PH 102 : Electromagnetism **Department of Physics**
Physics II **Indian Institute of Technology Guwahati**
Quiz-1 **30/01/18**
Total marks: 10 **Time Limit: 50 Minutes**

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1. An electrostatic field is given by $\vec{E} = \lambda(x\hat{x} + y\hat{y})$, where λ is a constant. Use Gauss's law to find the total charge enclosed by the surface shown in the figure consisting of S_1, the curved portion of the half cylinder $z = (r^2 - y^2)^{1/2}$ of length h and radius r; S_2 and S_3, the two semicircular plane end pieces; and S_4, the rectangular portion of the xy plane. The centre of the half cylinder is located at the origin. Express your answer in terms of λ, h and r.

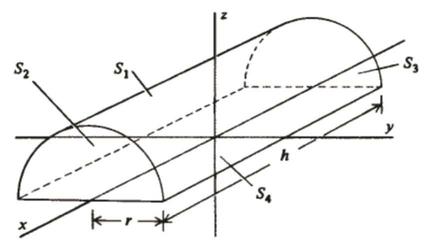

Figure 1: Gaussian surface for problem 1.

[6]

Solution:

Method 1: The surface integral has four parts.

Along S_1: The equation of the surface is $\varphi(z, y) = z^2 + y^2 = r^2$. Hence, the unit vector is

$$\hat{n} = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|} = \frac{z\hat{z} + y\hat{y}}{r}.$$

Therefore

$$\begin{aligned}
\int_{S_1} \vec{E}.d\vec{a} &= \int_{S_1} \vec{E}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{z}|} = \int_{S_1} \frac{\lambda y^2}{r}\frac{dxdy}{z/r} \\
&= \lambda \int_{x=-h/2}^{h/2} \int_{y=-r}^{r} \frac{y^2}{z} dxdy \\
&= \lambda h \int_{y=-r}^{r} \frac{y^2}{\sqrt{r^2 - y^2}} dy
\end{aligned}$$

Substituting $y = r\sin\theta$, the limits on y changes to $\theta : -\pi/2 \to \pi/2$. Hence, the integral reduces to

$$\begin{aligned}
\int_{S_1} \vec{E}.d\vec{a} &= \frac{\lambda h r^2}{2} \int_{-\pi/2}^{\pi/2} (1 - \cos 2\theta) d\theta \\
&= \frac{\pi \lambda h r^2}{2}
\end{aligned}$$

Along S_2: The surface element is $d\vec{a} = dydz\hat{x}$. Therefore

$$\int_{S_2} \vec{E}.d\vec{a} \;=\; \lambda \int_{S_2} xdydz = \frac{\lambda h}{2}\frac{\pi r^2}{2},$$

where, we have used $x = h/2$.

Along S_3: The surface element is $d\vec{a} = dydz(-\hat{x})$. Therefore

$$\int_{S_3} \vec{E}.d\vec{a} \;=\; -\lambda \int_{S_3} xdydz = \frac{\lambda h}{2}\frac{\pi r^2}{2},$$

where, we have used $x = -h/2$.

Along S_4: The surface element is $d\vec{a} = dxdy(-\hat{z})$. Therefore

$$\int_{S_4} \vec{E}.d\vec{a} = 0.$$

Hence the total flux through all the surfaces

$$\oint_{\mathcal{S}} \vec{E}.d\vec{a} = \lambda h \pi r^2.$$

Applying Gauss's law, the closed surface integral is related to the charge enclosed as

$$\begin{aligned} \oint_{\mathcal{S}} \vec{E}.d\vec{a} &= \frac{q_{\text{enc}}}{\epsilon_0} \\ &= \lambda h \pi r^2 \\ \implies q_{\text{enc}} &= \lambda h \pi r^2 \epsilon_0 \end{aligned}$$

Method 2: The surface integral can also be done using cylindrical coordinate, however, the transformation rule will be a little different due to the alignment of the cylindrical surface and will be as follows:

$$z = s\sin\phi,\; y = s\cos\phi,\; x = x.$$

Hence,

$$\hat{s} = \cos\phi\hat{y} + \sin\phi\hat{z},\; \hat{\phi} = -\sin\phi\hat{y} + \cos\phi\hat{z},\; \hat{x} = \hat{x}.$$

Therefore the electric field:

$$\vec{E} = \lambda(x\hat{x} + y\hat{y}) = \lambda x\hat{x} + \lambda s\cos\phi(\cos\phi\hat{s} - \sin\phi\hat{\phi})$$

Hence, Along S_1:

$$\begin{aligned}\int_{S_1}\vec{E}.d\vec{a} &= \int_{S_1}(\lambda x\hat{x}+\lambda s\cos\phi(\cos\phi\hat{s}-\sin\phi\hat{\phi})).(sd\phi dx\hat{s})\\ &= \lambda r^2\int_{x=-h/2}^{h/2}\int_{\phi=0}^{\pi}\cos^2\phi d\phi dx = \frac{\lambda r^2\pi h}{2},\end{aligned}$$

where, we have used $s=r$ for the cylindrical surface.

Along S_2:

$$\begin{aligned}\int_{S_2}\vec{E}.d\vec{a} &= \int_{S_1}(\lambda x\hat{x}+\lambda s\cos\phi(\cos\phi\hat{s}-\sin\phi\hat{\phi})).(sdsd\phi\hat{x})\\ &= \lambda\int_{s=0}^{r}\int_{\phi=0}^{\pi}xsdsd\phi = \frac{\lambda h\pi r^2}{4},\end{aligned}$$

where, we have used $x=h/2$.

Similarly, Along S_3:

$$\int_{S_3}\vec{E}.d\vec{a} = \frac{\lambda h\pi r^2}{4},$$

where, we have used $x=-h/2$.

Along S_4: The area element is $dxdy(-\hat{z})$. Note that $\hat{z}=\sin\phi\hat{s}+\cos\phi\hat{\phi}$, therefore $\int_{S_4}\vec{E}.d\vec{a}=0$.

Hence the total flux through all the surfaces

$$\oint_{\mathcal{S}}\vec{E}.d\vec{a} = \lambda h\pi r^2.$$

Applying Gauss's law, the closed surface integral is related to the charge enclosed as

$$\begin{aligned}\oint_{\mathcal{S}}\vec{E}.d\vec{a} &= \frac{q_{\text{enc}}}{\epsilon_0}\\ &= \lambda h\pi r^2\\ \implies q_{\text{enc}} &= \lambda h\pi r^2\epsilon_0\end{aligned}$$

If you choose the cylindrical surface in a conventional way $y=s\sin\phi,\ x=s\cos\phi,\ z=z$, you will get wrong answer, unless you also change the surface equation for the cylinder.

Method 3: One may also use Divergence theorem to evaluate the closed surface integral.

$$\int_{\mathcal{V}}\vec{\nabla}.\vec{E}d\tau = \oint_{\mathcal{S}}\vec{E}.d\vec{a}$$

Now, $\vec{\nabla}.\vec{E}=2\lambda$.

Therefore

$$\int_{\mathcal{V}} \vec{\nabla}.\vec{E} d\tau = 2\lambda \int_{\mathcal{V}} d\tau = 2\lambda \frac{\pi r^2 h}{2} = \lambda \pi r^2 h$$

Applying Gauss's law, the closed surface integral is related to the charge enclosed as

$$\begin{aligned} \oint_{\mathcal{S}} \vec{E}.d\vec{a} &= \int_{\mathcal{V}} \vec{\nabla}.\vec{E} d\tau = \frac{q_{\text{enc}}}{\epsilon_0} \\ &= \lambda h \pi r^2 \\ \implies q_{\text{enc}} &= \lambda h \pi r^2 \epsilon_0 \end{aligned}$$

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2. Let a vector field be $\vec{A} = (x - y)\hat{x} + (x + y)\hat{y}$. Find the line integral $\oint_C \vec{A}.d\vec{r}$ around the closed curve C as shown in the figure 2. From the result obtained, comment on the curl of the vector field, without doing an explicit calculation.

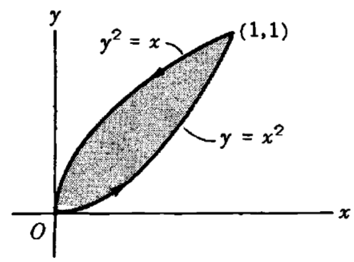

Figure 2: Closed path for problem 2.

[3]

Solution:

Along $y = x^2$: $dy = 2xdx$. Hence $\int \vec{A}.d\vec{r} = \int_{x=0}^{1} (x + x^2 + 2x^3)dx = 4/3$

Along $y^2 = x$: $dx = 2ydy$. Hence $\int \vec{A}.d\vec{r} = \int_{y=1}^{0} (2y^3 - y^2 + y)dy = -2/3$.

Hence, $\oint \vec{A}.d\vec{r} = 4/3 - 2/3 = 2/3$.

From Stoke's theorem $\int_{\mathcal{S}} \vec{\nabla} \times \vec{A}.d\vec{a} = \oint_C \vec{A}.d\vec{r}$. Therefore, the above result suggest $\vec{\nabla} \times \vec{A} \neq 0$.

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3. Evaluate the integral

$$J = \int_{\mathcal{V}} e^{-r} \left(\vec{\nabla}.\frac{\hat{r}}{r^2} \right) d\tau,$$

where $\mathcal{V}$ is a sphere of radius R centred at the origin.

[1]

Solution:

We know that $\vec{\nabla}.\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$.

Hence the integral is

$$J = \int_{\mathcal{V}} e^{-r}\left(\vec{\nabla}.\frac{\hat{r}}{r^2}\right) d\tau = \int_{\mathcal{V}} e^{-r} 4\pi\delta^3(\vec{r}) d\tau = 4\pi\frac{1}{e^0} = 4\pi.$$