1. The cycle-average theorem: show that if A and B are two complex quantities which both vary with time as $e^{-i\omega t}$ then the average over a cycle of the oscillation of the product of the real parts of A and B is given by
$$\langle Re[A] \; \times Re[B] \rangle = \frac{1}{2} Re(AB^{\star}),$$
where the star denotes complex conjugation.

2. In free space, an electric field is given as
$$\mathbf{E}\,(r,\,\theta,\,\phi,\,t) = \frac{A\ sin\theta}{r}\left(cos(kr-\omega t) - \frac{sin(kr-\omega t)}{kr}\right)\hat{e}_{\phi}$$
with $k = \omega/c$.
   (a) Show that the electric field **E** obeys Gauss's law
   (b) The magnetic field induction **B**
   (c) Verify the magnetic field induction **B** is solenoidal
   (d) Determine the poynting vector

3. A uniform plane wave in air with$\mathbf{E} = 8\ \hat{e}_y\ e^{-i(\omega\, t - 4\, x - 3\, z)}$ V/m is incident on a dielectric slab ($z \geq 0$) with $\mu_r = 1.0$, $\epsilon_r = 2.5$, $\sigma = 0$. Find
   (a) The polarization of the wave
   (b) The angle of incidence
   (c) The reflected **E** field
   (d) The transmitted **H** field.

4. Light of angular frequency $\omega$ passes from medium 1, through a slab (thickness d) of medium 2 and into medium 3 (for instance, from water through glass into air as shown in the figure). All three media are linear and homogeneous. Assume $\mu_1 = \mu_2 = \mu_3 = \mu_0$. Show that the transmission coefficient for normal incidence is given by
$$T = \left[\frac{1}{4n_1n_2}\left\{(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)\,(n_3^2-n_2^2)}{n_2^2} sin^2\left(\frac{n_2\omega d}{c}\right)\right\}\right]^{-1}$$

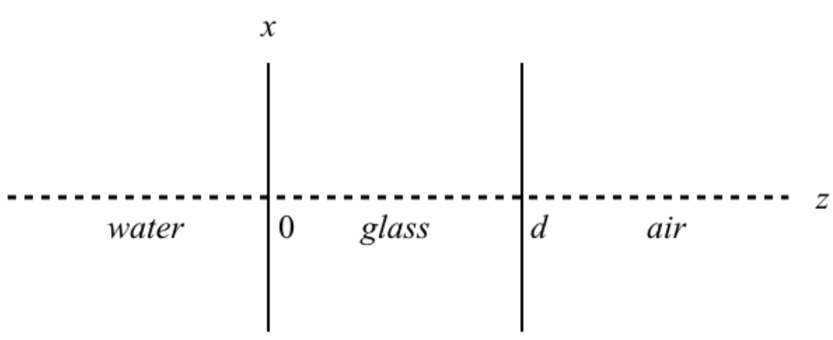



Figure 1: Problem 4

- Take home exercise from D J Griffiths's 3rd edition book
  [1] G 8.9 [2] G 8.11(a) [3]G 9.7 [4]G 9.8 [5]G 9.9 [6]G 9.11 [7]G 9.13 [8]G 9.14

Solution 1:

- The cycle-average theorem: Show that if A and B are two complex quantities which both vary with time as $\exp(-i\omega t)$ then the average over a cycle at the oscillation of the product of the real parts A & B is given by

$$\overline{\mathrm{Re}\,A \times \mathrm{Re}\,B} = \frac{1}{2}\mathrm{Re}\,(AB^*)$$

Let

$$A = (A_1 + iA_2)\,e^{-i\omega t}$$

$$B = (B_1 + iB_2)\,e^{-i\omega t}$$

$$\mathrm{Re}\,(A) = A_1 \cos\omega t + A_2 \sin\omega t$$

$$\mathrm{Re}\,(B) = B_1 \cos\omega t + B_2 \sin\omega t$$

$$\mathrm{Re}(A) \times \mathrm{Re}(B) = (A_1 \cos\omega t + A_2 \sin\omega t) \times (B_1 \cos\omega t + B_2 \sin\omega t)$$

$$= (A_1 \times B_1)\cos^2\omega t + (A_2 \times B_2)\sin^2\omega t + [(A_1 \times B_2)\cos\omega t \sin\omega t] + [(A_2 \times B_1)\cos\omega t \sin\omega t]$$

Reaion (2)

$$\mathrm{Re}\,A \times \mathrm{Re}\,B = (A_1 \times B_1)\cos^2\omega t + (A_2 \times B_2)\sin^2\omega t + \left[(A_1 \times B_2) + (A_2 \times B_1)\right]\sin\omega t\cos\omega t$$

Average over a cycle

$$\overline{\mathrm{Re}\,A \times \mathrm{Re}\,B} = \frac{1}{2}\left[A_1 \times B_1 + A_2 \times B_2\right]$$

$$= \frac{1}{2}\mathrm{Re}\,(A \times B^*) \quad \text{Ans}$$

Averge values of

$\cos^2\omega t = \sin^2\omega t = \frac{1}{2}$

$\sin\omega t = \cos\omega t = 0$

Let's check

$\mathrm{Re}\,(AB^*)$

$$AB^* = (A_1 + iA_2)e^{-i\omega t} \times (B_1 - iB_2)e^{i\omega t}$$

$$= (A_1 + iA_2) \times (B_1 - iB_2)$$

$$= A_1B_1 - iA_1B_2 - i^2 A_2B_2 + iA_2B_1$$

$$AB^* = A_1B_1 + A_2B_2 + i(A_2B_1 - A_1B_2)$$

$$\mathrm{Re}\,(AB^*) = A_1B_1 + A_2B_2$$

Hence

$$\boxed{\overline{\mathrm{Re}\,A \times \mathrm{Re}\,B} = \frac{1}{2}\mathrm{Re}\,(AB^*)}$$

Solution 2:

$$\vec{E}(r,\theta,\phi,t) = E_\phi \hat{e}_\phi, \quad \text{where}$$

$$E_\phi = \frac{A\sin\theta}{r}\left(\cos\psi - \frac{\sin\psi}{kr}\right)$$

$$\psi = kr - \omega t$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}E_\phi = 0$$

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E} = -\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 0 & r\sin\theta E_\phi \end{vmatrix}$$

$$= -\frac{1}{r^2\sin\theta}\left\{\hat{e}_r\frac{\partial}{\partial\theta}(r\sin\theta E_\phi) - r\hat{e}_\theta\frac{\partial}{\partial r}(r\sin\theta E_\phi)\right\}$$

$$= -\frac{\hat{e}_r}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{A\sin^2\theta}{r}\left(\cos\psi - \frac{\sin\psi}{kr}\right)\right]$$

$$+\frac{\hat{e}_\theta}{r}\frac{\partial}{\partial r}\left[A\sin\theta\left(\cos\psi - \frac{\sin\psi}{kr}\right)\right]$$

$$\frac{\partial\vec{B}}{\partial t} = -\frac{2A\cos\theta}{r^2}\left(\cos\psi - \frac{\sin\psi}{kr}\right)\hat{e}_r + \frac{A\sin\theta}{r}$$

$$\left(-k\sin\psi + \frac{\sin\psi}{kr^2} - \frac{\cos\psi}{r}\right)\hat{e}_\theta$$

Reaion (2)

$$\vec{B} = \frac{2A\cos\theta}{\omega r^2}\left(\sin\psi + \frac{\cos\psi}{kr}\right)\hat{e_r} + \frac{A\sin\theta}{\omega r}\left(-k\cos\psi + \frac{\cos\psi}{kr^2} + \frac{\sin\psi}{r}\right)\hat{e_\theta}$$

$$\vec{B} = B_r\,\hat{e_r} + B_\theta\,\hat{e_\theta}$$

On Comparing

$$B_r = \frac{2A\cos\theta}{\omega r^2}\left(\sin\psi + \frac{\cos\psi}{kr}\right)$$

$$B_\theta = \frac{A\sin\theta}{\omega r}\left(-k\cos\psi + \frac{\cos\psi}{kr^2} + \frac{\sin\psi}{r}\right)$$

(c) $\vec{\nabla}\cdot\vec{B} = 0$ (Condition of Solenoidal field)

$$\vec{\nabla}\cdot\vec{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, B_\theta)$$

$$= \frac{1}{r^2}\,\frac{2A\cos\theta}{\omega}\left(k\cos\psi - \frac{\cos\psi}{kr^2} - \frac{\sin\psi}{r}\right)$$

$$+ \frac{1}{r\sin\theta}\,\frac{2A\sin\theta\cos\theta}{\omega r}\left(-k\cos\psi + \frac{\cos\psi}{kr^2} + \frac{\sin\psi}{r}\right)$$

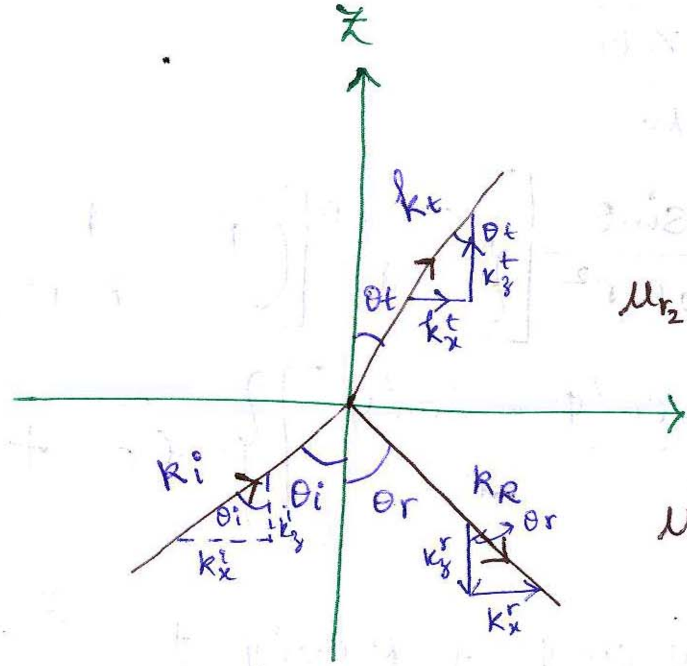
$$= 0$$

(d) $\vec{S}$ = Poynting Vector

$$= \frac{\vec{E} \times \vec{B}}{\mu_0}$$

$$= \frac{A^2 \sin\theta}{\mu_0 \omega r^2} \left[ \left\{ \frac{2\cos\theta}{r} \left[ \left( 1 - \frac{1}{k^2 r^2} \right) \sin\psi \cos\psi + \frac{\cos^2\psi - \sin^2\psi}{kr} \right] \right\} \hat{e}_\theta + \sin\theta \left[ \left( -\frac{2}{r} + \frac{1}{k^2 r^3} \right) \sin\psi \cos\psi + k \cos^2\psi + \frac{\sin^2\psi - \cos^2\psi}{kr^2} \right] \hat{e}_r \right.$$

## Solution :




$\mu_{r_2} = 1.0, \quad \epsilon_{r_2} = 2.5$

$\mu_r' = 1.0, \quad \epsilon_r' = 1.0$

Argument of the incident plane wave is

$\vec{k} = |k|\,\hat{s}$

$k = \dfrac{n\omega}{c}$

$= \dfrac{\omega}{c} \quad (\text{Air})$

$\omega = ck = 5c$

$$\boxed{\omega = 15 \times 10^8 \text{ rad/s}}$$

$(\omega t - 4x - 3z) =$

$= \left(\omega t - 5\times\frac{4}{5}x - 5\times\frac{3}{5}z\right)$

$= \left(\omega t - 5\left(\frac{4}{5}x + \frac{3}{5}z\right)\right)$

Comparing this with

$\left(\omega t - |k|\left(\sin\theta_i\, x + \cos\theta_i\, z\right)\right)$

so $|k| = 5$ & $\vec{k} = 4\hat{e}_x + 3\hat{e}_z$

$\sin\theta_i = \dfrac{4}{5}$

$\cos\theta_i = \dfrac{3}{5}$

(a) The plane of incidence is $xz$-plane. $\hat{e}_z$ is the unit normal vector to the interface ($z=0$).

Since $\vec{E}_i$ is normal to the plane of incidence, therefore we have $\perp$ polarization.

(b) The angle of incidence

$$\sin\theta_i = \frac{4}{5}$$

$$\boxed{\theta_i = 53.13^\circ}$$

(c)

$$\vec{E}_R = \vec{R}\, e^{-i\tau_R}$$

$$\tau_R = \omega\left(t - \frac{x\sin\theta_r - z\cos\theta_r}{v_r}\right) \qquad \boxed{v_r = c}$$

We know that

$$\theta_i = \theta_r = 53.13^\circ$$

$$k_{x_R} = \sin\theta_R \qquad k_{z_R} = -\cos\theta_R$$

$$\vec{k}_R = 4\hat{e}_x - 3\hat{e}_z$$

$$=$$

$$\tau_R = \omega\left(t - \frac{0.8x - 0.6z}{c}\right) \rightarrow \text{Argument part of the reflected wave}$$

Now from Fresnel's formulae

$$R_{\parallel} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} A_{\parallel}$$

$$R_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} A_{\perp}$$

Now from Snell's Law

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{1}{\sqrt{2.5}}\sin(53.13^\circ)$$

$$\sin\theta_t = \frac{n_1}{n_2}\sin(53.13^\circ)$$

$$\theta_t = 30.39^\circ$$

$\theta_i = 53.13^\circ$

$\theta_t = 30.39^\circ$

Since $A_{\parallel} = 0$

so $R_{\parallel} = 0$

& $A_{\perp} = 8$

$$R_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} A_{\perp}$$

$$\boxed{R_{\perp} = -3.112}$$

Reflected wave (Electric part)

$$\vec{E_R} = -3.112\,\hat{e_y}\, e^{-i(\omega t - \vec{K_R}\cdot\vec{r})}$$

where

$$\vec{K_R} = 4\,\hat{e_x} - 3\,\hat{e_z}$$

$$\vec{r} = x\,\hat{e_x} + z\,\hat{e_z}$$

d) The argument part of the disturbance

$$\tau_t = \omega\left(t - \frac{x\sin\theta_t + z\cos\theta_t}{V_2}\right)$$

$V_2$ = Propagation speed in medium 2

$$= \frac{c}{n}$$

$$= \frac{c}{\sqrt{\epsilon_2 \mu_2}}$$

$$= \frac{c}{\sqrt{2.5}}$$

$$\Rightarrow \quad |k_t| = \frac{n\omega}{c} = \omega \qquad = 7.906$$

$$k_{x_t} = \sin\theta_t\, K_t \qquad = 4$$

$$\vec{K_t} = 4\,\hat{e_x} + 6.819\,\hat{e_z}$$

Notice that $K_x^{(i)} = K_x^{(R)} = K_x^{(T)}$ as expected

$$T_{\parallel} = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} A_{\parallel}$$

$$= 0$$

$$T_{\perp} = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)} A_{\perp}$$

$$= 0.611 \times 8$$

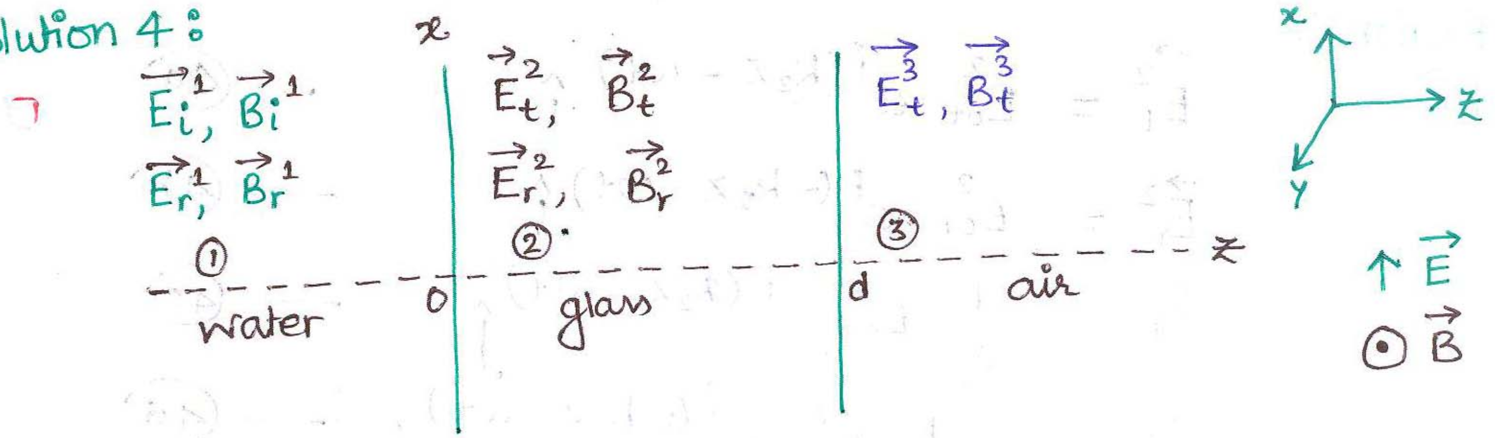
$$= 4.888$$

$$\vec{E}_t = 4.88\, \hat{e}_y \exp[-i(\omega t - 4x - 6.819z)] \;(V/m)$$

$$\vec{H}_t = \frac{1}{\mu_2 \omega} \vec{k}_t \times \vec{E}_t$$

$$= (-17.69\, \hat{e}_x + 10.37\, \hat{e}_z)\, e^{-i(\omega t - 4x - 6.819z)} \;\left(\frac{mA}{m}\right)$$

Solution 4 :




Given $\mu_1 = \mu_2 = \mu_3 = \mu_0$

Using the boundary conditions

$$\left.\begin{aligned} E_1^{\parallel} &= E_2^{\parallel} \\ B_1^{\parallel} &= B_2^{\parallel} \end{aligned}\right\} \text{ across (1) \& (2) interface} \quad —①$$

&

$$\left.\begin{aligned} E_2^{\parallel} &= E_3^{\parallel} \\ B_2^{\parallel} &= B_3^{\parallel} \end{aligned}\right\} \text{ across (2) \& (3) interface} \quad —②$$

Region (1)
$$\vec{E}_i^1 = E_{oi}^1 \, e^{i(k_1 z - \omega t)} \, \hat{i} \quad —(3a)$$
$$\vec{E}_r^1 = E_{or}^1 \, e^{i(-k_1 z - \omega t)} \, \hat{i} \quad —(3b)$$
$$\vec{B}_i^1 = \frac{1}{\omega} \vec{k}_1 \times \vec{E}_i^1 = \frac{k_1}{\omega} \hat{k}_1 \times \vec{E}_i^1$$
$$= \frac{1}{v_1} E_{oi}^1 \, e^{i(k_1 z - \omega t)} \, \hat{j} \quad —(3c)$$

Similarly, $\vec{B}_r^1 = -\frac{1}{v_1} E_{or}^1 \, e^{i(-k_1 z - \omega t)} \, \hat{j} \quad —(3d)$

Region (2)

$$\vec{E}_t^{\,2} = E_{ot}^2 \, e^{i(k_2 z - \omega t)} \, \hat{i} \quad \text{— (4a)}$$

$$\vec{E}_r^{\,2} = E_{or}^2 . e^{i(-k_2 z - \omega t)} \, \hat{i} \quad \text{— (4b)}$$

$$\vec{B}_t^{\,2} = \frac{1}{v_2} E_{ot}^2 \, e^{i(k_2 z - \omega t)} \, \hat{j} \quad \text{— (4c)}$$

$$\vec{B}_r^{\,2} = -\frac{1}{v_2} E_{or}^2 \, e^{i(-k_2 z - \omega t)} \, \hat{j} \quad \text{— (4d)}$$

Region (3)

$$\vec{E}_t^{\,3} = E_{ot}^3 \, e^{i(k_3 z - \omega t)} \, \hat{i} \quad \text{— (5a)}$$

$$\vec{B}_t^{\,3} = \frac{1}{v_3} E_{ot}^3 \, e^{i(k_3 z - \omega t)} \, \hat{j} \quad \text{— (5b)}$$

Using the boundary conditions (1) & (2)

We get from (1a), (3b), (4a) & (4b) at $z = 0$

$$E_{oi}^1 + E_{or}^1 = E_{ot}^2 + E_{or}^2 \quad \text{— (6a)}$$

$$\frac{1}{v_1} E_{oi}^1 - \frac{1}{v_1} E_{or}^1 = \frac{1}{v_2} E_{ot}^2 - \frac{1}{v_2} E_{or}^2$$

or

$$E_{oi}^1 - E_{or}^1 = \frac{v_1}{v_2}\left(E_{ot}^2 - E_{or}^2\right)$$

$$= \beta_1 \left(E_{ot}^2 - E_{or}^2\right) \quad \text{— (6b)}$$

$$\boxed{\beta_1 = \frac{v_1}{v_2}} \quad \text{— (7)}$$

Using 2 at $z = d$, we get from 4a, 4b & 5a

$$E_{ot}^{2} e^{ik_2 d} + E_{or}^{2} e^{-ik_2 d} = E_{ot}^{3} e^{ik_3 d}$$

& from 4c, 4d & 5b

$$\frac{1}{v_2} E_{ot}^{2} e^{ik_2 d} - \frac{1}{v_2} E_{or}^{2} e^{-ik_2 d} = \frac{1}{v_3} E_{ot}^{3} e^{ik_3 d}$$

or

$$E_{ot}^{2} e^{ik_2 d} - E_{or}^{2} e^{-ik_2 d} = \frac{v_2}{v_3} E_{ot}^{3} e^{ik_3 d}$$

$$= \beta_2 E_{ot}^{3} e^{ik_3 d} \quad \text{— (8b)}$$

$$\beta = \frac{v_2}{v_3} \quad \text{— (9)}$$

We have to evaluate

$$T = \frac{I_{ot}^{3}}{I_{oi}^{1}} = \frac{\epsilon_3 v_3}{\epsilon_1 v_1} \frac{|E_{ot}^{3}|^2}{|E_{oi}^{1}|^2} \quad \text{— (10)}$$

Adding 6a & 6b

$$2 E_{oi}^{1} = (1+\beta_1) E_{ot}^{2} + (1 - \beta_1) E_{or}^{2} \quad \text{— (11a)}$$

Adding 8a & 8b

$$2 E_{ot}^{2} e^{ik_2 d} = (1+\beta_2) E_{ot}^{3} e^{ik_3 d} \quad \text{— (11b)}$$

Substracting (8b) from (8a)

$$2 E_{or}^{2} e^{-ik_2 d} = (1-\beta_2) E_{ot}^{3} e^{ik_3 d} \quad \text{—— (11c)}$$

Substituting (11b) & (11c) in (11a), we get

$$2 E_{oi}^{1} = \frac{1}{2}(1+\beta_1) e^{ik_3 d - ik_2 d} (1+\beta_2) E_{ot}^{3} + \frac{1}{2}(1-\beta_1) e^{ik_3 d + ik_2 d} (1-\beta_2) E_{ot}^{3}$$

$$= \frac{1}{2} e^{ik_3 d} \left\{ (1+\beta_1)(1+\beta_2) e^{-ik_2 d} + (1-\beta_1)(1-\beta_2) e^{ik_2 d} \right\} E_{ot}^{3}$$

or $$\frac{E_{oi}^{1}}{E_{ot}^{3}} = \frac{1}{4} e^{ik_3 d} \left\{ [1+2\beta_1\beta_2 + (\beta_1+\beta_2)] e^{-ik_2 d} + [1+2\beta_1\beta_2 - (\beta_1+\beta_2)] e^{ik_2 d} \right\}$$

$$= \frac{1}{4} e^{ik_3 d} \left\{ (e^{-ik_2 d} + e^{ik_2 d}) + 2\beta_1\beta_2 (e^{-ik_2 d} + e^{ik_2 d}) + (\beta_1+\beta_2)(e^{-ik_2 d} - e^{ik_2 d}) \right\}$$

$$= \frac{1}{4} e^{ik_3 d} \left\{ 2\cos k_2 d + 2\beta_1\beta_2 \cos k_2 d - 2i(\beta_1+\beta_2) \sin k_2 d \right\}$$

$$\frac{|E^1_{0i}|^2}{|E^3_{0t}|^2} = \frac{1}{4}\left\{\left[\cos k_2 d + \beta_1\beta_2 \cos k_2 d - i(\beta_1+\beta_2)\sin k_2 d\right]\right.$$

$$\left. \times \left[\cos k_2 d + \beta_1\beta_2 \cos k_2 d + i(\beta_1+\beta_2) \sin k_2 d\right]\right\}$$

$$= \frac{1}{4}\left\{(1+\beta_1\beta_2)^2 \cos^2 k_2 d + (\beta_1+\beta_2)^2 \sin^2 k_2 d\right\}$$

$$= \frac{1}{4}\left\{(1+\beta_1\beta_2)^2 + \left[(\beta_1+\beta_2)^2 - (1+\beta_1\beta_2)^2\right]\sin^2 k_2 d\right\}$$

$$(\because \cos^2\alpha = 1 - \sin^2\alpha)$$

$$= \frac{1}{4}\left\{(1+\beta_1\beta_2)^2 + \left[(\beta_1^2-1) + \beta_2^2(1-\beta_1^2)\right]\sin^2 k_2 d\right\}$$

$$= \frac{1}{4}\left\{(1+\beta_1\beta_2)^2 - (1-\beta_1^2)(1-\beta_2^2)\sin^2 k_2 d\right\} \quad \text{— (12)}$$

$$\therefore \quad T = \left(\frac{\epsilon_3 v_3}{\epsilon_1 v_1}\right)\left\{\left|\frac{E^1_{0t}}{E^3_{0t}}\right|^2\right\}^{-1}$$

$$= \frac{(1/v_3^2\mu_3)\,v_3}{(1/v_1^2\mu_1)\,v_1}\left[\frac{1}{4}\left\{\left(1+\frac{v_1}{v_3}\right)^2 - \left(1-\frac{v_1^2}{v_2^2}\right)\left(1-\frac{v_2^2}{v_3^2}\right)\sin^2 k_2 d\right\}\right]^{-1}$$

$$= \frac{v_1}{v_3}\left[\frac{1}{4}\left\{\left(1+\frac{n_3}{n_1}\right)^2 - \left(1-\frac{n_2^2}{n_1^2}\right)\left(1-\frac{n_3^2}{n_2^2}\right)\sin^2 k_2 d\right\}\right]^{-1}$$

$$= \frac{n_3}{n_1}\left[\frac{1}{4}\left\{\frac{(n_1+n_3)^2}{n_1^2} - \frac{(n_1^2-n_2^2)(n_2^2-n_3^2)}{n_1^2\, n_2^2}\sin^2 k_2 d\right\}\right]^{-1}$$

$$\boxed{T = \left[\frac{1}{4 n_1 n_3}\left\{(n_1+n_3)^2 - \frac{(n_1^2-n_2^2)(n_2^2-n_3^2)}{n_2^2}\sin^2 k_2 d\right\}\right]^{-1}}$$