# MA 102 (Ordinary Differential Equations)

**IIT Guwahati**

Tutorial Sheet No. 1 **Date:** March 8, 2018

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**Basics of ODEs, Picard's Theorem.**

(1) Determine the *order* and *degree* of the following differential equations. Also, state whether they are *linear* or *nonlinear*.
$(a)$ $\frac{d^4y}{dx^4}+19\left(\frac{dy}{dx}\right)^2 = 11y$; $(b)$ $\frac{d^2y}{dx^2}+x\sin y = 0$; $(c)$ $\frac{d^2y}{dx^2}+y\sin x = 0$; $(d)$ $(1+\frac{dy}{dx})^{\frac{1}{2}} = x\frac{d^2y}{dx^2}$;
$(e)$ $\frac{d^6y}{dx^6} + \left(\frac{d^4y}{dx^4}\right)\left(\frac{d^3y}{dx^3}\right) + y = x$; $(f)$ $x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + y = e^x$.

**Solution:** $(a)$ O=4, D=1, nonlinear; $(b)$ O=2, D=1, nonlinear; $(c)$ O=2, D=1, linear; $(d)$ O=2, D=2, nonlinear; $(e)$ O=6, D=1, nonlinear; $(f)$ O=3, D=1, linear.

(2) Eliminating the arbitrary constants $c_1, c_2$, obtain the differential equation satisfied by the following functions.
$(a)$ $y = c_1e^{-x} + c_2e^{2x}$; $(b)$ $x^2 + c_1y^2 = 1$; $(c)$ $y = c_1x - c_1^3$.

**Solution:** $(a)$ $y'' - y' - 2y = 0$; $(b)$ $y' = (xy)/(x^2 - 1)$; $(c)$ $y = xy' + (y')^3$.

(3) Consider the equation $y'(x) = cy(x)$, $0 < x < \infty$, where $c$ is a real constant. Then
$(a)$ Show that if $\phi$ is any solution and $\psi(x) = \phi(x)e^{-cx}$ then $\psi(x)$ is a constant.
$(b)$ If $c < 0$, show that every solution tends to zero as $x \to \infty$.
$(c)$ If $c > 0$, prove that the magnitude of every non-trivial solution tends to $\infty$ as $x \to \infty$.
$(d)$ When $c = 0$, what can be said about the magnitude of the solution?

**Solution:** $(a)$ $\psi'(x) = e^{-cx}(\phi'(x) - c\phi(x)) = 0 \Longrightarrow \psi(x) = \text{constant}$.
$(b)$ $y(x) = Ce^{cx}$, $C$ is an arbitrary constant. For $c < 0$, $y(x) \longrightarrow 0$ as $x \longrightarrow \infty$.
$(c)$ For $c > 0$, $|y(x)| \longrightarrow \infty$ as $x \longrightarrow \infty$.
$(d)$ When $c = 0$, $y(x) = C$ for $0 < x < \infty$.

(4) Find all real valued $C^1$ solutions $y(x)$ of the differential equation $xy'(x) + y(x) = x,\ x \in (1, 2)$.

**Solution:** Write $y' = 1 - y/x = f(x, y)$ and note that $f(x, y)$ is homogeneous function of degree 0 ($f(tx, ty) = f(x, y)$). The substitution $y(x) = v(x)x$ transform into a separable form as $dv/(1 - 2v) = dx/x$. Integrating we obtain $(1 - 2v) = 1/(c^2x^2)$. Solving for $y$, we obtain $y = x/2+c_1/x$. The $C^1$ solution $y(x)$ is obtained by putting $c_1 = 0$, i.e., $y(x) = x/2$.

(5) Discuss the existence and uniqueness of a solution of the following initial value problems (IVP) in the region $R : |x| \leq 1\ |y| \leq 1$.
$(a)$ $\dfrac{dy}{dx} = 3y^{2/3}$, $y(0) = 0$; $(b)$ $\dfrac{dy}{dx} = \sqrt{|y|}$, $y(0) = 0$;
$(c)$ $\dfrac{dy}{dx} = x^2 + y^2$, $y(0) = 0$.

**Solution:** $(a)$ $f(x, y) = 3y^{2/3} \in C(R)$, hence IVP has a solution. But, $f$ does not satisfy a Lipschitz condition on $R$, since

$$\frac{|f(x, y) - f(x, 0)|}{|y - 0|} = \frac{3y^{2/3}}{y} = \frac{3}{y^{1/3}}$$

is unbounded in every neighborhood of the origin. Thus, IVP has a solution that is not unique. Note that $y_1(x) = 0$ and $y_2(x) = x^3$ are two different solutions.

$(b)$ $f(x, y) = \sqrt{|y|} \in C(R)$, hence IVP has a solution. But, the solution is not unique as for $y > 0$

$$\frac{|f(x,y) - f(x,0)|}{|y - 0|} = \frac{\sqrt{y}}{y} = \frac{1}{y^{1/2}}$$

is unbounded in every neighborhood of the origin. Note that $y_1(x) = 0$ and $y_2(x) = \begin{cases} x^2/4, & x \geq 0, \\ -x^2/4, & x < 0. \end{cases}$ are two different solutions.

$(c)$ Note that $f, \frac{\partial f}{\partial y} \in C(R)$. By the corollary to Picard's theorem IVP has a unique solution which is certain to exists in $|x| \leq 1/2$.

(6) Show that the equation $|y'(x)| + |y(x)| + 1 = 0$ has no real solutions.

**Solution:** suppose $\phi(x)$ is a real solution on some interval $I$. Then $|\phi'(x)| + |\phi(x)| + 1 = 0,\ \forall x \in I$. But, $|\phi'(x)|, |\phi(x)| \geq 0$. So, $|\phi'(x)| + |\phi(x)| + 1 \geq 1,\ \forall x \in I$, which leads to a contradiction.

(7) A point $P$ is dragged along the $xy$ plane by a string $PT$ of length $a$. If $T$ starts at the origin and moves along the positive $y$ axis, and if $P$ starts at $(a, 0)$, what is the path of $P$? Assume here that the string is always tangent to the curve traced by the point $P$.

**Solution:** The differential of the path is

$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}.$$

Separating variables and integrating and using the condition $y(a) = 0$ yields

$$y(x) = a \log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}.$$

*** End ***