# MA 102 (Mathematics II)
# Depertment of Mathematics, IIT Guwahati
Tutorial Sheet 2

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**Solution:** 1.(a) $f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2-y^2}, & x \neq \pm y \\ 0, & otherwise. \end{cases}$

Consider the sequence $\{(\frac{1}{n}, \frac{1}{n} + \frac{1}{n^2})\}$. Then,

$$\begin{aligned} f\left(\frac{1}{n}, \frac{1}{n} + \frac{1}{n^2}\right) &= \lim_{n\longrightarrow\infty} \frac{\frac{1}{n^3}\left[1+(1+\frac{1}{n})^3\right]}{\frac{1}{n^2}\left[1-(1+\frac{1}{n})^2\right]} \\ &= \lim_{n\longrightarrow\infty} \frac{1+(1+\frac{1}{n})^3}{n\left[-\frac{2}{n}-\frac{1}{n^2}\right]} \\ &= \lim_{n\longrightarrow\infty} \frac{1+(1+\frac{1}{n})^3}{-2-\frac{1}{n}} \\ &= \frac{2}{-2} = -1. \end{aligned}$$

But $\lim_{n\to\infty} f(\frac{1}{n}, \frac{1}{n}) = 0$. Thus limit does not exist.

**Solution:** 1.(b) Now $\frac{|x^2-y^2|}{|x^2+y^2|} \leq \frac{x^2+y^2}{x^2+y^2} = 1$.
Thus $\left|xy\frac{x^2-y^2}{x^2+y^2}\right| \leq |xy| \leq x^2+y^2$.
Thus, given $\epsilon > 0$ take $\delta = \sqrt{\epsilon}$.

**Solution:** 1.(c) $f(xy) = \frac{sinxy}{x^2+y^2}$.
Along $y = mx (m \neq 0)$

$$\begin{aligned} \lim_{x\longrightarrow 0} f(x,mx) &= \frac{sin(mx^2)}{(mx^2)} . \frac{mx^2}{x^2(1+m^2)} \\ &= \frac{m}{1+m^2} \end{aligned}$$

Therefore, limit does not exist.

**Solution:** 1.(d) $f(x,y) = \frac{|x|}{y^2} e^{-\frac{|x|}{y^2}}$
Along $y = \sqrt{x}$,

$$\lim_{x\longrightarrow 0+} f(x,\sqrt{x}) = \lim_{x\longrightarrow 0+} \frac{x}{x} e^{-\frac{x}{x}} = e^{-1}$$

Along $y = 2\sqrt{x}$,

$$\lim_{x\longrightarrow 0+} f(x,2\sqrt{x}) = \lim_{x\longrightarrow 0+} \frac{x}{4x} e^{-\frac{x}{4x}} = \frac{1}{4} e^{-\frac{1}{4}}$$

Therefore, limit does not exist.

**Solution:** 1.(e)

$$\begin{aligned}
&\lim_{(x,y)\longrightarrow(0,0)} \frac{1-cos(x^2+y^2)}{(x^2+y^2)} \\
&= \lim_{r\longrightarrow 0} \frac{1-cosr^2}{r^4} \\
&= \lim_{r\longrightarrow 0} \frac{2rsinr^2}{4r^3} \quad \text{[L' Hospital's rule]} \\
&= \lim_{r\longrightarrow 0} \frac{1}{2}\frac{sinr^2}{r^2} \\
&= \frac{1}{2}
\end{aligned}$$

**Solution:** 2.(a) $|f(x,y)| = |xycos(\frac{1}{x})| \leq |x||y| \leq x^2+y^2$
Thus given $\epsilon > 0$, if we chose $\delta = \sqrt{\epsilon}$.
Then,

$$|f(xy)| < \epsilon \quad \text{whenever} \sqrt{x^2+y^2} < \delta.$$

Hence the given function is continuous.

**Solution:** 2.(b) $f(0,0) = 0$
But $f(x,y) = 1$ along $y = x^3$ for $x < 1$.
Thus $f$ is not continuous at (0,0).

**Solution:** 2.(c) $|f(x,y)| = \dfrac{|x|^3}{|x^2+y^2|} \leq \dfrac{|x|^3}{|x|^2} = |x| \leq \sqrt{x^2+y^2}$
Thus for given $\epsilon > 0$, take $\delta = \epsilon$.

**Solution:** 2.(d)

$$\begin{aligned}
|f(x,y)| &= \frac{|x^3y|}{x^4+y^2} \\
&= \frac{|x||x^2y|}{x^4+y^2} \\
&\leq |x|\frac{(x^4+y^2)}{x^4+y^2} \\
&= |x| \leq \sqrt{x^2+y^2}
\end{aligned}$$

Take $\delta = \epsilon$.

**Solution:** 2.(e) $f(x,y) = \dfrac{sin(x+y)}{|x|+|y|}$
Along $y = x$,

$$\lim_{x\longrightarrow 0+} f(x,x) = \lim_{x\longrightarrow 0+} \frac{sin(2x)}{2x} = 1$$

Along $y = -x$,

$$\lim_{x\longrightarrow 0+} f(x,-x) = \lim_{x\longrightarrow 0+} \frac{sin(x-x)}{2x} = 0$$

Therefore, limit does not exist.

**Solution:** 2.(f) $f(x,y) = xyln(x^2+y^2) \quad (x,y) \neq (0,0)$
Taking $x = rcos\theta$, $y = rsin\theta$ we get, $f(r,\theta) = r^2 cos\theta sin\theta . lnr^2$.
Therefore,

$$|f(r,\theta)| \leq r^2 |lnr^2| \longrightarrow 0 \quad as \quad r \longrightarrow 0.$$

Thus $f$ is continuous at (0,0).

**Solution:** (3). Since $|f(X_0)| > 2$, $\exists \epsilon' > 0$ such that, $|f(X_0)| = 2+\epsilon'$.
Let $\epsilon = \frac{\epsilon'}{2}$. Then for this $\epsilon$, there exists $\delta > 0$ [since $f$ is continuous at $X_0$] such that,

$$|f(X) - f(X_0)| < \epsilon \quad \text{whenever} \quad ||X - X_0|| < \delta.$$

Now, $|f(X_0)| - |f(X)| \leq |f(X) - f(X_0)|$.
Thus,

$$\begin{aligned} |f(X)| &\geq |f(X_0)| - |f(X) - f(X_0)| \\ &> 2 + \epsilon' - \epsilon \\ &= 2 + \epsilon' - \frac{\epsilon'}{2} = 2 + \frac{\epsilon'}{2} \quad \text{whenever} \quad ||X - X_0|| < \delta. \end{aligned}$$

i.e, $|f(X)| > 2$ whenever $||X - X_0|| < \delta$.

**Solution:** (4). Take $(x,y)$ such that $x \neq 0, y \neq 0$.
***Case(I):*** $(x,y) \notin \mathbb{Q} \times \mathbb{Q}$. Then $f(x,y) = xy \neq 0$.
Take a sequence $\{(x_n,y_n)\} \subset \mathbb{Q} \times \mathbb{Q}$ such that $\{(x_n,y_n)\} \longrightarrow (x,y)$.
Then $f(x_n,y_n) = 0 \quad \forall n$.
Thus $\lim_{n \longrightarrow \infty} f(x_n,y_n) = 0 \neq f(x,y)$.
***Case(II):*** $(x,y) \in \mathbb{Q} \times \mathbb{Q}$.
Take a sequence $\{(x_n,y_n)\} \subset \mathbb{Q}^c \times \mathbb{Q}^c$ such that $\{(x_n,y_n)\} \longrightarrow (x,y)$. Then $x_n y_n \longrightarrow xy \neq 0$.

Thus, $f(x_n,y_n) = x_n y_n \not\longrightarrow 0 = f(x,y)$.
Now take $(x,y)$ where $y = 0$. Then $f(x,y) = 0$.
Now for any sequence $\{(x_n,y_n)\}$ such that $(x_n,y_n) \longrightarrow (x,0)$,

$$|f(x_n,y_n)| \leq |x_n y_n| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Thus, $f$ is continuous on $x - axis$.
Similarly, $f$ is continuous on $y - axis$.

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