PH 102: Physics II

Lecture 15 (Spring 2018)

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LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	6-3-2018	Lorentz Force, Biot-Savart law	5.1, 5.2	1	I, II (5-5:55 pm)
Lec 15	7-3-2018	Lorentz Force, Biot-Savart law	5.1, 5.2	1	III, IV (9-9:55 am)
Tut 8	13-3-2018	Lec 15			
Lec 16	13-3-2018	Divergence & Curl of Magnetostatic Fields, Applications of Ampere's law	5.3	1	I, II (5-5:55 pm)
Lec 16	14-3-2018	Divergence & Curl of Magnetostatic Fields, Applications of Ampere's law	5.3	1	III, IV (9-9:55 am)
Lec 17	14-3-2018	Magnetic Vector Potential, Force & torque on a magnetic dipole	5.4	1	I, II (4-4:55 pm)
Lec 17	15-3-2018	Magnetic Vector Potential, Force & torque on a magnetic dipole	5.4	1	III, IV (10-10:55 am)
Lec 18	15-3-2018	Lec 16+Lec 17 Continues		1	I, II (3-3:55 pm)
Lec 18	16-3-2018	Lec 16+Lec 17 Continues		1	III, IV (11-11:55 am)
Tut 9	20-3-2018	Lec 16+Lec 17+Lec 18			
Lec 19	21-3-2018	Magnetic Materials, Magnetization	6.1	1	I, II, III, IV
Lec 20	23-3-2018	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II, III, IV
Tut 10	3-4-2018	Quiz II			
Lec 21	4-4-2018	Ohm's law, motional emf, electromotive force	7.1	1	I, II, III, IV
Lec 22	6-4-2018	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II, III, IV
Tut 11	10-4-2018	Lec 21+Lec 22			
Lec 23	11-4-2018	Maxwell's equations	7.3	1	I, II, III, IV
Lec 24	13-4-2018	Continuity equation, Poynting Theorem	8.1	1	I, II, III, IV
Tut 12	17-4-2018	Lec 23+Lec 24			
Lec 25	18-4-2018	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II, III, IV
Lec 26	20-4-2018	L11+Electromagnetic waves in matter	9.3	1	I, II, III, IV
Tut 13	24-4-2018	Lec 25+ Lec 26			
Lec 27	25-4-2018	Reflection and transmission: Normal & Oblique Incidence	9.3, 9.4	1	I, II, III, IV
Lec 28	27-4-2018	Lec 27+Discussions	9.3, 9.4	1	I, II, III, IV

Static Charge

- In electrostatics, the field created by a charge q is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

- The force felt by a test charge Q is $\vec{F} = Q\vec{E}$

- The torque on an electric dipole is $\vec{\tau} = \vec{p} \times \vec{E}$

- Force on the test charge due to the presence of several charges, in accordance with the principle of superposition is $\vec{F} = \sum_i Q\vec{E}_i$

Charges in Motion

- What if the charges are in motion?
- While static charges produce only electric field, moving ones produce a magnetic field too.
- Such magnetic fields can be easily detected by a magnetic needle (compass).
- A current carrying wire creates its own magnetic field. This can exert a magnetic force on a charge in motion or another current carrying wire.
- Wires carrying current (charges in motion) in opposite (same) directions repel (attract).

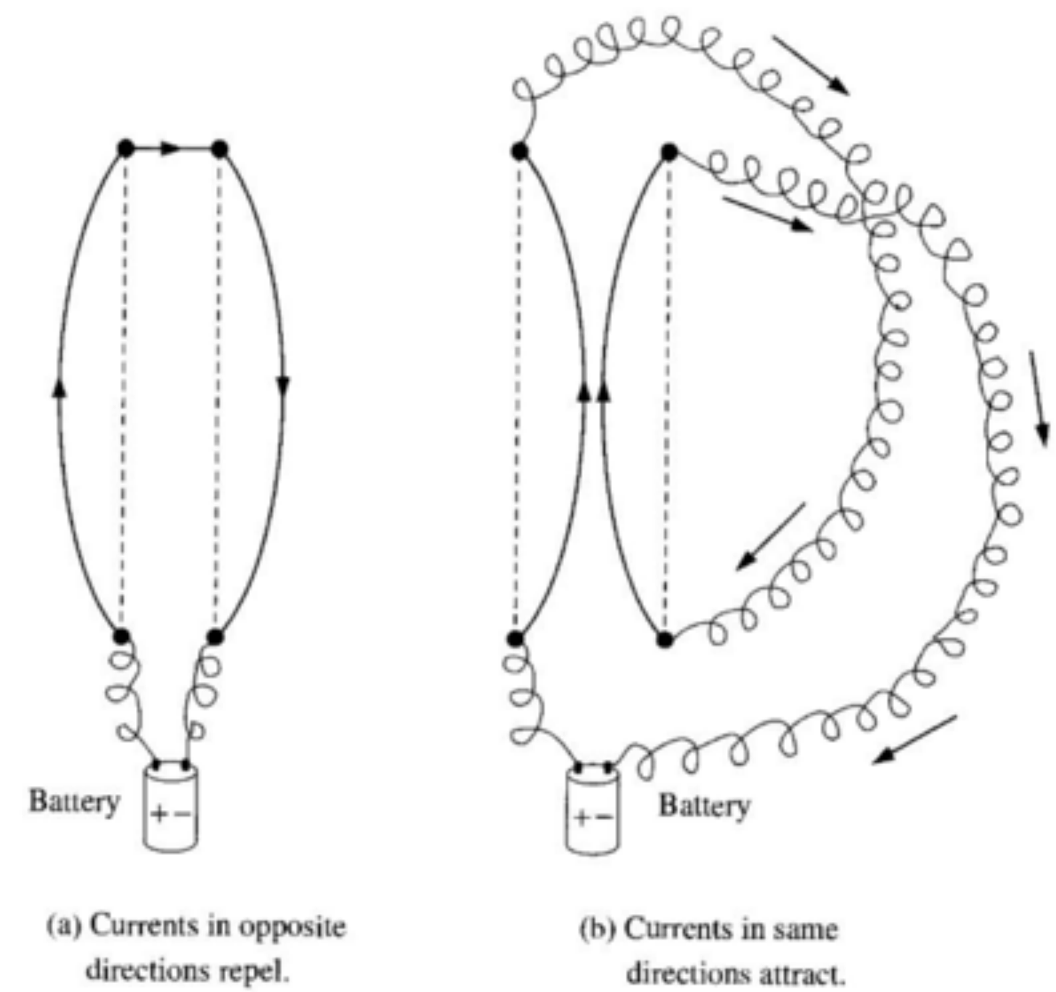
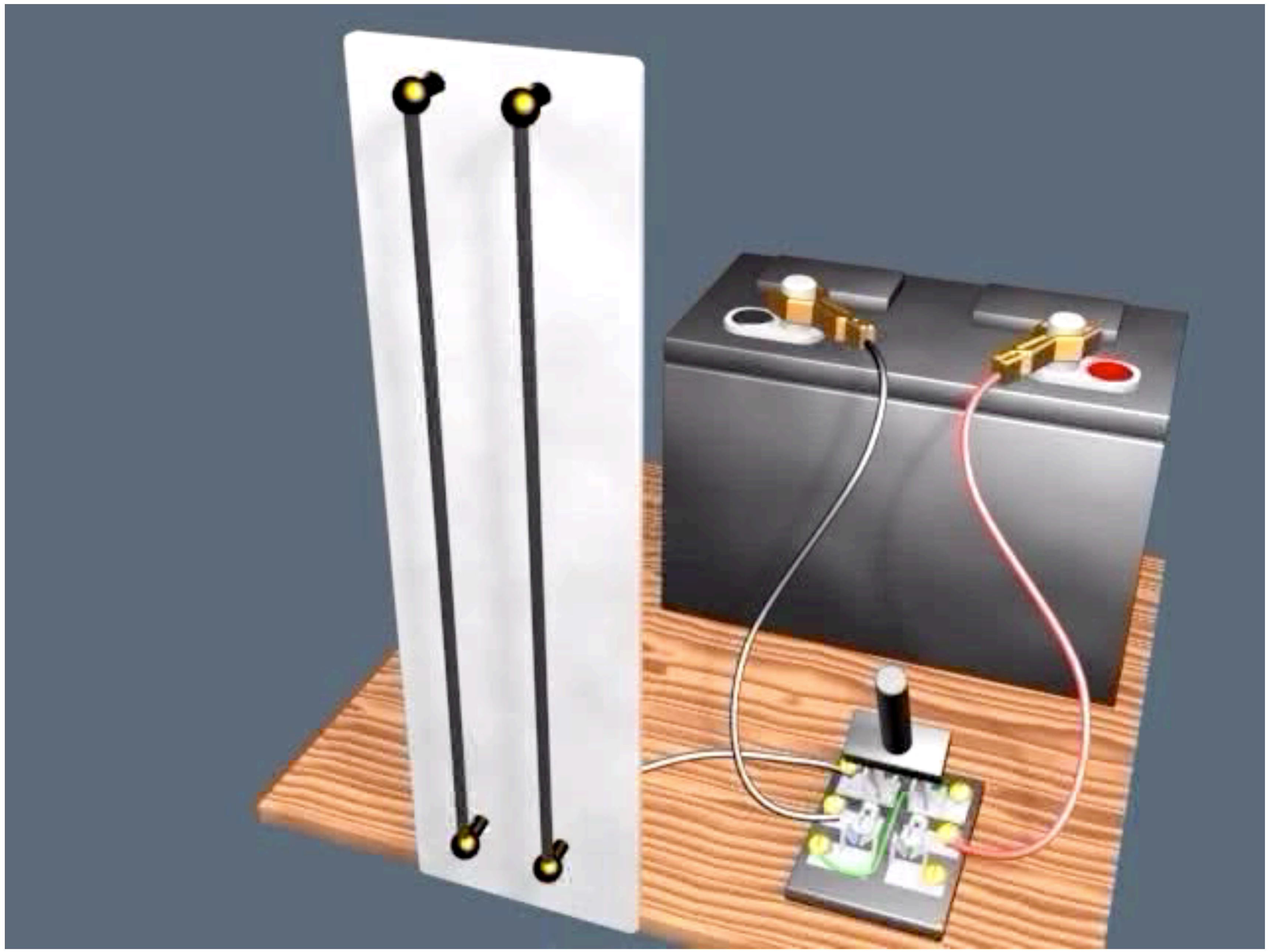


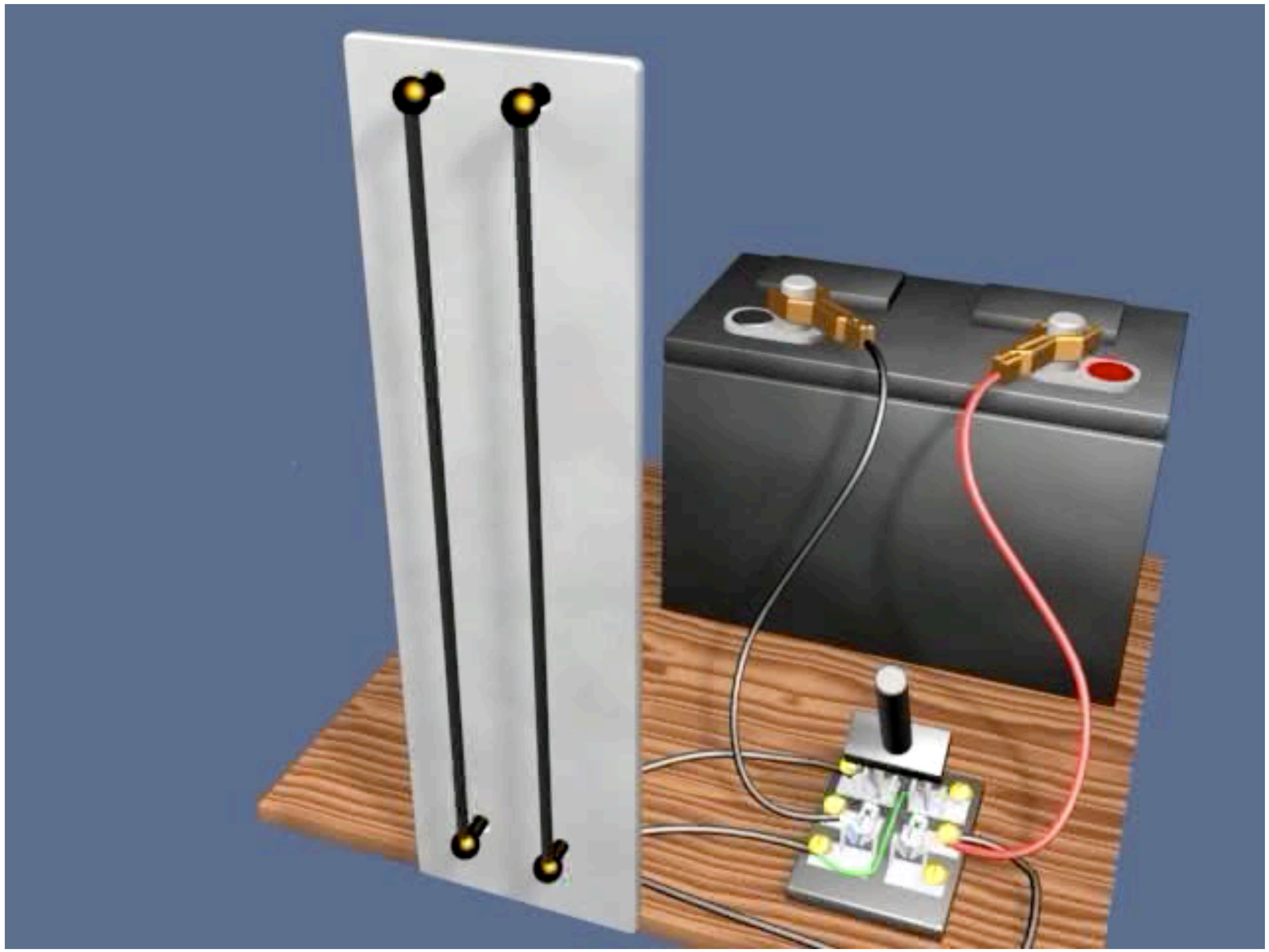
Fig. 5.2 (Introduction to Electrodynamics, D. J. Griffiths)

Parallel currents



Visualisation credit: MIT

Anti-parallel currents

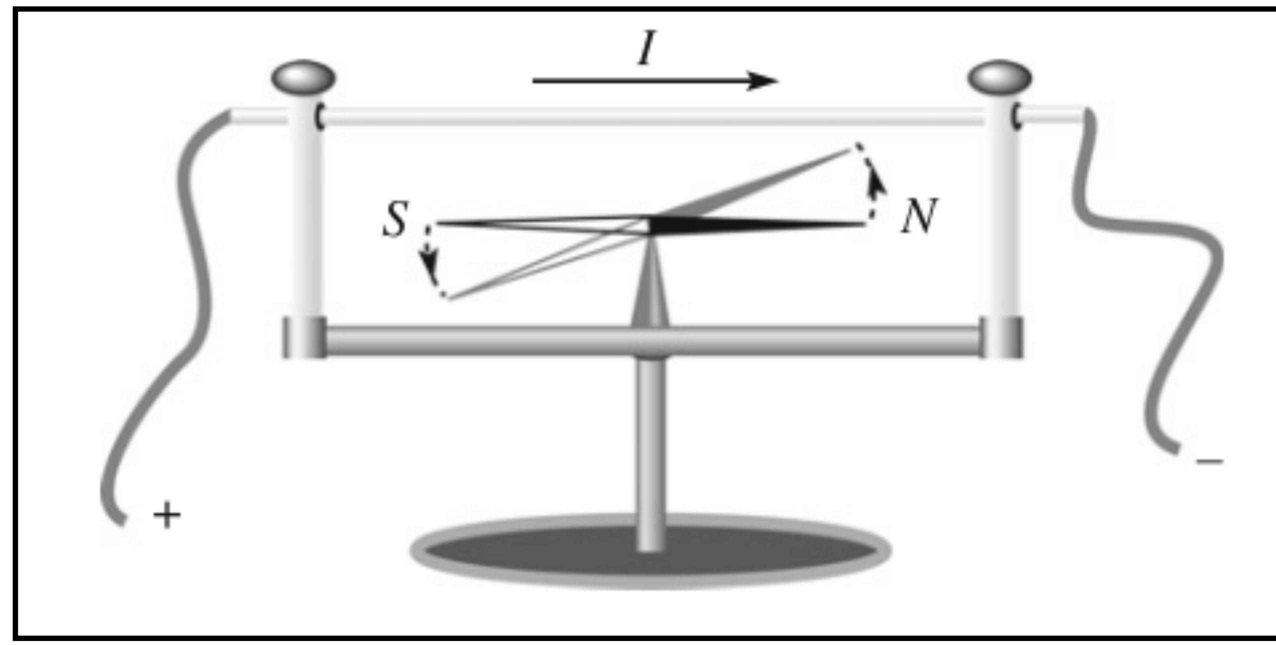


Visualisation credit: MIT

Orsted Experiment

Electric current produces magnetic field (H C Orsted, 1820).

Magnetic needle was found to change its equilibrium position when the current is turned on.



Orsted experiment, Credit: Springer

Magnetic Force

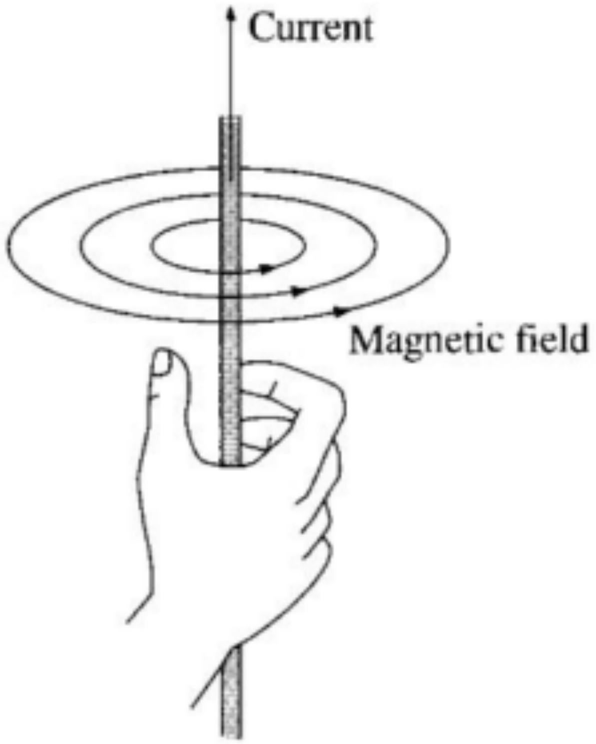
- Magnetic force on a charged particle having charge Q moving with velocity $\vec{v}$ in the presence of a magnetic field $\vec{B}$ is

$$\vec{F}_{\text{magnetic}} = Q(\vec{v} \times \vec{B})$$

- In the presence of both electric and magnetic field the net force is

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$ Lorentz Force Law

See Example 5.1, 5.2 (Introduction to Electrodynamics, D. J. Griffiths)



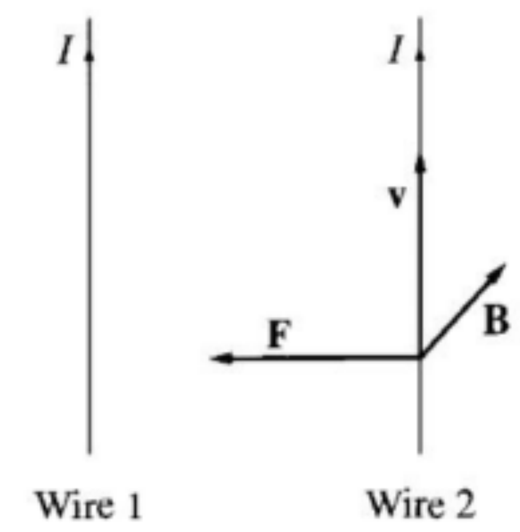
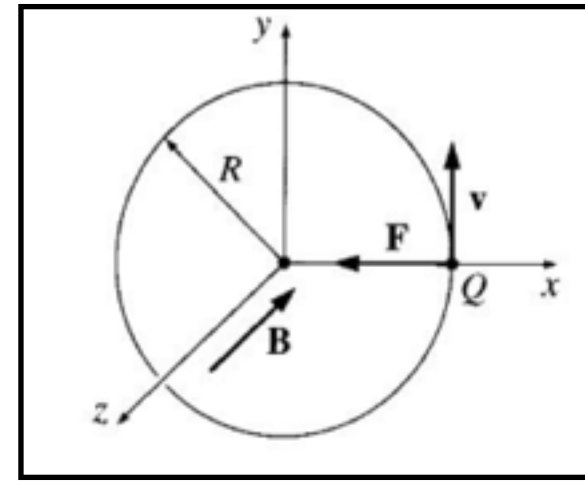


Fig. 5.3, 5.4 (Introduction to Electrodynamics, D. J. Griffiths)

Cyclotron: Motion of a charged particle in a magnetic field

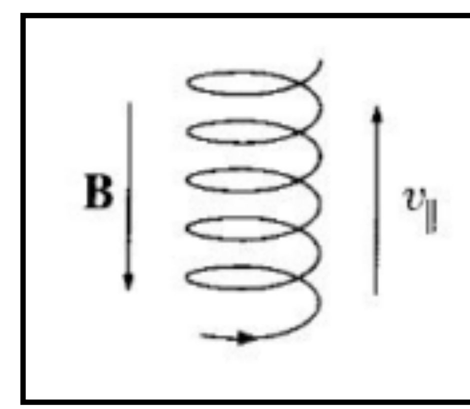


For $\vec{v} \perp \vec{B}$ we can write

$$QvB = m\frac{v^2}{R} \implies p = QBR$$

which is used for measuring particle momentum. If velocity has some component parallel to the magnetic field, then magnetic field does not affect the parallel motion.

The particle then moves in a helix.



$$Qv_\perp B = m\frac{v_\perp^2}{R}$$

The cyclotron is a charged particle accelerator invented by Ernest Orlando Lawrence (USA, 1901–1958) in 1932.

Fig. 5.5, 5.6 (Introduction to Electrodynamics, D. J. Griffiths)

Cyclotron

Charged particle can be accelerated to a very high speed as they complete a large number of round trips across the gap maintained at some fixed potential difference V.

Gain in speed in one trip across the gap is: $v = \sqrt{2qV/m}$

Gain in speed in n trips:

$$v_n = \sqrt{2nqV/m}$$

1 T magnetic field can accelerate a proton to a speed ~ 30% of c with just 100 volts peak voltage in fraction of a second. An electrostatic accelerator will need millions of volts to achieve the same.

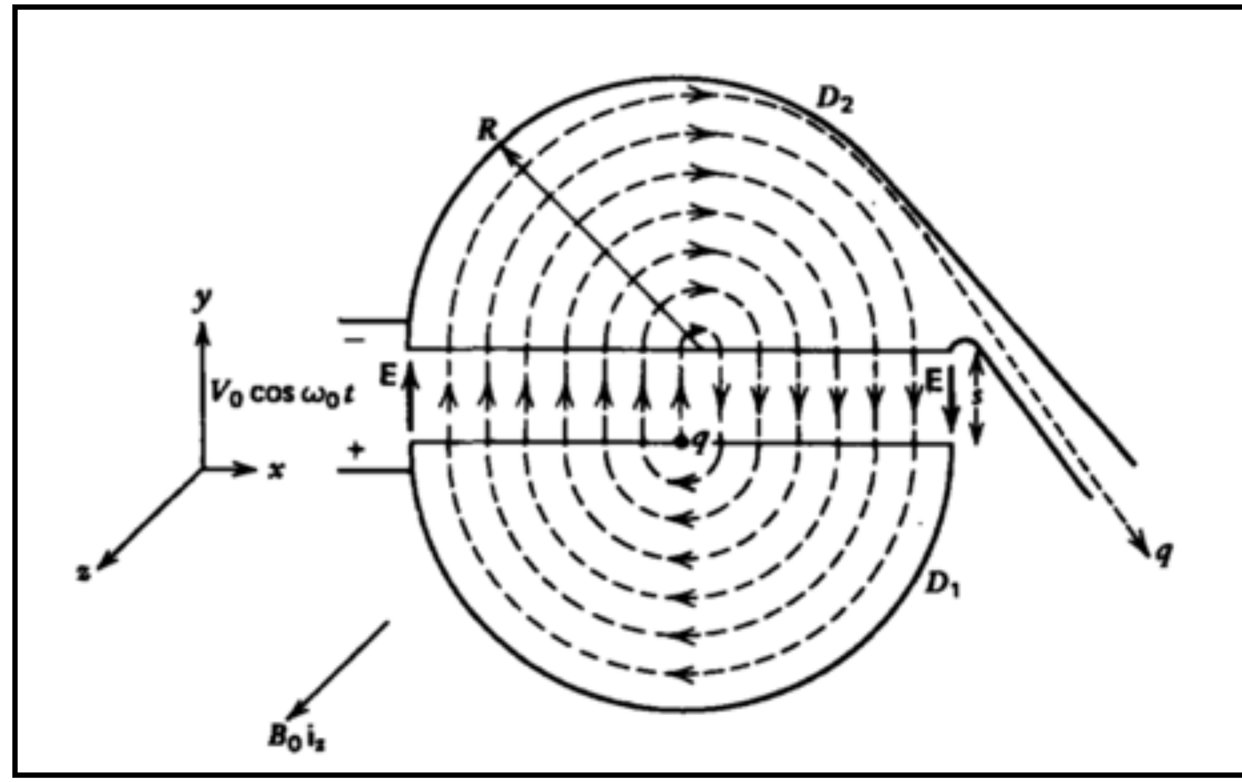


Image credit: Wiley

Another interesting application of Lorentz force law is Hall Effect: See tutorial 8

Cycloid Motion:

Motion of charged particles in simultaneous presence of electric and magnetic fields at right angles to each other.

For $\vec{E} = E\hat{z}, \vec{B} = B\hat{x}$, the charged particle initially at the origin will move in the y-z plane and hence its velocity is $\vec{v} = \dot{y}\hat{y} + \dot{z}\hat{z}$

Why no motion along x-direction?

The Lorentz Force law says:

$$F = Q(\vec{E} + \vec{v} \times \vec{B}) = Q(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}) = m\vec{a} = m(\ddot{y}\hat{y} + \ddot{z}\hat{z})$$

which gives the equations of motion as: $m\ddot{y} = QB\dot{z}, \;\; m\ddot{z} = QE - QB\dot{y}$

Denoting $\omega = QB/m$, the equations are: $\ddot{y} = \omega\dot{z}, \;\; \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$

Differentiating the 1st and then using the second equation:

$$\dddot{y} = \omega^2\left(\frac{E}{B} - \dot{y}\right) \implies y(t) = C_1 \cos\omega t + C_2 \sin\omega t + (E/B)t + C_3$$

Using y(t) in the second equation:

$$\ddot{z} = \omega(E/B + C_1\omega\sin\omega t - C_2\omega\cos\omega t - E/B) = \omega^2(C_1\sin\omega t - C_2\cos\omega t)$$
$$\implies z(t) = C_2\cos\omega t - C_1\sin\omega t + C_4$$

Using the initial conditions: $\dot{y}(0) = 0, \dot{z}(0) = 0, y(0) = 0, z(0) = 0$
one can find $C_3 = -C_1, C_2 = -\frac{E}{\omega B}, C_4 = -C_2, C_1 = 0$

The final solutions are: $y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t), \;\; z(t) = \frac{E}{\omega B}(1 - \cos\omega t)$

which is a circle $(y - R\omega t)^2 + (z - R)^2 = R^2, R = E/(\omega B)$ whose centre moves in the y-direction with speed $v = \omega R = E/B$

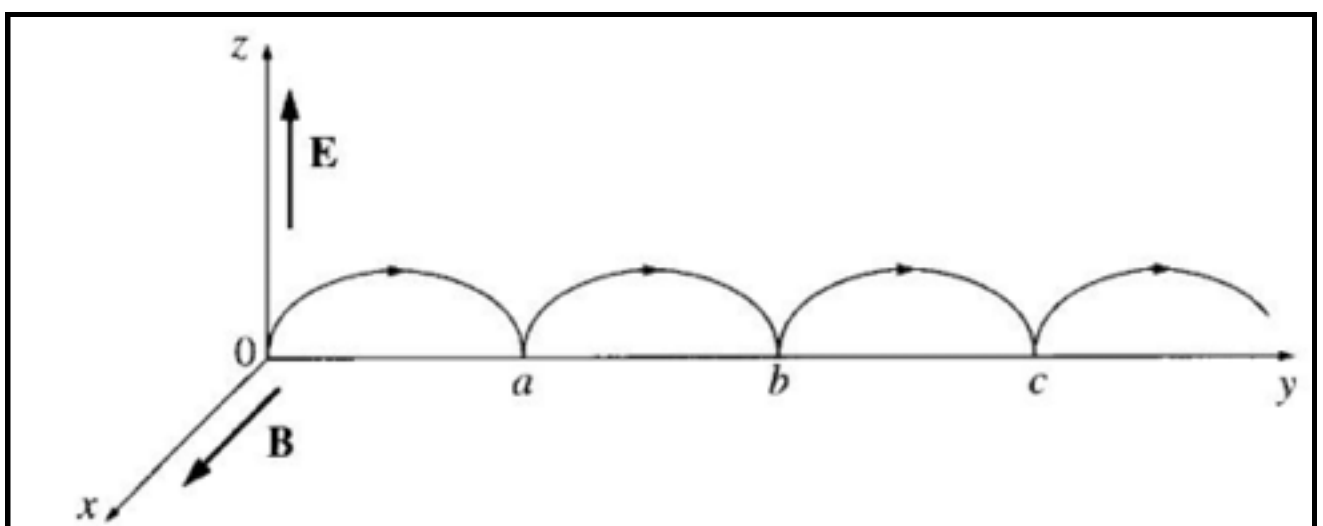


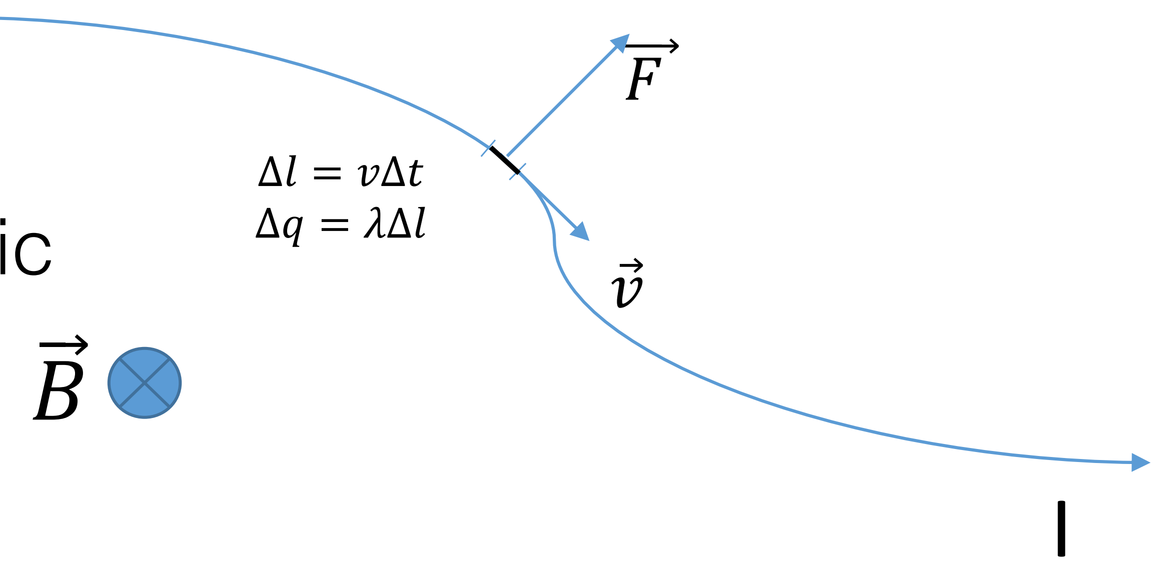
Fig. 5.7 (Introduction to Electrodynamics, D. J. Griffiths)

Motion is similar to that of a spot on the rim of a wheel rolling in y direction: **Cycloid motion**

Magnetic Force on current carrying wire

- Current passing through a wire can be written as $\vec{I} = \lambda \vec{v}$ where λ is the charge per unit length and $\vec{v}$ is the velocity.

- The magnetic force on a segment of this wire in the presence of a magnetic field B is



$$\vec{F} = \int (\vec{v} \times \vec{B}) dq = \int (\vec{I} \times \vec{B}) dl$$

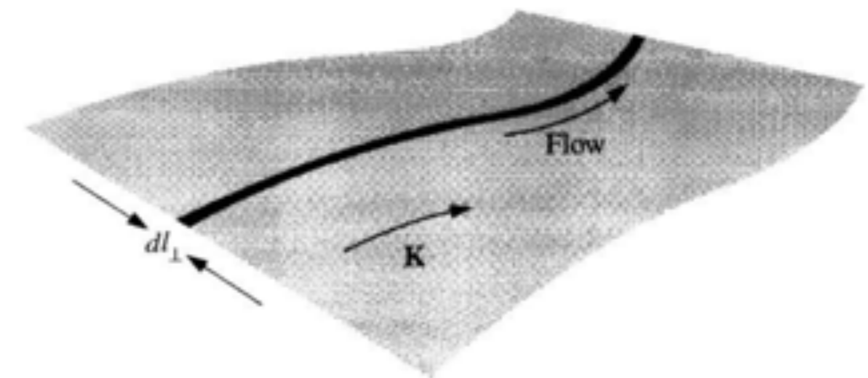
(Using $dq = \lambda dl, \lambda\vec{v} = \vec{I}$)

- As I and dl point in the same direction we can write it as $\vec{F} = \int I(\vec{dl} \times \vec{B})$

- For constant current $\vec{F} = I \int (\vec{dl} \times \vec{B})$

Magnetic Force on surface & volume current

- Let the dI be the current in a ribbon of infinitesimal width $dl_\perp$. The surface current density is $\vec{K} \equiv \frac{d\vec{I}}{dl_\perp}$.
- If the surface charge density is σ and its velocity is $\vec{v}$ then the surface current density is $\vec{K} = \sigma\vec{v}$.
- The magnetic force on the surface current is given by $\vec{F} = \int (\vec{v} \times \vec{B})\sigma da = \int (\vec{K} \times \vec{B}) da$
- If current in a tube of infinitesimal cross section $da_\perp$ is $d\vec{I}$, the volume current density is $\vec{J} \equiv \frac{d\vec{I}}{da_\perp}$
- If the volume charge density is ρ and the velocity is $\vec{v}$ then $\vec{J} = \rho\vec{v}$



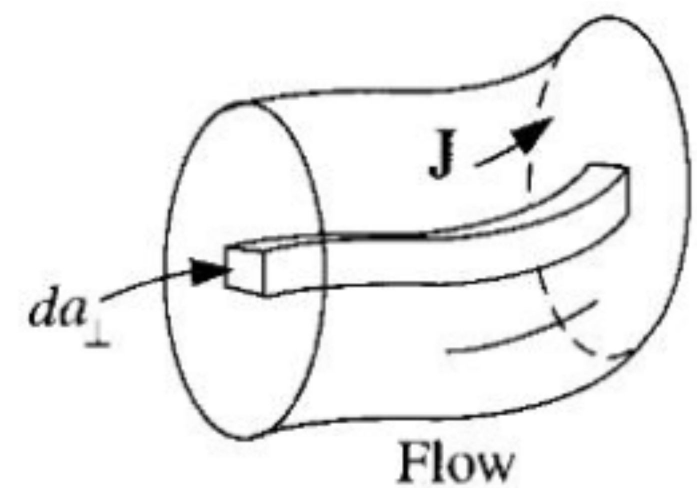


Fig. 5.13, 5.14 (Introduction to Electrodynamics, D. J. Griffiths)

- The magnetic force on a volume current is $\vec{F} = \int (\vec{v} \times \vec{B})\rho d\tau = \int (\vec{J} \times \vec{B}) d\tau$

Continuity Equation

- Since $\vec{J} \equiv \frac{d\vec{I}}{da_\perp}$, the current crossing a surface S can be written as
$$I = \int_S J da_\perp = \int_S \vec{J} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{J}) d\tau$$

- Since electromagnetic charge is conserved, the net flow of current through the surface should be equal to the rate of change of charge density inside the volume enclosed by the surface that is,

$$\int_V (\vec{\nabla} \cdot \vec{J}) d\tau = -\frac{d}{dt} \int_V \rho d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

- Since this applies to any volume, we can write

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

Continuity Equation

Magnetic Forces do no work!

- Displacement of a moving charged particle in time dt is $d\vec{l} = \vec{v}dt$

- The work done is therefore

$$dW_{\mathrm{magnetic}} = \vec{F}_{\mathrm{magnetic}} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}dt = 0$$

- Therefore, magnetic force may alter the direction in which the charged particle moves, but it can not alter its speed.

See example 5.3 (Introduction to Electrodynamics, D J Griffiths)

Magnetostatics

- Stationary charges $\longrightarrow$ Constant electric fields: Electrostatics. Steady current $\longrightarrow$ Constant magnetic fields: Magnetostatics.

- Steady current: continuous flow of current without change and without charge piling up anywhere.

- A moving point charge can not constitute a steady current.

- When a steady current flows through a wire, its magnitude I must be same all along the line so that there is no piling up of charges anywhere.

- In magnetostatics: $$\frac{\partial \rho}{\partial t} = 0 \implies \vec{\nabla} \cdot \vec{J} = 0$$

Magnetostatics

The magnetic field of a steady line current is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathfrak{r}}}{\mathfrak{r}^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\vec{l'}\times\hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

Biot-Savart Law

where the integration is along the current path and μ_0 is the permeability of free space

$$\mu_0 = 4\pi\times 10^{-7}\mathrm{N/A^2}$$

Unit of B (from Lorentz force law) is = Newton/(Coulomb metre/second)=Newton/(Ampere metre). $[\mathrm{B}] = \left[\mathrm{FQ^{-1}V^{-1}}\right] = \left[\mathrm{MT^{-1}Q^{-1}}\right]$

1 N/(A. m) = 1 Tesla (T) = 10000 Gauss (G)

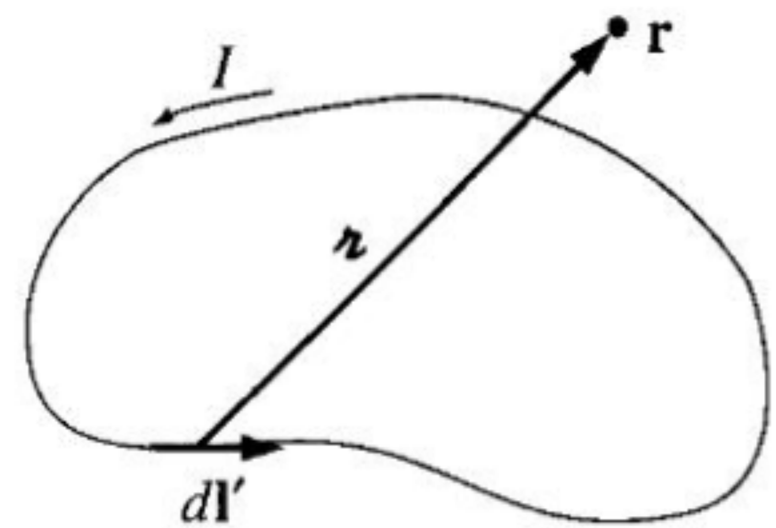
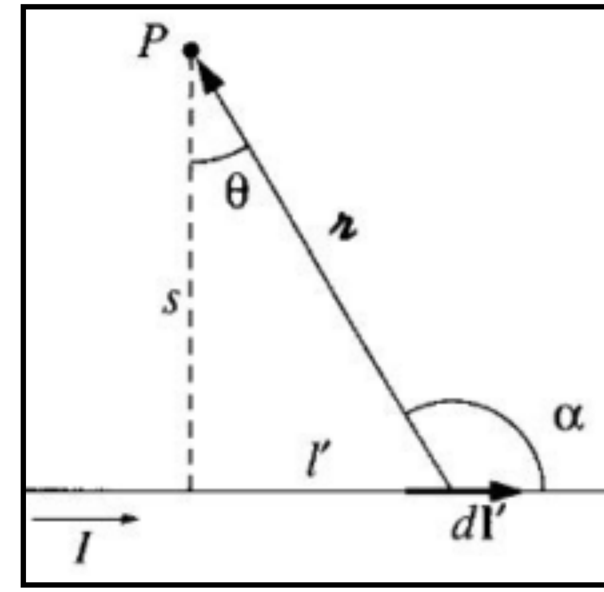


Fig. 5.17 (Introduction to Electrodynamics, D. J. Griffiths)

$$\vec{\mathfrak{r}} = \vec{r} - \vec{r'}$$

Example 5.5 (Introduction to Electrodynamics, D. J. Griffiths):
Find the magnetic field at a distance s from a long straight wire carrying a steady current I.



Using Biot-Savart law: $B(\vec{s}) = \frac{\mu_0}{4\pi} I \int \frac{|d\vec{l'} \times \hat{\mathfrak{r}}|}{\mathfrak{r}^2}, \vec{\mathfrak{r}} = \vec{s} - \vec{l'}$

Using $|d\vec{l'} \times \hat{\mathfrak{r}}| = dl' \sin\alpha = dl' \cos\theta$

$$l' = s\tan\theta \implies dl' = \frac{s}{\cos^2\theta} d\theta \qquad s = \mathfrak{r}\cos\theta \implies \frac{1}{\mathfrak{r}^2} = \frac{\cos^2\theta}{s^2}$$

we can write,

$$B(\vec{s}) = \frac{\mu_0}{4\pi} I \int_{\theta_1}^{\theta_2} \left(\frac{s}{\cos^2\theta} \cos\theta d\theta\right) \left(\frac{\cos^2\theta}{s^2}\right)$$

$$\implies B(\vec{s}) = \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta d\theta$$

$$\implies B(\vec{s}) = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 - \sin\theta_1)$$

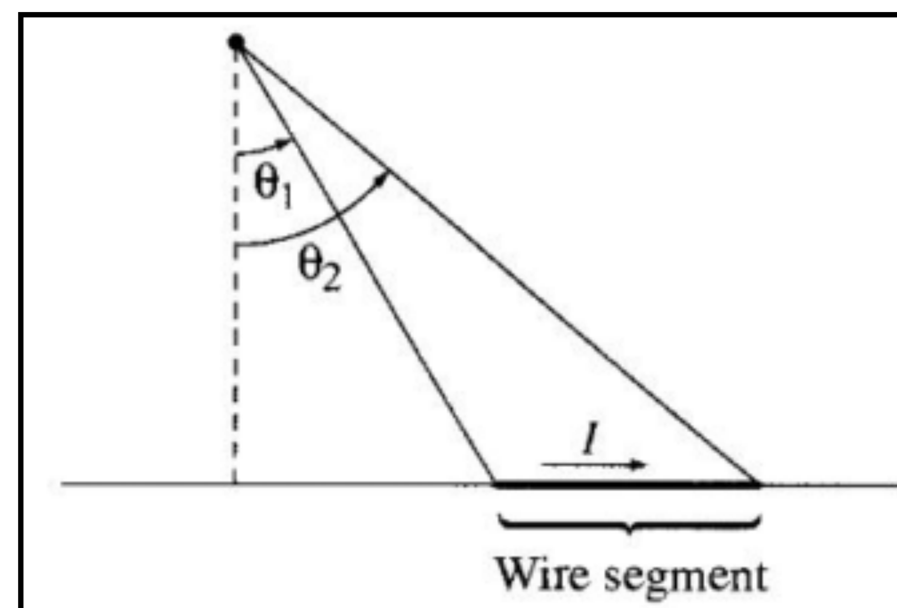


Fig. 5.18, 5.19 (Introduction to Electrodynamics, D. J. Griffiths)

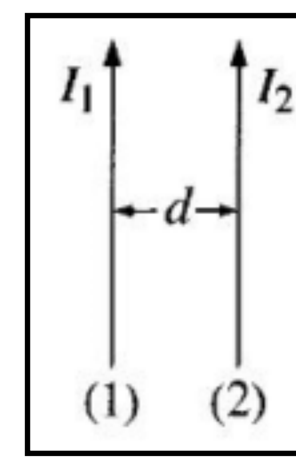
The magnetic field due to a straight wire carrying current I at a distance s is $B(\vec{s}) = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$

For an infinite wire, $\theta_1 = -\pi/2, \theta_2 = \pi/2$, therefore, $B(\vec{s}) = \frac{\mu_0 I}{2\pi s}$

Force between two parallel wires carrying current $I_{1,2}$ **and separation d**:

The field due wire 1 at the location of wire 2: $B = \frac{\mu_0 I_1}{2\pi d}$

The force on the wire 2 in the presence of this field is:



$$F = I_2 \int |d\vec{l} \times \vec{B}| = I_2 \left(\frac{\mu_0 I_1}{2\pi d}\right) \int dl$$

Thus, the force per unit length on the wire 2 is:

$$f = \frac{\mu_0 I_1 I_2}{2\pi d}$$

which is (repulsive) attractive is the currents are (anti) parallel

Example 5.6 (Introduction to Electrodynamics, D. J. Griffiths): Find the magnetic field a distance z above the centre of a circular loop of radius R, which carries a steady current I.

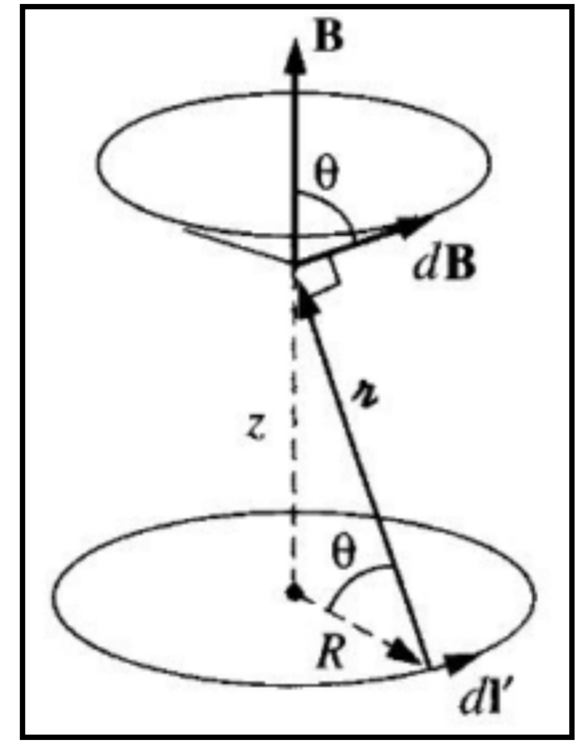


Fig. 5.21, (Introduction to Electrodynamics, D. J. Griffiths)

Field due to an elemental current element is:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l'} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

Taking the vertical components only and integrating:

$$B(z) = \frac{\mu_0 I}{4\pi} \int \frac{dl'}{\mathfrak{r}^2} \cos\theta = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{\mathfrak{r}^2} \int dl'$$

$$B(z) = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{\mathfrak{r}^2} 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$

where we have used $\cos\theta = \frac{R}{\mathfrak{r}}, \;\; \mathfrak{r} = (R^2 + z^2)^{1/2}$

Biot-Savart Law: Summary

- For line current: $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} dl'$$
- For surface current: $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} da'$$
- For volume current: $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau'$$

$$\vec{\mathfrak{r}} = \vec{r} - \vec{r'} \qquad \vec{I} = \lambda\vec{v}, \vec{K} = \sigma\vec{v}, \vec{J} = \rho\vec{v}$$

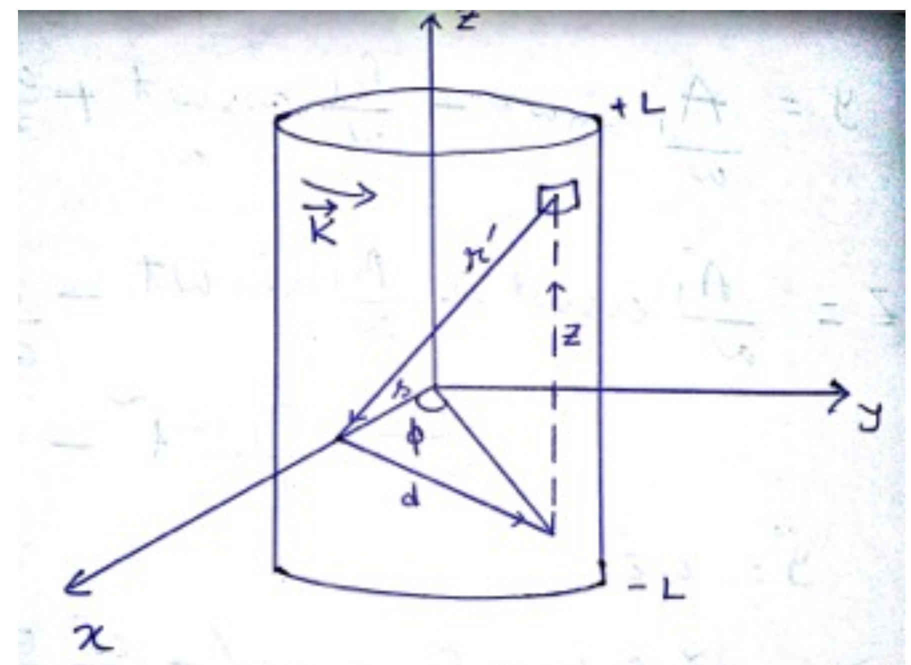
- **Superposition principle in magnetostatics**: For a collection of source currents, the net field is the vector sum of the fields due to each of them taken separately.

Apply to conductors with holes to find B everywhere

Problem 5.44 ((Introduction to Electrodynamics, D. J. Griffiths): Use the Biot-Savart law for surface currents to find the field inside and outside an infinitely long solenoid of radius R, with n turns per unit length, carrying a steady current I.

The surface current is $\vec{K} = K\hat{\phi} = K(-\sin\phi\hat{x} + \cos\phi\hat{y})$

Biot-Savart law says: $\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\times\hat{r'}}{(r')^2}da$



$$da = Rd\phi dz$$

$$\vec{r'} = -\vec{d} - z\hat{z} = -(\vec{R} - \vec{s}) - z\hat{z}$$

$$\implies \vec{r'} = (s - R\cos\phi)\hat{x} - R\sin\phi\hat{y} - z\hat{z}$$

$$\vec{K}\times\vec{r'} = K\Big[(-z\cos\phi)\hat{x} + (-z\sin\phi)\hat{y} + (R - s\cos\phi)\hat{z}\Big]$$

$$(r')^2 = d^2 + z^2 = z^2 + R^2 + s^2 - 2Rs\cos\phi$$

$$B_z = \frac{\mu_0 KR}{4\pi}\int \frac{R - s\cos\phi}{(z^2+d^2)^{3/2}}d\phi dz$$

$$= \frac{\mu_0 KR}{4\pi}\int_0^{2\pi}(R - s\cos\phi)\left[\int_{-\infty}^{+\infty}\frac{dz}{(z^2+d^2)^{3/2}}\right]d\phi$$

x,y components vanish
due to symmetry about z axis

Using $\int_{-\infty}^{+\infty} \frac{dz}{(z^2+d^2)^{3/2}} = \frac{2}{d^2}$ we get

$$B_z = \frac{\mu_0 K R}{2\pi} \int_0^{2\pi} \frac{R - s\cos\phi}{(R^2 + s^2 - 2Rs\cos\phi)} d\phi$$
$$= \frac{\mu_0 K R}{2\pi} \frac{1}{2R} \left[(R^2 - s^2) \int_0^{2\pi} \frac{1}{(R^2 + s^2 - 2Rs\cos\phi)} d\phi + \int_0^{2\pi} d\phi \right]$$

Using $\int_0^{2\pi} \frac{1}{a + b\cos\phi} d\phi = \frac{2\pi}{(a^2 - b^2)^{1/2}}$ $\quad a = R^2 + s^2, \quad b = -2Rs$

$$B_z = \frac{\mu_0 K}{4\pi} \left[\frac{R^2 - s^2}{|R^2 - s^2|} 2\pi + 2\pi \right] = \frac{\mu_0 K}{2} \left[\frac{R^2 - s^2}{|R^2 - s^2|} + 1 \right]$$

Inside the solenoid s<R, $B_z = \frac{\mu_0 K}{2}(1+1) = \mu_0 K = \mu_0 n I$

Outside the solenoid s>R, $B_z = \frac{\mu_0 K}{2}(-1+1) = 0$

The same can be found at one step using Ampere's law (to be discussed later).