Name:
Roll No:

Indian Institute of Technology Guwahati

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End-Semester Examination (Total Marks 50)

PH 102 - PHYSICS II

2-5 PM, May 3, 2016

Read these instructions carefully

a) All questions are compulsory.

b) You **must** write all parts of the answers in the space provided for the given question. It is *advised* that you solve the problems in the roughbook (usual answerbook) and then copy the key steps in the space provided for that problem.

c) Write **main key steps** instead of just answers. There is partial credit for intermediate steps. Just answers without justifications may not be awarded marks.

d) **Write your name and roll number on each page**.

Useful Formulae:

a) $\nabla \cdot \mathbf{A} = \dfrac{1}{r^2}\dfrac{\partial\left(r^2 A_r\right)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}\left(A_\theta \sin\theta\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_\varphi}{\partial\varphi}$ in spherical coordinates.

b) $\nabla \cdot \mathbf{A} = \dfrac{1}{\rho}\dfrac{\partial\left(\rho A_\rho\right)}{\partial\rho} + \dfrac{1}{\rho}\dfrac{\partial A_\varphi}{\partial\varphi} + \dfrac{\partial A_z}{\partial z}$ in cylindrical coordinates.

c) $\nabla \times \mathbf{A} = \frac{1}{\rho}\begin{bmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{bmatrix}$ in cylindrical coordinates.

d) Electric and magnetic fields produced by electric dipole $\mathbf{p}$ and magnetic dipole $\mathbf{m}$ are $\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3}\left[3\left(\mathbf{p}.\hat{r}\right)\hat{r} - \mathbf{p}\right]$ and $\mathbf{B} = \frac{\mu_0}{4\pi r^3}\left[3\left(\mathbf{m}.\hat{r}\right)\hat{r} - \mathbf{m}\right]$, respectively.

e) The solution to the second order inhomogeneous linear differential equation of type

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = B$$

is $x(t) = B/\omega^2 + C_1\sin(\omega t) + C_2\cos(\omega t)$ where B, C_1 C_2 and ω are constants.

Examiner's Signature	Student's Signature	Invigilator's Signature

Question 1: Answer the following short questions:

a) Use divergence theorem for the vector field $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is a constant vector and T is a scalar field, to prove

$$\int_V (\nabla T)\, d\tau = \oint_S T d\mathbf{a}.$$

The surface S encloses the volume V and $d\mathbf{a}$ is elemental vector area. [2]

Solution:
By divergence theorem

$$\begin{aligned}
\int_V (\nabla \cdot (\mathbf{c}T))\, d\tau &= \oint_S \mathbf{c}T \cdot d\mathbf{a} \\
\int_V (\mathbf{c} \cdot \nabla T + T\nabla \cdot \mathbf{c})\, d\tau &= \mathbf{c} \cdot \oint_S T d\mathbf{a} \\
\mathbf{c} \cdot \int_V \nabla T d\tau &= \mathbf{c} \cdot \oint_S T d\mathbf{a} \qquad \because \nabla \cdot \mathbf{c} = 0
\end{aligned}$$

Since $\mathbf{c}$ is completely arbitrary vector (though constant) field, the required identity is true. [Alternatively, choose $\mathbf{c} = \hat{\mathbf{i}}$ to prove x component of the identity and similarly the other components]

b) Find the charge and current distribution represented by the scalar and vector potentials given by $\Phi(\mathbf{r}, t) = 0, \quad \mathbf{A}(\mathbf{r}, t) = -\frac{qt}{4\pi\epsilon_0 r^2}\hat{r}$. [2]

Solution:
Electric Field $\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$, magnetic field induction $\mathbf{B} = \nabla \times \mathbf{A} = 0$. These are the fields produced by a stationary point charge q at the origin.

c) The electric potential at a point, is given by

$$V(r,\theta) = \begin{cases} \frac{k}{3\epsilon_0} r\cos\theta & r < R \\ \frac{k}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta & r > R \end{cases}$$

where r and θ are spherical coordinates of the point. k and R are positive real constants. Find surface charge density at the surface $r = R$. [2]

Solution:
The surface charge density is given by

$$\begin{aligned}
\sigma(\theta) &= -\epsilon_0 \frac{\partial}{\partial r} V(r > R)\Big|_{r=R} + \epsilon_0 \frac{\partial}{\partial r} V(r < R)\Big|_{r=R} \\
&= -\frac{k}{3}\cos\theta \left(-\frac{2R^3}{r^3}\right)_{r=R} + \frac{k}{3}\cos\theta \\
&= k\cos\theta
\end{aligned}$$

d) A spherical conductor of radius a, carries a charge Q. It is surrounded by a linear dielectric material of susceptibility χ_e, with outer radius b. Find the electrostatic energy of this configuration. [2]

Solution:
The Electric field $E(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & b < r \\ \frac{Q}{4\pi\epsilon r^2}\hat{r} & a < r < b \end{cases}$. The energy of this system $W = \frac{1}{2}\int D \cdot E\, dv = \frac{Q}{8\pi\epsilon_0}\left[\frac{1}{(1+\chi_e)}(\frac{1}{a} - \frac{1}{b}) + \frac{1}{b}\right]$

e) Consider a point charge q placed at the origin of an infinite dielectric medium of permittivity $(\varepsilon = \varepsilon_0\varepsilon_r)$. Find the polarization $\mathbf{P}$, the polarization (bound) charges and the total charge. [2]

Solution:
Gauss Law:$\nabla \cdot \mathbf{D} = \rho_\mathbf{f} \Rightarrow \mathbf{D} = \frac{q}{4\pi r^2}\hat{r}$, polarization $\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} \Rightarrow \mathbf{P} = \frac{\mathbf{q}}{\mathbf{4\pi r^2}}\left(\mathbf{1} - \frac{\epsilon_0}{\epsilon}\right)\hat{r}$
Volume bound charge $\rho_b = -\nabla \cdot \mathbf{P} = -q\left(1 - \frac{\epsilon_0}{\epsilon}\right)\delta^3(\mathbf{r})$, Total charge $Q = -q\left(1 - \frac{\epsilon_0}{\epsilon}\right) + q = \frac{q}{\epsilon_r}$

f) A region $z > 0$ has $\mu_r = 4$, while the region $z < 0$ has $\mu_r = 1$. The magnetic field $\mathbf{B}$ is uniform for $z > 0$ and is given by $\mathbf{B} = \frac{B_0}{2}\left(\sqrt{\frac{3}{2}}\hat{x} + \sqrt{\frac{3}{2}}\hat{y} + \hat{z}\right)$. Find $\mathbf{B}$ and $\mathbf{H}$ for $z < 0$ assuming that there is no free current at the interface $z = 0$. [2]

Solution:
By using boundary conditions, we get

$$B_z\,(z < 0) = \frac{B_0}{2}$$

and

$$B_\parallel\,(z < 0) = \frac{B_0}{8}\left(\sqrt{\frac{3}{2}}\hat{x} + \sqrt{\frac{3}{2}}\hat{y}\right)$$

Thus

$$\mathbf{B}\,(z < 0) = \frac{B_0}{8}\left(\sqrt{\frac{3}{2}}\hat{x} + \sqrt{\frac{3}{2}}\hat{y} + 4\hat{z}\right)$$

and

$$\mathbf{H} = \mathbf{B}/\mu_0.$$

g) In *free space*, the magnetic field is given by

$$\mathbf{H} = r\left(\sin\phi\,\hat{r}\ +\ 2\,\cos\phi\,\hat{\phi}\right)\cos(\omega\,t)$$

where r and ϕ are cylindrical coordinates and ω is a constant. Find the corresponding displacement current density $\mathbf{J_D}$. [2]

Solution:
magnetic field $\mathbf{H} = r\left(sin\phi\,\hat{r}\ +\ 2\,cos\phi\,\hat{\phi}\right)cos\,(\omega\,t)$, displacement current density $\mathbf{J}_D = \nabla \times H = 3\hat{z}\,cos\,\phi\,cos\,\omega t$

h) In a dielectric medium($z < 0,\ \epsilon = 9\epsilon_0,\ \mu = \mu_0$) a plane wave with

$$\mathbf{E} = 8\left(\sqrt{8}\hat{\mathbf{x}} - \hat{\mathbf{z}}\right)e^{-i\left(10^9 t - bx - \sqrt{8}bz\right)}$$

is incident on the boundary with air at $z = 0$. Find the angle of reflection θ_R and the angle of transmission θ_T. b is a positive constant. [2]

Solution:
Electric field $E = 8\left(\sqrt{8}\hat{\mathbf{x}} - \hat{\mathbf{z}}\right)e^{-i\left(10^9 t - \zeta x - \sqrt{8}\zeta z\right)}$ $\quad tan\theta_I = \frac{k_x^I}{k_z^I} = \frac{b}{\sqrt{8}b}$ $\quad \Rightarrow \theta_I = tan^{-1}\left(\frac{1}{\sqrt{8}}\right) =$ 19.47 and from Snell's law $sin\theta_T = \frac{n_1}{n_2}sin\theta_I = 1$ $\quad \Rightarrow \theta_T = \pi/2$.

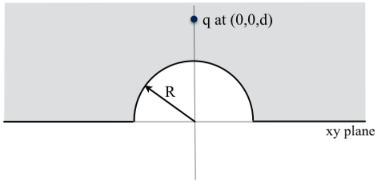
Question 2: A conducting grounded sheet lies in the xy plane with a hemispherical boss (radius R) centered at the origin (see figure). A point charge q is placed at the distance $d(> R)$ on the z axis.

[2+1+3]

a) Write down the image charges and their locations. (Don't derive, just guess on the basis of the classroom examples and write down the answer.)

b) Write down the potential at a general point $\mathbf{r} \equiv (x, y, z)$ in the shaded region ($z > 0$ and $r > R$).

c) Find the induced surface charge density and total charge induced on the *flat surface* of the conductor.



Solution:

a) $q_1 = -q$ at $(0, 0, d)$, $q_{2/3} = \mp q\frac{R}{d}$ at $\left(0, 0, \pm\frac{R^2}{d}\right)$ 2 marks here.

b) The potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\left(x^2+y^2+(z-d)^2\right)^{1/2}} + \frac{(-q)}{(x^2+y^2+(z+d)^2)^{1/2}} + \frac{(-qR/d)}{\left(x^2+y^2+(z-R^2/d)^2\right)^{1/2}} + \frac{(qR/d)\left(R^2/d\right)}{\left(x^2+y^2+(z+R^2/d)^2\right)^{1/2}}\right]$$

c) Take a point on the flat section of the conductor. The coordinates are $(x, y, 0)$. The induced charge density is

$$\begin{aligned}
\sigma(x,y) &= \epsilon_0 \mathbf{E}\cdot\hat{\mathbf{z}} \\
&= \frac{1}{4\pi}\left[\frac{q(-d)}{(x^2+y^2+d^2)^{3/2}} + \frac{(-q)d}{(x^2+y^2+d^2)^{3/2}} + \frac{(-qR/d)\left(-R^2/d\right)}{(x^2+y^2+R^4/d^2)^{3/2}} + \frac{(qR/d)\left(R^2/d\right)}{(x^2+y^2+R^4/d^2)^{3/2}}\right] \\
&= \frac{q}{4\pi}\left[-\frac{2d}{(x^2+y^2+d^2)^{3/2}} + \frac{2R^3}{d^2\,(x^2+y^2+R^4/d^2)^{3/2}}\right]
\end{aligned}$$

And the total charge

$$\begin{aligned}
Q_{ind} &= \frac{q}{4\pi}\left[-\int_R^\infty \frac{2d(2\pi s ds)}{(s^2+d^2)^{3/2}} + \int_R^\infty \frac{2R^3(2\pi s ds)}{d^2\,(x^2+y^2+R^4/d^2)^{3/2}}\right] \\
&= q\left[-d\frac{1}{\sqrt{R^2+d^2}} + \frac{R^3}{d^2\sqrt{R^2+R^4/d^2}}\right] \\
&= -\frac{q\left(d^2-R^2\right)}{d\sqrt{R^2+d^2}}
\end{aligned}$$

Question 3: An electron is injected with a velocity $\mathbf{u}_0 = \hat{y}\, u_0$ at the origin into a region where an electric field $\mathbf{E} = \hat{z} E_0$ and a magnetic field $\mathbf{B} = \hat{x}\, B_0$, both exist. [3+3+1]

a) Write down the equation of motion of the electron.

b) Obtain the trajectory equation.

c) What is the trajectory if $E_0 = u_0 B_0$?

Solution:

a) By Lorentz force law (electronic charge, $-e$)

$$\begin{aligned} m\dot{\mathbf{v}} &= -e\left(\hat{\mathbf{z}}E_0 + \mathbf{v}\times\hat{\mathbf{x}}B_0\right) \\ &= -\left(eE_0 - ev_yB_0\right)\hat{\mathbf{z}} - ev_zB_0\hat{\mathbf{y}} \end{aligned}$$

Then, clearly $v_x = 0$ at all times and

$$\begin{aligned} \dot{v}_y &= -\frac{eB_0}{m}v_z \\ \dot{v}_z &= -\frac{eE_0}{m} + \frac{eB_0}{m}v_y \end{aligned}$$

b) Taking one more derivative of v_z, we get (let $\omega = eB_0/m$)

$$\ddot{v}_z = \omega\dot{v}_y = -\omega^2 v_z.$$

Using initial conditions $v_z = 0$ and $\dot{v}_z = -\frac{eE_0}{m} + \frac{eB_0}{m}u_0$ at $t = 0$,

$$v_z = A\sin\omega t$$

where

$$A = -\frac{E_0}{B_0} + u_0.$$

Putting in the first equation and using initial condition,

$$\begin{aligned} \dot{v}_y &= -A\omega\sin\omega t \\ \implies \quad v_y &= \left(u_0 - \frac{E_0}{B_0}\right)\cos\omega t + \frac{E_0}{B_0}. \end{aligned}$$

Then,

$$\begin{aligned} x(t) &= 0, \\ y(t) &= \frac{E_0}{B_0}t + \frac{1}{\omega}\left(u_0 - \frac{E_0}{B_0}\right)\sin\omega t, \\ z(t) &= \frac{1}{\omega}\left(u_0 - \frac{E_0}{B_0}\right)\left(1 - \cos\omega t\right). \end{aligned}$$

Thus, the trajectory is

$$\left(y - \frac{E_0}{B_0}t\right)^2 + \left(z - \frac{1}{\omega}\left(u_0 - \frac{E_0}{B_0}\right)\right)^2 = \frac{1}{\omega^2}\left(u_0 - \frac{E_0}{B_0}\right)^2.$$

c) If $u_0 = E_0/B_0$, then the trajectory is $y(t) = E_0 t/B_0$.

Question 4: A sphere of radius R has the magnetization $\mathbf{M} = \hat{\mathbf{z}}\, a\, (\hat{\mathbf{z}} \cdot \mathbf{r})^2$ where a is a constant. Find the following: [1+1+3+2]

a) Bound volume current density in the sphere.

b) Bound surface current density on the sphere.

c) Magnetic moment of the sphere.

d) Magnetic vector potential $\mathbf{A}$ at a distance $r \gg R$.

Solution:

a) Note that $\mathbf{M} = az^2\hat{\mathbf{z}}$. Then, the bound volume current density $\mathbf{J}_b = \nabla \times \mathbf{M} = 0$.

b) The bound surface current density $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$. Here $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. Then $\mathbf{K}_b = aR^2 \cos^2\theta \sin\theta \hat{\phi}$.

c) Consider a ring of thickness $Rd\theta$ in a plane perpendicular to z axis. The current through the ring is $dI = K_b R d\theta = aR^3 \cos^2\theta \sin\theta$. The magnetic moment of this ring is

$$d\mathbf{m} = \pi\,(R\sin\theta)^2\, dI\hat{\mathbf{z}}$$

And the total moment of the sphere would be

$$\begin{aligned} \mathbf{m} &= \pi a R^5 \int_0^{\pi} \cos^2\theta \sin^3\theta \, d\theta \hat{\mathbf{z}} \\ &= \frac{4}{15}\pi a R^5 \hat{\mathbf{z}} \end{aligned}$$

d) Magnetic vector potential at a distance $r \gg R$ is given by

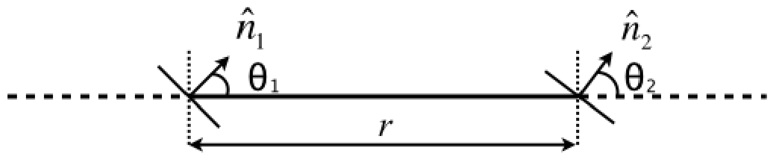
$$\begin{aligned} \mathbf{A} &= \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} \\ &= \frac{\mu_0}{15}\frac{aR^5}{r^2}\sin\theta\hat{\phi} \end{aligned}$$

Question 5: Two plane currents loops, each with area A and carrying current I, are placed a distance r apart. The normal to the current loops, $\hat{n}_1$ and $\hat{n}_2$ make angles θ_1 and θ_2 with the line joining the loops. The vectors $\hat{n}_1$ and $\hat{n}_2$ and the line joining the centers are co-planar as shown in figure. [3+3+1]

a) Find the mutual inductance $\mathcal{M}$ of this system of current loops. Assume the radius of each loop is much smaller than the distance between loops.

b) Find the force $\mathbf{F}$ between the two loops.

c) How would the force be different if the currents were reversed in one or both of the current loops ?



Solution:

a) Choose the z direction along $\hat{n}_1$. Magnetic moment of loop 1 is $IA\hat{\mathbf{z}}$. Then the field due to loop 1 at loop 2 is

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{IA}{r^3}\left(2\cos\theta_1\hat{\mathbf{r}} + \sin\theta_1\hat{\theta}\right)$$

Then the flux through the loop 2 is

$$\begin{aligned}
\phi_{21} &= A\hat{n}_2\cdot\mathbf{B} \\
&= \frac{\mu_0}{4\pi}\frac{A^2}{r^3}\left(2\cos\theta_1\hat{n}_2\cdot\hat{\mathbf{r}} + \sin\theta_1\hat{n}_2\cdot\hat{\theta}\right)I \\
&= \frac{\mu_0}{4\pi}\frac{A^2}{r^3}\left(2\cos\theta_1\hat{n}_2\cdot\hat{\mathbf{r}} + \sin\theta_1\hat{n}_2\cdot\hat{\theta}\right)I \\
&= \frac{\mu_0}{4\pi}\frac{A^2}{r^3}\left(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right)I
\end{aligned}$$

Thus the mutual inductance is

$$\begin{aligned}
\mathcal{M} &= \frac{\mu_0}{4\pi}\frac{A^2}{r^3}\left(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) \\
&= \frac{\mu_0 A^2}{8\pi r^3}\left[3cos(\theta_1+\theta_2) + cos(\theta_1-\theta_2)\right]
\end{aligned}$$

b) The force on the second loop due to the first one is given by

$$\begin{aligned}
\mathbf{F} &= \nabla\left(\hat{\mathbf{m}}_2\cdot\mathbf{B}\right) \\
&= \nabla\left(\frac{\mu_0}{4\pi}\frac{I^2A^2}{r^3}\left(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right)\right) \\
&= \frac{\mu_0 I^2A^2}{4\pi}\left(-\frac{3\hat{\mathbf{r}}}{r^4}\left(2\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right)\right. \\
&\quad \left.-\frac{\hat{\theta}}{r^4}\left(2\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\right)\right)
\end{aligned}$$

c) The direction of F is reversed if (only) one of the currents is reversed.

Question 6: In free space, an electric field is given by [2+2+2+1]

$$\mathbf{E} = \hat{x}E_0 \left[\cos\left(k\,z - \omega\,t\right) + \cos\left(k\,z + \omega\,t\right)\right].$$

a) Find the magnetic field **B**.

b) Determine the poynting vector and its time average over one cycle.

c) Find the time averged electric and magnetic energy densities.

d) Does this field represent a wave? If so, what kind of wave does it represent?

Solution:

a) Electric field $\mathbf{E} = \hat{\mathbf{x}}E_0 \left[\cos\left(kz - \omega t\right) + \cos\left(kz + \omega t\right)\right] = \hat{\mathbf{x}}2E_0 \cos kz \cos\omega t$.
The magnetic field is given by

$$\begin{aligned} B &= \int dt\,(-\nabla \times E) \\ &= \hat{\mathbf{y}} \int dt\,(2E_0 k \sin kz\ \cos\omega t) \\ &= 2E_0 \frac{k}{\omega} \sin kz\ \sin\omega t\hat{\mathbf{y}} \\ \therefore \qquad H &= \frac{2}{c\mu_0} E_0 \sin kz\ \sin\omega t\hat{\mathbf{y}} \end{aligned}$$

b) The poynting vector

$$\mathbf{S} = \frac{1}{\mu_0 c} E_0^2 \sin 2kz\ \sin 2\omega t\hat{\mathbf{z}}$$

and

$$\langle S\rangle = \frac{1}{T}\int_0^T \mathbf{S}dt = 0$$

c) The electric and magnetic energy densities are

$$\langle u_E\rangle = \epsilon_0 E_0^2 \cos^2 kz$$

and

$$\langle u_M\rangle = \frac{E_0^2}{c^2\mu_o}\sin^2 kz = \epsilon_0 E_0^2 \sin^2 kz$$

d) Yes. Its a standing wave.