PH 102 : Electromagnetism **Department of Physics**
Tutorial 3 **Indian Institute of Technology Guwahati**

Please solve the star ($\star$) marked problems first and discuss the rest if time permits.

1. $\star$ Compute the line integral of $\vec{F} = r\cos^2\theta\hat{r} - r\cos\theta\sin\theta\hat{\theta} + 3r\hat{\phi}$ around the path shown in Figure 5. Do it either in cylindrical or in spherical coordinates. Check your answer using Stokes' theorem.

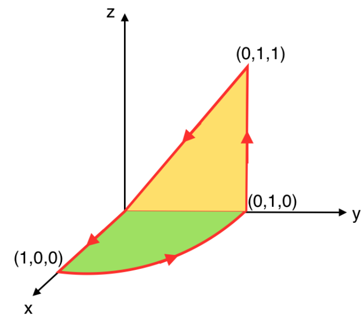

Figure 1: The path

Solution:

Stoke's theorem states that $\int_s(\vec{\nabla}\times\vec{F}).d\vec{a} = \oint_c \vec{F}.d\vec{r}$. We will calculate the line integral first using spherical polar coordinates.

Along $(0,0,0)\to(1,0,0)$: $\theta = \frac{\pi}{2}, \phi = 0$ are constants, $r : 0 \to 1$. Hence, $\vec{F}.d\vec{r} = (r\cos^2\theta)(dr) = 0 \implies \int\vec{F}.d\vec{r} = 0$.

Along $(1,0,0)\to(0,1,0)$: $\theta = \frac{\pi}{2}, r = 1$ are constants, $\phi : 0 \to \frac{\pi}{2}$. Hence, $\vec{F}.d\vec{r} = (3r)(r\sin\theta d\phi) = 3d\phi \implies \int\vec{F}.d\vec{r} = 3\int_0^{\frac{\pi}{2}} d\phi = \frac{3\pi}{2}$.

Along $(0,1,0)\to(0,1,1)$: $\phi = \frac{\pi}{2}$ is constant. Also, $y = r\sin\theta\sin\phi = r\sin\theta = 1$ is constant. Hence, $dr = -\frac{r\cos\theta d\theta}{\sin\theta} = -\frac{\cos\theta d\theta}{\sin^2\theta}$. Now, $\theta : \frac{\pi}{2} \to \frac{\pi}{4}$.

$$\begin{aligned}
\vec{F}.d\vec{r} &= (r\cos^2\theta\hat{r} - r\cos\theta\sin\theta\hat{\theta}).(dr\hat{r} + rd\theta\hat{\theta}) \\
&= r\cos^2\theta dr - r^2\cos\theta\sin\theta d\theta \\
&= -\left(\frac{\cos^3\theta}{\sin^3\theta} + \frac{\cos\theta\sin\theta}{\sin^2\theta}\right)d\theta = -\frac{\cos\theta}{\sin^3\theta}d\theta
\end{aligned}$$

Hence, $\int\vec{F}.d\vec{r} = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos\theta}{\sin^3\theta}d\theta = \left.\frac{1}{2\sin^2\theta}\right|_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{1}{2}$.

Along $(0,1,1)\to(0,0,0)$: $\theta = \frac{\pi}{4}, \phi = \frac{\pi}{2}$ are constants, $r : \sqrt{2} \to 0$. Hence, $\vec{F}.d\vec{r} = r\cos^2\theta dr = \frac{r}{2}dr \implies \int\vec{F}.d\vec{r} = \int_{\sqrt{2}}^0 \frac{r}{2}dr = -\frac{1}{2}$.

Hence, total contributions adds to $\oint\vec{F}.d\vec{r} = 0 + \frac{3\pi}{2} + \frac{1}{2} - \frac{1}{2} = \frac{3\pi}{2}$.

Using the expression for curl in spherical polar coordinates, $\vec{\nabla}\times\vec{F} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right)\hat{\phi}$, we obtain $\vec{\nabla}\times\vec{F} = 3\cot\theta\hat{r} - 6\hat{\theta}$.

There are two surfaces: The one at the bottom (the green one) is where $\theta = \frac{\pi}{2}$ is constant. Hence the elementary area is obtained $d\vec{a} = dr\hat{r} \times r\sin\theta d\phi\hat{\phi} = -r\sin\theta dr d\phi\hat{\theta} = -rdrd\phi\hat{\theta}$. Hence, $(\vec{\nabla} \times \vec{F}).d\vec{a} = 6rdrd\phi$. Hence,

$$\int (\vec{\nabla} \times \vec{F}).d\vec{a} = \int_{r=0}^{1} 6rdr \int_{\phi=0}^{\frac{\pi}{2}} d\phi = 3\frac{\pi}{2}$$

The other surface is along $y-z$ plane (the yellow one) on which, $\phi = \frac{\pi}{2}$ is constant. Area element $d\vec{a} = rd\theta\hat{\theta} \times dr\hat{r} = -rdrd\theta\hat{\phi}$. Hence, $(\vec{\nabla} \times \vec{F}).d\vec{a} = 0 \implies \int(\vec{\nabla} \times \vec{F}).d\vec{a} = 0$. Hence, Stoke's theorem is verified.

2. ⋆ Calculate the flux of $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$ through the surface defined by a cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$.

Solution:

Divergence theorem states $\int_{\mathcal{V}}(\vec{\nabla}.\vec{F})d\tau = \oint_{\mathcal{S}} \vec{F}.d\vec{a}$. The divergence of the field is $\vec{\nabla}.\vec{F} = 3$. The integral here can be performed either by using spherical polar coordinates or in cylindrical polar coordinates. We will use cylindrical polar coordinates as an illustration.

$$\begin{aligned}
\int \vec{\nabla}.\vec{F}d\tau &= 3\int_{\phi=0}^{2\pi}\int_{s=0}^{a}\int_{z=\frac{hs}{a}}^{h} sdsdzd\phi \\
&= 3 \times 2\pi \int_{s=0}^{a} [h - \frac{hs}{a}]sds \\
&= 3\frac{1}{3}\pi h a^2 \\
&= \pi
\end{aligned}$$

h is the height of the cone and a is the radius of the cone in the upper surface. As the top plane is defined by $z = 1$, we will choose $h = 1$. Also as the equation of the conic surface is $z = \sqrt{x^2 + y^2} = s$, we will have $h = a = 1$ at the upper plane.
The surface integral involves two surfaces: (i) One at the top of the cone. Here $\int_{(i)} \vec{F}.d\vec{a} = \int hdxdy = h\pi a^2 = \pi$ as $(h = a = 1)$. (ii) The second one is the conical surface. The equation of the surface is given by $\phi(s, z, \phi) = z - s = 0$. Hence the unit perpendicular to the surface is given by $\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{\hat{z}-\hat{s}}{\sqrt{2}}$. Hence $\vec{F}.\hat{n} = (s\hat{s} + z\hat{z}).\frac{\hat{z}-\hat{s}}{\sqrt{2}} = \frac{z-s}{\sqrt{2}} = 0$ as $(z = s)$. Hence, $\int_{(ii)} \vec{F}.d\vec{a} = \int_{(ii)} \vec{F}.\hat{n}da = 0$. Hence, $\int_{\mathcal{V}}(\vec{\nabla}.\vec{F})d\tau = \oint_{\mathcal{S}} \vec{F}.d\vec{a} = \pi$

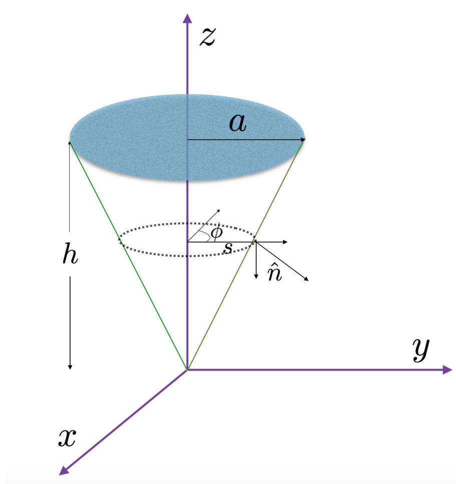

Figure 2: Cone

3. $\star$ Evaluate the following integrals:
   (a) $\int_{-1}^{2}[\sin x\delta(x+2) - \cos x\delta(x)]dx$.
   (b) $\int_{-3}^{2}(x^3 - 2x^2 + 3x + 1)\delta(x+2)dx$
   (c) $\int_{\mathcal{V}} \vec{r}.(\vec{d} - \vec{r})\delta^3(\vec{e} - \vec{r})d\tau$, where $\vec{d} = (1, 2, 3)$, $\vec{e} = (3, 2, 1)$ and $\mathcal{V}$ is the volume of a sphere of radius 1.5 units centred at $(2, 2, 2)$.
   (d) Show that $x\frac{d}{dx}\delta(x) = -\delta(x)$.

Solution:

(a) The first term in the integration involves $\delta(x+2)$ which peaks at $x = -2$ that lies outside the range of integration and hence yields zero. The second term however has $\delta(x)$ which has a peak at $x = 0$, therefore the value of integral is $\int_{-1}^{2} -\cos x\delta(x)dx = -\cos 0 = -1$.

(b) $\int_{-3}^{2}(x^3 - 2x^2 + 3x + 1)\delta(x+2)dx = (-2)^3 - 2(-2)^2 + 3(-2) + 1 = -21$.

(c) Let us first evaluate whether the delta function lies within the integral limit. For that we need to evaluate $(\vec{e}-\vec{r})^2 = ((3\hat{x}+2\hat{y}+\hat{z})-(2\hat{x}+2\hat{y}+2\hat{z}))^2 = (\hat{x}+0\hat{y}-\hat{z})^2 = 1+1 = 2 < (1.5)^2$. Therefore the delta function lies within the volume of the sphere of radius 1.5 units. Hence $\int_{\mathcal{V}} \vec{r}.(\vec{d}-\vec{r})\delta^3(\vec{e}-\vec{r})d\tau = \vec{e}.(\vec{d}-\vec{e}) = (3\hat{x}+2\hat{y}+\hat{z}).(\hat{x}+2\hat{y}+3\hat{z}-(3\hat{x}+2\hat{y}+\hat{z})) = (3\hat{x}+2\hat{y}+\hat{z}).(-2\hat{x}+0\hat{y}+2\hat{z}) = -6+2 = -4$

(d) In this problem we will use the fact that if there are two delta functions $D_1(x)$ and $D_2(x)$ following

$$\int_{\infty}^{\infty} f(x)D_1(x)dx = \int_{\infty}^{\infty} f(x)D_2(x)dx$$

for all possible normal function $f(x)$, then $D_1(x) = D_2(x)$, the delta functions must be equal. If the delta functions are different, suppose one peaks at $x = 15$ and the other at $x = 0$, then the integral would have been different as it would only yield the value of the function $f(x_0)$ where the delta function peaks (x_0). Now, consider the integral involving an ordinary function $f(x)$ as

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)x\frac{d}{dx}\delta(x)dx &= xf(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}(xf(x))\delta(x)dx \\
&= 0 - \int_{-\infty}^{\infty}(f(x) + x\frac{df}{dx})\delta(x)dx \\
&= 0 - f(0) - 0 \\
&= \int_{-\infty}^{\infty} f(x)[-\delta(x)]dx
\end{aligned}$$

$$\therefore x\frac{d}{dx}\delta(x) = -\delta(x)$$

where in the first step we used integration by parts; in the second step, the first component is zero as $\delta(x) = 0$ at $x = \pm\infty$. The rest follows procedure as has already been established.

4. Imagine four unit charges nailed to four corners of a square of side 2 units, with the North-East corner being at $(x = 1, y = 1)$. Draw pictures whenever appropriate.
   (a) Show that a charge -1 unit placed at the origin is in equilibrium, i.e., has no net force on it using symmetry arguments.
   (b) Now consider the stability of this equilibrium by lifting the charge at the centre slightly out of the plane by a tiny amount δ. Show that there is a restoring force $-k\delta$ and find k. (Use Taylor series. Since you need the force only for small displacement, drop any thing in the formula that goes like (displacement)^2 or higher.)
   (c) Find the angular frequency (ω) of small oscillations if the charge has mass m.
   (d) With what speed will it cross the origin if released from $z = \delta$?
   (e) Establish next the instability under displacements in the plane by choosing δ to be along the x-axis and showing $k = -1/(4\pi\epsilon_0\sqrt{2})$.

Solution:

(a) The force of each unit charge q at the corners exerted on the charge $-q$ in the center is $|\vec{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}$, where r is the distance between the charge at centre (yellow one) and the corner charge (red ones). Note also that according to the geometry of the problem, $r = \sqrt{2}$ here. As seen in the picture on the side, forces from charges at opposite corners cancel each other, resulting in a zero net force.

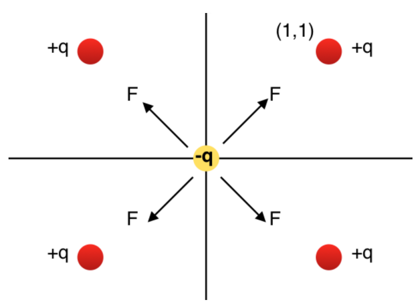

Figure 3: Figure for part 5(a)

(b) Let us write explicitly the force vectors for this case. As always, according to Coulomb's law, the force from the corner charge i $(i : 1 \to 4)$ pointing towards the centre is $\vec{F_i} = F\hat{r}_i$ where $F = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}$ and $\hat{r}$ is the unit vector pointing from the charge i towards the centre charge $-q$. According to our choice of origin, the centre charge is situated at $(0, 0, 0)$ and let the corner charges have coordinates (x, y, z). Then $\hat{r}_i = \frac{1}{r}(x\hat{x} + y\hat{y} + z\hat{z})$, where $r = \sqrt{x^2 + y^2 + z^2}$. So, when we displace the central charge by an amount δ in $+z$ direction, then $\hat{r}_i = \frac{1}{r}(\pm1\hat{x} + \pm1\hat{y} + \delta\hat{z})$ and $r = \sqrt{2 + \delta^2}$ and the $\pm$ sign depends on which charge we consider.

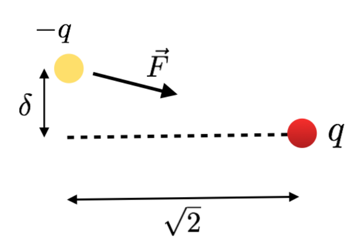

Figure 4: Figure for part 5(b)

From symmetry considerations we immediately see that the resulting force will have a component in the z-direction only.

$$\begin{aligned}\sum_{i=1}^{4} \vec{F}_i &= 4\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{-q^2}{r^3}\right)\delta\hat{z} = 4\left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{-q^2}{(2+\delta^2)^{3/2}}\right)\delta\hat{z},\\ &= 4\left(\frac{1}{4\pi\epsilon_0}\right)\frac{-q^2}{2^{3/2}}\left(1-\frac{3}{4}\delta^2+\mathcal{O}(\delta^4)\right)\delta\hat{z},\end{aligned}$$

where for the last equality we have used the Taylor expansion (Note to tutor: Taylor expansion was discussed in the class in the previous week). Neglecting all terms of order 2 or higher in δ, we get

$$\vec{F} = \sum_{i=1}^{4} \vec{F}_i = -\frac{1}{4\pi\epsilon_0}\sqrt{2}q^2\delta\hat{z}.$$

Therefore there is indeed a restoring force $-k\delta$ (we have encountered the spring problems in PH101), with the "spring constant" k being $\frac{\sqrt{2}q^2}{4\pi\epsilon_0}$.

(c) From PH101, we know that the angular frequency of a spring-mass system was simply given by $\omega = \sqrt{\frac{k}{m}}$, so here $\omega = \sqrt{\frac{\sqrt{2}q^2}{4\pi\epsilon_0 m}}$.

(d) To determine the speed with which the charge will cross the origin, we will use conservation of energy principle $E_{kin} = E_{pot} \implies \frac{1}{2}mv^2 = \frac{1}{2}kz^2 = \frac{1}{2}\frac{\sqrt{2}q^2}{4\pi\epsilon_0}\delta^2$. Therefore: $v = \sqrt{\frac{\sqrt{2}q^2}{4\pi\epsilon_0 m}}\delta = \omega\delta$.

(e) For this part, we can use the same approach as was taken for (b). We can write explicitly the forces as

$$\begin{aligned}\vec{F}_1 &= \frac{1}{4\pi\epsilon_0}\frac{-q^2}{r_1^3}((x+\delta)\hat{x}+y\hat{y}) = \frac{1}{4\pi\epsilon_0}\frac{-q^2}{r_1^3}((-1-\delta)\hat{x}+\hat{y})\\ \vec{F}_2 &= \frac{1}{4\pi\epsilon_0}\frac{-q^2}{r_2^3}((1-\delta)\hat{x}+\hat{y})\\ \vec{F}_3 &= \frac{1}{4\pi\epsilon_0}\frac{-q^2}{r_2^3}((1-\delta)\hat{x}-\hat{y})\\ \vec{F}_4 &= \frac{1}{4\pi\epsilon_0}\frac{-q^2}{r_1^3}((-1-\delta)\hat{x}-\hat{y}),\end{aligned}$$

where $r_1 = \sqrt{(x+\delta)^2+y^2} = \sqrt{(1+\delta)^2+1}$ and $r_2 = \sqrt{(1-\delta)^2+1}$. Taylor expansion of $\frac{1}{r_1^3}$ and $\frac{1}{r_2^3}$ up to first order in δ gives

$$\begin{aligned}\frac{1}{r_1^3} &= \frac{1}{((1+\delta)^2+1)^{3/2}} \sim \frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{2}}\delta\\ \frac{1}{r_2^3} &= \frac{1}{((1-\delta)^2+1)^{3/2}} \sim \frac{1}{2\sqrt{2}} + \frac{3}{4\sqrt{2}}\delta.\end{aligned}$$

As expected by symmetry, when summing up all four forces only a component in x-direction remains:

$$
\begin{aligned}
\vec{F} &= \sum_{i=1}^{4} \vec{F}_i = \frac{1}{4\pi\epsilon_0} q^2 2 \left(\frac{1-\delta}{r_2^3} + \frac{-1-\delta}{r_1^3} \right) \hat{x} \\
&= \frac{1}{4\pi\epsilon_0} q^2 2 \left[(1-\delta) \left(\frac{1}{2\sqrt{2}} + \frac{3}{4\sqrt{2}}\delta \right) + (-1-\delta) \left(\frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{2}}\delta \right) \right] \hat{x} \\
&= \frac{1}{4\pi\epsilon_0} q^2 \left(\frac{3}{\sqrt{2}}\delta - \frac{2}{\sqrt{2}}\delta \right) \hat{x} = \frac{q^2 \delta}{4\pi\epsilon_0 \sqrt{2}} \hat{x}.
\end{aligned}
$$

Thus, a displacement in the positive x direction will cause an instability and the "spring constant" in this case would be $\kappa = -\frac{q^2}{4\pi\epsilon_0\sqrt{2}}$.

5. A semi-infinite rod extending from the origin up the y-axis carries a line charge density λ. Find the field at the point $x = a$, $y = 0$. Show how you could relate the x-component of the field to the result for an infinite rod.

Solution:

The y-component of the electric field for the semi-infinite rod with line charge density λ is given by

$$
\begin{aligned}
E_y &= \vec{E}.\hat{y} = \frac{1}{4\pi\epsilon_0} \int_{y=0}^{\infty} \frac{dq}{r^2} (\hat{r}.\hat{y}) \\
&= \frac{1}{4\pi\epsilon_0} \int_{y=0}^{\infty} \frac{\lambda dy}{r^2} \left(\frac{-y}{r} \right) \\
&= -\lambda \frac{1}{4\pi\epsilon_0} \int_{y=0}^{\infty} \frac{y}{r^3} dy = \lambda \frac{1}{4\pi\epsilon_0} \int_{y=0}^{\infty} \frac{y}{(a^2+y^2)^{3/2}} dy \\
&= -\frac{\lambda}{4\pi\epsilon_0 a}
\end{aligned}
$$

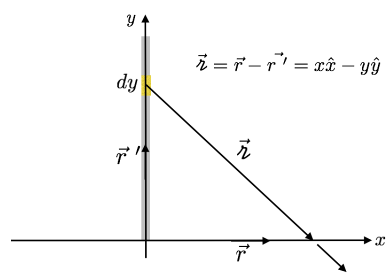

Figure 5: Semi-infinte rod

We could explicitly calculate the $x-$component by integrating the appropriate function, but it is much easier to note that the $x-$component of the field of an infinite rod (discussed in class) is twice the $x-$component of the field of the semi infinite rod. Thus

$$
E_x = \frac{1}{2} E_{x,\infty} = \frac{\lambda}{4\pi\epsilon_0 a}.
$$

6. $\star$ A charge q is at the center of a unit cube. What is the flux through one of its faces? Now consider the charge to be at one corner of the cube, as shown. What is the flux through the shaded side?

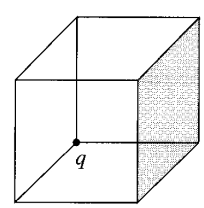

Figure 6: Prob. 6

Solution:

By Gauss's law, $\oint \vec{E}.d\vec{a} = \frac{1}{\epsilon_0}Q_{\text{enc}}$. For the first part of the problem, where the charge is at the centre, the flux through all the six surfaces of the cube will be $\frac{q}{\epsilon_0}$. Hence, by the symmetry of the cube, flux through any one of the faces is given by $\frac{q}{6\epsilon_0}$.

For the second part, where charge is at one of the corners, we can think of the cube as one of the 8 surrounding the charge q. Then, there will be 24 squares which will make up the surface of this larger cube as shown in figure. Each of these surfaces gets the same flux as every other one. So

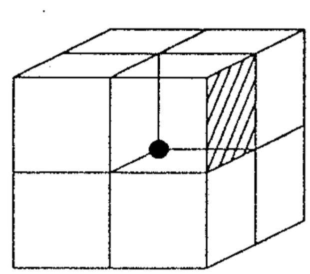

Figure 7: Figure for prob. 7

$$\int_{\text{one face}} \vec{E}.d\vec{a} = \frac{1}{24}\int_{\text{large cube}} \vec{E}.d\vec{a}$$

The latter is $\frac{1}{\epsilon_0}q$ by Gauss's law. Therefore, the flux through the shaded face will be given by $\int_{\text{one face}} \vec{E}.d\vec{a} = \frac{q}{24\epsilon_0}$.

7. $\star$ A solid sphere of radius R has uniform charge density ρ. A hole of radius $R/2$ is scooped out of it as shown in Figure. Show that the field inside the hole is uniform and along the x-axis and of magnitude $\rho R/6\epsilon_0$.

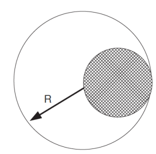

Figure 8: Sphere with a hole

Solution:

We can examine this situation as a solid sphere of uniform charge density ρ and radius R superimposed with a solid sphere of uniform charge density $-\rho$ with radius $\frac{R}{2}$. Now, let $\vec{r}_1$ denotes the vector from the centre of the larger sphere to a point within the smaller sphere, and let $\vec{r}_2$ denotes the vector from the centre of the smaller sphere to that same point. Let us first calculate the field $\vec{E}_+$ due to the larger sphere. At a distance r_1, the charge enclosed is $Q_{\text{enc}} = \frac{4}{3}\pi r_1^3\rho$. Thus the field $\vec{E}_+$ is given by $\vec{E}_+ = \frac{\rho r_1}{3\epsilon_0}\hat{r}_1 = \frac{\rho\vec{r}_1}{3\epsilon_0}$.

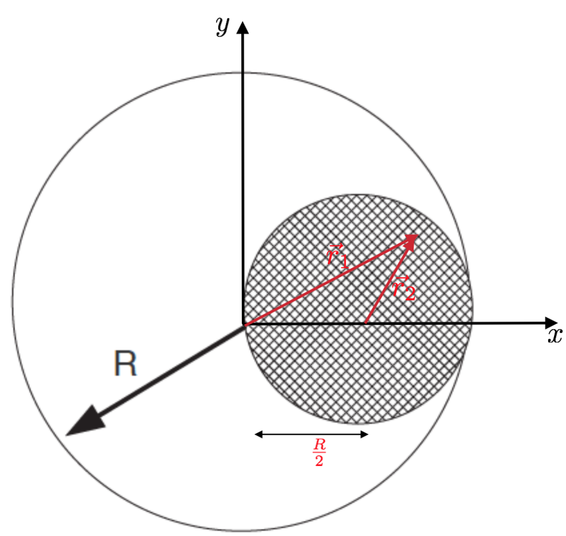

Figure 9: Sphere with a hole

Similarly, the field $\vec{E}_-$ due to the smaller, negatively charged sphere is $\vec{E}_- = -\frac{\rho \vec{r}_2}{3\epsilon_0}$. Summing together the two contributions to find the total field in the cavity, we get

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{3\epsilon_0}(\vec{r}_1 - \vec{r}_2)$$

But, we can see from the figure that $\vec{r}_1 - \vec{r}_2 = \frac{R}{2}\hat{x}$. Thus, $\vec{E} = \frac{\rho R}{6\epsilon_0}\hat{x}$, which describes a uniform field in the $\hat{x}$ direction.