MA 102 (Mathematics II)

Department of Mathematics, IIT Guwahati

Tutorial Sheet No. 1

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(1) Consider the Euclidean norm $\|X\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and show that $|\,\|X\| - \|Y\|\,| \leq \|X - Y\|$ for $X, Y \in \mathbb{R}^n$. Show that the vectors X and Y are orthogonal if and only if

$$\|X + Y\|^2 = \|X\|^2 + \|Y\|^2.$$

(2) Let $(X_k) \subset \mathbb{R}^n$ and $X \in \mathbb{R}^n$. Show that $X_k \to X$ in $\mathbb{R}^n$ if and only if for every $Y \in \mathbb{R}^n$ the sequence $(\langle X_k,\, Y\rangle) \subset \mathbb{R}$ converges to $\langle X,\, Y\rangle$, that is, $\langle X_k,\, Y\rangle \to \langle X,\, Y\rangle$ in $\mathbb{R}$.

(3) Let $(X_k) \subset \mathbb{R}^n$ be such that $X_k \to X$ for some $X \in \mathbb{R}^n$. Show that the sequence $(\|X_k\|) \subset \mathbb{R}$ converges to $\|X\|$. Additionally suppose that $X \neq 0$ and $X_k \neq 0$ for all k, and define $Y_k := X_k/\|X_k\|$ and $Y := X/\|X\|$. Show that $Y_k \to Y$.

(4) Let $(X_k) \subset \mathbb{R}^n$ and $X, Y \in \mathbb{R}^n$. Suppose that $X_k \to X$ and that $\langle X_k,\, Y\rangle = 0$ for all k. Show that $\langle X,\, Y\rangle = 0$.

(5) Convert from rectangular coordinates $(x,\ y,\ z)$ to spherical coordinates $(\rho,\ \phi,\ \theta)$.
(a) $(1,\ \sqrt{3},\ -2)$, (b) $(1,\ -1,\ \sqrt{2})$.

(6) Convert from spherical coordinates $(\rho,\ \phi,\ \theta)$ to rectangular coordinates $(x,\ y,\ z)$.
(a) $(5,\ \pi/6,\ \pi/4)$, (b) $(7,\ \pi/2,\ \pi/2)$.

(7) Convert from cylindrical coordinates $(r,\ \theta,\ z)$ to spherical coordinates $(\rho,\ \phi,\ \theta)$.
(a) $(\sqrt{3},\ \pi/6, 3)$, (b) $(1,\ \pi/4,\ -1)$.

(8) Convert from spherical coordinates $(\rho,\ \phi,\ \theta)$ to cylindrical coordinates $(r,\ \theta,\ z)$.
(a) $(5,\ \pi/4,\ 2\pi/3)$, (b) $(1,\ \pi/2,\ 7\pi/6)$.

(9) For each of the following sets in their mentioned spaces, identify (i) interior points, (ii) limit points, (iii) boundary points, (iv) Closure of the set.
(a) Space $= \mathbb{R}$, $S = \{1,\ 2,\ 3,\ 4\}$
(b) Space $= \mathbb{R}$, $S = \mathbb{Q}$
(c) Space $= \mathbb{R}$, $S = \{x \in \mathbb{R}\ :\ 0 < x < 1\}$
(d) Space $= \mathbb{R}^2$, $S = \{(x,\ y) \in \mathbb{R}^2\ :\ 0 < x < 1 \text{ and } y = 0\}$
(e) Space $= \mathbb{R}^2$, $S = \{(x,\ y) \in \mathbb{R}^2\ :\ 0 < x < 1 \text{ and } y \in \mathbb{R}\}$
(f) Space $= \mathbb{R}^2$, $S = \mathbb{Q}^c \times \mathbb{Q}$
(g) Space $= \mathbb{R}^2$, $S = \{(x,\ y) \in \mathbb{R}^2\ :\ x^2 + y^2 < 100\}$
(h) Space $= \mathbb{R}^3$, $S = \{\left(\frac{1}{n},\ 0,\ 0\right) \in \mathbb{R}^3\ :\ n \in \mathbb{N}\}$
(i) Space $= \mathbb{R}^n$, $S = \{(k,\ 0,\ 0,\ \cdots,\ 0) \in \mathbb{R}^n\ :\ k = 1,\ 2,\ \cdots,\ 10000000000000\}$

(10) For each of the following sets in their mentioned spaces, find out whether the given set is (i) open, (ii) closed, (iii) bounded .

(a) Space $= \mathbb{R}^2$, $S = \{(x,\ y) \in \mathbb{R}^2\ :\ xy > 0\}$

(b) Space $= \mathbb{R}^2$, $S = \{(x,\ y) \in \mathbb{R}^2\ :\ x^2 + y^2 \leq 1 \text{ and } y \geq 0\}$

(c) Space $= \mathbb{R}^2$, $S = \{(x,\ y) \in \mathbb{R}^2\ :\ y < 1\}$

(d) Space $= \mathbb{R}^3$, $S = \left\{ \left(\frac{1}{k},\ k,\ 0 \right) \in \mathbb{R}^3\ :\ k \in \mathbb{N} \right\}$