Higher Order Linear ODE: Existence and Uniqueness Results, Fundamental Solutions, Wronskian

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Differential Operators

Let I be an interval and n be a positive integer. We will now see what is meant by a differential operator from $C^n(I)$ to $C(I)$.
Consider the map $D : C^1(I) \to C(I)$ given by $D(f) = f'$. More generally, for any $k \in \{1, \dots, n\}$, consider the map $D^k : C^k(I) \to C(I)$ given by $D^k(f) = f^{(k)}$, where $f^{(k)}$ denotes the k-th derivative of f.
Observe that $D^k = D \circ D \circ \cdots \circ D$ (k times). By convention, $D^0 = Id$ (the identity map).
The operators (or maps) D^k are called **differentiation operators**.
Definition: A **differential operator** from $C^n(I)$ to $C(I)$ is a map $L : C^n(I) \to C(I)$ which can be expressed as a function of the differentiation operator D.
For example: Take $L = D^n$ or $L = e^D$ or
$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 D^0$, where
$a_0, a_1, \dots, a_n \in C(I)$.

Linear ODEs

Definition The differential operator $L : C^n(I) \to C(I)$ is said to be **linear** if for any $y(x), y_1(x), y_2(x) \in C^n(I)$ and $c \in \mathbb{R}$,

- $L(y_1 + y_2) = L(y_1) + L(y_2)$, and $L(cy) = cL(y)$.

Linear ODE: An ODE given by $F(x, y, y', \ldots, y^{(n)}) = 0$ on an interval I is said to be linear if it can be written as $L(y)(x) = g(x)$, where $L : C^n(I) \to C(I)$ is a linear differential operator.

Example: Consider $y'' + 3xy' + xy = x$, this is a linear ODE. Note that $L(y)(x) := y'' + 3xy' + xy$ is linear.

Non-linear ODE: A non-linear ODE involves higher powers of y and/or derivatives of y.

Example: $y'' + xy'^2 + xy^3 = x$ is a non-linear ODE. Note that $L(y)(x) := y'' + xy'^2 + xy^3$ is not linear.

- FACT: A general n-th order linear ODE can be represented as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x),$$

  where a_i and g are given functions of x, $a_n(x) \neq 0$.
- CHECK THAT: $L : C^n(I) \to C(I)$ given by $L(y)(x) := a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x)$ is a linear differential operator.
- When $g(x) = 0$, $L(y)(x) = 0$ is called homogeneous differential equation.

Existence and Uniqueness Results

Theorem: (Existence and uniqueness theorem for linear IVP of order n)
Suppose that $a_j(x), g(x) \in C(I)$ and $a_n(x) \neq 0$ for all $x \in I$. Let $x_0 \in I$. Then the initial value problem (IVP)

$$(Ly)(x) = g(x), \;\; y^{(j)}(x_0) = \alpha_j, \; j = 0, \ldots, n-1,$$

where $\alpha_j \in \mathbb{R}$ and $L(y)(x) :=$
$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x)$, has a unique solution $y(x)$ for all $x \in I$.

In particular, if g=0 and $\alpha_j = 0, \; j = 0, \ldots, n-1,$ then $y(x) = 0$ for all $x \in I$.

Example:

- The IVP
$(1+x^2)y'' + xy' - y = \tan x,\ y(1) = 1,\ y'(1) = 2$ has a unique solution which exists on $(-\pi/2, \pi/2)$).
- The IVP $y'' + 3x^2y' + e^x y = \sin x,\ y(0) = 1,\ y'(0) = 0$ has a unique solution which exists on $(-\infty, \infty)$).
- The IVP $y'' - y = 0,\ y(1) = 0,\ y'(1) = 0$ has a trivial solution $y(x) = 0$ for all $x \in \mathbb{R}$.

Theorem:(Superposition principle for homogeneous equation)

Let $y_i \in C^n(I),\ i = 1, \cdots, n$ be any solutions of $L(y)(x) = 0$ on I. Then $y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$, where $c_i,\ i = 1, \cdots, n$ are arbitrary constants, is also a solution on I.

Example: $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$ are two solutions of $y'' - 4y' + 4y = 0$. Note that $y(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution of $y'' - 4y' + 4y = 0$.

Theorem:(Superposition principle for non-homogeneous equation)

Let $y_{p_i} \in C^n(I)$ be solutions of $L(y)(x) = g_i(x)$ for each $i = 1, \cdots, n$ on I. Then

$$y_p(x) = c_1 y_{p_1}(x) + c_2 y_{p_2}(x) + \cdots + c_n y_{p_n}(x),$$

where $c_i,\ i = 1, \cdots, n$ are arbitrary constants, is a solution of $L(y)(x) = \sum_{i=1}^{n} c_i g_i(x)$ on I.

Example: Note that $y_{p_1}(x) = e^x$ is solution of $y'' - 2y' + 2y = e^x$ and $y_{p_2}(x) = x^2$ is a solution of $y'' - 2y' + 2y = 2 - 4x + 2x^2$. Then $10e^x + 7x^2$ is a solution of $y'' - 2y' + 2y = 10e^x + 7(2 - 4x + 2x^2)$.

Solution of linear ODE:

Consider the linear differential operator L where

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y,$$

where $a_i : I \to \mathbb{R}$ are given functions.

Problem: Given $g \in C(I)$, find $y \in C^n(I)$ such that $L(y) = g$.

Since $L : C^n(I) \to C(I)$ is a linear transformation, the solution set of

$$L(y) = g$$

is given by

$$\mathrm{Ker}(L) + y_P,$$

where y_p is a particular solution (PS) satisfying $L(y_P) = g$ and $Ker(L) = \{y \in C^n(I) \,|L(y) = 0\}$.

Note that $Ker(L)$ is a vector space.

If $\{y_1, \ldots, y_n\} \subset C^n(I)$ is a basis of $\mathrm{Ker}(L)$, then the general solution (GS) of $L(y) = g$ is given by

$$y = c_1 y_1 + \cdots + c_n y_n + y_P.$$

Moral: (The GS of $L(y) = g$) $=$ (The GS of $L(y) = 0$)
$+$ (a PS y_p satisfying $L(y_p) = g$)

Theorem: We have $\dim(\mathrm{Ker}(L)) = n$.

Proof: Choose $x_0 \in I$. Define $T : \mathrm{Ker}(L) \to \mathbb{K}^n$ by

$$Ty := (y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)).$$

Here, $\mathbb{K}$ is either the field of real numbers or the field of complex numbers.
Then T is linear. By uniqueness theorem, $T(y) = \mathbf{0}$ implies $y = 0$. Therefore, T is one-to-one. The existence of solution shows that T is onto. Thus, T is bijective. Hence $\dim(\mathrm{Ker}(L)) = n$.