Thoughts on Unit Q:

→ It is solid on intuition and concepts

→ But weak on math.

Good physics is the union of both...
→ with experiment deciding which intuitions are correct!

Plan:

Let Unit Q section 1 → 10 drive the course, at a leisurely pace.

Supplement with additional material

→ Feynman Lectures will be minimally used (can return the book if you want.)

HWs: Unit Q problems

+ my own Problems
(good exam prep)

~~Next~~ Thur. Lecture: Prof. Erbacher

Colloquium

Waves: disturbance that moves through a medium

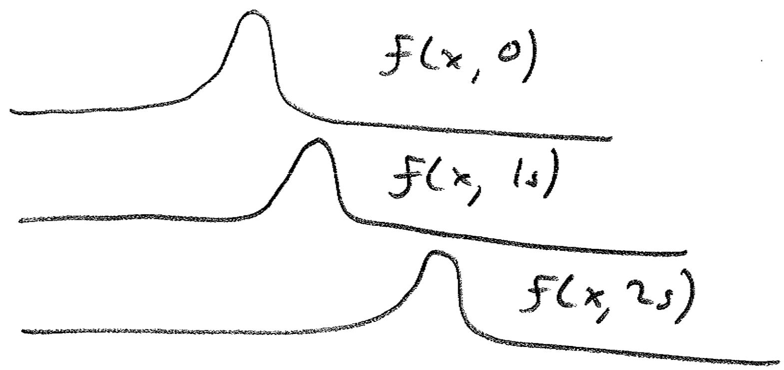
→ many different contexts:

Tension wave (taught string)

Sound wave (pressure)

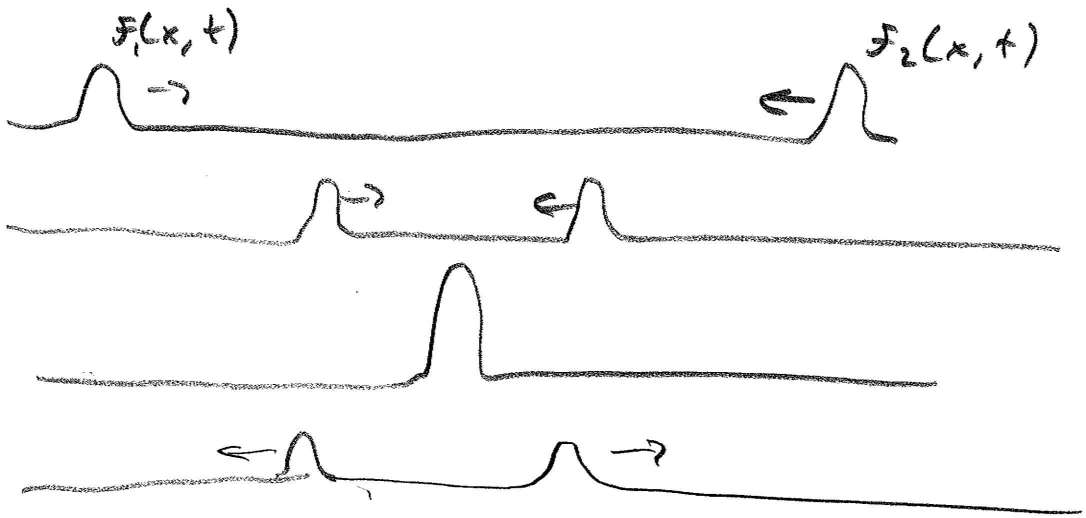
↳ 1-D waves described by a function

$f(x,t)$



Superposition:

The combination of two waves is just the simple algebraic sum:



(Waves pass through one another.)

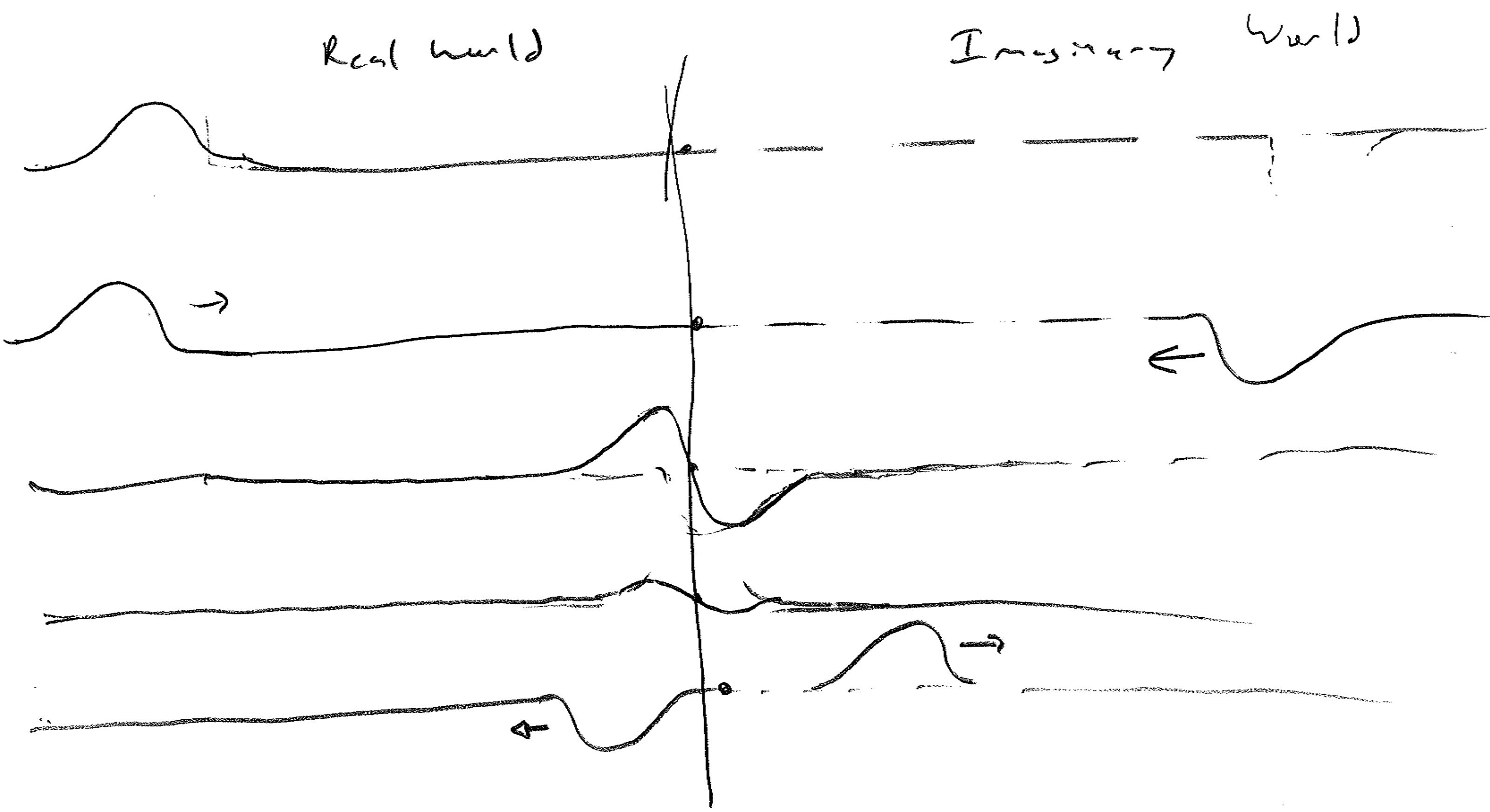
Reflections:

(We'll do the math next week!!)

We can impose boundary conditions on our string, e.g.

On the face of it, this seems impossible to predict what will happen.

But superposition principle <u>alone</u> lets us predict what happens...

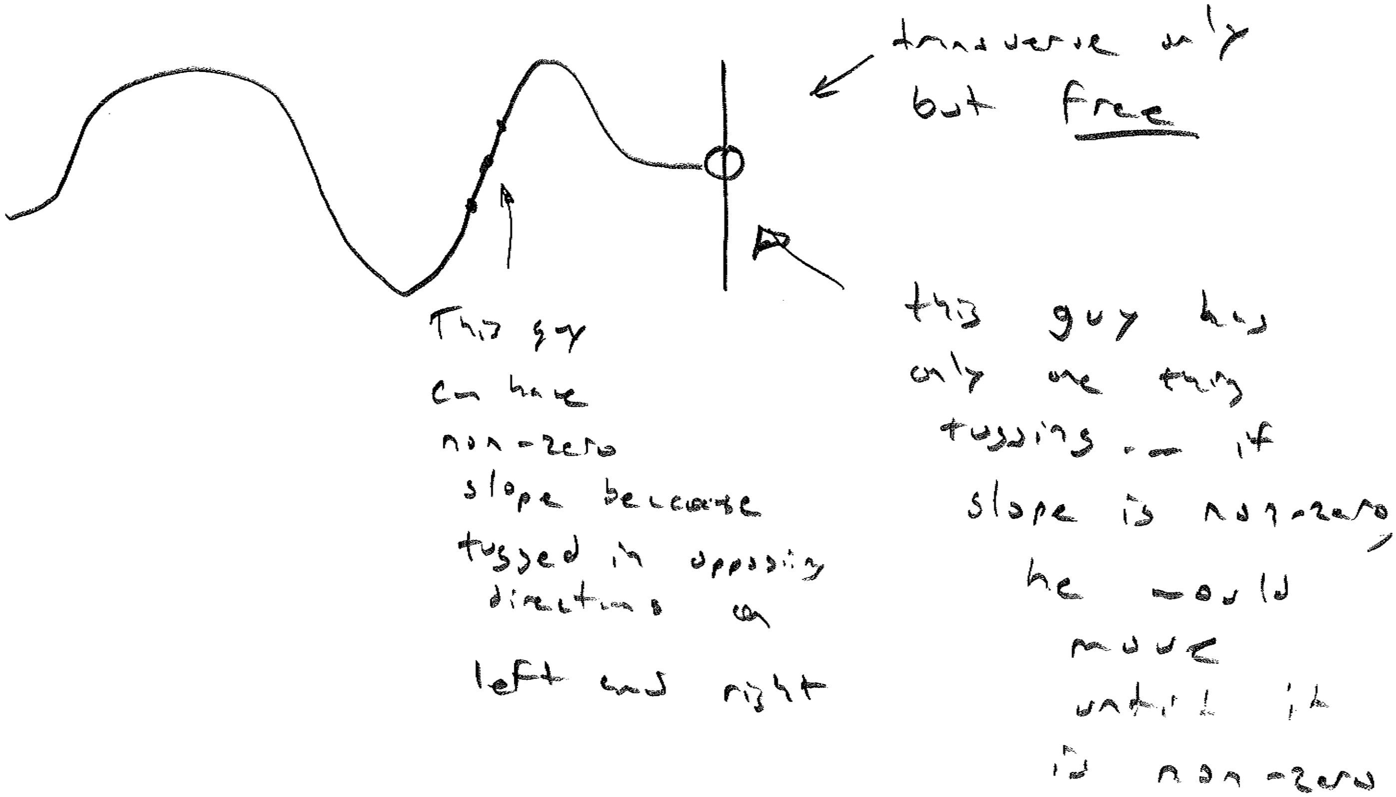


I said superposition principle is
all we need, but we need one
more concept

"unique solution";

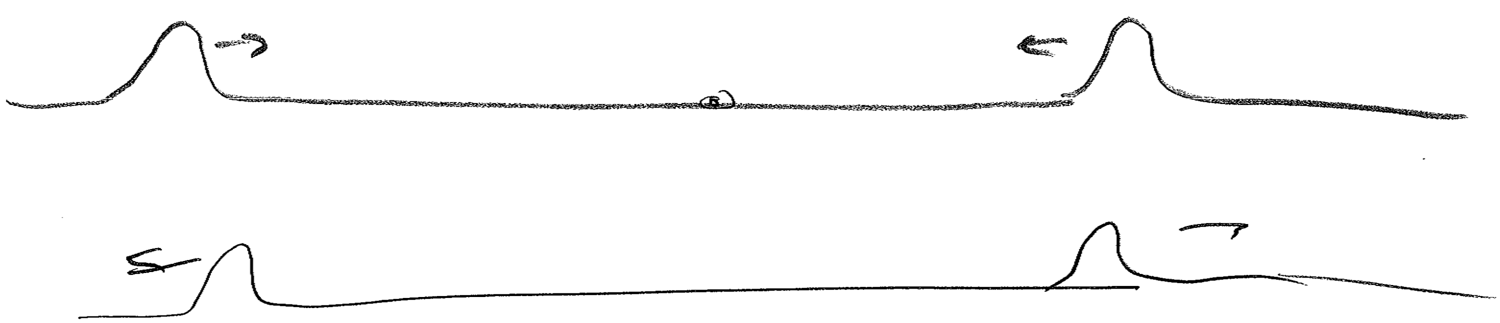
* a solution to the wave equation (to be defined)
that satisfies boundary conditions
is a <u>unique</u> solution

Free - End Reflection:



Slope is zero at free end.

Use superposition again



A particular type of wave
(Very Important, as we will see!)

are sinusoidal waves!

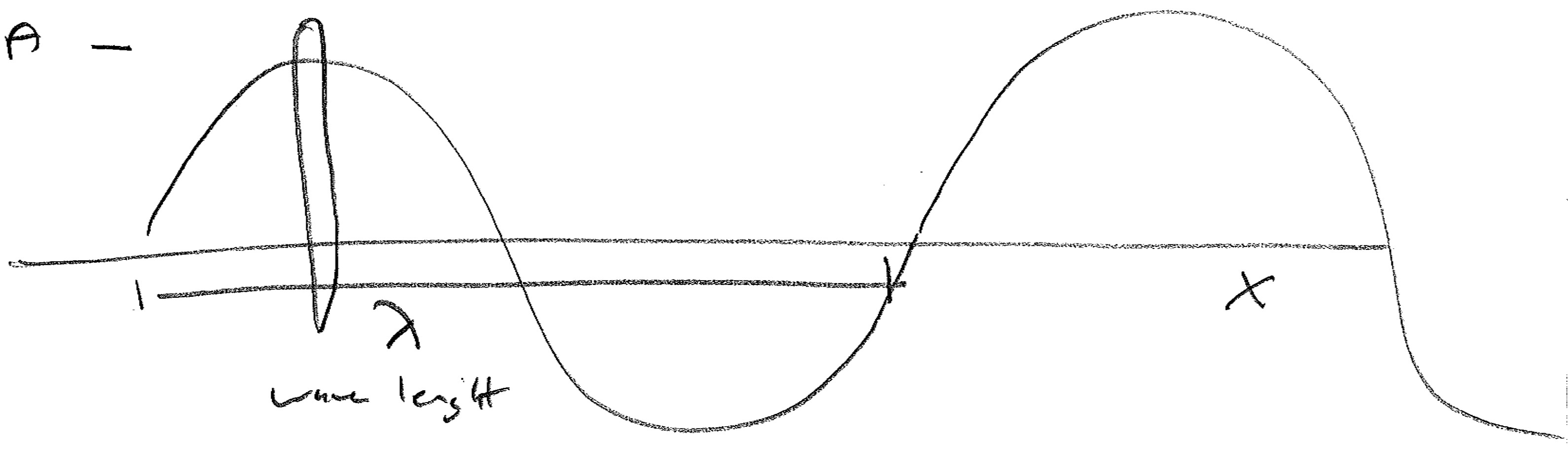
$$F(x,t) = A \sin(kx - \omega t)$$

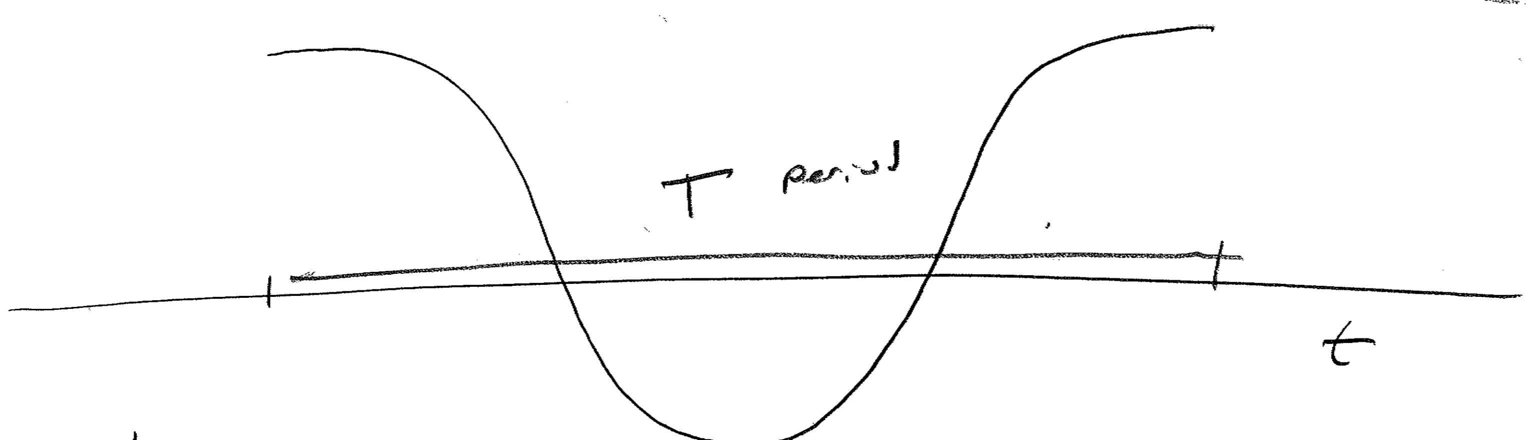
Amplitude

wave number

angular frequency

$[k] = \left[\frac{1}{x}\right]$ $\quad [\omega] = \left[\frac{1}{t}\right]$





Q: Relate k and ω to λ and T

A) $\sin(\theta + 2\pi) = \sin(\theta)$

$\therefore$ (Set $t = 0$)

Want $\sin(k\lambda + \phi) = \sin(k0 + \phi)$

True if $k\lambda = n \cdot 2\pi$

(λ is <u>one cycle only</u>, so corresponds to $n = 1$)

$$k\lambda = 2\pi \Rightarrow k = \frac{2\pi}{\lambda}$$

Similarly:

$$\omega T = 2\pi \Rightarrow \omega = \frac{2\pi}{T}$$

$$f(x,t) = A \sin(kx - \omega t) = A \sin\left(\frac{2\pi}{\lambda} x - \frac{2\pi}{T} t\right)$$

(The latter makes perfect sense, but would be a pain to write!!!)

One more common definition

Frequency of wave

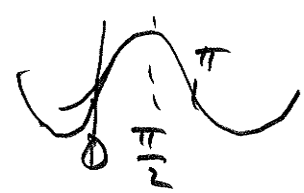
$$f \equiv \frac{1}{T}$$

$$\boxed{\omega = 2\pi f}$$

Q: What is the speed of a wave

$$A \sin(kx - \omega t)$$

A: Trick: watch a crest move!



$$kx_c - \omega t = \frac{\pi}{2}$$

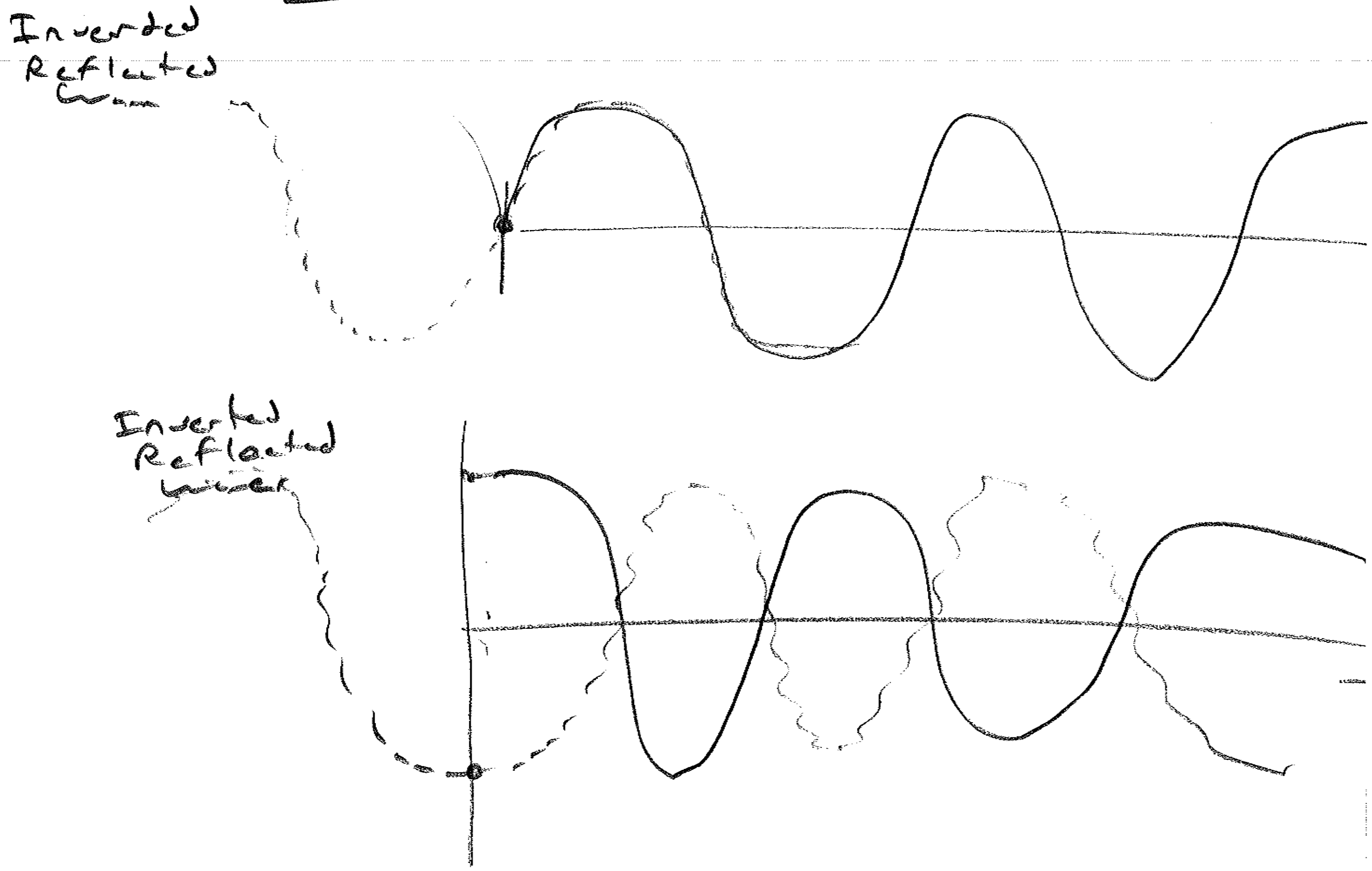
$$x_c = \frac{\pi}{2}\frac{1}{k} + \frac{\omega}{k} t$$

$$x_c = \frac{\lambda}{4} + \frac{\omega}{k} t$$

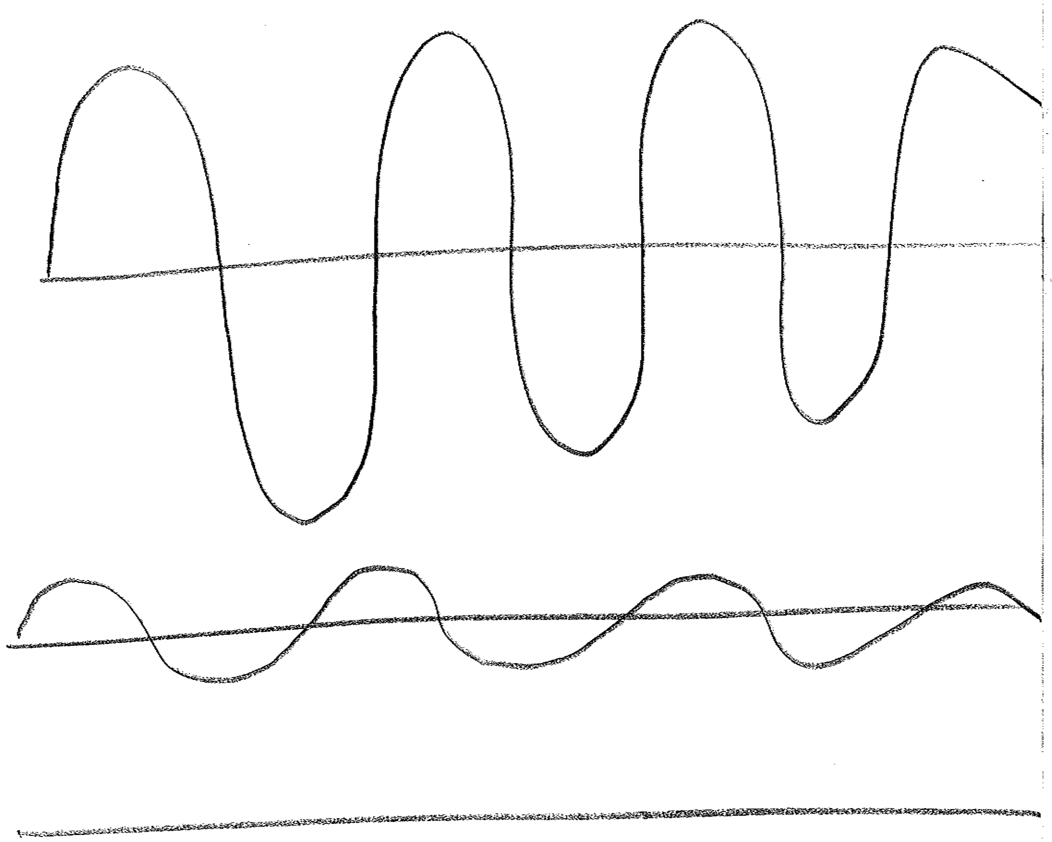
$$v_{wave} = \boxed{\frac{\omega}{k}}$$ most useful!

Also $$v_{wave} = \frac{\frac{2\pi}{T}}{\frac{2\pi}{\lambda}} = \boxed{\frac{\lambda}{T}}$$ makes sense!

Picture



Result:



Math

$$f(x,t) = A \sin(kx+\omega t)$$

Add reflected $x \to -x$, inverted $A \to -A$

wave

$$f(x,t) = A\sin(kx+\omega t) - A\sin(-kx+\omega t)$$

$$= A\sin(kx+\omega t) + A\sin(kx-\omega t)$$

Recall: $\sin(\alpha \pm B) = \sin\alpha\cos B \pm \cos\alpha \sin B$

$$f(x,t) = A\Big(\sin kx \cos\omega t + \cancel{\cos kx \sin\omega t} + \sin kx \cos(\omega t) - \cancel{\cos kx \sin\omega t}\Big)$$

$$\boxed{f(x,t) = 2A \cdot \sin kx \cdot \cos\omega t}$$

STANDING WAVE

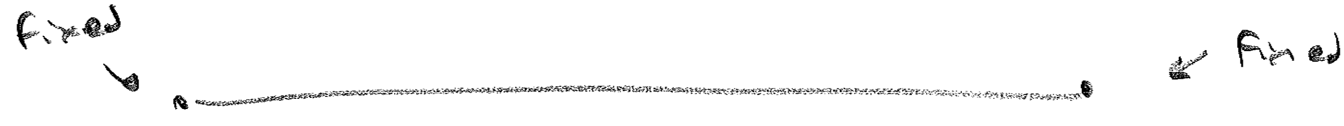
You Learned that

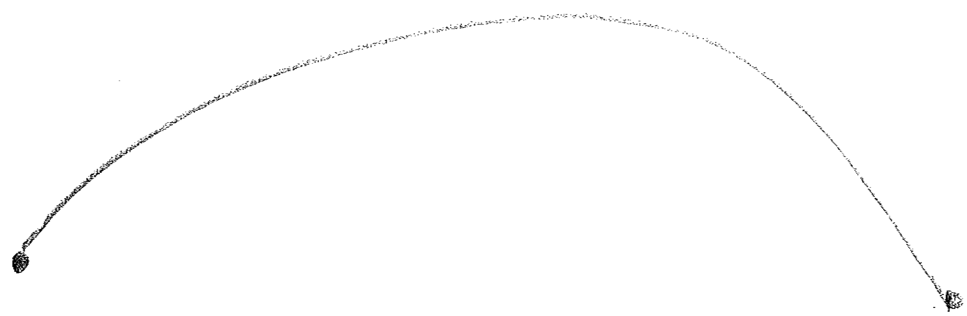
Stationary Solutions are

separable Solutions

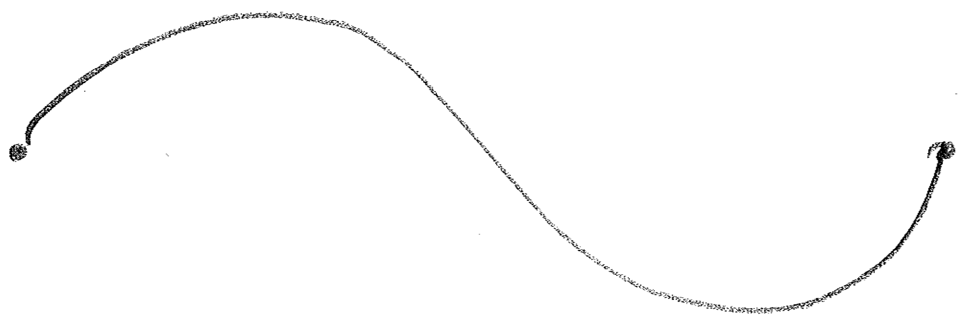
$$\boxed{f(x,t) = g(x) \cdot h(t)}$$

Stationary Solutions on a Fixed interval

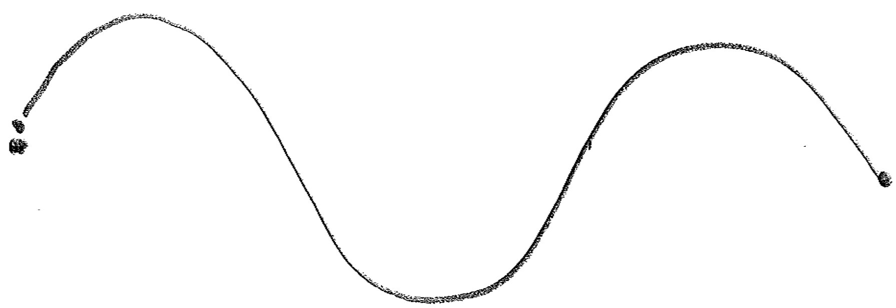




$\lambda = 2L$



$\lambda = L$

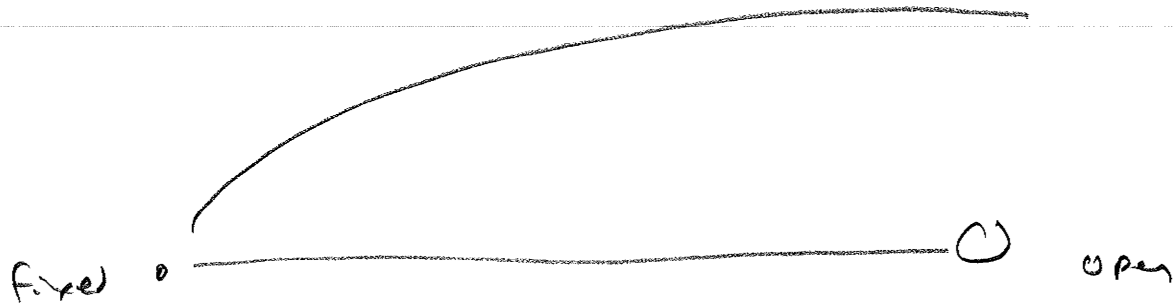


$\lambda = \frac{2L}{3}$

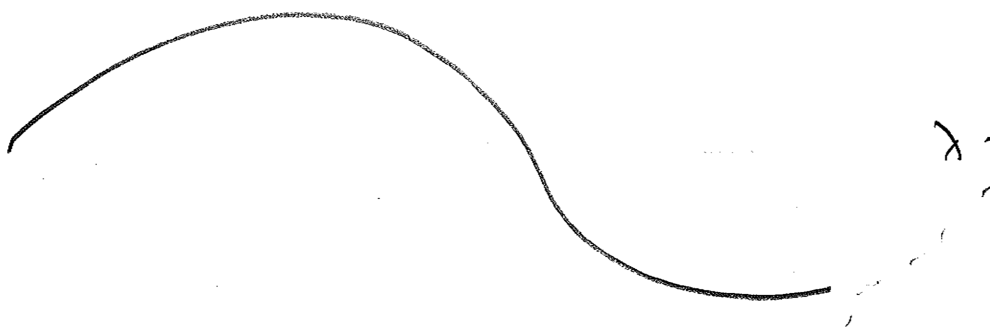
n

$$n\lambda = 2L \qquad n = 1, 2, 3, \ldots$$

* Interesting... so the standing wave solutions are discrete... quantized... ... I wonder if that is important?



$\lambda = 4L$



$\lambda = \frac{4L}{3}$

$n\lambda = 4L \qquad n = 1, 3, 5$

Recap

$n\lambda = 2L \qquad n = 1, 2, 3, 4$ (fixed - fixed) (open - open)

$n\lambda = 4L \qquad n = 1, 3, 5$ (fixed - open)

To reproduce book equations

$$v = \frac{\lambda}{T} = \lambda f \quad \Rightarrow \quad \lambda = \frac{v}{f}$$

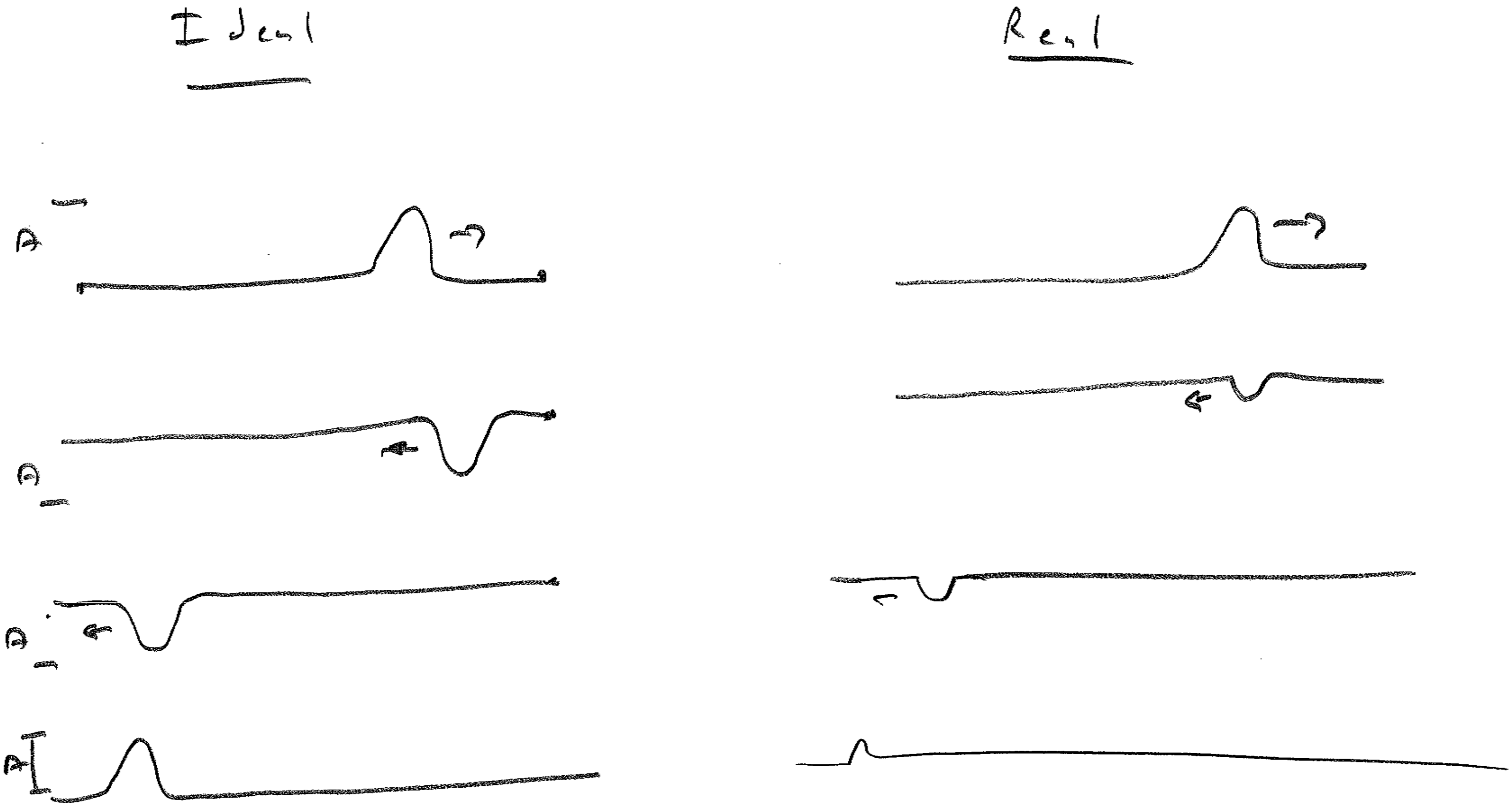
$$n\frac{v}{f} = 2L \quad \Rightarrow \quad f = \frac{nv}{2L} \qquad n = 1, 2, 3, 4$$

(f-f, o-o)

$$n\frac{v}{f} = 4L \quad \Rightarrow \quad f = \frac{nv}{4L} \qquad n = 1, 3, 5, \ldots$$

(f - o)

Resonance:



Now, standing waves also attenuate, but less quickly... because the wave function is always a nice smooth sine function going to <u>zero</u> at the fixed end.

Pulses on the other hand, nearly disappear during the reflection, with all of there energy stored in very extreme <u>kinky</u> (literally!) shapes near the fixed end... much more susceptible to <u>non</u>-<u>linearity</u>

(* I suppose this is why strings tend to break near where they are fixed off)

Closing Thought 2: FT

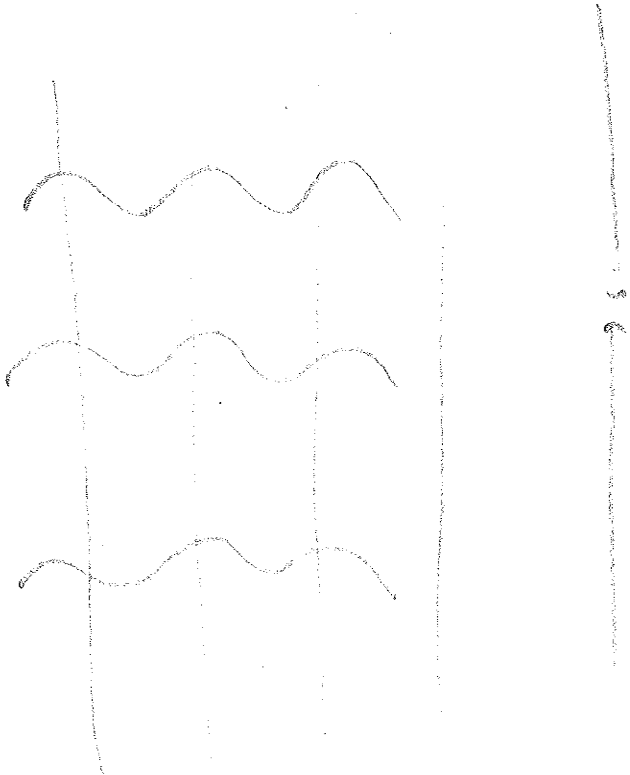
Any periodic function with frequency f can be approximated as <u>accurately</u> <u>as you wish</u>

by adding sine and cosines with frequencies

$f, 2f, 3f, 4f, 5f, \ldots$

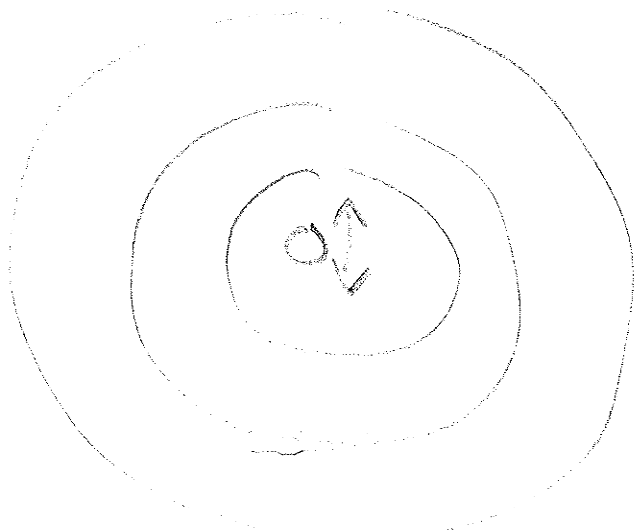
(See Wikipedia "Fourier Transform" for some beautiful illustrations)

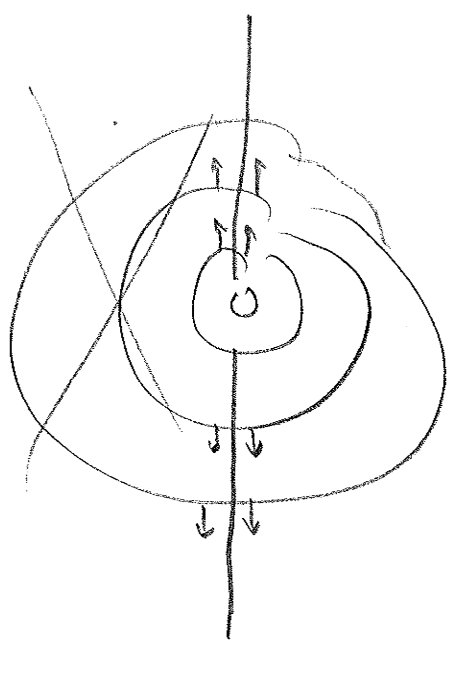
Q2



?

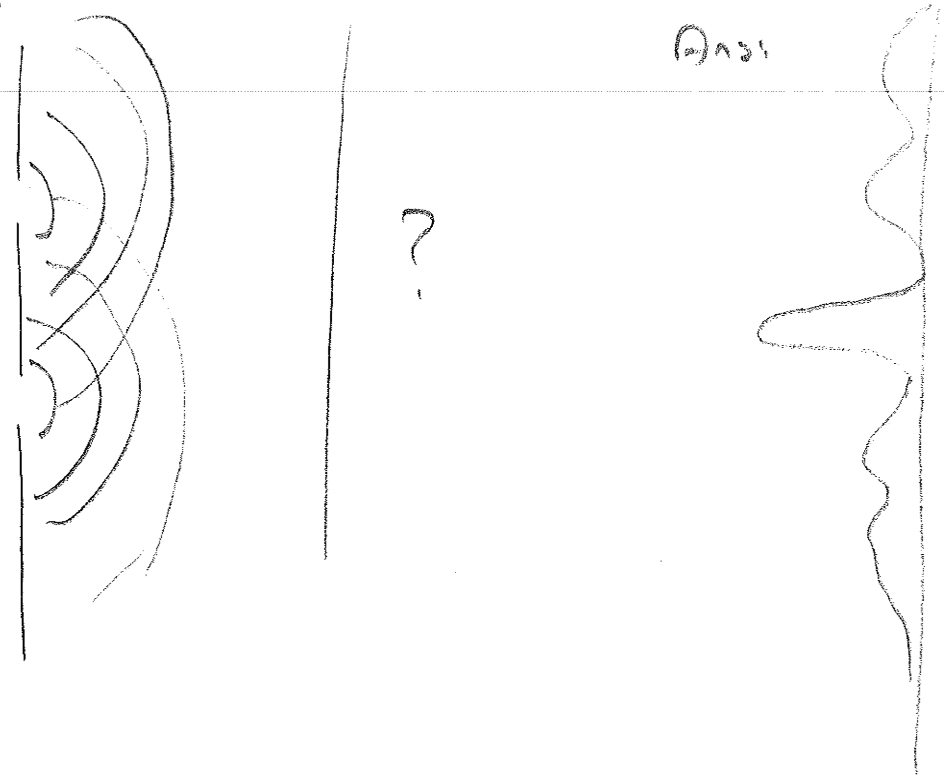
Trick:





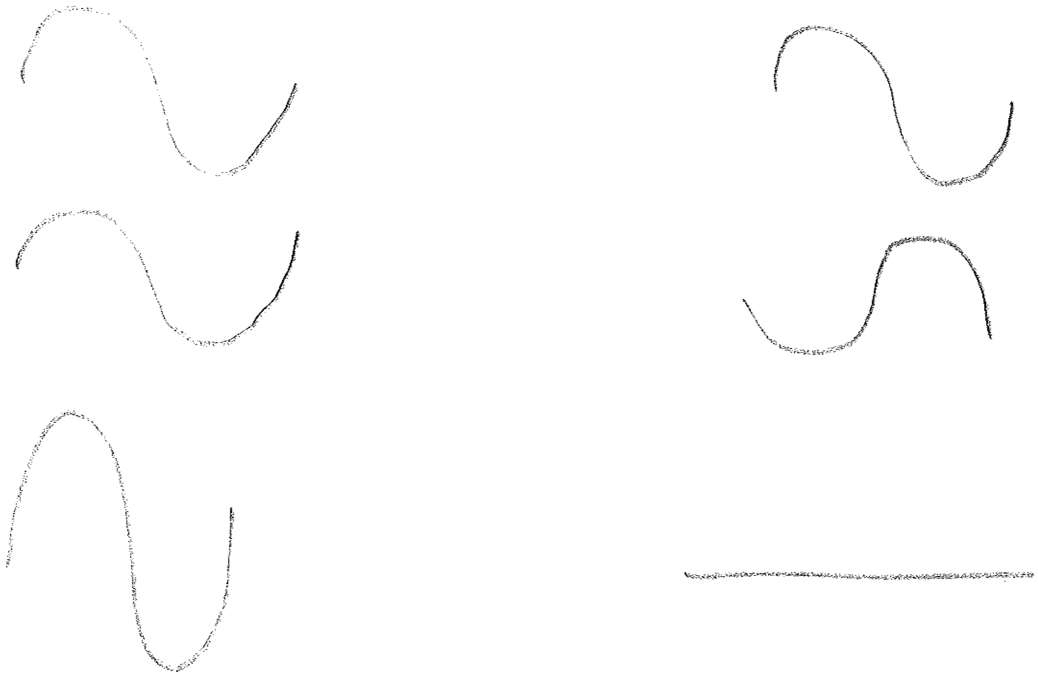
circular wave

2 slits



* We will derive the funky shape on Thursday ... today focus on its key features..

Superposition Principle $\Rightarrow$ Constructive + Destructive Interference

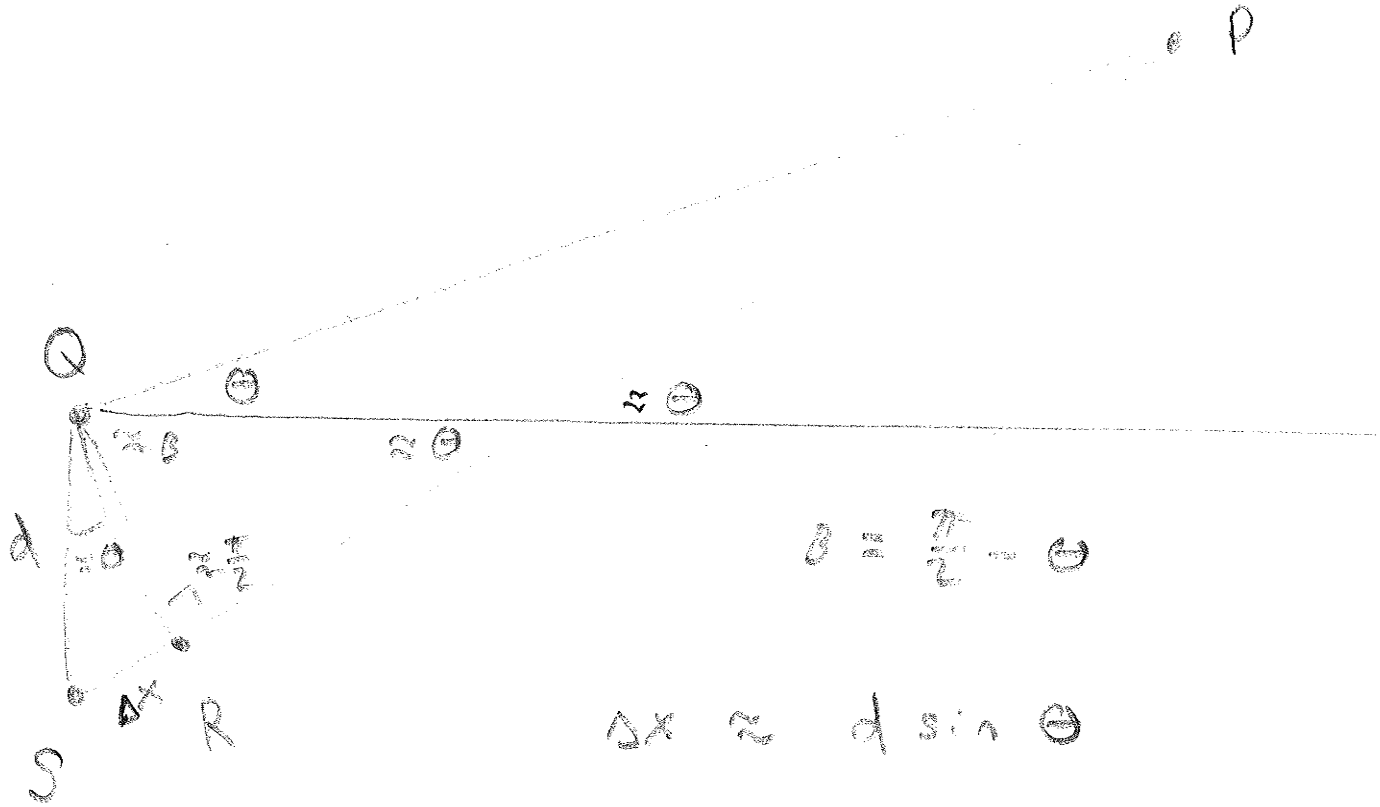


* What matters is relative <u>phase</u> of the sinusoidal waves...

$\Delta\phi = 0, 2\pi, \ldots$ constructive

$\Delta\phi = \pi, 3\pi, \ldots$ destructive

Constructive Interference



$$\beta = \frac{\pi}{2} - \theta$$

$$\Delta x \approx d \sin\theta$$

(Keep the figure)

Constructive Interference

$$k \Delta x = 2\pi n$$

$$\frac{2\pi}{\lambda} \Delta x = 2\pi n$$

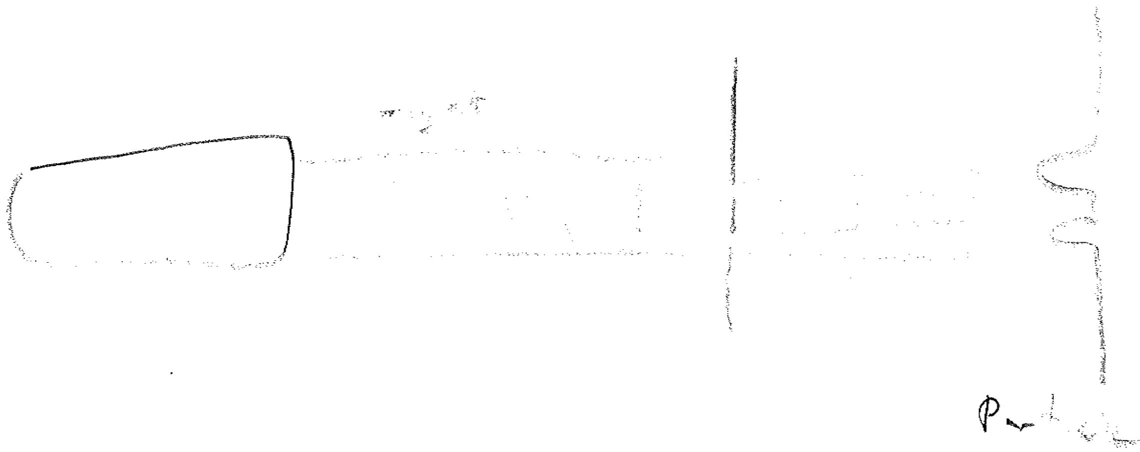
$$\Rightarrow \Delta x = n\lambda \approx d \sin\theta$$

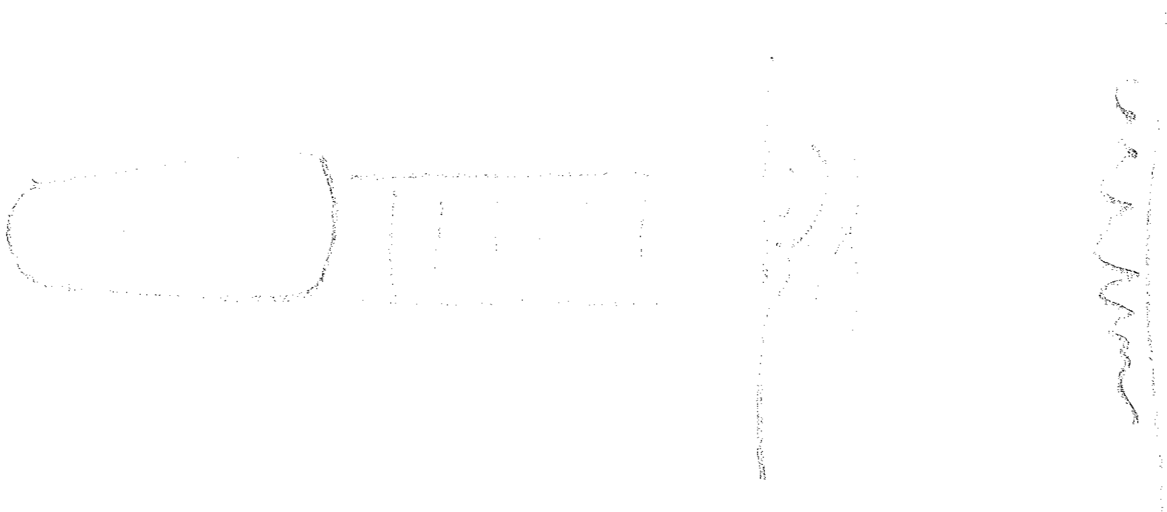
$$d \sin\theta_{nc} = n\lambda$$

quantum number, constructive

$$\theta_{nc} = \sin^{-1}\left(\frac{n\lambda}{d}\right)$$

Wave Nature of Light:



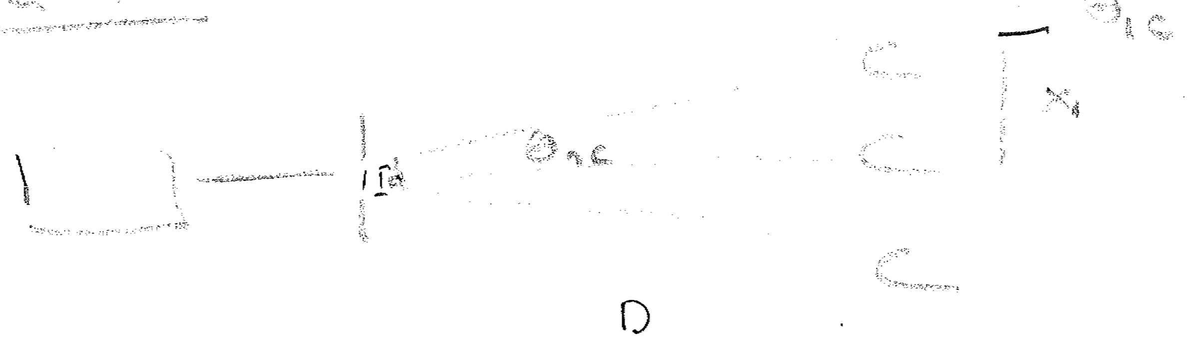


What we see:

Problems

Q: Why does spacing between peaks increase when distance between slits decreases.

Q: Q 2.2



$$\tan^{-1}\theta_{nc} \approx \theta_{nc} = \frac{x_n}{D}$$

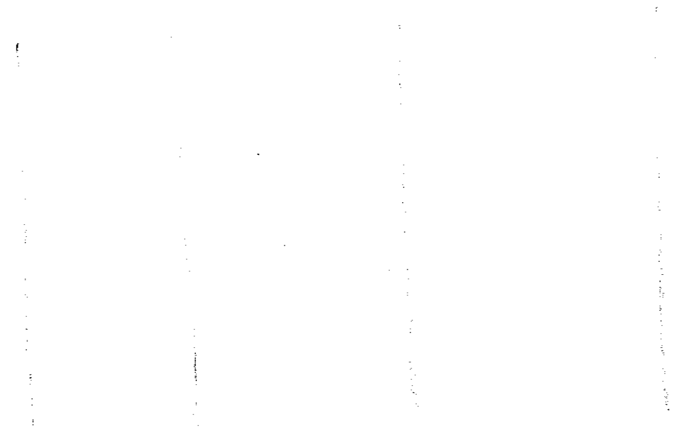
$$\frac{n\lambda}{d} = \sin\theta_{nc} \approx \theta_{nc} \approx \frac{x_n}{D}$$

$$x_n = \frac{D}{d}\cdot\lambda n$$

$$x_{n+1} - x_n = \frac{D}{d}\lambda$$ evenly spaced!

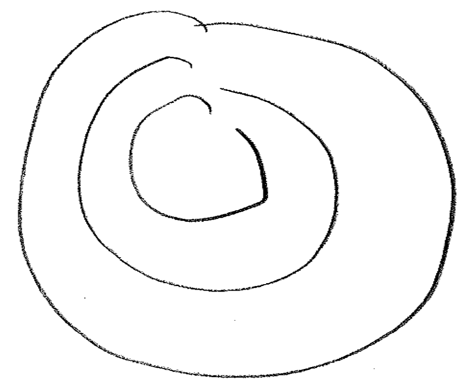
$$\boxed{\lambda = \frac{d}{D}\Delta x}$$

Huygen's Principle



(Equivalent because of
Superposition principle)

Circular Aperture



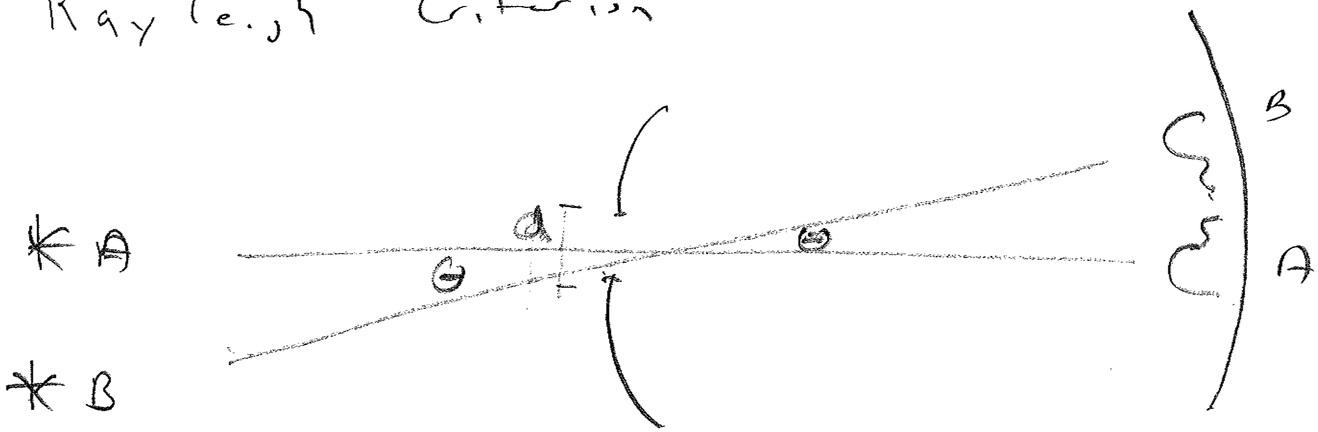
$$\sin\theta_{1d} = 1.22\frac{\lambda}{a}$$

"1.22" ???

- There's a joke that we use up all the fun physics problems on undergraduates
- So all there's left for graduate students is long hard slogs...
- Even graduate students don't calculate that value, "1.22" !!!

Circ...

Rayleigh Criterion



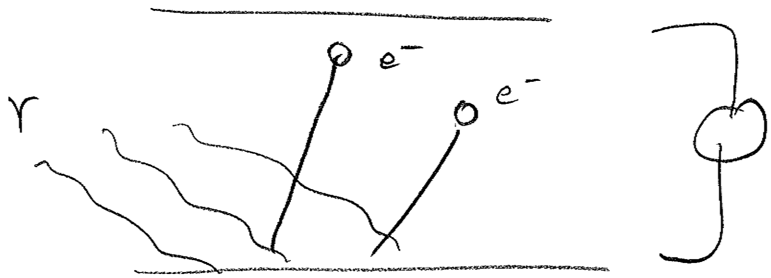
$$\boxed{\theta > \theta_{id} \approx \sin^{-1} 1.22\frac{\lambda}{a}}$$

Deep thought:

You taught yourself optics as a baby!

Measurements

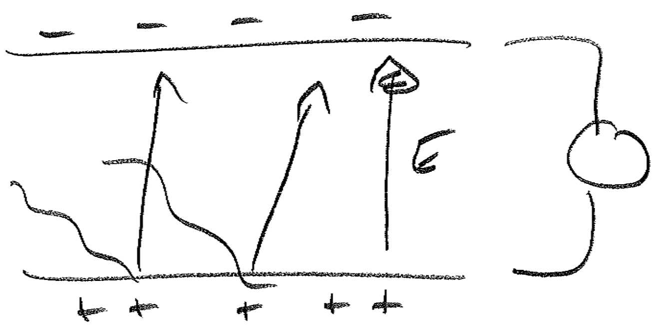
Q3



Current meter

(Current = q/t)

(Explain: $F = qE$)



Volt meter

(Max Kinetic Energy)

→ Measure rate

→ measure max kinetic energy

Wave-Model Predictions

$$\frac{\#\ \text{Electrons}}{\text{Time}} = \frac{(\text{energy / electron}) / \text{time}}{\text{energy ejection}}$$

$$= \frac{\dfrac{\text{energy delivered}}{\text{time area}}, \ \text{area of } e^-}{\text{energy ejection}}$$

$$\boxed{\text{rate} = \frac{I\, d^2}{E_{eject}}}$$

1) Rate will increase with I.

→ At low I, significant delays between illumination and first emission are possible.

2) Rate might vary from metal to metal as complicated function of frequency (resonance)

3) Max kinetic energy likely to increase with I.

(Time Scale ???)

Einstein model

$$E = hf = \frac{hc}{\lambda}$$

$$\boxed{K = \frac{hc}{\lambda} - W}$$

4) Cuttoff is $\frac{hc}{\lambda} < W$

$\rightarrow$ No ejection

5) Because $K = \frac{hc}{\lambda} - W$ (ind of I)

6) $K = \frac{hc}{\lambda} - W \sim f$

3) I const but increase f means

<u>fewer</u> <u>photons</u>

2) Sure, even 1 photon can do it!

1) Sure, number of photons is still
proportional to "I"!

Q3.4

--- experiments show;

✓ 1) At high I, rate $\propto$ I as expected!

? 2) But electrons emitted instantaneously, even at low I!

? 3) If I held constant, number of electrons ejected decreases with increasing frequency in same way for all metals.

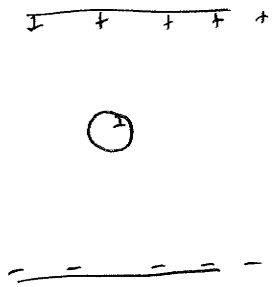
XX 4) If frequency below cutoff value, no emission at any I!

X 5) If frequency held constant, maximum kinetic energy does not depend on intensity

X 6) K $\propto$ F with same proportion for all metals

Millikan Oil Drop Experiment!

$\boxed{Q4}$



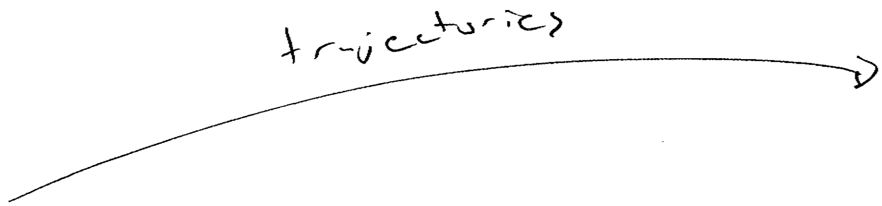
← Quantization of electric charge

Implicit: e^- is a particle

Bubble Chambers / Cloud Chamber

Compressed Gas Near Boiling point

Supersaturated Vapor
Clear condensation



Clearly e^-, p, etc are <u>particles</u>, ...

de Broglie Hypothesis

$$E = hf$$

$$E = \frac{hc}{\lambda}$$

Warning: we use SR units to write e.g.

$$E = p \quad \text{(for light)}$$

but SR units has

$$[x] = [t]$$
$$[c] = 1$$

We have to add the c back in

$$[E] = [mv^2]$$
$$[p] = [mv]$$

$$\boxed{E = pc} \quad \text{for light}$$

$$E = \frac{hc}{\lambda} = pc$$

$$\Rightarrow \boxed{p = \frac{h}{\lambda}} \quad \text{(for light)}$$

de Broglie

Relativity places Space + Time, Mass + Energy
on equal footing, ...

Why should particles w/ mass (e^-)
be different than particles of light
w/ mass = 0 (γ)

Hypothesis:

$$p = \frac{h}{\lambda}$$

For particles too!!!

What does it mean???

Diffraction when $\lambda \sim a$

Do I have a de broglie wavelength?

Well I guess ...

Say $m \sim 100$ kg

Let's put range of speeds from $v = 0$

to $v = 4 \times 10^6$ m/s (fastest satellite ever built!)

$P_{min} = 0$ $\qquad P_{max} = 100 \text{ kg} \times 4 \times 10^6 \text{ m/s}$

$= 4 \times 10^8 \text{ kg m/s}$

$$\lambda = \frac{h}{p}$$

$$\lambda_{max} = \frac{h}{0} = \infty$$

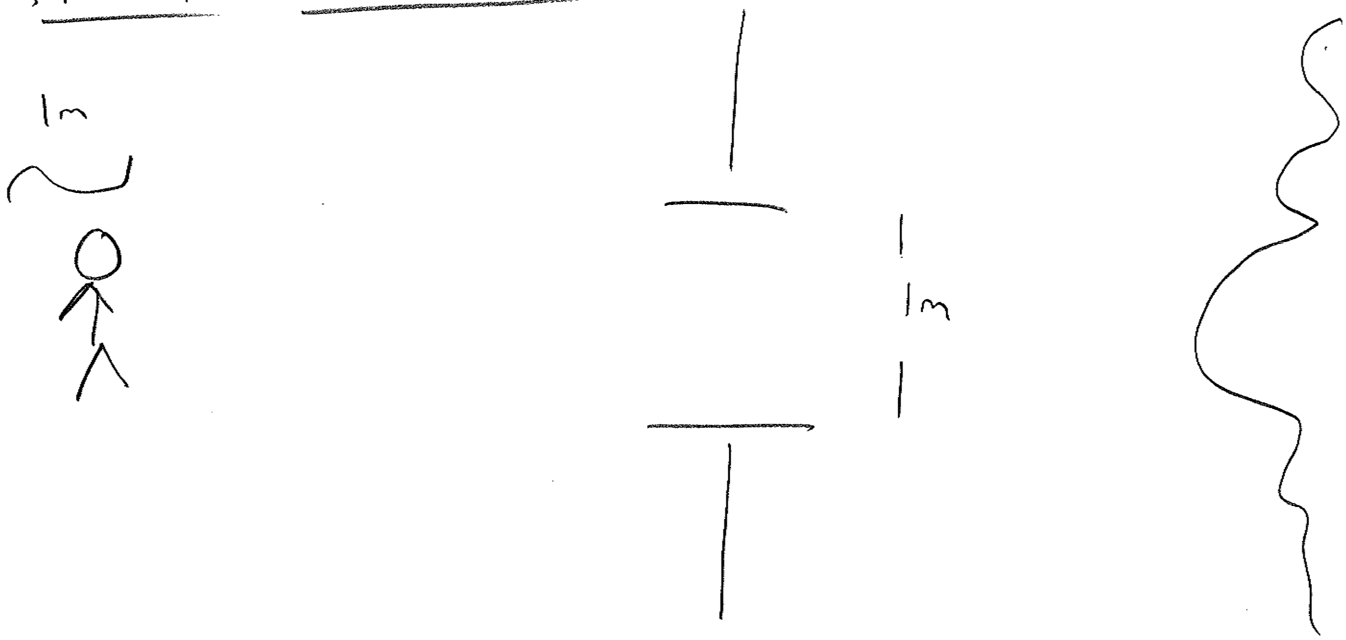
$$\lambda_{min} = \frac{h}{p} = \frac{6.6 \times 10^{-34} \text{ Js}}{4 \times 10^8 \text{ m/s}} = 1.6 \times 10^{-42}$$

So at a glance its a wide range

∞ to 1.6×10^{-42}

Why can't we see this wavelength?
Can we do human diffraction patterns?

Human Diffraction



$$p = \frac{h}{\lambda} = mv \quad \Rightarrow \quad v = \frac{h}{m\lambda} \quad \text{(non-relativistic)}$$

$$= \frac{6.6 \times 10^{-34}\ \text{J s}}{100\ \text{kg}\ \ 1\ \text{m}}$$

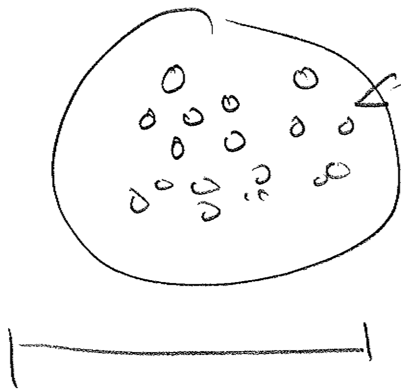
$$= 6.6 \times 10^{-36}\ \text{m/s}$$

Time to travel 10 m $\sim \boxed{10^{36}\ \text{s}}$

Current Age of the universe

$$\boxed{10^{17}\ \text{s}}$$

Soot Particles



10^9 C atoms

100 nm

total $m = 2 \times 10^{-17}$ kg

$$v_p = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2 \times 1.38 \times 10^{-23}\ \text{J/K} \cdot 300\ \text{K}}{2 \times 10^{-17}\ \text{kg}}}$$

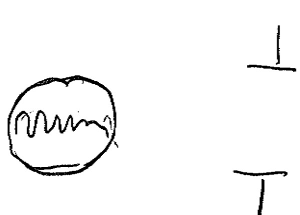
$$= 0.02\ \text{m/s}$$

$$p = 4 \times 10^{-19}\ \text{kg m/s}$$

$$\lambda = \frac{h}{p} = \frac{6.6 \times 10^{-34}\ \text{J}\cdot\text{s}}{4 \times 10^{-19}\ \text{kg m/s}}$$

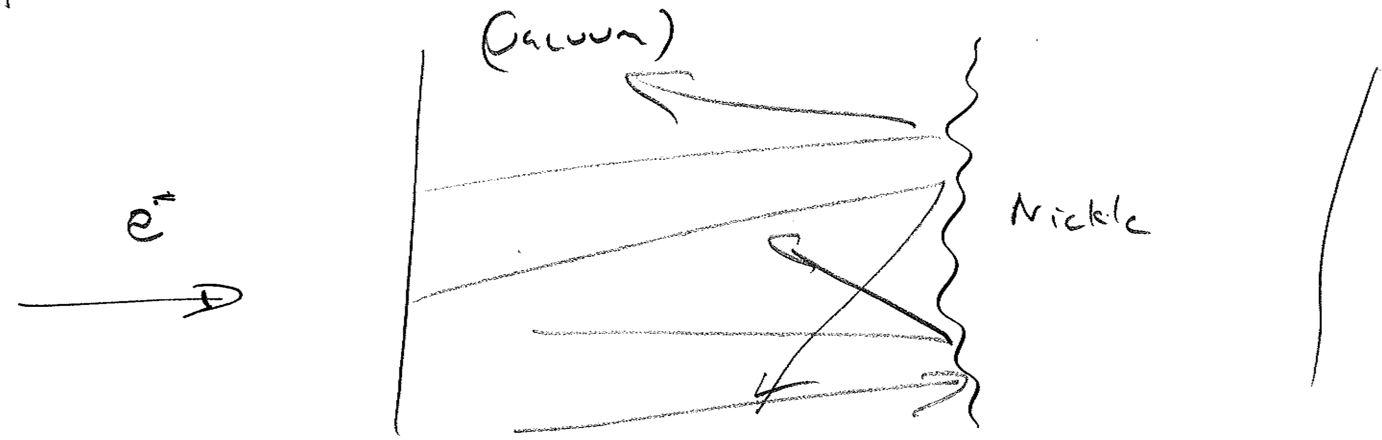
$$= 1.65 \times 10^{-15}\ \text{m}$$

* In general λ for macroscopic particles moving at "any" appreciable speed is small compared to their size



(This is math's way of telling us model doesn't apply here)

Davisson - Germer "Accidental" Discovery

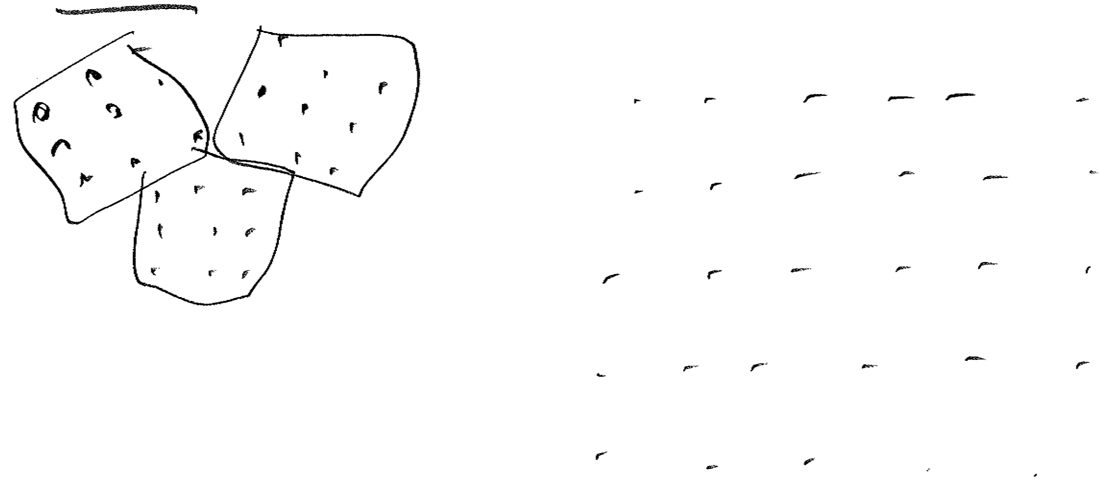


Diffuse reflection
due to rough surface

Accident Broke Vacuum, oxidized nickle...

To clean it, they heated sample
to high temperature...

"poly crystaline" structure converted to
<u>single</u> <u>crystals</u> over beam width



Turned out they built perfect
e^- interference experiment!!!

Useful recast:

$$\lambda = \frac{h}{p} \qquad K = \frac{p^2}{2m} \quad \text{(non-relativistic!)}$$

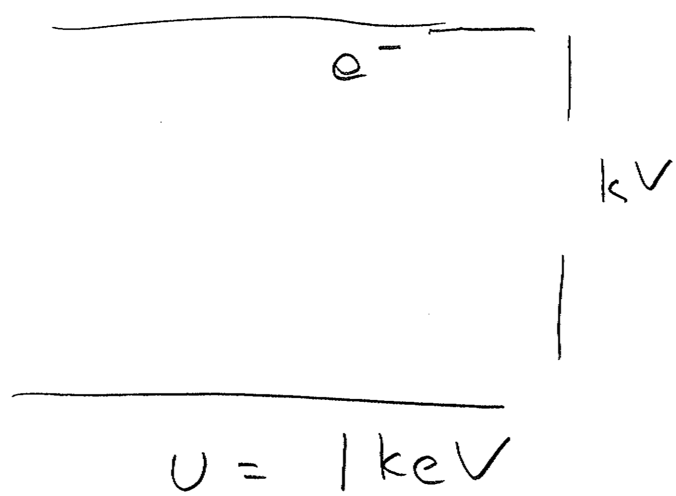
$$\Rightarrow p = \sqrt{2mK}$$

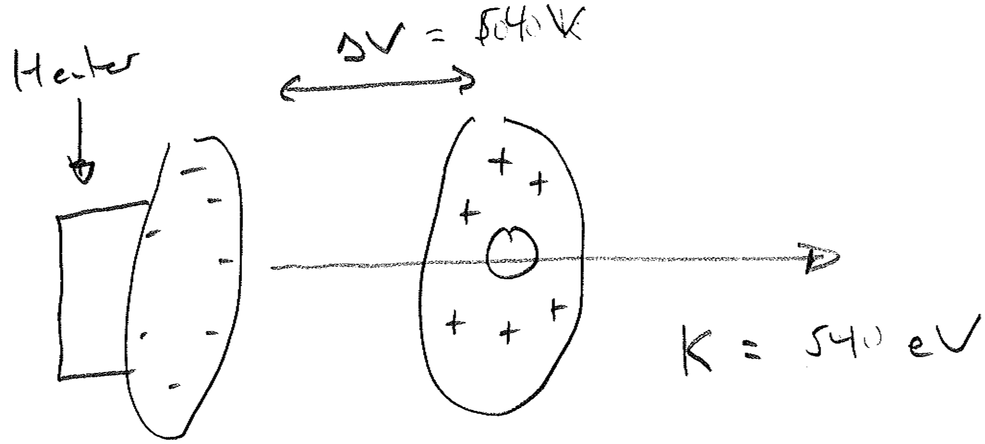
$$\lambda = \frac{h}{\sqrt{2mK}} = \frac{hc}{\sqrt{2mc^2 K}}$$

Why? $hc = 1.240 \text{ eV} \cdot \text{nm}$

$m_e = 511 \text{ keV}$

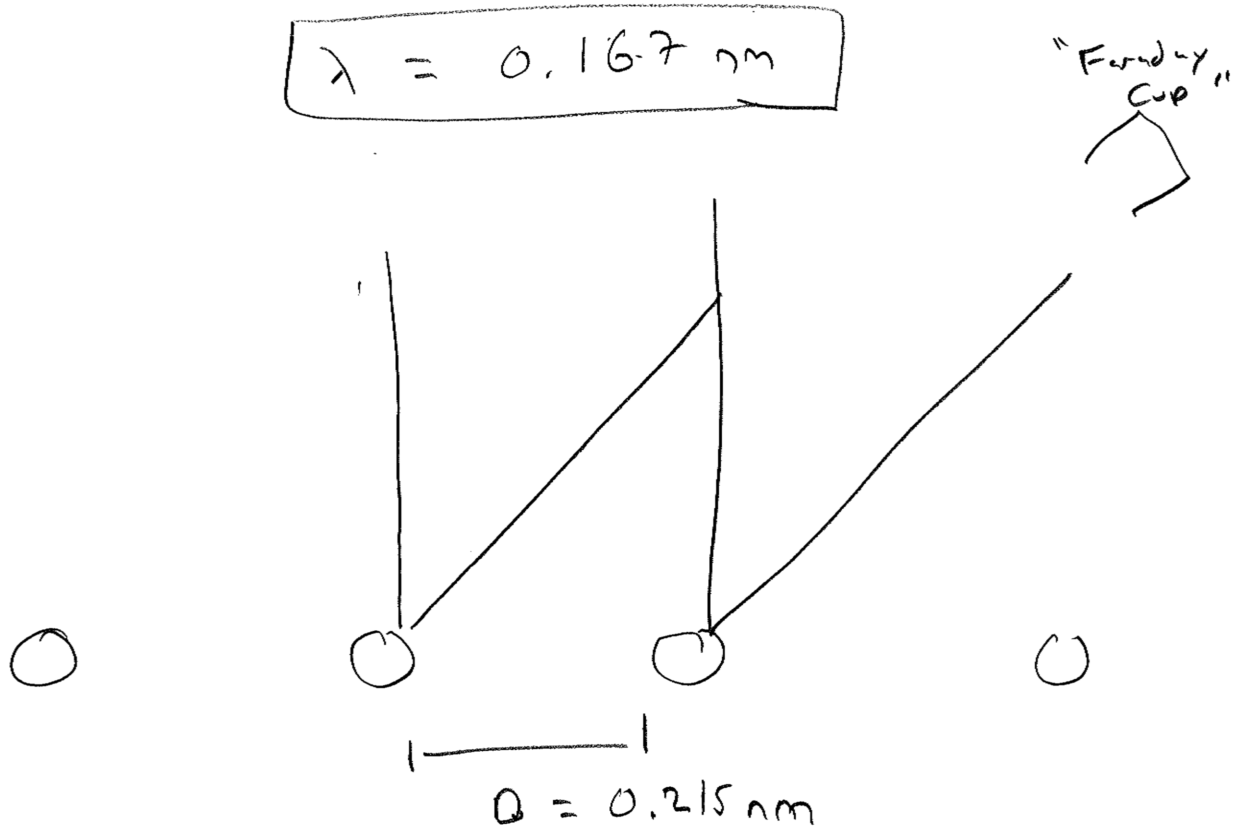
$K = ?$





$$\lambda = \frac{hc}{\sqrt{2Kmc^2}} = \frac{1240 \text{ eV nm}}{\sqrt{2\,(54\text{ eV})\,(511{,}000\text{ eV})}}$$

$$\boxed{\lambda = 0.167 \text{ nm}}$$



$$\frac{\lambda}{D} = 0.77 \qquad \theta_1 = \sin^{-1}\left(\frac{\lambda}{D}\right) = 0.87 = 50^\circ$$

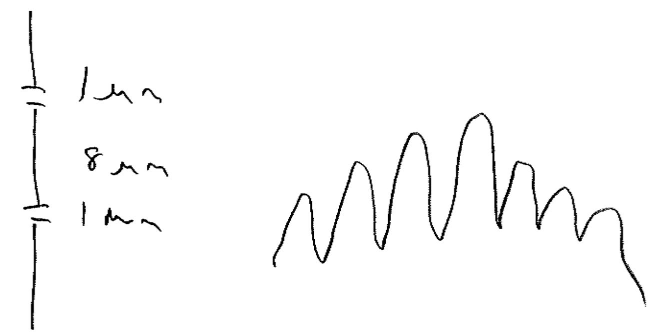
$\Rightarrow$ only one peak

More Experiments

Carnal + Mlynek

He

$\lambda = 0.1\,nm$



LQ5

Recap Experiments:

Classical Behavior:

- Light Interference patterns
- Electron charges trajectories

New Behavior

Electron Interference Patterns

Light Quantization

→ B.B. radiation

→ Photo-Electric Effect

→ Compton Scattering

* Puzzling New Behavior...

→ People puzzled over the implications

→ made incremental progress

→ And mistakes...

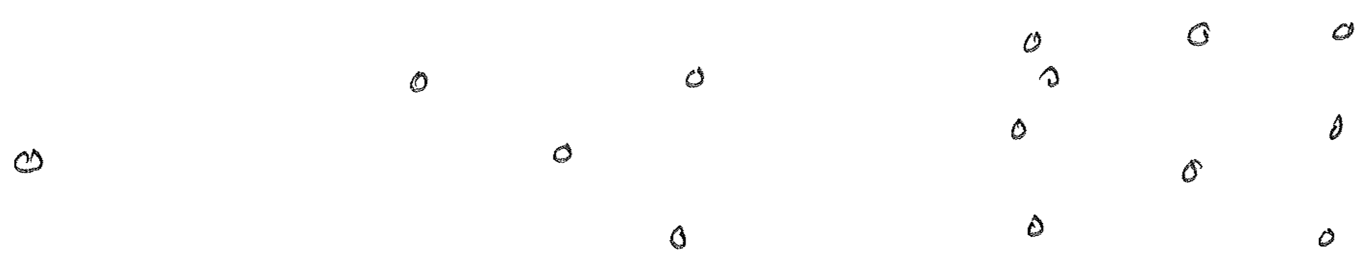
→ We get to skip to the answer!

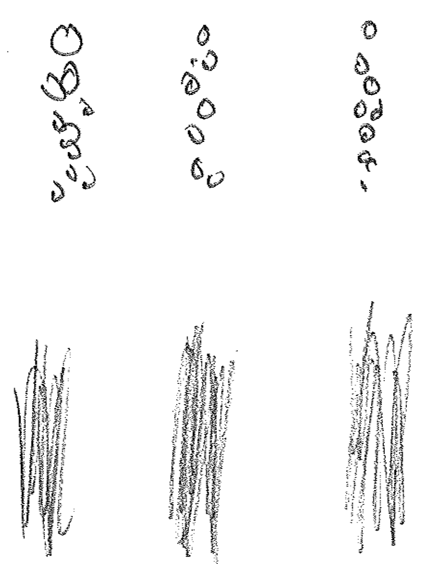
* → Suspend Skepticism, understand the new theory, see if it makes new, correct predictions! *

Quanton: like all forms of matter/energy, shows properties of particle and wave (not widely used!)

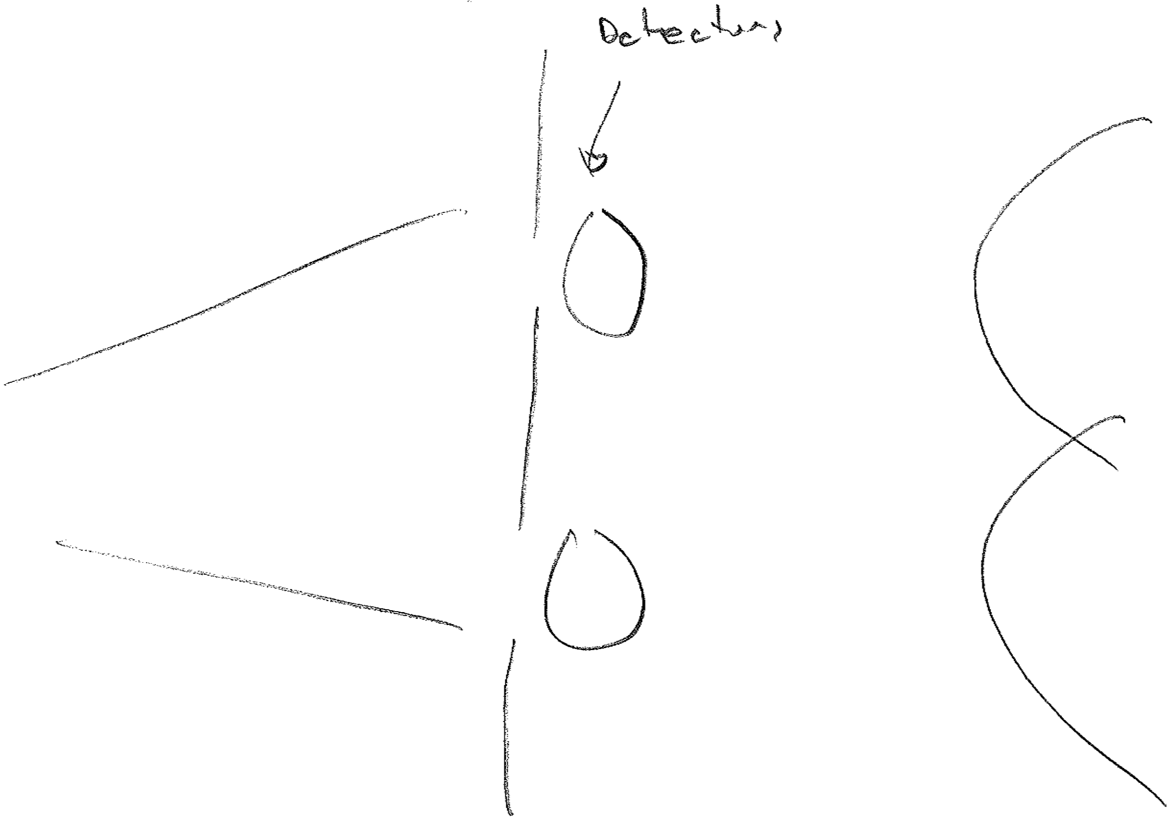
⇒ Technically "particle" refers now to both γ's, e's, etc... but it is a loaded term ——

Single Q Interference:



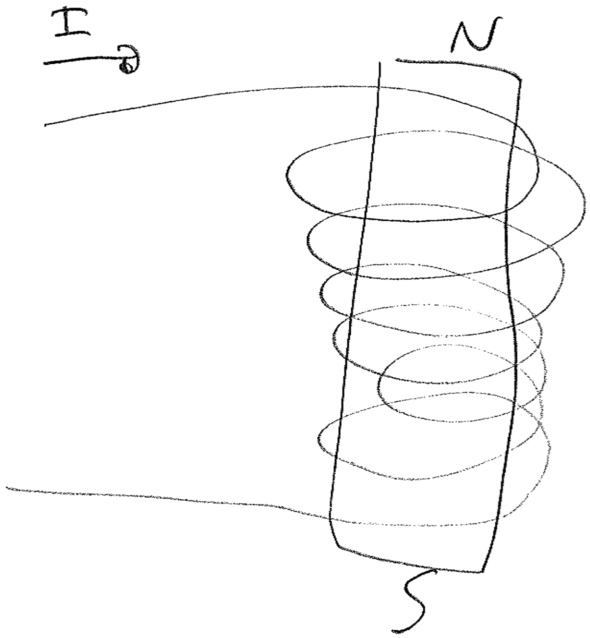


Neither wave or particle model works!



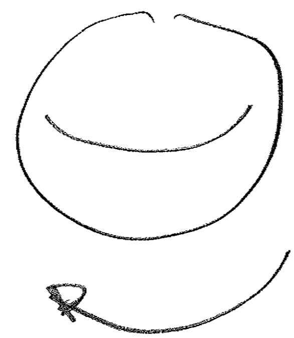
Attempting to measure which slit causes the interference pattern to dissappear ...

"Spin Hand Wave"



"Magnetic moment"

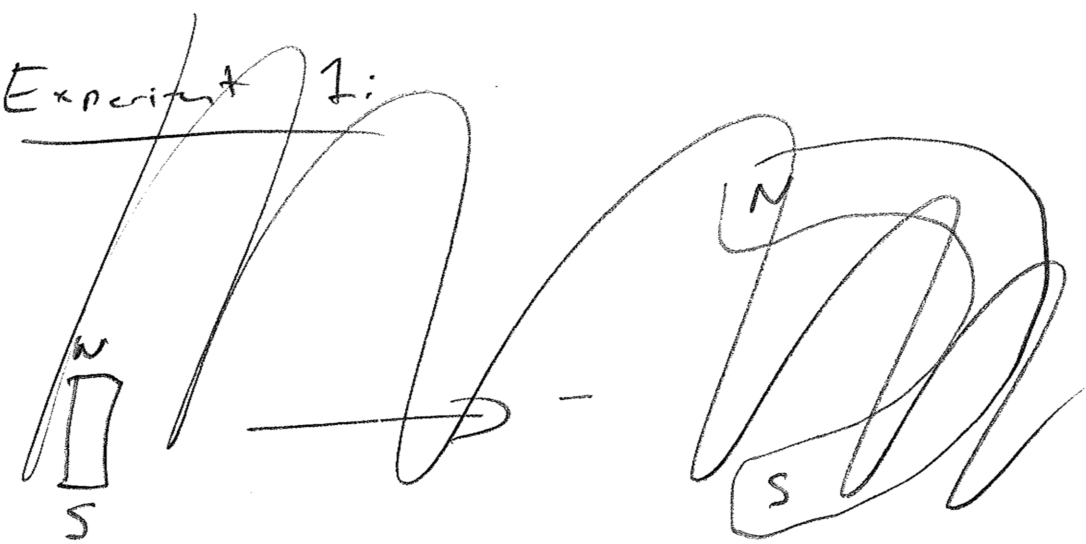
Likewise:



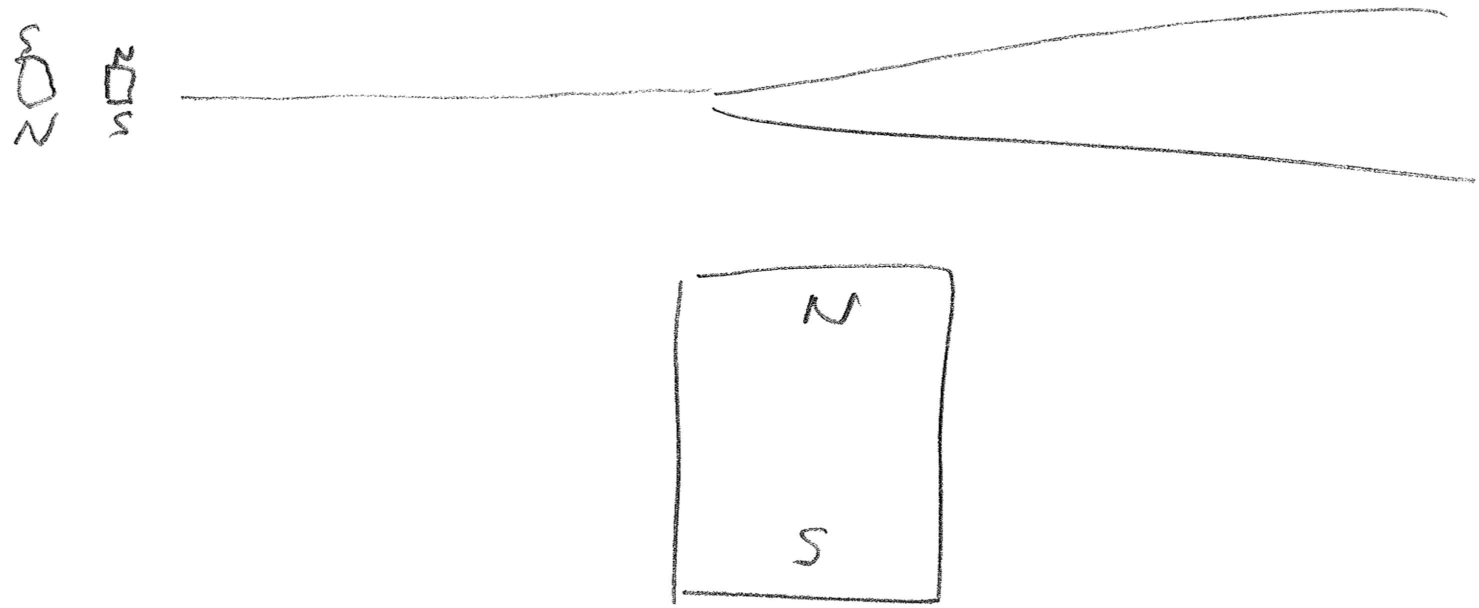
Net Charge
+ Angular momentum

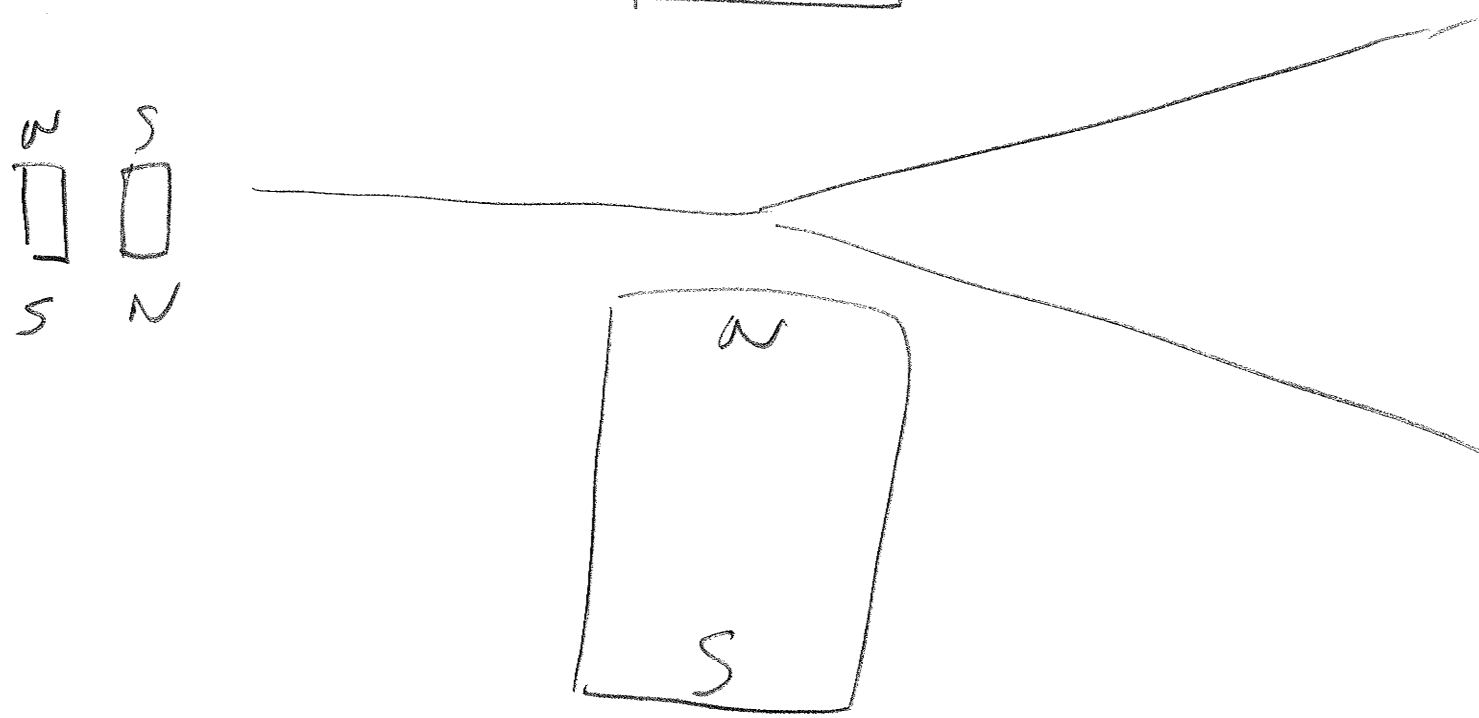
= magnetic moment

Experiment 1:

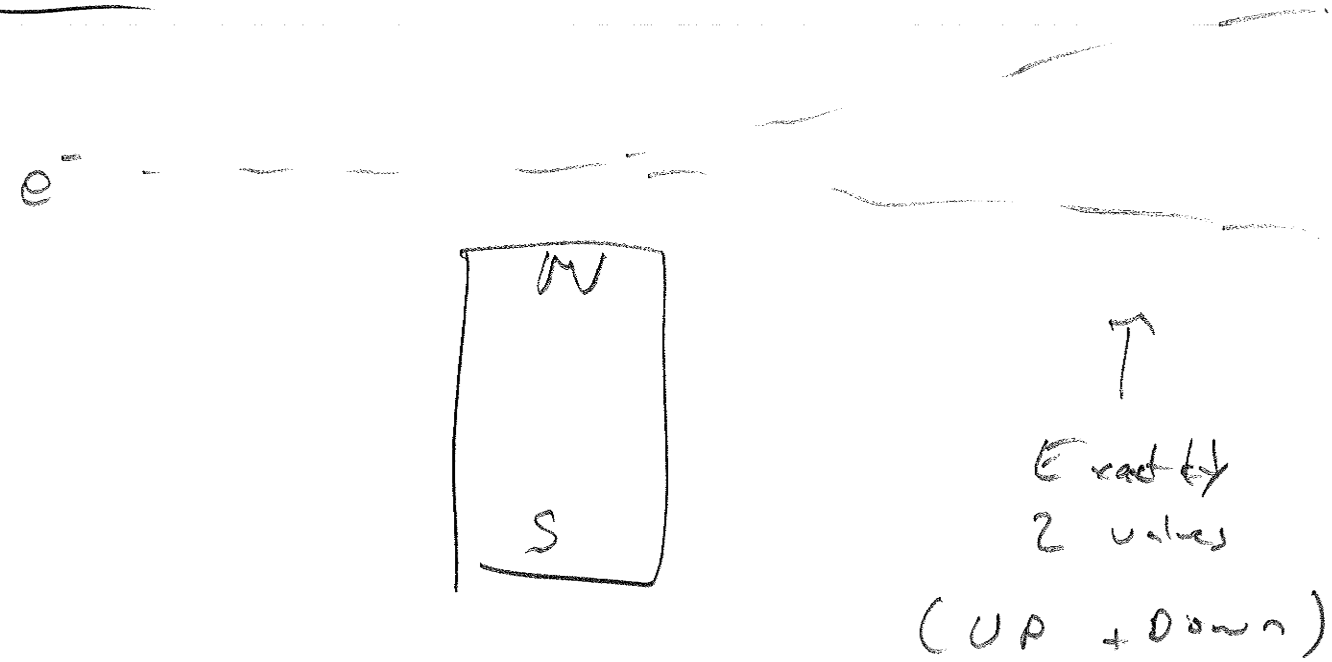


Exp 1





Exp 2



Q 6 - P1

A bit of philosophy:

Surprise! Turns out universe does not work in ways that are obvious and intuitive to our "ape-brains" that evolved to find the most bananas.

I guess atoms dont act like bananas...

Here's the amazing thing:

→ We developed logic + math + scientific method

→ So far that still works !!!

→ Which is the only reason our ape-brains can comprehend: black-holes, Higgs Boson's, Dark matter, ...

* State theory in Mathematical Terms
→ Test experimentally

QM predicts probabilities of possible outcomes of repeatable experiments that measure "observable" properties of a quanton

→ These observable properties are any measurement we might want to make e.g.

S_x, x, p, L, E

→ Differ from intrinsic properties of a quanton such as electron's charge or mass (which are same for all such quantons)

QM accomplishes this by providing a complete description of state of a quanton which we call the "state-vector" generally written

$$|\psi\rangle$$

Eigen Values + Eigen-Vectors

Our experiments showed that repeated measurements of the same observable always gave the same result. Conclusion:

Let $|a\rangle$ be the state vector after a measurement yields value "a". Since repeated measurements of $|a\rangle$ yield same result, we write

$$O|a\rangle = a|a\rangle$$

"Observable O acting on state-vector $|a\rangle$ yields the eigen value "a" times the state-vector $|a\rangle$."

We say that $|a\rangle$ is an eigen vector with eigen-value "a".

("Eigen" is german for "characteristic".)

* If outcome had been "b" quanton would instead be in eigen-vector $|b\rangle$, and

$$O|b\rangle = b|b\rangle$$

"Before" the measurement --.

→ First measurement had many possible outcomes

⇒ NOT AN EIGEN-VECTOR.

* But * turns out that quite generally, the eigen-vectors are "complete", (like sine and cosine adding to make any function) and we can write any state vector of our quanton as

$$|\psi\rangle = c_a|a\rangle + c_b|b\rangle + c_c|c\rangle + c_d|d\rangle$$

complex number coefficients

(→ Exactly which observables provide the complete description of the state varies from case to case... in fact there are generally multiple possible choices!)

Once we choose though, we can write

$$|\psi\rangle = \begin{bmatrix} c_a \\ c_b \\ c_c \\ c_d \end{bmatrix} \qquad |a\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad |b\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Measurements

When we make a measurement O of our state

$$|\psi\rangle = c_a|a\rangle + c_b|b\rangle + c_c|c\rangle + c_d|d\rangle$$

the probability of each outcome is the absolute square of the coefficient

$$Pr(\text{outcome "a"}) = |c_a|^2$$

$$Pr(\text{outcome "b"}) = |c_b|^2$$

We write this as "inner product"

$$Pr(a) = \left|\langle a|\psi\rangle\right|^2$$

Backwards sets complex conjugated.

$$= \left| \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{*} \cdot \begin{bmatrix} c_a \\ c_b \\ c_c \\ c_d \end{bmatrix} \right|^2$$

$$= |c_a|^2$$

Collapse

The outcome of the measurement of $\mathcal{O}$ determines the observable value by "collapsing" the state-vector to one of the eigen-vectors.

$$|\psi\rangle \longrightarrow |a\rangle \quad \text{w/ prob } |c_a|^2$$

$$|\psi\rangle \longrightarrow |b\rangle \quad \text{w/ prob } |c_b|^2$$

...

Self Consistency

Note this is self consistent ... after measuring "a" outcome we have state

$$|a\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Future measurements of O yield

$$Pr(a) = |\langle a|a\rangle|^2 = \left| \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^* \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right|^2$$

$$= 1$$

"a" with 100 % probability ...

Superposition Rule:

If we preform a measurement, but keep only certain outcomes (e.g. "a" and "b") then recombine in a way that it is impossible to know which occurred,

$$|\psi_{rc}\rangle = c_a' |a\rangle + c_b' |b\rangle$$

$$c_a' = \frac{c_a}{|c_a|^2 + |c_b|^2} \qquad c_b' = \frac{c_b}{|c_a|^2 + |c_b|^2}$$

(where $c_a = \langle a | \psi \rangle$) (where $c_b = \langle b | \psi \rangle$)

QGA.1

S.G. Example

S_z outcome is up or down, given (for now) values $+1$ or -1.

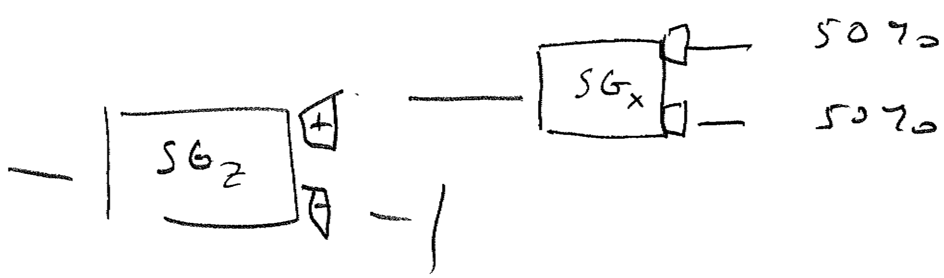
$|+z\rangle$, $|-z\rangle$ are the eigenvectors, with

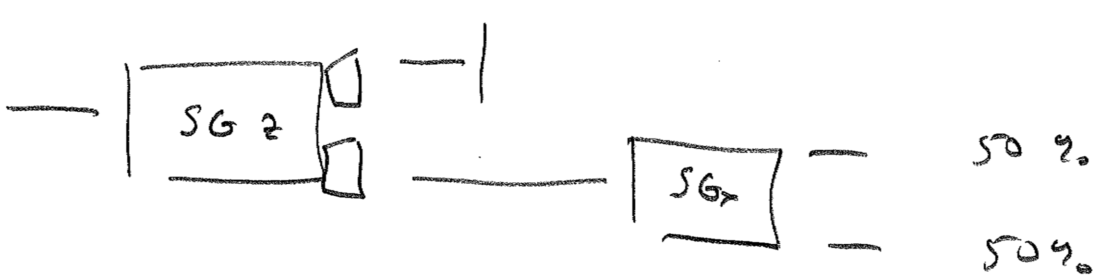
$$S_z\,|+z\rangle = (+1)\,|+z\rangle$$
$$S_z\,|-z\rangle = (-1)\,|+z\rangle$$

$$|+z\rangle = \begin{bmatrix}1\\0\end{bmatrix} \qquad |-z\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

Now what about S_x?

Recall:





This tells us;

$$|\langle x\uparrow | z\uparrow\rangle|^2 = |\langle x\downarrow | z\uparrow\rangle|^2 = \frac{1}{2}$$

$$|\langle x\uparrow | z\downarrow\rangle|^2 = |\langle x\downarrow | z\downarrow\rangle|^2 = \frac{1}{2}$$

$$|x\uparrow\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \langle x\uparrow | x\downarrow\rangle = ac + bd = 0$$

$$|x\downarrow\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\langle x\uparrow | z\uparrow\rangle = \begin{bmatrix} a \\ b \end{bmatrix}^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a^* = \cancel{\tfrac{1}{2}}$$

$$\langle x\downarrow | z\uparrow\rangle = \begin{bmatrix} c \\ d \end{bmatrix}^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c^*$$

$$\langle x\uparrow | z\downarrow\rangle = \begin{bmatrix} a \\ b \end{bmatrix}^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b^*$$

$$\langle x\downarrow | z\downarrow\rangle = \begin{bmatrix} c \\ d \end{bmatrix}^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = d^*$$

$$|a|^2 = |b|^2 = |c|^2 = |d|^2 = \frac{1}{2}$$

Will show in HW, that

$$|\psi'\rangle = e^{i\phi}|\psi\rangle$$

yields same predictions as $|\psi\rangle$...

So we are free to choose overall phase of $|x\downarrow\rangle$, so choose

$$\cancel{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}} \qquad \cancel{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{i\alpha} \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\beta} \\ 1 \end{bmatrix} \qquad a = e^{i\alpha}$$

$$\langle x\uparrow | x\downarrow\rangle = \frac{1}{2}\begin{bmatrix} e^{i\beta} \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ e^{i\alpha} \end{bmatrix} \Rightarrow e^{-i\beta} + e^{i\alpha} = 0$$

$$1 \quad - 1 \quad = 0$$

$$i \quad - i \quad = 0$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(Now Play)

Recap:

For any measurement we make, there are eigenvectors associated with each possible outcomes. If O has a, b, c, d as outcomes

$$O|a\rangle = a|a\rangle$$

("Measuring O for state a results in answer "a" and leaves state unchanged")

Any state can generally be written

$$|\psi\rangle = C_a|a\rangle + C_b|b\rangle + C_c|c\rangle + C_d|d\rangle$$

The probability that it will be found in state $|a\rangle$ is $|C_a|^2$.

If we choose the states $|a\rangle, |b\rangle, |c\rangle, |d\rangle$ as basis vectors we can write

$$|\psi\rangle = \begin{pmatrix} C_a \\ C_b \\ C_c \\ C_d \end{pmatrix} \qquad |a\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C_a = \langle a|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} C_a \\ C_b \\ C_c \\ C_d \end{pmatrix} = C_a$$

$$\boxed{C_a = \langle a|\psi\rangle}$$

Normalization:

$$\langle a|a\rangle \equiv \begin{bmatrix}1\\0\\0\\0\end{bmatrix}^* \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 1$$

$$\langle b|b\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}^* \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 1$$

$$\langle \psi|\psi\rangle = \begin{bmatrix}c_a\\c_b\\c_c\\c_d\end{bmatrix}^* \begin{bmatrix}c_a\\c_b\\c_c\\c_d\end{bmatrix}$$

$$|c_a|^2 + |c_b|^2 + |c_c|^2 + |c_d|^2 = 1$$

We normalize all state vectors, but interpretation varies:

$\langle a|a\rangle = \langle b|b\rangle = 1$ means prob $|a\rangle$ is in state $|a\rangle$ is 100%

$\langle \psi|\psi\rangle = |c_a|^2 + |c_b^2| + |c_c|^2 + |c_d|^2 = 1$

$\Rightarrow$ prob in any state is 100%

↳ (Only true if <u>complete</u> set of unit vectors is used...)

A property of inner products ...

$$\langle \psi | \phi \rangle = \langle \phi | \psi \rangle^*$$

Proof:

$$\langle \psi | \phi \rangle = \begin{bmatrix} c_a \\ c_b \\ c_c \\ c_d \end{bmatrix}^* \begin{bmatrix} d_a \\ d_b \\ d_c \\ d_d \end{bmatrix}$$

$$= \begin{bmatrix} d_a \\ d_b \\ d_c \\ d_d \end{bmatrix} \begin{bmatrix} c_a \\ c_b \\ c_c \\ c_d \end{bmatrix}^*$$

$$= \left[\begin{bmatrix} d_a \\ d_b \\ d_c \\ d_d \end{bmatrix}^* \begin{bmatrix} c_a \\ c_b \\ c_c \\ c_d \end{bmatrix} \right]^*$$

$$= \left[\langle \phi | \psi \rangle \right]^*$$

$$= \langle \phi | \psi \rangle^*$$

The continuum ...

In discrete case we could write

$$|\psi\rangle = c_a|a\rangle + c_b|b\rangle + c_c|c\rangle + c_d|d\rangle$$

$$= \sum_i |i\rangle \cdot c_i$$

where $c_i = \langle i|\psi\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}^* \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{bmatrix} \left(= c_i\right)$

So $|\psi\rangle = \sum_i |i\rangle \langle i|\psi\rangle$

Now suppose the measurement we want to make is position along x. x can take on any value along the x-axis ... any real number

$\sum_i$ only counts integers

$$\sum_i \longrightarrow \int dx$$

So $|\psi\rangle = \int dx\ |x\rangle \langle x|\psi\rangle$

what is this ?

What is $\langle x|\psi\rangle$???

1) Where as $\langle i|\psi\rangle$ gives "n" complex coefficients,

$\langle x|\psi\rangle$ is defined for any x ...

Its a complex valued function

$$\boxed{\langle x|\psi\rangle \equiv \psi(x)}$$

2) By analogy with $c_a = \langle a|\psi\rangle$

tempted to say

$|\langle x|\psi\rangle|^2$ is prob at position x..

That's mostly correct ... with some clean up!

Problem 1:

Since x is continuous, the probability that we are at any exact position "x" is zero ...

Resolution:

$$|\psi\rangle = \sum_i |i\rangle\langle i|\psi\rangle \qquad |\psi\rangle = \int dx\, |x\rangle\langle x|\psi\rangle$$

$$Pr(i) = |\langle i|\psi\rangle|^2 \qquad Pr(x_1 < x < x_2) = \int_{x_1}^{x_2} dx\, |\langle x|\psi\rangle|^2$$

$$\psi(x) \equiv \langle x|\psi\rangle$$

$|\psi(x)|^2$ is PDF in x

Normalization ...

$$\langle\psi|\psi\rangle = 1$$

$$|\psi\rangle = \int_{-\infty}^{\infty} dx \; |x\rangle\langle x|\psi\rangle$$

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx \; \langle\psi|x\rangle\langle x|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx \; \psi^*(x)\,\psi(x)$$

$$= \int_{-\infty}^{\infty} dx \; |\psi(x)|^2 \qquad = 1$$

Our interpretation: probability that quanton is <u>somewhere</u> is <u>100%</u>!

A bit more on time-dependence ...

E is an observable, and there are therefore energy eigenvectors.
The measurement we make (operator) is called "the Hamiltonian" and energy eigenstates are

$$\hat{H}|E_i\rangle = E_i|E_i\rangle$$

The equivalent of Newton's Law for QM is

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

For energy eigenstates this is easy!

$$i\hbar\frac{\partial}{\partial t}|E_i\rangle = \hat{H}|E_i\rangle = E_i|E_i\rangle$$

$$\Rightarrow \frac{d}{dt}|E_i\rangle = -\frac{E_i}{\hbar}i\,|E_i\rangle$$

$$\Rightarrow \boxed{|E_i(t)\rangle = |E_i(0)\rangle\, e^{-i\frac{E_i t}{\hbar}}}$$

Time-Evolution Rule

One of the possible observables is always Energy of the quantum "E", so if we know state vector at

$$t = 0$$

$$|\psi(0)\rangle = c_1 |E_1\rangle + c_2 |E_2\rangle + \cdots$$

Time evolution is just

$$|\psi(t)\rangle = c_1 e^{-iE_1 t/\hbar} |E_1\rangle + c_2 e^{-iE_2 t/\hbar} |E_2\rangle + \cdots$$

Problem 2:

Dimensions:

$\langle \psi | \psi \rangle = 1$ dimensions of $|\psi\rangle$ and $\langle \psi|$

$\langle a | a \rangle = 1$ must be "1".

$$|\psi\rangle = \int dx \, |x\rangle \underbrace{\langle x|\psi\rangle}$$

dimensions of "x"

must each have dimension $\frac{1}{\sqrt{x}}$

Dimensions of discrete eigenvectors are 1.

But dimensions of continuous variables (x) must be $\frac{1}{\sqrt{x}}$

So $|\langle x|\psi\rangle|^2$ has dimensions $\frac{1}{x}$

Can't be a probability!

This is just like the Boltzman Distribution: not a probability, but a probability distribution

Free - Particles

De Broglie :
$$\lambda = \frac{h}{p}$$

$$\sin\left(2\pi \frac{x}{\lambda}\right) = \sin\left(\frac{2\pi}{h} p x\right)$$

$$\left(\hbar \equiv \frac{h}{2\pi}\right) \qquad = \sin\left(\frac{px}{\hbar}\right)$$

ψ is complex valued, so prefer to use

$$e^{i\theta} = \cos\theta + i\sin\theta$$

So wave equation should be

$$\psi(x) = \exp\left(i \frac{px}{\hbar}\right)$$

But momentum Eigenstate must also be <u>energy</u> Eigenvector, since

$$E = \frac{p^2}{2m}$$ for free particle!

So

$$\psi(x,t) = \psi(x) \cdot \exp\left(-i \frac{Et}{\hbar}\right)$$

$$= \exp\left(i\left(\frac{px - Et}{\hbar}\right)\right)$$

$$= \exp\left(i\left(kx - \omega t\right)\right)$$

$$k = \frac{p}{\hbar} \qquad \omega = \frac{E}{\hbar}$$

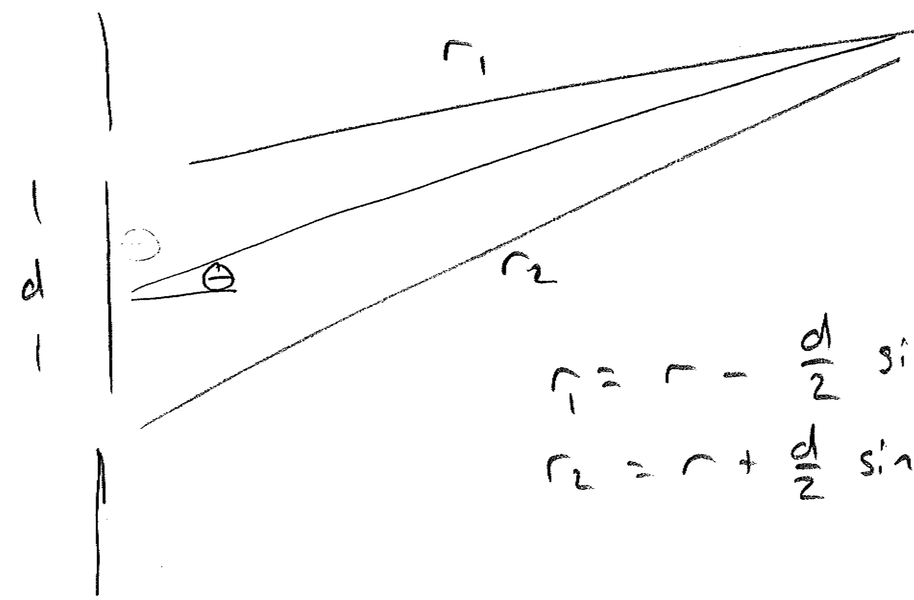
Interesting: $\omega = 2\pi f = \frac{E}{\hbar} = 2\pi \frac{E}{h}$

$$\boxed{E = hf}$$

So time-dependence reproduces Einstein's law for quantization of photons!

Double Slit Wave Function

Recall:



$$r_1 = r - \frac{d}{2}\sin\theta$$
$$r_2 = r + \frac{d}{2}\sin\theta$$

Assume spherical waves: $\psi_1 = \frac{A}{r_1}\exp\left(i(kr_1 - \omega t)\right)$

$$\psi = \left[\frac{A}{r_1}\exp(ikr_1) + \frac{A}{r_2}\exp(ikr_2)\right]\exp(-i\omega t)$$

$r_1 \sim r_2 \sim r$ (but not in periodic functions!)

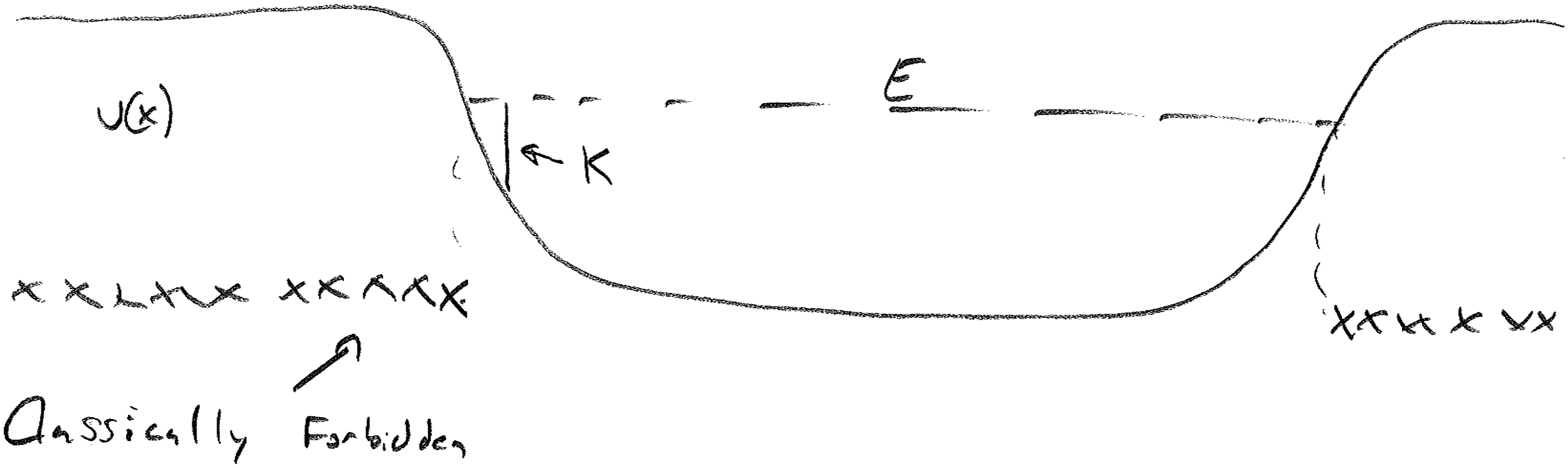
$$\psi \sim \frac{A}{r}\exp\left(i(kr - \omega t)\right)\left[\exp\left(ik\frac{d}{2}\sin\theta\right) + \exp\left(-ik\frac{d}{2}\sin\theta\right)\right]$$

$$= \frac{A}{r}\exp\left(i(kr-\omega t)\right)\cos\left(k\frac{d}{2}\sin\theta\right)$$

$$\psi^*\psi = \left(\frac{A}{r}\right)^2\cos^2\left(k\frac{d}{2}\sin\theta\right)$$

Q7

<u>Bound Systems:</u> Single Quanton in



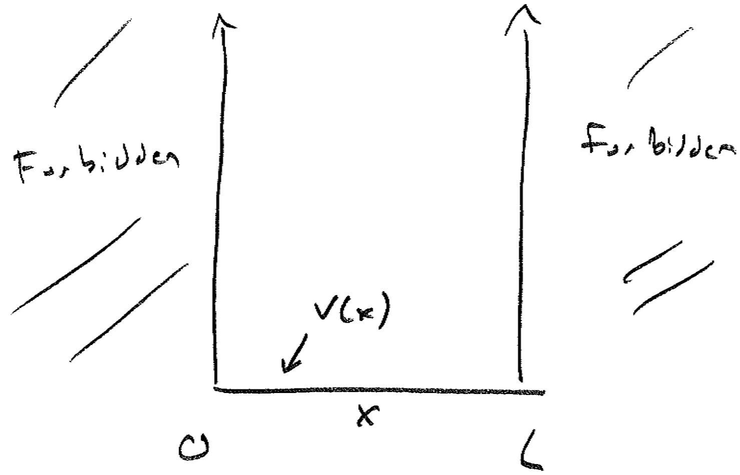
Most fruitful approach to QM bound systems is to look at the Energy Eigenstates

1) Easy to tell if bound or not!
2) Crucial observable (spectrum, next chapter)
3) Time evolution singles out E as important
4) E states are "stationary - states", e.g. $\sin(kx)\, e^{i\omega t}$. No change in $|\psi(x)|^2$ as function of time... Therefore, no <u>radiation</u> <u>of</u> <u>energy</u>.

Note that does change! $A \sin k_1 x\, e^{i\omega_1 t} + B \sin k_2 x\, e^{i\omega_2 t}$

Quanton in a box

Model:



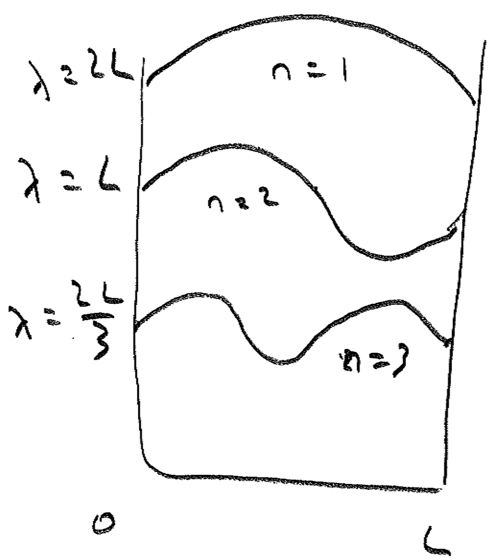
Insite:

1) Energy Eigenstates have fixed wavelength given by de Broglie relation

$$\lambda = h/p$$

2) $\psi(x)$ must vanish at boundaries

$$\psi(0) = \psi(L) = 0$$



Solution:

$$\psi(x) = A \sin kx$$

$$\lambda_n = \frac{2L}{n}$$

$$k = \frac{2\pi}{\lambda} = \frac{\pi n}{L}$$

What is Energy of each state?

$$E = \frac{p^2}{2m} \qquad p = \frac{h}{\lambda} = \frac{nh}{2L}$$

$$E = \frac{1}{2m}\left(\frac{nh}{2L}\right)^2 = \boxed{\frac{h^2 n^2}{8mL^2}} = E_n$$

Classical S.H.O.

$$F = -kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\,x(t)$$

Define $\omega = \sqrt{\frac{k}{m}}$

$$\frac{d^2x}{dt^2} = -\omega^2\,x(t)$$

Solution: $x = x_{tp}\cos(\omega t + \Delta\phi)$

$x_{tp} = \,?$

At t.p. $E = V = -\int F\,dx = \int kx\,dx = \frac{1}{2}kx^2$

$\omega^2 = k/m \Rightarrow k = m\omega^2$

$$E = \frac{1}{2}m\omega^2 x_{tp}^2$$

$$x_{tp} = \pm\sqrt{\frac{2E}{m\omega^2}}$$

* Unlike Particle in Box, t.p. not the same for all energy.

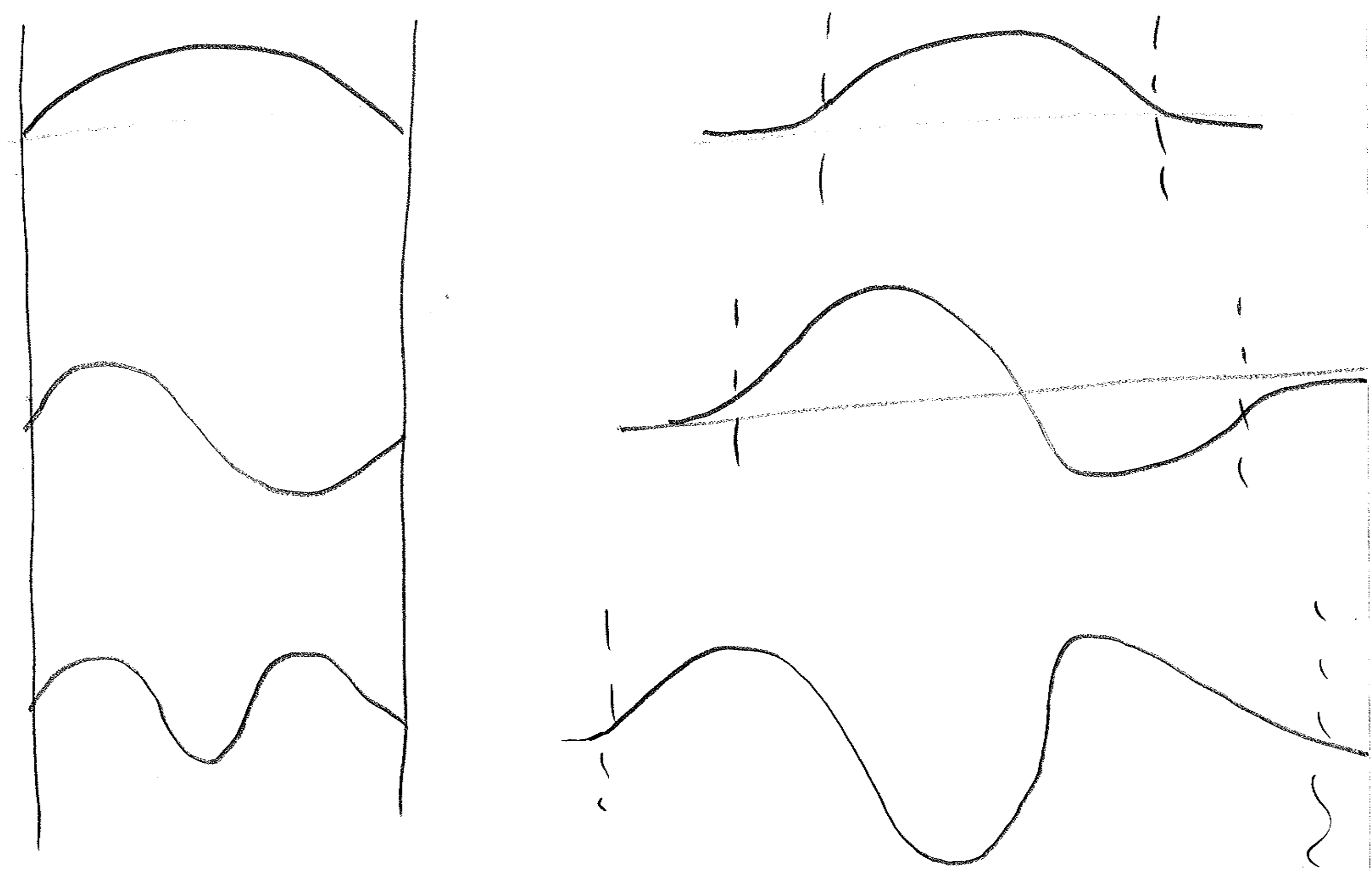
QM. Solution

Has to wait for chap Q10 ...

But analogous to particle in box except:

→ tp wider apart at higher energy

→ p is not constant, distorts sine waves



QM. Solution:

$$\boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right)}$$

3) Bohr's Beautiful Insight

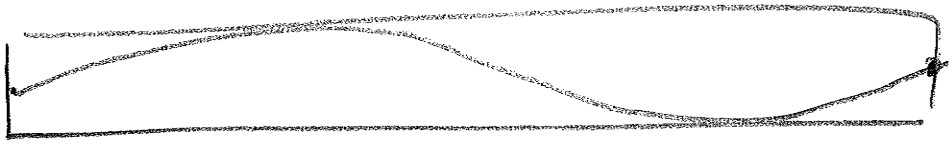
→ In QM position of particle is probabilistic, must be described by a wave function

→ For circular orbit, r is fixed, only Θ varies.

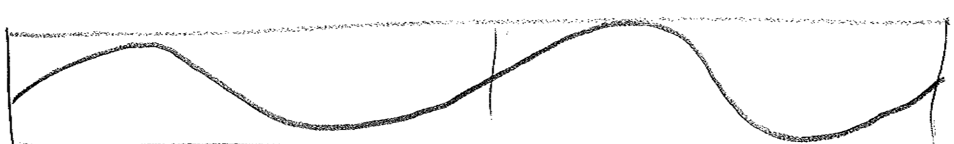
→ * Wave function must be single valued as we orbit complete the entire orbit.

$$\psi(\Theta) = \psi(\Theta + 2\pi n)$$

$$\Rightarrow \boxed{2\pi r = n\lambda}$$



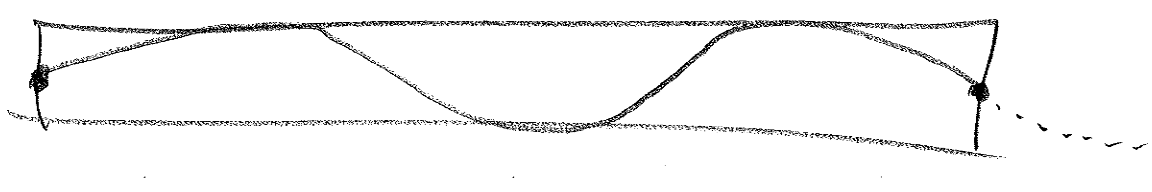
$n = 1$



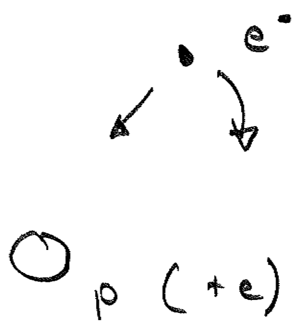
$n = 2$

. . .

Q: Why not



Bohr Model



$$F_{net} = -\frac{ke^2}{r^2}\hat{r}$$

(Coulomb's Law)
(like gravity ...)

A surprisingly <u>accurate</u>, highly intuitive, highly instructive model for hydrogen

1) Assume circular orbits, and governed by Classical Mechanics

$$|F_{net}| = \frac{ke^2}{r^2} = ma = mv^2/r$$

(both in $-\hat{r}$ direction)

$$\boxed{mv^2 = \frac{ke^2}{r}}$$

(SAVE)

2) De Broglie Relation:

$$\boxed{\lambda = h/p}$$

Combine (2) and (3)

$$2\pi r = n\lambda = nh/p$$

$$\Rightarrow \quad p = \frac{nh}{2\pi r} = \boxed{\frac{n\hbar}{r} = p} \qquad (4)$$

(SAVE!)

Combine (4) and (1):

$$mv^2 = k\frac{e^2}{r}$$

$$p^2 = m^2v^2 = mk\frac{e^2}{r} = \left(\frac{n\hbar}{r}\right)^2$$

$$\Rightarrow \quad \boxed{r_n = \frac{n^2\hbar^2}{mke^2} \equiv a_0 n^2}$$

Something Else Crucial:

For Energy Eigen States, angular momentum is well defined: (See (4))

$$\frac{n\hbar}{r} = p$$

$$L = |\vec{p} \times \vec{r}| = pr = n\hbar$$

Total Angular Momentum is just

$$L_n = n\hbar$$

Astounding!

Summary:

If we label Eigen states by "n", we have

$$\hat{H} |n\rangle = E_n |n\rangle$$

$$\hat{L} |n\rangle = n\hbar |n\rangle$$

$$\left(E_n = - \frac{mk^2e^4}{2n^2\hbar^2} \right)$$

Total Energy

Notice:

$$K = -\frac{1}{2} V(r) = \frac{1}{2} |V(r)|$$

So

$$E_{TOT} = K + V = \frac{1}{2} V(r)$$

$$E_{TOT} = -\frac{1}{2} K$$

$$E_{TOT} = -\frac{ke^2}{2r}$$

$$E_n = -\frac{ke^2}{2} \left(\frac{mke^2}{n\hbar} \right)$$

$$\boxed{E_n = -\frac{mk^2e^4}{2n^2\hbar^2}} \quad \left(= -\frac{ke^2}{2a_0 n^2} \right)$$

Spectra

Q8

One of our greatest experimental handles on the atomic scale world...

Visualization: Energy-Level Diagrams

$$E_n = \frac{h^2 n^2}{8mL^2}$$

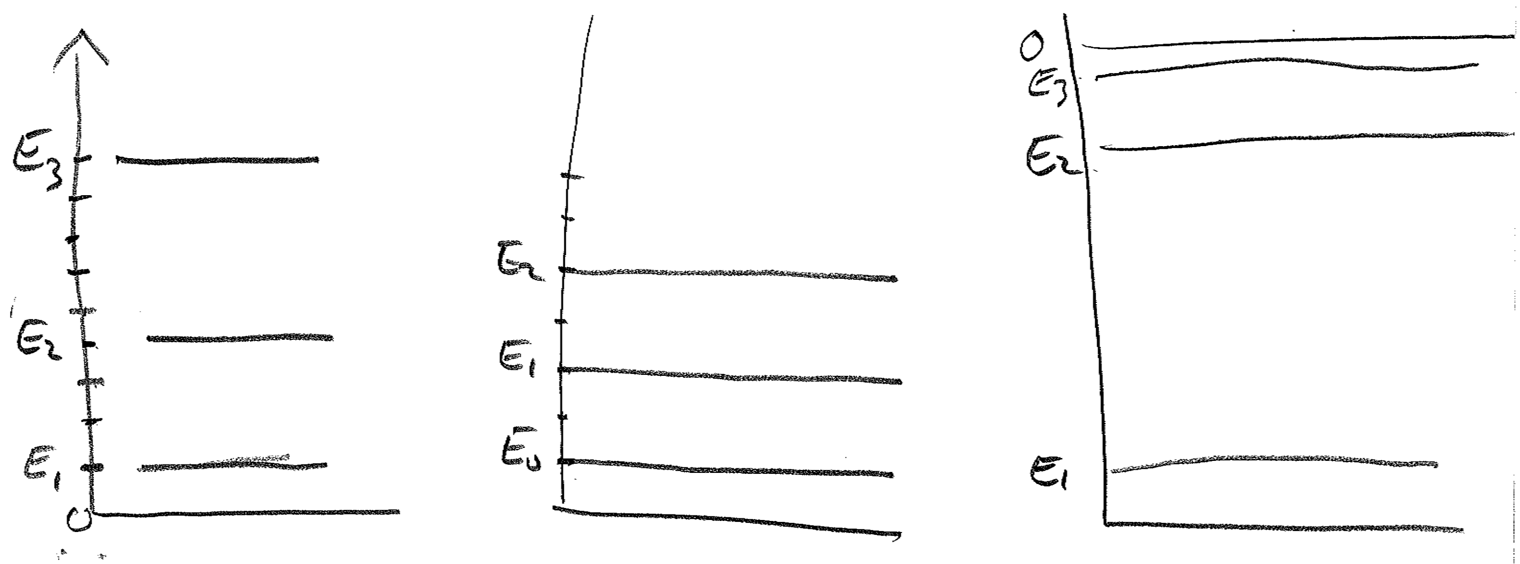
(Box)

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

(SHO)

$$E_n = -\frac{ke^2}{2a_0 n^2}$$

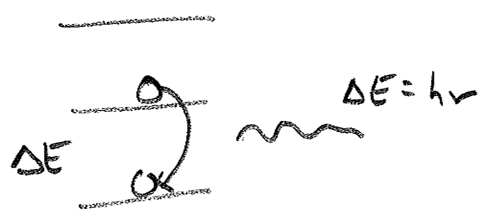
Bohr



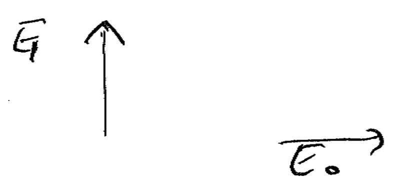
Spontaneous Emission

Charged quantons, if not in Energy Eigenstate, have wave function changing with time → radiate photons!

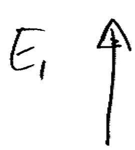
However, even charged quantons in an energy eigen state can spontaneously emit a photon:



How? Vector analogy

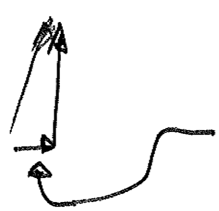


Pure



(zero prob of emission)

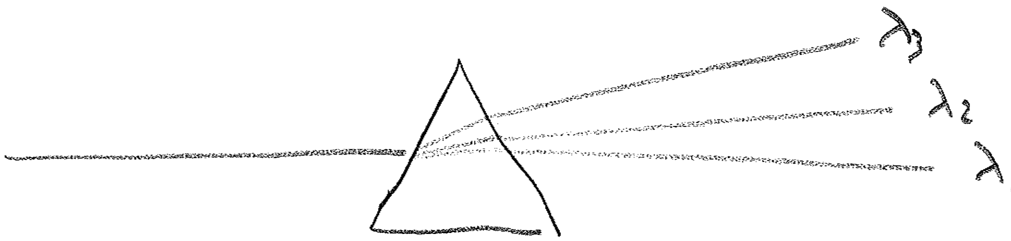
(small perturbation



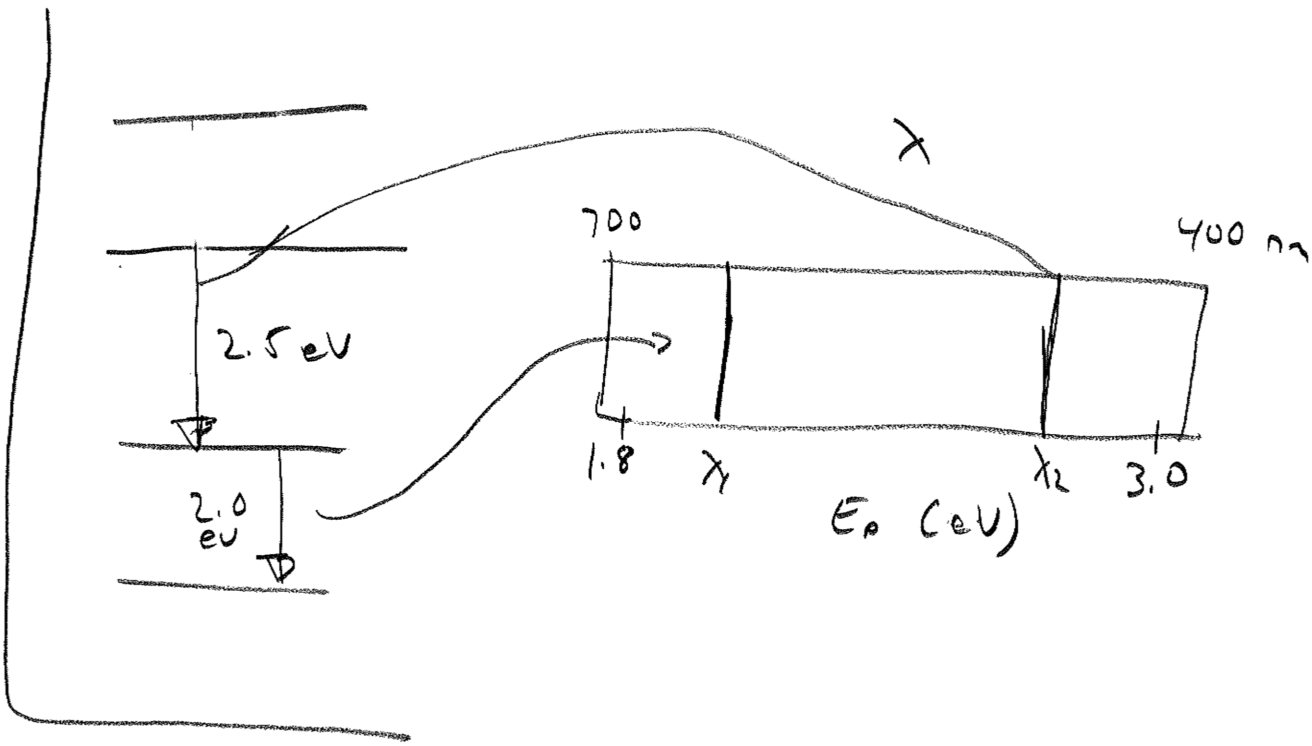
small prob of quantum being found in state E_0

Spectral Lines

Prism (Snells Law)

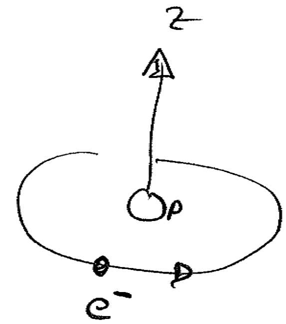


We can precisely measure wavelength of light.



Angular Momentum and Spin:

Bohr model $\Rightarrow$ $L_z = n\hbar$



(Recap:

$$n\lambda = 2\pi r$$

$$p = \frac{h}{\lambda} = \frac{nh}{2\pi r}$$

$$\Rightarrow pr = n\hbar \;)$$

Ours naive picture would have

$$L_y = L_x = 0 \quad \text{and} \quad L_z = n\hbar$$

Turns out this isn't the case ...

L_x, L_y are incompatible observables with L_z

(just like SG experiments showed, measuring L_x messes up L_z)

L $\quad$ L_z $\quad$ are compatible

l $\quad$ m $\quad$ (quantum numbers)

$$L = \sqrt{l(l+1)}\,\hbar \qquad l = 0, 1, 2, 3, \ldots$$

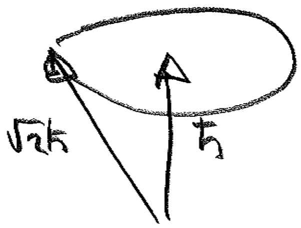
$$L_z = m\hbar \qquad m = -l, -l+1, \ldots, l$$

Examples:

$\ell = 1$

$L = \sqrt{\ell(\ell+1)}\,\hbar$

$= \sqrt{2}\,\hbar$



$m = 1, \quad L_z = \hbar$

$m = 0 \quad L_z = 0$

$m = -1 \quad L_z = -1$

$\ell = 2$

$m = 2 \quad L_z = 2\hbar$

$m = 1 \quad L_z = \hbar$

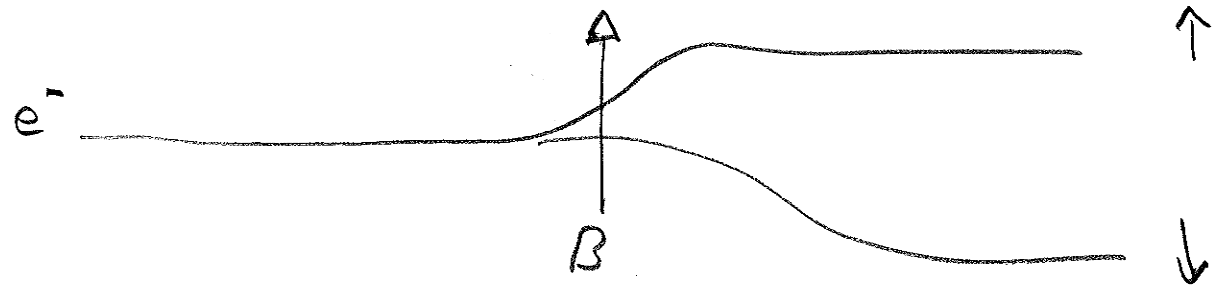
$m = 0 \quad L_z = 0$

$m = -1 \quad L_z = -\hbar$

$m = -2 \quad L_z = -2\hbar$

SG:

"Intrinsic" not "Orbital" angular momentum of electron:



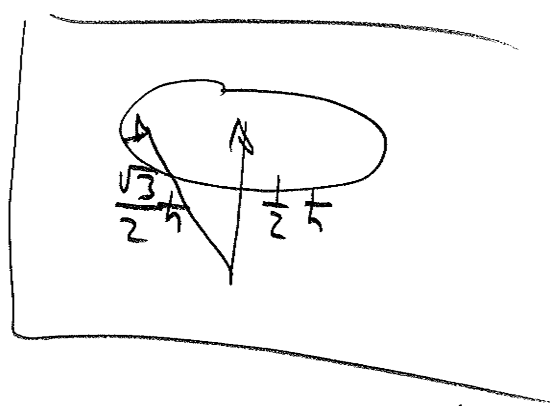
"Spin" has two possible z values ...
not possible using

$$m = -l, \ldots, l \qquad l = 0, 1, 2$$

$l = 0$ m: 0

$l = 1$ m: 3

$l = 2$ m: 5



Solution "s" $= \frac{1}{2}$

$$m_s = -s, \ldots, s = -\frac{1}{2}, \frac{1}{2}$$

$$S = \sqrt{s(s+1)} = \frac{\sqrt{3}}{2}$$

$$S_z = m_s \hbar = \pm \frac{1}{2}\hbar$$

Book makes different argument:

If phase of wave function is immaterial then one orbit only requires $\frac{\lambda}{2}$

$$\frac{n}{2}\lambda = 2\pi r$$

$$\Rightarrow \quad p = \frac{h}{\lambda} = \frac{n\hbar}{4\pi r}$$

$$pr = \frac{1}{2} n\hbar$$

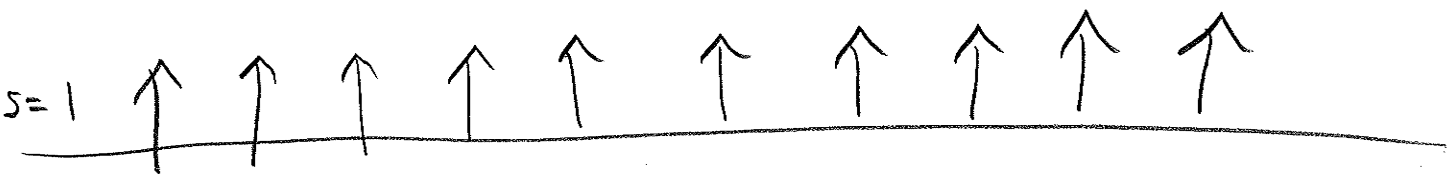
(I don't find this convincing..)

(But Hawking likes it, and he is way smarter than me!)

Bosons: $s = 0, 1, 2, \ldots$

Happily occupy the identical quantum state

$s = 1$



Spin of particles can be either

integer $s = 0, 1, 2, \ldots$

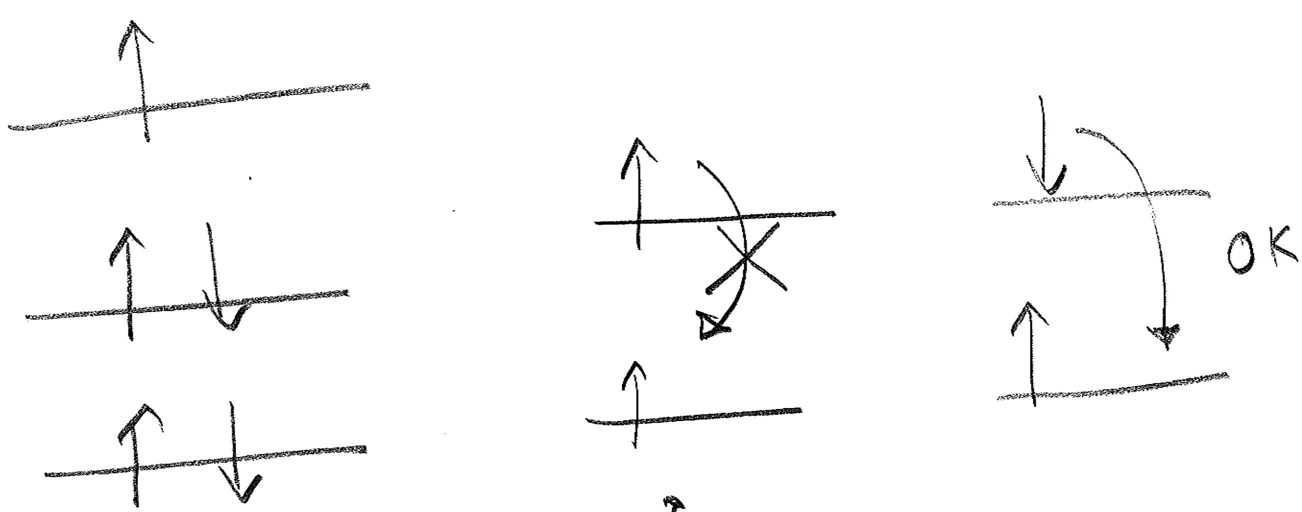
or half odd integer $s = \frac{1}{2}, \frac{3}{2}, \ldots$

Turns out they behave in

FUNDAMENTALLY different ways...

<u>FERMIONS:</u> $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

Obey Pauli Exclusion Principle



(spin flip
does occur
rarely)

Q9

The radial picture

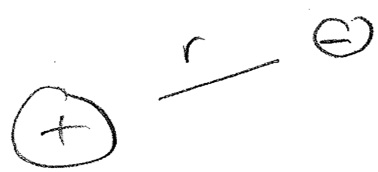
$$V(r) = -\frac{ke^2}{r}$$

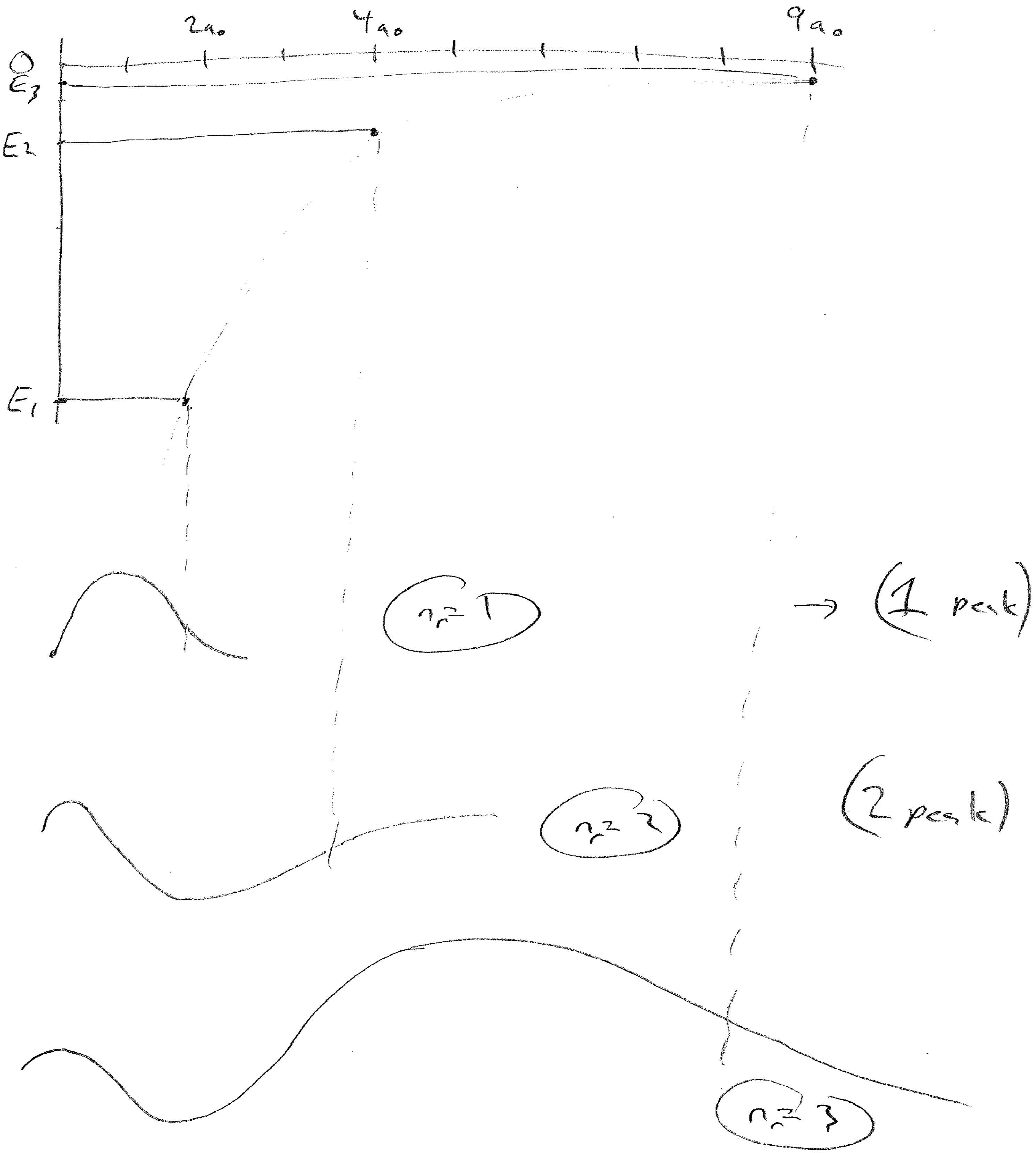
$$E_n = -\frac{ke^2}{2a_0 n^2}$$

Turning points:

$$V(r) = E_n$$

$$-\frac{ke^2}{r} = -\frac{ke^2}{2a_0 n^2} \Rightarrow \boxed{r = 2a_0 n^2}$$



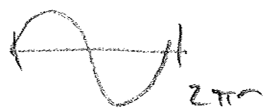


$2a_0$
$4a_0$
$9a_0$
0
E_3
E_2
E_1
$n_r = 1$
(1 peak)
$n_r = 2$
(2 peak)
$n_r = 3$

The angular part ...

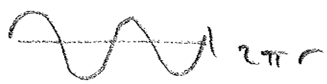
Recall $L_z = m\hbar$ $\quad (2\pi r = m\lambda)$

$m = 1$



(2 peaks)

$m = 2$



(4 peaks)

$m = 0$

(0 peaks)

* To have peaks in angular function, must have at least <u>one</u> peak in radial function...

$$n = n_r + l$$

$n = 1, 2, 3, \ldots$

$n_r = 1, 2, 3, \ldots$ (not zero)

$l = 0, 1, 2, \ldots$

$m = -l, \ldots, l$

Putting it together ...

Bohr found the energy quantum number "n" correctly ...

$n = 1, 2, 3, \ldots$ $\qquad E_n = -\dfrac{ke^2}{2a_0 n^2}$

We saw we can have peaks in radial position: (new quantum number)

$n_r = 1, 2, 3, \ldots$ (# radial peaks)

Or peaks in angular function, based on quantum number "m"

$\ell = 0, 1, 2$ $\qquad L = \sqrt{\ell(\ell+1)}\,\hbar$

$m = -\ell, \ldots, \ell$ $\qquad L_z = m\hbar$

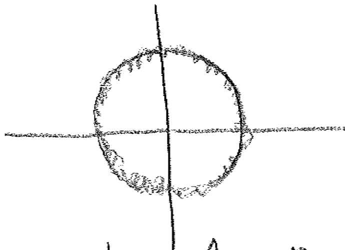
peaks $= 2 \times m$

$(2\pi r = m\lambda)$

...

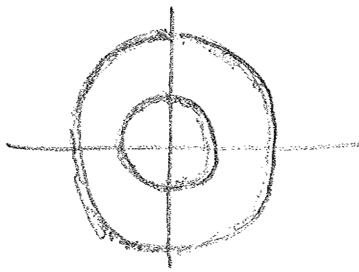
In pictures: (square of real part)

n = 1

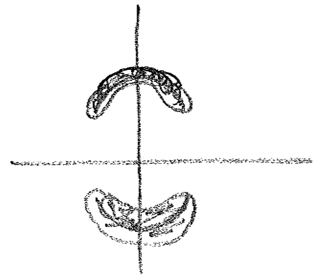


$n_r = 1 \quad \ell = 0$

n = 2

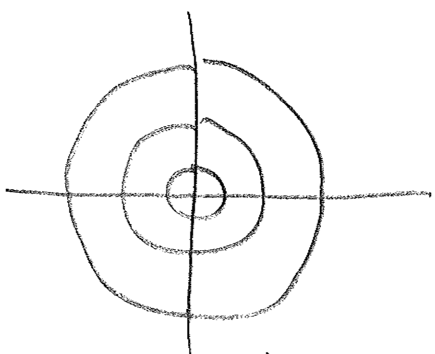


$n_r = 2 \quad \ell = 0$

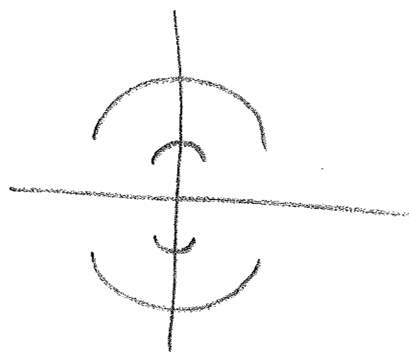


$n_r = 1, \ \ell = 1$

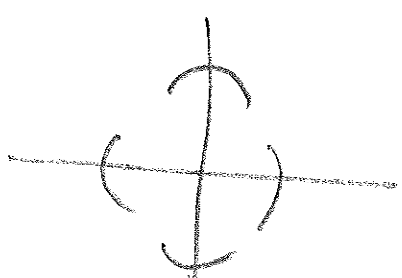
n = 3



$n_r = 3 \quad \ell = 0$



$n_r = 2 \quad \ell = 1$



$n_r = 1 \quad \ell = 2$

Why physics owns chemistry....

For every n, we have

$\ell = 0, \ldots, n-1$

$m = -\ell$ to ℓ $\quad$ ($2\ell+1$ states)

e^- spin $\uparrow$ or $\downarrow$ $\quad$ ($\times 2$ states)

"Notation"

$\ell = 0$ $\quad$ (s) $\quad \longrightarrow$ 2 states

$\ell = 1$ $\quad$ (p) $\quad \longrightarrow$ 6 states

$\ell = 2$ $\quad$ (d) $\quad \longrightarrow$ 10 states

$\ell = 3$ $\quad$ (f) $\quad \longrightarrow$ 14 states

$n=1$	$1s^1$							$1s^2$				
	$2s^1$	$2s^2$	$2p^1$	$2p^2$	$2p^3$	$2p^4$	$2p^5$	$2p^6$				
	$3s^1$	$3s^2$	$3p^1$	$3p^2$	$3p^3$	$3p^4$	$3p^5$	$3p^6$				
	$4s^1$	$4s^2$	$4p^1$	$4p^2$	$4p^3$	$4p^4$	$4p^5$	$4p^6$				
			$3d^1$	$3d^2$	$3d^3$	$3d^4$	$3d^5$	$3d^6$	$3d^7$	$3d^8$	$3d^9$	$3d^{10}$

(Turns out $E(3d) > E(4s)$)

(Look Familiar ?)

* Chemical Properties largely a function of which orbital "valence" electron occupies ...

Spin of Fundamental Particles:

e^- spin $s = \frac{1}{2}$ $m = -\frac{1}{2}, +\frac{1}{2}$ $\uparrow$ or $\downarrow$

p spin $s = \frac{1}{2}$ $m = -\frac{1}{2}, +\frac{1}{2}$ $\uparrow$ or $\downarrow$

n

Atomic Nuclei? Depends on # $p + n$, and how the momentum add...

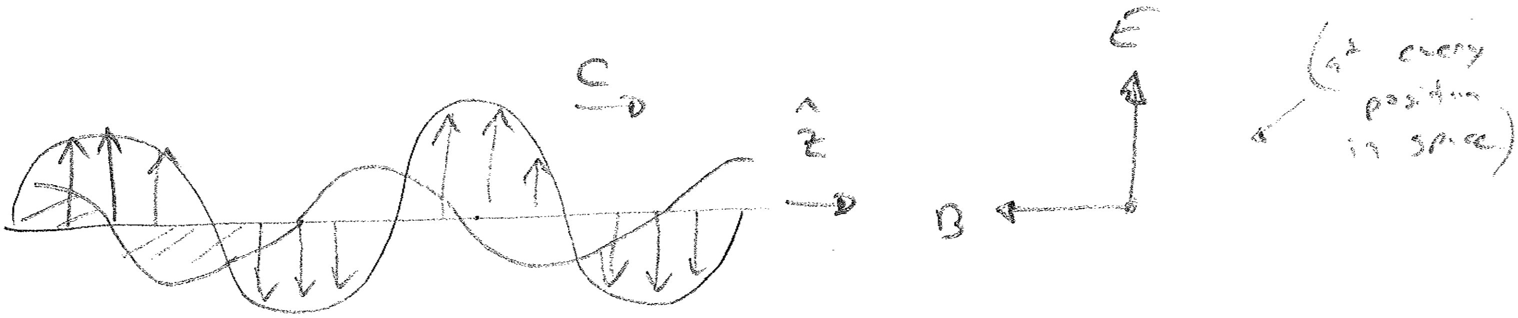
What about photons???

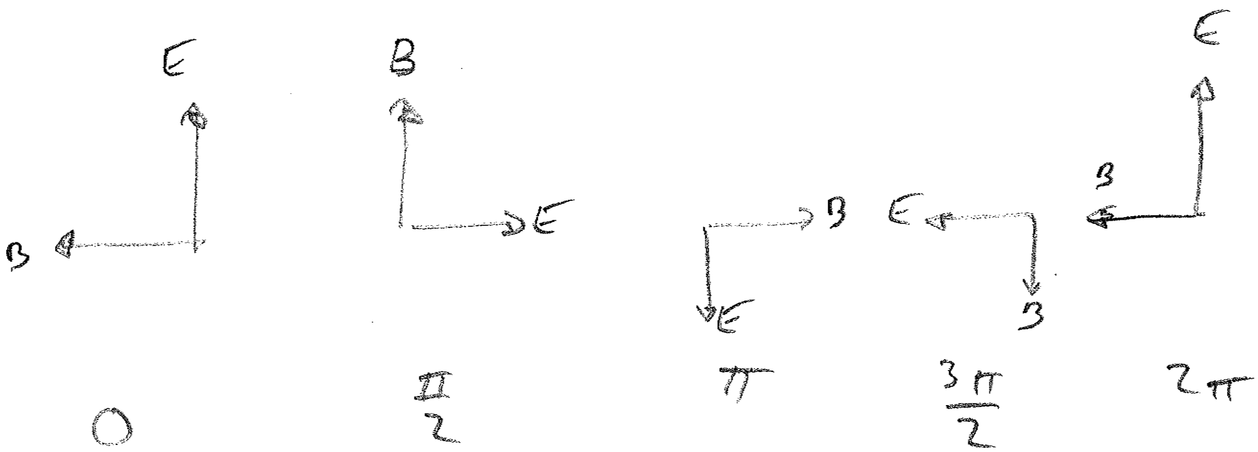
<u>Answer:</u> $\ell = 1$

<u>but</u> $m = -1$ or $+1$, never 0

(this is because it is <u>massless</u>)

Hand Wavy Argument:



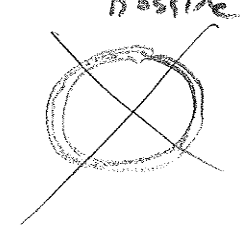


$$\left(2\pi r = l\lambda\right)$$

Why not $m = 0$?

Wave needs a disturbance to propagate, for light must be transverse to direction

$\longrightarrow$ No fully symmetric (in θ) wave function possible:

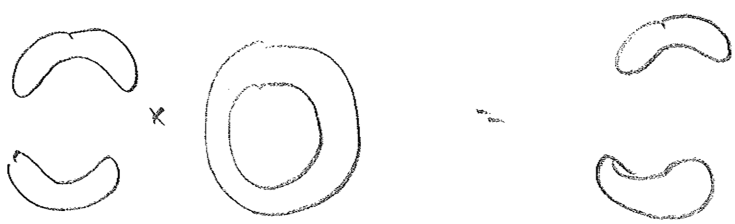


won't propagate ...

Selection Rules:

$$\boxed{\Delta l = \pm 1}$$

$\longrightarrow$ Book says this because photon has $l = 1$ ... I think this is hogwash.

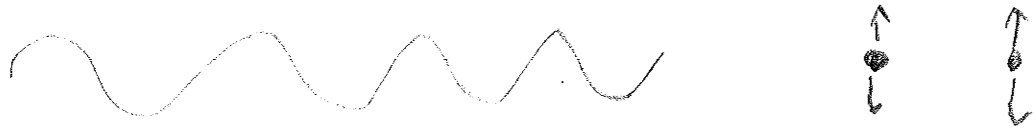


odd   even   odd

$l=1$ $l=0$ $l=1$

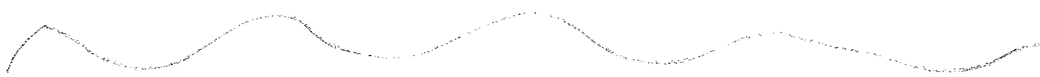
LASERS

Classical model



(dipole oscillating at resonant frequency)

Quantum Picture



photon absorbed (attenuates field)

photon emitted (enhances field!)

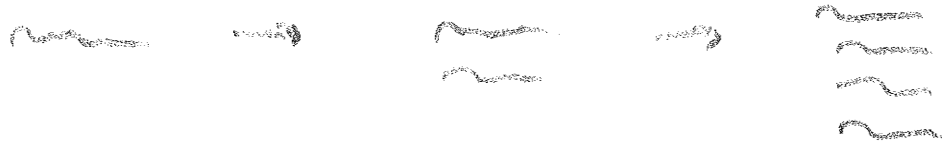
Normally, absorption wins...

Why? $\boxed{\exp(-E/kT)}$

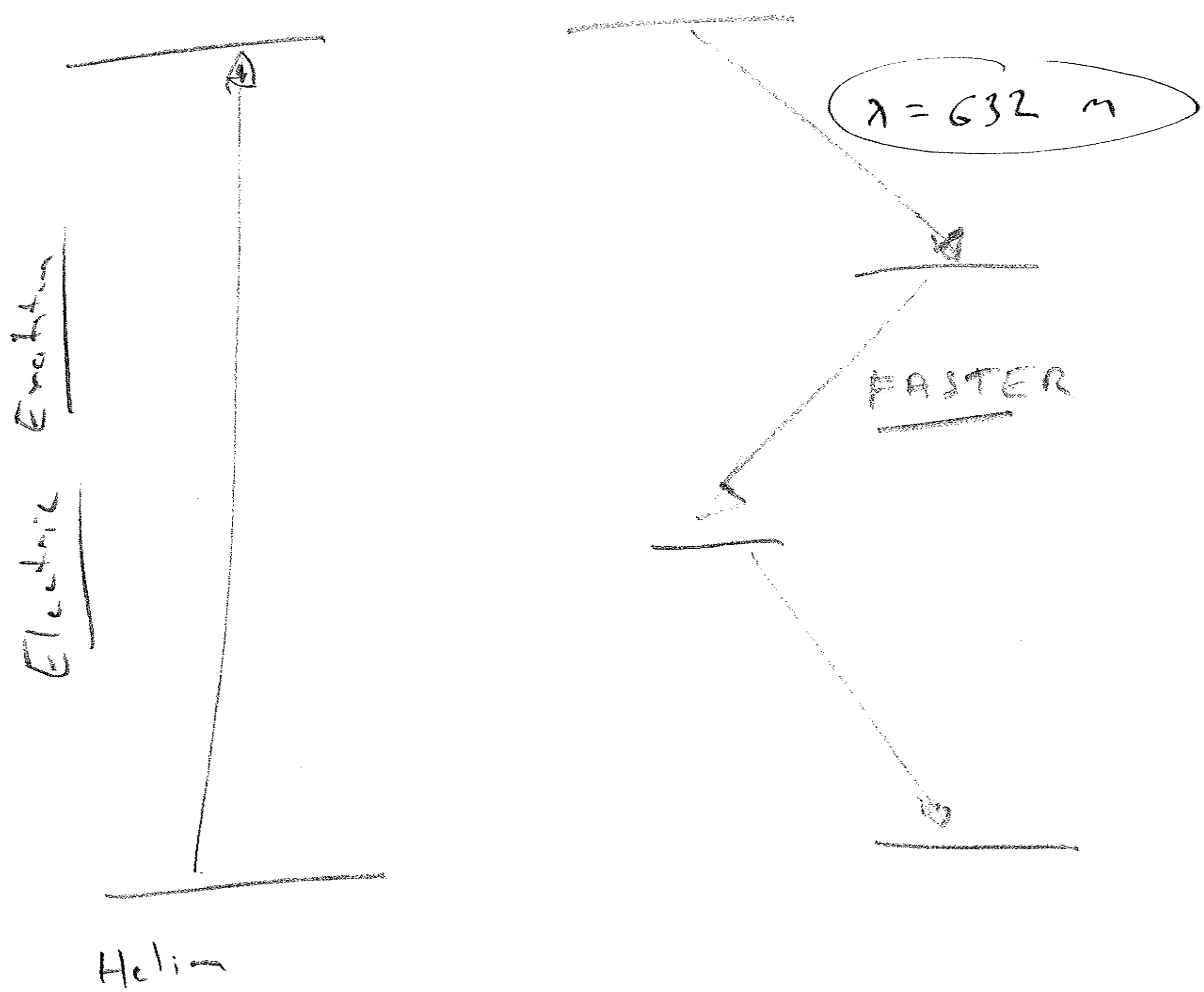
Unless, we can arrange

population inversion

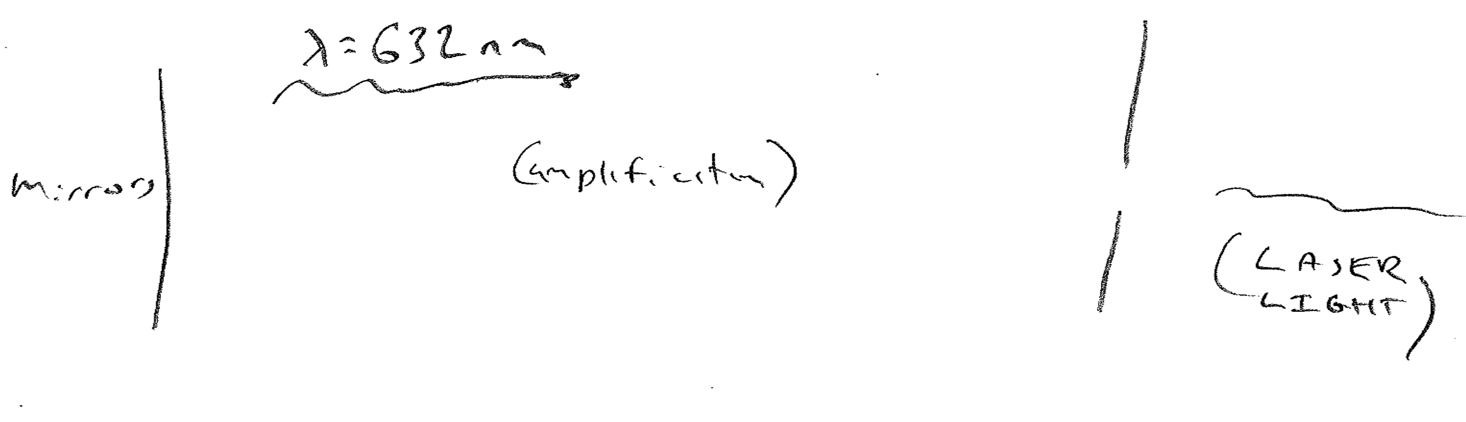
then



(LASER)



Population Inversion Achieved



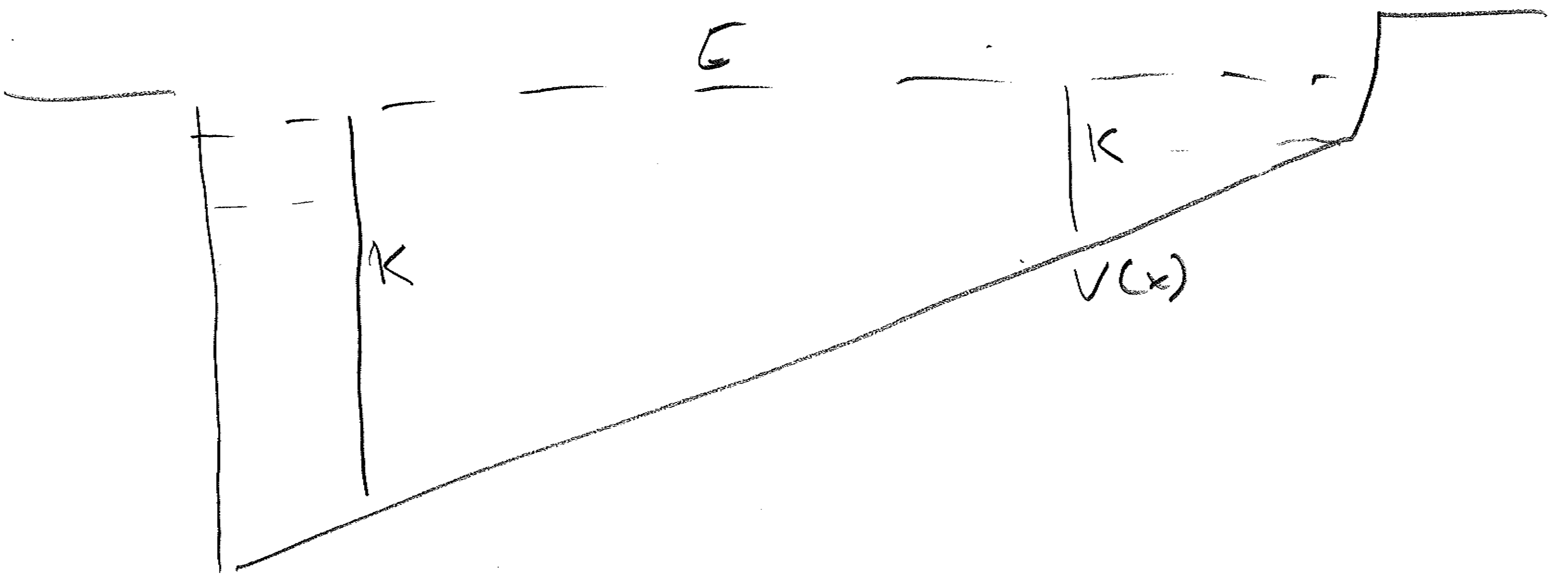
Q10

We've see great success with

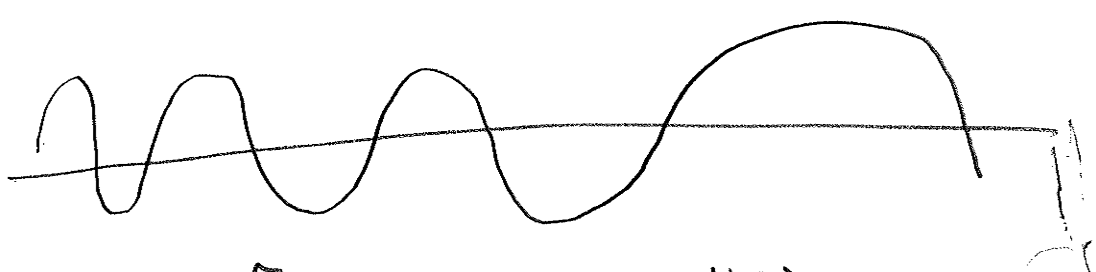
$$\lambda = \frac{h}{p}$$

Today generalize to non-constant $\underline{p}$.

Example



$$K = \frac{p^2}{2m} = \frac{h^2}{2m}\cdot\frac{1}{\lambda^2} \qquad K\downarrow \quad \lambda\uparrow$$



← Idea for $\psi(x)$

That's a great intuition, but for quantitative work, there's a problem...

K can be different at every x

$\Rightarrow$ λ different at every x...

Q: How can we define λ if it changes before 1 cycle is complete!?

Idea: $$\psi(x) = \sin\left(\frac{2\pi}{\lambda}x\right)$$

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -\left(\frac{2\pi}{\lambda}\right)^2 \psi(x)$$

$$\Rightarrow \frac{1}{\lambda^2} = \underbrace{\frac{-1}{4\pi^2}\frac{\partial^2\psi}{\partial x^2}\frac{1}{\psi(x)}}_{\text{Totally local generalization of "}\lambda\text{"}}$$

Now whole point was to relate this to <u>changing</u>

$$K = E - V(x)$$

$$K = \frac{p^2}{2m} = \frac{\left(\frac{h}{\lambda}\right)^2}{2m} = \frac{h^2}{2m}\frac{1}{\lambda^2}$$

$$K = \frac{h^2}{2m}\frac{1}{\lambda^2} = \frac{h^2}{2m}\left(\frac{-1}{4\pi^2}\frac{\partial^2\psi}{\partial x^2}\frac{1}{\psi(x)}\right) = E - V(x)$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi(x) = E\,\psi(x)}$$

<u>Time Independent</u> <u>Schrodinger Equation</u>

Q10

Notes:

$$\frac{\partial^2 f}{\partial x^2} = -k^2 f \quad \rightarrow \quad \sin(kx)$$

$$\frac{\partial^2 f}{\partial x^2} = k^2 f \quad \rightarrow \quad \exp(kx)$$

One oscillitory, one not, why?

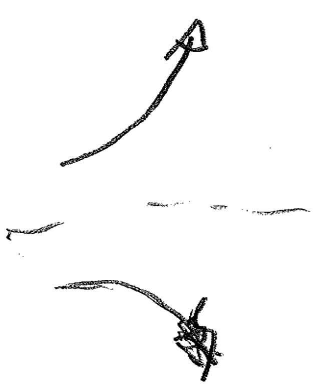
$$\frac{\partial^2 f}{\partial x^2} = -k^2 f \quad \Rightarrow \quad f > 0 \quad \frac{\partial^2 f}{\partial x^2} < 0 \quad \downarrow$$

$$f < 0 \quad \frac{\partial^2 f}{\partial x^2} > 0 \quad \uparrow$$

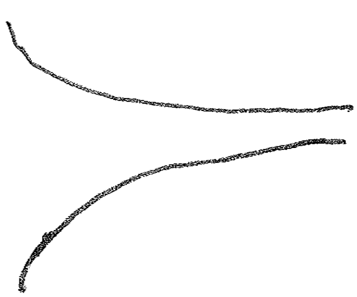
$\Rightarrow$

$$\frac{\partial^2 f}{\partial x^2} = k^2 f \qquad f > 0 \Rightarrow \frac{\partial^2 f}{\partial x^2} > 0 \quad \uparrow$$

$$f < 0 \Rightarrow \frac{\partial^2 f}{\partial x^2} < 0 \quad \downarrow$$



or



$$\frac{d^2 \psi}{dx^2} = -\frac{2m}{\hbar^2}\left[E - V(x)\right] \psi$$

$E > V(x) \Rightarrow$ oscillatory

$E < V(x) \Rightarrow$ exponential

(Simple Harmonic Oscillator)

SHO

So far, only told

$$E_n = \hbar\omega\left(\tfrac{1}{2} + n\right) \qquad n = 0, 1, 2, \ldots$$

Now we have S.E.! For SHO

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_E}{dx^2} + \left[V(x) - E\right]\psi_E = 0$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_E}{dx^2} + \left[\tfrac{1}{2}m\omega^2x^2 - E\right]\psi_E = 0$$

"Guess": $\psi_E = A e^{-bx^2}$

$$\frac{d\psi}{dx} = -2Abx\, e^{-bx^2}$$

$$\frac{d^2\psi}{dx^2} = -2Ab\, e^{-bx^2} + 4Ab^2x^2 e^{-bx^2}$$

$$= -2b\psi + 4b^2x^2\psi$$

Plugging into S.E

$$-\frac{\hbar^2}{2m}\left[-2b\psi + 4b^2x^2\psi\right] + \left[\tfrac{1}{2}m\omega^2x^2 - E\right]\psi = 0$$

$$\underbrace{\left[\frac{\hbar^2 b}{m} - E\right]}_{=0}\psi + \underbrace{\left(\tfrac{1}{2}m\omega^2 - \frac{2b^2\hbar^2}{m}\right]}_{=0}x^2\psi = 0$$

Or

$$\frac{1}{2} m\omega^2 = \frac{2 b^2 \hbar^2}{m}$$

$$b^2 = \frac{m^2 \omega^2}{4 \hbar^2}$$

$$b = \frac{m\omega}{2\hbar}$$

(why not $b < 0$?
Ans: $e^{-bx^2} \to \infty$)

$$E = \frac{\hbar^2 b}{m} = \frac{\hbar \omega}{2}$$

(This is $n = 0$ in $\hbar\omega\left(\frac{1}{2} + n\right)$!)

$$\psi(x) = A x e^{-bx^2}$$

$$\frac{d\psi}{dx} = A e^{-bx^2} - 2Ab x^2 e^{-bx^2}$$

$$\frac{d^2\psi}{dx^2} = -2Abx e^{-bx^2} - 4Abx e^{-bx^2} + 4Ab^2 x^3 e^{-bx^2}$$

$$= -6b\psi + 4b^2x^2\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = \left\{\frac{3\hbar^2 b}{m} - \frac{2b^2\hbar^2}{m}x^2\right\}\psi$$

$$= \left\{E - \frac{1}{2}m\omega^2x^2\right\}\psi$$

$$\Rightarrow \quad \frac{2b^2\hbar^2}{m} = \frac{1}{2}m\omega^2 \quad \Rightarrow \quad b^2 = \frac{m^2\omega^2}{4\hbar^2}$$

$$b = +\frac{m\omega}{2\hbar}$$

$$E = \frac{3\hbar^2 b}{m} = \frac{3\hbar^2}{m}\left[\frac{m\omega}{2\hbar}\right] = \boxed{\frac{3}{2}\hbar\omega}$$

$n=1$ in $E = \hbar\omega\left[n+\frac{1}{2}\right]$

Next: ?

$4y^2 - 2$ $(n = 2)$

$8y^3 - 12y$ $(n = 3)$

$16y^4 - 48y^2 + 12$ $(n = 4)$

Q11

SE rearranged

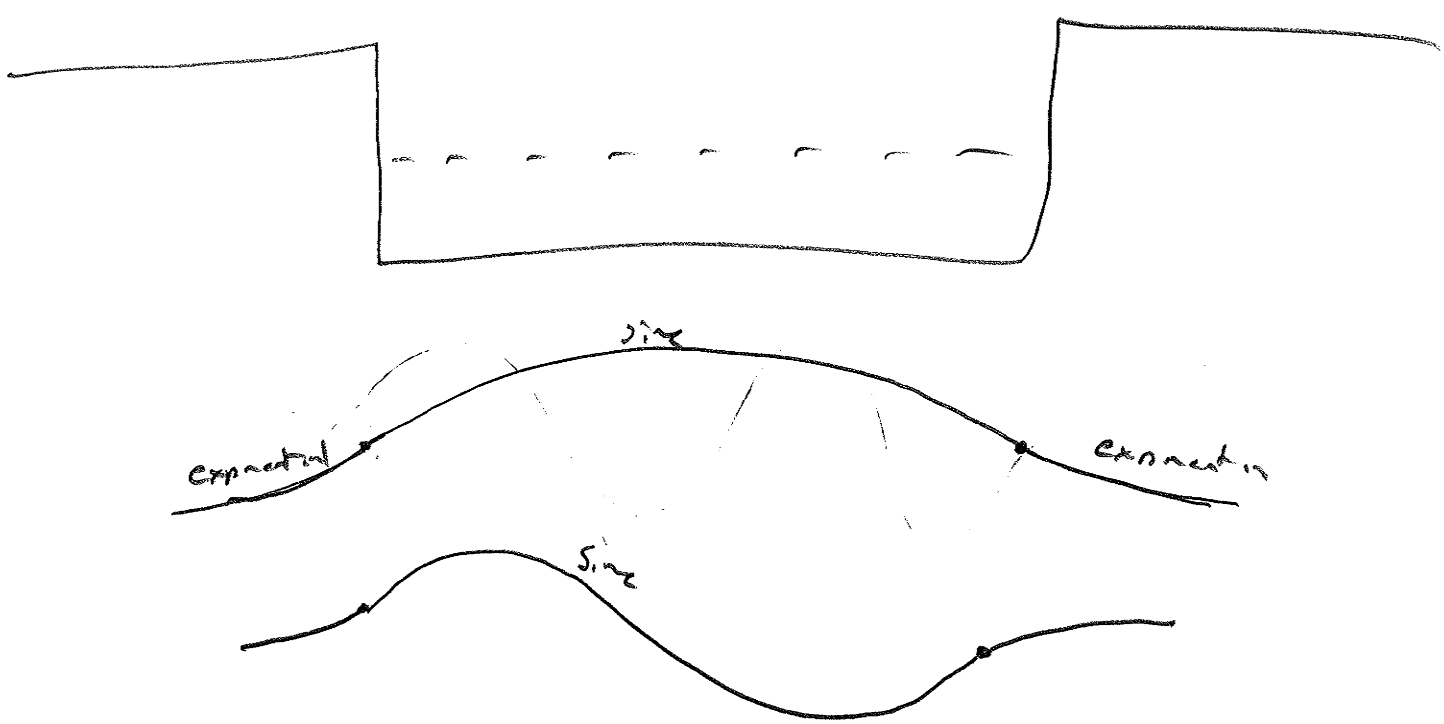
$$\frac{d^2\psi_E}{dx^2} = -\frac{2m}{\hbar^2}\left[E - U(x)\right]\psi_E$$

$E > V(x)$ classically allowed

$f'' = -|\alpha| f \rightarrow$ oscillatory solution ($e^{\pm ikx}$)

$E < U(x)$

$f'' = |\alpha| f \Rightarrow$ exponentials $e^{\pm kx}$



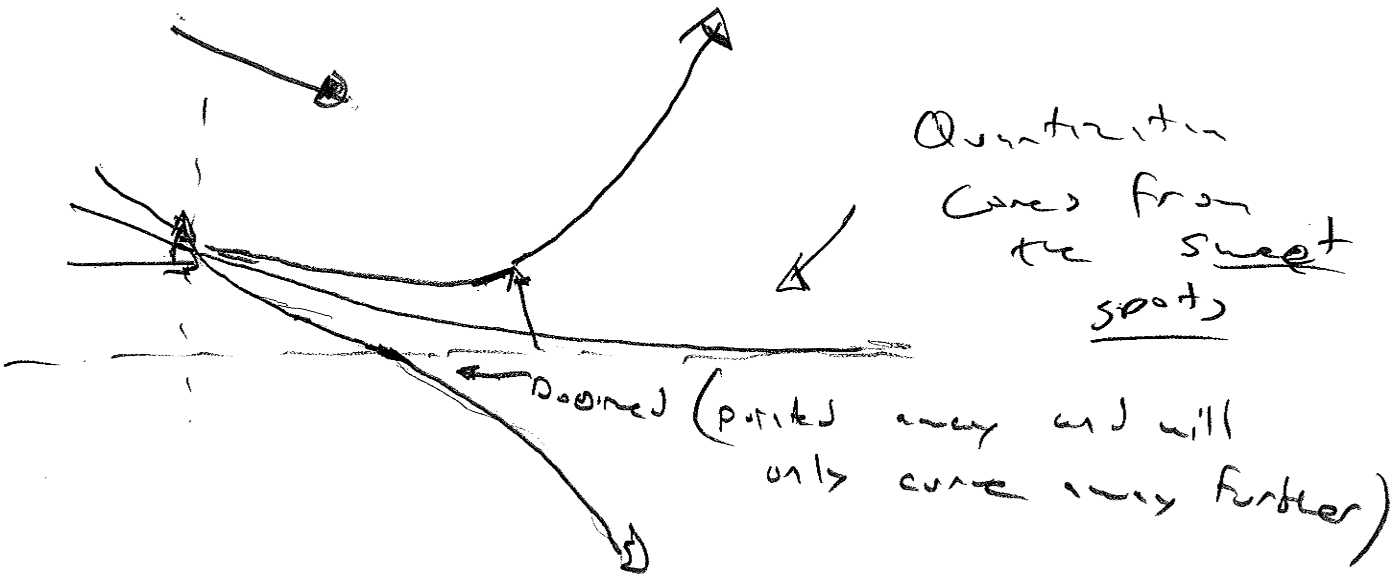
Q: What about complex part of ψ?

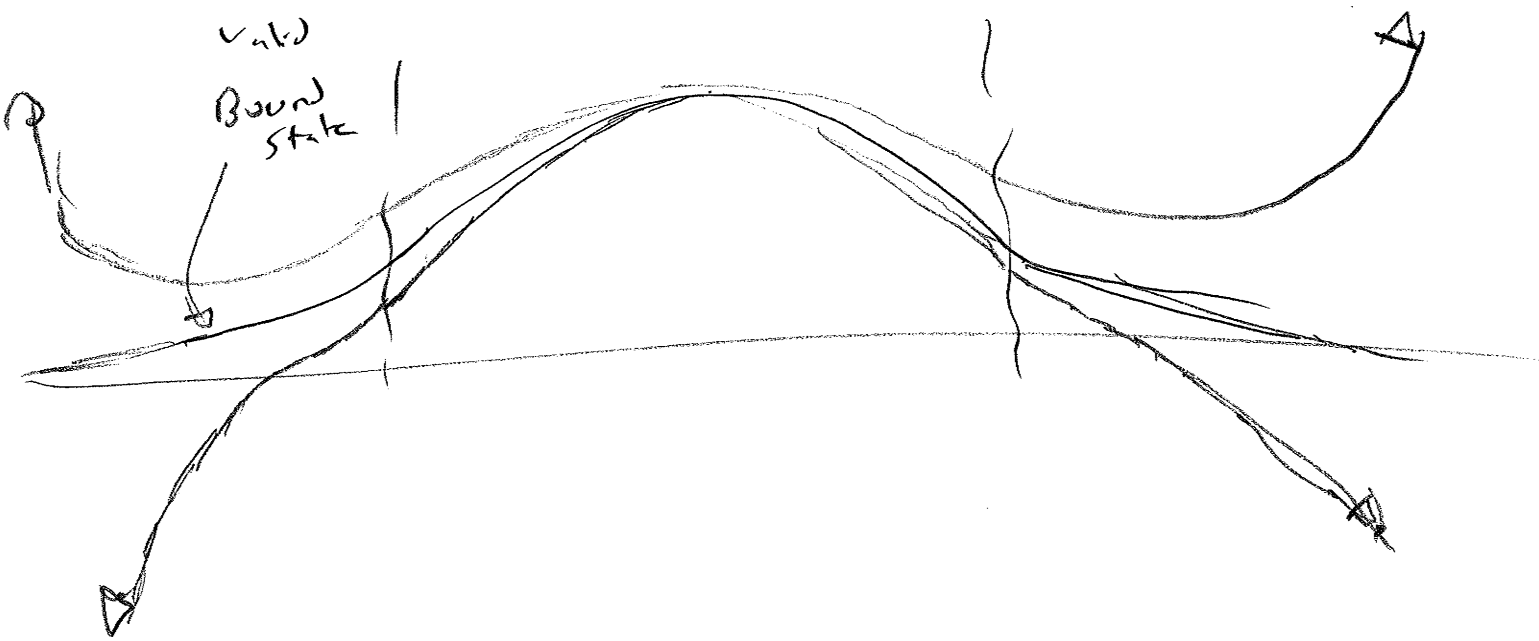
A: Turns out we can always find real solutions (no imaginary part) for bound states!

Q: But you told me solutions to S.E. for atoms had real and imaginary parts

A: You got me... this gets complicated... those solutions were degenerate: multiple solutions with same energy... We chose to use Quantum Numbers of n, l, and m. We could have used other, less useful quantum numbers and had purely real solutions, but they would not be eigenstates of L^2 and L_z too! (less useful)

Why Are Bound States Quantized?
(Even if potential not infinite!)





Sketching Wave Functions

1) Solutions curve toward x-axis in classically allowed regions (oscillate) and away in forbidden (exp)

2) Magnitude of curvature increases with $|E - U(x)|$ $\left\{ \begin{array}{l} \rightarrow \text{shorter wavelengths} \\ \rightarrow \text{faster exponential decay, growth} \end{array} \right.$

3) Follows from (2): $\lambda \downarrow$ as $|E - U(x)| \uparrow$

4) Solutions are continuous and smooth (no kinks) if $U(x)$ is not infinite

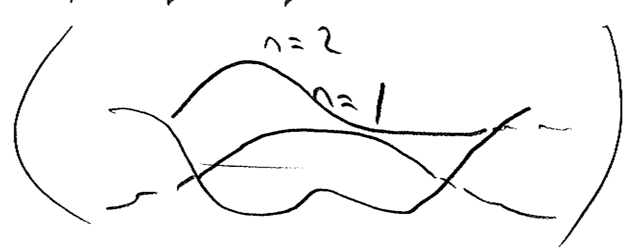
discontinuities
"no kinks" $\rightarrow$ $\frac{d\psi}{dx}$ is continuous.

5) ψ must remain finite as $|x| \rightarrow \infty$ (otherwise, not normalizable) (indeed $\psi \rightarrow 0$ for truly physical states)

$\rightarrow$ Only certain Energies E_n have solutions

6) $n = 1, 2, 3, 4$ $E = E_1, E_2, E_3, \ldots$

n = # bumps!



Review of Spin Eigen-vectors

$$|+z\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |-z\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\langle +x|+z\rangle|^2 = |\langle +x|-z\rangle|^2 = |\langle -x|+z\rangle|^2 = |\langle -x|-z\rangle|^2$$

$+ \quad 4 \quad (y/z)$

$+ \quad 4 \quad (x/y)$

$$\langle +x|-x\rangle = 0 \quad , \quad \langle +y|-y\rangle = 0$$

$$|+x\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \qquad |-x\rangle = \begin{bmatrix} c \\ d\alpha \end{bmatrix}$$

a, b, c, d non-negative reals, $|\alpha| = 1$ but complex ($\alpha = e^{i\theta}$).

$$|\langle +x|+z\rangle|^2 = a^2 = \frac{1}{2} \Rightarrow a, b, c, d = \frac{1}{\sqrt{2}}$$
$$|\langle +x|-z\rangle|^2 = b^2 = \frac{1}{2}$$

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad |-x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$$

$$\langle +x|-x\rangle = \frac{1}{2}(1+\alpha) = 0 \Rightarrow \alpha = -1$$

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad |-x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$$

$$|+y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \alpha \end{bmatrix} \qquad |-y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \beta \end{bmatrix}$$

$$\langle +y|-y\rangle = \frac{1}{2}\left(1+\alpha^*\beta\right) = 0$$

$$\Rightarrow \alpha^*\beta = -1$$

$$\boxed{\beta = -1/\alpha^*}$$

$$|\langle +x|+y\rangle|^2 = \frac{1}{4}|1+\alpha|^2 = \frac{1}{2}$$

$$|1+\alpha|^2 = 2$$

$$(1+\alpha)(1+\alpha^*) = 2$$

$$1+\alpha+\alpha^*+|\alpha|^2 = 2 \qquad (|\alpha|^2 = 1)$$

$$\alpha+\alpha^* = 0$$

$$\alpha = \pm i$$

$$\alpha = +i \qquad \beta = \frac{-1}{-i} = -i$$

$$|+y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad |-y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\left(\text{Book: } |-y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}\right)$$

Suppose we want to know

$$\Pr(+y) \quad \text{at} \quad t=0$$

given that, at $t=0$, the state vector is

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}|+z\rangle + \sqrt{\frac{2}{3}}\,|-z\rangle \qquad (1)$$

So we have:

$$|\psi(0)\rangle = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix}$$ Column vector version of (1).

And

$$|+y\rangle = \begin{bmatrix} \sqrt{1/2} \\ i\sqrt{1/2} \end{bmatrix}$$ From Table Q6.1 (p 102)

$$\langle +y| = \begin{bmatrix} \sqrt{1/2} \\ -i\sqrt{1/2} \end{bmatrix}$$ Take complex conjugate

$$\langle +y|\psi(0)\rangle = \begin{bmatrix} \sqrt{1/2} \\ -i\sqrt{1/2} \end{bmatrix} \begin{bmatrix} \sqrt{1/3} \\ \sqrt{2/3} \end{bmatrix}$$

$$= \sqrt{\frac{1}{6}} - i\sqrt{\frac{2}{6}}$$

$$|\langle +y|\psi(0)\rangle|^2 = \left(\sqrt{\frac{1}{6}} - i\sqrt{\frac{2}{6}}\right)\left(\sqrt{\frac{1}{6}} + i\sqrt{\frac{2}{6}}\right)$$

$$= \frac{1}{6} + \frac{2}{6} = \frac{3}{6} = \frac{1}{2}$$

Eg 2:

$$|\psi(0)\rangle = |+y\rangle = \begin{bmatrix} \sqrt{1/2} \\ i\sqrt{1/2} \end{bmatrix}$$

$$\langle +y|\psi(0)\rangle = \begin{bmatrix} \sqrt{1/2} \\ -i\sqrt{1/2} \end{bmatrix} \begin{bmatrix} \sqrt{1/2} \\ i\sqrt{1/2} \end{bmatrix}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$|\psi(0)\rangle = |-y\rangle = \begin{bmatrix} i\sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix}$$

$$\langle +y|\psi(0)\rangle = \begin{bmatrix} \sqrt{1/2} \\ -i\sqrt{1/2} \end{bmatrix} \begin{bmatrix} i\sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix} = 0$$