Homework Assignment 8
"A spectrum of possibilities"

Practice Problems

These problems are graded on effort only.

Griffiths: P3.16, P3.32, P3.33

Additional Problems

Problem 1: Suppose that an observable $\hat{Q}$ has a discrete spectrum of eigenvalues $\{\lambda_i\}$ with corresponding eigenvectors $|\lambda_i\rangle$ so that:

$$\hat{Q}\,|\lambda_i\rangle = \lambda_i\,|\lambda_i\rangle$$

and at $t = 0$ the state vector is:

$$|\Psi_0\rangle = \sqrt{\frac{2}{5}}\,|\lambda_1\rangle + i\sqrt{\frac{2}{5}}\,|\lambda_2\rangle + k\,|\lambda_3\rangle$$

where k is a positive real number.

(A) What is the value of k?

(B) What is the expectation value of the observable $\hat{Q}$? (Write your answer in terms of λ_1, λ_2, and λ_3.)

(C) Suppose the spectrum is non-degenerate. What is the probability that a measurement of the observable $\hat{Q}$ would result in the outcome λ_1?

(D) Suppose the measurement in (C) was made with outcome λ_1. What is the state vector $|\Psi_1\rangle$ immediately after the measurement?

(E) Suppose the spectrum is degenerate with $\lambda_1 = \lambda_2$ and $\lambda_1 \neq \lambda_3$ and the state vector is $|\Psi_0\rangle$ above. What is the probability that a measurement of the observable $\hat{Q}$ would result in the outcome λ_1?

(F) Suppose the measurement in (E) was made with outcome λ_1. What is the state vector $|\Psi_1\rangle$ immediately after the measurement?

Problem 2: Suppose that an observable $\hat{Q}$ has a discrete non-degenerate spectrum of eigenvalues $\{\lambda_i\}$ for $i = 1, 2, 3, \ldots$ with corresponding eigenvectors $|\lambda_i\rangle$ so that:

$$\hat{Q}\,|\lambda_i\rangle = \lambda_i\,|\lambda_i\rangle$$

Also suppose that the Hamiltonian $\hat{H}$ has a discrete non-generate spectrum of eigenvalues $\{E_i\}$ for $i = 1, 2, 3, \ldots$ with corresponding stationary states $|E_i\rangle$ so that:

$$\hat{H}\,|E_i\rangle = E_i\,|E_i\rangle$$

Last, suppose we know the inner products:

$$\langle E_1|\lambda_1\rangle = a, \qquad \langle E_2|\lambda_1\rangle = b, \qquad \langle E_i|\lambda_1\rangle = 0 \qquad (i > 2)$$

and:

$$\langle E_1|\lambda_2\rangle = c, \qquad \langle E_2|\lambda_2\rangle = d, \qquad \langle E_i|\lambda_2\rangle = 0 \qquad (i > 2)$$

(A) Suppose at $t = 0$, a measurement of $\hat{Q}$ has outcome λ_1. What is the state vector at time $t = 0$? (You may choose the overall phase factor to be something convenient, like 1)

(B) Determine the time-dependent state vector $|\Psi(t)\rangle$ appropriate for later times. Write your answer in terms of a, b, E_i, and the stationary state vectors $|E_i\rangle$. **Hint:** You do not know the time-dependence of the $|\lambda_i\rangle$ but you do know the time-dependence of $|E_i\rangle$. So write your answer from (A) in terms of $|E_i\rangle$, using Fourier's trick to determine each coefficient, then just slap on the time-dependence.

(C) Suppose the state vector from part (B) evolves until time t. What is the probability that a measurement of $\hat{Q}$ will measure value λ_2? **Hint:** use Fourier's trick!

Problem 3: There's a bit more integration in this problem, but the insights are totally worth it! For a Gaussian PDF:

$$G(y;\sigma) \equiv \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

you may assume use the following Gaussian integrals without calculating them again:

$$\begin{aligned}
\int_{-\infty}^{+\infty} G(y;\sigma)\,dy &= 1 \\
\int_{-\infty}^{+\infty} y\,G(y;\sigma)\,dy &= 0 \\
\int_{-\infty}^{+\infty} y^2\,G(y;\sigma)\,dy &= \sigma^2
\end{aligned}$$

Consider a stationary Gaussian wave packet:

$$\psi(x) = \left(\frac{1}{2\pi\sigma_x^2}\right)^{\frac{1}{4}} \exp\left(-\frac{x^2}{4\sigma_x^2}\right)$$

(A) Using the Gaussian integrals above, confirm that $\langle x^2\rangle - \langle x\rangle^2 = \sigma_x^2$. (If it were otherwise, the choice of parameter name σ_x would have been very confusing!)

(B) Calculate the "momentum space wave function" $\widetilde{\psi}(p)$ which is the Fourier transform of $\psi(x)$ but in terms of momentum $p = \hbar k$ instead of just k:

$$\widetilde{\psi}(p) \equiv \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \psi(x)\, e^{-ipx/\hbar}\, dx$$

Hint: use the technique of completing the square (used previously) to calculate this integral.

(C) Show that:

$$|\widetilde{\psi}(p)|^2 = G(p; \sigma_p)$$

for a suitable choice of σ_p.

(D) Calculate the uncertainty product $\sigma_x \sigma_p$ and compare your result to the uncertainty principle.