Physics II: Electromagnetism (PH102)

Lecture 12

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Electric fields in matter

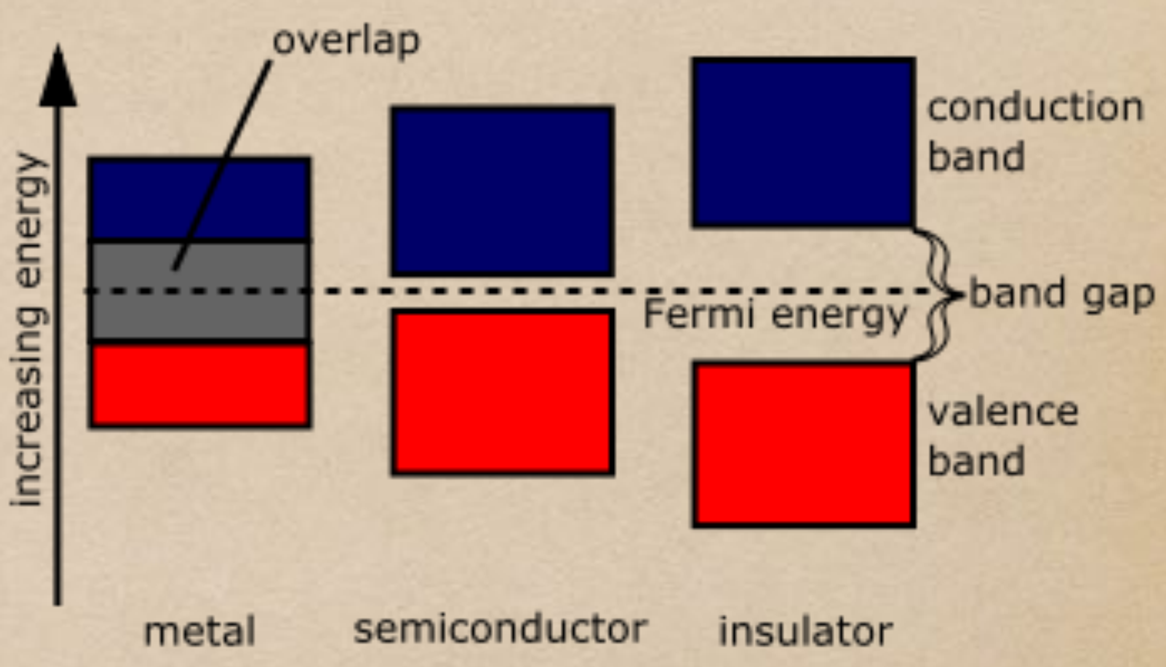
Quantum mechanical band theory classifies materials into different categories

Insulators (Dielectrics)

Semiconductors

Conductors

Superconductors

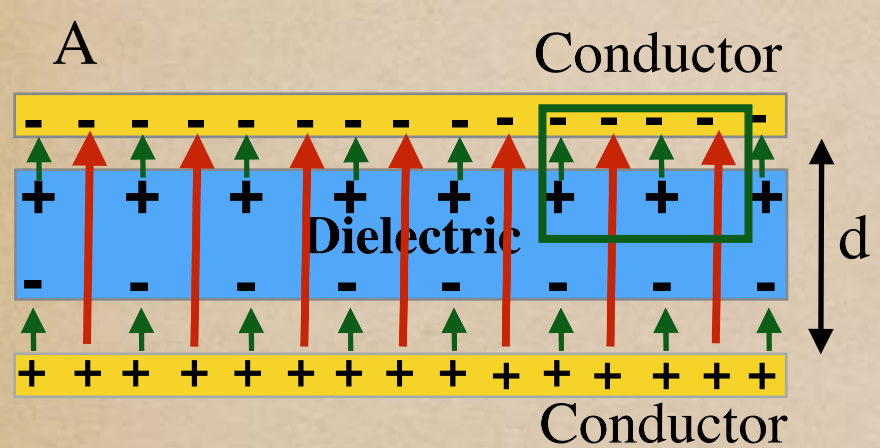


- We have already talked about conductors
- Will talk about Insulators or dielectric materials

What does an insulator do ?

Experiment: Whenever we insert a dielectric material within a capacitor, the capacitance increases by an amount K, which is an inherent property of the material

We know that the capacitance of a parallel plate capacitor is



$$C = \frac{\epsilon_0 A}{d} \qquad \text{where,} \qquad Q = CV$$

If C increases then V must be decreasing !

This means electric field must be decreasing as well. But how ?

The only way electric field can reduce if the net charge inside the Gaussian surface is lower than it would be without the material.

Hence there must be opposite charges on the surface of the dielectric. Since the field is reduced, but not zero, we would expect this positive charge to be smaller than the negative charges on the conductor!

For dielectric also, opposite charges are induced to nearby surface

Induced dipole

What happens to a neutral atom when it is placed in an electric field ?

- Due to the presence of a positively charged core in an atom with electrons surrounding it, the nucleus is pushed towards the electric filed.

The two opposing forces : electric field pulling the electron and nucleus apart and their mutual attractions drawing them together reach a balance :

Atom is polarised

With plus and minus charges shifted slightly, results in a dipole moment $\vec{p}$ pointing in the same direction as of the electric field.

$$\vec{p} = \alpha \vec{E}$$

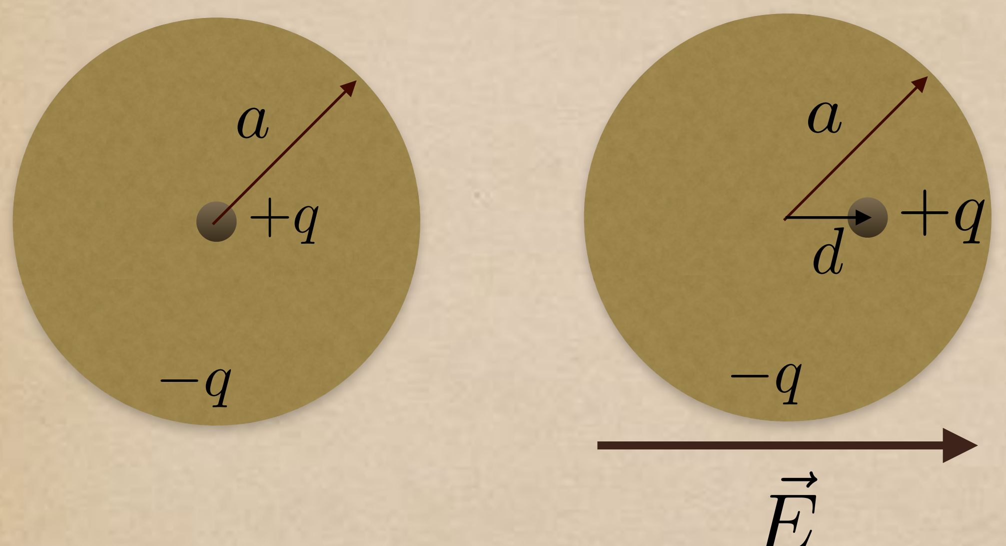
Atomic polarisability

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.76	0.396	24.1	1.64	43.4	59.6

Table 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3).

A quick example on atomic polarisability

A primitive atomic model: A point nucleus of charge +q surrounded by a uniformly charged sphere of charge -q: Crude approximation. Then apply Electric field to the atom.



- We assume that the electron cloud remains same in external field
- Suppose equilibrium occurs when the nucleus is `d' distance apart

At this point, the external field pushing the nucleus will balance the internal field pulling it to left $\vec{E} = \vec{E}_e$

The field at a distance d inside a uniformly charged sphere:

$$\vec{E}_e = \frac{\rho d}{3\epsilon_0}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}\hat{r} = \vec{E} \longrightarrow \boxed{E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}}$$

Dipole moment: $p = qd = (4\pi\epsilon_0 a^3)E = \alpha E$

Atomic polarisability : $\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 V$

V is volume of the atom

Example: Hydrogen atom

Electron cloud in Hydrogen atom has charge density q is the charge of electron and a is Bohr radius. Find atomic polarisability.

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

Field at a distance r $\oint_{sur} \vec{E}.d\vec{a} = \oint_{sur} |\vec{E}|da = |\vec{E}| \oint_{sur} da = |\vec{E}|4\pi r^2 = \frac{Q_{enc}}{\epsilon_0}$

$$\begin{aligned} Q_{\text{enc}} &= \int_0^r \rho\, d\tau = \frac{4\pi q}{\pi a^3} \int_0^r e^{-2\bar{r}/a} \bar{r}^2\, d\bar{r} = \frac{4q}{a^3} \left[-\frac{a}{2} e^{-2\bar{r}/a} \left(\bar{r}^2 + a\bar{r} + \frac{a^2}{2} \right) \right] \Big|_0^r \\ &= -\frac{2q}{a^2} \left[e^{-2r/a} \left(r^2 + ar + \frac{a^2}{2} \right) - \frac{a^2}{2} \right] = q \left[1 - e^{-2r/a} \left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2} \right) \right]. \end{aligned}$$

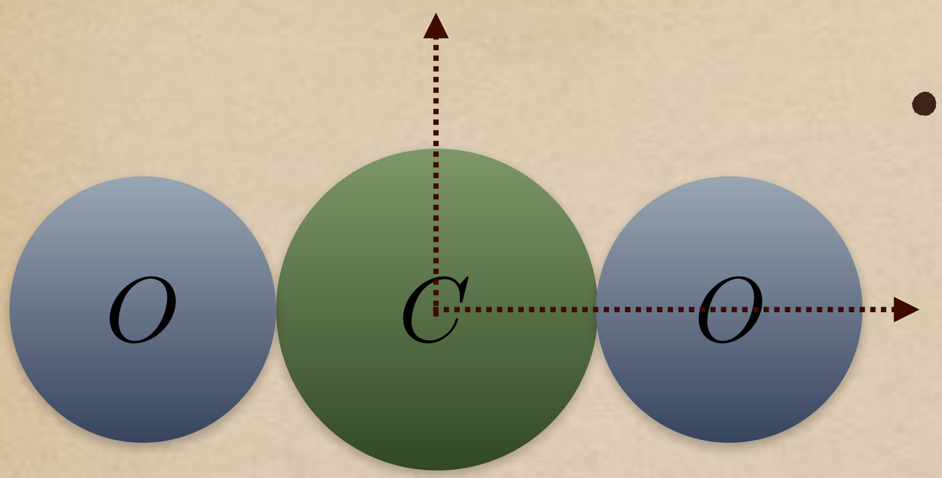
Following that external field balances internal one at a distance d : $\boxed{\vec{E} = \vec{E}_e}$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left[1 - e^{-2d/a} \left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right) \right] \longrightarrow E = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left(\frac{4}{3} \frac{d^3}{a^3} \right) = \frac{1}{4\pi\epsilon_0} \frac{4}{3a^3} (qd) = \frac{1}{3\pi\epsilon_0 a^3} p.$$

$$\begin{aligned} e^{-2d/a} &= 1 - \left(\frac{2d}{a}\right) + \frac{1}{2}\left(\frac{2d}{a}\right)^2 - \frac{1}{3!}\left(\frac{2d}{a}\right)^3 + \cdots = 1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots \\ 1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right) &= 1 - \left(1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots\right)\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right) \\ &= \cancel{1} - \cancel{1} - \cancel{2\frac{d}{a}} - \cancel{2\frac{d^2}{a^2}} + \cancel{2\frac{d}{a}} + \cancel{4\frac{d^2}{a^2}} + \cancel{4\frac{d^3}{a^3}} - \cancel{2\frac{d^2}{a^2}} - \cancel{4\frac{d^3}{a^3}} + \frac{4}{3}\frac{d^3}{a^3} + \cdots \\ &= \frac{4}{3}\left(\frac{d}{a}\right)^3 + \text{higher order terms.} \end{aligned}$$

$$\alpha = 3\pi\epsilon_0 a^3$$

The case of molecular polarisability



- For molecules situations are more complex.

Polarisability of a molecule often depend on the direction of electric field, subject to the orientation of atoms.

CO_2 has polarisability $4.5 \times 10^{-40} \quad C^2.m/N$ when the field is along the axis of the molecule; $2 \times 10^{-40} \quad C^2.m/N$ when the field is perpendicular.

$$\longrightarrow \{\alpha_\perp, \alpha_\parallel\}$$

- When the field is at some angle, one must resolve it in parallel and perpendicular components

$$\longrightarrow \vec{p} = \alpha_\perp \vec{E}_\perp + \alpha_\parallel \vec{E}_\parallel$$

For a completely asymmetric molecule:

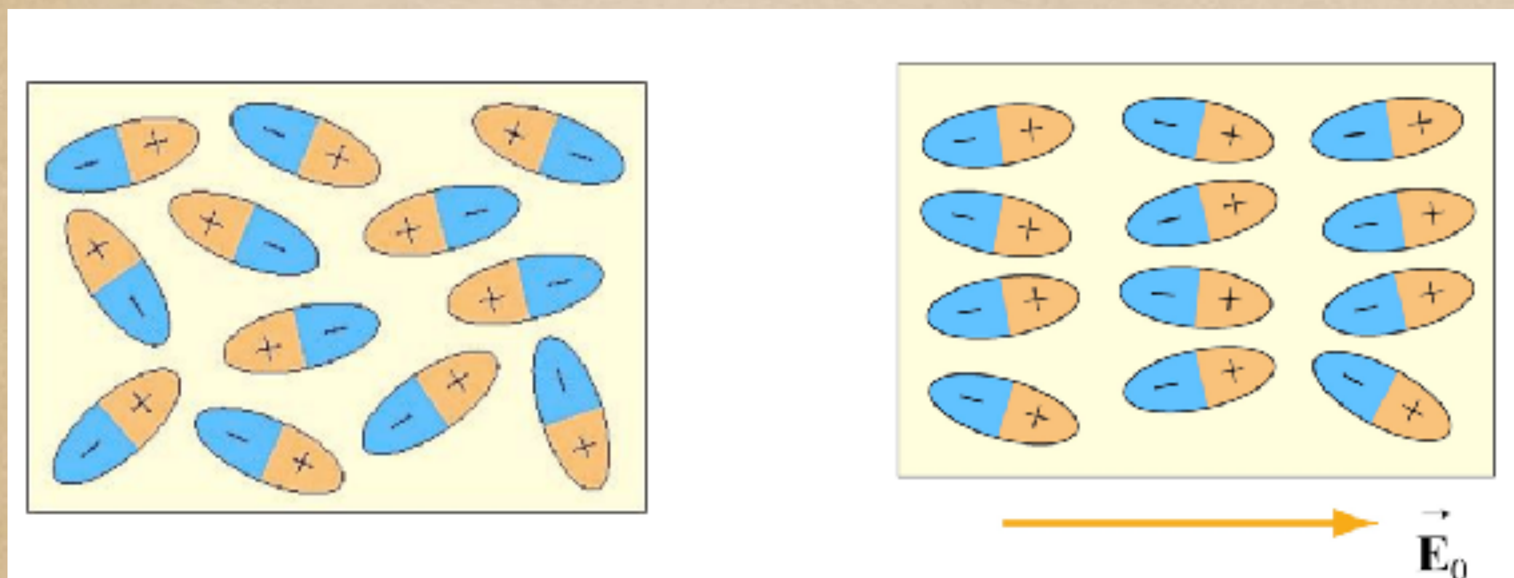
$$p_x = \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z$$
$$p_y = \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z$$
$$p_z = \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z$$

α_{ij} forms the components of polarisability tensor.

Any object with more than three components can be thought as tensors.

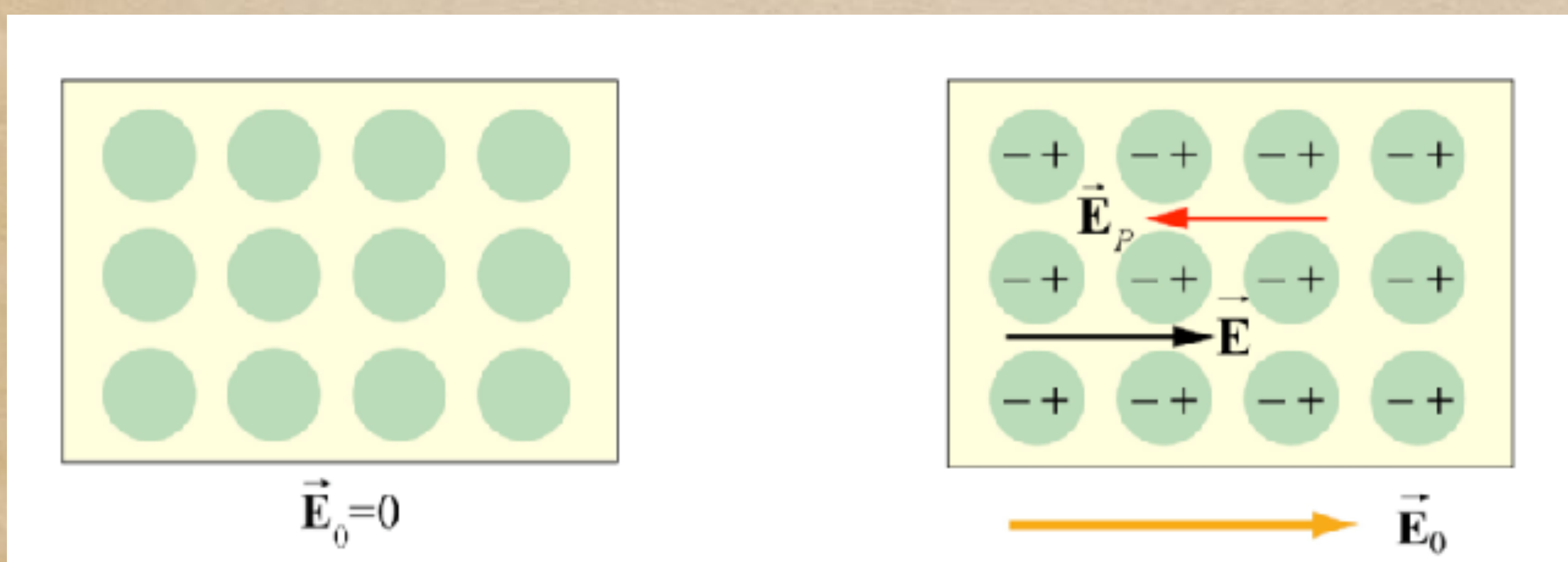
Two types of dielectrics

Polar dielectrics - having permanent electric dipole moments. (Example: water)



The alignment of molecules generate an electric field which is smaller in magnitude.

Non-Polar dielectrics - No permanent electric dipole moments.

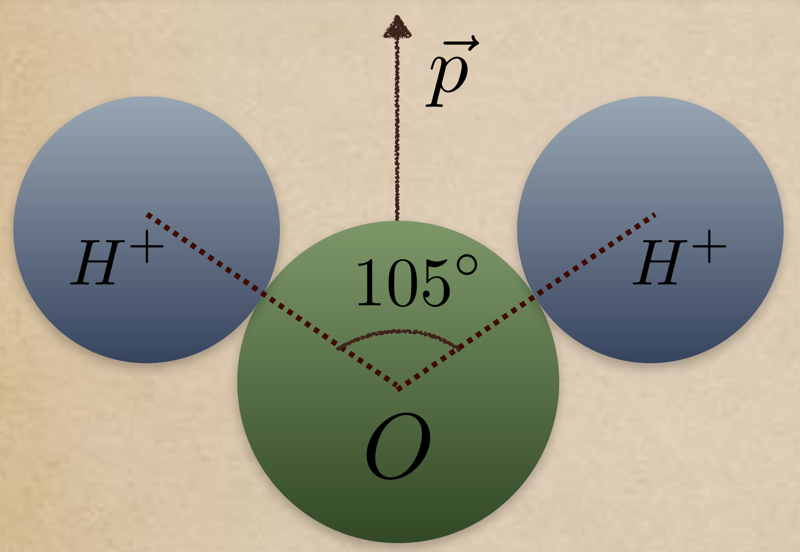


$$\vec{E} = \vec{E}_0 + \vec{E}_p$$

$$|\vec{E}| < |\vec{E}_0|$$

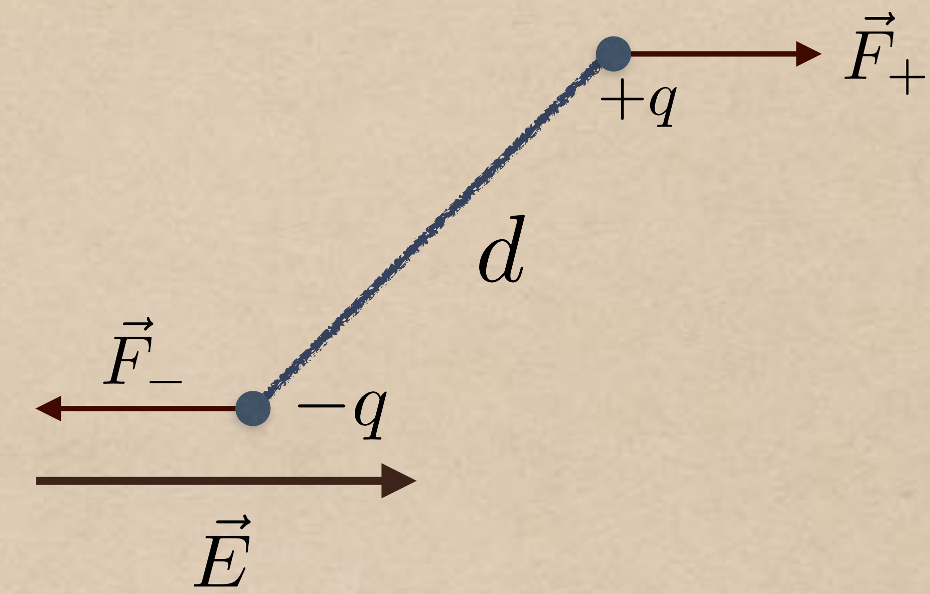
Polar molecules in electric field

Consider molecules which has built in dipole moment. Ex: Water molecule



$p = 6.1 \times 10^{-30}\ C.m$

What happens when we bring such polar molecules in electric field ?



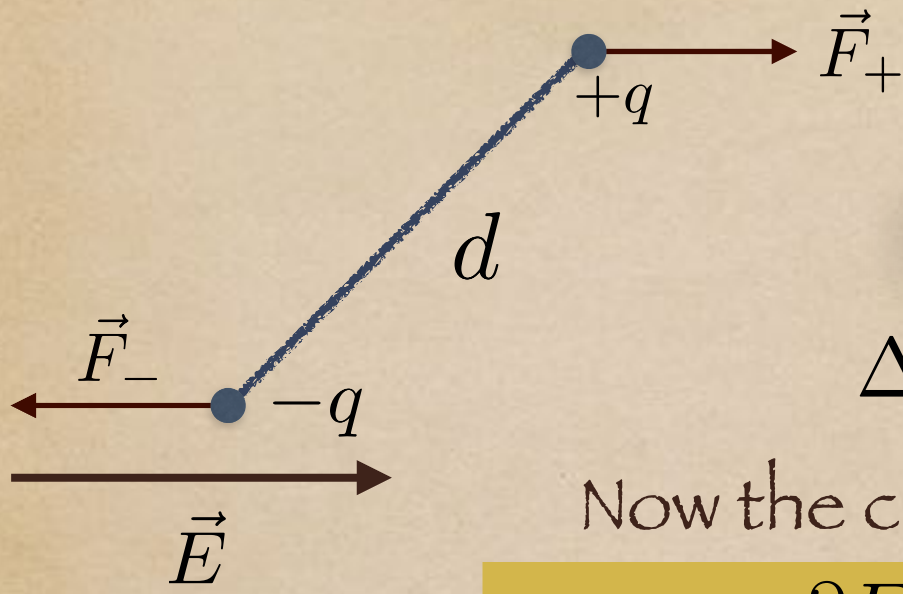
In case the field is uniform the force cancel at both end, with a residual torque on the dipole

$$\vec{N} = [(\vec{d}/2) \times (q\vec{E})] + [(-\vec{d}/2) \times (-q\vec{E})]$$
$$= q\vec{d} \times \vec{E} = \vec{p} \times \vec{E}$$

$$\vec{p} = q\vec{d}$$

Dipole moment

Polar molecule in non-uniform field



There will be a net force on the dipole

$$\vec{F} = \vec{F}_+ + \vec{F}_- = q(\vec{E}_+ - \vec{E}_-) = q(\Delta\vec{E})$$

$\Delta\vec{E}$ is the difference in electric field at both ends.

Now the change in x-component of the field :

$$\Delta E_x = \frac{\partial E_x}{\partial x}\Delta x + \frac{\partial E_x}{\partial y}\Delta y + \frac{\partial E_x}{\partial z}\Delta z = (\vec{d}.\vec{\nabla})E_x$$

- Above formula works for a very small dipole : $(d_x \sim \Delta x, \quad d_y \sim \Delta y, \quad d_z \sim \Delta z)$
- Similarly one may write the change in the electric field in y and z directions

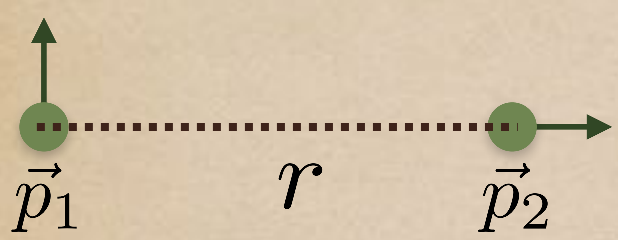
Hence, one may write $\Delta\vec{E} = (\vec{d}.\vec{\nabla})\vec{E} \longrightarrow \vec{F} = q(\vec{d}.\vec{\nabla})\vec{E} = (\vec{p}.\vec{\nabla})\vec{E}$

For a perfect dipole in non-uniform field, torque about the centre of dipole remains same $\vec{N} = \vec{p} \times \vec{E}$

- However, the torque about an arbitrary point becomes $\vec{N} = (\vec{p} \times \vec{E}) + (\vec{r} \times \vec{F})$

Example...

What is the torque on $\vec{p}_1$ due to $\vec{p}_2$ and on $\vec{p}_2$ due to $\vec{p}_1$?



Although it might seem to be the same, it is actually not !

Electric field due to $\vec{p}_1$ at the position of $\vec{p}_2$: $\boxed{\vec{E}_1 = \frac{p_1}{4\pi\epsilon_0 r^3}\hat{\theta}}$ points down

Recall, $\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$ and use $\theta = 90^\circ$

Hence, torque on $\vec{p}_2$ due to $\vec{p}_1$:

$$\vec{N} = \vec{p}_2 \times \vec{E}_1 = p_2 E_1 \sin 90^\circ(-\hat{z}) = \boxed{\frac{p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{z})}$$

points into the page

Electric field due to $\vec{p}_2$ at the position of $\vec{p}_1$: $\boxed{\vec{E}_2 = \frac{p_2}{4\pi\epsilon_0 r^3}(-2\hat{r})}$

Again, use: $\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$ with $\theta = \pi$

Hence, torque on $\vec{p}_1$ due to $\vec{p}_2$:

$$\vec{N} = \vec{p}_1 \times \vec{E}_2 = p_1 E_2 \sin 90^\circ(-\hat{z}) = \boxed{\frac{2p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{z})}$$

Twice than the other one

Polarisation

- What happens to a piece of dielectric material in an external electric field ?

What we said so far:

Presence of a atom or molecule in external field will induce a tiny dipole moment aligned in the direction of the field

If the material is a polar object it will feel a torque to align the dipole along the external field

Hence, we can summarise, that a material placed in external field will produce a lot of tiny little dipoles along the direction of the field: Material is polarised

We define hence, a parameter called Polarisation as

$$\vec{P} = \text{dipole moment per unit volume}$$

We will first study the field a polarised material itself produces and then study the effect of such material in external electric field

The field of a polarised object

Suppose we have a polarised object with dipole moment per unit volume is $\vec{P}$. What is the electric field it produces ?

Strategy: We know the field of an individual dipole, so we can chop the material up into infinitesimal dipoles and integrate to get the total field.

Potential due to dipole: $\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}.\hat{\imath}}{\imath^2}}$

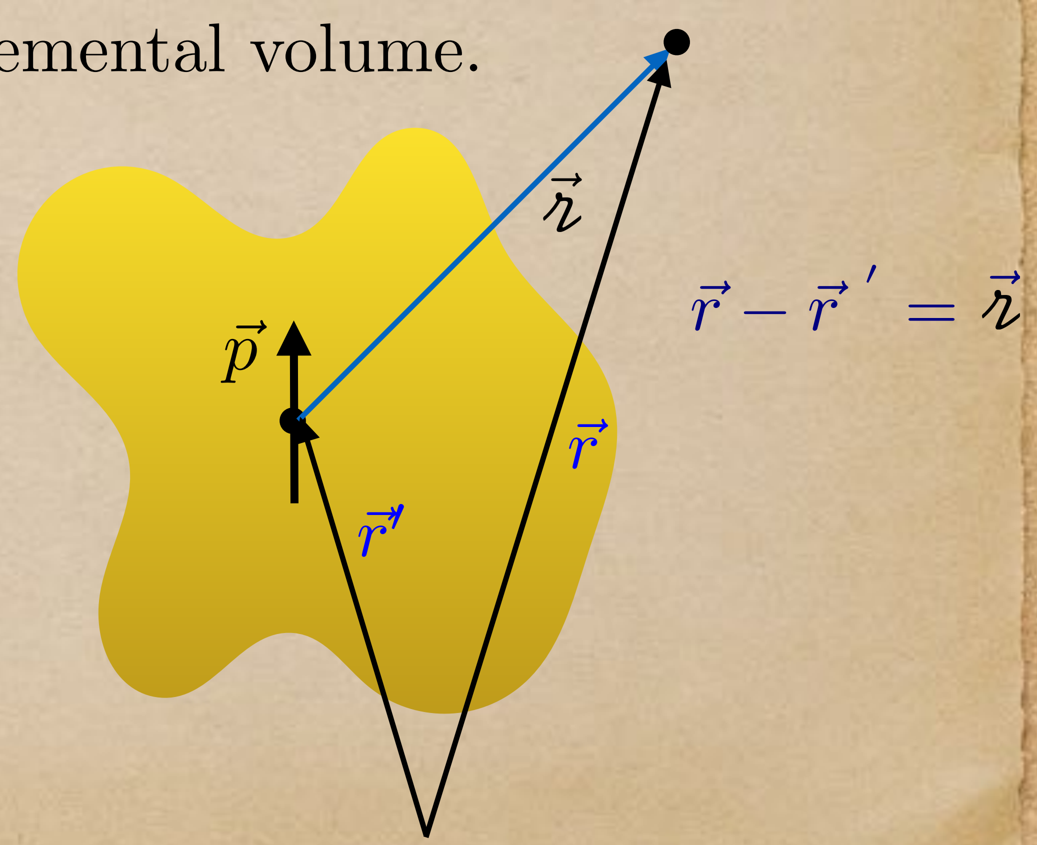
The polarisation is given by $\vec{P}(\vec{r}\,')d\tau'$ in an elemental volume.

Total potential due to the whole volume

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\vec{P}(\vec{r}\,').\hat{\imath}}{\imath^2} d\tau'$$

Recall, $\vec{\nabla}'\left(\frac{1}{\imath}\right) = \frac{\hat{\imath}}{\imath^2}$

i.e. $\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$



The field of a polarised object

Therefore: $$\frac{\vec{P}(\vec{r}\,').\hat{\imath}}{\imath^2} = \vec{P}(\vec{r}\,').\vec{\nabla}'\left(\frac{1}{\imath}\right) = \vec{\nabla}'.\left(\frac{\vec{P}(\vec{r}\,')}{\imath}\right) - \frac{1}{\imath}\left(\vec{\nabla}'.\vec{P}(\vec{r}\,')\right)$$

Here, we have used the vector identity

$$\vec{\nabla}.(f\vec{A}) = f\vec{\nabla}.\vec{A} + (\vec{\nabla}f).\vec{A}$$

Hence the potential is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\vec{P}(\vec{r}\,').\hat{\imath}}{\imath^2}d\tau'$$

$$\implies V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}}\vec{\nabla}'.\left(\frac{\vec{P}(\vec{r}\,')}{\imath}\right)d\tau' - \int_{\mathcal{V}}\frac{1}{\imath}(\vec{\nabla}'.\vec{P}(\vec{r}\,'))d\tau'\right]$$

Using divergence theorem

Looks like potential for a volume charge

$$= \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}}\frac{1}{\imath}\vec{P}(\vec{r}\,').\hat{n}'da' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{1}{\imath}(\vec{\nabla}'.P(\vec{r}\,'))d\tau'$$

Looks like potential for a surface charge

$$= \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}}\frac{\sigma_b(\vec{r}\,')}{\imath}da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\rho_b(\vec{r}\,')}{\imath}d\tau'$$

Bound volume charge density

Bound surface charge density

$$\boxed{\sigma_b = \vec{P}(\vec{r}).\hat{n}}$$

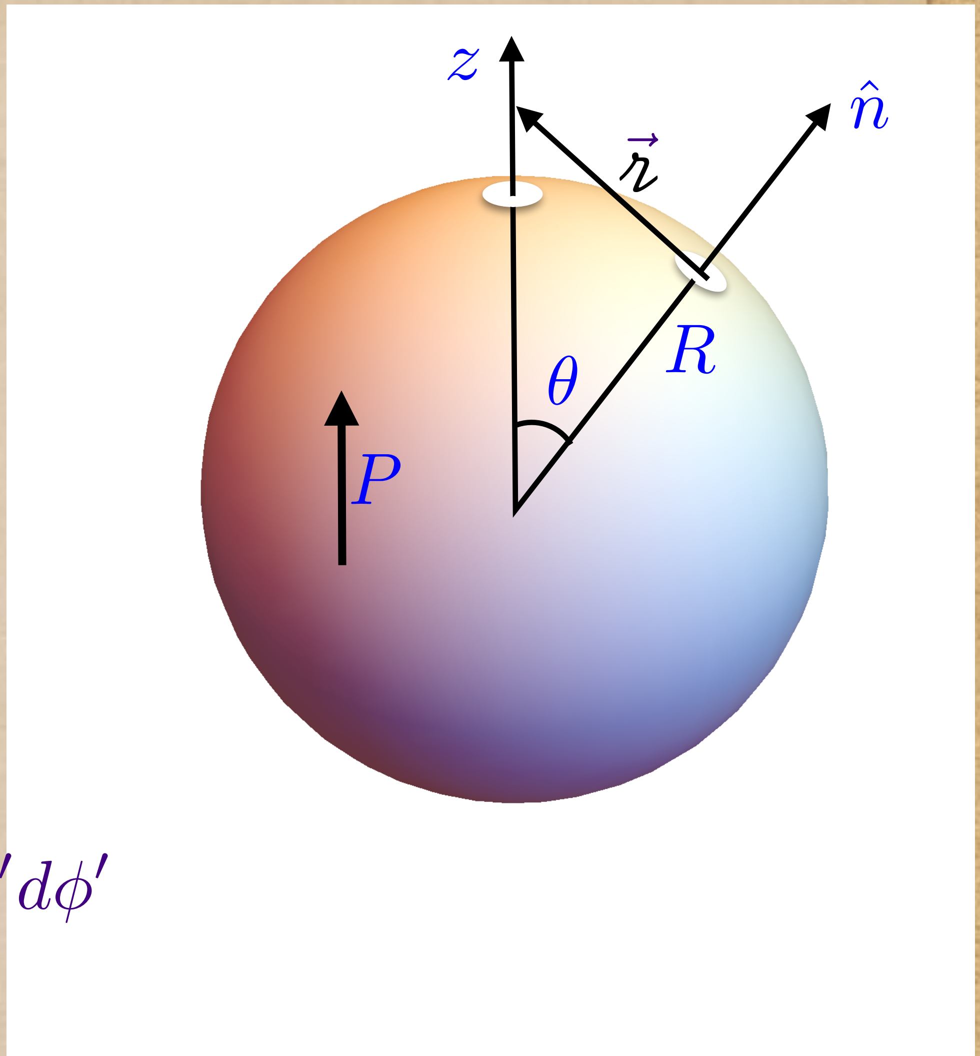
$$\boxed{\rho_b = -\vec{\nabla}.P(\vec{r})}$$

Potential of a polarised object can be identified as created by bound surface charge density and bound volume charge density

Electric field due to uniformly polarised sphere

Let $\vec{P} = P\hat{z}$ $\longrightarrow$ $\rho_b = -\vec{\nabla}.\vec{P} = 0$

Surface bound charge density $\sigma_b = \vec{P}.\hat{n} = P\cos\theta$



$$\therefore V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b(\vec{r}\,')}{\mathscr{r}} da' + \cancel{\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b(\vec{r}\,')}{\mathscr{r}} d\tau'}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b(\vec{r}\,')}{\mathscr{r}} da'$$

$$= \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{P\cos\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}} R^2 \sin\theta' d\theta' d\phi'$$

$$= \frac{PR^2}{4\pi\epsilon_0}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{\cos\theta' \sin\theta' d\theta' d\phi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

Substitute: $R^2 + z^2 - 2Rz\cos\theta' = t^2$

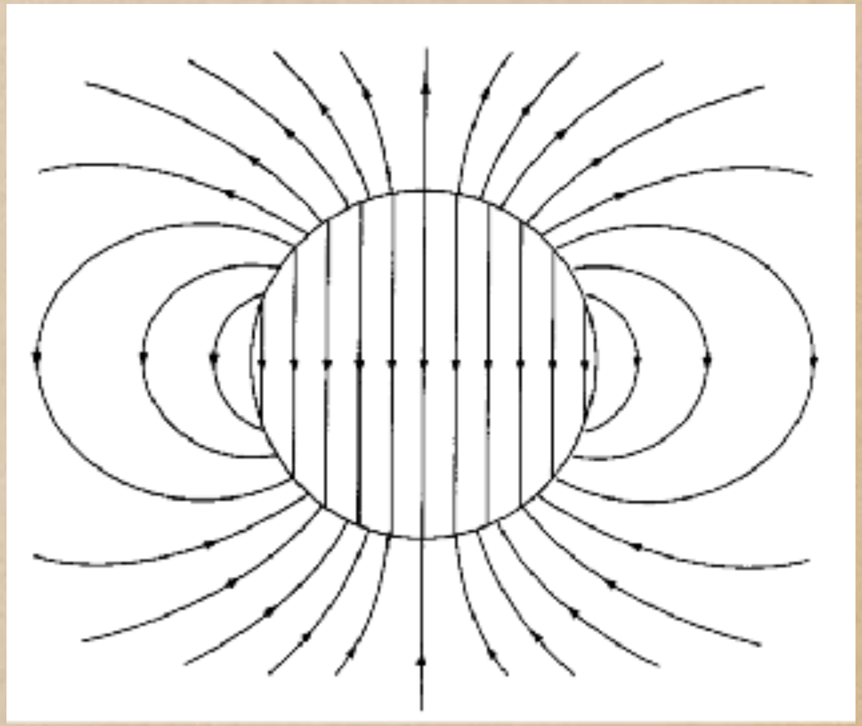
$$V(\vec{r}) = \frac{PR^2}{2\epsilon_0}\int_{R-z}^{R+z} \frac{(R^2 + z^2 - t^2)}{2(Rz)^2} dt$$

Electric field due to uniformly polarised sphere

After the integration: $V(r,\theta) = \begin{cases} \frac{P}{3\epsilon_0} r\cos\theta & \text{if } r \leq R; \\ \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta & \text{if } r \geq R. \end{cases}$ (Remember: $z = r\cos\theta$)

Inside the sphere: $\vec{E} = -\vec{\nabla}V = -\frac{P}{3\epsilon_0}\hat{z} = -\frac{\vec{P}}{3\epsilon_0}$ for $r < R$.

Electric field is uniform within uniformly polarised sphere



Also note, the potential outside ($r \geq R$)

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\frac{4}{3}\pi R^3 P}{r^2}\cos\theta$$

$$= \boxed{\frac{1}{4\pi\epsilon_0}\frac{\vec{p}.\hat{r}}{r^2}}.$$

where $p = Pd\tau = \frac{4}{3}\pi R^3 P$ is the total dipole moment of the sphere

i.e. Outside the sphere the potential is identical to that of a perfect dipole at the origin, whose dipole moment ($\vec{p}$) is the total dipole moment of the sphere!

$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

Summary

- Neutral Atom gets polarised in external electric field due to charge separation. The dipole moment induced is proportional to atomic polarisability, a property of the atom.
- For asymmetric molecules, polarisability is a matrix with nine components (rank two tensor).
- Polar molecules feel a torque in external field and also a residual force in varying electric field.
- Dielectric materials get polarised when placed in external field.
- Polarisation is defined by dipole moment per unit volume.
- The field of polarised object can be calculated by bound surface charge density and bound volume charge density.