Homework Assignment 6
Scatter There Thy Cheerful Beams!

Practice Problems

These problems are graded on effort only.

Griffiths: P2.23, P2.25, P2.58
Hint for 2.23b: use integration by parts to move the derivative off of $\theta(x)$.

Additional Problems

Problem 1: Consider the vector space of 2x1 column vectors, for example the vectors x and y:

$$x = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad\qquad y = \begin{pmatrix} c \\ d \end{pmatrix}$$

with transpose:

$$x^T = \begin{pmatrix} a & b \end{pmatrix}$$

We can create an inner product space by defining the inner product as:

$$\langle x|y\rangle \;\equiv\; (x^*)^T\, y \;=\; a^*\, c + b^*\, d$$

(A) Verify that this definition does satisfy the properties of an inner product:

I1	$\forall x, y \in H$	$\langle x\|y\rangle = \langle y\|x\rangle^*$
I2	$\forall x, y, z \in H$and $\forall \alpha \in \mathbb{C}$	$\langle x\|\alpha y\rangle = \alpha\,\langle x\|y\rangle$
I3	$\forall x, y, z \in H$	$\langle x+y\|z\rangle = \langle x\|z\rangle + \langle y\|z\rangle$
I4	$\forall x \in H$	$\langle x\|x\rangle \geq 0$
I5	$\forall x \in H$	$\langle x\|x\rangle = 0$ if and only if $x = 0$

By definition, operators return a new vector for a given vector. In this finite-dimensional vector space, any linear operator O can be represented as a 2x2 matrix:

$$O\,x = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} O_{11}\,a + O_{12}\,b \\ O_{21}\,a + O_{22}\,b \end{pmatrix}$$

(B) Show that the hermetian adjoint of O is given by:

$$O^\dagger = (O^*)^T = \begin{pmatrix} O_{11}^* & O_{21}^* \\ O_{12}^* & O_{22}^* \end{pmatrix}$$

from our definition:

$$\langle x|O^\dagger y\rangle = \langle Ox|y\rangle$$

Hint: calculate:

$$\langle Ox|y\rangle$$

then do whatever it takes to bring the action over onto the y instead, and then read off $O^\dagger$. You may use the property of matrices A and B that:

$$(AB)^T = B^T A^T$$

An operator U is unitary if

$$U^\dagger U \; = \; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv I$$

Note that:

$$Ix = x$$

for any vector x.

(C) Show that the rotation matrix:

$$R \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is unitary.

(D) Show that for a unitary matrix U:

$$\langle Ux|Uy\rangle = \langle x|y\rangle$$

Problem 2: In lecture we studied the scattering states (with $E > 0$) of the delta-function potential:

$$V(x) = -\alpha\delta(x)$$

We found that the general solution:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x \leq 0 \\ Fe^{ikx} + Ge^{-ikx} & x \geq 0 \end{cases}$$

has boundary conditions:

$$F + G = A + B$$

and

$$F - G = A(1 + 2i\beta) - B(1 - 2i\beta)$$

where:

$$\beta = \frac{m\alpha}{\hbar^2 k}$$

Notice that the waves (with coefficients) A and G are incoming, while the waves B and F are outgoing. They are connected by the scattering matrix S:

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$$

(A) Calculate the S-matrix from the boundary conditions.

Hint: Put B and F on the LHS of the boundary conditions and A and G on the RHS. Eliminate F and solve for B in terms of A and G. Then read off:

$$B = S_{11}\,A + S_{12}\,G$$

(B) Show that the S-matrix you calculated in (A) is unitary.

(C) Calculate the probability current:

$$J(x) = \frac{i\hbar}{2m}\left(\psi\frac{d\psi^*}{dx} - \psi^*\frac{d\psi}{dx}\right)$$

for $x < 0$.
Hint: this explodes a bit but you should get nice cancellation leaving you just two terms in your answer.

(D) Also calculate the probability current $J(x)$ for $x > 0$.

Hint: do not calculate this again from scratch! Use your results from (C) and substitution!

(E) For normalizable solutions, to the SE, we showed that:

$$J(a) = J(-a)$$

in the limit $a \to \infty$. These are not normalizable solutions, but let's assume still that the probability current you calculated in (C) equals the probability current you calculated in (D). Calculate a condition on $|A|^2$, $|B|^2$, $|F|^2$, and $|G|^2$ that results from this.

(F) Define the outgoing waves O and the incoming waves I as:

$$O = \begin{pmatrix} B \\ F \end{pmatrix}, \qquad\qquad I = \begin{pmatrix} A \\ G \end{pmatrix}$$

Calculate a condition on $|A|^2$, $|B|^2$, $|F|^2$, and $|G|^2$ from:

$$\langle O|O\rangle = \langle I|I\rangle$$

Compare to your condition in (E).

(G) Show that:

$$\langle O|O\rangle = \langle I|I\rangle$$

implies that S is unitary.

Hint: use $O = SI$.

Problem 3: Consider the potential:

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$$

and assume $E > V_0$.

(A) Show that the general solutions to the SE can be written as:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x \leq 0 \\ Fe^{i\eta kx} + Ge^{-i\eta kx} & x \geq 0 \end{cases}$$

where:

$$\eta \equiv \sqrt{\frac{E - V_0}{E}}$$

(B) Determine the boundary conditions at $x = 0$ and use these conditions to determine the scattering matrix S defined by:

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$$

(C) Is S unitary? Is there anything unphysical about this potential which might explain this?

(D) Calculate the reflection from the left:

$$R = \frac{|B|^2}{|A|^2}\bigg|_{G=0}$$

(E) And transmission from the left by:

$$T = 1 - R$$

(F) Calculate:

$$\frac{|F|^2}{|A|^2}\bigg|_{G=0}$$

Is this the same as T?