Homework Assignment 3
Infinite Square Well Potential

Practice Problems

These problems are graded on effort only.

Griffiths: P1.17, P2.1bc, P2.3

Additional Problems

Problem 1: Suppose that $\psi_n(x)$ is a solution to the TISE:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n}{\partial x^2} + V(x)\,\psi_n(x) = E_n\psi_n(x)$$

for

$$n = 1, 2, 3, \ldots$$

Show that:

$$\Psi(x,t) \equiv \sum_{n=1}^{\infty} c_n \exp\left(-\frac{i\,E_n\,t}{\hbar}\right)\psi_n(x)$$

is a solution to the SE:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\,\Psi$$

Problem 2: We found the complete set of orthonormal solutions to the TISE for the infinite square well potential to be:

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}) & 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases} \qquad n = 1, 2, 3, \ldots \tag{1}$$

Show that if the particle is in any stationary state $\psi_n(x)$ then the expectation value of the position is:

$$\langle x\rangle = \frac{a}{2}$$

You may use the definite integral:

$$\int_0^{n\pi} x\,\sin^2 x\,dx = \frac{n^2\,\pi^2}{4}$$

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Problem 3: Suppose at time t=0, a particle is in the state

$$\Psi(x,0) = \frac{1}{\sqrt{2}}\psi_\alpha(x) + \frac{1}{\sqrt{2}}\psi_\beta(x)$$

where $\psi_\alpha(x)$ and $\psi_\beta(x)$ are two specific $\psi_n(x)$ from Problem 2 with definite energies E_α and E_β and $\alpha \neq \beta$. Remember that $\psi_\alpha(x)$ and $\psi_\beta(x)$ are real functions.

(A) Write down the time-dependent wave function $\Psi(x,t)$ (You verified in Problem 1 that this will be a solution to the SE.)

(B) Calculate $|\Psi|^2$. You should find:

$$|\Psi|^2 = \frac{|\Psi_\alpha|^2 + |\Psi_\beta|^2}{2} + \ldots$$

where the remaining bit is due to interference.

(C) Calculate the expectation value of the position $\langle x \rangle$ at any time t. Make sure you use the results from Problem 2 so you should just have one not-so-bad integral to compute. You may use:

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}$$

to help compute the integral.

Problem 4: The axiomatic definition of a vector space V and inner product space H over the real numbers $\mathbb{R}$ is detailed in Table 1 on the next page.

(A) Show that D1 follows from A1-5 and M1-5.

Hint: we already know for the scalars that $0 + 0 = 0$

(B) Show that D2 follows from A1-5, M1-5, and D1.

Hint: you need to show $x + (-1)x = 0$. And we already know that $1 + (-1) = 0$.

(C) Show that D3 follows from I1 and I2.

(D) Show that D4 follows from I1 and I3.

Table 1: Here we define the properties of a vector space V and an inner product space H. Note that no complex conjugation appears in these definitions as the scalar field is the real numbers numbers.

Useful Math Symbols:

$\forall\, x \in V$	for all x in V (for any vector x)
$\forall\, \alpha \in \mathbb{R}$	for all α in $\mathbb{R}$ (for any real number α)
$\exists!\, y$	there exists unique y
s.t.	such that

Properties of Addition:

A1	**Closure**	$\forall x, y \in V$	$(x + y) \in V$
A2	**Commutative**	$\forall x, y \in V$	$x + y = y + z$
A3	**Associative**	$\forall x, y, z \in V$	$(x + y) + z = x + (y + z)$
A4	**Zero**	$\exists!\ 0$ s.t. $\forall x \in V$	$x + 0 = x$
A5	**Inverse**	$\forall x \in V \exists!\ (-x) \in V$ s.t.	$x + (-x) = 0$

Properties of Scalar Multiplication:

M1	**Closure**	$\forall x \in V$ and $\forall \alpha \in \mathbb{R}$	$\alpha x \in V$
M2	**Identity**	$\forall x \in V$	$1x = x$
M3	**Associative**	$\forall x \in V$ and $\forall \alpha, \beta \in \mathbb{R}$	$\alpha(\beta x) = (\alpha\beta)x$
M4	**Distributive**	$\forall x, y \in V$and $\forall \alpha \in \mathbb{R}$	$\alpha(x + y) = \alpha x + \alpha y$
M5	**Distributive**	$\forall x \in V$and $\forall \alpha, \beta \in \mathbb{R}$	$(\alpha + \beta)x = \alpha x + \beta x$

Deducible Properties:

D1	$\forall x \in V$	$0x = 0$
D2	$\forall x \in V$	$(-1)x = (-x)$

Properties of Inner Products:

I1	$\forall x, y \in H$	$\langle x\|y\rangle = \langle y\|x\rangle$
I2	$\forall x, y, z \in H$and $\forall \alpha \in \mathbb{R}$	$\langle x\|\alpha y\rangle = \alpha \langle x\|y\rangle$
I3	$\forall x, y, z \in H$	$\langle x + y\|z\rangle = \langle x\|z\rangle + \langle y\|z\rangle$
I4	$\forall x \in H$	$\langle x\|x\rangle \geq 0$
I5	$\forall x \in H$	$\langle x\|x\rangle = 0$ if and only if $x = 0$

Deducible Properties:

D3	$\forall x, y \in H$and $\forall \alpha \in \mathbb{R}$	$\langle \alpha x\|y\rangle = \alpha \langle x\|y\rangle$
D4	$\forall x, y, z \in H$	$\langle x\|y + z\rangle = \langle x\|y\rangle + \langle x\|z\rangle$