PH 102 : Electromagnetism **Department of Physics**
Tutorial 5 **Indian Institute of Technology Guwahati**

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1. Image charge for a grounded conducting spherical shell:

   (a) A point charge $-q$ is located at $x = a$, and a point charge Q is located at $x = A$. Show that the locus of points with $V = 0$ is a circle in the xy plane (and hence a spherical shell in space).

   (b) What must be the relation among q, Q, a, and A, so that the center of the circle is located at $x = 0$?

   (c) Assuming that the relation you found in part (b) holds, what is the radius of the circle in terms of a and A?

   (d) Explain why the previous results imply the following statement: if a charge Q is externally located a distance $A > R$ from the centre of a grounded conducting spherical shell with radius R, then the external field due to the shell is the same as the field of an image point charge $-q = -QR/A$ located a distance $a = R^2/A$ from the centre of the shell; see Fig. The total external field is the sum of this field plus the field from Q. (The internal field is zero, by the uniqueness theorem.)

   (e) Likewise for the following statement: if a charge $-q$ is internally located a distance $a < R$ from the centre of a grounded conducting spherical shell with radius R, then the internal field due to the shell is the same as the field of an image point charge $Q = qR/a$ located a distance $A = R^2/a$ from the centre of the shell; see Fig. The total internal field is the sum of this field plus the field from q. (The external field is zero, because otherwise the shell would not have the same potential as infinity. Evidently a charge $+q$ flows onto the grounded shell.)

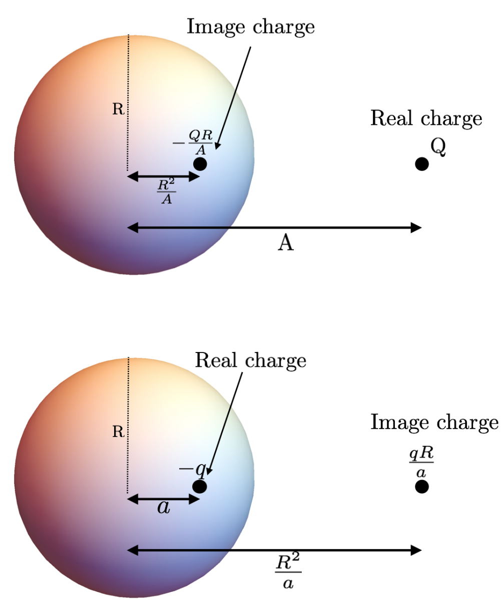

Figure 1

Solution:

(a) The potential at an arbitrary point in the xy plane is

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{\sqrt{(x-A)^2+y^2}} - \frac{q}{\sqrt{(x-a)^2+y^2}}\right)$$

Setting this equal to zero, putting one term on either side of the equation, and squaring gives

$$\begin{aligned} Q^2(x^2 - 2ax + a^2 + y^2) &= q^2(x^2 - 2Ax + A^2 + y^2) \\ (Q^2 - q^2)(x^2 + y^2) - 2x(Q^2 a - q^2 A) &= q^2A^2 - Q^2a^2 \\ x^2 + y^2 - 2\left(\frac{Q^2 a - q^2 A}{Q^2 - q^2}\right)x &= \frac{q^2A^2 - Q^2a^2}{Q^2 - q^2}. \end{aligned}$$

Since the coefficients of x^2 and y^2 are equal, this equation describes a circle. More precisely, the equation has been written in the form of $x^2 + y^2 - 2Bx = C$, which in turn can be written as $(x - B)^2 + y^2 = C + B^2$, by completing the square. This equation describes a circle with its centre at $(B, 0)$ and with radius $\sqrt{C + B^2}$. For our purpose, $B = \frac{Q^2a - q^2A}{Q^2 - q^2}$ and $C = \frac{q^2A^2 - Q^2a^2}{Q^2 - q^2}$.

(b) From the above discussion, it is clear that the centre of the circle will be at $x = 0$ if the coefficient of x i.e. B is zero. That is only possible if $Q^2a = q^2A$.

Alternative argument: We can work in terms of the angle θ shown in Fig. Using the law of cosines to determine the distances from a point P on the circle to the two charges (assuming the centre is located at $x = 0$), we see that the potential at P is zero if

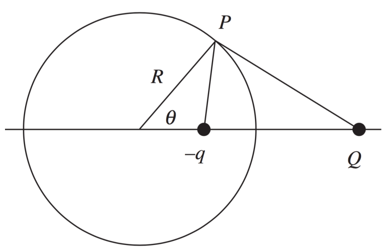

Figure 2

$$\frac{Q}{\sqrt{R^2 + A^2 - 2RA\cos\theta}} = \frac{q}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}}$$

$$\implies Q^2(R^2 + a^2 - 2Ra\cos\theta) = q^2(R^2 + A^2 - 2RA\cos\theta) \ (1)$$

If this equation is to be true for all values of θ, then the coefficient of $\cos\theta$ must be the same on both sides. This yields $Q^2a = q^2A$.

(c) If $Q^2a = q^2A$, then the radius of the circle is given by

$$R^2 = \frac{q^2A^2 - Q^2a^2}{Q^2 - q^2} = \frac{(Q^2a/A)A^2 - Q^2a^2}{Q^2 - (Q^2a/A)} = aA.$$

Alternatively, we can work in terms of the angle θ shown in the figure. If $Q^2a = q^2A$, then Eqn. (1) gives $Q^2(R^2 + a^2) = q^2(R^2 + A^2) \implies R^2 = aA$.

(d) Having derived $R^2 = aA$, we can eliminate a from the relation $Q^2a = q^2A$ to obtain $Q^2(R^2/A) = q^2A \implies q = QR/A$. Putting all the results together, we see that if we have a charge Q at $x = A$ and a charge $-q = -QR/A$ at $x = a = R^2/A$, then the entire surface of the sphere of radius R centred at the origin will be at zero potential. But this is exactly the boundary condition for a grounded conducting sphere. The uniqueness theorem therefore tells us that the two setups (point charge outside grounded conducting sphere, and point charge near image charge) have exactly the same field in the exterior of the sphere. (This reasoning doesn't apply to the interior, because the setups are different there; one contains an image charge, the other doesn't. The uniqueness theorem requires the same charge distribution in the relevant region in both setups.) The results for this problem look a little cleaner if we let $A = nR$, where n is a numerical factor. The image charge then has the value $-q = -Q/n$ and is located at radius R/n.

(e) Again using $R^2 = aA$, we can eliminate A from the relation $Q^2 a = q^2 A$ to obtain $Q^2 a = q^2(R^2/a) \implies Q = qR/a$. Putting all the results together, we see that if we have a charge $-q$ at $x = a$ and a charge $Q = qR/a$ at $x = A = R^2/a$, then the entire surface of the sphere of radius R centred at the origin will be at zero potential. As above, we conclude that the two setups (point charge inside grounded conducting sphere, and point charge near image charge) have the same field in the interior of the sphere. If we let $a = R/n$, then the image charge has the value $Q = nq$ and is located at radius nR.

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2. Consider a generalisation of the above problem. Suppose that the point charge Q is situated at a distance A from the centre of a conducting sphere of radius R kept at a finite potential. The same basic model discussed in the previous problem will handle the case of a sphere at any finite potential V_0 (relative to infinity) with the addition of a second image charge. What charge should you use, and where should you put it ? Find the force of attraction between a point charge Q and a neutral conducting sphere.

Solution:

From the previous problem, we already know that the presence of an image charge $q = -\frac{RQ}{A}$ at a distance $a = \frac{R^2}{A}$ from the centre of the sphere will mimic the situation of grounded conducting sphere with $V = 0$ on the surface of the sphere. Hence, placing an additional charge q' at the centre will create a non-zero potential on the surface:

$$V_0 = \frac{1}{4\pi\epsilon_0}\frac{q'}{R} \implies q' = 4\pi\epsilon_0 V_0 R$$

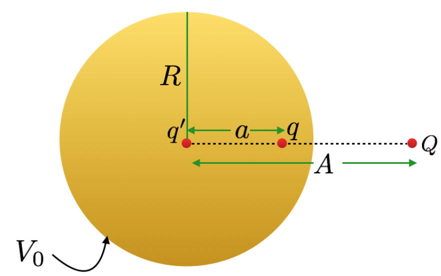

Figure 3

For the conducting sphere to be neutral, one additionally requires the charge on them to be zero. Hence,

$$q + q' = 0 \implies q' = \frac{RQ}{A}$$

Force between the test charge Q with the conducting sphere kept at a potential V_0 can then be identified as the force between q with the image charges q and q'. Therefore,

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0} Q \left(\frac{q'}{A^2} + \frac{q}{(A-a)^2} \right) = \frac{Qq}{4\pi\epsilon_0} \left(-\frac{1}{A^2} + \frac{1}{(A-a)^2} \right) \\ &= \frac{Qq}{4\pi\epsilon_0} \frac{a(2A-a)}{A^2(A-a)^2} \\ &= -\frac{Q^2}{4\pi\epsilon_0} \left(\frac{R}{A} \right)^3 \frac{(2A^2 - R^2)}{(A^2 - R^2)^2} \end{aligned}$$

where in the last step we used, $q = -\frac{RQ}{A}$ and $a = \frac{R^2}{A}$. Negative sign indicates the force of attraction.

3. Two similar charges ($+q$) are placed at a distance $2b$ apart.

(a) Find, approximately, the minimum radius a (assume $a << b$) of a grounded conducting sphere placed mid way between them that would neutralise their mutual repulsion (*Hint: Expand the expression of force in terms of* a/b).

(b) What is the force on each of the two charges, if the same sphere, with the radius determined in part 3(a) above, is now kept at a constant potential V.

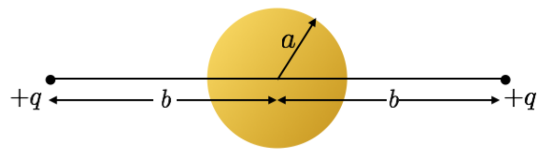

Figure 4

Solution:

We will again use method of images to solve the problem. For one real charge $+q$ at $x = b$ requires the presence of one imaginary charge $-q'$ at $x = r$ to make a sphere of radius a (centred at origin) equipotential. Choose the origin of the coordinates as the origin of the sphere and neglect z dependence from the symmetry. q' and r can be found as follows from previous problem:

$$q' = q(a/b), \quad r = a^2/b.$$

In similar argument, presence of $+q$ at $x = -b$ requires the presence of one imaginary charge $-q' = -q(a/b)$ at $x = -r = -a^2/b$ to make a sphere of radius a (centred at origin) equipotential. From superposition principle, it is then straightforward to assume that presence of two $+q$ charges at $x = \pm b$ require the imagination of two $-q' = -q(a/b)$ charges at $x = \pm r = \pm a^2/b$ as shown in Fig.

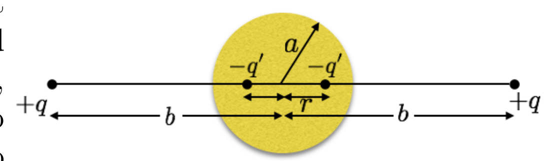

Figure 5

Mutual repulsion of the $+q$ charges in presence of a grounded conducting sphere will then simply be the mutual Coulomb forces due to all the charges in above configuration. If there is no net force on any of the charges:

$$\frac{1}{4\pi\epsilon_0}\left[\frac{q^2}{4b^2} - \frac{q^2(a/b)}{(b-(a^2/b))^2} - \frac{q^2(a/b)}{(b+(a^2/b))^2}\right] = 0$$

$$\implies \frac{q^2}{4b^2} = \frac{q^2(a/b)}{(b-(a^2/b))^2} - \frac{q^2(a/b)}{(b+(a^2/b))^2}$$

$$\implies \frac{q^2}{4b^2} = \frac{2q^2a}{b^3}\left[1 + 3\left(\frac{a}{b}\right)^4 + \cdots\right]$$

$$\therefore a = \frac{b}{8}$$

keeping the zeroth order term in the expansion of $\frac{a}{b}$.

(b) When the sphere is kept at constant potential V, we need charge Q at centre, so that

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{a} = \frac{1}{4\pi\epsilon_0}\frac{Q}{b/8}$$
$$\implies Q = \frac{\pi\epsilon_0 bV}{2}$$

Hence, a resultant force on any $+q$ charge will act as

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{b^2}\hat{x} = \frac{1}{4\pi\epsilon_0}\frac{q(\pi\epsilon_0 bV/2)}{b^2}\hat{x} = \frac{qV}{8b}\hat{x}$$

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4. Two semi infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q situated as shown in Fig. Set up the image configuration, and calculate the potential in this region - what charge do you need, and where should they be located ? What is the force on q ? Suppose the planes met at some angle other than 90°; would you still be able to solve the problem by method of images ? If not (in general), for what particular angles does the method work ?

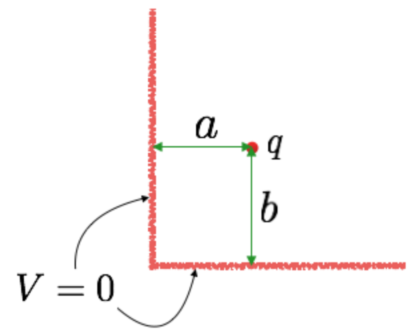

Figure 6

Solution:

The image charge configuration of the system can be seen as in Fig 7. We just need to place equal and opposite charge in the same distance on the other side of the grounded conductor to show that the conducting planes (along x and y axis) will have zero potentials. Hence potential at any point (x, y, z) will be

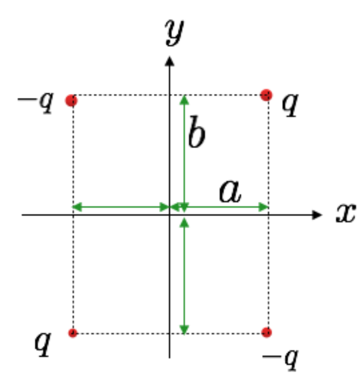

Figure 7

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right]$$

The force on q due to conducting planes can be calculated using the force exerted by other image charges simply as follows:

$$\vec{F} = -\frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{b^2}\hat{y} + \frac{1}{a^2}\hat{x} - \left(\frac{a}{(a^2+b^2)^{3/2}}\hat{x} + \frac{b}{(a^2+b^2)^{3/2}}\hat{y}\right)\right].$$

where the last term in parenthesis indicates the force of repulsion between similar charges placed at opposite corner of the rectangle. The method of images work for the conductors making the angle θ, divisor of 180°. Hence, $\theta = \{45°, 60°, 90°, 180°\}$. For example, if the conductors are placed at 45°, the image configuration can be as shown in Fig. 8. The equal and opposite image charges are equidistant from each other along the imaginary axis drawn all at 45° to x and y axis. This can make the potential zero along x axis and along the one at 45° shown in red. One can also argue in the similar line, while for example, if the conductors make an angle 135°, image charge configuration cant be made. One of the reasons, is to make the surfaces equipotential, one is forced to imagine an image charge in the original region of interest, which is not allowed.

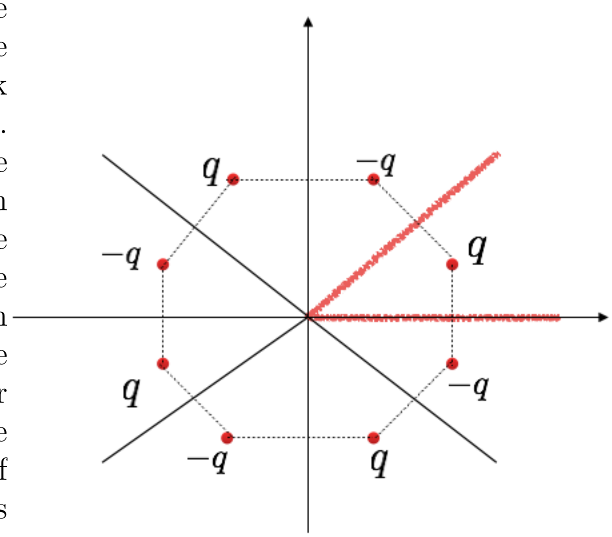

Figure 8

5. A rectangular pipe, running parallel to the z axis (from $-\infty \to +\infty$), has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specific potential $V_0(y)$.
   (a) Develop a general formula for the potential within the pipe.
   (b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).

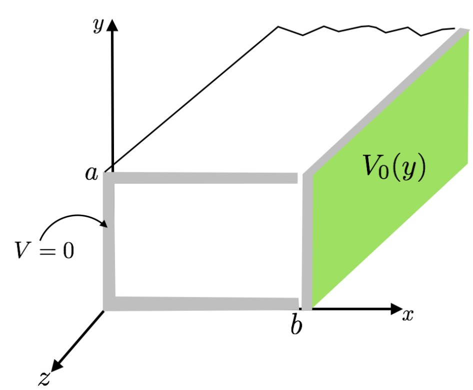

Figure 9

Solution:

The potential has to be evaluated in charge free region following Laplace's equation:

$$\nabla^2 V = 0 \implies \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

where we have used the symmetry of the configuration to interpret that the potential will only have x, y components as the pipe runs along z axis. Boundary conditions are :

$$\begin{aligned} (i)\ V(x,0) &= 0, \\ (ii)\ V(x,a) &= 0, \\ (iii)\ V(0,y) &= 0, \\ (iv)\ V(b,y) &= V_0(y) \end{aligned}$$

Separation of variables (as done in class) yields:

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

Using the boundary conditions (i): $D = 0$, (iii): $B = -A$, (ii): $ka = n\pi$. Hence,

$$V(x,y) + AC(e^{n\pi x/a} - e^{-n\pi x/a})\sin(n\pi y/a) = 2AC\sinh(n\pi x/a)\sin(n\pi y/a)$$

Evidently, the most general solution will be a linear combination of all the above solutions for different values of n

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi x/a)\sin(n\pi y/a)$$

where C_n is a constant $\equiv AC$ in expression above and can be evaluated using the boundary condition (iv) as follows:

$$\sum_{n=1}^{\infty} C_n \sinh(n\pi b/a)\sin(n\pi y/a) = V_0(y)$$

Using Fourier's trick:

$$C_n \sinh(n\pi b/a) = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)dy \implies C_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a V_0(y)\sin(n\pi y/a)dy$$

This integral above can only be evaluated provided a definite function $V_0(y)$ is given.
(b) For $V_0(y) = V_0$,

$$C_n = \frac{2V_0}{a\sinh(n\pi b/a)}\int_0^a \sin(n\pi y/a)dy = \frac{2V_0}{a\sinh(n\pi b/a)} \times \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{2a}{n\pi} & \text{for } n \text{ odd} \end{cases}$$

Hence,

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5..}^{\infty} \frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}$$

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6. Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$. The left end, at $x = 0$, consists of two metal strips: one, from $y = 0$ to $y = \frac{a}{2}$, is held at a constant potential V_0, and the other, from $y = \frac{a}{2}$ to $y = a$, is at potential $-V_0$. Find the potential in the infinite slot.

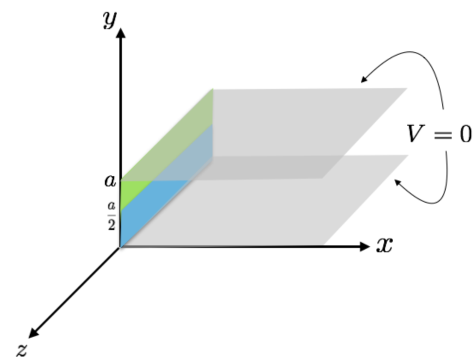

Figure 10

Solution:

We have seen in class that

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

where

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)dy$$

Now, we are given a specific potential configuration as

$$V_0(y) = \begin{cases} +V_0 & \text{for} \quad 0 < y < \frac{a}{2} \\ -V_0 & \text{for} \quad \frac{a}{2} < y < a \end{cases}$$

Hence,

$$\begin{aligned} C_n &= \frac{2}{a}V_0\left[\int_0^{a/2}\sin(n\pi y/a)dy - \int_{a/2}^{a}\sin(n\pi y/a)dy\right] \\ &= \frac{2V_0}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right] \\ &= \frac{2V_0}{n\pi}[1 + (-1)^n - 2\cos(\frac{n\pi}{2})] \end{aligned}$$

Following a little algebra, we can find that

$$C_n = \begin{cases} \frac{8V_0}{n\pi} & \text{for} \quad n = 2, 6, 10, 14... \ (4j+2) \text{ with } j = 0, 1, 2.. \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

Hence,

$$V(x,y) = \frac{8V_0}{\pi} \sum_{n=2,6,10,14..} \frac{e^{-n\pi x/a} \sin(n\pi y/a)}{n} = \frac{8V_0}{\pi} \sum_{j=0,1,2..} \frac{e^{-(4j+2)\pi x/a} \sin((4j+2)\pi y/a)}{n}$$