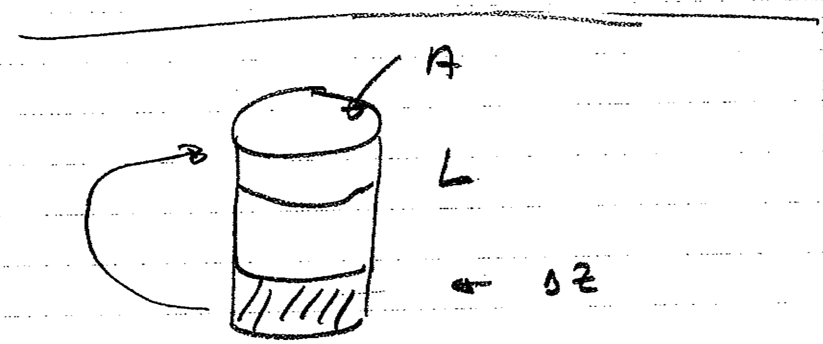
Proof of Archimedes principles



$$\Delta U = \Delta m \, g \, L = (A \cdot \Delta z \, \rho) \, g \cdot L$$

$$\frac{\Delta U}{\Delta z} = \rho \, g \, L \cdot A$$

$$\boxed{F = \rho g \cdot L \cdot A = W_{water}}$$

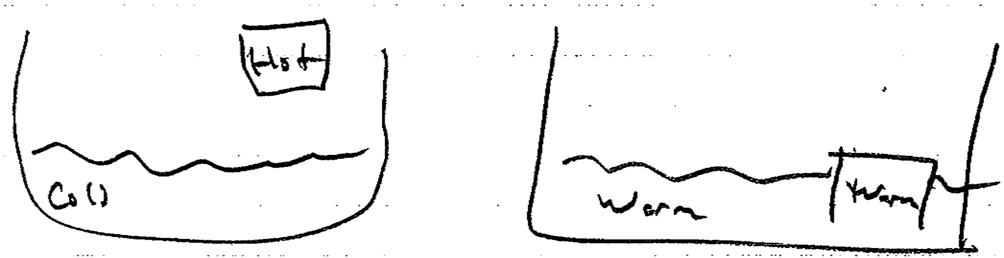
Goal: Define temperature

Problem: We don't yet have the understanding to do it properly yet... a state we are often (maybe always) in...

So we proceed empirically at first. We notice things.

Thing 1:

When hot and cold are in contact, the hot gets colder and the cold gets hotter, until they stop.



Q: What do we mean by thermal contact?

For now: touching!

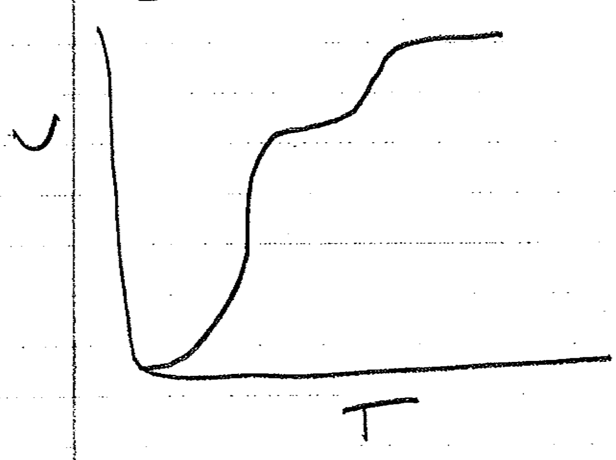
(Will find more ways.)

T1 T? 4

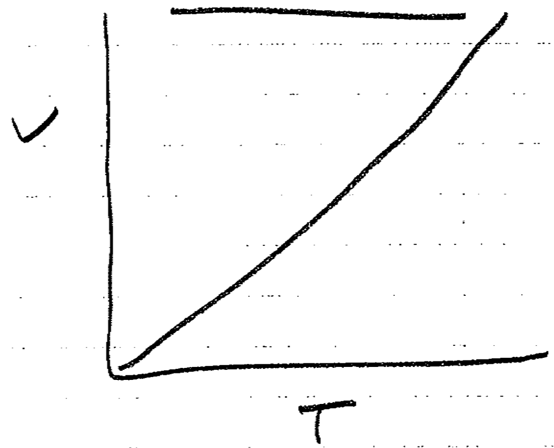
Thing 2: Many phenomena has a **functional** relationship with hotness or coldness!

→ resistances increase with hotness
→ pressure of a gas increases

Thermoscope



Thermometer



We don't know enough yet to define a standard thermometer, but for now we only need a thermoscope to notice

T1 5.2

$$T \rightarrow -T$$

$$\Rightarrow \frac{dU}{dT} = mc$$

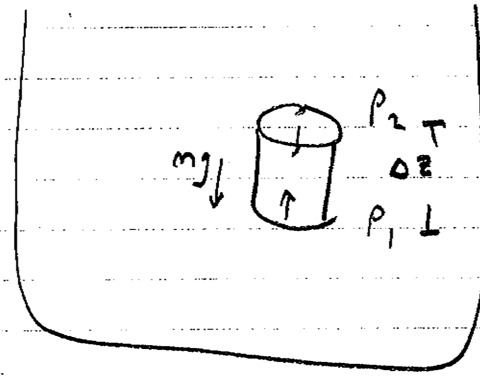
$$\rightarrow \frac{dU}{dT} = -mc$$

Heat capacities would be negative ...

$$m v_x^2 = k_B T$$

$$\rightarrow \quad m v_x^2 = -k_B T$$

T1 S.3



$$0 = mg + P_2 \cdot A - P_1 \cdot A$$

$$\Rightarrow P_1 - P_2 = \frac{mg}{A}$$

$$= \frac{m}{A \Delta z} \cdot \Delta z \cdot g = \frac{m}{V} \Delta z \, g$$

$$\boxed{P_1 - P_2 = \rho \, \Delta z \, g}$$

Lecture T2

Ideal Gas Model:

1) $N >> 1$

2) Molecules are point-like.

3) Molecules obey Newton's laws (definitely not true)

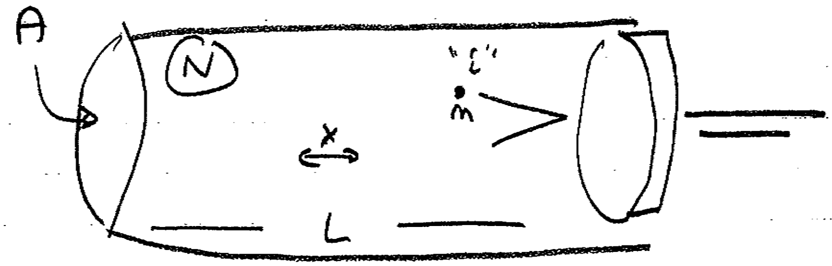
4) Molecules don't interact with one another

5) Collisions with wall are elastic

6) Motion of molecules is entirely random.

We need to make a lot of unnecessary assumptions here to make the math easier.

Nowadays, you can write a computer simulation to verify this model. If you know how to program, and are interested, I will help you write a model.
(No Extra Credit → But Very Fun).



$$\Delta \vec{P}_i = \Delta P_{i,\text{final}} - \Delta P_{i,\text{initial}}$$

$$\Delta P_y = \Delta P_z = 0$$

$$\Delta P_x = -m|u_{ix}| - m|u_{ix}|$$

$$= -2m|u_{ix}|$$

On piston: $\Delta P_x = \underline{\underline{+}}\,2m|u_x|$

$$\Delta t = 2L / u_{ix}$$

$$F_i = \frac{\Delta P_x}{\Delta t} = \frac{2 m u_{ix}}{2L/u_{ix}} = \frac{m u_{ix}^2}{L}$$

$$P_i = \frac{F_i}{A} = \frac{m u_{ix}^2}{A \cdot L} = \frac{m u_{ix}^2}{V}$$

$$P_{TOT} = \sum_i \frac{m_i u_{ix}^2}{V} = \frac{N}{V}\left(\frac{1}{N}\sum_i m_i u_{i,x}^2\right)$$

$$\boxed{P = \frac{N}{V}\,[m u_x^2]_{avg}}$$

Q: Why $\underline{\underline{u_x^2}}$ (One factor ΔP, One factor Δt!)

Now our definition of T in terms of a gas thermoscope can be seen in a new light.

$$P = C T$$

$$+ \quad P = \frac{N}{V}\left[m v_x^2\right]$$

$$\Rightarrow \quad m v_x^2 = \left(\frac{V}{N} C\right) \cdot T$$

$$m v_x^2 = C' \; T$$

Now at this stage, C' can depend on V, N, type of gas, --

But what if it didn't.

B~~IG~~ CLUE we are on the right path.

$$\boxed{C' = k_b = 1.38 \times 10^{-23} \; J/K}$$

$$P = \frac{N}{V} k_B \; T$$

T2 S. 2

Boyles Law $\quad P \sim \frac{1}{V}$ at const V

Gay-Lussac's Law $\quad V \sim T$ at const P

Avogadro $\quad V \sim N$ at const P and T.

B: $PV = N k_B T \Rightarrow PV =$ const at const N, T

$$\Rightarrow P \sim \frac{1}{V}$$

G.L.: $PV = N k_B T$

$$\Rightarrow P = \left(\frac{N k_B}{V}\right) T$$

$P \sim T$ at const N, V

Avogadro:

$$V = N \left(\frac{k_B T}{P}\right)$$

Temperature and Energy

Randomness:

$$[m v_x^2]_{avg} = [m v_y^2]_{avg} = [m v_z^2]_{avg}$$

$$= k_B T$$

$$3 k_B T = [m v_x^2]_{avg} + [m v_y^2]_{avg} + [m v_z^2]_{avg}$$

$$= \frac{1}{N} \sum_i m(v_x^2 + v_y^2 + v_z^2)$$

$$= 2 \cdot K_{avg}$$

$$\Rightarrow \boxed{K_{avg} = \frac{3}{2} k_b T}$$

WOW! We are definitely on the right path.

→ T is proportional to average kinetic energy of molecule in a gas.

Heat Capacity of a gas

$$\Delta U = m c \Delta T$$

$$U = \frac{3}{2} N k_B T$$

$$\frac{dU}{dT} = \frac{3}{2} N k_b \quad (\equiv m c)$$

$$\left(c = \frac{3}{2}\left(\frac{N}{m}\right) k_b \right)$$

Non-Ideal Gases:

$$U = \frac{f}{2} N k_B T$$

$f = 3$ for monatomic gases

$f = 5$ for diatomic gases

$f > 6$ for poly-atomic gases

Those Pesky Chemists...

$$PV = N k_B T$$

$$\rightarrow PV = \frac{N}{N_A} (N_A \cdot k_B) T$$

$$= n R T \qquad \left(n \equiv \frac{N}{N_A} \right)$$

$$n = \frac{M}{M_A} \begin{array}{l} \leftarrow \text{mass of sample} \\ \leftarrow \text{mass of } N_A \text{ molecules, } (m \cdot N_A) \end{array}$$

$$\left(n = \frac{M/m}{N_A} \right) \quad \text{(Conceptually)}$$

T2 S.3

a) Assume for simplicity $m_i = m$

$$\langle x \rangle = \sum m_i x_i \Big/ \sum m_i = \frac{1}{N}\sum_i x_i$$

$$\frac{d\langle x \rangle}{dt} = 0 \Rightarrow \frac{d}{dt}\sum_i x_i = \frac{1}{N}\sum_i v_i$$

$$\Rightarrow \boxed{\langle v \rangle = 0}$$

b) Simple example: if every particle has a duplicate with $v \to -v$

$$\langle v \rangle = \frac{1}{2N}\left(\sum_i v_i + (-v_i)\right)$$

$$\langle v^2 \rangle = \frac{1}{2N}\sum_i v_i^2 + (-v_i)^2$$

$$> 0$$

T2 S,4

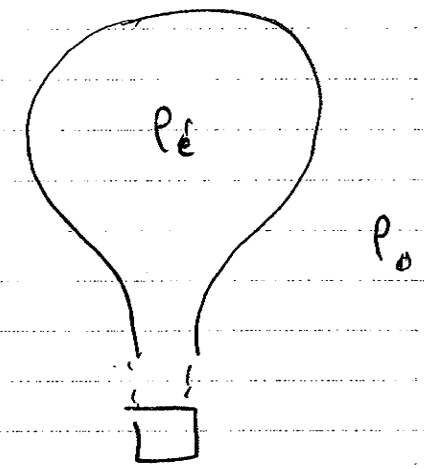
$\Delta p = 2 m v_0$

$N = r \cdot \Delta t \cdot A$ (defines "r")

$$P = N \frac{\Delta p}{\Delta t} \frac{1}{A} = \left(\frac{N}{\Delta t A}\right) \Delta p = \boxed{r \, 2 m v_0}$$

T2 5.5

Boyant Force is



$$F = \rho_o \cdot V \cdot g$$

Gravity is

$$F = -\rho_i V \cdot g - Mg$$

Floating has

$$\rho_o V g + (-\rho_i V g) - Mg = 0$$

$$\Rightarrow \boxed{\rho_o - \rho_i = \frac{M}{V}}$$

b) $\rho = \dfrac{N \cdot m}{V}$ $\qquad m = \dfrac{M_A}{N_A}$

$$= \frac{N}{N_A} \cdot \frac{M_A}{V} = \frac{n}{V} M_A$$

$$pV = nRT \quad \Rightarrow \quad \frac{n}{V} = \frac{p}{RT}$$

$$\Rightarrow \quad \boxed{\rho = \frac{p\, M_A}{R\, T}}$$

c) $\rho_o - \rho_i = \frac{M}{V}$

$$\Rightarrow V = M \Big/ (\rho_o - \rho_i)$$

$$= \frac{M}{\rho_o} \frac{1}{(1 - \rho_i/\rho_o)}$$

From (b) $\rho \sim \frac{1}{T}$

$$V = \frac{M}{\rho_o} \frac{1}{(1 - T_o/T_i)}$$

Lecture T3

Review Reading

$$\boxed{\Delta U = Q + W}$$

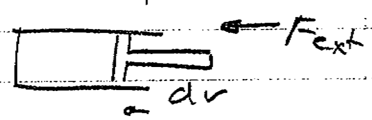
Heat (Flows due to ΔT) — Everything Else

$$\boxed{dW = -P\,dV}$$

Why?

$$W = F \cdot dr$$
$$= \frac{F}{A} \cdot dr \cdot A$$
$$W = -P \cdot dV$$

Q: ? Why Negative



Does Positive work, but $dV < 0$

P, V, T, U, m, N

$$N = \frac{m}{m_A} \qquad PV = N k_B T \qquad U = \frac{f}{2} N k_B T$$

$\Rightarrow$ 3 independent state variables only
If 1 held constant (e.g. N)

Value of PV uniquely defines state

Computing Work

Isobaric

$$W = -\int_{V_I}^{V_F} p\, dV = -p(V_F - V_I)$$

Isothermal

$$p = \frac{NkT}{V} = \frac{p_i V_i}{V}$$

$$W = -NkT \int \frac{dV}{V} = -NkT \log\left(V_F/V_I\right)$$

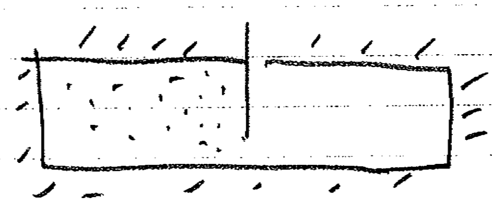
$$= -p_i V_i \log\left(V_F/V_I\right)$$

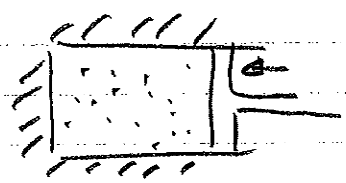
Isochoric

$$W = 0 \quad !$$

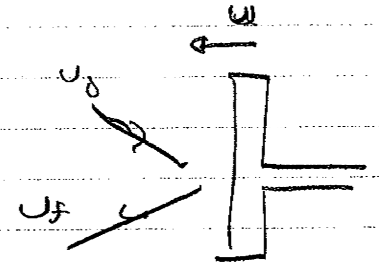
Adiabatic?

Introduce,





Work on Adiabatic Compression of Gas



1) $u_f = |u_{ix}| + 2u$ (why?)

2) $\Delta K = \frac{1}{2} m \left(|u_{ix}| + 2u \right)^2 - \frac{1}{2} m u_{ix}^2$

$\sim 2m|u_{ix}|\, u$

3) $\frac{dK_i}{dt} = \frac{2m\,|u_{ix}|\,u}{2L/|u_i|} = \frac{m\,u_{ix}^2\; u}{L}$

4) $\frac{dU}{dt} = \sum_i \frac{dK_i}{dt} = \frac{\left(\sum m u_{ix}^2\right) \cdot u}{L}$

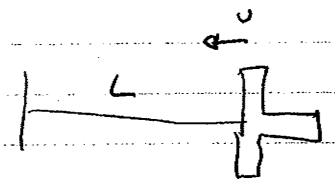
$$\boxed{\frac{dU}{dt} = \frac{(NkT) \cdot u}{L}}$$

5) $U = \frac{f}{2} NkT$

$$\frac{dU}{dt} = \frac{f}{2} Nk \frac{dT}{dt} = \frac{NkT \cdot u}{L}$$

$$\boxed{\frac{1}{T}\frac{dT}{dt} = \frac{2}{f}\frac{u}{L}}$$

6) $u = -\frac{dL}{dt}$



$$\Rightarrow \frac{u}{L} = -\frac{1}{L}\frac{dL}{dt} = -\frac{1}{L \cdot A}\frac{d(LA)}{dt} = -\frac{1}{V}\frac{dV}{dt}$$

7) $$\boxed{\frac{1}{T}\frac{dT}{dt} = -\frac{2}{f}\frac{1}{V}\frac{dV}{dt}}$$

Differential Equation!

(See Better Solution)

8) How to solve ~~without~~ magic

$$\frac{1}{T}\frac{dT}{dt} + \frac{2}{f}\frac{1}{V}\frac{dV}{dt} = 0$$

Goal: Write as

$$\frac{1}{(?)}\frac{d(?)}{dt} = 0$$

why? $\Rightarrow$ $(?) = \text{const}$

How: Looks like a chain rule...

$$X = TV \qquad \frac{1}{X}\frac{dX}{dt} = \frac{1}{T}\frac{dT}{dt} + \frac{1}{V}\frac{dV}{dt}$$

missing a factor!

$$X = TV^{\alpha}$$

$$\frac{1}{X}\frac{dX}{dt} = \frac{1}{T}\frac{dT}{dt} + \alpha\cdot\frac{V^{\alpha-1}}{V^{\alpha}}\frac{dV}{dt}$$

$$= \frac{1}{T}\frac{dT}{dt} + \alpha\frac{1}{V}\frac{dV}{dt}$$

$$\alpha = \frac{2}{f}$$

$$\frac{1}{TV^{2/f}}\frac{d}{dt}\left(TV^{2/f}\right) = \frac{1}{dt}\frac{dT}{dt} + \frac{2}{f}\frac{1}{V}\frac{dV}{dt}$$

$$= 0$$

9) $\frac{d}{dt}\left(T V^{2/f}\right) = 0$

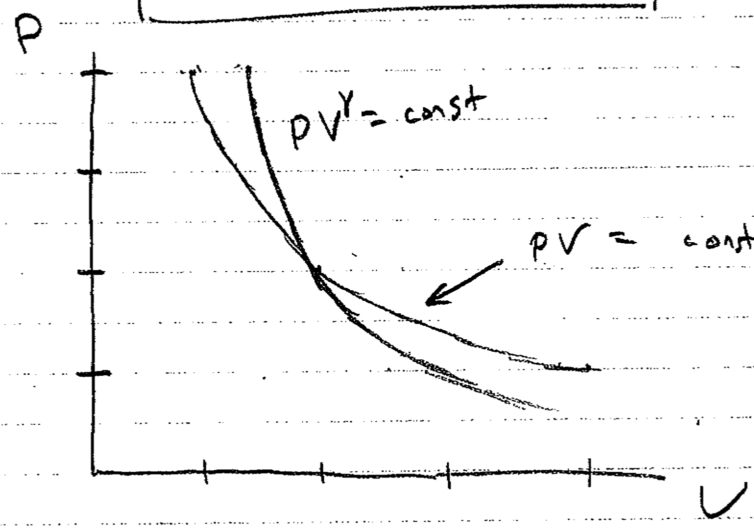
Means? $T V^{2/f} = \text{const}$

10) $T \sim PV$

$\Rightarrow \quad P V^{1+2/f} = \text{const}$

Define $\gamma = 1 + 2/f$

$$\boxed{\begin{aligned} PV^{\gamma} &= \text{const} \\ TV^{\gamma-1} &= \text{const} \end{aligned}}$$



(Better Solution to Diff Eq)

$$0 = \frac{1}{T}\frac{dT}{dt} + \frac{2}{f}\frac{1}{V}\frac{dV}{dt}$$

$$-\frac{1}{T}\frac{dT}{dt} = \frac{2}{f}\frac{1}{V}\frac{dV}{dt}$$

Integrate both sides wrt dt

$$-\int \frac{1}{T}\frac{dT}{dt}\,dt = \frac{2}{f}\int \frac{1}{V}\frac{dV}{dt}\,dt$$

$$-\int \frac{1}{T}\,dT = \frac{2}{f}\int \frac{dV}{V}$$

$$-\log T + C = \frac{2}{f}\log V$$

$$C = \log V^{2/f}\cdot T$$

$$\exp(C) = \text{const} = V^{2/f}\,T \quad !!!$$

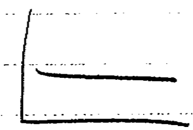
Have Students
Do it...

P-V Diagrams

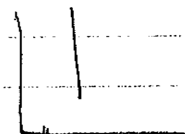
Draw Process going through

$P = 8\ kPa$; $U = 8\ m^3$

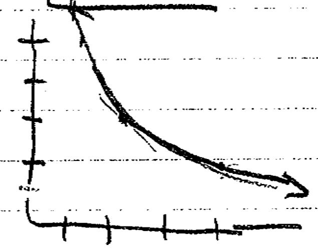
- isochoric (const V)



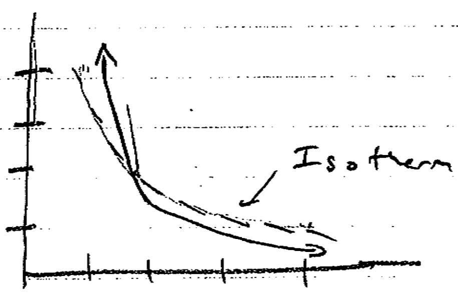
- isobaric (const P)



- Isothermal (const T)



- adiabatic ($Q = 0$)

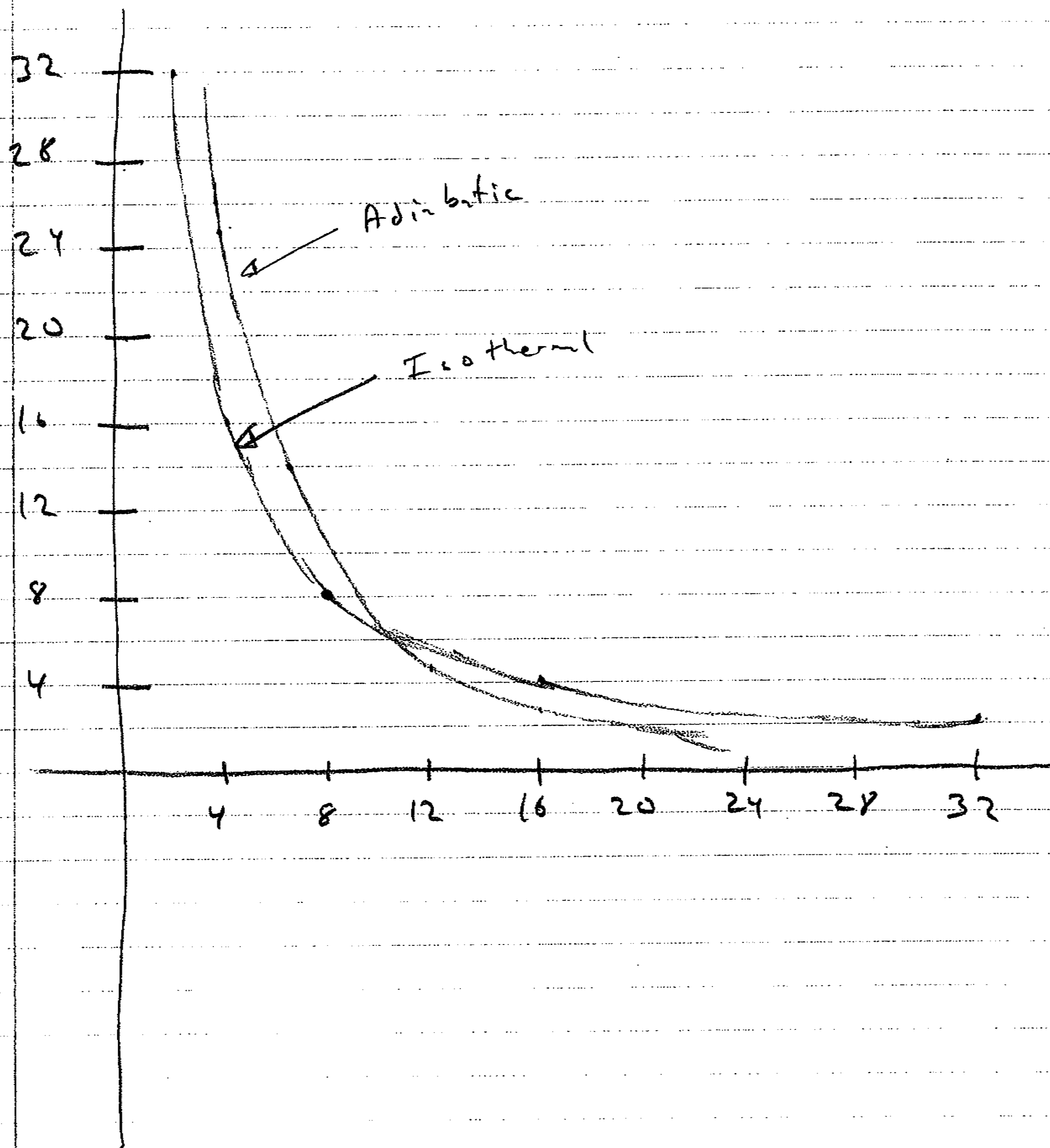


Explicit E.g. comparing Adiabatic / Isot

V	$p \sim \frac{1}{V}$	$p \sim \frac{1}{V^{5/3}}$
8	8 ($\equiv$)	8 ($\equiv$)
6	10	13
4	16	25
2	32	80
1	64	256

$$\boxed{\begin{array}{l} \underline{\text{Calc:}} \quad P = \dfrac{8 \cdot 8}{V} = \dfrac{64}{V} \\ \\ \quad\quad\quad\;\; P = \dfrac{8 \cdot 8^{5/3}}{V^{5/3}} = \dfrac{256}{V^{5/3}} \end{array}}$$

10	6.4	5.5
12	5.3	4.1
14	4.6	3.1
16	4	2.5
32	2	0.8



Adiabatic
Isothermal
32
28
24
20
16
12
8
4
4
8
12
16
20
24
28
32

IN CLASS

T3 5.4

	Q	W	ΔU
$A \to B$	+	$-$	+
$B \to C$	+	0	+
$C \to A$	$-$	+	$-$

A → B:

$$W = -\int P\,dV = -P\Delta V < 0$$

$$PV = NkT \Rightarrow V \sim T$$

$$\Rightarrow T_F > T_I \Rightarrow U_F > U_I \Rightarrow dU > 0$$

Since $\Delta U = Q + W$, and $\Delta U > 0$ but $W < 0$

$$Q > 0$$

B → C

$$W = 0$$

$$PV = NkT \Rightarrow P \sim T$$

$$P_F > P_I \Rightarrow T_F > T_i \Rightarrow U_F > U_i$$

$$\Rightarrow \Delta U > 0$$

$C \to A$

ΔU for the cycle must be 0,
so $\Delta U = -$.

W is negative. In fact,

$|W_{C \to A}| > |W_{A \to B}|$

Implying that W for the entire cycle is $+$.

Since $\Delta U = 0 = Q + W$ for the whole cycle, Q for the cycle must be negative. But Q for $A \to B$, $B \to C$ is positive, so Q for $C \to A$ must be negative

Alternate:

$P_I V_I > P_F V_F$ for $C \to A$

$\Rightarrow T_I > T_F \Rightarrow \boxed{dU < 0}$

As $dV < 0$, $\boxed{W > 0}$

Since $dU < 0$ and $W > 0$, $\boxed{Q < 0}$

(HW)

T3 5.5

	Q	W	ΔU
$A \to B$	0	$-$	$-$
$B \to C$	$-$	$+$	0
$C \to D$	$+$	0	$+$

$A \to B$

$W = -\int P\,dV < 0$

Adiabatic $\Rightarrow Q = 0$
Hence $\Delta U = W < 0$

Alternative: Since $T_B = T_C$, $T_A > T_B \Rightarrow U_A > U_B$

$\Rightarrow dU < 0$

$B \to C$

Isothermal $\Rightarrow T = \text{const} \Rightarrow U = \text{const} \Rightarrow \Delta U = 0$

$W = -\int P\,dV > 0$

$\Delta U = Q + W = 0 \Rightarrow Q = -W$

$C \to A$

$W = 0$

Since $V = \text{const}$, and $PV = NkT$, $T_A > T_C$
so $U_A > U_C \Rightarrow dU > 0$.

Alternatively! Since ΔU for cycle $= 0$, $\Delta U > 0$,

Since $W = 0$, $\Delta U = Q$, $Q > 0$

T35.6 $T\uparrow$, so $\Delta U > 0$

$V = \text{const} \Rightarrow W = 0$, so $Q = \Delta U > 0$

$pV \Rightarrow NkT \Rightarrow p \sim T$

$$\frac{T_f}{T_i} = \frac{p_f}{p_i} \Rightarrow T_f = \frac{p_f}{p_i} T_i = \frac{50}{30}\ 290\ K$$

From

$$pV = NkT \Rightarrow N = \frac{p_i V_i}{k T_i} \left(= \frac{(30\,kPa)\ (1\,m^3)}{k \quad 290\ K} \right)$$

$$dU = \frac{3}{2} N k\ dT$$

$$= \frac{3}{2} \frac{p_i V}{T_i} (T_f - T_i)$$

T3 S.7 $\qquad TV^{\gamma-1} = \text{const}$

$$PV^{\gamma} = \text{const}$$

$$P_i V_i^{\gamma} = P_f V_f^{\gamma}$$

$$\frac{V_i^{\gamma}}{V_f^{\gamma}} = \frac{P_f}{P_i}$$

$$\frac{V_i}{V_f} = \left(\frac{P_f}{P_i}\right)^{\frac{1}{\gamma}}$$

$$T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$

$$\Rightarrow \quad T_f = \left(\frac{V_i}{V_f}\right)^{\gamma-1} \cdot T_i$$

$$= \left(\frac{P_f}{P_i}\right)^{\frac{\gamma-1}{\gamma}} \cdot T_i$$

$$= \left(\frac{1}{3}\right)^{\frac{\gamma-1}{\gamma}} T_i$$

For air $\quad \gamma \sim 1,4$

T3 5.8 $\quad \gamma \sim 1.4$

$$T V^{\gamma-1} = \text{const}$$
$$P V^{\gamma} = \text{const}$$

We are given $P_f / P_i \sim 0.045$

$$P_i V_i^{\gamma} = P_f V_f^{\gamma}$$
$$\Rightarrow V_f / V_i = \left(P_i / P_f \right)^{1/\gamma}$$

$$T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$
$$T_f = T_i \left(\frac{V_i}{V_f} \right)^{\gamma-1}$$
$$= T_i \left(\frac{P_f}{P_i} \right)^{\left(\frac{\gamma-1}{\gamma} \right)}$$

T3 S.9 For adiabatic process:

$$T V^{\gamma-1} = \text{const}$$
$$P V^{\gamma} = \text{const} \Rightarrow P = P_I V_I^{\gamma} V^{-\gamma}$$

$$W = -\int P\,dV = -P_I V_I^{\gamma} \int_{V_I}^{V_F} V^{-\gamma}\,dV$$

$$= \frac{-P_I V_I^{\gamma}}{1-\gamma} V^{1-\gamma} \Big|_{V_I}^{V_F}$$

$$= \frac{P_I V_I^{\gamma}}{\gamma-1} \left(V_F^{1-\gamma} - V_I^{1-\gamma}\right)$$

!!!

$$= \frac{P_I V_I}{\gamma - 1} \left(\frac{V_I^{\gamma-1}}{V_F^{\gamma-1}} - 1\right)$$

$$= \frac{N k T_I}{\gamma - 1} \left(\frac{T_F}{T_I} - 1\right)$$

$$= \left(\frac{1}{\gamma-1}\right) N k \left(T_F - T_I\right)$$

$$= \frac{f}{2} N k\, dT$$

///
///

T3 S10

a) $PV = NkT$

$P\frac{dV}{dT} + V\frac{dP}{dT} = Nk$

$$\boxed{PdV + VdP = Nk\,dT}$$

b) For adiabatic process, $Q = 0$, so

$\Delta U = Q + W \Rightarrow \Delta U = W$

But $dW = -PdV$

And $dU = \frac{f}{2}Nk\,dT$

$\Rightarrow PdV = -\frac{f}{2}Nk\,dT$

c) $Nk\,dT = -\frac{2}{f}PdV$

$\Rightarrow PdV + VdP = -\frac{2}{f}PdV$

$VdP + P\left(1 + \frac{2}{f}\right)dV = 0$

$VdP + \gamma PdV = 0 \qquad \gamma \equiv 1 + \frac{2}{f}$

$$\boxed{\frac{dP}{dV} + \gamma\frac{P}{V} = 0}$$

d) $\frac{d}{dV} V^\gamma = \gamma V^{\gamma-1}$

So, as we have a term

$\gamma \frac{p}{V} \rightarrow \gamma V^{\gamma-1} \cdot p \rightarrow p \frac{d}{dV}(V^\gamma)$

$(\times V^\gamma)$

So multiply above by V^γ

$$V^\gamma \frac{dp}{dV} + \gamma p V^{\gamma-1} = 0$$

$$V^\gamma \frac{dp}{dV} + p \frac{d}{dV} V^\gamma = 0$$

$$\frac{d}{dV}\left(p V^\gamma\right) = 0$$

$$\Rightarrow \boxed{p V^\gamma = \text{const}}$$

T3 8.2 In Class:

Let's say you are clever:

$$Q = 0 \Rightarrow dU = W$$

(HW solution!)

$$U = \frac{f}{2} N k T$$

$$dU = \frac{f}{2} N k \, dT$$

$$W = \Delta U = \frac{f}{2} N k \, \Delta T = \frac{f}{2} N k (T_f - T_i)$$

Now imagine you aren't smart:

$$W = -\int P \, dV$$

$$P V^{\gamma} = \text{const} = P_I V_I^{\gamma}$$

$$\Rightarrow P = P_I V_I^{\gamma} V^{-\gamma}$$

$$W = -\int_{V_I}^{V_F} P \, dV = -P_I V_I^{\gamma} \int_{V_I}^{V_F} V^{-\gamma} dV$$

$$= \frac{P_I V_I^{\gamma}}{(\gamma - 1)} \left(\frac{1}{V_F^{\gamma-1}} - \frac{1}{V_I^{\gamma-1}} \right)$$

We also know:

$$T V^{\gamma-1}$$

$$\Rightarrow T_I V_I^{\gamma-1} = T_F V_F^{\gamma-1}$$

$$\Rightarrow \frac{V_I^{\gamma-1}}{V_F^{\gamma-1}} = \frac{T_F}{T_I}$$

$$W = \frac{p_I V_I}{\gamma-1}\left(\frac{T_F}{T_I} - 1\right)$$

$$p_I V_I = N k T_I$$

$$W = \frac{N k}{\gamma-1}\left(T_F - T_I\right)$$

$$\frac{1}{\gamma-1} = \frac{1}{1 + 2/f - 1} = \frac{f}{2}$$

$$\boxed{W = \frac{f}{2} N k \left(T_F - T_I\right)}$$

!!!

Easy Way

T3 S.2

$Q = 0 \quad \Rightarrow \quad dU = W$

$U = \frac{f}{2} NkT$

$\Rightarrow \quad dU = \frac{f}{2} Nk \, dT$

$W = \Delta U = \frac{f}{2} Nk \left(T_f - T_i\right)$

HARD WAY !!!

T3 S.2 Work done during an adiabatic change in volume is: :

$$W = -\int P\,dV$$

$$PV^{\gamma} = \text{const} = P_I V_I^{\gamma}$$

$$\Rightarrow P = P_I V_I^{\gamma} V^{-\gamma}$$

$$W = -\int_{V_I}^{V_F} P\,dV = -P_I V_I^{\gamma} \int_{V_I}^{V_F} V^{-\gamma}\,dV$$

$$= -P_I V_I^{\gamma} \left. \frac{V^{1-\gamma}}{1-\gamma} \right|_{V_I}^{V_F}$$

$$= \frac{P_I V_I^{\gamma}}{(\gamma-1)} \left(\frac{1}{V_F^{\gamma-1}} - \frac{1}{V_I^{\gamma-1}} \right)$$

$$\boxed{W = \frac{P_I V_I}{(\gamma-1)} \left(\frac{V_I^{\gamma-1}}{V_F^{\gamma-1}} - 1 \right)}$$

Cont. ...

T3 5.2 ... cont ...

$$TV^{\gamma-1} = \text{const}$$

$$\Rightarrow T_I V_I^{\gamma-1} = T_F V_F^{\gamma-1}$$

$$\Rightarrow V_I^{\gamma-1} / V_F^{\gamma-1} = T_F / T_I$$

$$W = \frac{P_I V_I}{(\gamma-1)} \left(\frac{T_F}{T_I} - 1 \right)$$

$$\boxed{W = \frac{N k T_I}{(\gamma - 1)} \left(\frac{T_F}{T_I} - 1 \right)}$$

$$\frac{1}{\gamma - 1} = \frac{1}{1 + 2/f - 1} = \boxed{\frac{f}{2}}$$

$$\boxed{W = \frac{f}{2} N k \left(T_F - T_I \right)}$$

T3 S.3 (HW)

*** Isothermal process for an ideal gas at constant N is has $U = NkT = \text{const} \Rightarrow \Delta U = 0$.

$$\Delta U = 0 \quad \text{as} \quad U = \frac{f}{2} NkT \quad ***$$

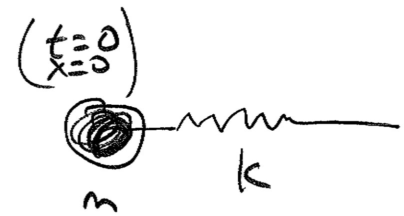
$$\Delta U = Q + W$$

$$\Rightarrow \boxed{Q = -W = +NkT \ln \frac{V_f}{V_i}}$$

Classical Mass on Spring

$\boxed{T4}$

$$F = -kx = ma = m\frac{d^2x}{dt^2}$$



$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Hes solution: $x = A\sin\omega t + B\cos\omega t$

But $x = 0$ at $t = 0$ $\Rightarrow$ $B = 0$

$$x = A\sin\omega t$$

$$\frac{dx}{dt} = \omega A\cos\omega t$$

$$\frac{d^2x}{dt^2} = -\omega^2 A\sin\omega t = -\frac{k}{m}x \quad \text{if} \quad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

Energy of spring?

Quick solution: potential energy = 0 at $x = 0$

$$\frac{dx}{dt} = \omega A\cos\omega t = \omega A \quad \text{at } x = 0$$

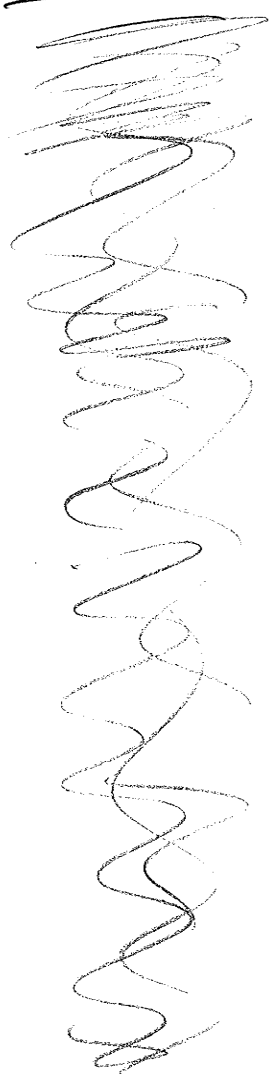
$$K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega^2 A^2 = \boxed{\frac{1}{2}kA^2}$$

Note:

$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2$ when $\frac{dx}{dt} = 0$ and x is maximal

Classically, thats it, we can choose any value of A, and therefore system energy can take any value (at least until spring break?!!)

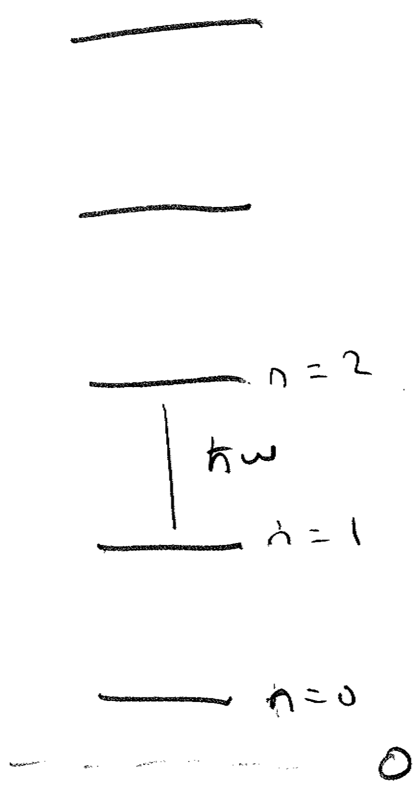
System E:



* Q.M. the system energy is in discrete units

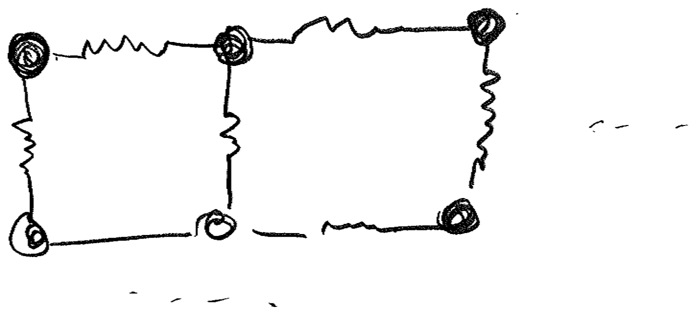
$$E = \hbar\omega\left(n + \frac{1}{2}\right)$$

planck's constant — Quantum Number



* For our purbos.

Einstein model for a Solid:



Each atom can oscillate in x, y, or z dimensions $\Rightarrow$ 3 quantum numbers for each atom

$$U_{solid} = \sum_{i=1}^{3N} \varepsilon\left(n_i + \cancel{\tfrac{1}{2}}\right) = \sum_{i=1}^{3N} \varepsilon n_i$$

$\uparrow$ Not accessible

T4S8:

M oscillators with total energy $E = \varepsilon q$

$(q = 0, 1, 2, \ldots)$

Say $m = 4$ $q = 7$

3 0 4 0   000||0000|

2 2 2 1

4 3 0 0

1 1 1 4

Need q marbles and $M-1$ matches

$$\Omega = \frac{(q+M-1)!}{q!\,(M-1)!}$$

For Einstein Solid. $M = 3N$

$$\boxed{\Omega = \frac{(3N+q-1)!}{q!\,(3N-1)!}}$$

$q = E/\varepsilon$

T4S.1

Ω_A for $U_A = 4\varepsilon$, $N = 3$

1	2	3
4	0	0
0	4	0
0	0	4
3	1	0
3	0	1
0	3	1
1	3	0
1	0	3
0	1	3
2	2	0
2	0	2
0	2	2
2	1	1
1	2	1
1	1	2

15

$U_A = 7\varepsilon$

1	2	3
7	0	0
0	7	0
0	0	7
6	1	0
6	0	1
0	6	1
1	6	0
1	0	6
0	1	6
5	2	0
5	0	2
0	5	2
2	5	0
2	0	5
0	2	5
5	1	1
1	5	1
1	1	5
4	3	0
4	0	3
0	4	3
3	4	0
3	0	4
0	3	4

3	3	1
3	1	3
1	3	3
3	2	2
2	3	2
2	2	3
4	2	1
4	1	2
1	4	2
2	4	1
2	1	4
1	2	4

T4 5.2 (HW)

a)

1	2	3	4	5	6	
2	0	0	0	0	0	
		...				6
0	0	0	0	0	2	
1	1	0	0	0	0	
		...				5
1	0	0	0	0	1	
0	1	1	0	0	0	
		...				4
0	1	0	0	0	1	

$$\Omega_B = 6 + 5 + 4 + 3 + 2 + 1$$
$$= 21$$

b) $$\Omega(N,U) = \frac{(q+3N-1)!}{q!\,(3N-1)!}$$

$U_B = 6\varepsilon$ $\quad U_B = 9\varepsilon$ $\quad 3N = 6$

$$\frac{(6+6-1)!}{6!\,5!} \qquad \frac{(9+6-1)!}{9!\,5!}$$

$$\frac{11!}{6!\,5!} \qquad \frac{14!}{9!\,5!}$$

$$\frac{11 \cdot \cancel{10}^{2} \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} \qquad \frac{14 \cdot 13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10}}{\cancel{5}\ \cancel{4}\ \cancel{3}\ \cancel{2}}$$

$66 \cdot 7 = \boxed{462}$

14
13
42
140
182
11
182
182

$\boxed{2002}$

(1-HW)

T4 S.3

a)

(1) U_A	(2) U_B	Ω_A	Ω_B	Ω_{AB}
6	0	28	1	28
5	1	21	6	126
4	2	15	21	315
3	3	10	56	560
2	4	6	126	756
1	5	3	252	756
0	6	1	462	462

b) $U_A = 1 \quad U_B = 5 \qquad \frac{U_A}{N_A} = 1 \quad \frac{U_B}{N_B} = 2.5$

$U_A = 2 \quad U_B = 4 \qquad \frac{U_A}{N_A} = 2 \quad \frac{U_B}{N_B} = 2$

#45.4

U_A	U_B	Ω_A	Ω_B	Ω_{AB}
9	0	55	1	55
8	1	45	6	270
7	2	36	21	756
6	3	28	56	1568
5	4	21	126	2646
4	5	15	252	3780
3	6	10	462	4620
2	7	6	792	4752
1	8	3	1287	3861
0	9	1	2002	2002

$$\frac{U_A}{N_A} = 2 \qquad \frac{U_B}{N_B} = \frac{7}{2} = 3.5$$

T4 5.5

a)

U_A	U_B	Ω_A	Ω_B	Ω_{AB}
9	0	2002	1	2002
8	1	1287	6	7722
7	2	792	21	16632
6	3	462	56	25872
5	4	252	126	31752
4	5	126	252	31752
3	6	56	462	25872
2	7	21	792	16632
1	8	6	1287	7722
0	9	1	2002	2002

b) $\frac{5}{2} = 2.5$ $\frac{4}{2} = 2$

4.5.6 (HW)

a) (1) 2.828×10^{13}
(2) 10, 10 3.847×10^{12}
(3) 1 and 2
(4) 0.3 ~ 0.7

b) (1) 3×10^{144}
(2) 100, 100 1.3×10^{143}
(3) 2 2
(4) 0.45 - 0.55

c) # states increases,
states become more concentrated toward

TY S.7

a) $U_A / N_A = U_B / N_B$

Not Always, -- eg $39/20$

b) Hold true.

$\boxed{T5}$

Recup w/ Stat mech

N_A	N_B	U_A	U_B
2	3	3	5

$$\Omega_A = \frac{(3\cdot 2 + 3 - 1)!}{(3\cdot 2 - 1)!\,(3!)} = \frac{8!}{5!\,3!}$$

$$= \frac{8\cdot 7\cdot \cancel{6}}{\cancel{3\cdot 2}} = \boxed{56}$$

$$\Omega_B = \frac{(3\cdot 3 + 5 - 1)!}{(3\cdot 3 - 1)!\;5!} = \frac{13!}{8!\,5!}$$

$$= \frac{13\cdot \cancel{12}\cdot 11\cdot \cancel{10}\cdot 9}{\cancel{8}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}$$

$$= 13\cdot 11\cdot 9$$

2
99
1 × 13
297
990
$\boxed{1287}$

N_A	N_B	U
10	10	10

→ Notice that equal "shares" of energy are most probable . . .

→ Show effect gets more dramatic as $N \to \infty$ (10, 100, 100)

2nd Law of Thermodynamics

The entropy of an isolated system never <u>decreases</u>.

Interesting Cases

→ Small # Atoms, lots of energy
5 5 1000 → Not Narrow

→ Large # Atoms, limited energy
1000, 1000, 10

Lecture T5 + T6

Caused confusion — What is A vs B? Micro or Macro...

$$S = k \log \Omega$$

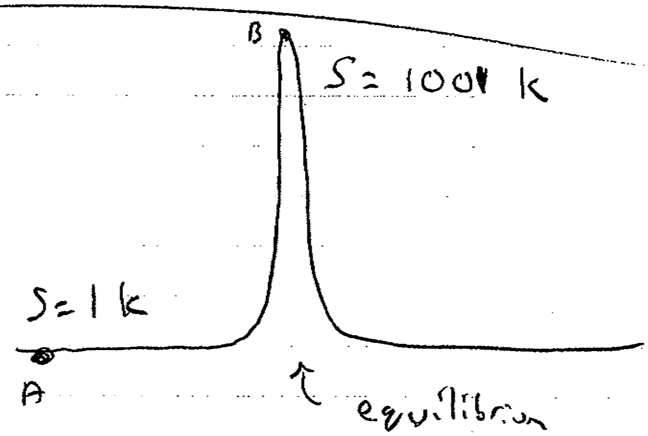
$$\Rightarrow \quad S_{AB} = S_A + S_B$$

Q: $$\frac{Prob(A)}{Prob(B)} = \frac{\Omega_A}{\Omega_B} = \frac{\exp(S_A/k)}{\exp(S_B/k)}$$

$$= \exp\left(\frac{S_A - S_B}{k}\right)$$

Irreversibility

$$\frac{P(A)}{P(B)} = \exp(-1000)$$



At later time, much much much more likely to observe is state B than A...

<u>2nd Law</u>: The entropy of an isolated system never decreases.

What is Entropy?

$$S = k \ln \Omega$$

Order vs Disorder

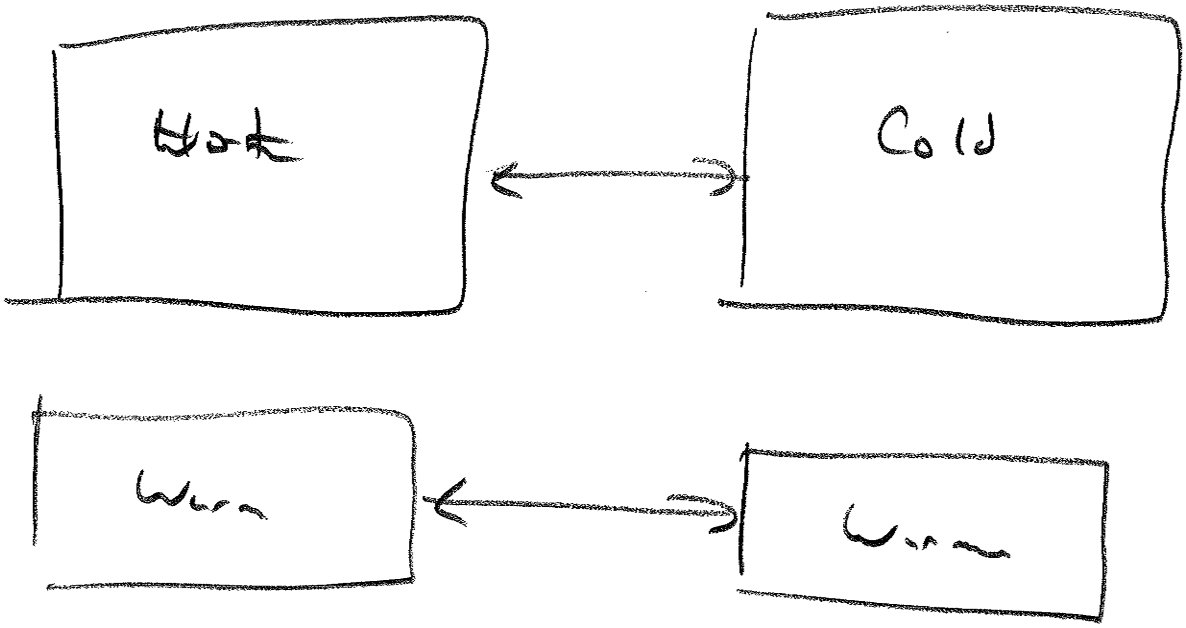
$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}$ $\qquad \Omega = 1 \qquad S = 0$

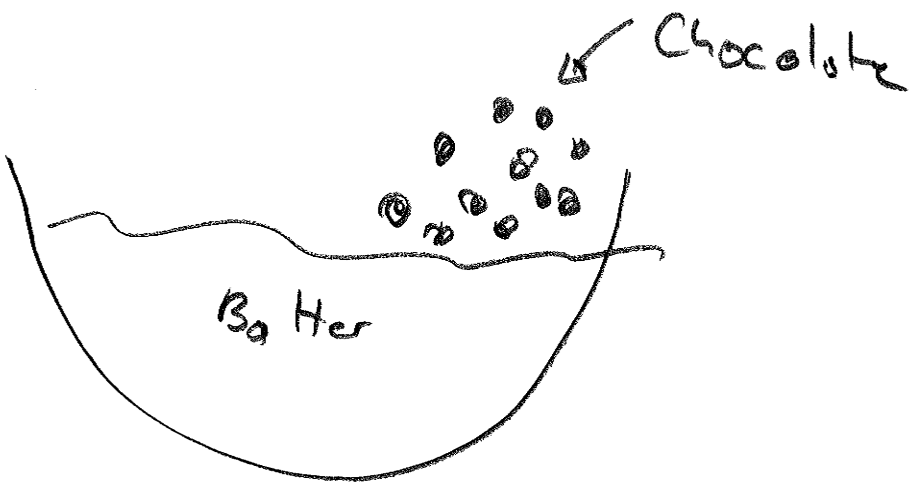
$\boxed{3}$

$\boxed{1}\ \boxed{2}\ \quad\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}$ $\qquad \Omega = 8 \qquad S \sim 2k$

$\boxed{2}\ \boxed{5}\ \boxed{1}\ \boxed{4}\ \boxed{3}\ \boxed{6}\ \boxed{7}$ $\qquad S >> 1$

Examples





Which has higher Entropy?

Answer(s):

(1) One must add Q to ice in order to make water ...
More energy generally $\Rightarrow$ more Ω

(2) $\frac{1}{T} = \frac{\partial S}{\partial U} \Rightarrow \Delta S = T_{freeze} \cdot \Delta Q$

$\boxed{\text{Q2:}}$ Does answer depend on T?

$\boxed{\text{A:}}$ Entropy of system (G + Reservoir) Yes!
Entropy of water ... No!

T6

Temperature and Entropy

New definition:

Equilibrium maximizes entropy of system, ie.

$$\frac{dS_{AB}}{dU_A} = 0$$

$$S_{AB} = S_A + S_B$$

$$\Rightarrow \quad 0 = \frac{dS_A}{dU_A} + \frac{dS_B}{dU_A}$$

$$U_A + U_B = \text{const} \quad \Rightarrow \quad dU_A = -dU_B$$

$$0 = \frac{dS_A}{dU_A} - \frac{dS_B}{dU_B}$$

$$\Rightarrow \quad \boxed{\frac{dS_A}{dU_A} = \frac{dS_B}{dU_B}}$$

* The quantity

$$\frac{dS}{dU} \text{ is the same for } A + B$$

at equilibrium.

Q What else is the same? T !!!

Pick

$$\frac{1}{T} = \frac{dS}{dU}$$

Is consistent → higher temp, higher energy

(Will show later for ideal gas)

~~For now look at Einstein solid~~

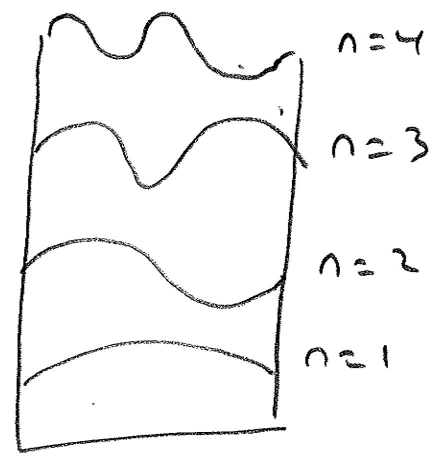
Particle in a Box → Ideal Gas

Recall: (Don't show to students they don't have QM yet...)

$$\lambda = \frac{h}{p}$$

$$\lambda = \frac{2L}{n} = \frac{h}{p}$$

$$n \sim p \Rightarrow U \sim p^2 \sim n^2$$



What we say:

QM tells us for particle constrained in a volume

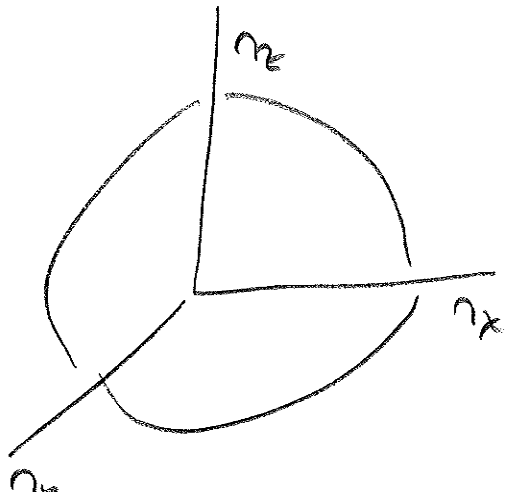
$$U = \varepsilon\left(n_x^2 + n_y^2 + n_z^2\right)$$

Quantum Numbers which enumerate microstates.

For now just assume this is true.

So

$$\frac{U}{\varepsilon} = n_x^2 + n_y^2 + n_z^2 \equiv n^2 \quad \Rightarrow \quad n \sim \sqrt{\frac{U}{\varepsilon}}$$



Consider 1 particle of a gas with Energy U. This particle can occupy any state from those with $n = 0$ to $n = \sqrt{\frac{U}{\varepsilon}}$

i.e. $\Omega_1 \sim n^3 \sim \left(U/\varepsilon\right)^{3/2}$

$$\boxed{\Omega_N \sim \left(U/\varepsilon\right)^{3/2 \cdot N}}$$

$$S = k \log \Omega_N \sim k \log U^{3/2 N}$$

$$\sim \frac{3}{2} N k \log\left(U/\varepsilon\right)$$

$$\frac{1}{T} \equiv \frac{\partial S}{\partial U} = \frac{3}{2} N k \frac{(1/\varepsilon)}{(U/\varepsilon)} = \frac{3}{2} N k \frac{1}{U}$$

$$\Rightarrow \boxed{U = \frac{3}{2} N k T}$$

(Temperature of Einstein Solid)

Gongping's Trick (Gongping Niu)

$$\Omega = \frac{(3N-1+q)!}{(3N-1)!\; q!}$$

$$\log \Omega = \log(3N-1+q)! - \log q! - \log(3N-1)!$$

To use Stirling's Approximation, we only need $q \gg 1$
(the last term is just a constant)

$$\log \Omega = (3N-1+q)\log(3N-1+q) - q\log q + \text{Const}$$

Gongping's trick:

$$= (3N-1)\log(3N-1+q) + q\log(3N-1+q) - q\log q + \text{Const}$$

$$= (3N-1)\log(3N-1+q) + \log\left(\frac{3N-1+q}{q}\right)^q$$

$$\lim_{q\to\infty}\left(\frac{3N-1+q}{q}\right)^q = \lim_{q\to\infty}\left(1+\frac{3N-1}{q}\right)^q$$

$$= \exp(3N-1)$$

Hence:

$$\boxed{\log \Omega = (3N-1)\log(3N-1+q) + \text{Const}}$$

$$\frac{1}{T} = \frac{k}{\varepsilon}\frac{\partial}{\partial q}\log \Omega$$

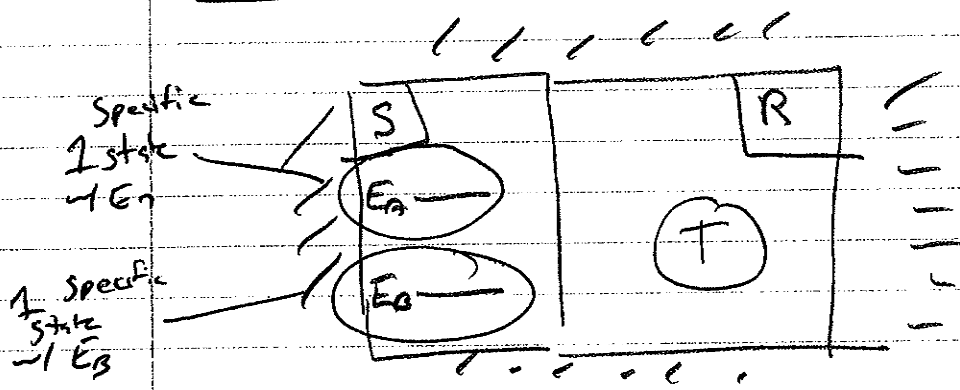
$$= \frac{k}{\varepsilon}\frac{3N-1}{3N-1+q} = \frac{k}{\varepsilon q}\cdot\frac{3N-1}{1+\left(\frac{3N-1}{q}\right)}$$

$$\boxed{\frac{1}{T} = \frac{k}{U}(3N-1)}$$

(True for both large and small N, large q!)

For large N this is

Boltzmann Factor



→ $P(A) / P(B) = ?$

→ In thermal contact w/ reservoir at T
→ Isolated
→ Reservoir:
- large system
- T is constant, even if heat flows in/out of reservoir.

$$\frac{P(A)}{P(B)} = \frac{\Omega_A^{S+R}}{\Omega_B^{S+R}} = \frac{\Omega_A^S}{\Omega_B^S} \cdot \frac{\Omega_A^R}{\Omega_B^R} = \frac{\Omega_A^R}{\Omega_B^R}$$

$\Omega_A^S / \Omega_B^R = 1/1 = 1$ (by construction!)

$$\frac{P(A)}{P(B)} = \frac{\exp(S_A^R/k)}{\exp(S_B^R/k)} = \exp\left(\frac{S_A^R - S_B^R}{k}\right)$$

Discuss!

rel prob of 2 configs of S

Entropy difference of R

For the reservoir

$$\frac{\partial S}{\partial U} = \frac{1}{T} = \underline{\text{const}},$$

$$\Rightarrow \Delta S^R = \Delta U^R / T$$

$$S_A^R - S_B^R = (U_A^R - U_B^R)/T$$

Isolated sos

$$E_A + U_A^R = E_B + U_B^R$$

$$\Rightarrow U_A^R - U_B^R = -(E_A - E_B)$$

Finally:

$$\boxed{\frac{p(A)}{p(B)} = \exp\left(\frac{-(E_A - E_B)}{kT}\right)}$$

Not possible to overstate the importance ...

Normalization

$$P(E_i) = C \cdot \exp\left(-E_A / kT\right)$$

$$\frac{P(E_A)}{P(E_B)} = \frac{C \cdot \exp\left(-E_A / kT\right)}{C \cdot \exp\left(-E_B / kT\right)}$$

$$= \exp\left(\frac{-(E_A - E_B)}{kT}\right)$$

But what is C ?

$$\sum_i P(E_i) = 1$$

$$\sum_i C \cdot \exp\left(-E_i / kT\right) = 1$$

$$\frac{1}{C} = \sum_i \exp\left(-E_i / kT\right) \equiv Z$$

$$\boxed{\begin{aligned} P(E_i) &= \frac{1}{Z} \exp\left(-E_i / kT\right) \\ Z &= \sum_i \exp\left(-E_i / kT\right) \end{aligned}}$$

Alternate Solution to TGS. 9

Define ground state by all protons
(lowest energy configuration)

The energy of a state with N_n protons
is therefore:

$$U = \varepsilon N_n \quad \Rightarrow \quad N_n = \frac{U}{\varepsilon}$$

We also must have

$$N = N_n + N_p \quad \Rightarrow \quad N_p = \left(N - \frac{U}{\varepsilon}\right)$$

The multiplicity of states is

$$\Omega = \frac{(N_n + N_p)!}{N_n! \; N_p!}$$

$$S = k \log \Omega = k \log (N_n + N_p)! - k \log N_n! - k \log N_p!$$

Use Stirling's approximation:

$$N! = N \log N - N \qquad (\text{for large } N)$$

$$S = k(N_n + N_p)\log(N_n + N_p) \cancel{-kN_n} \cancel{-kN_p}$$

$$- kN_n \log N_n \cancel{+kN_n} - kN_p \log N_p \cancel{+kN_p}$$

The first term is constant, since $N = N_n + N_p$ is conserved. So we can write as

$$S = S_0 - kN_n \log N_n - kN_p \log N_p$$

Using $N_n = \frac{U}{\varepsilon}$ $\quad N_p = N - \frac{U}{\varepsilon}$

$$S = S_0 - k\frac{U}{\varepsilon}\log\frac{U}{\varepsilon} - \left(N - \frac{U}{\varepsilon}\right)k\log\left(N - \frac{U}{\varepsilon}\right)$$

$$\frac{\partial S}{\partial U} = -\frac{k}{\varepsilon}\log\frac{U}{\varepsilon} + \frac{k}{\varepsilon}\log\left(N - \frac{U}{\varepsilon}\right)$$

$$\underbrace{-k\frac{U}{\varepsilon}\frac{1}{\left(\frac{U}{\varepsilon}\right)}}_{=-1} \quad \underbrace{-k\left(N - \frac{U}{\varepsilon}\right)\frac{(-1)}{\left(N - \frac{U}{\varepsilon}\right)}}_{=+1}$$

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{k}{\varepsilon}\log\left(\frac{N - \frac{U}{\varepsilon}}{\frac{U}{\varepsilon}}\right)$$

$$\boxed{\frac{\varepsilon}{kT} = \log\left(\frac{N_p}{N_n}\right)}$$

Stirling Approximation

$\log(n!) = \log(1) + \log(2) + \cdots + \log(n)$

Trick: RHS looks like an integral except coefficients of $\log(1)$ and $\log(2)$ ...

$$\log(n!) - \frac{1}{2}\cancel{\log(1)}^{\,0} - \frac{1}{2}\cancel{\log(n)}^{\,\log n \ll n}$$

$$= \frac{\log(1) + \log(2)}{2} + \frac{\log(2) + \log(3)}{2} + \cdots + \frac{\log(n-1) + \log(n)}{2}$$

$$\approx \int_1^n \log x \, dx$$

What is $\int \log x \, dx$?

skip!

Integration by parts:

$$(uv)' = u'v + v'u$$

$$d(uv) = du\,v + u\,dv$$

$$\boxed{u\,dv = uv - v\,du}$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\,dx \qquad v = x$

$$\int \log x \, dx = x \ln x - \int x \frac{1}{x} \cdot dx = x \log x - x$$

$$\Rightarrow \boxed{\log(n!) = n\log n - n}$$

Other Useful trick

$\exp(x)$ where x = big number...

Trick: IF find y s.t. $x = y \log 10$

then $x = \log 10^y$

$\exp(x) = \exp(\log 10^y) = 10^y$

So, given x, calculate $y = x / \log 10$

$$\boxed{\exp(x) = 10^{x/\log 10}}$$

Maxwell - Boltzmann Distribution

Consider a particle to be small system in equilibrium with reservoir at temperature T (the temperature of the gas)

$$Pr(\text{one particular microstate with speed } v)$$

$$\propto e^{-E/kT} = e^{-\frac{mv^2}{2kT}}$$

$$\sqrt{\frac{2kT}{m}} \equiv v_p$$

$$Pr(v) \propto e^{-v^2/v_p^2}$$

But there might be a <u>different</u> <u>number</u> of microstates at each value of v.

what to do?

Recall: Confined particle has

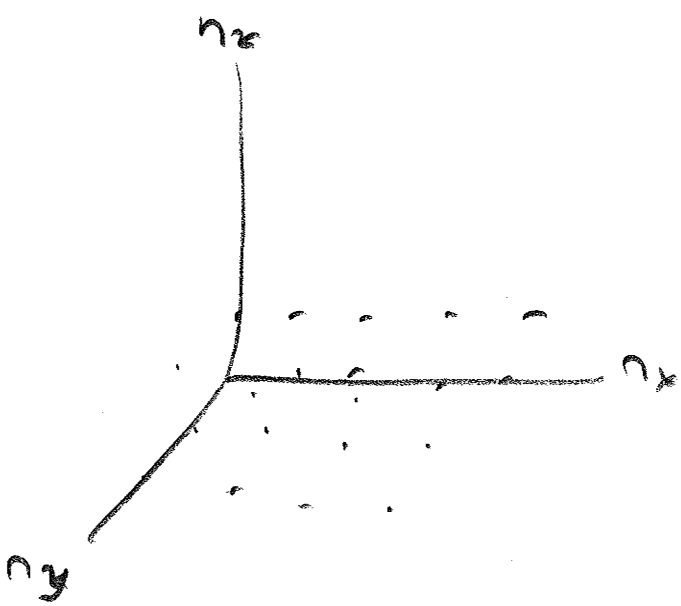
$$E = \varepsilon(n_x^2 + n_y^2 + n_z^2) = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2)$$

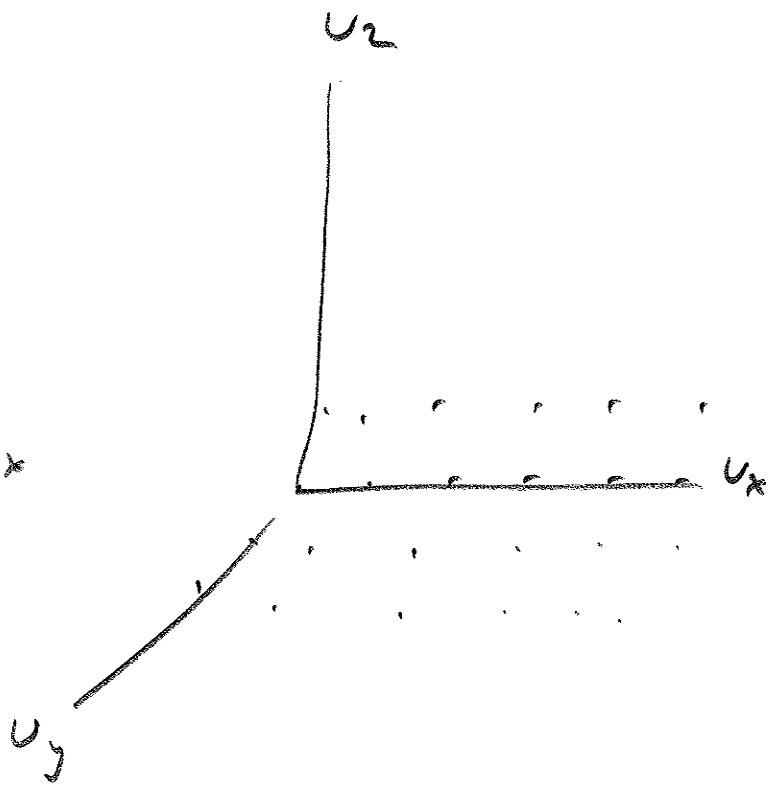
It turns out that in fact

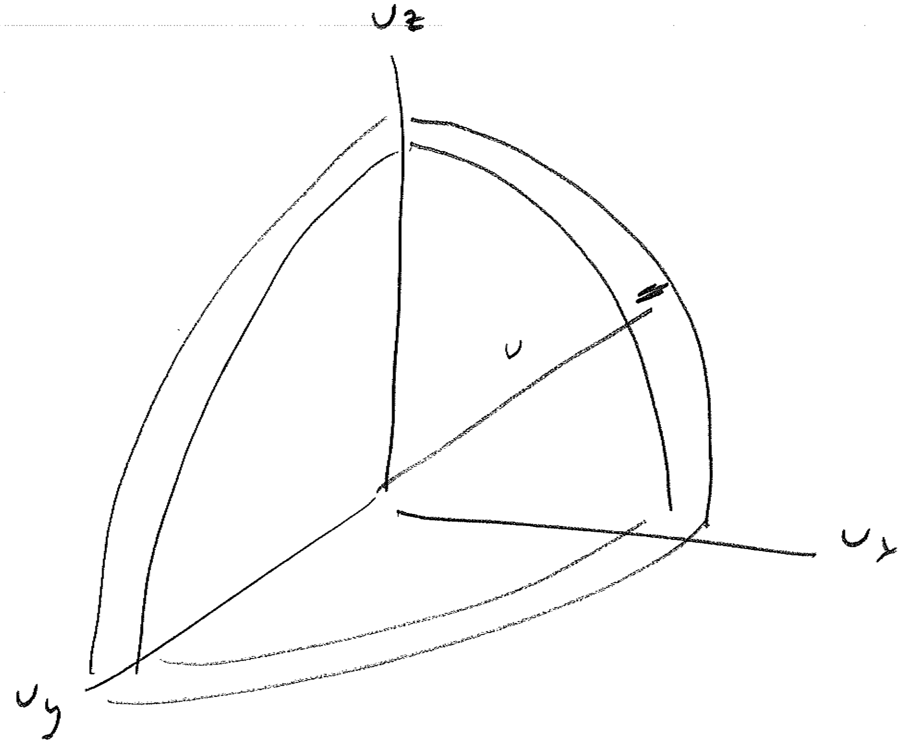
$$|v_x| \propto n_x \qquad |v_y| \propto n_y \qquad |v_z| \propto n_z$$

Important point:

Microstates are uniform in <u>velocity space</u>







$\#$ states $\propto 4\pi v^2\, dv$

(Book says $\frac{1}{8} \cdot 4\pi v^2\, dv$ since $|v_x| > 0$, etc)

But all we need is

$$n(v) \propto v^2\, dv \propto \left(\frac{v}{v_p}\right)^2 \frac{dv}{v_p}$$

So Finally:

$$D(v) = N \cdot \left(\frac{v}{v_p}\right)^2 \frac{dv}{v_p} \exp\left(-\left(v/v_p\right)^2\right)$$

$$1 \equiv \int_0^\infty D(v)\, dv = N \cdot \int x^2\, dx\, \exp(-x^2)$$

$$= N \cdot \frac{\sqrt{\pi}}{4}$$

$$\Rightarrow \quad N = \frac{4}{\sqrt{\pi}}$$

$$Pr(v) = \frac{4}{\sqrt{\pi}} \left(\frac{v}{v_p}\right)^2 \left(\frac{dv}{v_p}\right) \exp\left(-\left(\frac{v}{v_p}\right)\right)^2$$

$$\boxed{v_p \cdot \frac{dP}{dv} = \frac{4}{\sqrt{\pi}} \left(\frac{v}{v_p}\right)^2 \exp\left(-\left(\frac{v}{v_p}\right)^2\right)}$$

To get probabilities you have to either integrate over some range or approximate

$$P = \frac{dP}{dv} \cdot \Delta v$$

1) $I = \int x e^{-x^2} dx$

Let $y = x^2 \quad dy = 2x\,dx$

$I = \frac{1}{2} \int 2x\,dx\, e^{-x^2} = \frac{1}{2} \int dy\, e^{-y}$

$= -\frac{1}{2} e^{-y} = -\frac{1}{2} e^{-x^2}$

$$\boxed{\int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2}}$$

2) $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$

$= \iint e^{-(x^2+y^2)} dx\, dy$

$= \iint e^{-r^2} r\, dr\, d\theta$

$= -\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \cdot 2\pi$

$= \pi$

$\Rightarrow I = \sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}$

So $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

3) $I = \int_0^\infty x^2 e^{-x^2} dx$

$$u = x \qquad dv = x e^{-x^2} dx$$

$$du = dx \qquad v = -\frac{1}{2} e^{-x^2}$$

$$\int_0^\infty x^2 e^{-x^2} = x\left(-\frac{1}{2} e^{-x^2}\right)\Big|_0^\infty + \frac{1}{2}\int_0^\infty e^{-x^2} dx$$

$$= 0 + \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$$

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$$

Calculating Entropy Changes $\boxed{T8}$

$$\frac{1}{T} \equiv \frac{\partial S}{\partial U} \quad \text{(constant } V, N)$$

$$\Rightarrow dS = \frac{dU}{T} \quad \text{(at constant } V, N)$$

Easy cases:

1) ΔT for ideal gas (constant V, N)

$$U = \frac{3}{2} N k T$$

$$dU = \frac{3}{2} N k \, dT$$

$$dS = \frac{3}{2} N k \frac{dT}{T}$$

$$S = \frac{3}{2} N k \int_{T_1}^{T_2} \frac{dT}{T} = \frac{3}{2} N k \ln\left(T_2/T_1\right)$$

2) $dU = mc \, dT$

$$S = mc \ln\left(\frac{T_2}{T_1}\right)$$

<u>Recall</u>: $\Delta U = Q + W$

Essentially we've calculated ΔS
in case where $W = 0$.

Now lets consider

$$Q = 0 \quad W \neq 0$$

But to make life easier, lets look at quasistatic work first
(work done slowly, so system in equilibrium at every point.)

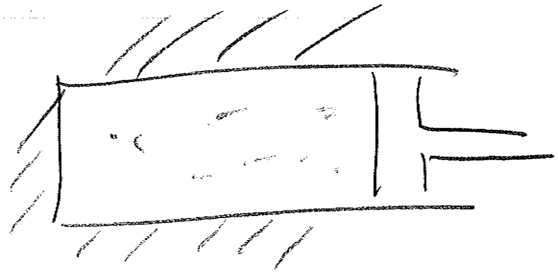
$$\Delta U = \cancel{Q} + W_{qs} + \cancel{W_{nqs}}$$

$(Q = 0)$ $\qquad$ $(W_{nqs} = 0)$

$\boxed{T8}$

Quasistatic, Adiabatic

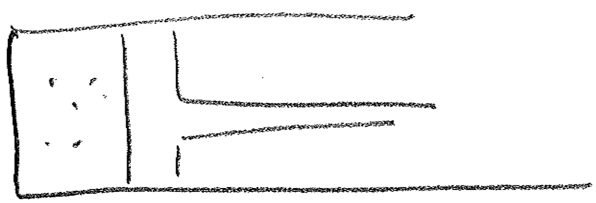
(I)



$T_1 \; V_1$

$\longleftarrow W_1$

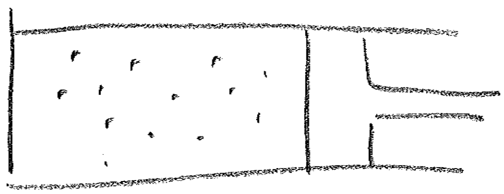
(II)



$T_2 \; V_2$

$\longleftarrow W_2$

(III)



Return to V_1, T_3

Q: What is T_2? $\quad T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$

$$\Rightarrow \quad T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

Q: What is T_3? $\quad T_1 V_1^{\gamma-1} = T_3 V_1^{\gamma-1}$

$$\Rightarrow \quad \boxed{T_1 = T_3}$$

$$\Delta U = 0$$

$$\Rightarrow \quad W_1 + W_2 = 0$$

<u>Reversible</u>: Can return to original state with <u>zero</u> net work!

For monatomic gas

T 8.7

$$S \sim \ln V^{2/3} U$$
$$\sim \ln V^{2/3} T$$

During adiabatic compression of a gas $V^{2/3} T = \text{const}$

i.e. $\Delta S = 0$ during adiabatic compression or expansion of a gas!!!

Is there a more general argument?

1st Try:

$$I \longrightarrow II \quad \text{reversible}$$

$$I \rightarrow II \rightarrow I \quad \text{possible}$$

$$\Delta S \left(I \rightarrow II \rightarrow I \right) = 0$$

$$\Delta S \left(I \rightarrow II \right) > 0$$

$$\Delta S \left(II \rightarrow I \right) < 0 \quad \text{(violates 2nd Law ???)}$$

$\therefore \quad \Delta S = 0$ for reversible processes.

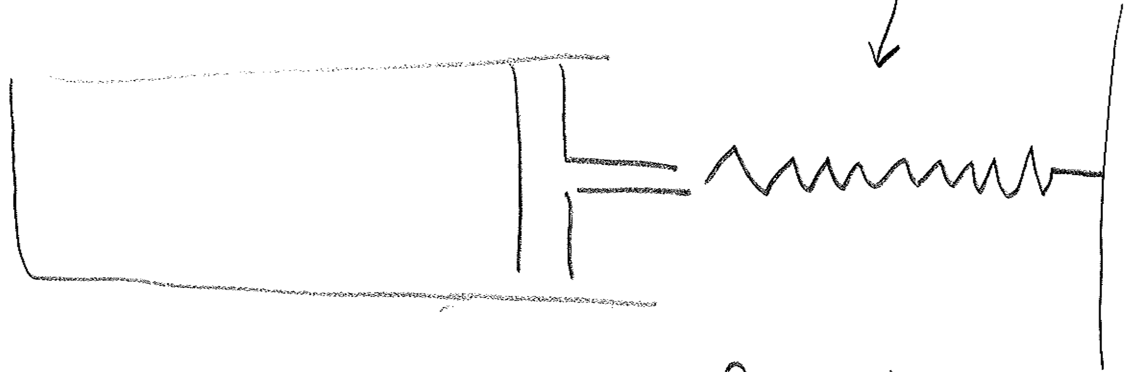
Not quite so easy, because we can have work during

$$I \xrightarrow[W_1]{} II \xrightarrow[W_2]{} I$$

i.e. Not An isolated system !!!

Patch-up Job:

Perfect Spring



Does it have Entropy?

For $U_{spring} = \frac{1}{2} k x^2$

there is only one state of spring:

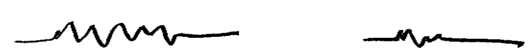
$\Omega = 1$

$S = 0$!!!

So patch up our argument

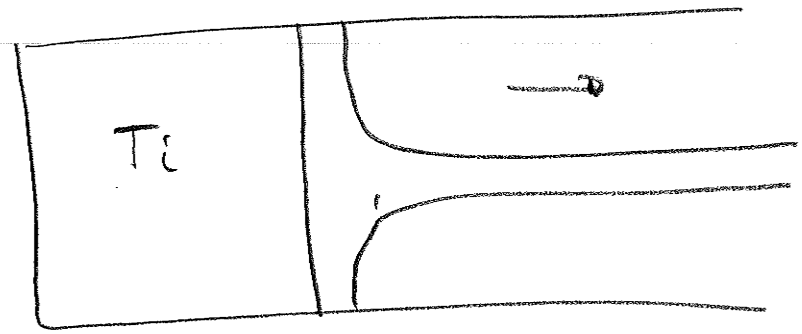
$\mathrm{I} \longrightarrow \mathrm{II}$ w/ work

becomes (original system) Spring System $(S=0)$

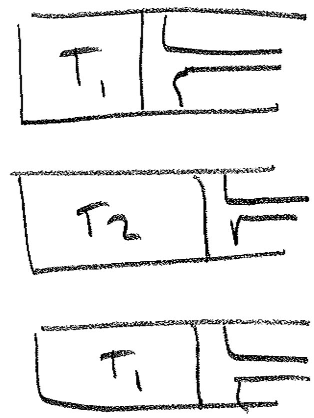
$\mathrm{I} \longrightarrow \mathrm{II}$ + 

$\mathrm{II} \longrightarrow \mathrm{I}$ + 

3)



Adiabatic Quasistatic Expansion, then heat!



$$TV^{2/f} = \text{const}$$

$$T_1 V_1^{2/f} = T_2 V_2^{2/f}$$

$$T_1 V^{2/f} = T_2 (2V)^{2/f}$$

$$\left(\frac{1}{2}\right)^{2/f} T_1 = T_2$$

$$\Delta S = \int \frac{dU}{T} = \int_{T_2}^{T_1} \frac{\frac{f}{2} N k\,dT}{T}$$

$$= \frac{f}{2} N k \ln\left(T_1/T_2\right)$$

$$T_1/T_2 = 2^{2/f}$$

$$= \boxed{Nk \ln 2}$$

Problems

T8B4 $\Delta S = \frac{Q}{T} = \frac{-35\,J}{(273+22)K} = -0.12\ JK$

T8B7 $\Delta Q = mL$

$$\Delta S = \frac{\Delta Q}{T} = \frac{mL}{T} = \frac{(0.125\,kg)(2256\ kJ/kg)}{(273+100)\,K}$$

$$= 760\ J/K$$

T8S9 If this were done quasistatically

$\Delta T = 0$, $\Delta S = 0$

In this case, you would work harder to compress the gas, as the particle density would be higher near the cylinder.

So $\Delta U = |W_1| - |W_2|$ where $|W_1| > |W_2|$

So $\Delta U > 0$, so $\boxed{\Delta T > 0}$.

This is consistent with $\boxed{\Delta S > 0}$