MA 102 (Mathematics II)

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Tutorial Sheet No. 2 January 18, 2016

Continuity of functions of several variables

(1) Find the natural domains of the following functions:
(a) $f(x,y) := x\ln(y^2 - x)$ (b) $f(x,y) := \frac{xy}{x^2-y^2}$ (c) $f(x,y) := xy\ln(x^2+y^2)$.

(2) Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,

(a) $f(x,y) := \begin{cases} xy\cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ (b) $f(x,y) := \begin{cases} 1 & \text{if } x > 0 \ \&\ 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$

(c) $f(x,y) := \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ (d) $f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(e) $f(x,y) := \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ (f) $f(x,y) := \begin{cases} \frac{x^3y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(g) $f(x,y) := \begin{cases} \frac{\sin(x+y)}{|x|+|y|} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ (h) $f(x,y) := \begin{cases} xy\ln(x^2+y^2) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(3) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function at $x_0 \in \mathbb{R}^2$ and that $|f(x_0)| > 2$. Show that there is a $\delta > 0$ such that $|f(x)| > 2$ whenever $||x - x_0|| < \delta$.

(4) $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 0$ if $x \in \mathbb{Q}, y \in \mathbb{Q}$ and $f(x,y) = xy$ otherwise. Find all the points in $\mathbb{R}^2$ where f is continuous.

(5) If the functions $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuous then show that the functions $F, G : \mathbb{R}^n \to \mathbb{R}$ given by $H(x) := \max(f(x), g(x))$ and $G(x) := \min(f(x), g(x))$ are continuous.

(6) Prove or disprove (giving proper justification) the following statements.
(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. If $(x_k) \subset \mathbb{R}^n$ is a Cauchy sequence then $(f(x_k)) \subset \mathbb{R}$ is a Cauchy sequence.
(b) Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be such that for every Cauchy sequence $((x_n, y_n)) \subset \mathbb{R}^2$ the sequence $(f(x_n, y_n)) \subset \mathbb{R}$ is also a Cauchy sequence. Then f is continuous on A.
(c) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}^2$ is bounded then the image $f(A) \subset \mathbb{R}$ is bounded.

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