Lecture Slides 5:
Differentiability of functions of several variables

Department of Mathematics
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Differential Calculus for $f : \mathbb{R}^n \to \mathbb{R}$

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- f is differentiable at $\mathbf{a} \Rightarrow f$ is continuous at $\mathbf{a}$.
- Sum, product and chain rules hold for $\mathrm{D}f(\mathbf{a})$.
- Mean Value Theorem and Taylor's Theorem hold for f.

Differentiability of $f : (c, d) \subset \mathbb{R} \to \mathbb{R}$

- Conventional: f is differentiable at $a \in (c, d)$ if there exists $\alpha \in \mathbb{R}$ such that
$$\alpha = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

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- Smart: f is differentiable at $a \in (c, d)$ if there exists a linear map $L : \mathbb{R} \to \mathbb{R}$ such that
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Differentiability of $f : U \subset \mathbb{R}^n \to \mathbb{R}$

Smart: Let $U \subset \mathbb{R}^n$ be open. Then $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in U$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\mathbf{h}\to 0} \frac{|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})|}{\|\mathbf{h}\|} = 0. \qquad (*)$$

The linear map L is called the derivative of f at $\mathbf{a}$ and is denoted by $\mathrm{D}f(\mathbf{a})$, that is, $L = \mathrm{D}f(\mathbf{a})$.

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Fact: If $L : \mathbb{R}^n \to \mathbb{R}$ is linear then $L(\mathbf{x}) = \mathbf{p} \bullet \mathbf{x} = \langle \mathbf{x}, \mathbf{p} \rangle$ for some $\mathbf{p} := (L(\mathbf{e}_1), \ldots, L(\mathbf{e}_n)) \in \mathbb{R}^n$.

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• Considering $\mathbf{h} := t\mathbf{e}_i$ for $t \in \mathbb{R}$ in $(*)$ and letting $t \to 0$, we have

$$\mathbf{p} = (\partial_1 f(\mathbf{a}), \ldots, \partial_n f(\mathbf{a})).$$

Theorem: If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in U$ then partial derivatives $\partial_1 f(\mathbf{a}), \ldots, \partial_n f(\mathbf{a})$ exist and the derivative $\mathrm{D}f(\mathbf{a}) : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\mathrm{D}f(\mathbf{a})\mathbf{h} = \nabla f(\mathbf{a}) \bullet \mathbf{h} = \langle \mathbf{h}, \nabla f(\mathbf{a}) \rangle,$$

where $\nabla f(\mathbf{a}) := (\partial_1 f(\mathbf{a}), \ldots, \partial_n f(\mathbf{a}))$ is called the gradient of f at $\mathbf{a}$.

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• Conventional: $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in U$ if $\nabla f(\mathbf{a}) := (\partial_1 f(\mathbf{a}), \ldots, \partial_n f(\mathbf{a}))$ exists and

$$\lim_{\mathbf{h} \to 0} \frac{|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \bullet \mathbf{h}|}{\|\mathbf{h}\|} = 0.$$

Examples

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x,y) := xy\dfrac{x^2 - y^2}{x^2 + y^2}$ if $(x,y) \neq (0,0)$. Then

- f is continuous at $(0,0)$ and $\nabla f(0,0) = (0,0)$.

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- Now

$$\frac{|f(h,k) - f(0,0) - \nabla f(0,0) \bullet (h,k)|}{\|(h,k)\|} \leq \frac{|hk|}{\|(h,k)\|} \to 0$$

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Consider $g : \mathbb{R}^3 \to \mathbb{R}$ given by $g(x,y,z) := 3x + 5y - z$. Then g is differentiable. Find $\mathrm{D}g(x,y,z)$.

Affine approximation

Define the error function $e : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$e(\mathbf{h}) := \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \bullet \mathbf{h}}{\|\mathbf{h}\|}.$$

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- Then f is differentiable at $\mathbf{a}$ if and only if

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \bullet \mathbf{h} + e(\mathbf{h})\|\mathbf{h}\|$$

and $e(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$.

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- The affine function $y = f(\mathbf{a}) + \nabla f(\mathbf{a}) \bullet \mathbf{h}$ approximates $f(\mathbf{a}+\mathbf{h})$ for small $\|\mathbf{h}\| \iff f$ is differentiable at $\mathbf{a}$.

Geometric interpretation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in \mathbb{R}^n$. Then

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- For $n = 2$: a plane $z = f(a, b) + f_x(a, b)x + f_y(a, b)y$ passing through $(0, 0, f(a, b)) \in \mathbb{R}^3$ that approximates $f(a + x, b + y)$.

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- For $n \geq 3$: a hyperplane $y = f(\mathbf{a}) + \partial_1 f(\mathbf{a})x_1 + \cdots + \partial_n f(\mathbf{a})x_n$ passing through $(\mathbf{0}, f(\mathbf{a})) \in \mathbb{R}^{n+1}$ that approximates $f(\mathbf{a} + \mathbf{x})$.

Implications of differentiability

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$.

- If f is differentiable at $\mathbf{a}$ then f is continuous at $\mathbf{a}$.
- If f is differentiable at $\mathbf{a}$ then directional derivatives exist for all $\mathbf{u} \in \mathbb{R}^n$ and

$$\mathrm{D}_{\mathbf{u}} f(\mathbf{a}) = \mathrm{D} f(\mathbf{a})\mathbf{u} = \nabla f(\mathbf{a}) \bullet \mathbf{u}.$$

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Proof: Use

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \bullet \mathbf{h} + e(\mathbf{h})\|\mathbf{h}\|$$

and the fact that $e(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$.

Example

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x,y) := \dfrac{x^2y}{x^4+y^2}$ if $(x,y) \neq (0,0)$. Then

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- $\mathrm{D}_{\mathbf{u}} f(0,0)$ exists for all $\mathbf{u} \in \mathbb{R}^2$ and $\nabla f(0,0) = (0,0)$.
- For $\mathbf{u}$ such that $u_1 u_2 \neq 0$, we have

$$\mathrm{D}_{\mathbf{u}} f(0,0) = u_1^2/u_2 \neq \nabla f(0,0) \bullet \mathbf{u}.$$

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Moral: The equality $\mathrm{D}_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \mathbf{u}$ may not hold if f is NOT differentiable at $\mathbf{a}$.

Properties of derivative

Fact: Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in \mathbb{R}^n$. Then

- $\mathrm{D}(f + \alpha g)(\mathbf{a}) = \mathrm{D}f(\mathbf{a}) + \alpha \mathrm{D}g(\mathbf{a})$.
- $\mathrm{D}(fg)(\mathbf{a}) = \mathrm{D}f(\mathbf{a})g(\mathbf{a}) + f(\mathbf{a})\mathrm{D}g(\mathbf{a})$.

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Proof: Use $\nabla(fg)(\mathbf{a}) = f(\mathbf{a})\nabla g(\mathbf{a}) + g(\mathbf{a})\nabla f(\mathbf{a})$ and

$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \bullet \mathbf{h} + e(\mathbf{h})\|\mathbf{h}\|$$

and the fact that $e(\mathbf{h}) \to 0$ as $\|\mathbf{h}\| \to 0$.

Sufficient condition for differentiability

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$. If $\partial_i f(\mathbf{x})$ exists for $i = 1, 2, \ldots, n$, and are continuous on $B(\mathbf{a}, \epsilon)$ for some $\epsilon > 0$, then f is differentiable at $\mathbf{a}$.

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Remark: f differentiable at $\mathbf{a} \not\Rightarrow \partial_i f(\mathbf{x})$ is continuous at $\mathbf{a}$.

Example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x,y) := (x^2 + y^2)\sin(1/(x^2 + y^2))$ if $(x,y) \neq (0,0)$.

*** End ***