k\pi}{n})\right],\ k = 0, 1, \ldots, n-1.$

Since $z = (\cos\pi + i\sin\pi)$, we obtain

$$z^{1/4} = \left[\cos(\frac{\pi + 2k\pi}{4}) + i\sin(\frac{\pi + 2k\pi}{4})\right],\ k = 0, 1, 2, 3.$$

Thus, the roots are

$$\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i,\ -\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i.$$

The GS is

$$\begin{aligned} y(x) &= e^{\frac{\sqrt{2}}{2}x}\left[c_1\sin\frac{\sqrt{2}}{2}x + c_2\cos\frac{\sqrt{2}}{2}x\right] \\ &+e^{-\frac{\sqrt{2}}{2}x}\left[c_3\sin\frac{\sqrt{2}}{2}x + c_4\cos\frac{\sqrt{2}}{2}x\right]. \end{aligned}$$

(b) The AE is $r^5 - 2r^4 + r^3 = 0 \Rightarrow r^3(r-1)^2 = 0$. The GS is

$$y(x) = (c_1 + c_2x + c_3x^2) + (c_4 + c_5x)e^x.$$

(c) The AE is $(r^2+1)(r-1) = 0$. The G.S. is $y(x) = c_1e^x + c_2\sin x + c_3\cos x$.

(d) The AE is $(r+1)^5 = 0$. The G.S. is $y(x) = (c_1 + c_2x + c_3x^2 + c_4x^3 + c_4x^4)e^{-x}$.

(9) Solve the following initial-value problems:

(a) $y'' - 2y' + y = 2xe^{2x} + 6e^x;\ y(0) = 1,\ y'(0) = 0.$

(*b*) $y''(x) + y(x) = 3x^2 - 4\sin x,\ y(0) = 0,\ y'(0) = 1.$

Solution: (*a*) $y_h(x) = c_1e^x + c_2xe^x$. $y_p = Axe^{2x} + Be^{2x} + Cx^2e^x$. Now, $Ly_p = 2xe^{2x} + 6e^x$ yields $A = 2$, $B = -4$, $C = 3$. The GS is given by

$$y(x) = c_1e^x + c_2xe^x + 2xe^{2x} - 4e^{2x} + 3x^2e^x.$$

Using the IC $y(0) = 1$ and $y'(0) = 0$, we obtain the particular solution

$$y(x) = (x + 5)e^x + 3x^2e^x + 2xe^{2x} - 4e^{2x}.$$

(*b*) $y_h(x) = c_1 \sin x + c_2 \cos x$. $y_p(x) = Ax^2 + Bx + C + Dx\sin x + Ex\cos x$. Then $Ly_p = 3x^2 - 4\sin x$ yields $A = 3$, $B = 0$, $C = -6$, $D = 0$ and $E = 2$. Thus, $y_p = 3x^2 - 6 + 2x\cos x$. The GS is

$$y(x) = c_1 \sin x + c_2 \cos x + 3x^2 - 6 + 2x\cos x.$$

Applying IC we obtain $c_1 = -1$ and $c_2 = 6$. The particular solution is

$$y(x) = 6\cos x - \sin x + 3x^2 - 6 + 2x\cos x.$$

(10) Use the method of undermined coefficients to find a particular solution to the following differential equations:
(*a*) $y'' - 3y' + 2y = 2x^2 + 3e^{2x}$.
(*b*) $y''(x) - 3y'(x) + 2y(x) = xe^{2x} + \sin x$.

Solution: (*a*) $y_p(x) = Ax^2 + Bx + C + Dxe^{2x}$. $y_p' = 2Ax + B + 2Dxe^{ex} + De^{2x}$. $y_p'' = 2A + 4Dxe^{2x} + 4De^{2x}$. Substituting in the differential equation and solving for A, B, C and D, we obtain $A = 1$, $B = 3$, $C = 7/2$ and $D = 3$. So, $y_p = x^2 + 3x + 7/2 + 3xe^{2x}$.

(*b*) $y_p(x) = Ax^2e^{2x} + Bxe^{2x} + C\sin x + D\cos x$. Proceed as in (*a*), we determine $A = 1/2$, $B = -1$, $C = 1/10$ and $D = 3/10$.

(11) Use the annihilator method to determine the form of a particular solution for the equations:
(*a*) $y''(x) - 5y'(x) + 6y(x) = \cos(2x) + 1$.
(*b*) $y''(x) - 5y'(x) + 6y(x) = e^{3x} - x^2$.

Solution: (*a*) Here $L(y) = (D^2 - 5D + 6)(y) = \cos 2x + 1$. Note that $(D^2 + 4)\cos(2x) = 0$ and $D(1) = 0$. So, $Q = D(D^2 + 4)$ annihilates $\cos(2x) + 1$. Thus,

$$QL(y) = D(D^2 + 4)(D^2 - 5D + 6)(y) = D(D^2 + 4)(\cos 2x + 1) = 0.$$

The AE of $D(D^2+4)(D^2-5D+6)(y) = 0$ is $r(r^2+4)(r-3)(r-2) = 0$. The GS to $QL(y) = 0$ is

$$y(x) = c_1e^{2x} + c_2e^{3x} + c_3\cos(2x) + c_4\sin(2x) + c_5.$$

The GS to $L(y) = 0$ is $y_h(x) = c_1e^{2x} + c_2e^{3x}$. The GS to $L(y) = \cos 2x + 1$ is $y(x) = y_h(x) + y_p(x) = c_1e^{2x} + c_2e^{3x} + y_p(x)$. Comparing, we find that

$$y_p(x) = c_3\cos(2x) + c_4\sin(2x) + c_5.$$

(*b*) $e^{3x} - x^2$ is annihilated by $Q = D^3(D - 3)$. The GS of $QL(y) = D^3(D - 3)^2(D - 2)(y) = 0$ is

$$y(x) = c_1e^{2x} + c_2e^{3x} + c_3xe^{3x} + c_4x^2 + c_5x + c_6.$$

Since the GS of $L(y) = 0$ is $y_h(x) = c_1e^{2x} + c_2e^{3x}$, $y(x) = y_h(x) + y_p(x)$ yields

$$y_p(x) = c_3xe^{3x} + c_4x^2 + c_5x + c_6.$$