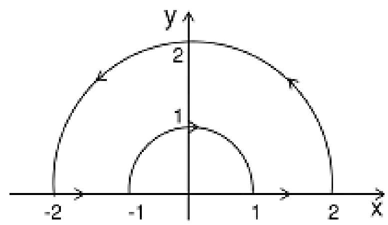
Mid Sem Exam Solutions : PH 102

Note Title 2/17/2013

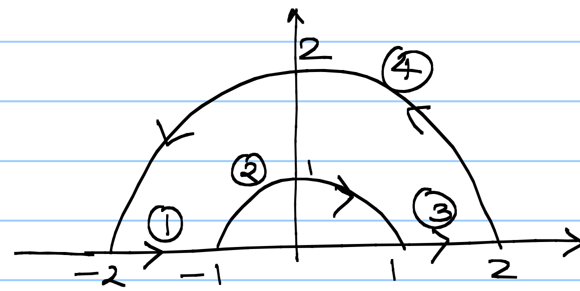
Q.1

Let $\vec{A} = r\sin\phi\,\hat{r} + r^2\,\hat{\phi}$. (a) Evaluate $\oint \vec{A}.\vec{dl}$ along the closed path shown below. (b) Verify the above result using the Stoke's theorem. [3+3=6]



Solutions:

(a) $\oint \vec{A}.\vec{dl}$



along path 1

$\vec{dl} = dr\,\hat{r}, \quad \phi = \pi$

$\int \vec{A}.\vec{dl} = 0$

along path 2 $\quad \vec{dl} = r\sin\theta\, d\phi\,\hat{\phi}, \quad \theta = \pi/2$

$= r\, d\phi\,\hat{\phi}, \quad r = 1$

$\int \vec{A}.\vec{dl} = \int_{\pi}^{0} r^3\, d\phi = -\pi$

along path 3, $\vec{dl} = dr\,\hat{r}$, $\int \vec{A}.\vec{dl} = 0$

along path 4, $dl = r\sin\theta\, d\phi\,\hat{\phi}$

$= 2\, d\phi\,\hat{\phi}$

$\int \vec{A}.\vec{dl} = 8\int_{0}^{\pi} d\phi = 8\pi$

Hence $\oint \vec{A}.\vec{dl} = 8\pi - \pi = 7\pi$

(b) $$\vec{\nabla}\times\vec{A} = \frac{1}{h_2h_3}\left\{\frac{\partial}{\partial u_2} h_3 A_{u_3} - \frac{\partial}{\partial u_3} h_2 A_{u_2}\right\}\hat{u}_1 + \frac{1}{h_3h_1}\left\{\frac{\partial}{\partial u_3} h_1 A_{u_1} - \frac{\partial}{\partial u_1} h_3 A_{u_3}\right\}\hat{u}_2 + \frac{1}{h_1h_2}\left\{\frac{\partial}{\partial u_1} h_2 A_{u_2} - \frac{\partial}{\partial u_2} h_1 A_{u_1}\right\}\hat{u}_3$$

$$= \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial\theta} r\sin\theta A_\phi - \frac{\partial}{\partial\phi} r A_\theta\right\}\hat{r} + \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\phi} A_r - \frac{\partial}{\partial r} r\sin\theta A_\phi\right\}\hat{\theta} + \frac{1}{r}\left\{\frac{\partial}{\partial r} r A_\theta - \frac{\partial}{\partial\theta} A_r\right\}\hat{\phi}$$

$$\vec{da} = r\sin\theta\, d\phi\, dr\,(-\hat{\theta})$$

$$\int(\vec{\nabla}\times\vec{A})\cdot\vec{da} = -\iint \frac{1}{r\sin\theta}\left\{r\cos\phi - \sin\theta\, 3r^2\right\} r\sin\theta\, d\phi\, dr$$

$$= -\int_1^2 r\,dr \int_0^\pi \cos\phi\, d\phi + 3\int_1^2 r^2 dr \int_0^\pi d\phi$$

$$= -\left[\frac{r^2}{2}\Big|_1^2 \sin\phi\Big|_0^\pi + 3\,\frac{r^3}{3}\Big|_1^2\, \phi\Big|_0^\pi\right]$$

$$= 0 + (8-1)\pi = 7\pi$$

Hence $$\int_{\text{surface}}(\vec{\nabla}\times\vec{A})\cdot\vec{da} = \oint_C \vec{A}\cdot\vec{dl}$$

Q.2

A spherical conducting shell of radius 'a', centred at the origin has a potential

$$V(r) = \begin{cases} V_0 & \text{if } r \leq a \\ \frac{V_0 a}{r} & \text{if } r > a \end{cases}$$

with the reference point at infinity. (a) Obtain the expression of stored energy for a volume enclosing all space. (b) What is the net charge Q on the spherical shell? [4+2=6]

(a) $\vec{E} = -\vec{\nabla} V$

$$= -\left\{ \frac{1}{h_1}\frac{\partial V}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\hat{u}_3 \right\}$$

$$= -\left\{ \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi} \right\}$$

$= 0$ if $r \leq a$

$= \frac{V_0 a}{r^2}\hat{r}$ if $r > a$

Expression of energy in all space

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau \qquad d\tau = r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{\epsilon_0}{2}\left[\int_{r \leq a} E^2 d\tau + \int_{r > a} E^2 d\tau \right]$$

$$= \frac{\epsilon_0}{2}\left[0 + \int_a^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{V_0 a}{r^2}\right)^2 r^2 \sin\theta \, dr \, d\theta \, d\phi \right]$$

$$= \frac{\epsilon_0}{2}(V_0 a)^2 \int_a^\infty \frac{dr}{r^2} \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{V_0^2 a^2 \epsilon_0}{2} \left(-\frac{1}{r}\right)\Big|_a^\infty \times \left(-\cos\theta\right)\Big|_0^\pi \times 2\pi$$

$$= 2 V_0^2 a \epsilon_0 \pi$$

(b) Net charge Q

$$\oint \vec{E} \cdot d\vec{a} = Q/\epsilon_0$$

over the sphere of radius $r > a$

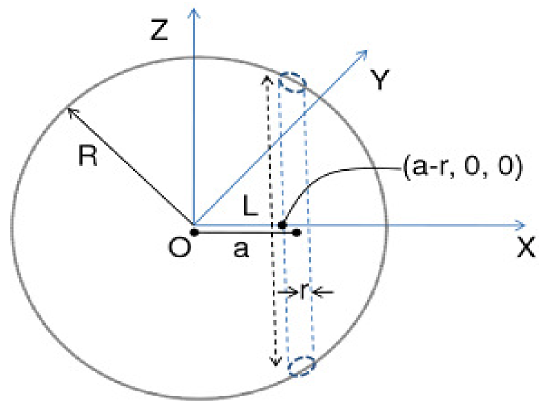
$$\Rightarrow \frac{V_0 a}{r^2} \times 4\pi r^2 = Q/\epsilon_0$$

$$\Rightarrow Q = 4\pi V_0 a \epsilon_0$$

Note Title 2/16/2013

Q.3

A cylindrical slot of radius 'r' is removed from a charged sphere of radius 'R' having uniform volume charge density 'ρ'. The axis of the cylindrical slot of mean length 'L', is parallel to the Z axis and is intersecting the X axis at a distance 'a' from the origin, O, as shown in the figure below. Assuming L≫a and L≫r, find (a) the electric field, and (b) the potential, at the point $(a-r, 0, 0)$ located on the surface of the cylindrical slot. Take the origin O as the reference to describe the potential. [3+3=6]



Solution:

(a) Electric field at a point P (a-r, 0, 0) follows the superposition principle.

Let $\vec{E}_S$ = electric due to the entire solid sphere of radius R having charge density ρ.

$\vec{E}_s$ = electric field due to the solid sphere without the cylindrical slot.

$\vec{E}_c$ = electric field due to a solid cylinder with charge density ρ that will fit into the empty slot.

According to the superposition principle

$$\vec{E}_S\Big|_P = \vec{E}_s\Big|_P + \vec{E}_c\Big|_P$$

$$\vec{E}_S = \left(\tfrac{4\pi}{3} r^3 \rho\right) \Big/ \left(\epsilon_0 4\pi r^2\right) \hat{r} = \frac{\rho}{3\epsilon_0} r \hat{r}$$

$$\vec{E}_S\Big|_P = \frac{\rho}{3\epsilon_0} (a-r) \hat{r}$$

$$\vec{E}_c\Big|_P = \frac{\pi r_0^2 \rho}{\epsilon_0\, 2\pi r_0}\left(-\hat{r}\right) = -\frac{\rho r_0}{2\epsilon_0}\,\hat{r}$$ [since $L \gg r_0, a$]

Hence $$\vec{E}_s\Big|_P = \vec{E}_S\Big|_P - \vec{E}_c\Big|_P$$

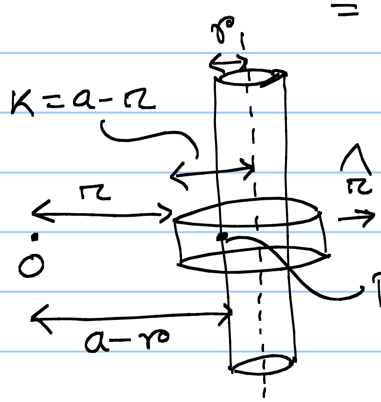
$$= \frac{\rho}{6\epsilon_0}\left\{2(a-r_0) + 3r_0\right\}\hat{r} = (2a+r_0)\frac{\rho}{6\epsilon_0}\,\hat{r}$$

(b) Similarly superposition also applies to potential.

Hence at P $\qquad V_S = V_s + V_c$

$$V_S = -\int_0^{a-r_0} \vec{E}_S \cdot d\vec{r} = -\int_0^{a-r_0} \frac{\rho r}{3\epsilon_0}\,dr$$

$$= -\frac{\rho}{3\epsilon_0}\,\frac{r^2}{2}\Bigg|_0^{a-r_0} = -\frac{\rho}{3\epsilon_0}\,\frac{(a-r_0)^2}{2}$$



using the gaussian surface shown $$\vec{E}_c = -\frac{r_0^2 \rho}{2\epsilon_0 K}\,\hat{r}$$

$$V_c = -\int_O^P \vec{E}_c \cdot d\vec{r} = -\frac{r_0^2\rho}{2\epsilon_0}\int_a^{r_0}\frac{dK}{K}$$ [$dK = -dr$]

$$= -\frac{r_0^2\rho}{2\epsilon_0}\,\ln K\Bigg|_a^{r_0}$$

$$= -\frac{r_0^2\rho}{2\epsilon_0}\,\ln\frac{r_0}{a}$$

Hence

$$V_s = V_S - V_C$$

$$= -\frac{\rho}{6\epsilon_0}\left\{(a-r_0)^2 + 3r_0^2 \ln\left(\frac{a}{r_0}\right)\right\}$$

N.B.! Problem 3(b) can also be solved by finding potential at P or O due to a disc at a height z (considering infinity to be the reference) and integrating over z from $-L/2$ to $L/2$

$$V_{c,P} = r^2\rho/4\epsilon_0 \ln\left(1 + L^2/r^2\right)$$

$$V_{c,O} = r^2\rho/4\epsilon_0 \ln\left(1 + L^2/a^2\right)$$

$$V_c = V_{c,P} - V_{c,O}$$

$\boxed{Q.4}$

Consider two long concentric circular cylinders of radius r=1 mm and r=20 mm. For the inner cylinder potential, $V = 0$ while for the outer cylinder $V = 150$ V. Find (a) the potential, $V(r)$, and (b) the electric field for the region between between the two cylinders (i.e. 1 mm<r< 20 mm). [4+2=6]

(a) Laplace's equation

$$\nabla^2 V = 0$$

$$\Rightarrow \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right\} = 0$$

$$\Rightarrow \frac{1}{s}\left\{\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{s}\frac{\partial V}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(s\frac{\partial V}{\partial z}\right)\right\} = 0$$

$$\Rightarrow \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Since there is no ϕ and z dependance

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

$$\Rightarrow \frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0$$

$$\Rightarrow s\frac{dV}{ds} = c_1$$

$$\Rightarrow dV = c_1\frac{ds}{s}$$

$$\Rightarrow V = c_1 \ln s + c_2$$

$V(s=0.001\,m) = 0$

$\Rightarrow 0 = c_1(-6.90776) + c_2$

$\Rightarrow V(s = 0.02\,m) = 150\,V$

$\Rightarrow 150 = -c_1 3.912 + c_2$

$\Rightarrow 150 = c_1(6.90776 - 3.912) = c_1 2.99576$

$\Rightarrow c_1 = 150/2.99576 = 50.07$

$C_2 = 345.87$

hence $V(s) = 50.07 \ln s + 345.87$ (volt)

(b) $\vec{E} = -\vec{\nabla} V$

$$= -\left(\frac{\partial V}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z} \right)$$

$$= -50.07 \frac{1}{s}\hat{s} \quad (\text{volt}/m)$$

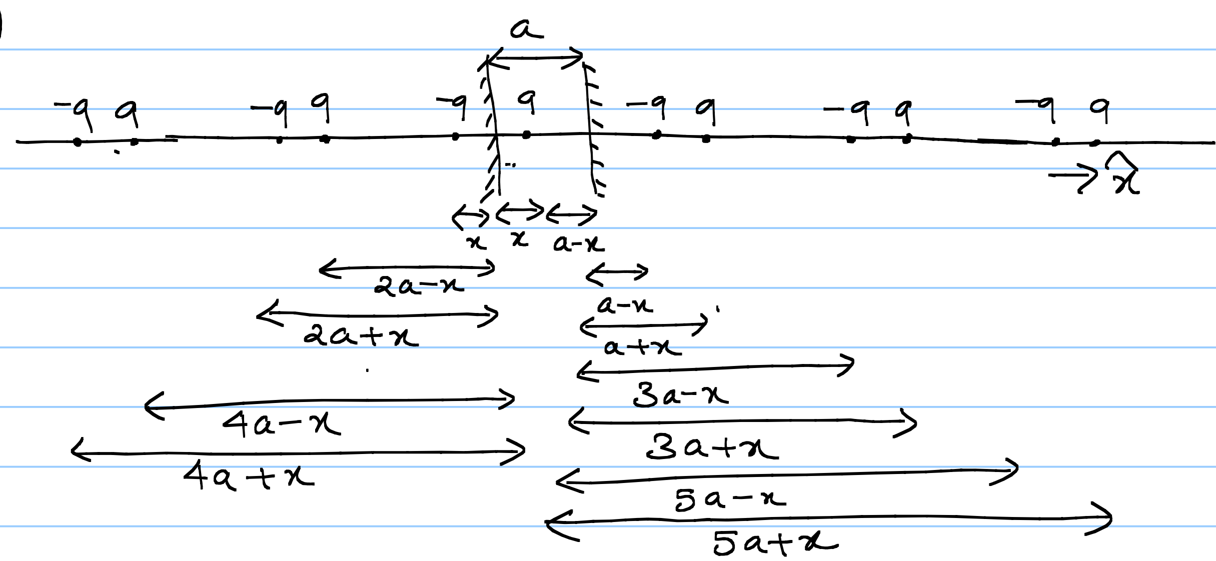
$\boxed{\text{Q. 5}}$

Two infinite grounded conducting planes are held a distance 'a' apart. A point charge 'q' is placed between them at a distance x from one plane. (a) What is the number of image charges? (b) Obtain the expression of force on q. (c) Show that as $a \to \infty$, the force on 'q' is same as due to a single grounded conducting plane at a distance x. [1+3+2=6]

Solution :

(a) Using the method of images, for a charge q situated at distance d from an infinite grounded conducting plane, the image charge $(-q)$ is situated at a distance d behind the plane. Hence for the given configuration number image charge is infinity.

(b)



$$\vec{F} = \frac{q^2}{4\pi\epsilon_0}\,\frac{1}{4}\left[\left\{\frac{1}{(a-x)^2} + \frac{1}{(2a-x)^2} + \frac{1}{(3a-x)^2} + \cdots\right\} + \left\{\frac{1}{a^2} + \frac{1}{(2a)^2} + \cdots\right\} - \left\{\frac{1}{x^2} + \frac{1}{(a+x)^2} + \frac{1}{(2a+x)^2} + \cdots\right\} - \left\{\frac{1}{a^2} + \frac{1}{(2a)^2} + \cdots\right\}\right]\hat{x}$$

$$= \frac{q^2}{16\pi\epsilon_0}\left[\left\{\frac{1}{(a-x)^2} + \frac{1}{(2a-x)^2} + \frac{1}{(3a-x)^2} + \cdots\right\} - \left\{\frac{1}{x^2} + \frac{1}{(a+x)^2} + \frac{1}{(2a+x)^2} + \cdots\right\}\right]\hat{x}$$

(c) In the limit $a \to \infty$, all the denominators containing 'a' will become infinite.

Hence $\vec{F} = -\frac{q^2}{16\pi\epsilon_0}\frac{1}{x^2}\hat{x} = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2x)^2}\hat{x}$

which is the force on q when only one plane is situated at a distance x from the charge.