MA 102 (Mathematics II)

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Quiz 1 Solution

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(1) Examine if the limit as $(x, y) \to (0, 0)$ exist for the following function:

$$f(x,y) = \begin{cases} \frac{x^4+y^4}{x^2-y^2} & x \neq \pm y \\ 0 & x = \pm y \end{cases}$$

[Marks 3]

Solution. Fix $m \neq 0$. Let us approach the origin along the curve $y = +\sqrt{x^2 - mx^4}$; here $y \to 0$ as $x \to 0$.

$$f(x,y) = \frac{x^4 + (x^2 - mx^4)^2}{mx^4} = \frac{1 + (1 - mx^2)^2}{m} \longrightarrow \frac{2}{m}, \qquad \text{as } x \to 0$$

Thus the limit depends on m and hence is different for different values of m. Hence $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Alternative Solution: Consider the sequence $\{(x_n, y_n)\}$ where $x_n = 1/n$, $y_n = 1/n + 1/n^3$. Then,

$$f(x_n, y_n) = \frac{1 + (1 + \frac{1}{n})^4}{-(2 + \frac{1}{n^2})} \to -1, \quad \text{as } n \to \infty$$

But if we take the sequence $\{(1/n, 1/n)\}$ then $f(1/n, 1/n) \to 0$. Hence we have two distinct sequences both of which converges to $(0, 0)$ but the functional limit is different. Thus $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. $\square$

(2) Examine the continuity of the following function at $(0, 0)$:

$$g(x,y) = \begin{cases} \frac{x^6-2y^4}{(x^2+y^2)^{3/2}} & x^2 + y^2 \neq 0 \\ 0 & x = 0, y = 0 \end{cases}$$

[Marks 2]

Solution. Let $\epsilon > 0$ be given. Then notice that

$$\begin{aligned} |f(x,y) - f(0,0)| = |f(x,y)| &\leq \frac{|x|^6 + 2|y|^4}{(x^2+y^2)^{3/2}} \leq \frac{(\sqrt{x^2+y^2})^6 + 2(\sqrt{x^2+y^2})^4}{(\sqrt{x^2+y^2})^3} \\ &\leq (\sqrt{x^2+y^2})^3 + 2\sqrt{x^2+y^2} \\ &\leq 3\sqrt{x^2+y^2} \end{aligned}$$

Thus choosing $\delta = \frac{\epsilon}{3}$ we have

$$|f(x,y) - f(0,0)| < \epsilon \quad \text{whenever} \quad \sqrt{x^2+y^2} < \delta.$$

Hence f is continuous at $(0,0)$.

Alternative Solution: We use polar coordinates to show $\lim_{(x,y)\to(0,0)} \frac{x^6-2y^4}{(x^2+y^2)^{3/2}} = 0$. Let $x = r\cos\theta, y = r\sin\theta$. Note that $(x,y)\to(0,0)$ is equivalent to $r\to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^6-2y^4}{(x^2+y^2)^{3/2}} = \lim_{r\to 0} \frac{r^4(r^2\cos^6\theta - 2\sin^2\theta)}{r^3} = \lim_{r\to 0} r(r^2\cos^6\theta - 2\sin^2\theta) = 0$$

Therefore $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. Hence f is continuous at $(0,0)$. □

(3) Consider the curve parametrized by $\Gamma(t) = (2\cos^3 t, 2\sin^3 t)$ for $0 \le t \le \pi/2$. Find its curvature at $t = \pi/4$. [Marks 3]

Solution. Clearly $\Gamma'(t) = (-6\cos^2 t\sin t, 6\sin^2 t\cos t) \neq 0$ for $t \in (0,\pi/2)$. Thus $\|\Gamma'(t)\| = 6|\sin t||\cos t| = 6\sin t\cos t \neq 0$ for $t\in(0,\pi/2)$. Thus we have

$$T(t) = \frac{\Gamma'(t}{\|\Gamma'(t)\|} = \left(-\frac{\cos^2 t\sin t}{\sin t\cos t}, \frac{\sin^2 t\cos t}{\sin t\cos t}\right) = (-\cos t, \sin t)$$
$$T'(t) = (\sin t, \cos t) \implies \|T'(t)\| = 1$$

Thus the curvature $\kappa(t) = \frac{\|T'(t)\|}{\|\Gamma'(t)\|} = \frac{1}{6\sin t\cos t}$. Hence curvature at $t = \pi/4$ is $\kappa(\pi/4) = 1/3$.

Alternative Solution: Again for $t \in (0,\pi/2)$ we have

$$\Gamma''(t) = (12\cos t\sin^2 t - 6\cos^3 t, 12\sin t\cos^2 t - 6\sin^3 t)$$

$$\Gamma'(t)\times\Gamma''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6\cos^2 t\sin t & 6\sin^2 t\cos t & 0 \\ 12\cos t\sin^2 t - 6\cos^3 t & 12\sin t\cos^2 t - 6\sin^3 t & 0 \end{vmatrix}$$
$$= -36\sin^2 t\cos^2 t\hat{k}$$
$$\kappa(\pi/4) = \frac{|\Gamma'(\pi/4)\times\Gamma''(\pi/4)|}{\|\Gamma'(\pi/4)\|^3} = \frac{36\times\frac{1}{4}}{3^3} = \frac{1}{3}$$

Hence curvature at $t = \pi/4$ is $\kappa(\pi/4) = \frac{1}{3}$. □

(4) a. Convert the following point from rectangular coordinates $(x,\ y,\ z)$ to spherical coordinates $(\rho,\ \phi,\ \theta)$,

$$(1,\ \sqrt{3},\ -2)\,.$$

b. Convert the following point from spherical coordinates $(\rho,\ \phi,\ \theta)$ to rectangular coordinates $(x,\ y,\ z)$,

$$(2, \pi/2, \pi/3)\,.$$

c. Convert the following point from spherical coordinates $(\rho,\ \phi,\ \theta)$ to cylindrical coordinates $(r,\ \theta,\ z)$,

$$(8, (2\pi)/3, \pi/3)\,.$$

d. Convert the following point from cylindrical coordinates $(r,\ \theta,\ z)$ to spherical coordinates $(\rho,\ \phi,\ \theta)$

$$(4, \pi/2, 5)\,.$$

[Marks 0.5+0.5+0.5+0.5=2]

Solution. (a) Given that $x = 1, y = \sqrt{3}, z = -2$. Then $\rho = \sqrt{1+3+4} = 2\sqrt{2}$ and $\tan\phi = \frac{r}{z} = \frac{2}{-2} = -1$. Thus $\phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. Again $\tan\theta = \frac{\sqrt{3}}{1}$ gives $\theta = \pi/3$. Thus $(\rho, \phi, \theta) = (2\sqrt{2}, 3\pi/4, \pi/3)$.

(b) Given that $\rho = 2, \phi = \pi/2, \theta = \pi/3$. Using the relation between spherical coordinates and cartesian coordinates we get $x = r\cos\theta = \rho\cos\theta\sin\phi = 2\cdot\cos\pi/3\cdot\sin\pi/2 = 1$ and $y = r\sin\theta = \rho\sin\theta\sin\phi = 2\cdot\sin\pi/3\cdot\sin\pi/2 = \sqrt{3}$. Again $z = \rho\cos\phi = 0$. Thus $(x, y, z) = \big(1, \sqrt{3}, 0\big)$.

(c) Given that $\rho = 8, \phi = \frac{2\pi}{3}, \theta = \pi/3$. Using the relation between spherical coordinates and cylindrical coordinates we get $r = \rho\sin\phi = 4\sqrt{3}$ and $z = \rho\cos\phi = -4$. Thus $(r, \theta, z) = \big(4\sqrt{3}, \frac{\pi}{3}, -4\big)$.

(d) Given that $r = 4, \theta = \pi/2, z = 5$. Then we have $\phi = \sqrt{r^2 + z^2} = \sqrt{41}$ and $\tan\phi = \frac{r}{z} = \frac{4}{5}$ gives $\phi = \tan^{-1}(\frac{4}{5})$. Thus $(\rho, \phi, \theta) = \big(\sqrt{41}, \tan^{-1}(\frac{4}{5}), \frac{\pi}{2}\big)$.

$\square$