RLC

circuit

analysis

(~ 3 weeks)

Intro:

Physics 116A - Analog Electronics

Grading:

ME 1 - 15%

ME 2 - 15%

FE - 20%

Lab - 30%

HW - 20%

→ Late HW can be turned in (self-graded) for 50% credit at end of term.

→ Drop worst

Text: The Art of Electronics

→ You'll love it eventually, but don't be alarmed if you don't follow everything!

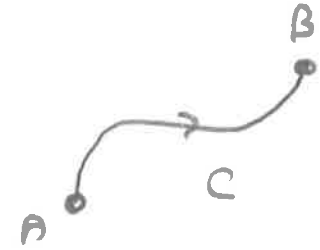
Lectures: Will loosely follow text.

You should take notes while reading, in case exams are open note.

Review:

Change in potential energy for a conservative force is

$$\Delta U\,(A\to B) \equiv -W_{csv} = -\int_C \vec{F}_{csv}\cdot dx$$



If we choose a reference point "A" as $U=0$, we can define a potential

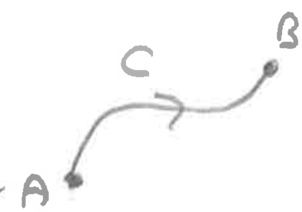
$$U(\vec{x}) = -\oint_A^{\vec{x}} \vec{F}_{csv}\cdot d\vec{x}$$

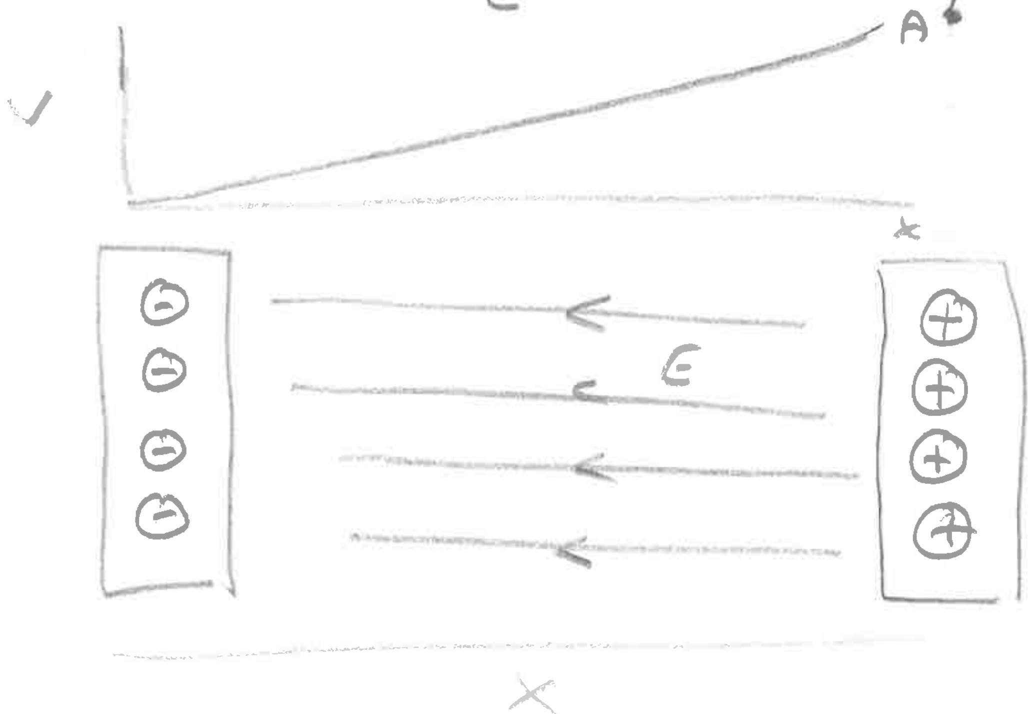
(Conservative if $\oint_{closed} \vec{F}_{csv}\cdot d\vec{x} = 0$)

From Electrostatics, we have

$$\vec{E} \equiv \frac{\vec{F}}{\Delta q} \qquad V(\vec{x}) \equiv \frac{U(\vec{x})}{\Delta q} \qquad \lim \Delta q \to 0$$

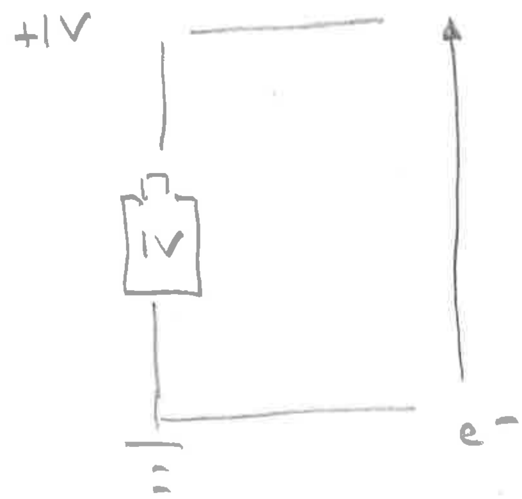
$$V(\vec{x}) = -\int_C \vec{E}\cdot d\vec{x}$$





⊖ seek high V

⊕ seek low V



e^- 1.6×10^{-19} C

$$K = 1.6 \times 10^{-19}\ \text{J} \equiv 1\ \text{eV}$$

Volt is at useful scale in many contexts.

- your car battery (12), toaster (120V)
- Kinetic energy of photo-electron (~ 1eV)
- CMB energy density 0.25 eV/cm^3

Current: $I = \dfrac{\Delta q}{\Delta t}$ $\qquad 1\,A = C/s$

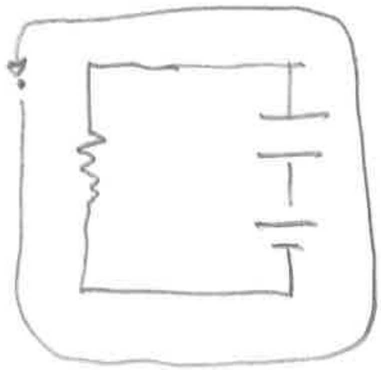
Power: $W = \int V\,dq = \int V \frac{dq}{dt}\,dt$

$$= \int V I\,dt$$

$$\frac{dW}{dt} = \boxed{VI = P}$$

Conservation Laws for circuits

KVL



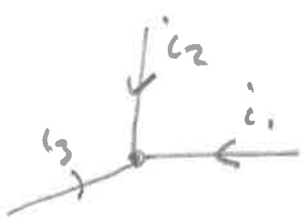
$\Delta V = 0$

$$\sum_{source} \Delta V = \sum_{load} \Delta V$$

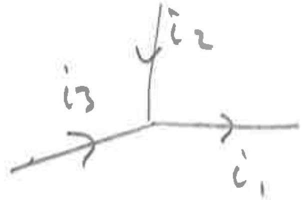
$$0 = \sum_{source} \Delta V + \sum_{load} (-\Delta V)$$

"Source" vs "Load"

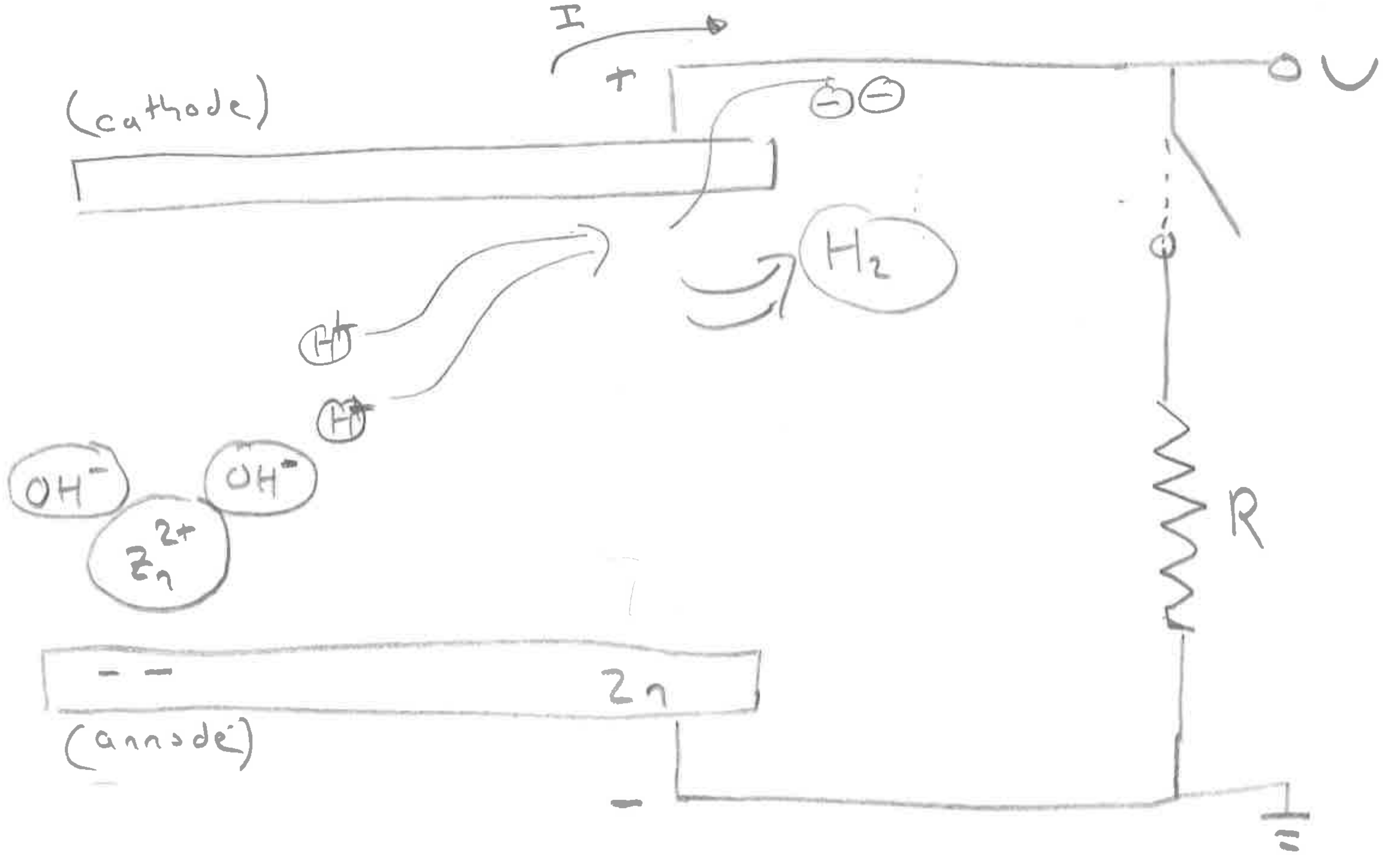
KCL



$i_1 + i_2 + i_3 = 0$



$i_1 + i_2 = i_3$



- Dissolving Zn in water is energetically favorable:

$$Zn(s) \rightarrow Zn(aq) + 2e^- \qquad E_0 = -0.763\,V$$

$$\Delta E = 0.763\text{ eV} \qquad \text{"reduction potential"}$$

- As H^+ ions accumulate in brine, e^- on Zn, the voltage increases.
- When $V \sim 0.763\,V$ dissolving Zn no longer energetically favorable (equilibrium)
- When switch closed, electrons can flow through circuit to reach Cu cathode, where they combine w/ H^+ to make H_2

$$2H_+ + 2e^- \rightarrow H_2(g) \qquad E_0 = 0 \text{ (ref)}$$

- Now Zn is continuously dissolved, and H_2 gas bubbles out, producing the energy that drives the circuit!

(Extra Note)

Chemistry: (Not for lecture)

Oxidation is loss of electrons
Reduction is gain of electrons
("OIL RIG")

Reduction

$$CuO + Mg \longrightarrow Cu + MgO$$

Oxidation

oxidation

$$Cu^{2+} + Mg \longrightarrow Cu + Mg^{2+}$$

reduction

Reduction Potentials:

	$E^\circ (V)$
$2H^+ + 2e^- \rightarrow H_2(g)$	0 (reference)
$Zn^{2+}(aq) + 2e^- \rightarrow Zn(s)$	-0.763 V

* Actual value will vary with pH.
(Nernst Formula) *

Simple Drude Model

$\langle p_0 \rangle = \text{const}$

$\langle \vec{p}_0 \rangle = 0$ (1)

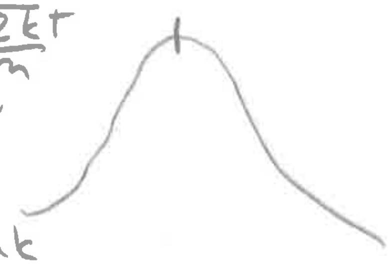
$\langle t - t_0 \rangle \equiv \tau$ (2)

time since last collision

$v_{peak} = \sqrt{\frac{2kT}{m}}$

$\neq \langle v \rangle$!

$\langle v \rangle = \frac{2}{\sqrt{\pi}} v_{peak}$



$$\vec{p} = \vec{p}_0 + qE(t - t_0)$$

$$\langle \vec{p} \rangle = \langle \vec{p}_0 \rangle + qE\langle t - t_0 \rangle$$

$$\langle \vec{p} \rangle = qE\tau$$

$$\boxed{\langle \vec{v} \rangle = \frac{q}{m}\vec{E}\tau}$$

(1) Clear simplifying assumption, since E-field imposes preferred direction... Assumption is that Thermal Equilibrium drives $\langle \vec{p}_0 \rangle = 0$.

(2) Factor of two confusion... might reason on $\langle t - t_0 \rangle = \frac{\tau}{2}$ if τ is mean time between collisions... we avoid by defining $\langle t - t_0 \rangle \equiv \tau$

Relaxation Time Version:

$$\frac{d\vec{p}}{dt} = 0 \quad \rightarrow \quad \frac{d\vec{p}}{dt} = -\frac{\vec{p}}{\tau}$$

Where τ is relaxation time

$$\vec{p}(t) = \vec{p_0} \exp(-t/\tau)$$

In presence of $\vec{E}$-field

$$\frac{d\vec{p}(t)}{dt} + \frac{\vec{p}}{\tau} = q\vec{E}$$

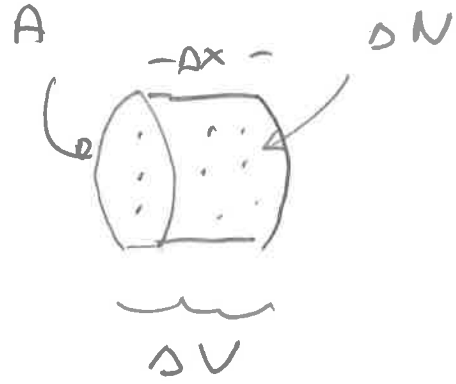
$$\frac{d}{dt}\langle\vec{p}\rangle + \frac{\langle p\rangle}{\tau} = q\vec{E}$$

Steady state: $\frac{d}{dt}\langle\vec{p}\rangle = 0$

$$\langle\vec{p}\rangle = q\vec{E}\tau$$

$$\langle\vec{v}\rangle = \frac{q\tau}{m}\vec{E}$$

Current Density



$$I = \frac{\Delta Q}{\Delta t}$$

$$= q \frac{\Delta N}{\Delta t}$$

$$= q n \frac{\Delta V}{\Delta t}$$

$$= q n A \frac{\Delta x}{\Delta t}$$

$$= q n A v_x$$

$$\frac{I}{A} = q n v_x$$

$$\boxed{\vec{J} \equiv \frac{I}{A} = q n \langle \vec{v} \rangle}$$

$$\vec{j} = qn\langle\vec{v}\rangle$$

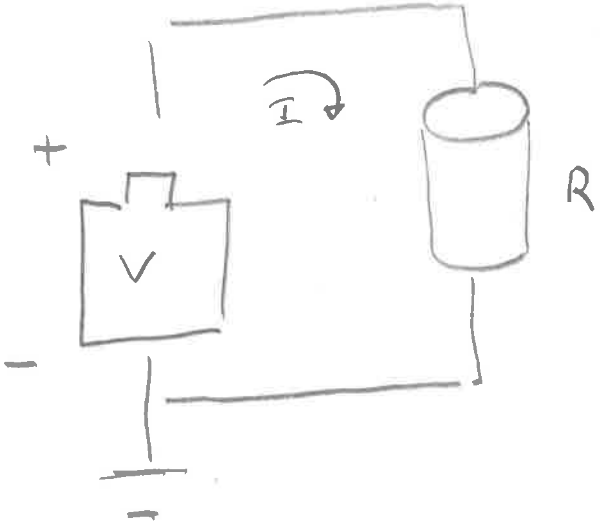
$$\langle\vec{v}\rangle = \frac{q}{m}\vec{E}\tau$$

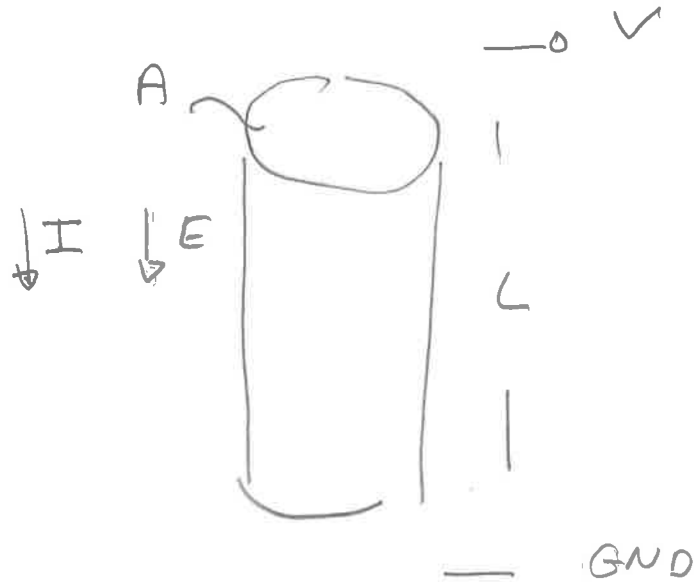
$$\vec{j} = \frac{q^2 n \tau}{m}\vec{E}$$

$$\boxed{\vec{j} = \sigma\vec{E}}$$

Linear Relationship between current and applied field ... (valid)

(Temperature dependence ... does not work
failure of Drude model was evidence for Quantum theory)





$V = -\int \vec{E} \cdot d\vec{l}$, so for E, I down in above diagram $V = EL$

$$j = \sigma E$$

$$\frac{I}{A} = \sigma \left(\frac{V}{L}\right)$$

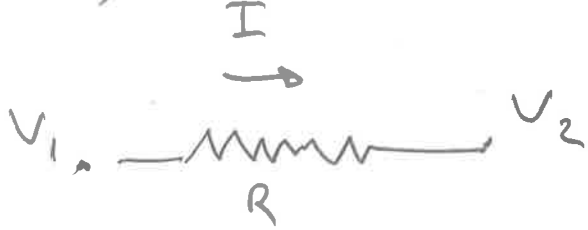
$$V = \frac{L}{\sigma A} I$$

$$\boxed{V = IR} \quad *$$

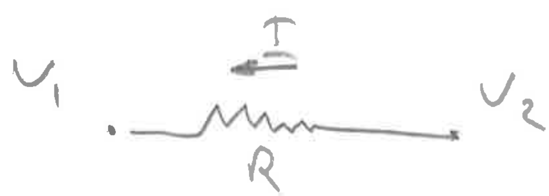
$([R] = \Omega = \frac{V}{A})$

V is a drop in direction of current

$$\Delta V = 0 - (V) = -V$$

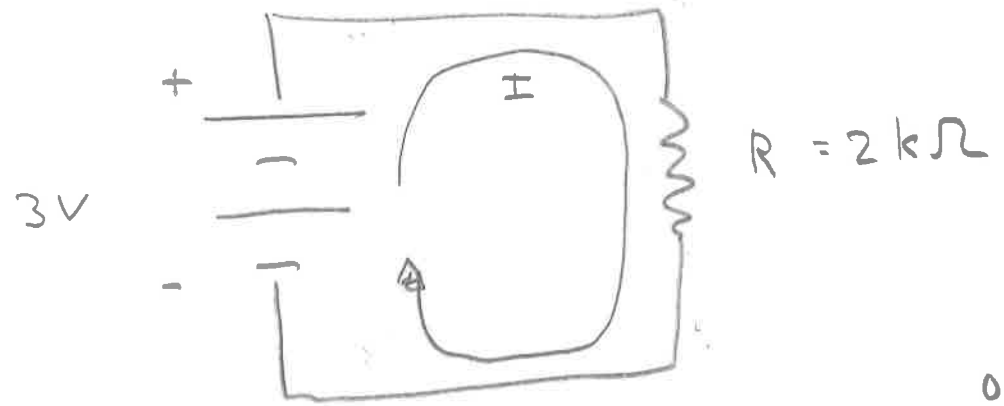


$$V_2 = V_1 - IR$$



$$V_2 = V_1 + IR$$

Exercise 1:



or $+V_{bat} = +IR$

Solution:

$$\Delta V_{loop} = 0 = (+V_{bat}) - IR$$

$$\Rightarrow I = \frac{V_{bat}}{R} = \frac{3V}{2k\Omega} = \frac{3}{2}\, mA$$

In other direction $\circlearrowleft I$

$$\Delta V_{loop} = 0 = (-V_{bat}) - IR$$

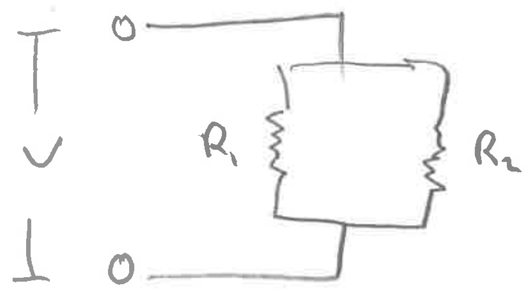
$$\Rightarrow I = -\frac{V_{bat}}{R} = -\frac{3}{2}\, mA$$

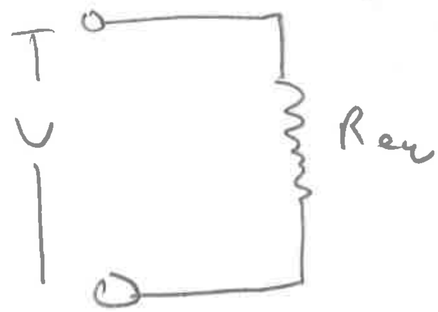
(best to draw positive currents when possible!)

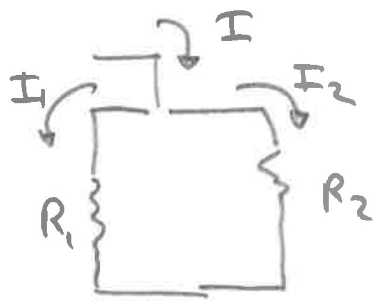
* Always draw current directions and note signs! *

Exercise 2:

HW







$$I = I_1 + I_2$$

$$V = I R_{eq} = I_1 R_1 = I_2 R_2$$

$$I R_{eq} = I_1 R_1$$

$$R_{eq} = \frac{I_1}{I} R_1$$

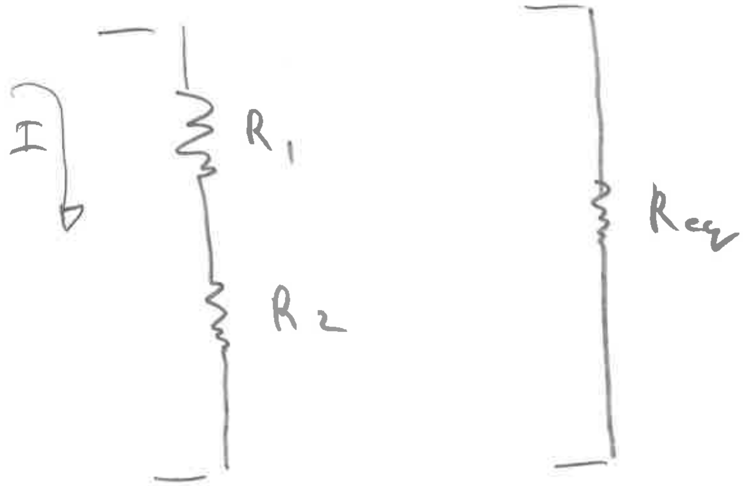
$$I_1 R_1 = I_2 R_2 = (I - I_1) R_2$$

$$I_1 = I \frac{R_2}{R_1 + R_2}$$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \left(+ \dots\right)$$

Exercise 3:



$$V = I R_{eq} = I R_1 + I R_2$$

$$\boxed{R_{eq} = R_1 + R_2}$$

Exercise 4: Show from Drude model:

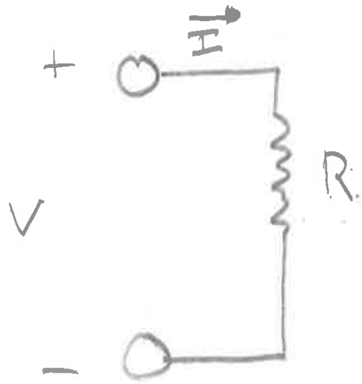
$$R \sim \frac{d}{A}$$

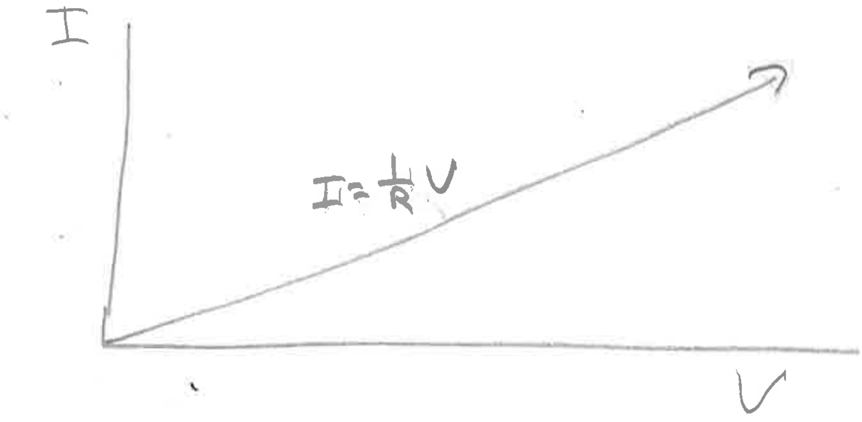
series: $R_{eq} \sim \dfrac{d_1 + d_2}{A} = R_1 + R_2$

parallel: $\dfrac{1}{R_{eq}} \sim \dfrac{A}{L} = \dfrac{A_1 + A_2}{L} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

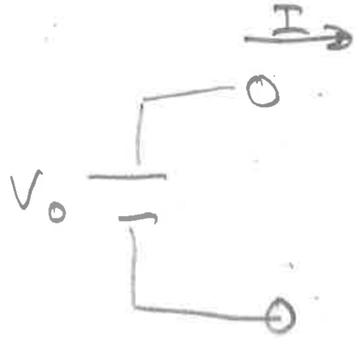
IV Curves and Operating Points

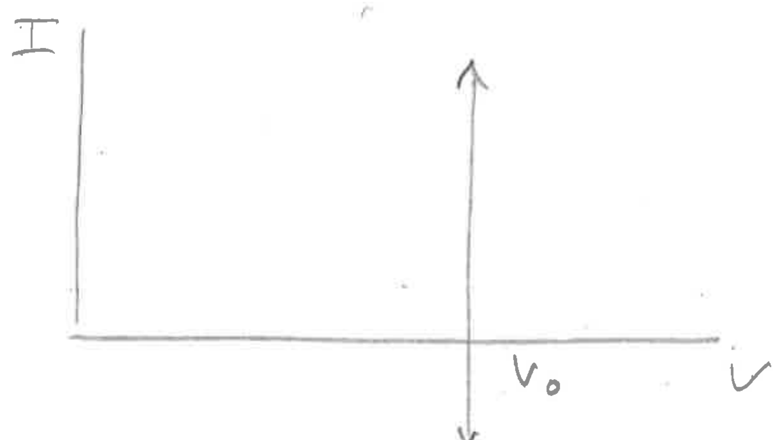
Load IV, assume our circuit consumes current supplied externally



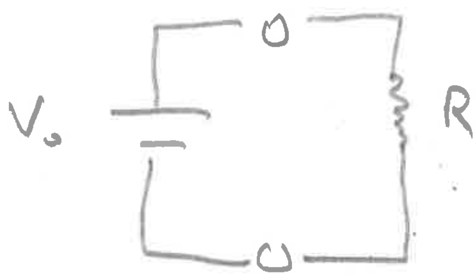


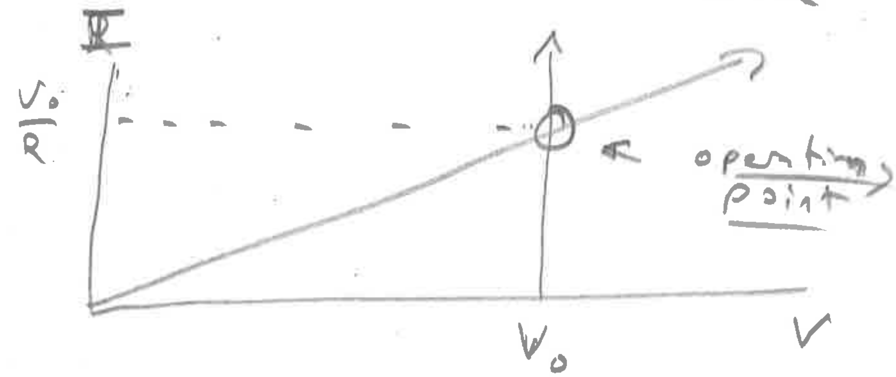
Source IV: now I is assumed to be provided to Load circuit



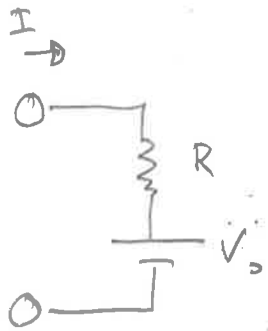


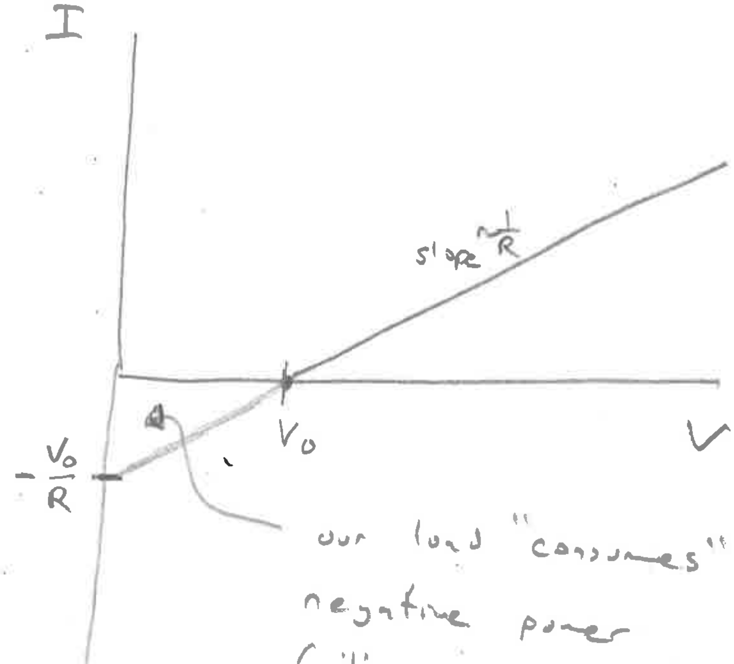
Operating Point is the intersection of supply IV curve with source IV curve



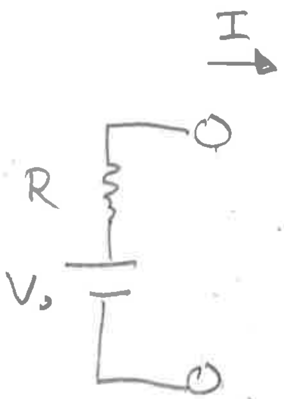


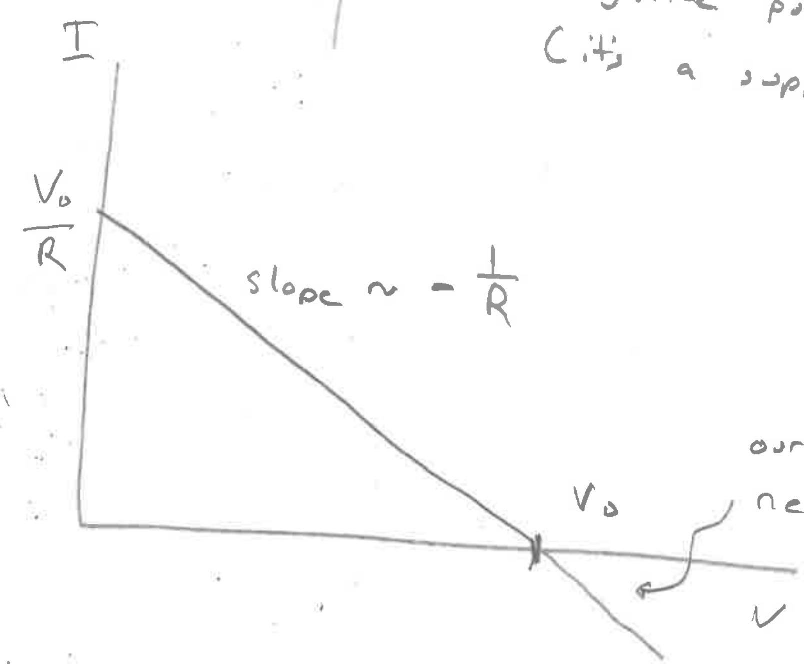
We can mix R's and V-sources and still have Load IV and Source IV





our load "consumes" negative power (it's a supply!)



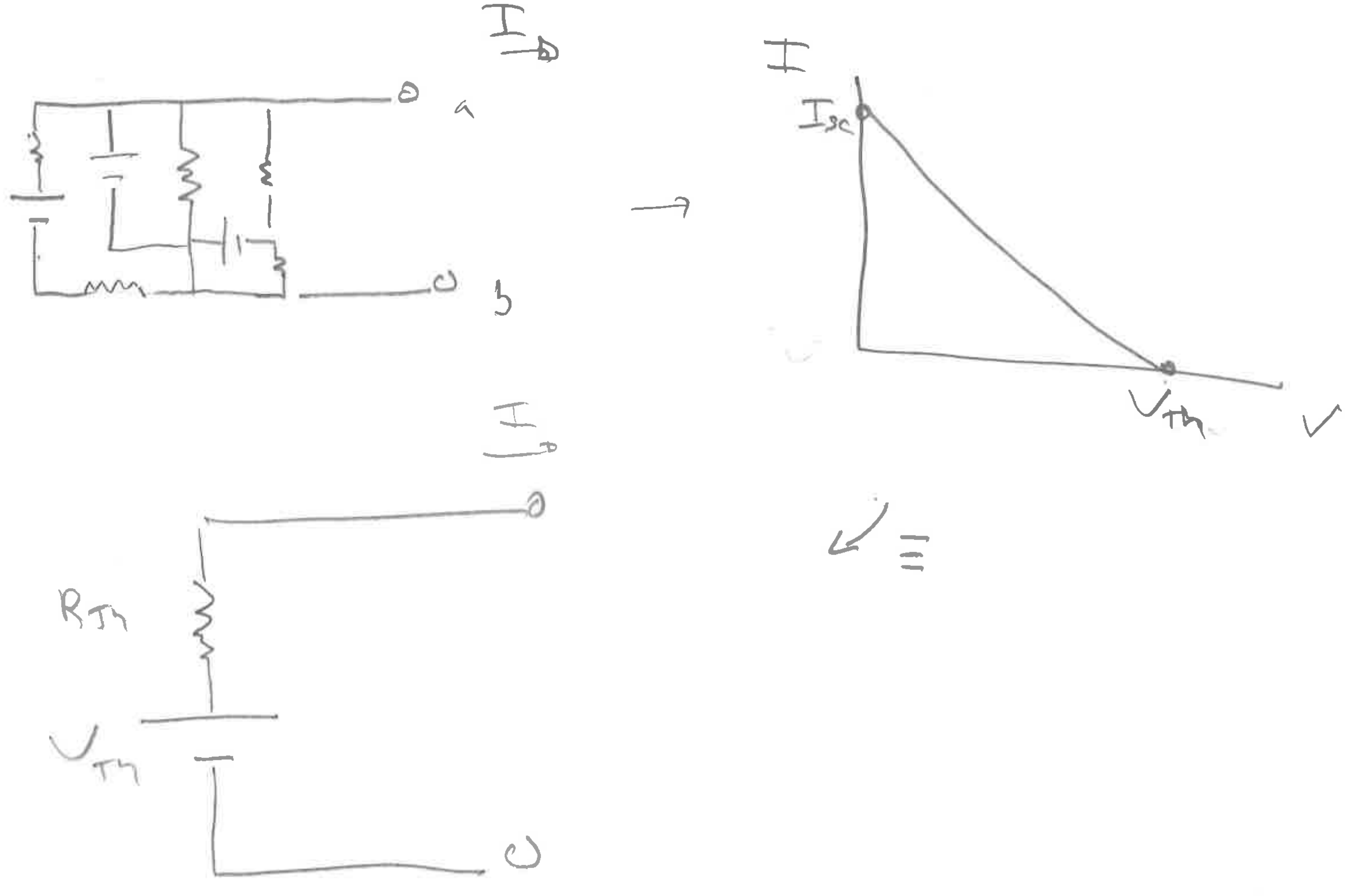


our source "provides" negative power (it's a load!)

(Notice that lower plot is just upper part reflected across V-axis, since they are same with $I \to -I$)

Thevenin Equivalent Circuits

Because of superposition principle of E+M; combining linear components always has a linear response.... all circuits consisting of R, V, and I-sources have a linear IV curve

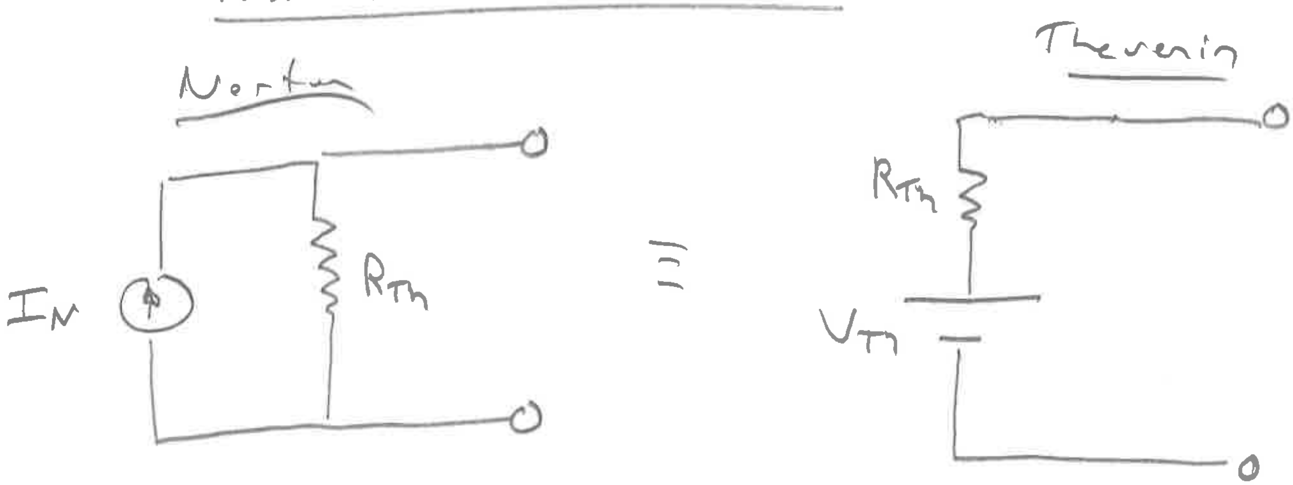


V_{th} is open circuit voltage

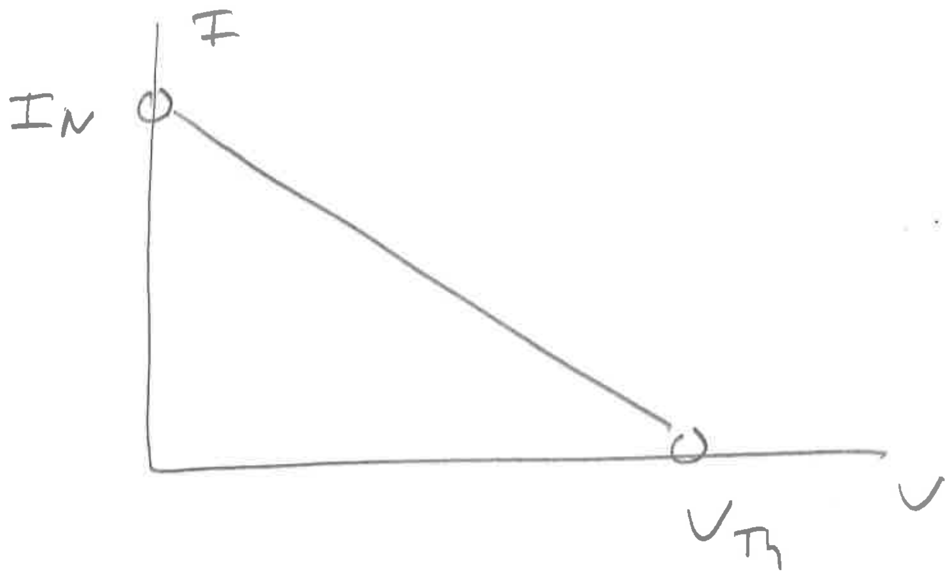
I_{sc} is short circuit current

$R_{Th} = V_{th} / I_{sc}$

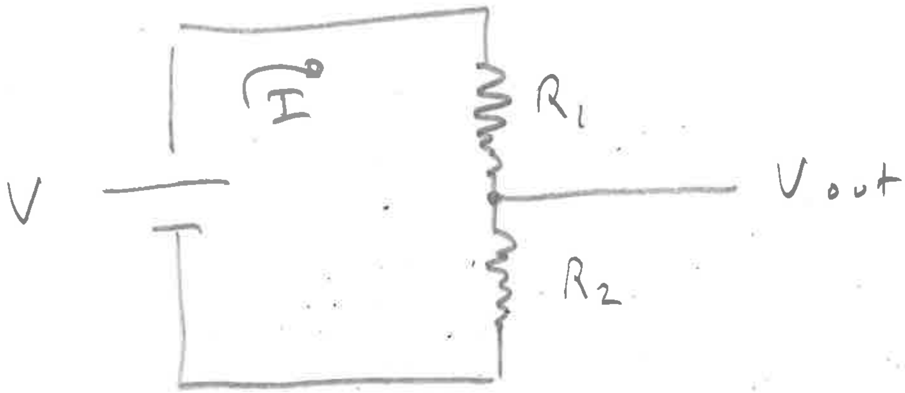
Norton Equivalent



I



Voltage Divider



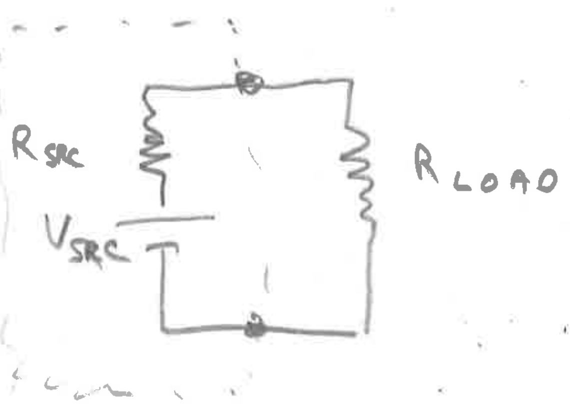
$$V = IR_1 + IR_2$$

$$V_{out} = IR_2$$

$$\frac{V_{out}}{V} = \frac{R_2}{R_1 + R_2}$$

$$V_{out} = \frac{R_2}{R_1 + R_2}$$

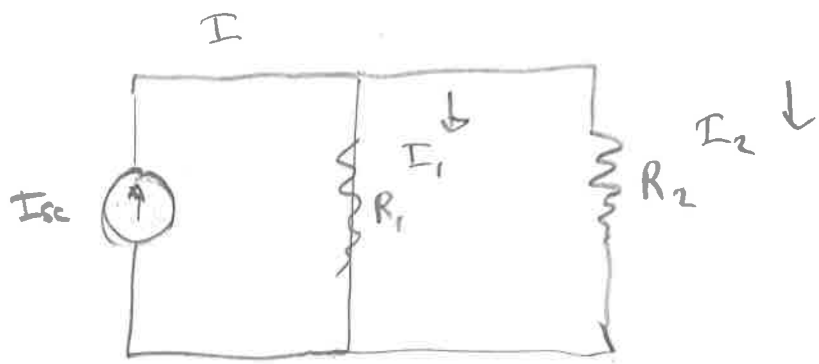
From TE., nearly any realistic source makes a voltage divider!



TE.

Good Designs have (usually) $R_{SRC} \ll R_{LOAD}$

Current Divider



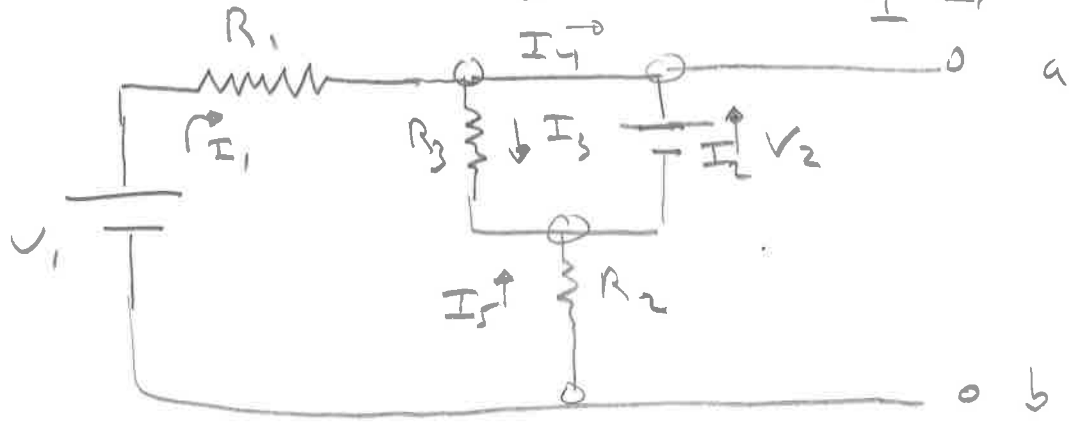
$$I = I_1 + I_2$$

$$V = I_1 R_1 = I_2 R_2$$

$$\frac{I_1 R_1}{I_2 R_2} = \frac{V}{V} = 1$$

$$\Rightarrow \boxed{\frac{I_1}{I_2} = \frac{R_2}{R_1}}$$

T. E.



For open-circuit voltage, our earlier calculation of I1 still holds:

$$U_{Th} = U_1 - I_1 R_1$$
$$= U_1 - \frac{U_1 - U_2}{R_1 + R_2} R_1$$

$$\boxed{U_{Th} = \frac{R_2 U_1 + R_1 U_2}{R_1 + R_2}}$$

But for I_{sc}, currents change $(I_1 \neq I_2 + I_3)$

$I_1 = I_3 + I_4$ $\qquad$ $I_4 + I_2 = I$

$I_2 = I_3 + I_5$

$I = I_1 + I_5$

$$V_1 - I_1 R_1 - V_2 + I_5 R_2 = 0$$

$$V_2 - I_5 R_2 = 0$$

(other loop I_5 not needed!)

$$I_5 = U_2 / R_2$$

$$V_1 - I_1 R_1 - V_2 + \left(\frac{U_2}{R_2}\right) R_2$$

$$I_1 = U_1 / R_1$$

$$I = I_1 + I_5 = \frac{U_1}{R_1} + \frac{U_2}{R_2}$$

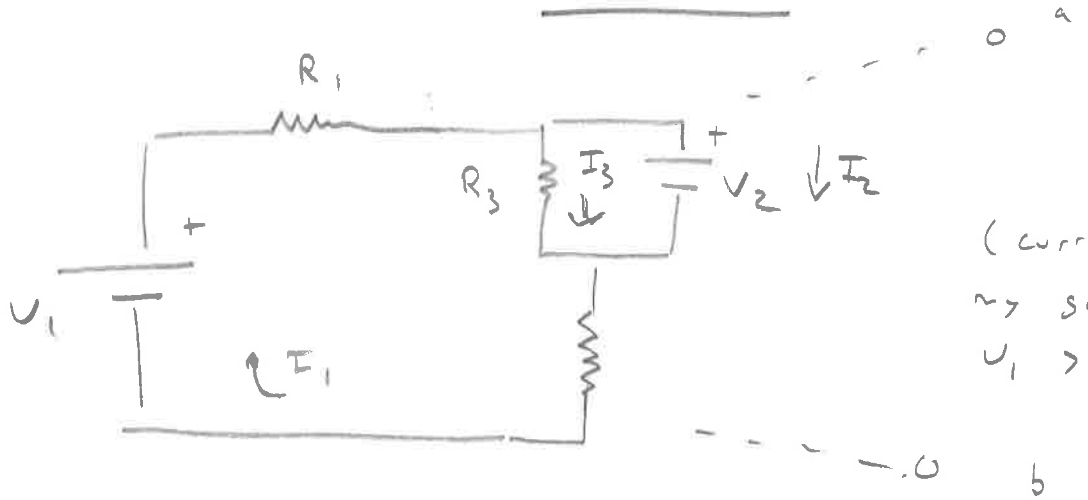
$$\boxed{I_{sc} = \frac{U_1 R_2 + U_2 R_1}{R_1 R_2}}$$

$$R_{Th} = \frac{U_{Th}}{I_{sc}} = \frac{\dfrac{R_2 U_1 + R_1 U_2}{R_1 + R_2}}{\dfrac{R_2 U_1 + R_1 U_2}{R_1 R_2}}$$

$$\boxed{R_{Th} = \frac{R_1 R_2}{R_1 + R_3}}$$

Easier ways?

Example



(currents drawn for my guess if $U_1 >> U_2$)

$$I_1 = I_2 + I_3$$

$$U_2 + (-I_3 R_3) = 0$$

$$U_1 - I_1 R_1 - U_2 - I_1 R_2 = 0$$

$$I_1 = \frac{U_1 - U_2}{R_1 + R_2}$$ (independent of R_3)

$$I_3 = V_2 / R_3$$

$$I_2 = I_1 - I_3 = \frac{U_1 - V_2}{R_1 + R_2} - \frac{U_2}{R_3}$$

$$= \frac{R_3 U_1 - (R_1 + R_2 + R_3) U_2}{(R_1 + R_2) R_3}$$

$R_3 = 0$, $I_2 \to \infty$ (supplying a S.C. !) ✓

$R_3 = \infty$ $I_2 \to \frac{U_1 - U_2}{R_1 + R_2} = I_1$ ✓

Claim

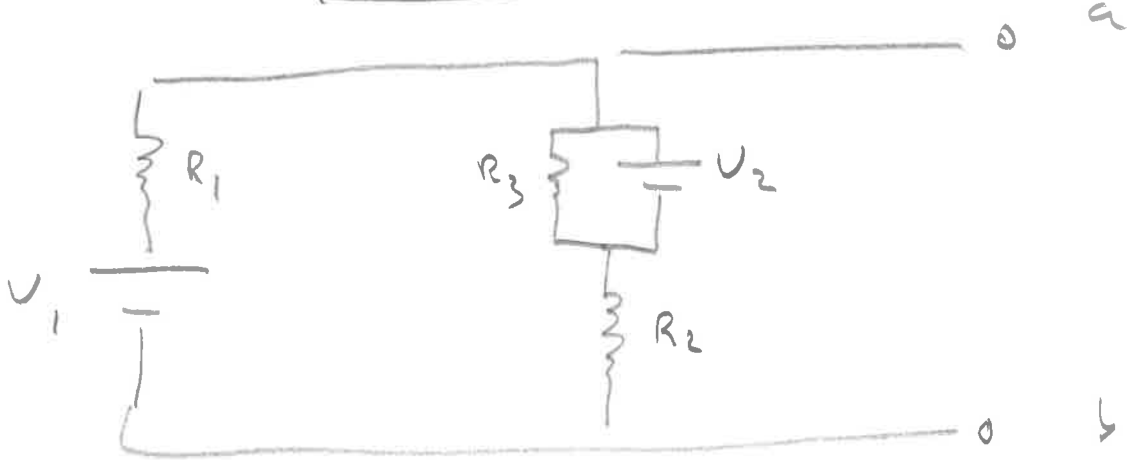
Because of superposition principle, you can solve any circuit by setting $V=0$ for all but one voltage source at a time, then add contributions ....

Corollary:

For TE resistance, you can short all voltage sources, and calculate effective resistance of network.

Lastly, you can replace a subnetwork (between two terminals) with its T.E. (provided you don't care about details of subnetwork)

Fast way



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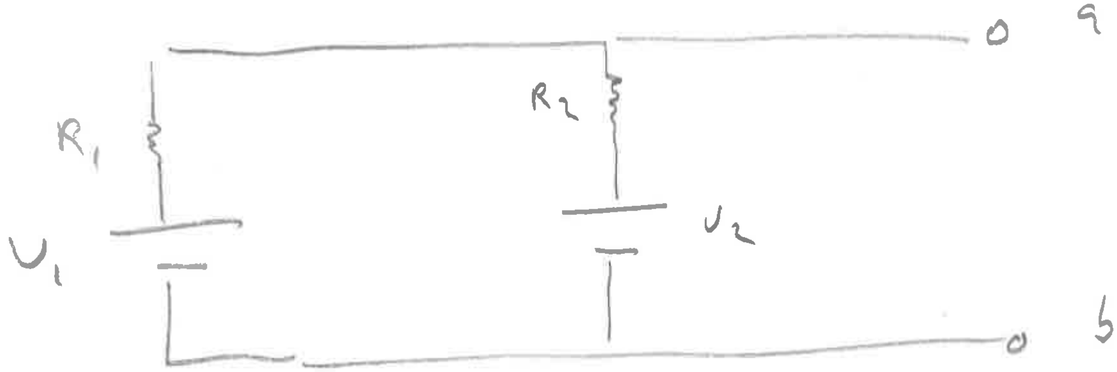


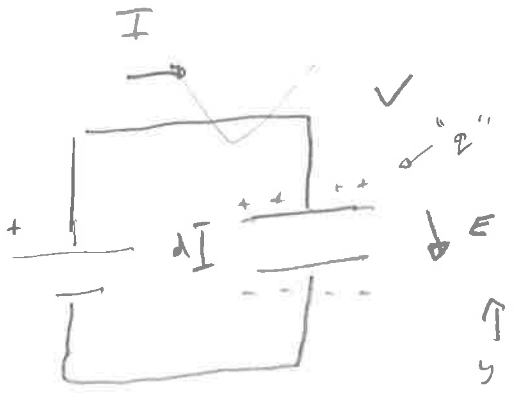
$$V_{Th} = V_{th}(V_2 = 0) + V_{Th}(V_1 = 0)$$

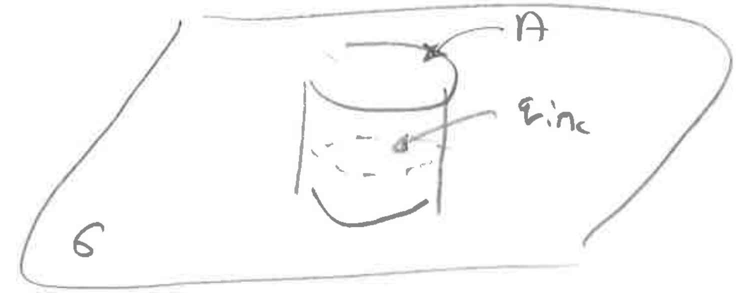
$$V_{Th} = V_1 \frac{R_2}{R_1 + R_2} + V_2 \frac{R_1}{R_1 + R_2}$$

$$= \frac{V_1 R_2 + V_2 R_1}{R_1 + R_2}$$

$$R_{Th} = \frac{R_1 R_2}{R_1 + R_2}$$

Capacitor Model





From: (im

$$E_y = -\frac{q_{inc}}{\varepsilon_0 A}$$

$$V = -\int_0^d dy \left(-\frac{q_{inc}}{\varepsilon_0 A}\right)$$

$$= \frac{d}{\varepsilon_0 A} q$$

$$q = \frac{\varepsilon_0 A}{d} V$$

$$\boxed{\begin{aligned} q &= CV \\ C &= \frac{\varepsilon_0 A}{d} \end{aligned}}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$$

$$\int \nabla \cdot \vec{E} \, dV = \int \frac{1}{\varepsilon_0} \rho \, dV$$

$$\int \vec{E} \cdot d\vec{A} = \frac{q_{inc}}{\varepsilon_0}$$

$$E A = \frac{q_{inc}}{\varepsilon_0}$$

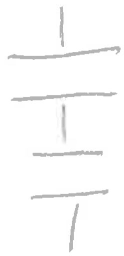
$$E = \frac{q_{inc}}{\varepsilon_0 A}$$

Units: $[F] = \frac{C}{V} = \frac{A\,s}{V} = \frac{s}{\Omega}$

$(F \cdot \Omega = s)$!!!

Exercise 1: From $C = \varepsilon_0 \frac{A}{d}$, deduce C_{eq} for capacitors in parallel and in series:

$A_{eq} = A$

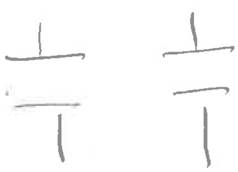


$d_{eq} = d_1 + d_2$

$$\varepsilon_0 C_{eq} = \varepsilon_0 \frac{A_{eq}}{d_{eq}} = \frac{\varepsilon_0 A}{d_1 + d_2}$$

$$\frac{1}{C_{eq}} = \frac{d_1}{\varepsilon_0 A} + \frac{d_2}{\varepsilon_0 A} = \frac{1}{C_1} + \frac{1}{C_2}$$

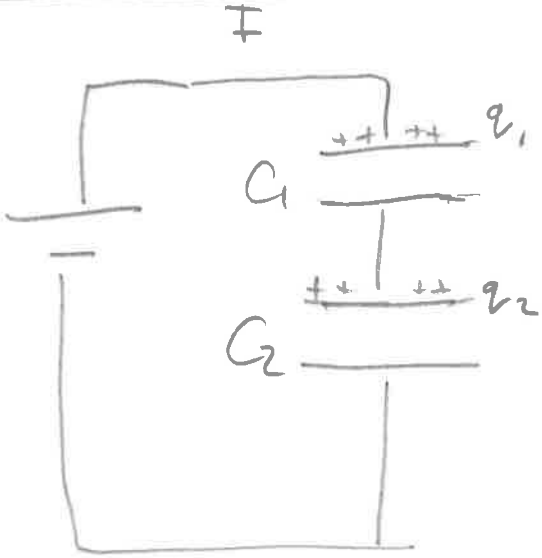
$A_{eq} = A_1 + A_2$

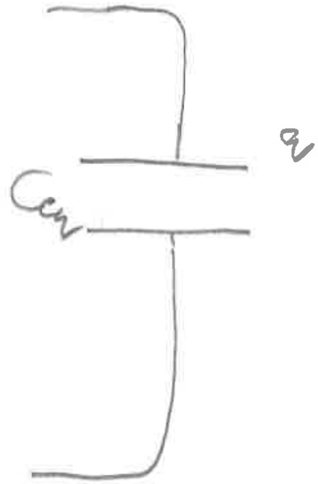


$d_{eq} = d$

$$C_{eq} = \varepsilon_0 \frac{A_{eq}}{d_{eq}} = \varepsilon_0 \frac{A_1 + A_2}{d} = C_1 + C_2$$

Exercise 2





$$q = \int I\,dt \qquad I = I_1 = I_2$$

$$\Rightarrow q = q_1 = q_2 \quad (*)$$

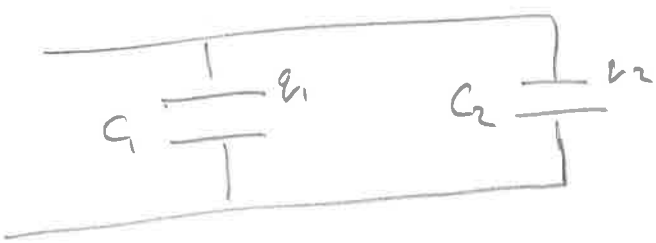
((*) assumes initially uncharged! But in real system, leaks will tend to equalize the capacitance.)

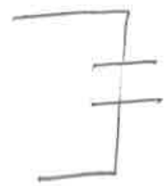
$$V = \frac{q}{C_{eq}} = \frac{q_1}{C_1} + \frac{q_2}{C_2} = \frac{q}{C_1} + \frac{q}{C_2}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Exercise 3

HW?





$I = I_1 + I_2 \Rightarrow q = q_1 + q_2$

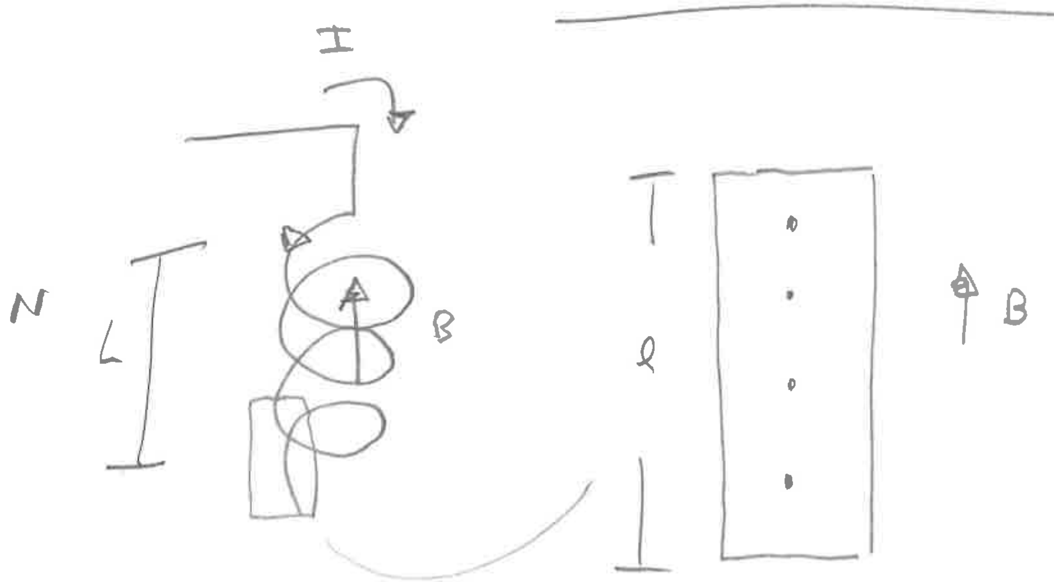
$V = \frac{q}{C}$

$\Rightarrow q = CV$

$C_{eq} V = C_1 V + C_2 V$

$$\boxed{C_{eq} = C_1 + C_2}$$

Current in Solenoid



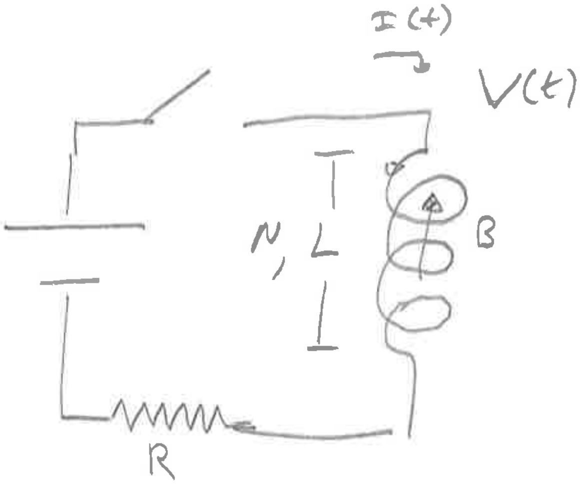
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\int \vec{\nabla} \times \vec{B} \; d\vec{A} = \mu_0 \int \vec{J} \cdot d\vec{A}$$

$$\int \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{A}$$

$$B l = \mu_0 \left(I \, l \cdot \frac{N}{L} \right)$$

$$B = \mu_0 \, I \, \frac{n}{L}$$

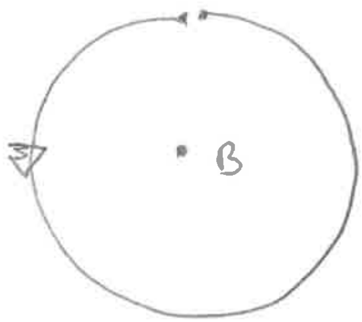


Self Inductance

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{dt}$$

$$\int \vec{\nabla} \times \vec{E} \cdot d\vec{A} = - \int \frac{d\vec{B}}{dt} \cdot d\vec{A}$$

$$- \int \vec{E} \cdot d\vec{l} = \int \frac{d\vec{B}}{dt} \cdot d\vec{A}$$



$$V_{loop} = - \frac{d}{dt} B A$$

$$= \frac{d}{dt} \left(\mu_0 \frac{N}{L} I \right) A$$

$$= \mu_0 \frac{A N}{L} \frac{dI}{dt}$$

$$V = N V_{loop} = \mu_0 \frac{A N^2}{L} \frac{dI}{dt}$$

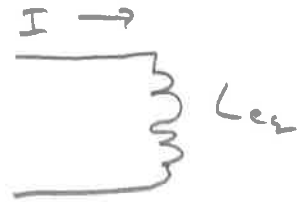
$$V = L \frac{dI}{dt}$$

$$L = \mu_0 \frac{N^2 A}{L}$$

$$[L] = H = \frac{V s}{A} = \Omega s$$

$$\left(H / \Omega = s \right)$$

Exercise 1

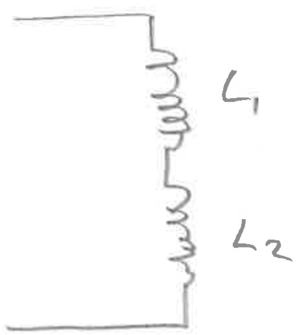


$$I = I_1 + I_2 \quad \Rightarrow \quad \frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

$$\left(V = L\,\frac{dI}{dt}\right) \quad \Rightarrow \quad \frac{V}{L_{eq}} = \frac{V}{L_1} + \frac{V}{L_2}$$

$$\Rightarrow \quad \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$$

Exercise 2



$$V = V_1 + V_2 \qquad L_{eq}\frac{dI}{dt} = L_1\frac{dI_1}{dt} + L_2\frac{dI_2}{dt}$$

$$\boxed{L_{eq} = L_1 + L_2}$$

Exercise: From

$$L \sim n^2 \frac{A}{d}$$

can you guess equivalent inductance for series inductors.

ANS: $$L \sim n^2 \frac{A}{d} = \cancel{\left(\frac{n}{d}\right)}^{\text{const}} n A$$

n adds linearly.

Inductors are like resistors re equivalence.

* Assuming no cross-inductance *

RLC Summary

$R \sim \frac{d}{A}$ $C \sim \frac{A}{d}$ $L \sim \frac{A}{d} n^2$

$V = IR$ $V = \frac{q}{C}$ $V = L \frac{dI}{dt}$

series = sum series = quad series = <u>sum</u>

Solutions to Linear Differential Equations

$$\underset{\text{given}}{\mathcal{L}}\ \overset{\text{solve}}{y(t)} = \underset{\text{given}}{F(t)}$$

Linear means:

$$\mathcal{L}\left[\alpha x(t) + \beta y(t)\right] = \alpha \mathcal{L} x(t) + \beta \mathcal{L} y(t)$$

Homogenous Diff Eqs

$$\mathcal{L} y(t) = 0$$

$$\mathcal{L} x(t) = 0$$

$$\Rightarrow \quad \mathcal{L}\left(\alpha x(t) + \beta y(t)\right) = 0$$

* "easy to solve for general case"

For Non-homogenous, suffices to find one solution

$$\mathcal{L} z(t) = F(t)$$

Then

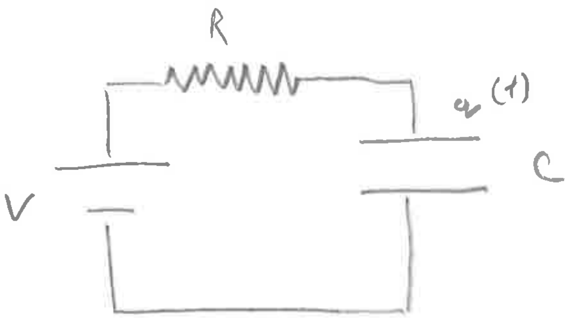
$$\mathcal{L}\left(z(t) + \alpha x(t) + \beta y(t)\right) = F(t)$$

** Also can do $\mathcal{L} z_1(t) = F_1(t)$, $\mathcal{L} z_2(t) = F_2(t)$ **

Maxwell Equations are linear (until material interactions are strong!)

* We'll see later, even non-linear problems can be linearized near the operating point! *

RC circuit



KVL: $0 = V - IR - \frac{q}{C}$ or $V = IR + \frac{q}{C}$

$$V = \frac{dq}{dt} R + \frac{q}{C}$$

q = const is specific solution: $V = \frac{q}{C} \Rightarrow q = CV$

Now solve $\frac{dq}{dt} R + \frac{q}{C} = 0$

$q(t) = A \exp^{Bt}$ $\qquad \frac{dq}{dt} = BA \exp^{Bt} = Bq$

$$BRq + \frac{q}{C} = 0 \Rightarrow B = -\frac{1}{RC}$$

General Solution:

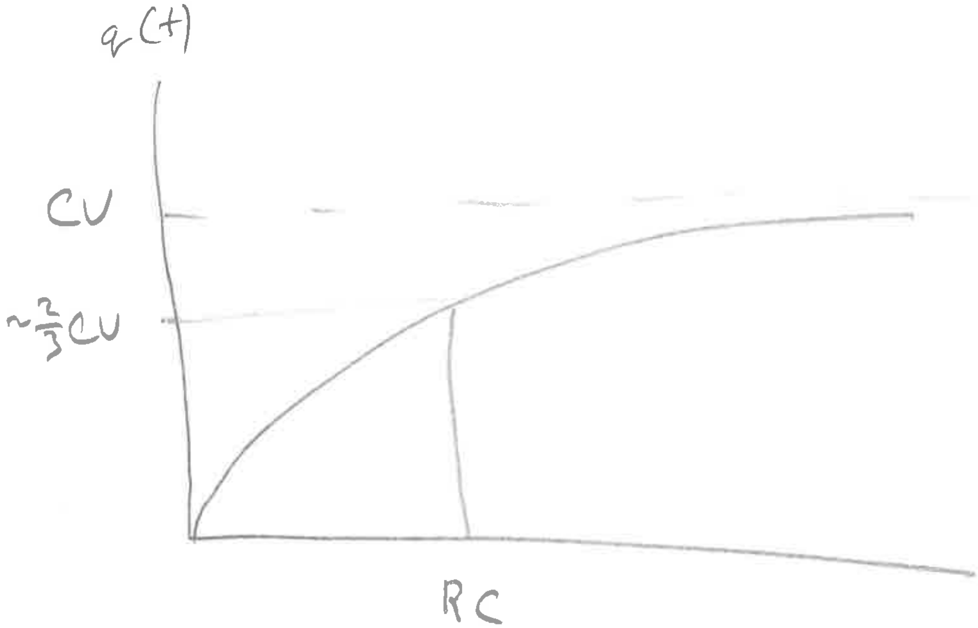
$$q(t) = CV + A \exp\left(-\frac{t}{RC}\right)$$

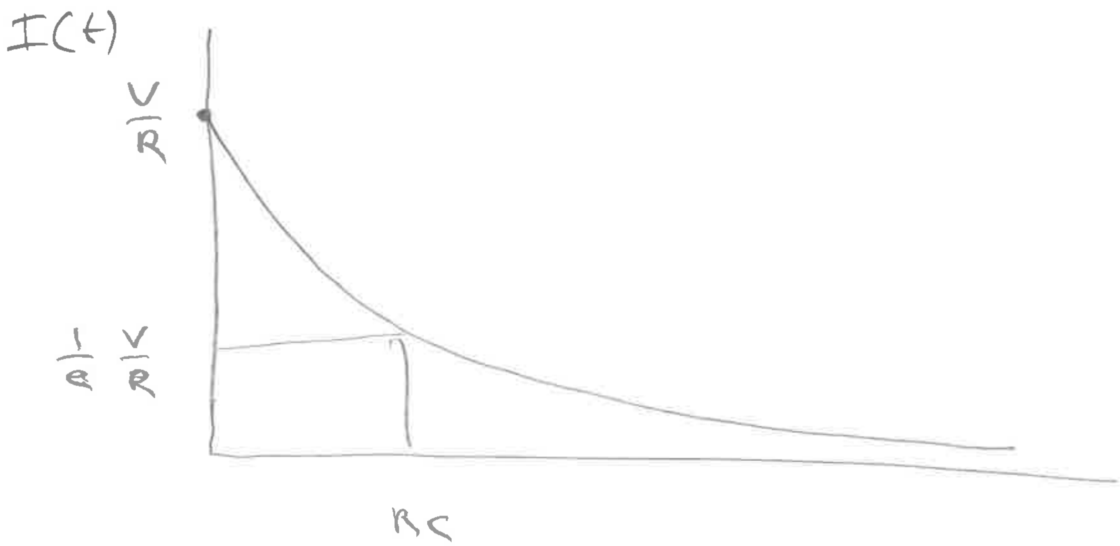
IF $q = 0$ at $t = 0$

$$q(t) = CV\left(1 - \exp\left(-\frac{t}{RC}\right)\right)$$

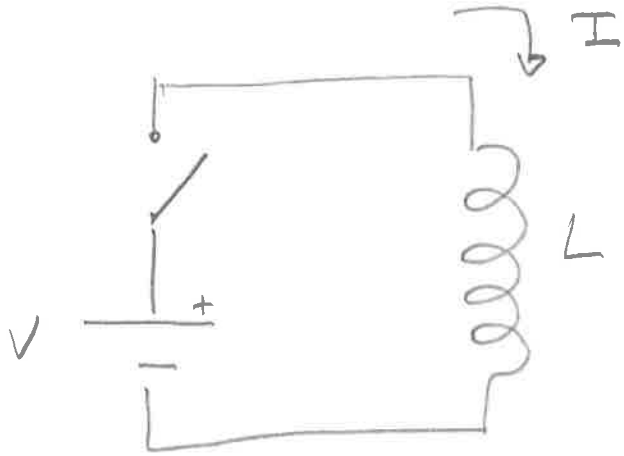
$$I(t) = \frac{dq}{dt} = \frac{CV}{RC} \exp\left(-\frac{t}{RC}\right)$$

$$= \frac{V}{R} \exp\left(-\frac{t}{RC}\right)$$

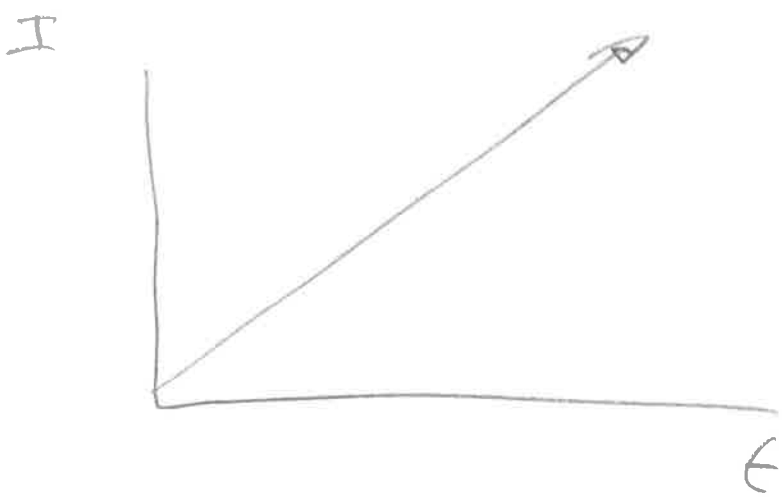

q(t)
CV
~2/3 CV
RC


I(t)
V/R
1/e V/R
RC

An L circuit



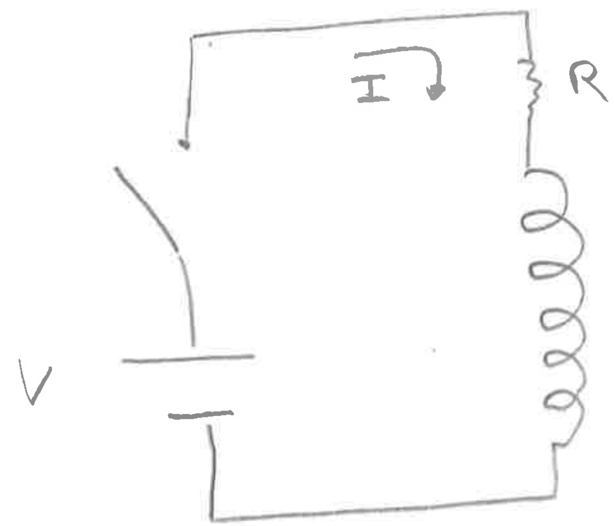
$$V = L\frac{dI}{dt} \Rightarrow I = \frac{V}{L}t \quad + \text{const}$$



$p = IV = \dfrac{V^2}{L}t$ is increasing . . .

. . . not very physical

An LR circuit



$$V = IR + L\frac{dI}{dt}$$

Special solution: $I = \text{const} \Rightarrow I = \frac{V}{R}$

Now solve: $0 = IR + L\frac{dI}{dt}$

$$I = A \exp Bt \qquad \frac{dI}{dt} = BI$$

$$0 = IR + LBI \Rightarrow B = -\frac{R}{L}$$

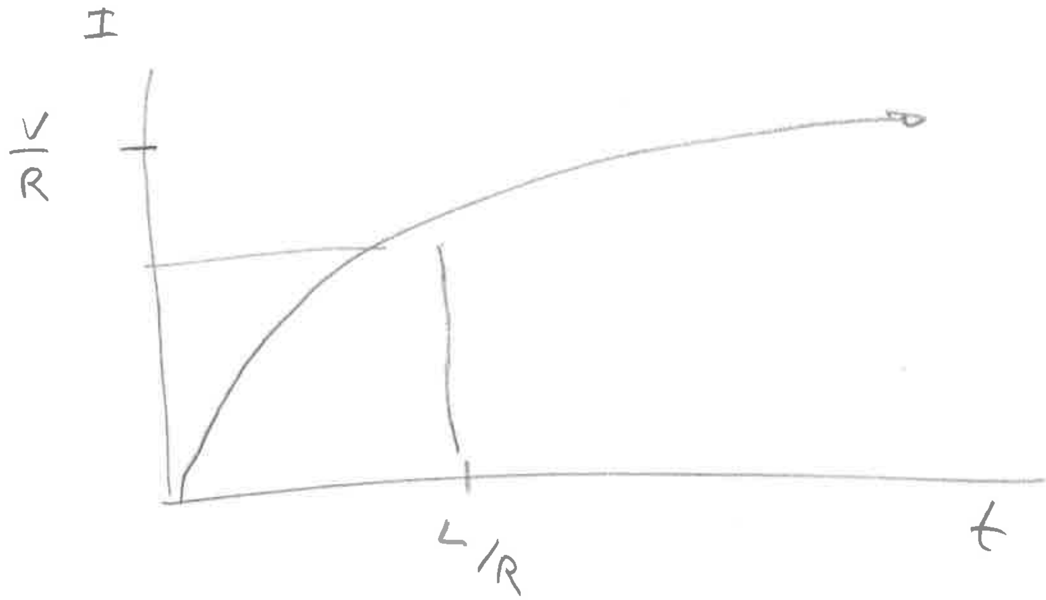
General Solution: $I = \frac{V}{R} + A \exp\left(-\frac{t}{L/R}\right)$

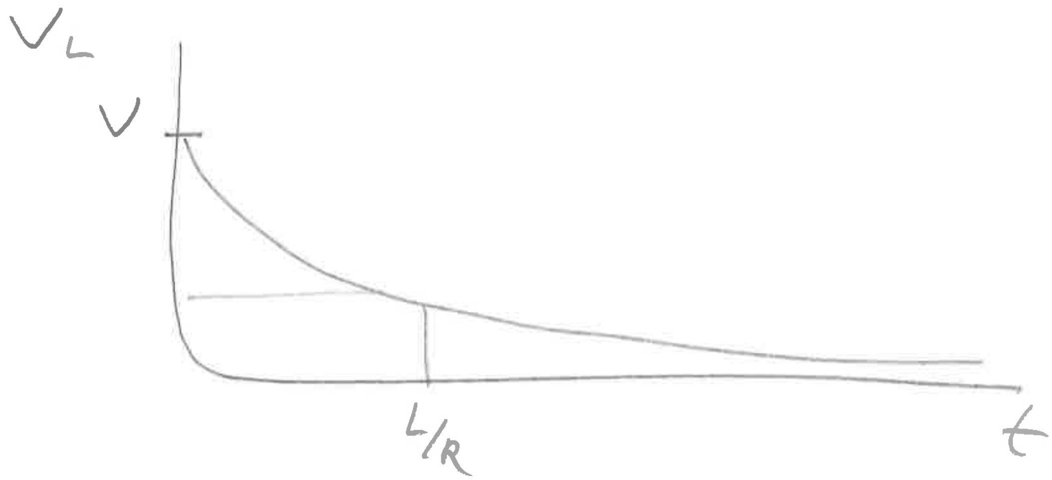
$I = 0$ at $t = 0$ (continuity)

$$\Rightarrow I(t) = \frac{V}{R}\left(1 - \exp\left(-\frac{t}{L/R}\right)\right)$$

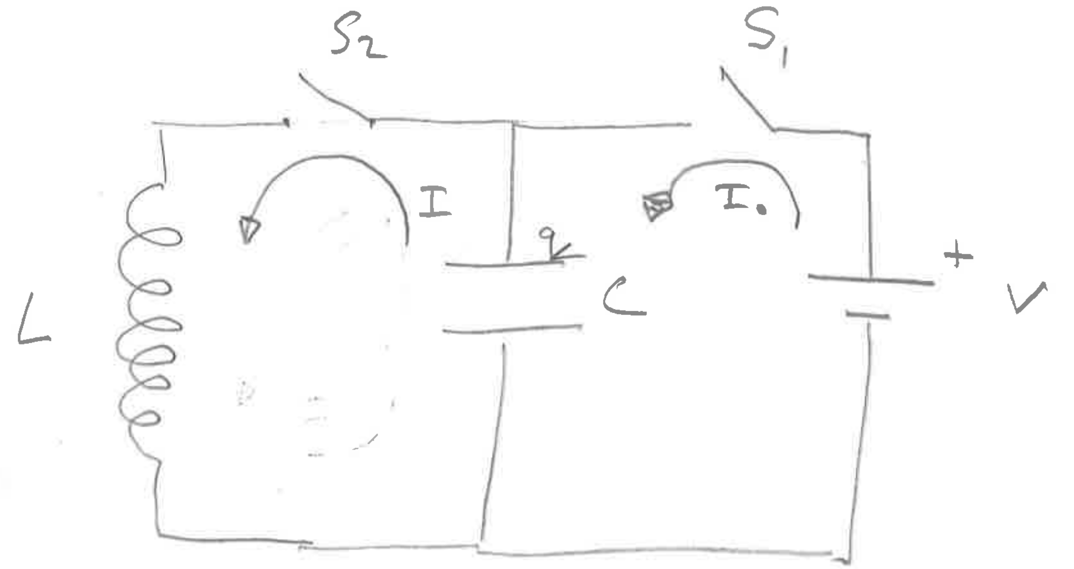
$$V_L = L\frac{dI}{dt} = V\exp\left(-\frac{t}{L/R}\right)$$

Note: small $t \ll L/R$, recover $I = \frac{V}{L}t$


I
V
R
L/R
t


V_L
V
L/R
t

LC circuit



S_1 is initially closed for long time, with S_2 open,
so capacitor charges to $q = CV$, and $V_{cap} = V$

$$\frac{q}{C} = L\frac{dI}{dt} \qquad I = -\frac{dq}{dt}$$

$$q = -LC\frac{d^2q}{dt^2}$$

$$q = A\cos\omega t + B\sin\omega t$$

$$\frac{d^2q}{dt^2} = -\omega^2 q \quad \Rightarrow \quad \omega = \sqrt{LC} = \frac{2\pi}{T}$$

$$I = -\omega A\sin\omega t + \omega B\cos\omega t$$

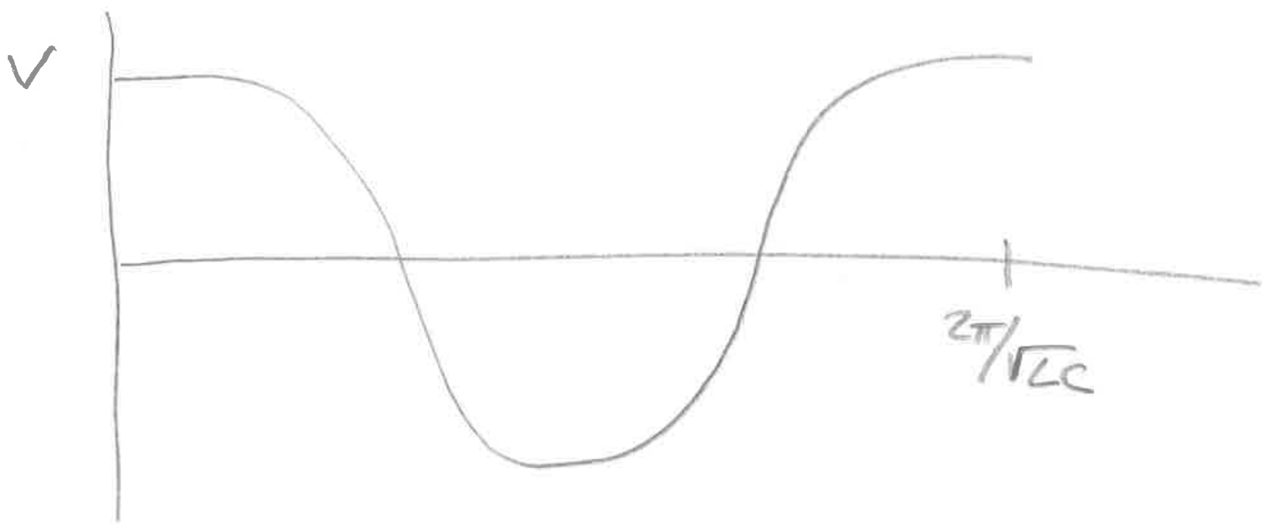
$I = 0$ at $t = 0 \Rightarrow B = 0$

$q = A = CV \qquad q(t) = CV\cos\sqrt{LC}\,t$

$$V_{cap}(t) = \frac{q(t)}{C} = V \cos t/\sqrt{LC}$$

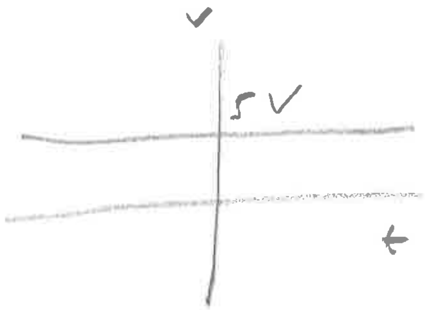
$$V_L(t) = L\frac{dI}{dt} = V \cos t/\sqrt{LC}$$

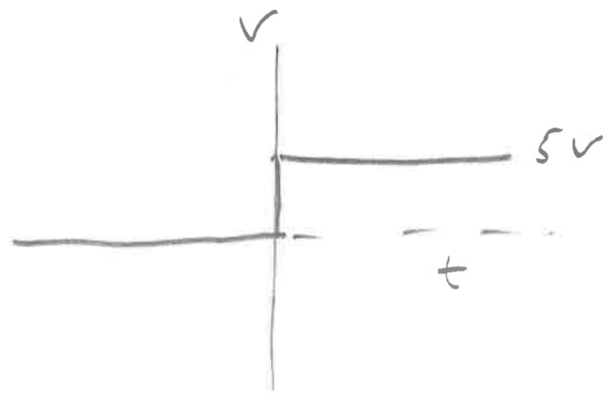
(as required by KVL)



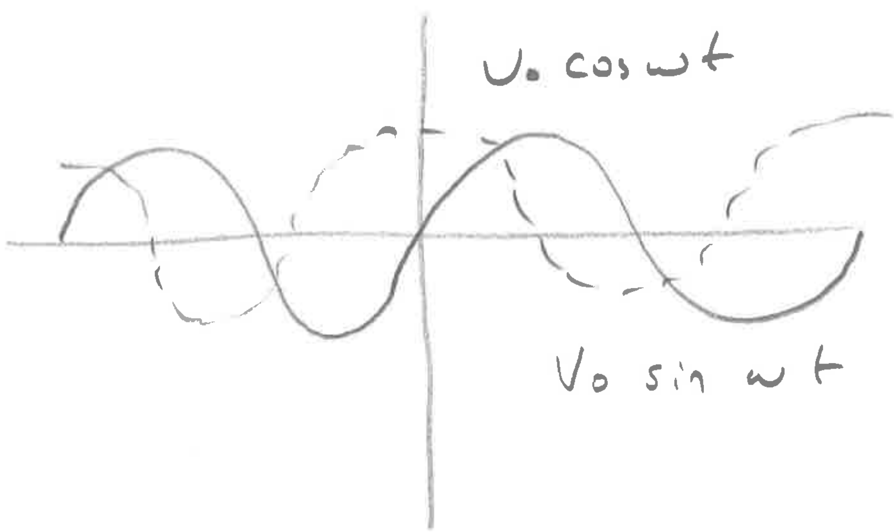
Alternating Current

So far we've dealt w/ direct current, possibly switched on/off:



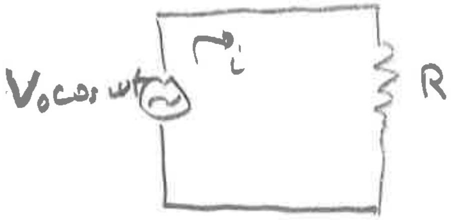


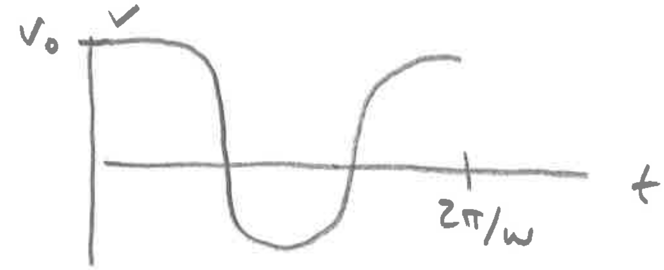
Now we'll consider time varying voltages, particularly, sines and cosines:



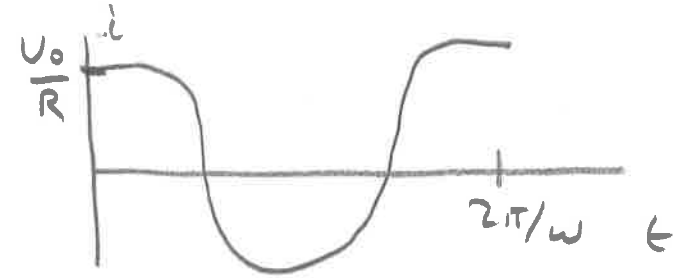
Convention of text: AC signals lower case.

AC circuits w/ R, C, L

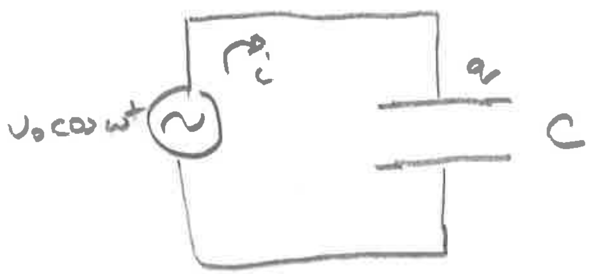


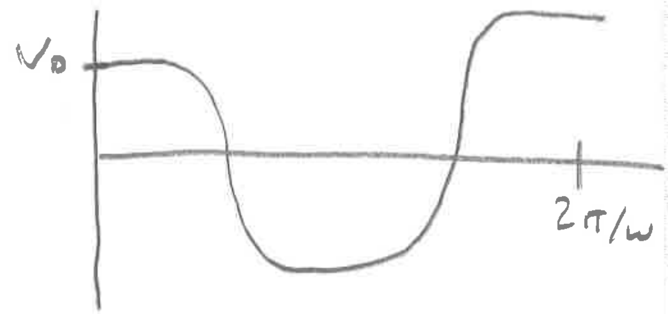


$$i = V/R = \frac{V_0}{R}\cos \omega t$$



$$\boxed{I_{max} = V_0/R \qquad \phi = 0}$$

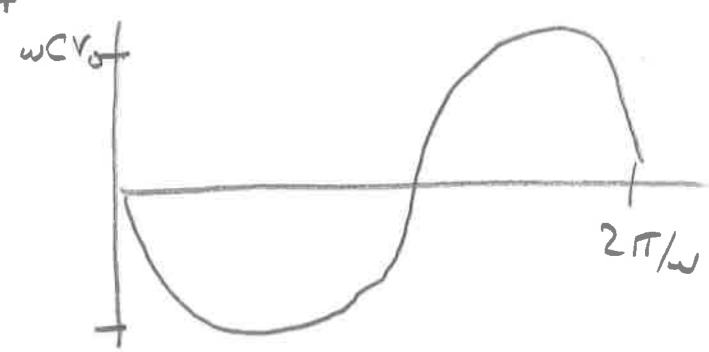




$$q(t) = C\,V(t) = C V_0 \cos \omega t$$

$$i = \frac{dq}{dt} = -\omega C V_0 \sin \omega t$$

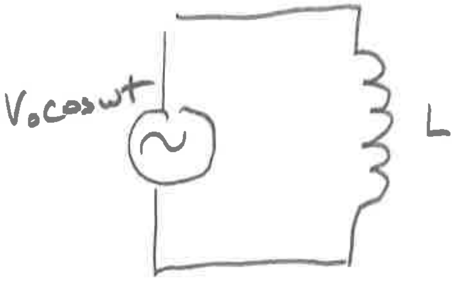
$$= \omega C V_0 \cos\left(\omega t + \frac{\pi}{2}\right)$$

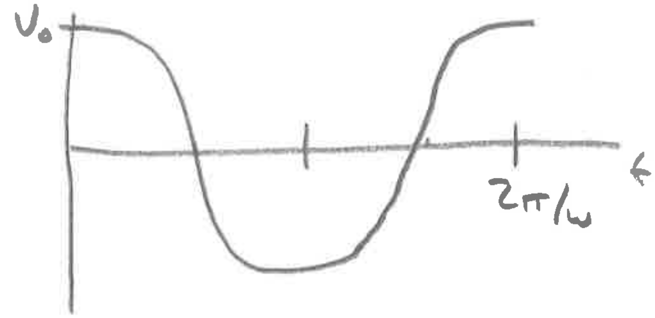


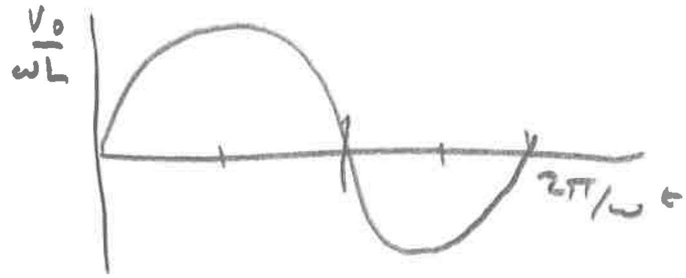
$$\boxed{\begin{aligned} I_{max} &= \omega C V_0 \\ \phi &= (-)^{*}\frac{\pi}{2} \end{aligned}}$$

(current leads voltage $\frac{1}{4}$ period)

*(ϕ of voltage relative to current)







$$v = V_0 \cos \omega t = L \frac{di}{dt}$$

$$\frac{di}{dt} = \frac{V_0}{L} \cos \omega t$$

$$i = \frac{V_0}{L} \int \cos \omega t \, dt$$

$$i = \frac{V_0}{\omega L} \sin \omega t + \cancel{C}$$

vanishes w/ any R!

$$= \frac{V_0}{\omega L} \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$I_{max} = \frac{V_0}{\omega L}$$

$$\phi = (+)^{*} \frac{\pi}{2}$$

(current lags voltage $\frac{1}{4}$ period)

*(ϕ of voltage relative to current)

To summarize:

Component	I_{max}	ϕ
R	V_0/R	0
C	$\omega C V_0$	$-\frac{\pi}{2}$
L	$V_0/\omega L$	$\frac{\pi}{2}$

R has resistance R

C has reactance $\frac{1}{\omega C}$

L has reactance ωL

But what about phase?

$\rightarrow$ Complex Plane!

"Phase Vectors" or "Phasors"

An AC signal:

$$V(t) = V_0 \cos(\omega t + \phi)$$

can be rearranged as

$$v(t) = \text{Re}\left\{ V_0 \, e^{j(\omega t + \phi)} \right\}$$

$$= \text{Re}\left\{ \left[V_0 e^{j\phi} \right] e^{j\omega t} \right\}$$

magnitude and phase of our specific signal.

generic AC w/ ang freq ω.

$\tilde{V}$ is a complex number, representing AC signal by

$$\boxed{v(t) \equiv \text{Re}\left\{ \tilde{V} e^{j\omega t} \right\}}$$

We call $\tilde{V}$ the "phase vector" or "phasor" for the AC signal $v(t)$.

<u>Given:</u> $V(t) = V_0 \cos(\omega t + \phi)$

$$\boxed{\tilde{V} \equiv V_0 \, e^{i\phi}}$$

Examples:

$\tilde{V} = V_0$

$$v(t) = \mathrm{Re}\left\{ \tilde{V} e^{j\omega t} \right\}$$
$$= \mathrm{Re}\left\{ V_0 \cos\omega t + j V_0 \sin\omega t \right\}$$
$$= V_0 \cos \omega t$$

$\tilde{V} = j V_0$

$$v(t) = \mathrm{Re}\left\{ (j V_0)\, e^{j\omega t} \right\}$$
$$= \mathrm{Re}\left\{ j V_0 \cos\omega t + j V_0 j \sin\omega t \right\}$$
$$= - V_0 \sin \omega t$$

$\tilde{V} = V_0 e^{j\phi}$

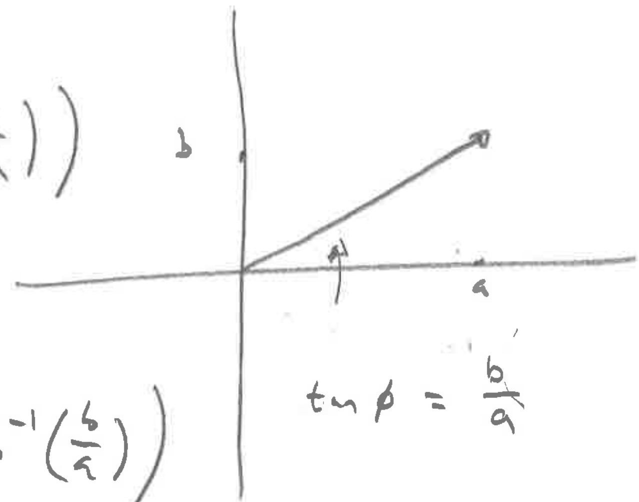
$$v(t) = \mathrm{Re}\left\{ V_0 e^{j\phi} \cdot e^{j\omega t} \right\}$$
$$= \mathrm{Re}\left\{ V_0 e^{j(\omega t + \phi)} \right\}$$
$$= \mathrm{Re}\left\{ V_0 \cos(\omega t + \phi) + j V_0 \sin(\omega t + \phi) \right\}$$
$$= V_0 \cos(\omega t + \phi)$$

* Nothing magical here, just a nice compact notation.... so far! *

* Important Example *

$$\tilde{V} = a + bj$$

$$= \sqrt{a^2+b^2}\ \exp\left(j \tan^{-1}\left(\frac{b}{a}\right)\right)$$



$$\tan\phi = \frac{b}{a}$$

$$v(t) = \sqrt{a^2+b^2}\ \cos\left(\omega t + \underbrace{\tan^{-1}\left(\frac{b}{a}\right)}_{\phi}\right)$$

So why is this useful?

$$x(t) = \mathrm{Re}\left\{ \tilde{X} e^{j\omega t} \right\}$$

$$\frac{dx}{dt} = \mathrm{Re}\left\{ \tilde{X} \frac{d}{dt} e^{j\omega t} \right\}$$

$$= \mathrm{Re}\left\{ (i\omega \tilde{X})\, e^{j\omega t} \right\}$$

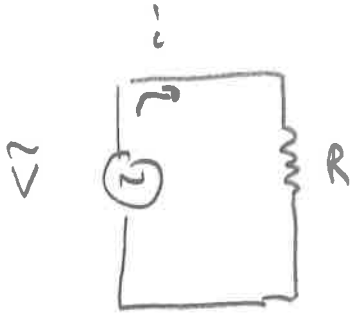
$$\int x \, dt = \mathrm{Re}\left\{ \tilde{X} \int dt \; e^{j\omega t} \right\}$$

$$= \mathrm{Re}\left\{ \frac{\tilde{X}}{i\omega} e^{j\omega t} + \underbrace{\cancel{C}}_{\text{(typically vanishes)}} \right\}$$

To take $\frac{d}{dt}$, multiply phasor by $j\omega$

To integrate, multiply phasor by $\frac{1}{j\omega}$

Phasors in RLC circuits

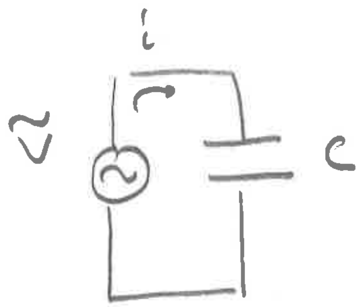


$$v(t) = \mathrm{Re}\left\{ \tilde{V} e^{j\omega t} \right\}$$

$$i(t) = \frac{v(t)}{R} = \mathrm{Re}\left\{ \frac{\tilde{V}}{R} e^{j\omega t} \right\}$$

$$\tilde{I} = \frac{\tilde{V}}{R} \qquad \boxed{\tilde{V} = R\tilde{I}}$$

Recover Ohm's Law



$$v(t) = \mathrm{Re}\left\{ \tilde{V} e^{j\omega t} \right\}$$

$$q(t) = Cv(t) = \mathrm{Re}\left\{ C\tilde{V} e^{j\omega t} \right\}$$

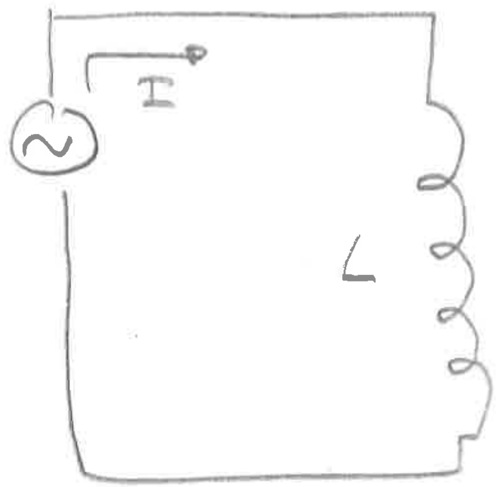
$$i(t) = \frac{dq}{dt} = \mathrm{Re}\left\{ j\omega C \tilde{V} e^{j\omega t} \right\}$$

$$\tilde{I} = (j\omega C)\tilde{V}$$

$$\tilde{V} = \left(\frac{1}{j\omega C} \right) \tilde{I}$$

Ohm's Law for a Capacitor!!!

Phasor in L circuit



$$V(t) = \mathrm{Re}\left\{ V_0 e^{j\phi} e^{j\omega t} \right\}$$

$$V = L \frac{dI}{dt}$$

$$\Rightarrow \frac{dI}{dt} = \frac{V}{L}$$

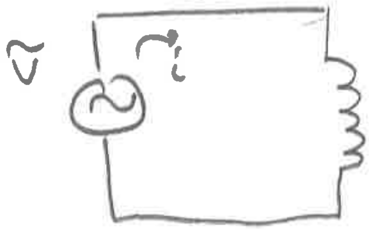
$$I = \frac{1}{L} \int V dt$$

$$= \mathrm{Re}\left\{ \frac{V_0}{L} e^{j\phi} \int dt\, e^{j\omega t} \right\}$$

$$= \mathrm{Re}\left\{ \frac{V_0 e^{j\phi}}{j\omega L} e^{j\omega t} \right\}$$

$$\tilde{I} = \frac{\tilde{V}}{j\omega L} \qquad \tilde{V} = (j\omega L)\tilde{I}$$

Phasor's in L circuit



$$v(t) = \mathrm{Re}\left\{ \tilde{V} e^{j\omega t} \right\}$$

$$v = L \frac{di}{dt}$$

$$\frac{di}{dt} = \frac{v}{L} = \mathrm{Re}\left\{ \frac{\tilde{V}}{L} e^{j\omega t} \right\}$$

$$i = \mathrm{Re}\left\{ \frac{\tilde{V}}{L} \int e^{j\omega t} dt \right\}$$

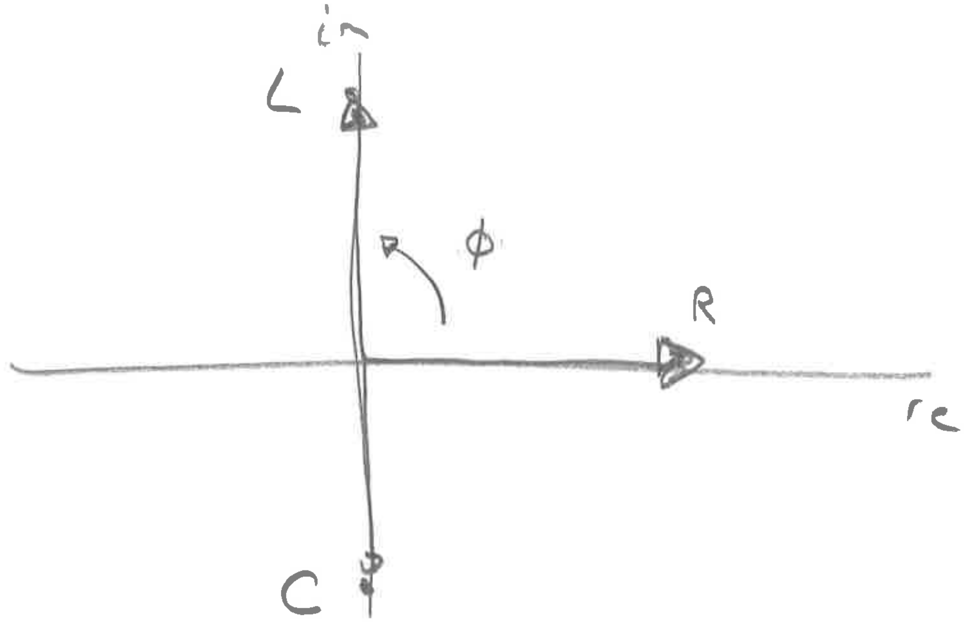
$$= \mathrm{Re}\left\{ \frac{\tilde{V}}{j\omega L} e^{j\omega t} + \cancel{c} \right\}$$

$$\tilde{I} = \frac{\tilde{V}}{j\omega L} \qquad \boxed{\tilde{V} = (j\omega L)\, \tilde{I}}$$

Ohm's Law for Inductor.

Summary

	Z	ϕ (V wrt I)
R	R	0
C	$\frac{1}{j\omega C}$	$-\frac{\pi}{2}$
L	$j\omega L$	$\frac{\pi}{2}$



$$\tilde{V} = Z \tilde{I}$$

To Recap:

$$\tilde{V} = R\,\tilde{I}$$

$$\tilde{V} = \frac{1}{j\omega C}\,\tilde{I}$$

$$\tilde{V} = j\omega L\,\tilde{I}$$

We replace diff eqs with complex algebra!

New Ohm's Law:

$$Z = A + Bj$$

Impedance

Resistance

Reactance

$$\boxed{\tilde{V} = Z\,\tilde{I}}$$

Because Impedance follows Ohm's law, calculate equivalent impedance exactly as for resistance

Decibels (dB)

Used often by Engineers, decibel initially related power to a reference power by a logarithmic scale:

$$P\,[dB] = 10 \log_{10}\left(P/P_0\right)$$

For electrical systems, we have

$$P = \frac{V^2}{R}$$

So

$$P\,[dBV] = 10 \log_{10}\left(P/P_0\right)$$

$$= 10 \log_{10}\left(\frac{V^2/R}{V_0^2/R}\right)$$

$$= 20 \log_{10}\left(V/V_0\right)$$

<u>Confusion:</u>

What is P_0 or V_0 ...

Factors of 20 / 10 ...

Bode Plot Cheat Sheet

$20 \log_{10} \sqrt{2} = 3$

$20 \log_{10} 10 = 20$

$20 \log_{10} 100 = 40$

$\tan^{-1}(1) = \frac{\pi}{4} = 45°$

$\tan^{-1}(0.1) = 6°$

$\tan^{-1}(10.0) = 90° - 6°$

Particularly interesting point (we'll see)

is when

$$P = \frac{1}{2} P_0 \qquad \text{or} \qquad V = \frac{1}{\sqrt{2}} V_0$$

i.e.

$$P[dB] = 10 \log_{10}\left(\frac{1}{2}\right)$$

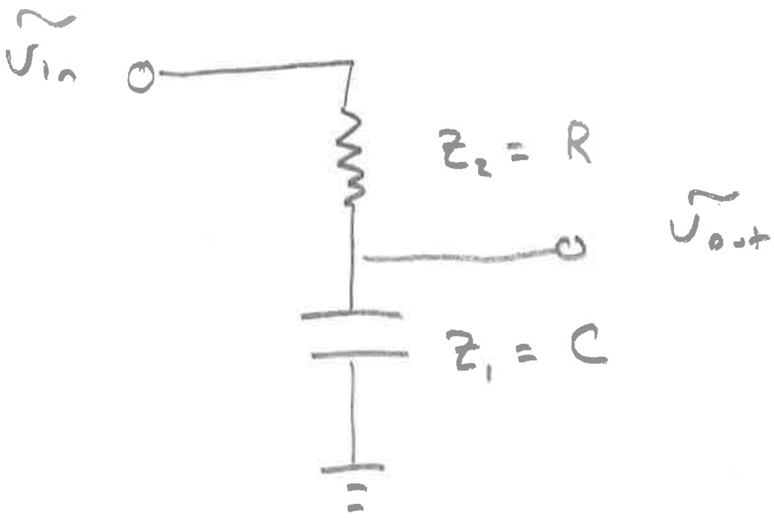
$$= -10 \log_{10} 2 = -3.01$$

$$= 20 \log_{10}\left(\frac{1}{\sqrt{2}}\right)$$

$$= -10 \log_{10}(2) = -3.01$$

The "-3 dB" is where power drops by 1/2, voltage by $1/\sqrt{2}$.

Lowpass Filter



$$H \equiv \frac{\tilde{v}_{out}}{\tilde{v}_{in}} = \frac{Z_1}{Z_1 + Z_2} = \frac{1/j\omega C}{R + 1/j\omega C}$$

$$= \frac{1}{1 + j\omega RC} = \frac{1}{1 + j(\omega/\omega_0)}$$

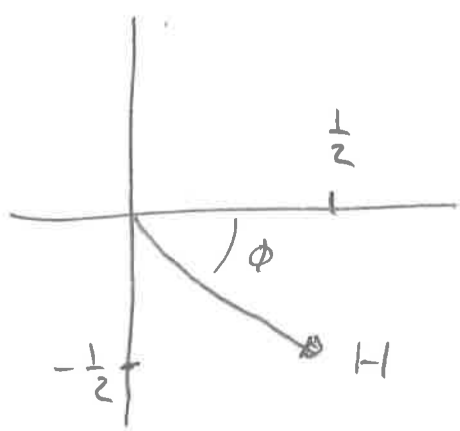
$\omega_0 \equiv 1/RC$

$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi RC}$

$$H = \frac{\tilde{v}_{out}}{\tilde{v}_{in}} = \frac{1 - j(\omega/\omega_0)}{1 + (\omega/\omega_0)^2}$$

At $\boxed{\omega = \omega_0}$:

$$H = \frac{1 - j}{2}$$



Gain:

$$G = \frac{|\tilde{V}_{out}|}{|\tilde{V}_{in}|} = \frac{V_{out}}{V_{in}} = |H| = \boxed{\frac{1}{\sqrt{2}}}$$

$$20 \log_{10} G = -3.01$$

ω_0 is the -3 dB point ...

$\boxed{\phi = -45^\circ}$ from plot or $\phi = \tan^{-1}\left(\frac{-1}{1}\right)$

Lowpass Filter

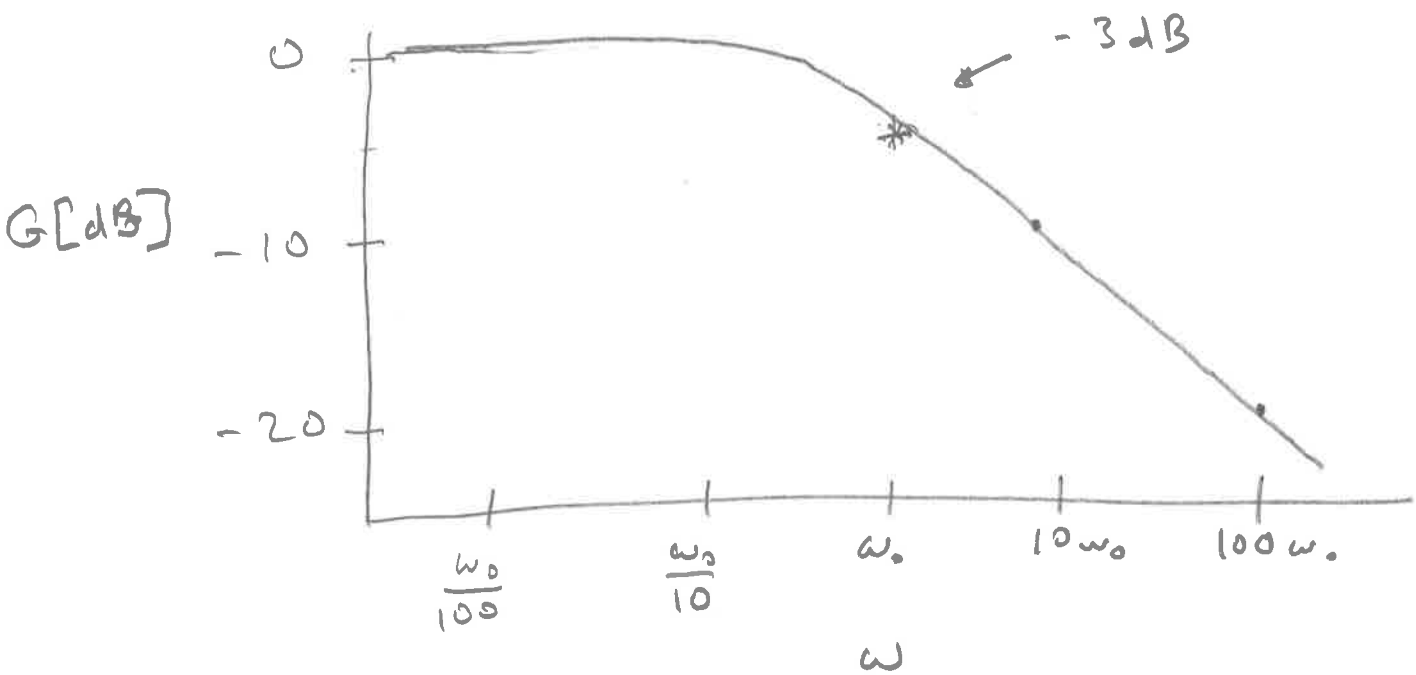
$$G = |H| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

On dB scale:

$$G\,[dB] = 20 \log_{10} G$$

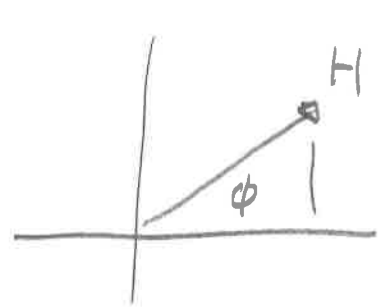
$$= -10 \log_{10} \left[1 + (\omega/\omega_0)^2\right]$$

ω/ω_0	G [dB]
1	-3
10	-20
100	-40
0	0



Phase of Lowpass Filter

$$\phi(H) = \tan^{-1}\left(\frac{Im(H)}{Re(H)}\right)$$

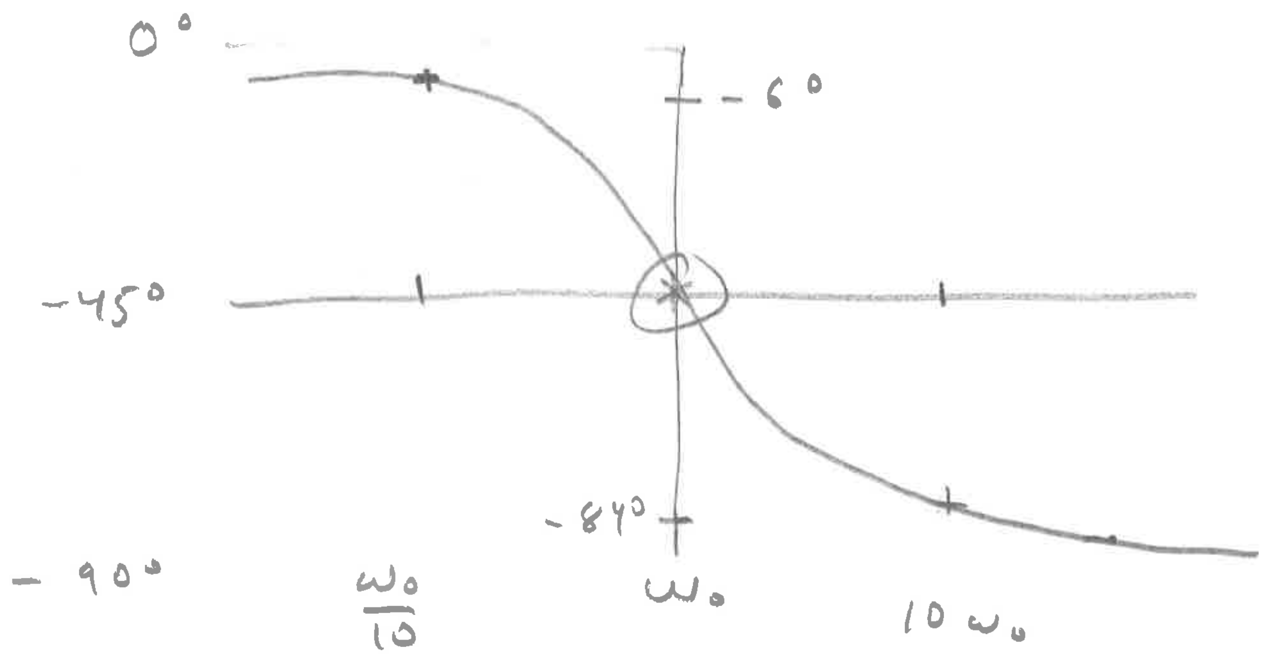


$$H = \frac{1 - j(\omega/\omega_0)}{1 + (\omega/\omega_0)^2}$$

$$\tan\phi = \frac{-(\omega/\omega_0)}{1}$$

$$\phi = -\tan^{-1}(\omega/\omega_0)$$

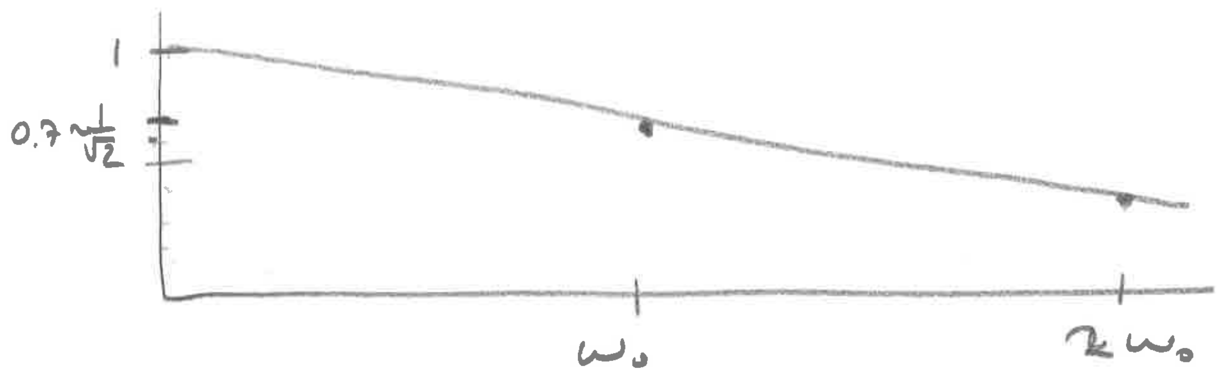
ω/ω_0	
1	-45°
0.1	-6°
10.0	$-(90^\circ - 6^\circ)$



In general:

$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1 - j(\omega/\omega_0)}{1 + (\omega/\omega_0)^2}$$

$$\frac{V_{out}}{V_{in}} = \frac{\sqrt{1 + (\omega/\omega_0)^2}}{1 + (\omega/\omega_0)^2} = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$



On log scale (in dB)

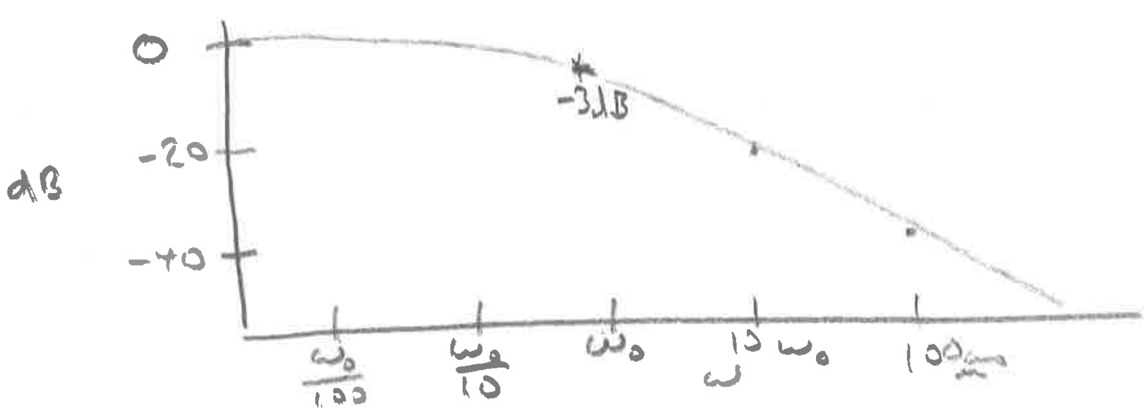
$$\frac{V_{out}}{V_{in}} [dB] = 20\log_{10} \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

$$= -10 \log_{10} 1 + (\omega/\omega_0)^2$$

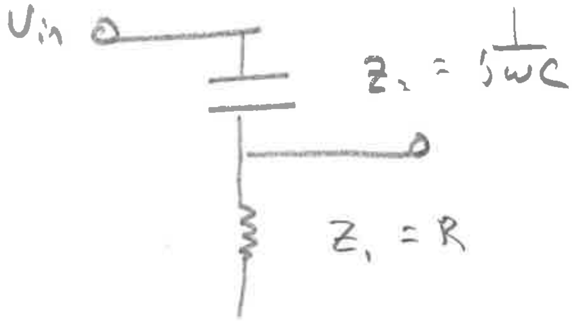
$\omega = \omega_0 \Rightarrow -3\text{ dB}$

$\omega = \omega_0 \times 10 \Rightarrow -20\text{ dB}$

$\omega = \omega_0 \times 100 \Rightarrow -40\text{ dB}$



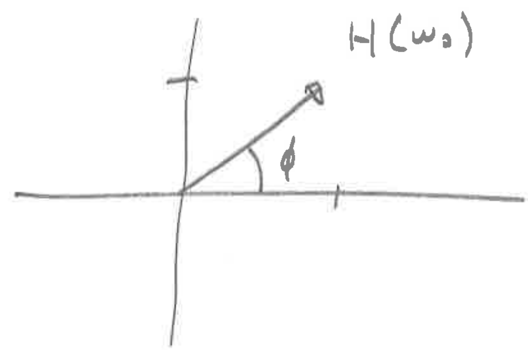
High pass Filter



$$H = \frac{Z_1}{Z_1 + Z_2} = \frac{R}{R + \frac{1}{j\omega C}}$$

$$= \frac{1}{1 - j\frac{1}{\omega RC}} = \frac{1}{1 - j\left(\frac{\omega_0}{\omega}\right)}$$

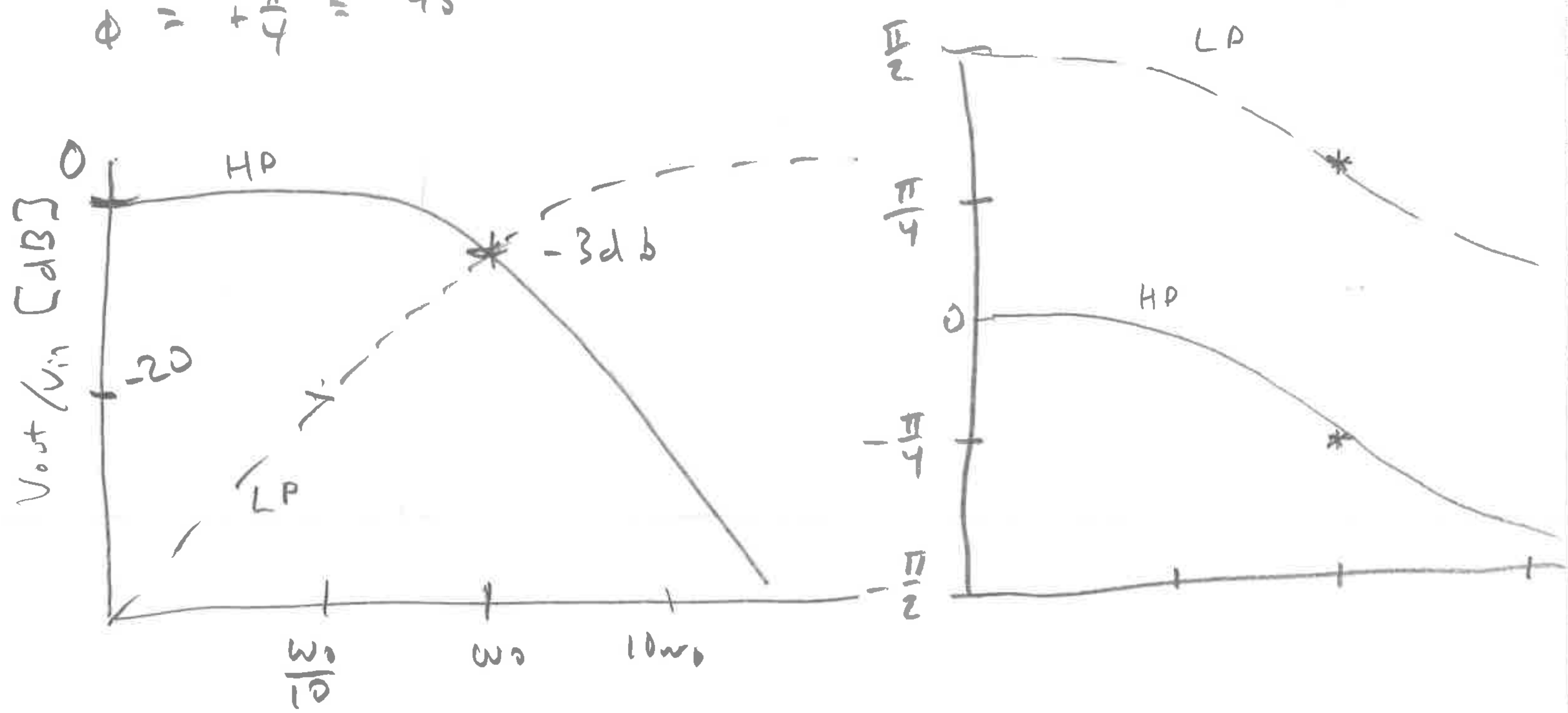
$$= \frac{1 + j\left(\frac{\omega_0}{\omega}\right)}{1 + \left(\frac{\omega_0}{\omega}\right)^2}$$



At ω_0

$$G = |H| = \frac{1+j}{2}$$

$$\phi = +\frac{\pi}{4} = 45^\circ$$

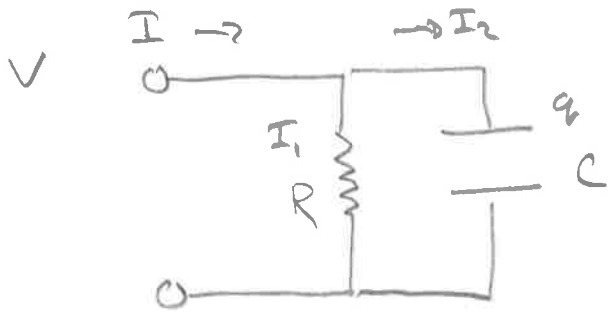


Not Included Yet ...

→ Power in RLC systems?

→ Transformers ?

DC IV curve for R + C : (steady state)



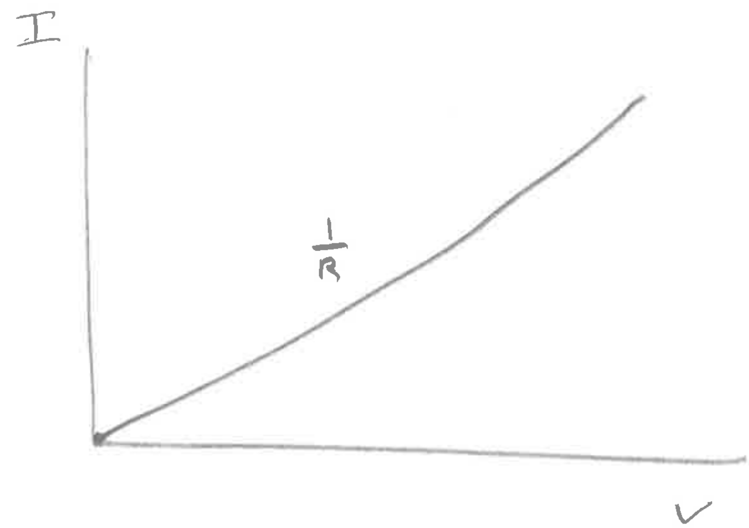
$$\frac{dq}{dt} = 0$$

$$\Rightarrow \quad I_2 = 0$$

$$q = CV$$

$$I_2 = 0 \Rightarrow I = I_1$$

$$V = I_1 R = I R$$

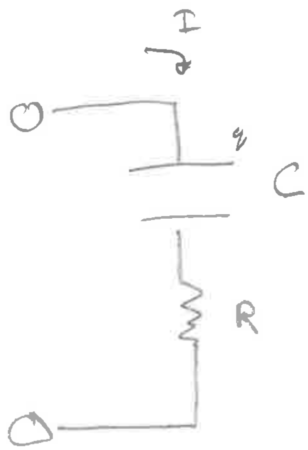


Looks just like

** DC steady-state only **

Much more interesting behavior from AC sources and transients

(Leave for After Lab 1)



$$\frac{dq}{dt} = 0$$

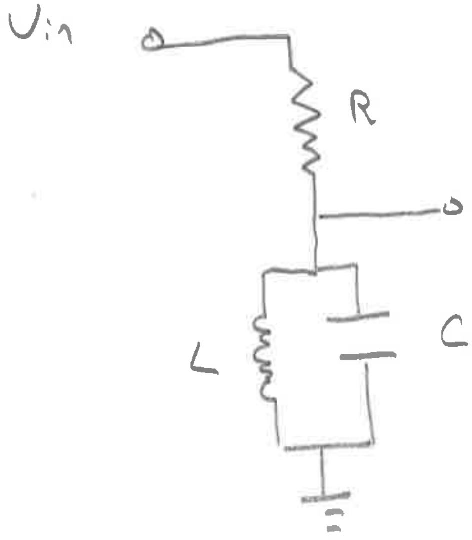
$$I = 0$$

Looks like

$R = \infty$

* For DC steady state, simply replace capacitors with an open circuit $(R = \infty)$

RLC Band pass Filter



$Z_2 = 2$

$Z_1 = L \parallel C$

$$H = \frac{Z_1}{Z_1 + Z_2} = \frac{1}{1 + Z_2/Z_1}$$

$$\frac{1}{Z_1} = \frac{1}{Z_C} + \frac{1}{Z_L} = j\omega C + \frac{1}{j\omega L}$$

$$\frac{Z_2}{Z_1} = j\,\frac{RC}{\omega}\left(\omega^2 - \frac{1}{LC}\right) \qquad \tau = RC \qquad \omega_0 = \frac{1}{LC}$$

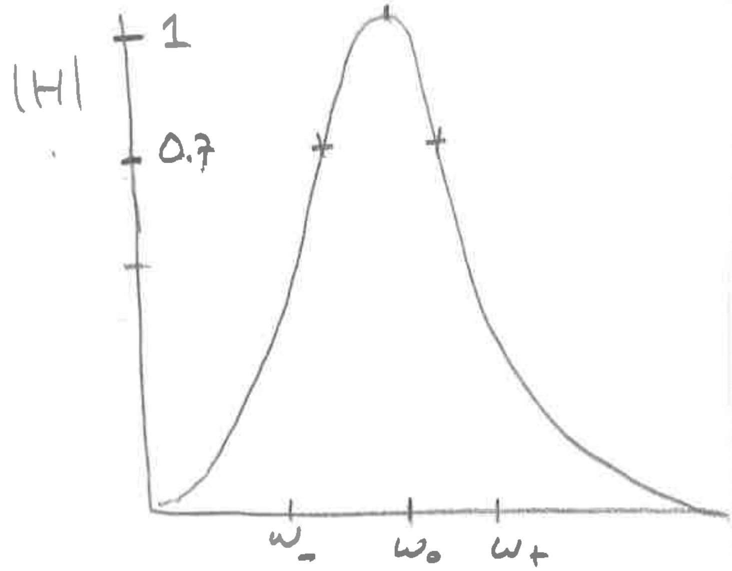
$$= j\,\frac{\tau}{\omega}\left(\omega^2 - \omega_0^2\right) = j(\tau\omega_0)\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$H(\omega = \omega_0) = \frac{1}{1+0} = 1$$

In phase, unit gain.

$$\boxed{H = \frac{1}{1 + j(\tau\omega_0)\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}}$$

$$|H|^2 = \frac{1}{1 + (\tau\omega_0)^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}$$



$$Q \equiv \frac{\omega_0}{\omega_+ - \omega_-}$$

At ω_+ and ω_-

$$|H|^2 = \frac{1}{2}$$

$$\Rightarrow (\tau\omega_0)^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 = 1$$

$$\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 = \frac{1}{(\tau\omega_0)^2}$$

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = \pm \frac{1}{\tau\omega_0}$$

$$\left(\frac{\omega}{\omega_0}\right)^2 \pm \frac{1}{\tau\omega_0}\left(\frac{\omega}{\omega_0}\right) = 1$$

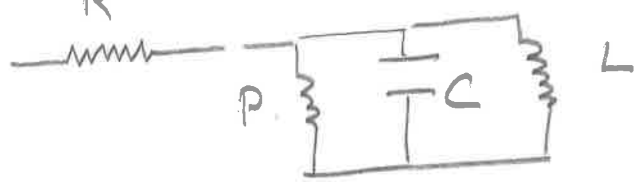
$$\left(\frac{\omega}{\omega_0} \pm \frac{1}{2\tau\omega_0}\right)^2 = 1 + \frac{1}{4(\tau\omega_0)^2} = 1 + \varepsilon$$

$$\frac{\omega}{\omega_0} = \sqrt{1 + \varepsilon} \pm \frac{1}{2\tau}$$

$$\omega_+ - \omega_- = \frac{1}{\tau}$$

$$Q \equiv \frac{\omega_0}{\omega_+ - \omega_-} = \boxed{\omega_0\tau}$$

Effect of Parasitic Parallel Resistance
(Not for lecture)



$$G = \frac{1}{1 + Z_2/Z_1}$$

$$\frac{1}{Z_1} = \frac{1}{P} + j\omega C + \frac{1}{j\omega L}$$

$$\frac{Z_2}{Z_1} = \frac{R}{P} + j\omega RC\left(1 - \frac{1}{\omega^2 LC}\right)$$

$$= \frac{R}{P} + j\,\omega\,\tau\left(1 - \frac{\omega_0^2}{\omega^2}\right) \qquad \tau \equiv RC \quad \omega_0^2 = \frac{1}{LC}$$

$$= \frac{R}{P} + j\,\omega_0\tau\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$G = \frac{1}{1 + \frac{R}{P} + j(\omega_0\tau)\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

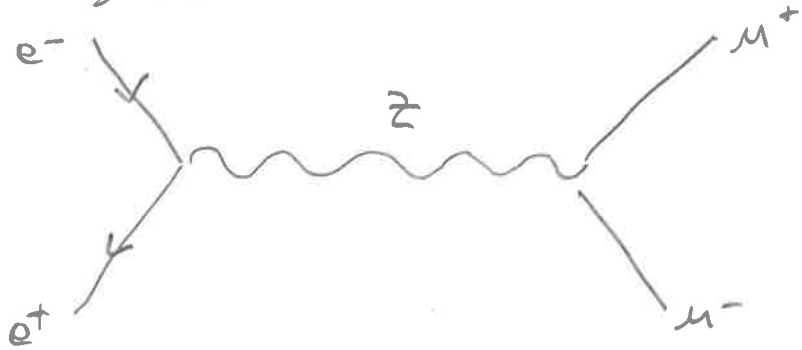
$$|G|^2 = \frac{1}{\left(1 + \frac{R}{P}\right)^2 + (\omega_0\tau)^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}$$

$$|G_{max}|^2 = \frac{1}{1 + (R/P)^2} \qquad \boxed{Q = \frac{\omega_0\tau}{1 + R/P}}$$

$$\frac{|G|^2}{|G_{max}|^2} = \frac{1}{1 + \left(\frac{\omega_0\tau}{1 + R/P}\right)^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}$$

Parallels with QFT

Feynman Diagrams



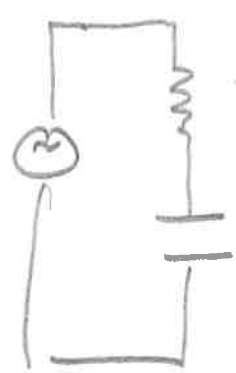
complicated many dimensional integrals ← (Feynman Diagram) → Complex Numbers (Amplitudes)

Rate of Process $\sim$ $|\text{Amplitude}|^2$

Just a bit like

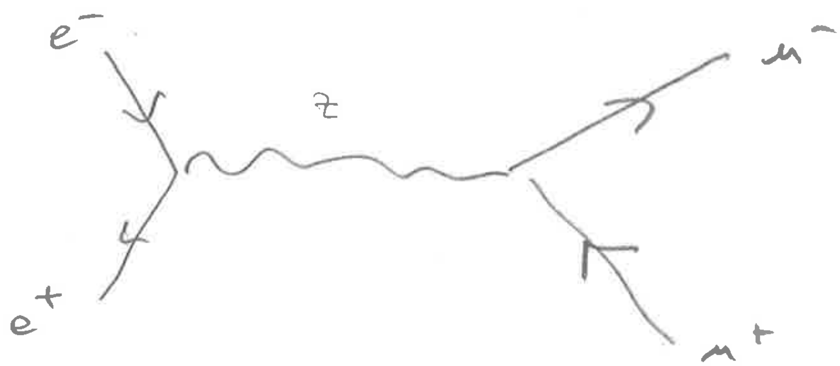
Diff Eq

$$V = \frac{dq}{dt} R + \frac{q}{C}$$



$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C}$$

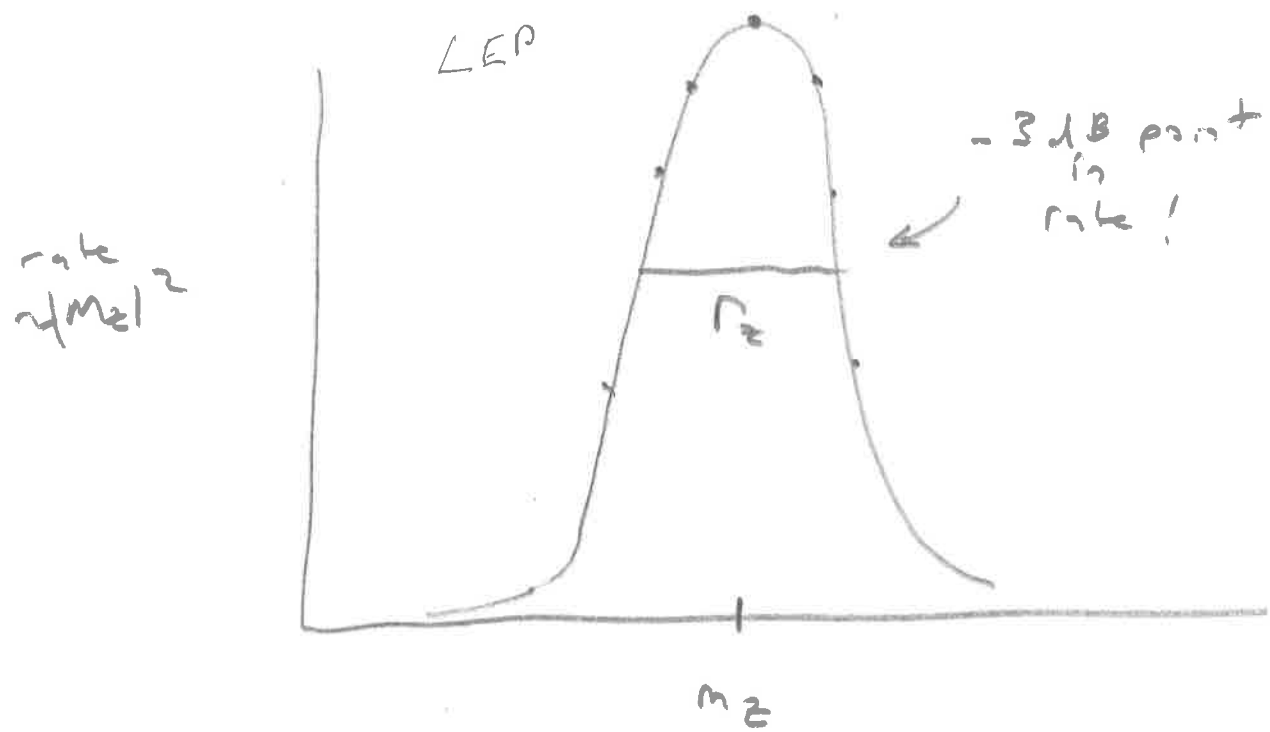


$s = E_{cms}^2$

$e^- \longrightarrow \quad \longleftarrow e^+$

LEP

$$M_Z \sim \frac{1}{(s - m_Z) + i\Gamma_Z} \qquad |M_Z|^2 = \frac{1}{\Gamma_Z^2 + (s - m_Z)^2}$$



LEP measurement of m_Z and Γ_Z remains unbeatable today... Resonance!

Interference

$$M_Z \sim \frac{1}{(s-m_Z^2)+i\Gamma_Z} \qquad M_\gamma \sim \frac{1}{s}$$

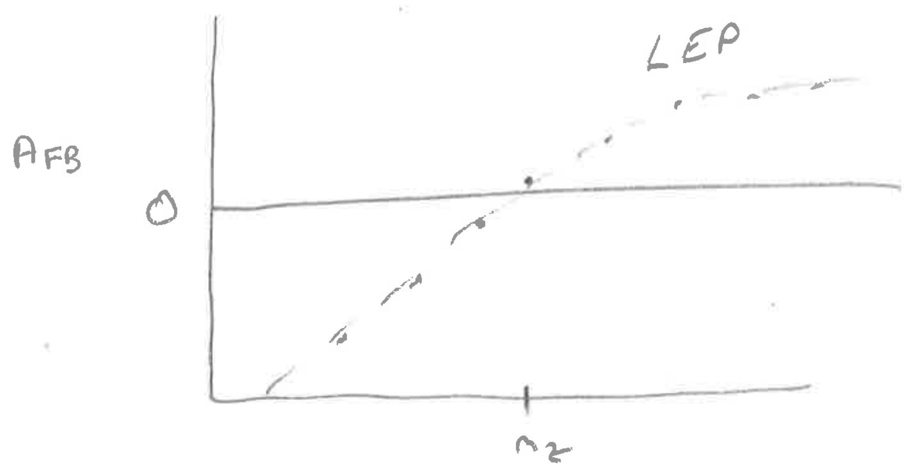
In QFT, like QM, processes interfere:

$$|M_{\gamma Z}|^2 = M_Z^* M_\gamma + M_Z M_\gamma^*$$

$$= \frac{(s-m_Z^2)}{s\left[(s-m_Z)^2+\Gamma_Z^2\right]}$$

At $s=m_Z$ interference goes to zero at $s=m_Z^2$

(LC circuit, phase goes to zero $\omega=\omega_0$)



→ Double Back:

⇒ Power in reactive circuits

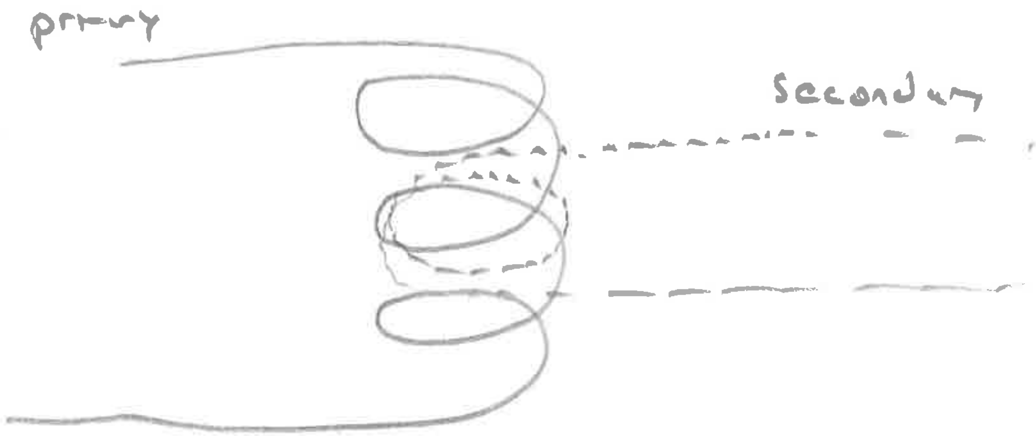
→ Goes to zero

⇒ Power company story

(Different pricing for large reactive loads ... capacitor banks at factories)

Transformer

A transformer consists of two intertwined inductors, with different number of turns



Trick is they share the same

Flux!

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{dt}$$

$$\int \vec{\nabla} \times \vec{E} \cdot d\vec{A} = -\int \frac{\partial \vec{B}}{dt} \cdot d\vec{A}$$

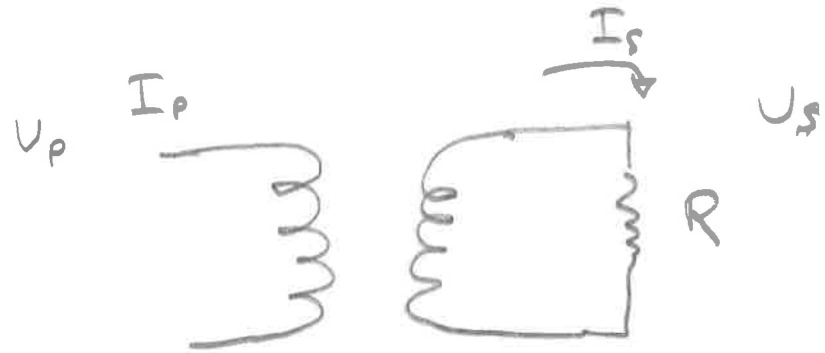
$$\int \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

$$E l \big|_p = -\frac{d}{dt} B A = E \cdot l \big|_s = V_{loop}$$

per loop

$$V_p = n_p V_{loop}$$
$$V_s = n_s V_{loop}$$

$$\Longrightarrow \boxed{\frac{V_p}{V_s} = \frac{n_p}{n_s} \equiv a}$$

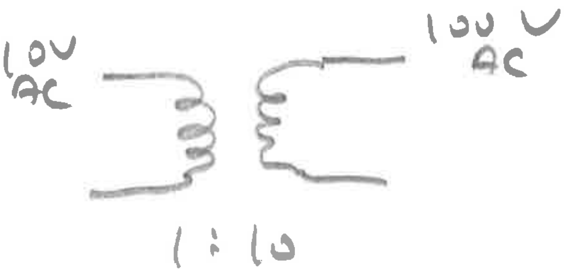


Apparent Load

$$R_a = \frac{V_p}{I_p} = \frac{a V_s}{I_s / a} = a^2 \frac{V_s}{I_s} = a^2 R$$

$$\boxed{R_a = a^2 R}$$

Step - Up



Step Down

Ideal Transformer : Power In = Power Out

$$\frac{V_p}{V_s} = \frac{n_p}{n_s} = a$$

$$P_{in} = V_p I_p = P_{out} = V_s I_s$$

$$\Rightarrow \quad \frac{I_p}{I_s} = \frac{V_s}{V_p} = \frac{1}{a}$$