Continuity of functions of several variables — Tutorial - 2

1 (a) $$f(x,y) := x \ln(y^2 - x)$$

So f is defined everywhere except at the points (x,y) where $y^2 - x \leq 0$.

OR Domain $= \{(x,y) \in \mathbb{R}^2 : \ y^2 > x\}$

(b) Domain $= \{(x,y) \in \mathbb{R}^2 : \ y^2 - x^2 \neq 0\}$

(c) Domain $= \mathbb{R}^2 \setminus \{(0,0)\}$

2 (a) Suppose (x_n, y_n) be a sequence in $\mathbb{R}^2$ s.t. $(x_n, y_n) \longrightarrow (0,0)$ then

$$\left| f(x_n, y_n) - 0 \right| = \left| x_n y_n \cos\left(\frac{1}{x_n}\right) \right| \leq |x_n y_n| \longrightarrow 0$$

So $f(x_n, y_n) \longrightarrow f(0,0)$

and therefore f is continuous at $(0,0)$

(b) f is not continuous

take $x_n = \frac{1}{\sqrt{n}}$ & $y_n = \frac{1}{2n}$

then $(x_n, y_n) \longrightarrow (0,0)$

$\because$ $y_n = \frac{1}{2n} < \frac{1}{n} = x_n^2$

$y_n < x_n^2$

and therefore $f(x_n, y_n) = 1 \neq 0$

and therefore f is not continuous

(c) f is continuous

take (x_n, y_n) sequence in $\mathbb{R}^2$ s.t. $(x_n, y_n) \longrightarrow (0,0)$

then $\left| f(x_n, y_n) - 0 \right| = \left| \frac{x_n^3}{x_n^2 + y_n^2} \right| \leq |x_n| \longrightarrow 0$

$f(x_n, y_n) \longrightarrow f(0,0)$

(d) choose $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$ then $(x_n, y_n) \longrightarrow (0,0)$

but $|f(x_n, y_n) - 0| = \frac{1}{2} \neq f(0,0)$

So f is not continuous in this case

Another method – choose path $y = mx$ passing through $(0,0)$

Then
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

and therefore this limit is different for different value of m and Hence limit does not exist.

$\Rightarrow$ f is not continuous.

(e)

choose $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right)$ then $(x_n, y_n) \longrightarrow (0,0)$

but $f\left(\frac{1}{n}, \frac{1}{n^2}\right) \longrightarrow \frac{1}{2} \neq 0$

and therefore f is not continuous at $(0,0)$

Another method - choose path $y = mx^2$ passing through $(0,0)$

then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \left(\frac{m x^4}{(1+m^2) x^4}\right) = \frac{m}{1+m^2}$

and therefore limit does not exists and Hence function is not continuous.

(f) Let (x_n, y_n) be sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$

then $$|f(x_n, y_n) - 0| = \left|\frac{x_n (2x_n^2 y)}{2(x_n^4 + y_n^2)}\right| \le \frac{x_n}{2}\left(\frac{x_n^4 + y_n^2}{x_n^4 + y_n^2}\right)$$

$$\le \frac{x_n}{2} \longrightarrow 0$$

So $f(x_n, y_n) \longrightarrow f(0,0)$.

and therefore function is continuous.

(g) take $(x_n, 0) = \left(\frac{1}{n}, 0\right)$

Then $\left| f(x_n, 0) - 0 \right| = \left| \frac{\sin \frac{1}{n}}{\frac{1}{n}} \right| \longrightarrow 1 \neq 0$

and therefore f is continuous.

(h) Suppose (x_n, y_n) be sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$

Then $$|f(x_n y_n)| \leq \left| \left(\frac{x_n^2 + y_n^2}{2}\right) \ln (x_n^2 + y_n^2) \right| \longrightarrow 0$$

so $$f(x_n, y_n) \longrightarrow f(0,0)$$

and therefore it shows that f is cont.

(3) choose $\epsilon = |f(x_0)| - 2$

Since f is continuous and therefore for this $\epsilon > 0$

$\exists$ $\delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

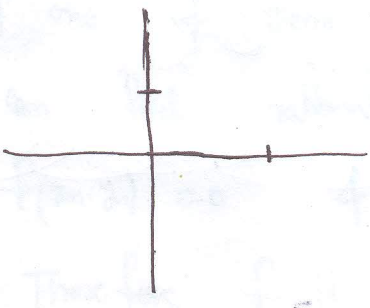
$$\Rightarrow -\epsilon + f(x) < f(x_0) < f(x) + \epsilon$$

$$\Rightarrow f(x) > f(x_0) - f(x_0) + 2$$

$$\Rightarrow f(x) > 2$$

(4)

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) \in Q \\ xy & \text{otherwise} \end{cases}$$



Now if (x, y) lies on coordinate axis then clearly

$$f(x,y) = 0 \text{ constant}$$

and hence continuous.

Now Subpose $(x,y) \in \mathbb{R}^2$ rational

Then we can find a irrational sequence (x_n, y_n) Such that $f(x_n, y_n) \rightarrow (x, y)$

$f(x_n, y_n) = x_n y_n \longrightarrow xy \neq 0$

$f(x, y) = 0$

so $f(x_n, y_n) \not\rightarrow f(x, y)$

$(x, y) \in \mathbb{R}^2$
irrational then

Then we can find rational sequence converging to

$(x_n, y_n) \longrightarrow (x, y)$

$f(x_n, y_n) = 0 \neq xy$

$f(x, y) = xy$

If one of them is rational and one is irrational then we can find rational sequence (x_n, y_n) s.t. $(x_n, y_n) \rightarrow (x, y)$

$f(x_n, y_n) = 0$ & $f(x, y) = xy \neq 0$

and Therefore f is not continu.

(5) Since we know that If f is continuous then so is $|f|$.

$$H(x) = \max(f(x), g(x)) = \frac{|f(x) - g(x)| + f(x) + g(x)}{2}$$

$$G(x) = \min(f(x), g(x)) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

and therefore $H(x)$ & $G(x)$ are continuous.

6 (a) $(x_k) \subseteq \mathbb{R}^n$ is Cauchy

$\Rightarrow$ (x_k) is convergent ($\because \mathbb{R}^n$ complete)

$x_k \longrightarrow x$

Since $f(\cancel{x_k})$ is continuous

$\Rightarrow$ $f(x_k) \longrightarrow f(x)$ $\Rightarrow$ $(f(x_k))$ convergent

and Hence $(f(x_k))$ Cauchy

Problem.6(b). *Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be such that for every cauchy sequence $((x_n, y_n)) \subset \mathbb{R}^2$ the sequence $(f(x_n, y_n)) \subset \mathbb{R}$ is also a cauchy sequence. Then f is continuous on A.*

Solution. *Let $(x_n) \to x$ be a convergent sequence in A. Then (x_n) is a cauchy sequence in $\mathbb{R}^2$ and hence $(f(x_n))$ is a cauchy sequence and hence convergent. Suppose $(f(x_n)) \to \alpha$. Now for showing continuity of f we need to show that $f(x) = \alpha$.*
Define

$$y_n = \begin{cases} x_n, & n \text{ odd} \\ x, & n \text{ even} \end{cases}$$

then $y_n \to x$ and hence cauchy. Therefore

$$z_n = f(y_n) = \begin{cases} f(x_n), & n \text{ odd} \\ f(x), & n \text{ even} \end{cases}$$

is cauchy sequence. Again $(f(x))$ is convergent subsequence of z_n and hence $z_n \to f(x)$ is convergent. Now since $f(x_n)$ is subsequence of z_n and hence $f(x_n) \to f(x) = \alpha$ (Since limit of sequence is unique).

Which shows that f is continuous $\square$

(c) If possible suppose $f(A)$ is not bounded then

$\exists$ a sequence (y_n) in $f(A)$ such that

$$|y_n| \geq n$$

$\Rightarrow$ $\exists$ $x_n \in \mathbb{R}^2$ such that $f(x_n) = y_n$

Now since A is bounded $\Rightarrow$ (x_n) is bounded

$\Rightarrow$ from Bolzano - weirstrass Theorem (x_n) has a convergent subsequence (x_{n_p})

$\Rightarrow$ $f(x_{n_p}) = (y_{n_p})$ is convergent

but $(y_{n_p}) \geq n_p$ contradiction