MA 102 (Ordinary Differential Equations)

IIT Guwahati

Tutorial Sheet No. 10 **Date:** March 28, 2016

Higher-order linear IVPs; Wronskian; Undetermined coefficients and the Annihilator method.

(1) Determine the largest interval (a, b) in which the given IVP is certain to have a unique solution:
(a) $e^x y'' - \frac{y'}{x-3} + 3y = \ln x,\ y(1) = 3,\ y'(1) = 2.$
(b) $(1-x)y'' - 3xy' + 3y = \sin x,\ y(0) = 1,\ y'(0) = 1.$
(c) $x^2 y'' + 4y = \cos x,\ y(1) = 0,\ y'(1) = -1.$

(2) Let y_1 and y_2 be two solutions of $y''(x) + p(x)y'(x) + q(x)y = 0$ defined in the interval $[a, b]$. Show that if their Wronskian $W(y_1, y_2) = 0$ at least one point in $[a, b]$ then $W(y_1, y_2) = 0$ for all $x \in [a, b]$.

(3) If y_1 and y_2 are linearly independent solutions of $xy'' + 2y' + xe^x y = 0$ and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.

(4) (a) Verify that the functions $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ are linearly independent solutions of the differential equation $x^2y'' - 4xy' + 6y = 0$ on $(-\infty, \infty)$; (b) Show that y_1 and y_2 are linearly dependent on $(-\infty, 0)$, but are linearly independent on $(-\infty, \infty)$; (c) Although y_1 and y_2 are linearly independent, show that $W(y_1, y_2) = 0$ for all $x \in (-\infty, \infty)$. Does this violate the fact that $W(y_1, y_2) = 0$ for every $x \in (-\infty, \infty)$ implies y_1 and y_2 are linearly dependent?

(5) Let $p(x), q(x) \in C(I)$. Assume that the functions $y_1,\ y_2 \in C^2(I)$ are solutions of the differential equations $y'' + p(x)y' + q(x)y = 0$ on an open interval I. Prove that (a) if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval; (b) if y_1 and y_2 have a common point of inflection x_0 in I, then they cannot be a fundamental set of solutions on that interval.

(6) Let $S = \{f : \mathbb{R} \to \mathbb{R}\,|\,L(f) = 0\}$, where $L(f) := f''' + f'' - 2$. Find the $Ker(L)$. Let $S_0 \subset Ker(L)$ be the subspace of solutions g such that $\lim_{x\to\infty} g(x) = 0$. Find $g \in S_0$ such that $g(0) = 0$ and $g'(0) = 2$.

(7) Find the general solution of the following differential equations.
(a) $\dfrac{d^4y}{dx^4} + y(x) = 0.$
(b) $\dfrac{d^5y}{dx^5} - 2\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} = 0.$
(c) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y(x) = 0.$
(d) $\dfrac{d^5y}{dx^5} + 5\dfrac{d^4y}{dx^4} + 10\dfrac{d^3y}{dx^3} + 10\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + y(x) = 0.$

(8) Solve the following initial-value problems:
(a) $y'' - 2y' + y = 2xe^{2x} + 6e^x;\ y(0) = 1,\ y'(0) = 0.$
(b) $y''(x) + y(x) = 3x^2 - 4\sin x,\ y(0) = 0,\ y'(0) = 1.$

(9) Use the method of undermined coefficients to find a particular solution to the following differential equations:
(a) $y'' - 3y' + 2y = 2x^2 + 3e^{2x}.$
(b) $y''(x) - 3y'(x) + 2y(x) = xe^{2x} + \sin x.$

(10) Use the annihilator method to determine the form of a particular solution for the equations:
(a) $y''(x) - 5y'(x) + 6y(x) = \cos(2x) + 1.$
(b) $y''(x) - 5y'(x) + 6y(x) = e^{3x} - x^2.$