Higher Order Linear ODE: Existence and Uniqueness Results, Fundamental Solutions, Wronskian

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Higher-Order ODEs

Recall a general n-th order ODE is often written as

$$F(x, y, y', \ldots, y^{(n)}) = 0, \quad y \in C^n(\mathbb{R}).$$

There are two types of ODE, namely, Linear ODE and Non-linear ODE.

Linear ODE: An ODE given by $F(x, y, y', \ldots, y^{(n)}) = 0$ is said to be linear if it can be written as $L(y) = g(x)$, where $L : C^n(\mathbb{R}) \to C(\mathbb{R})$ is a linear differential operator.

Definition The differential operator $L : C^n(\mathbb{R}) \to C(\mathbb{R})$ is said to be linear if for any $y(x), y_1(x), y_2(x) \in C^n(\mathbb{R})$ and $c \in \mathbb{R}$,

- $L(y_1 + y_2) = L(y_1) + L(y_2)$, and $L(cy) = cL(y)$.

Example: Consider $y'' + 3xy' + xy = x$, where $(Ly) := y'' + 3xy' + xy$ is a linear differential operator.

Non-linear ODE: A non-linear ODE involves higher powers of y and/or derivatives of y.

Example: $y'' + xy'^2 + xy^3 = x$ is a non-linear ODE. Note that $Ly := y'' + xy'^2 + xy^3$ is not linear.

- A general n-th order linear ODE is represented as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x),$$

where a_i and g are given functions, $a_n(x) \neq 0$.

- $Ly := a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y$ is called a linear differential operator.
- When $g(x) = 0$, $Ly = 0$ is called homogeneous differential equation.

Existence and Uniqueness Results

Theorem: (Existence and uniqueness theorem for linear IVP of order n)
Suppose that $a_j(x), g(x) \in C([a,b])$ and $a_n(x) \neq 0$ for all $x \in [a,b]$. Let $x_0 \in [a,b]$. Then the initial value problem (IVP)

$$(Ly)(x) = g(x), \;\; y^{(j)}(x_0) = \alpha_j, \; j = 0, \ldots, n-1,$$

where $\alpha_j \in \mathbb{R}$ has a unique solution $y(x)$ for all $x \in [a,b]$.

In particular, if g=0 and $\alpha_j = 0, \; j = 0, \ldots, n-1$, then $y(x) = 0$ for all $x \in [a,b]$.

Example:

- The IVP $(1+x^2)y'' + xy' - y = \tan x,\ y(1) = 1,\ y'(1) = 2$ has a unique solution exists on $(-\pi/2, \pi/2))$.
- The IVP $y'' + 3x^2y' + e^x y = \sin x,\ y(0) = 1,\ y'(0) = 0$ has a unique solution exists on $(-\infty, \infty))$.
- The IVP $y'' - y = 0,\ y(1) = 0,\ y'(1) = 0$ has a trivial solution $y(x) = 0$ for all $x \in \mathbb{R}$.

Theorem:(Superposition principle for homogeneous equation)

Let $y_i \in C^n([a, b]),\ i = 1, \cdots, n$ be any solutions of $Ly = 0$ on $[a, b]$. Then $y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$, where $c_i,\ i = 1, \cdots, n$ are arbitrary constants, is also a solution on $[a, b]$.

Example: $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$ are two solutions of $y'' - 4y' + 4y = 0$. Note that $y(x) = c_1y_1 + c_2y_2$ is also a solution of $y'' - 4y' + 4y = 0$.

Theorem:(Superposition principle for non-homogeneous equation)

Let $y_{p_i} \in C^n([a,b])$ be solutions of $L(y) = g_i(x)$ for each $i = 1, \cdots, n$ on $[a,b]$. Then

$$y_p(x) = c_1 y_{p_1}(x) + c_2 y_{p_2}(x) + \cdots + c_n y_{p_n}(x),$$

where c_i, $i = 1, \cdots, n$ are arbitrary constants, is also a solution of $L(y) = \sum_{i=1}^{n} c_i g_i(x)$ on $[a,b]$.

Example: Note that $y_{p_1}(x) = e^x$ is solution of $y'' - 2y' + 2y = e^x$ and $y_{p_2}(x) = x^2$ is a solution of $y'' - 2y' + 2y = 2 - 4x + 2x^2$. Then $10e^x + 7x^2$ is a solution of $y'' - 2y' + 2y = 10e^x + 7(2 - 4x + 2x^2)$.

Solution of linear ODE:

Consider the linear differential operator

$$Ly := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y,$$

where $a_i : \mathbb{R} \to \mathbb{R}$ are given functions.

Problem: Given $g \in C(\mathbb{R})$, find $y \in C^n(\mathbb{R})$ such that $Ly = g$.

Since $L : C^n(\mathbb{R}) \to C(\mathbb{R})$ is a linear transformation, the solution set of

$$Ly = g$$

is given by

$$\mathrm{Ker}(L) + y_P,$$

where y_p is a particular solution (PS) satisfying $Ly_P = g$ and $Ker(L) = \{y \in C^n(\mathbb{R}) \,|Ly = 0\}$.

Note that $Ker(L)$ is a vector space.

If $\{y_1, \ldots, y_n\} \subset C^n(\mathbb{R})$ is a basis of $\mathrm{Ker}(L)$, then the general solution (GS) of $Ly = g$ is given by

$$y = c_1 y_1 + \cdots + c_n y_n + y_P.$$

Moral: (The GS of $Ly = g$) = (The GS of $Ly = 0$)
+ (a PS y_p satisfying $Ly_p = g$)

The next result shows that the homogeneous equation $Ly = 0$ has n linearly independent solutions, that is, $\dim(\mathrm{Ker}(L)) = n$.

Theorem: We have $\dim(\mathrm{Ker}(L)) = n$.

Proof: Choose $x_0 \in [a, b]$. Define $T : \mathrm{Ker}(L) \to \mathbb{K}^n$ by

$$Ty := (y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)).$$

Then T is linear. By uniqueness theorem, $T(y) = \mathbf{0}$ implies $y = 0$. Therefore, T is one-to-one. The existence of solution shows that T is onto. Thus, T is bijective. Hence $\dim(\mathrm{Ker}(L)) = n$.

Recall that all solutions of $Ly = g$ are given by the affine subspace

$$\mathrm{Ker}(L) + y_P,$$

where $Ly_P = g$ is a particular solution.

Hence what we need to do is to find

- a basis $\{y_1, \ldots, y_n\}$ of $\mathrm{Ker}(L)$ and
- a particular solution y_P.

Then the general solution of $Ly = g$ is given by

$$y := c_1 y_1 + \cdots + c_n y_n + y_P.$$

Definition: If $\{f_1, \ldots, f_n\} \subset C^n(\mathbb{R})$, then

$$W(f_1, \cdots, f_n) := \begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ & & \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of $f_1, \ldots, f_n$.

Theorem: Let $y_1, y_2, \ldots, y_n \in C^n((a,b))$ be solution of $L(y) = 0$, where $a_i(x) \in C((a,b))$, $i = 0, \ldots, n$, and $a_n(x) \neq 0$. If

$$W(y_1, \ldots, y_n)(x_0) \neq 0$$

for some $x_0 \in (a,b)$, then every solution $y(x)$ of $L(y) = 0$ can be expressed in the form

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x),$$

where $C_1, \ldots, C_n$ are constants.

Example: The functions $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ are both solutions of $y'' - 4y = 0$ on $(-\infty, \infty)$. The Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4 \neq 0.$$

The general solution is $y = c_1 e^{2x} + c_2 e^{-2x}$.

Theorem: (**Abel's formula**) Let $y_1, \ldots, y_n$ be any n solutions to

$$Ly = y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y = 0$$

on (a, b). Then, for $x_0 \in (a, b)$, we have

$$W(y_1, \ldots, y_n)(x) = W(y_1, \ldots, y_n)(x_0) \exp\left(-\int_{x_0}^{x} p_1(t)\,dt \right).$$

Proof. Prove for $n = 2$.

Corollary: The Wronskian of solutions $W(y_1, \ldots, y_n)(x)$ is either identically zero or never zero on (a, b).

Definition: A set of n linearly independent solutions of $Ly = 0$ that spans $\mathrm{Ker}(L)$ are called fundamental solutions.

Fact: Let $y_1, y_2, \ldots, y_n \in C^n((a, b))$ be solutions of $L(y) = 0$. Then the following statements are equivalent:

- $\{y_1, y_2, \ldots, y_n\}$ is a fundamental solution set on (a, b).
- $\{y_1, y_2, \ldots, y_n\}$ are linearly independent on (a, b).
- $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ on (a, b).

Theorem: Let $y_p(x) \in C^n((a,b))$ be a particular solution to $L(y) = g(x)$ on (a,b) and let $\{y_1, y_2, \ldots, y_n\} \in C^n((a,b))$ be a fundamental solution set of $L(y) = 0$ on (a,b). Then every solution of $L(y) = g$ on (a,b) can be expressed in the form

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x) + y_p(x)$$

Example: Given that $y_p = x^2$ is a particular solution to $y'' - y = 2 - x^2$ and $y_1(x) = e^x$ and $y_2(x) = e^{-x}$ are solution to $y'' - y = 0$. A general solution is

$$y(x) = C_1 e^x + C_2 e^{-x} + x^2.$$

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