MA 102 (Mathematics II)

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Tutorial Sheet No. 3 January 27, 2017

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(1) Let D be a closed and bounded subset of $\mathbb{R}^n$. Prove that if $f : D \to \mathbb{R}$ is continuous then it is uniformly continuous.

(2) Show that the function $f : \mathbb{R}^n \to \mathbb{R}$ $(n > 1)$ defined by $f(X) = \sin(\|X\|^2)$ is not uniformly continuous on $\mathbb{R}^n$.

(Thus, the function $f(x, y) := \sin(x^2 + y^2)$ is not uniformly continuous on $\mathbb{R}^2$.)

(3) Prove that $\sqrt{x}$ is Lipschitz continuous on $[1, \infty)$.

(4) Let $F : S \subseteq \mathbb{R} \to \mathbb{R}^3$ and $G : S \subseteq \mathbb{R} \to \mathbb{R}^3$ where S is an open set in $\mathbb{R}$. Let $t_0 \in S$. Prove that $(F \times G)'(t_0) = (F'(t_0) \times G(t_0)) + (F(t_0) \times G'(t_0))$.

(The cross product of F and G at t is defined as the cross product of the vectors $F(t)$ and $G(t)$, that is, $(F \times G)(t) := F(t) \times G(t)$.)

(5) Find the arc length of the following curves.

(a) $r(\theta) = (2\cos^2\theta, 2\cos\theta\sin\theta),\ 0 \le \theta \le \pi$.

(b) $r(t) = (t^2, t^3)$, $1 \le t \le 2$.

(6) Reparametrize the following curves in terms of arc length.

(a) $r(t) = \frac{t^2}{2}\,\hat{i} + \frac{t^3}{3}\,\hat{k} \qquad (0 \le t \le 2)$

(b) $r(t) = (3\ \cos t^2)\,\hat{i} + (3\ \sin t^2)\,\hat{j} \qquad (0 \le t \le 2\pi)$.

(7) Find $T(t)$, $N(t)$ and κ for the circular helix $F(t) = (a\cos(t),\ \ a\sin(t),\ \ b\,t)$ in the space where $a > 0$ and $b > 0$.

(8) Consider the curve $r(t) = t\,\hat{i} + t^2\,\hat{j} + \frac{2}{3}t^3\,\hat{k}$. Find the equations of the unit tangent, principal normal, and binormal to this curve at the point $(1,\ 1,\ \frac{2}{3})$. For this curve show that the curvature $\kappa = 2(1 + 2t^2)^{-2}$.

(9) An object moves counterclockwise along a circle of radius $r_0 > 0$ with a constant speed $v_0 > 0$. Set up the coordinate system so that the circle lies in the xy-plane with the origin as its center and so that the object is on the positive x-axis at time 0 and moves counterclockwise around the circle. Show that the position vector of the object is given by $R(t) = r_0\left(\cos(v_0t/r_0)\,\hat{i} + \sin(v_0t/r_0)\,\hat{j}\right)$. Find formulas for the velocity and acceleration of the object.

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