MA 102 (Mathematics II)

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Tutorial Sheet No. 5

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(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(0, 0)$. Suppose that for $U := (3/5, 4/5)$ and $V := (1/\sqrt{2}, 1/\sqrt{2})$, we have $D_U f(0, 0) = 12$ and $D_V f(0, 0) = -4\sqrt{2}$. Then determine $f_x(0, 0)$ and $f_y(0, 0)$.

(2) Find the direction where the directional derivative is greatest for the function $f(x, y) = 3x^2y^2 - x^4 - y^4$ at the point $(1, 2)$.

(3) Let $f(x, y) = \frac{1}{2}\ln(x^2 + y^2) + tan^{-1}(\frac{y}{x})$, $P = (1, 3)$. Find the direction in which $f(x, y)$ is increasing the fastest at P. Find the derivative of $f(x, y)$ in this direction.

(4) A heat-seeking bug is a bug that always moves in the direction of the greatest increase in heat. Find the direction along which the heat-seeking bug will move when it is placed at the point $(2, 1)$ on a metal plate heated so that the temperature at (x, y) is given by $T(x, y) = 50y^2 e^{\frac{-1}{5}(x^2+y^2)}$.

(5) Let $f(x, y, z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of f at $(1, -1, 3)$ and the equations of the tangent plane and the normal line to the surface $f(x, y, z) = 7$ at $(1, -1, 3)$.

(6) Find $D_U f(2, 2, 1)$, where $f(x, y, z) = 3x - 5y + 2z$ and U is the unit vector in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$.

(7) Find equations for the tangent plane and the normal line to the level surface $x^2+y^2+z^2 = 4$ at the point $P_0 = (-1,\ 1,\ \sqrt{2})$

(8) Find equations for the tangent plane and normal line to the surface $z = 6 - 3x^2 - y^2$ at the point $P_0 = (1,\ 2,\ -1)$.

(9) Find the equation of the tangent plane to the graphs of the following functions at the given point:

(a) $f(x, y) = x^2 - y^4 + e^{xy}$ at the point $(1, 0, 2)$

(b) $f(x, y) = \tan^{-1}\frac{y}{x}$ at the point $(1, \sqrt{3}, \frac{\pi}{3})$.

(10) Check the following functions for differentiability, and then find the Jacobian Matrix.

(a) $f(x, y) = (e^{x+y} + y, xy^2)$ (b) $f(x, y) = (x^2 + \cos y, e^x y)$ (c) $f(x, y, z) = (ze^x, -ye^z)$.

(11) Let $z = x^2 + y^2$, and $x = 1/t, y = t^2$. Compute $\frac{dz}{dt}$ by (a) expressing z explicitly in terms of t and (ii) chain rule.

(12) Let $w = 4x + y^2 + z^3$ and $x = e^{rs^2}, y = \log\frac{r+s}{t}, z = rst^2$. Find $\frac{\partial w}{\partial s}$.

(13) If $w = \sqrt{x} + yz^3$, $x(r,\ s) = 1 + r^2 + s^2$, $y(r,\ s) = rs$, $z(r,\ s) = 3r$, then find $\partial w/\partial r$ and $\partial w/\partial s$ using the chain rule.

(14) For the following functions, compute the mixed partial derivatives at all points in $\mathbb{R}^2$. Further find out at each point, whether the mixed derivatives are equal or not?

(a) $f(x,\ y) = x\sin y + y\sin x + xy$

(b) $f(x,\ y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$ for $(x,\ y) \neq (0,\ 0)$ and $f(0,\ 0) = 0$

(15) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $F(x,\ y) = (\sin x\cos y,\ \sin x\sin y,\ \cos x\cos y)$. Show that F is differentiable in $\mathbb{R}^2$ and find its Jacobian matrix.

(16) Using Taylor's formula find the quadratic and cubic approximations of the function $f(x,\ y) = e^x\cos(y)$ near the origin.

(17) Find the first three terms in the Taylor's formula for the function $f(x,\ y) = \cos x\cos y$ at origin. Find a quadratic approximation of f near the origin. How accurate is the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$?