Yule-Walker Equations in matrix form

PRACTICAL TIME SERIES ANALYSIS

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Objectives

- Rewrite Yule – Walker equations in matrix form for AR(p) processes

AR(p) process

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots \phi_p X_{t-p} + Z_t$$

where

$$Z_t \sim Normal(0, \sigma_Z^2)$$

Note that, if we take expectation from both sides of the model

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots \phi_p X_{t-p} + Z_t$$

We get

$$\mu = \phi_0 + \phi_1 \mu + \phi_2 \mu + \cdots \phi_p \mu$$

Subtract side by side

$$X_t - \mu = \phi_1 (X_{t-1} - \mu) + \phi_2 (X_{t-2} - \mu) + \cdots \phi_p (X_{t-p} - \mu) + Z_t$$

If $\tilde{X}_t = X_t - \mu$, then $E[\tilde{X}_t] = 0$, and

$$\tilde{X}_t = \phi_1 \tilde{X}_{t-1} + \phi_2 \tilde{X}_{t-2} + \cdots \phi_p \tilde{X}_{t-p} + Z_t$$

AR(p) process with $\mu = 0$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots \phi_p X_{t-p} + Z_t$$

where

$$Z_t \sim Normal(0, \sigma_Z^2)$$

Yule –Walker equations

Autocorrelation function obeys

$$\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2) + \cdots + \phi_p\rho(k-p)$$

for $k \geq 1$, $\rho(0) = 1$ and $\rho(k) = \rho(-k)$ for $k < 0$.

Lets write them for $k = 1,2, \ldots, p$.

Yule Walker equations

$$\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2) + \cdots + \phi_p\rho(k-p)$$

For $k = 1,2,\ldots,p,$

$$\begin{gathered}
\rho(1) = \phi_1\rho(0) + \phi_2\rho(-1) + \phi_3\rho(-2) + \cdots + \phi_p\rho(1-p) \\
\rho(2) = \phi_1\rho(1) + \phi_2\rho(0) + \phi_3\rho(-1) + \cdots + \phi_p\rho(2-p) \\
\rho(3) = \phi_1\rho(2) + \phi_2\rho(1) + \phi_3\rho(0) + \cdots + \phi_p\rho(3-p) \\
\vdots \\
\rho(p-1) = \phi_1\rho(p-2) + \phi_2\rho(p-3) + \phi_3\rho(p-4) + \cdots + \phi_p\rho(1) \\
\rho(p) = \phi_1\rho(p-1) + \phi_2\rho(p-2) + \phi_3\rho(p-3) + \cdots + \phi_p\rho(0)
\end{gathered}$$

Recall $\rho(k) = \rho(-k)$.

i.e., $\rho(-1) = \rho(1),\ \rho(-2) = \rho(2),\ \ldots, \rho(2-p) = \rho(p-2), \rho(1-p) = \rho(p-1)$

$$\rho(-k) = \rho(k)$$

$$
\begin{aligned}
\rho(1) &= \phi_1\rho(0) + \phi_2\rho(1) + \phi_3\rho(2) + \cdots + \phi_p\rho(p-1) \\
\rho(2) &= \phi_1\rho(1) + \phi_2\rho(0) + \phi_3\rho(1) + \cdots + \phi_p\rho(p-2) \\
\rho(3) &= \phi_1\rho(2) + \phi_2\rho(1) + \phi_3\rho(0) + \cdots + \phi_p\rho(p-3)
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
\rho(p-1) &= \phi_1\rho(p-2) + \phi_2\rho(p-3) + \phi_3\rho(p-4) + \cdots + \phi_p\rho(1) \\
\rho(p) &= \phi_1\rho(p-1) + \phi_2\rho(p-2) + \phi_3\rho(p-3) + \cdots + \phi_p\rho(0)
\end{aligned}
$$

$\rho(0) = 1$

$$
\begin{gathered}
\rho(1) = \phi_1 + \phi_2\rho(1) + \phi_3\rho(2) + \cdots + \phi_p\rho(p-1) \\
\rho(2) = \phi_1\rho(1) + \phi_2 + \phi_3\rho(1) + \cdots + \phi_p\rho(p-2) \\
\rho(3) = \phi_1\rho(2) + \phi_2\rho(1) + \phi_3 + \cdots + \phi_p\rho(p-3) \\
\\
\vdots \\
\rho(p-1) = \phi_1\rho(p-2) + \phi_2\rho(p-3) + \phi_3\rho(p-4) + \cdots + \phi_p\rho(1) \\
\rho(p) = \phi_1\rho(p-1) + \phi_2\rho(p-2) + \phi_3\rho(p-3) + \cdots + \phi_p
\end{gathered}
$$

Matrix form: Yule- Walker equations

$$\underbrace{\begin{bmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \\ \vdots \\ \rho(p-1) \\ \rho(p) \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} 1 & \rho(1) & \rho(2) & & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \cdots & \rho(p-2) \\ \rho(2) & \rho(1) & 1 & & \rho(p-3) \\ & \vdots & & \ddots & \vdots \\ \rho(p-2) & \rho(p-3) & \rho(p-4) & \cdots & \rho(1) \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & & 1 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix}}_{\phi}$$

$$b = R\phi$$

$$R^{-1}\, b = \phi$$

Sample ACF $r_k \approx \rho(k)$

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{p-1} \\ r_p \end{bmatrix} = \begin{bmatrix} 1 & r_1 & r_2 & & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & 1 & & r_{p-3} \\ & \vdots & & \ddots & \vdots \\ r_{p-2} & r_{p-3} & r_{p-4} & & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix}$$

$\hat{b}$ $\hat{R}$ $\hat{\phi}$

$$\hat{b} = \hat{R}\hat{\phi}$$

$$\hat{R}^{-1}\,\hat{b} = \hat{\phi}$$

Matrices R and $\hat{R}$

- These matrices are symmetric matrices
- They are positive semidefinite matrices
- All eigenvalues are nonnegative
- Inverses of these matrices exist

$$\begin{bmatrix} 1 & r_1 & r_2 & & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & 1 & & r_{p-3} \\ & \vdots & & \ddots & \vdots \\ r_{p-2} & r_{p-3} & r_{p-4} & & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & \cdots & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \rho(1) & \rho(2) & & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \cdots & \rho(p-2) \\ \rho(2) & \rho(1) & 1 & & \rho(p-3) \\ & \vdots & & \ddots & \vdots \\ \rho(p-2) & \rho(p-3) & \rho(p-4) & & \rho(1) \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \cdots & 1 \end{bmatrix}$$

- $\hat{b} = \hat{R}\hat{\phi}$ has a unique solution

Example – AR(2)

We (will) estimate coefficients of the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

by first finding r_1, r_2 using acf() routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix}$$

Example – AR(3)

We (will) estimate coefficients of the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t$$

by first finding r_1, r_2, r_3, using acf() routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix}$$

What We've Learned

- Matrix form of Yule – Walker equations
- How to estimate the coefficients of an AR process using Yule-Walker equations