Basic Definitions, Existence and Uniqueness Results for First-Order IVP

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Texts/References:

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2. W. E. Boyce and R. C. Diprima, Elementary Differential Equations and Boundary Value Problems, John Wiley & Son, 2001.
3. E. A. Coddington, An Introduction to Ordinary Differential Equations, Prentice Hall India, 1995.
4. E. L. Ince, Ordinary Differential Equations, Dover Publications, 1958.

Definition: An equation containing the derivatives or differentials of functions is said to be a differential equation(DE).

Definition: A DE involving ordinary derivatives w.r.t a single independent variable is called an ordinary differential equation(ODE).

A general form of the nth order ODE:

$$F(x, y(x), y'(x), y''(x), \cdots, y^n(x)) = 0, \tag{1}$$

where $y'(x) = \frac{dy}{dx}$, $y''(x) = \frac{d^2y}{dx^2}, \cdots, y^n(x) = \frac{d^ny}{dx^n}$.

- The order of a DE is the order of the highest derivative that occurs in the equation.
- The degree of a DE is the power of the highest order derivative occurring in the differential equation.
- Eq. (1) is linear if F is linear in $y, y', y'', \ldots, y^n$, with coefficients depending on the independent variable x. Eq. (1) is called nonlinear if it is not linear.

Examples:

- $y''(x) + 3y'(x) + xy(x) = 0$
  (second-order, first-degree, linear)
- $y''(x) + 3y(x)y'(x) + xy(x) = 0$
  (second-order, first-degree, nonlinear)
- $(y''(x))^2 + 3y'(x) + xy^2(x) = 0$
  (second-order, second-degree, nonlinear)

Definition: A DE involving partial derivatives w.r.t more than one independent variable is called a partial differential equation(PDE).

A PDE for a function $u(x_1, x_2, \dots, x_n)$ $(n \geq 2)$ is a relation of the form

$$F(x_1, x_2, \dots, x_n, u, u_{x_1}, u_{x_2}, \dots, u_{x_1 x_1}, u_{x_1 x_2}, \dots,) = 0, \qquad (2)$$

where F is a given function of the independent variables $x_1, x_2, \dots, x_n$, and of the unknown function u and of a finite number of its partial derivatives.

Examples:

- $xu_x + yu_y = 0$ (first-order equation)
- $u_{xx} + u_{yy} = 0$ (second-order equation)

We shall consider only ODE.

Applications:

- Newton's second law can be applied to a falling object leads to the equation

$$m\frac{d^2h}{dt^2} = -mg,$$

where m is the mass of the object, h is its height above the ground, $\frac{d^2h}{dt^2}$ is its acceleration, $-mg$ is the force due to gravity.

Integrating twice w.r.t t, we obtain

$$h = h(t) = -\frac{1}{2}gt^2 + c_1 t + c_2,$$

where the integration constants c_1 and c_2 are determined if we know the initial height and initial velocity of the object.

- In case of radioactive decay, the rate of decay is proportional to the amount of radioactive substance present. This leads to the equation

$$-\frac{dA}{dt} = kA, \quad k > 0,$$

where $A(> 0)$ is the unknown amount of radioactive substance present at time t and k is the proportionality constant. Solving for A yields

$$A = A(t) = Ce^{-kt}.$$

The value of C is determined if the initial amount amount of radioactive substance is given.

Definition: A function $\phi(x) \in C^n((a, b))$ that satisfies

$$F(x, \phi(x), \phi'(x), \phi''(x), \cdots, \phi^n(x)) = 0, \quad x \in (a, b)$$

is called an explicit solution to the equation on (a, b).

Example: $\phi(x) = x^2 - x^{-1}$ is an explicit solution to

$$y''(x) - 2\frac{y}{x^2} = 0.$$

Note that $\phi(x)$ is an explicit solution on $(-\infty, 0)$ and also on $(0, \infty)$.

Definition: A relation $\psi(x, y) = 0$ is said to be an implicit solution to

$$F(x, y(x), y'(x), y''(x), \cdots, y^n(x)) = 0$$

on (a, b) if it defines one or more explicit solutions on (a, b).

Examples:

• $x + y + e^{xy} = 0$ is an implicit solution to

$$(1 + xe^{xy})y' + 1 + ye^{xy} = 0.$$

• $4x^2 - y^2 = c$, where c is an arbitrary constant, an implicit solution to $yy' - 4x = 0$.

Definition: (Initial Value Problem)
Find a solution $y(x) \in C^n((a,b))$ that satisfies

$$F(x, y, y'(x), \cdots, y^{(n)}(x)) = 0, \quad x \in (a,b)$$

and the n initial conditions(IC)

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1},$$

where $x_0 \in (a,b)$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.

First-order IVP: $F(x, y, y'(x)) = 0, \quad y(x_0) = y_0.$

Second-order IVP: $F(x, y, y'(x), y''(x)) = 0,$
$y(x_0) = y_0, \; y'(x_0) = y_1.$

Example: The function $\phi(x) = \sin x - \cos x$ is a solution to IVP: $y''(x) + y(x) = 0, \; y(0) = -1, \; y'(0) = 1.$

Consider the following IVPs:

$$|y'| + 2|y| = 0, \quad y(0) = 1 \text{ (no solution)}.$$

$$y'(x) = x, \quad y(0) = 1 \text{ (a unique solution } y = \frac{1}{2}x^2 + 1).$$

$$xy'(x) = y - 1, \quad y(0) = 1 \text{ (many solutions } y = 1 + cx).$$

Observation:
Thus, an IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

may have none, precisely one, or more than one solution.

Well-posed IVP

An IVP is said to be well-posed if

- it has a solution,
- the solution is unique and,
- the solution is continuously depends on the initial data y_0 and f.

Theorem (Peano's Theorem):

Let $R : |x - x_0| \leq a, \ |y - y_0| \leq b$ be a rectangle. If $f(x, y) \in C(R)$ then the IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

has at least one solution $y(x)$. This solution is defined for all x in the interval $|x - x_0| \leq h$, where

$$h = \min\{a, \ \frac{b}{K}\}, \qquad K = \max_{(x,y)\in R} |f(x, y)|.$$

Theorem **(Picard's Theorem):**

Let $f(x, y) \in C(R)$ and satisfy the Lipschitz condition with respect to y in R, i.e., there exists a number L such that

$$|f(x, y_2) - f(x, y_1)| \leq L|y_2 - y_1| \quad \forall (x, y_1), (x, y_2) \in R.$$

Then, the IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

has a unique solution $y(x)$. This solution is defined for all x in the interval $|x - x_0| \leq h$, where

$$h = \min\{a, \frac{b}{K}\}, \quad K = \max_{(x,y)\in R} |f(x, y)|$$

Example: Consider the IVP:

$$y'(x) = |y|, \quad y(1) = 1.$$

$f(x, y) = |y|$ is continuous and satisfies Lipschitz condition w.r.t y in every domain R of the xy-plane. The point $(1, 1)$ certainly lies in some such domain R. The IVP has a unique solution ϕ defined on some $|x - 1| \leq h$ about $x_0 = 1$.

Corollary to Picard's Theorem:

Let $f(x, y)$, $\frac{\partial f}{\partial y} \in C(R)$. Then the IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

has a unique solution $y(x)$. This solution is defined for all x in the interval $|x - x_0| \leq h$, where

$$h = \min\{a, \frac{b}{K}\}, \quad K = \max_{(x,y)\in R} |f(x, y)|.$$

Example: Let R : $|x| \leq 5,\ |y| \leq 3$ be the rectangle. Consider the IVP

$$y' = 1 + y^2, \quad y(0) = 0$$

over R.

Here, $a = 5,\ b = 3$. Then

$$\max_{(x,y)\in R} |f(x,y)| = \max_{(x,y)\in R} |1 + y^2| \leq 10(= K),$$

$$\max_{(x,y)\in R} \left| \frac{\partial f}{\partial y} \right| = \max_{(x,y)\in R} 2|y| \leq 6(= L).$$

$$\alpha = \min\{a, \frac{b}{K}\} = \min\{5,\ \frac{3}{10}\} = 0.3 < 5.$$

Note that the solution of the IVP is $y = \tan x$. This solution is valid in the interval $|x| \leq 0.3$ in stead of the entire interval $|x| \leq 5$.

Example**(Non-uniqueness)**: Consider the IVP:

$$y' = 3\,y^{2/3} \text{ for } x \in \mathbb{R}, \quad y(c) = 0.$$

The solutions are

$$y_c(x) = \begin{cases} 0 & \text{if} \quad x \leq c, \\ (x-c)^3 & \text{if} \quad x \geq c, \end{cases} \qquad .$$

where $c \geq 0$. For each real number $c \geq 0$, we have a solution $y_c(x)$ to the IVP. Therefore, this IVP has infinitely many solutions.

The Method of Successive Approximations

Consider the IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0. \tag{3}$$

Key Idea: Replacing the IVP (3) by an the equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt. \tag{4}$$

Note that (3) and (4) are equivalent.

A rough approximation to a solution is given by $y_0(x) = y_0$. A better approximation $y_1(x)$ is obtained as follows:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\, dt.$$

The next step is to use $y_1(x)$ to generate even better approximation $y_2(x)$ in the same way:

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt.$$

At the nth step, we have

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt.$$

This procedure is called Picard's method of successive approximations.

Example: Consider IVP: $y' = y, \quad y(0) = 1.$

$$y_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Note that $y_n(x) \to e^x$ as $n \to \infty$.

Facts:

- The sequence of approximation $y_n(x)$ converges to the exact solution of the IVP $y(x)$ uniformly.
- The main disadvantage of this method of successive approximations is that it leads to tedious and sometimes impossible calculations.
- Nevertheless, the method is of practical importance for the first few approximations alone are sometimes quite accurate.
- The principal use of the method of successive approximations is in proving existence and uniqueness results.

Direction Fields

- Useful in visualizing the solutions to a first-order DE.

$$(a)\ \ y' = -2y; \quad (b)\ y' = -y/x.$$

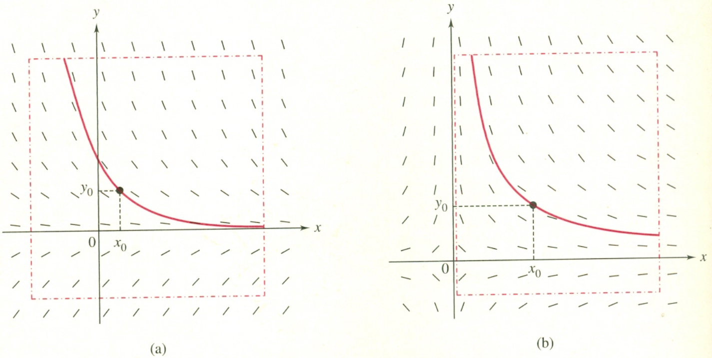

Figure 1.8 (a) A solution for $dy/dx = -2y$ (b) A solution for $dy/dx = -y/x$

For (a), choose a starting point x_0 and initial value $y(x_0) = y_0$. Since $f(x, y) = -2y \in C^1$ for all x, y, we can enclose (x_0, y_0) in a rectangle R and conclude that the IVP has one and only one solution curve passing through (x_0, y_0).

For (b), $f(x, y) = -y/x$ does not meet the continuity conditions when $x = 0$. However, for any $x_0 \neq 0$ and any initial value $y(x_0) = y_0$, we can enclose (x_0, y_0) in a rectangle of continuity that excludes the y-axis. Thus, we can be assured of a unique solution curve passing through (x_0, y_0).

*** End ***