First-Order ODE: Separable Equations, Exact Equations and Integrating Factor

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Separable Equations

Definition: A first-order equation $y'(x) = f(x, y)$ is separable if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y)$$

Method for solving separable equations: To solve the equation

$$\frac{dy}{dx} = g(x)p(y),$$

we write it as $h(y)dy = g(x)dx$, where $h(y) := \frac{1}{p(y)}$. Integrating both sides

$$\int h(y)dy = \int g(x)dx \implies H(y) = G(x) + C,$$

which gives an implicit solution to the differential equation.

Formal justification of method: Writing the equation in the form

$$h(y)\frac{dy}{dx} = g(x), \quad h(y) := \frac{1}{p(y)}.$$

Let $H(y)$ and $G(x)$ be such that

$$H'(y) = h(y), \quad G'(x) = g(x).$$

Then

$$H'(y)\frac{dy}{dx} = G'(x).$$

Since $\frac{d}{dx}H(y(x)) = H'(y(x))\frac{dy}{dx}$(by chain rule), we obtain

$$\frac{d}{dx}H(y(x)) = \frac{d}{dx}G(x) \Rightarrow H(y(x)) = G(x) + C.$$

Remark: In finding a one-parameter family of solutions in the separation process, we assume that $p(y) \neq 0$. Then we must find the solutions $y = y_0$ of the equation $p(y) = 0$ and determine whether any of these are solutions of the original equation which were lost in the formal separation process.

Example: Consider $(x-4)y^4dx - x^3(y^2-3)dy = 0$. Separating the variable by dividing x^3y^4, we obtain

$$\frac{(x-4)dx}{x^3} - \frac{(y^2-3)dy}{y^4} = 0$$

The general solution is $-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = C, \quad y \neq 0$

Note: $y = 0$ is a solution of the original equation which was lost in the separation process.

First-Order Linear Equations

A linear first-order equation can be expressed in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \tag{1}$$

where $a_1(x) \neq 0$, $a_0(x)$ and $b(x)$ depend only on the independent variable x, not on y.

Examples:

$(1 + 2x)\frac{dy}{dx} + 6y = e^x$ (linear)

$\sin x \frac{dy}{dx} + (\cos x)y = x^2$ (linear)

$\frac{dy}{dx} + xy^3 = x^2$ (not linear)

Theorem **(Existence and Uniqueness)**:

Suppose $a_1(x), a_0(x), b(x) \in C((a, b))$, $a_1(x) \neq 0$ and $x_0 \in (a, b)$. Then for any $y_0 \in \mathbb{R}$, there exists a unique solution $y(x) \in C^1((a, b))$ to the IVP

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \quad y(x_0) = y_0.$$

Observations:

I. If $a_0(x) = 0$, then Eq. (1) reduces to

$$a_1(x)\frac{dy}{dx} = b(x) \Rightarrow y(x) = \int \frac{b(x)}{a_1(x)}dx + C, \quad a_1(x) \neq 0.$$

II. If $a_0(x) = a_1'(x)$, then

$$a_1(x)y'(x) + a_0(x)y = a_1(x)y' + a_1'(x)y = \frac{d}{dx}\{a_1(x)y\}.$$

Therefore, Eq. (1) becomes

$$\frac{d}{dx}\{a_1(x)y\} = b(x).$$

The general solution is given by

$$y(x) = \boxed{\frac{1}{a_1(x)}\left\{\int b(x)dx + C\right\}}.$$

Integrating Factor(I.F.)

Rewriting $a_1(x)y' + a_0(x)y = b(x)$ as

$$y' + p(x)y = q(x), \quad p(x) = \frac{a_0(x)}{a_1(x)}, \quad q(x) = \frac{b_0(x)}{a_1(x)}.$$

Multiplying both side by $\mu(x)$ so that

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \frac{d}{dx}\{\mu(x)y\} = \mu(x)\frac{dy}{dx} + \mu'(x)y.$$

This yields

$$\mu'(x) = \mu(x)p(x) \;\Rightarrow\; \mu(x) = e^{\int p(x)dx}.$$

Thus,

$$\frac{d}{dx}\{\mu(x)y\} = \mu(x)q(x) \Rightarrow y(x) = \frac{1}{\mu(x)}\left\{\int \mu(x)q(x)dx + C\right\}.$$

Example: $y' + \frac{1}{x}y = 3x$. $(y(x) = x^2 + cx^{-1})$

Exact Differential Equation

Suppose $f(x,y) = c$ defines y implicitly as a differentiable function of x. Then, $y = y(x)$ satisfies a first-order DE

$$f_x(x,y) + f_y(x,y)y'(x) = 0,$$

which is an exact DE.

Definition: A first-order DE of the form

$$M(x,y) + N(x,y)y'(x) = 0$$

is an exact DE in a rectangle R if there is a function $f(x,y)$ such that

$$f_x(x,y) = M(x,y) \quad \text{and} \quad f_y(x,y) = N(x,y).$$

Note: If $f(x,y)$ is known then the general solution is given implicitly by $f(x,y) = c$.

$$\begin{aligned} \frac{d}{dx} f(x, y(x)) &= f_x(x,y) + f_y(x,y)y'. \\ &= M(x,y) + N(x,y)y' = 0. \end{aligned}$$

Theorem: Let $M(x,y), N(x,y) \in C^1(R)$. Then

$$M(x,y) + N(x,y)y' = 0 \text{ is exact } \iff M_y(x,y) = N_x(x,y)$$

for $(x,y) \in R$.

Example: Consider $4x + 3y + 3(x + y^2)y' = 0$.

Note that $M, N \in C^1(R)$ and $M_y = 3 = N_x$. Thus, there exists $f(x,y)$ such that $f_x = 4x + 3y$ and $f_y = 3x + 3y^2$.

$f_x = 4x + 3y \Rightarrow f(x,y) = 2x^2 + 3xy + \phi(y)$. Now,

$$3x + 3y^2 = f_y(x,y) = 3x + \phi'(y).$$

$$\Rightarrow \phi'(y) = 3y^2 \Rightarrow \phi(y) = y^3.$$

Thus, $f(x,y) = 2x^2 + 3xy + y^3$ and the general solution is given by

$$2x^2 + 3xy + y^3 = C$$

Definition: If the equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad (2)$$

is not exact, but the equation

$$\mu(x,y)\{M(x,y)dx + N(x,y)dy\} = 0 \qquad (3)$$

is exact then $\mu(x,y)$ is called an integrating factor of (2).

Example: The equation $(y^2 + y)dx - xdy = 0$ is not exact. But, when we multiply by $\frac{1}{y^2}$, the resulting equation

$$(1 + \frac{1}{y})dx - \frac{x}{y^2}dy = 0, \quad y \neq 0$$

is exact.

Remark: While (2) and (3) have essentially the same solutions, it is possible to lose or gain solutions when multiplying by $\mu(x,y)$.

Theorem: If $\frac{(M_y - N_x)}{N}$ is continuous and depends only on x, then

$$\mu(x) = \exp\left(\int \left\{\frac{M_y - N_x}{N}\right\} dx\right)$$

is an integrating factor for $Mdx + Ndy = 0$.

Proof. If $\mu(x, y)$ is an integrating factor, we must have

$$\frac{\partial}{\partial y}\{\mu M\} = \frac{\partial}{\partial x}\{\mu N\} \Rightarrow M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu.$$

If $\mu = \mu(x)$ then $\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$, where $(M_y - N_x)/N$ is just a function of x.

Example: Solve $(2x^2 + y)dx + (x^2y - x)dy = 0$.
The equation is not exact as $M_y = 1 \neq (2xy - 1) = N_x$. Note that

$$\frac{M_y - N_x}{N} = \frac{2(1 - xy)}{-x(1 - xy)} = \frac{-2}{x},$$

which is a function of only x, so an I.F $\mu(x) = x^{-2}$ and the solution is given by $2x - 2yx^{-1} + \frac{y^2}{2} = C$.

Remark. Note that the solution $x = 0$ was lost in multiplying $\mu(x) = x^{-2}$.

Theorem: If $\frac{N_x - M_y}{M}$ is continuous and depends only on y, then

$$\mu(y) = \exp\left(\int \left\{\frac{N_x - M_y}{M}\right\} dy\right)$$

is an integrating factor for $Mdx + Ndy = 0$.

Homogeneous Functions

If $M(x,y)dx + N(x,y)dy = 0$ is not a separable, exact, or linear equation, then it may still be possible to transform it into one that we know how to solve.

Definition: A function $f(x,y)$ is said to be homogeneous of degree n if

$$f(tx, ty) = t^n f(x,y),$$

where $t > 0$ and n is a constant.

Example:

1. $f(x,y) = x^2 + y^2 \log(y/x), \; x > 0, \; y > 0$ (homogeneous of degree 2)
2. $f(x,y) = e^{y/x} + \tan(y/x)$ (homogeneous of degree 0)

- If $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree then the substitution $y = vx$ transforms the equation into a separable equation.

Writing $Mdx + Ndy = 0$ in the form $\frac{dy}{dx} = -M/N = f(x,y)$. Then, $f(x,y)$ is a homogeneous function of degree 0. Now, substitution $y = vx$ transform the equation into

$$v + x\frac{dv}{dx} = f(1,v) \Rightarrow \frac{dv}{f(1,v) - v} = \frac{dx}{x},$$

which is in variable separable form.

Example: Consider $(x+y)dx - (x-y)dy = 0$.

Put $y = vx$ and separate the variable to have

$$\frac{(1-v)dv}{1+v^2} = \frac{dx}{x}$$

Integrating and replacing $v = y/x$, we obtain

$$tan^{-1}\frac{y}{x} = \log\sqrt{x^2+y^2} + C.$$

Substitutions and Transformations

- A first-order equation of the form

$$y' + p(x)y = q(x)y^{\alpha},$$

where $p(x), q(x) \in C((a, b))$ and $\alpha \in \mathbb{R}$, is called a Bernoulli equation.

The substitution $v = y^{1-\alpha}$ transforms the Bernoulli equation into a linear equation

$$\frac{dv}{dx} + p_1(x)v = q_1(x),$$

where $p_1(x) = (1 - \alpha)p(x),\ q_1(x) = (1 - \alpha)q(x)$.

Example: Consider $y' + y = xy^3$. The general solution is given by $\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$.

- An equation of the form

$$y' = p(x)y^2 + q(x)y + r(x)$$

is called Riccati equation.

If its one solution, say $u(x)$ is known then the substitution $y = u + 1/v$ reduces to a linear equation in v.

Remark: Note that if $p(x) = 0$ then it is a linear equation. If $r(x) = 0$ then it is a Bernoulli equation.

- A DE of the form $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy$, where a_i's, b_i's and c_i's are constants, can be transformed into the homogeneous equation by substituting

$$x = u + h \quad \text{and} \quad y = v + k,$$

where h, k are solutions of $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$.
If $a_2/a_1 = b_2/b_1 = k$, then substitution $z = a_1x + b_1y$ reduces the above DE to a separable equation in x and z.

- If a DE is in the special form

$$y(Ax^py^q + Bx^ry^s)dx + x(Cx^py^q + Dx^ry^s)dy = 0,$$

where A, B, C, D are constants, then it has an I.F. of the form $\mu(x, y) = x^ay^b$, where a and b are suitably chosen constants.

Orthogonal Trajectories

Suppose

$$\frac{dy}{dx} = f(x, y)$$

represents the DE of the family of curves. Then, the slope of any orthogonal trajectory is given by

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} \quad \text{or} \quad -\frac{dx}{dy} = f(x, y),$$

which is the DE of the orthogonal trajectories.

Example: Consider the family of circles $x^2 + y^2 = c^2$. Differentiate w.r.t x to obtain $x + y\frac{dy}{dx} = 0$. The differential equation of the orthogonal trajectories is $x + y\left(-\frac{dx}{dy}\right) = 0$. Separating variable and integrating we obtain $y = c\,x$ as the equation of the orthogonal trajectories.

*** End ***