148 → Ressler 152

Lecture 1

Single-Board Microcontroller

Single board:
microprocessor, I/O, clock, RAM

Arduino

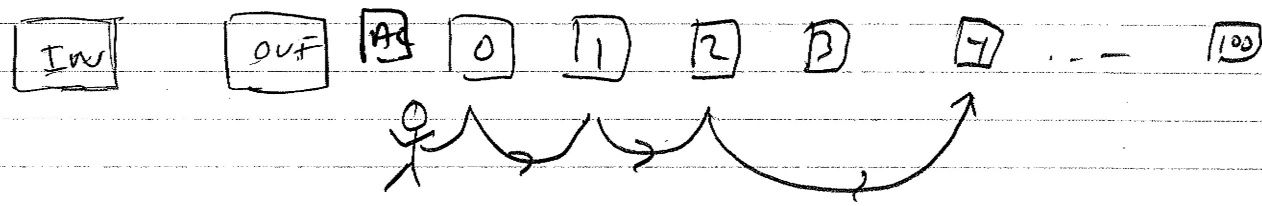
- Open-hardware
- Standard I/O arrangement → stackable shields
- digital I/O
- analog input
- analog output → DAC
  → PWM (Pulse width modulation)
- Atmel AVR (8-bit Reduced Instruction Single Chip) (Micro controller)

*** Programming : C/C++

setup () { ... }
loop () { ... }

\+ $\boxed{\text{Wiring}}$ (Library)

Little Man Computer (Toy)



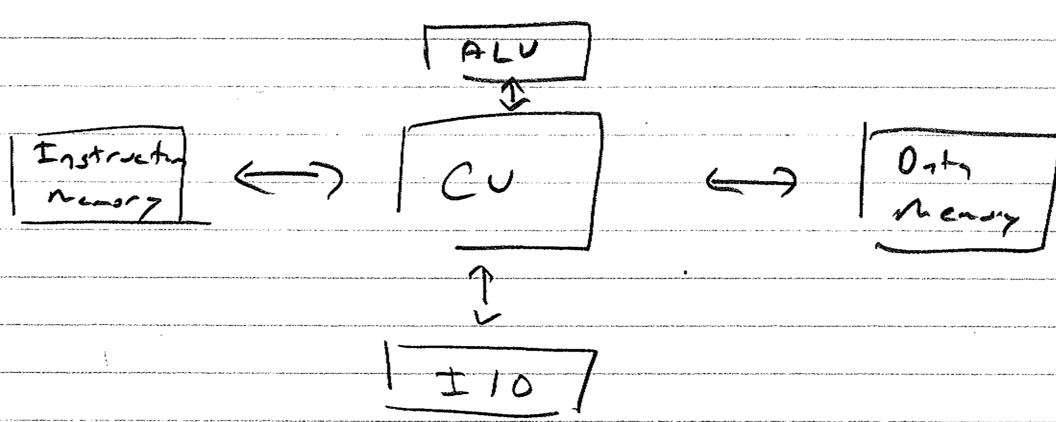
Show Instruction Set:

→ Add or Substract two numbers based on user input

→ Branching

→ By program modification!

⇒ Nowadays program modification more controlled: Harvard Architecture



(ARM: Cache is Harvard, Memory is V-N)

Microprocessor

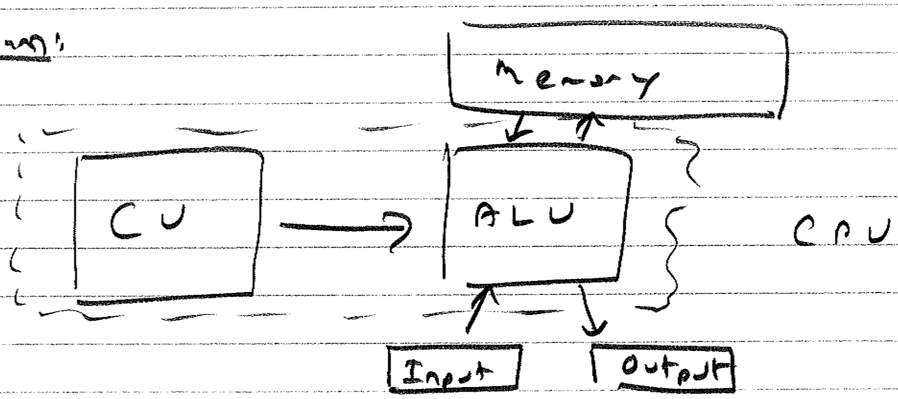
Q: What is a microprocessor?
All modern CPUs are microprocessors:

→ Single IC containing CPU

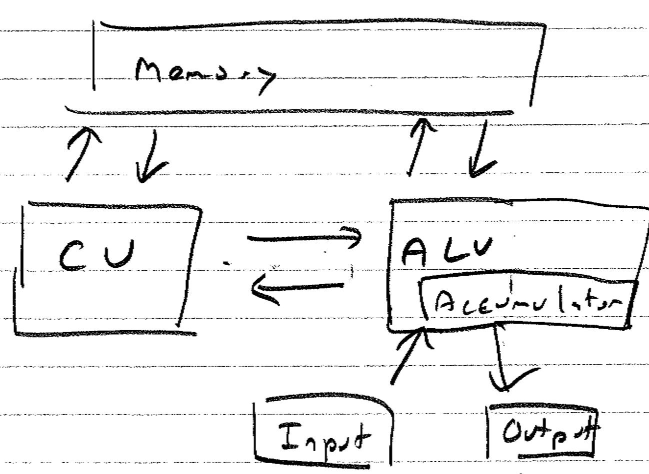
Q: What is a CPU?

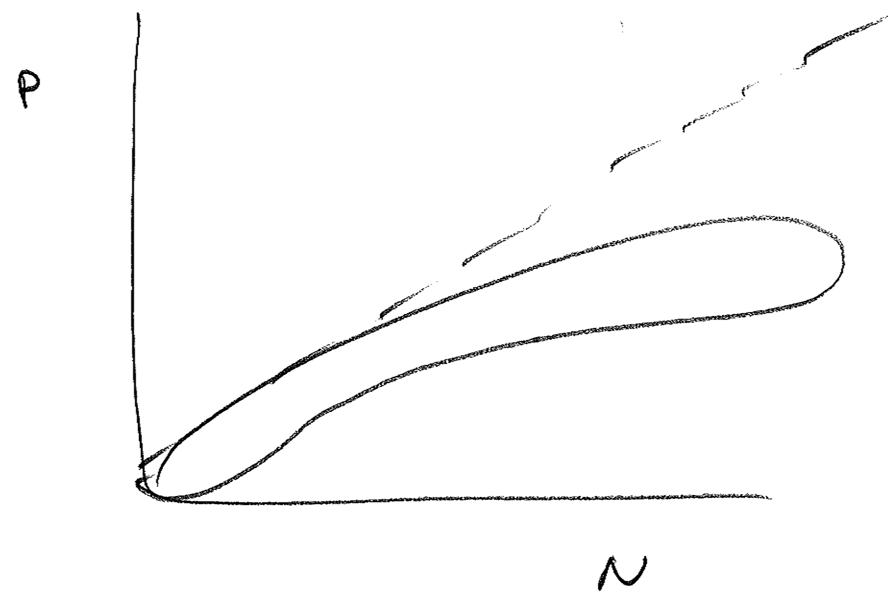
Control Unit + Arithmetic Logic Unit

Fixed Programs:



Von Neumann Architecture





PP (Not stable)

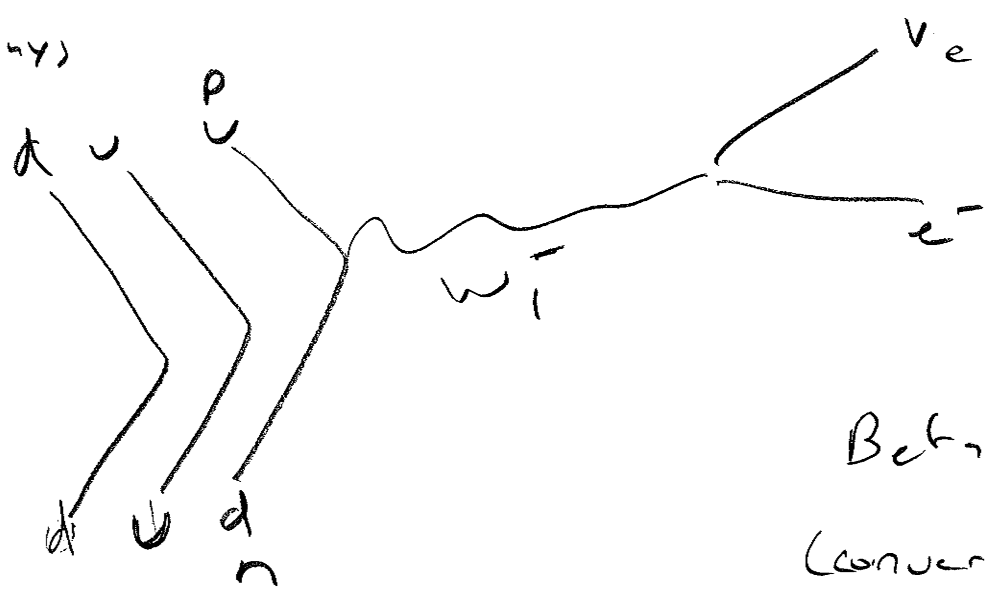
NP (stable) ~ P = N

NN (not stable)

As P+N increases, Coulomb Force is long range, Nuclear force is short range

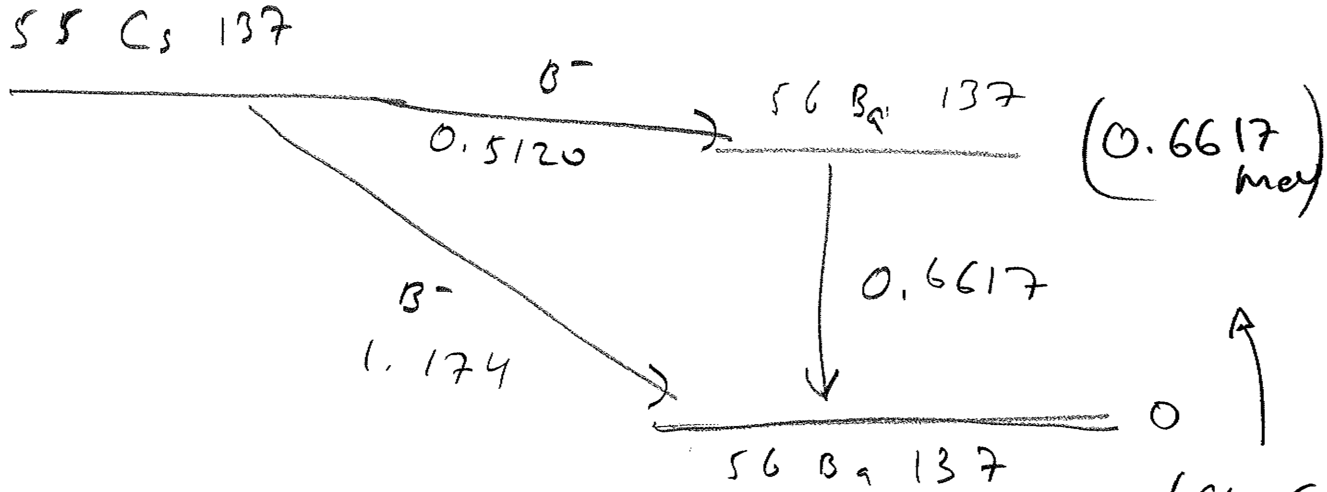
⇒ Need more N's than P's

Decays



Beta Decay

(convert N ⟷ P)



(slower, higher order, multipole emission)

55 Cs 137 : 136.90709 amu

56 Ba 137 : 136.90582 amu

$\Delta m = \sim 0.0012$ amu

$(1\ \text{amu})\,c^2 = 931 \times 10^6\ \text{eV}$

$\Rightarrow \sim 1.1\ \text{MeV}$

$.5\ \text{MeV}/c^2$

Radio-active Decay

→ Totally random which (when) a particular nucleus decays. (unlike humans!)

→ Equally probable that any nucleus will decay at any time (unlike humans)
(Nuclei don't get old and die!)

$$\Rightarrow \quad \frac{dN}{dt} = -kN$$

$$N(t) = N \exp(-kt)$$

$$N(0) = N_0$$

$$N(\tau_{1/2}) = N_0/2$$

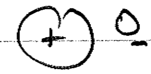
$$N(t) = N_0 \exp(-kt)$$

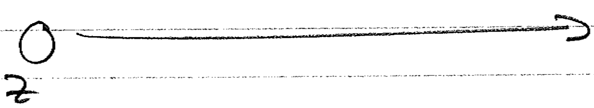
$$N(\tau_{1/2}) = N_0 \exp(-k\tau_{1/2}) = N_0/2$$

$$\left.\begin{aligned} \Rightarrow \quad 2 &= \exp(k\tau_{1/2}) \\ \log 2 &= k\tau_{1/2} \\ k &= \frac{1}{\log 2\, \tau_{1/2}} \end{aligned}\right\} \quad \text{(Fix)}$$

$$N(t) = N_0 \exp\left(\frac{-t}{\log(2)\,\tau_{1/2}}\right)$$

Passage of charged particles:





As charged particles move through medium, loose energy in a number of ways.

→ Ionize atoms by liberating electrons

→ Multible Scattering, from nuclei

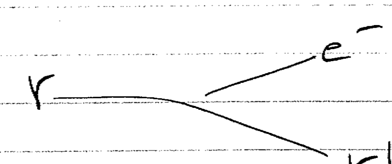
→ For light particles acceleration causes radiation (bremsstrahlung)

<u>Photons</u>

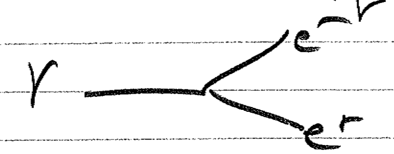
Energy

(a) Photo-electric effect

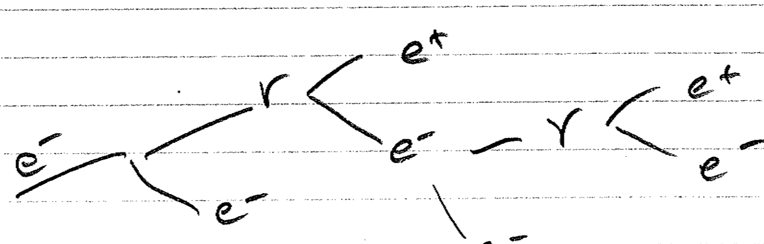
(b) Compton Scattering



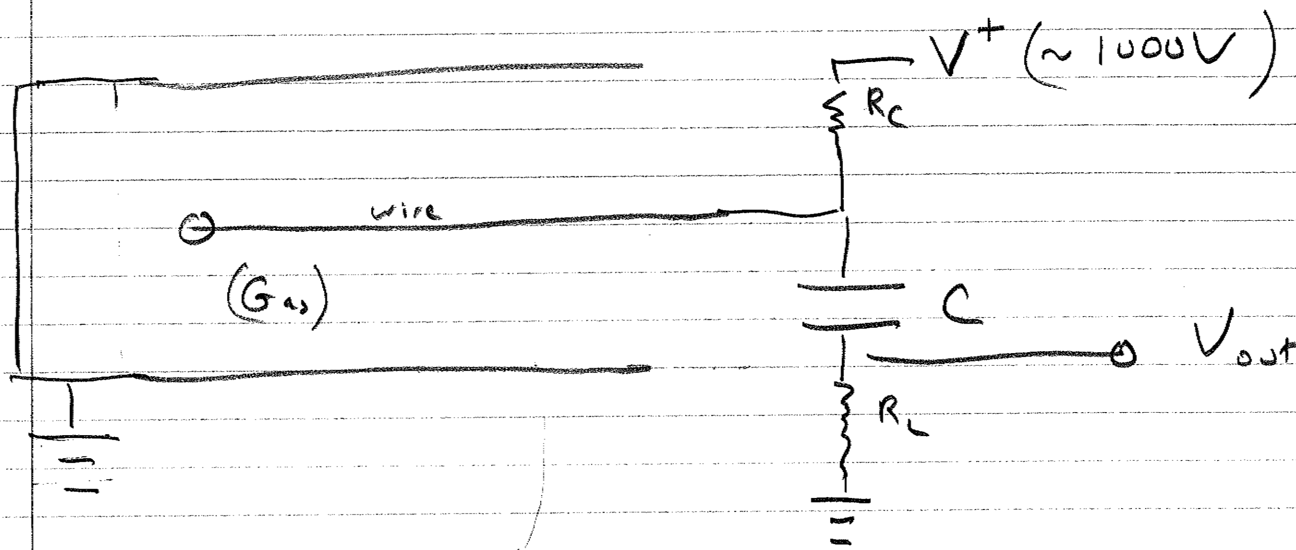
(c) Pair production



<u>Cascades:</u>



<u>Gaseous Ionization Detector</u>

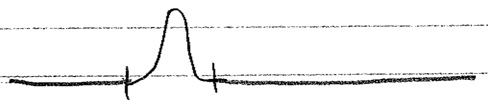


We'll look from two equivalent perspectives (equivalent due to superposition principle)

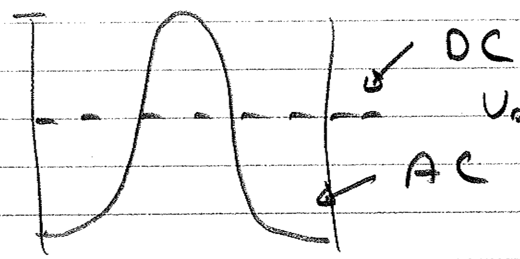
1. Capacitor charges (with time constant $\tau = C(R_C + R_L)$) to V^+
2. Ionizing particle passes through chamber, ionizing gas
3. electrons move toward positive voltage wire, + ions toward ground
4. These mobile charge carriers essentially short chamber wire to ground.
5. Capacitor therefore discharges (with time constant $CR_L \ll C(R_L + R_C)$)
6. $\therefore$ Current flows through R_L
   $\rightarrow$ See voltage at V_{out}
7. Ionization stops, V^+ restores cap to V^+, current flows opposite direction in R_L.

Equivalent way to think..

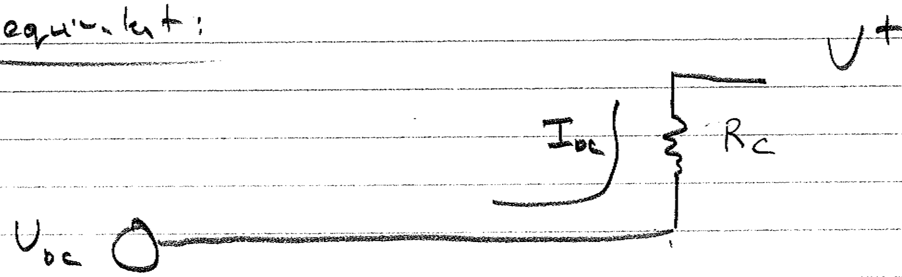
Our ionization causes a short-lived current pulse



we are only concerned with behavior near pulse (we know steady state behavior)

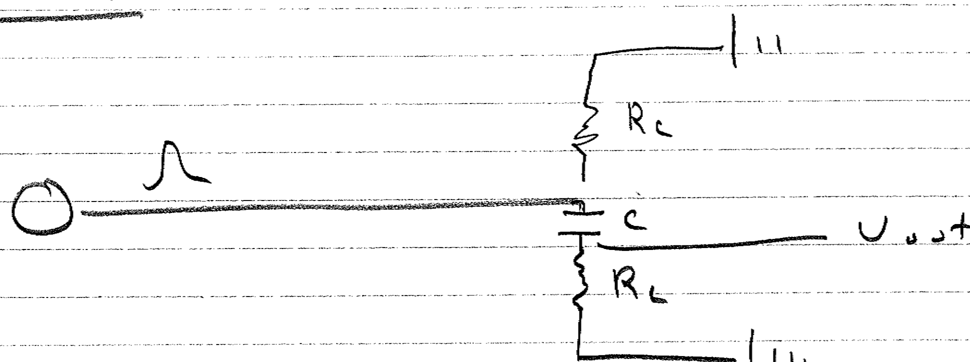


<u>D.C. equivalent:</u>

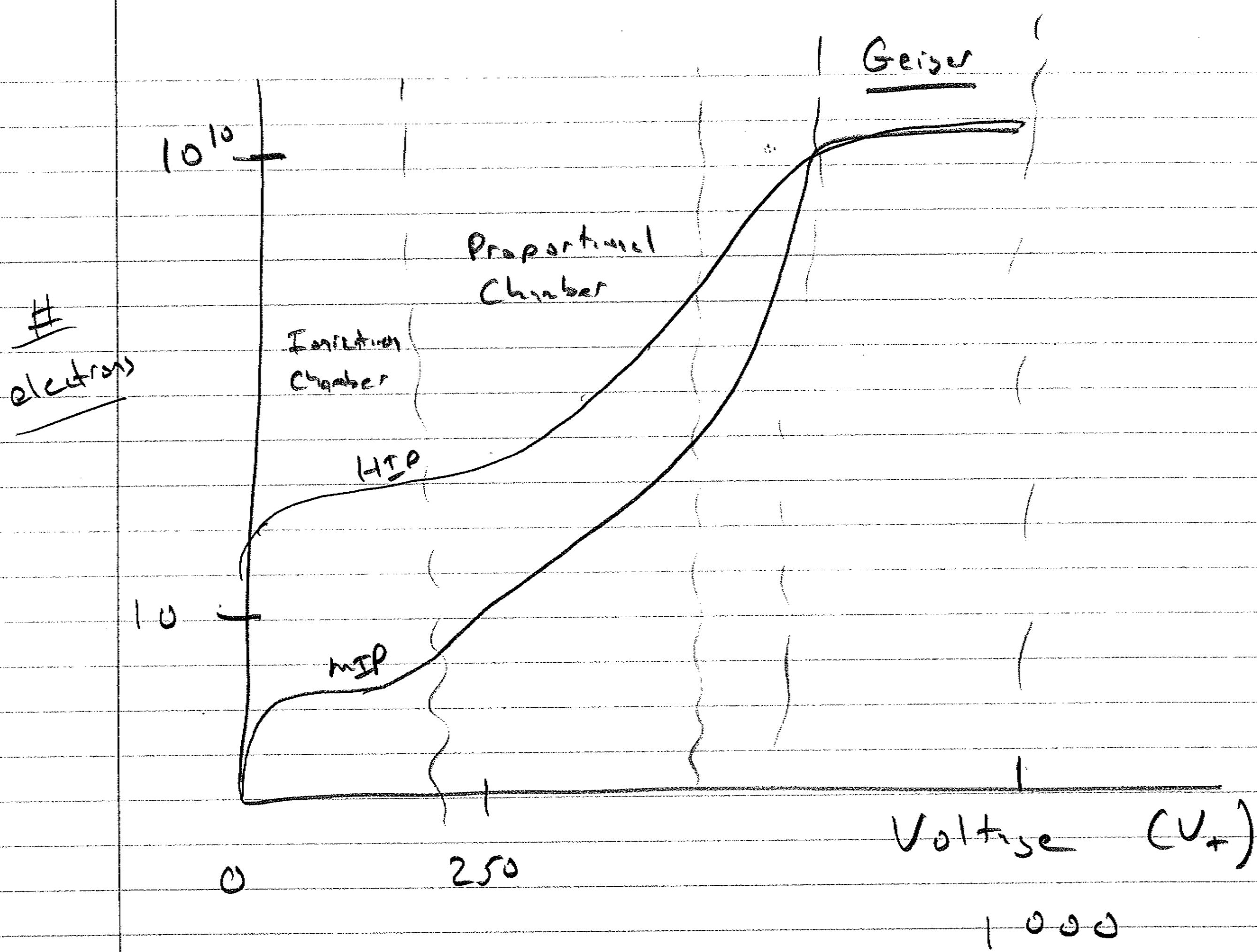


DC current flows through R_C ← Cap has ∞ impedance to DC

<u>A.C. equivalent</u>



AC sees voltage divider,
If frequency is high, $R_C \gg \frac{1}{\omega C} + R_L$ → See pulse at V_{out}



As voltage increases:

(1) Collect ionized charge

(2) Voltage accelerates electrons to high enough energy to cause additional collisions, liberating additional electrons

(3) Saturation

From plot

$$V = \frac{Q}{C} = \frac{1.6 \times 10^{-19} C \times 10^{10}}{10^{-9} F} = \boxed{1.6 V}$$

⇒ Proportion Chambers require fancy electronics.
⇒ Geiger counters are simple!

Chapters 1 – 3

Uncertainties

→ Scientist believe (or imagine) they are perfect!

Error ≠ Mistake

Error = Uncertainty

→ Why are uncertainties useful?

→ Report the precision of the measurement.

→ Measurement can be compared to other measurements

→ Measurement can be compared to theory predictions.

"Conservative" Error Estimation

Suppose we are very timid scientists that only report results for which we are <u>100%</u> <u>certain</u>.

In this case

$x \pm \delta x$ means true value of x lies in range $[x - \delta x, x + \delta x]$ with 100% certainty.

I.E.:

$$\delta x = \frac{x_{max} - x_{min}}{2}$$

$$\left(x = \frac{x_{max} + x_{min}}{2} \right)$$

$\underline{x+y}$?

$$(x+y)_{min} = (x - \delta x) + (y - \delta y)$$

$$(x+y)_{max} = (x + \delta x) + (y + \delta y)$$

$$\delta(x+y) = \frac{(x+y)_{max} - (x+y)_{min}}{2}$$

$$= \delta x + \delta y$$

$x - y$?

$$(x-y)_{min} = (x - \delta x) - (y + \delta y)$$

$$(x+y)_{max} = (x + \delta x) - (y - \delta y)$$

$$\delta(x+y) = \delta x + \delta y$$

x/y ?

$$x/y_{max} = \left(\frac{x+\delta x}{y-\delta y}\right)$$

$$x/y_{min} = \left(\frac{x-\delta x}{y-\delta y}\right) \qquad (\text{Assume } x/y > 0 \quad y > 0 \quad x > 0$$

$$\frac{\delta(x/y)}{x/y} = \left(\frac{1+\frac{\delta x}{x}}{1-\frac{\delta y}{y}}\right) - \left(\frac{1-\frac{\delta x}{x}}{1+\frac{\delta y}{y}}\right)$$

$$= \frac{\frac{\delta x}{x} + \frac{\delta y}{y}}{\left(1-\frac{\delta y}{y}\right)\left(1+\frac{\delta y}{y}\right)}$$

$$\frac{\delta\left(x/y\right)}{x/y} = \frac{\frac{\delta x}{x} + \frac{\delta y}{y}}{1-\left(\frac{\delta y}{y}\right)^2}$$

Denominator is generally valid note that $\frac{\delta y}{y} \sim 1$ should have $\delta\left(\frac{x}{y}\right) \sim \infty$!

But often $\frac{\delta y}{y} < .1$ so

$$\left(\frac{\delta y}{y}\right)^2 << \left(\frac{\delta y}{y}\right)$$

$$\delta\left(x/y\right) = \frac{\delta x}{x} + \frac{\delta y}{y}$$

Conventions:

best estimate $\pm$ uncertainty

(1) Round uncertainty to one significant figure (except 1 or 2)

(2) Value should be stated such that LSF is same order as uncertainty

Calculate 132.572 ± 0.361

Write: $\boxed{132.6 \pm 0.4}$

Now we need to understand what these uncertainties mean, in a practical sense.

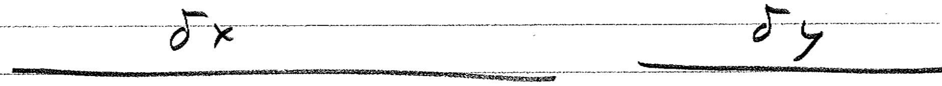
Less Conservative Error Estimation

In reality, we are never 100% certain.

$x \pm \delta x$ will mean true value is in $[x - \delta x, x + \delta x]$ with some probability (in fact 68.5%)

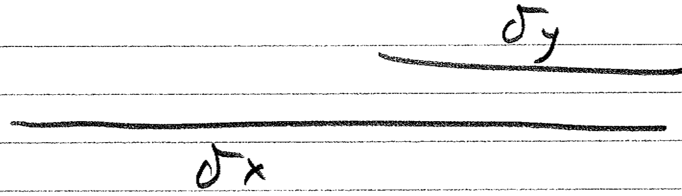
Hand waving answer:

Conservative



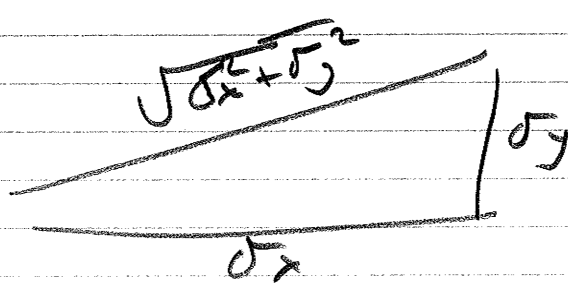
$$\delta(x+y) = \delta x + \delta y$$

Reckless



$$\delta(x+y) = |\delta x - \delta y|$$

Just Right



Adding in Quadrature:

$$\delta(x+y) = \sqrt{\delta x^2 + \delta y^2}$$

$$\delta(x-y) = \sqrt{\delta x^2 + \delta y^2}$$

$$\frac{\delta(x \cdot y)}{xy} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$$

$$\frac{\delta(x/y)}{x/y} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$$

Test consistency:

$\delta(x+y)$

$f(x,y) = x + y$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 1$$

$$\Rightarrow \quad \delta(x+y) = \sqrt{\delta_x^2 + \delta_y^2}$$

$\delta(x \cdot y)$

$f(x,y) = xy$

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x$$

$$\delta(xy) = \sqrt{\cancel{x}^2 \delta x^2 + x^2 \delta y^2}$$

$$\frac{\delta(xy)}{xy} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$$

$\sigma(x/y) = ?$

$f(x,y) = x/y$

$\frac{\partial f}{\partial x} = \frac{1}{y}$ $\quad\quad$ $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$

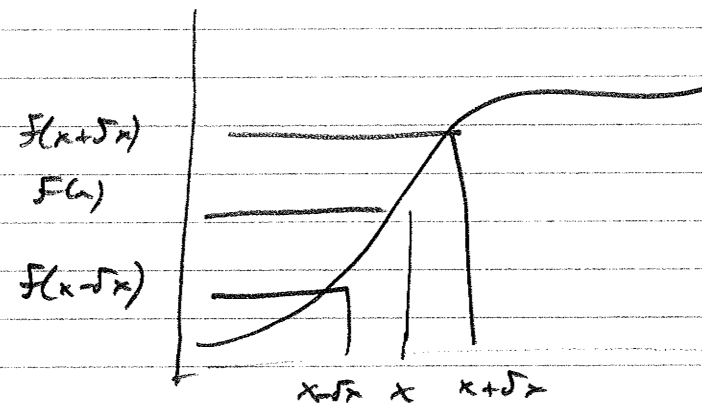
$$\sigma\left(\frac{x}{y}\right) = \sqrt{\frac{1}{y^2}\sigma x^2 + \frac{x^2}{y^4}\sigma y^2}$$

$$\frac{\sigma\left(\frac{x}{y}\right)}{\frac{x}{y}} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

Error Propagation

x : $x \pm \delta x$

$f(x)$?



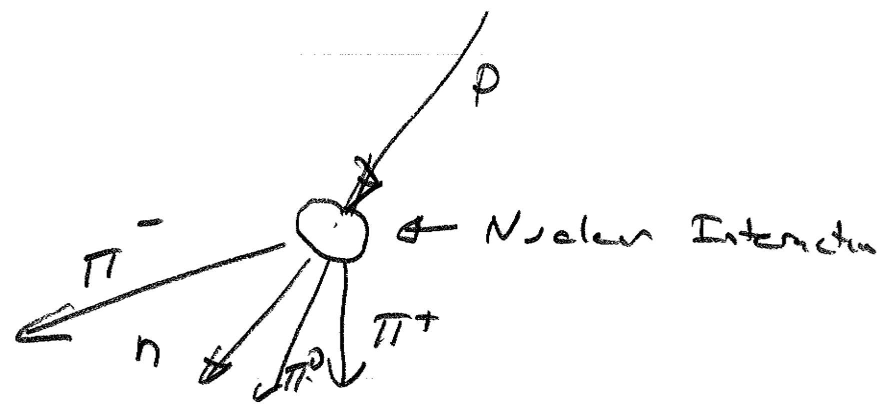
$$f(x \pm \delta x) \sim f(x) \pm f'(x)\,\delta x$$

$$\delta(f(x)) = f'(x)\ \delta x$$

Combine with addition in quadrature rule

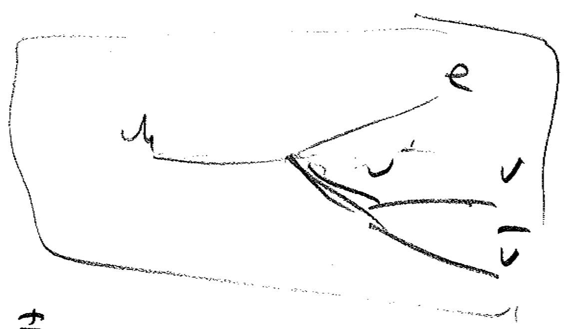
$$\delta(f(x,y)) = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 (\delta x)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \delta y^2}$$

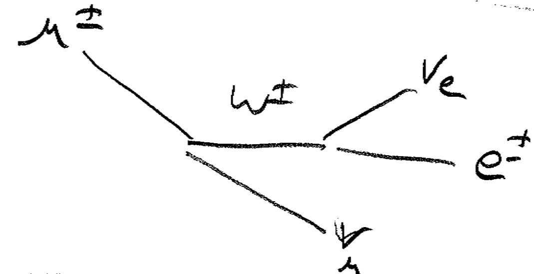
Muon Production + Decay



$\pi^\pm \rightarrow \mu^\pm \nu_\mu$ $\qquad\qquad$ $\pi^0 \rightarrow \gamma\gamma$

$\mu^\pm \rightarrow e^\pm$





$\boxed{1/\text{minute} \cdot \text{cm}^2 \quad \text{at} \quad E \sim 4\,\text{GeV}}$

↳ Prof of relativity

Muon Lab Lecture

Decay Rate:

μ is fundamental particle
→ discussed radioactive decay, it has no "age" and therefore decays at a <u>fixed</u> <u>rate</u>

$$dN = -N(t)\,\lambda\,dt$$

$$\Rightarrow \quad N(t) = N_0 \exp(-t/\tau)$$

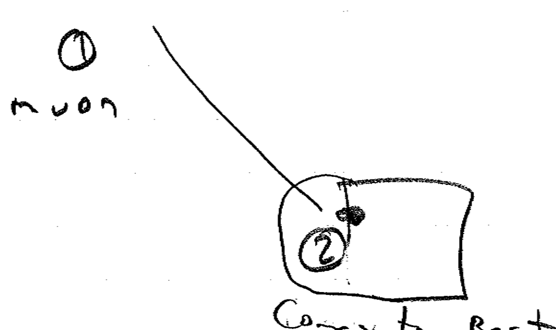
$$\tau \equiv \text{lifetime} = \frac{1}{\lambda}$$

(Relate to half-life):

$$N(t_{1/2}) = \frac{1}{2}N_0 = N_0 \exp(-t_{1/2}/\tau)$$

$$\boxed{\tau \cdot \log(2) = t_{1/2}}$$

Plan:



Q: Don't know how long muon took to here

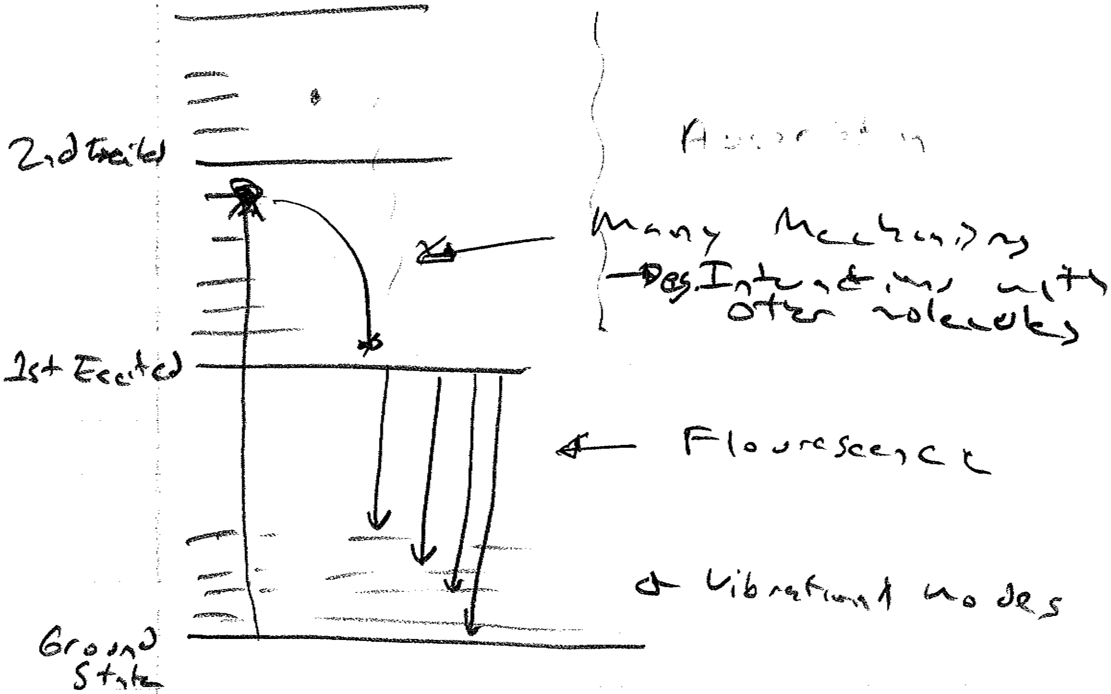
Does it matter?

$$dN = -\lambda N_0 \exp(-\lambda t)$$

$$\boxed{dN = -\lambda \exp(-\lambda t)}$$

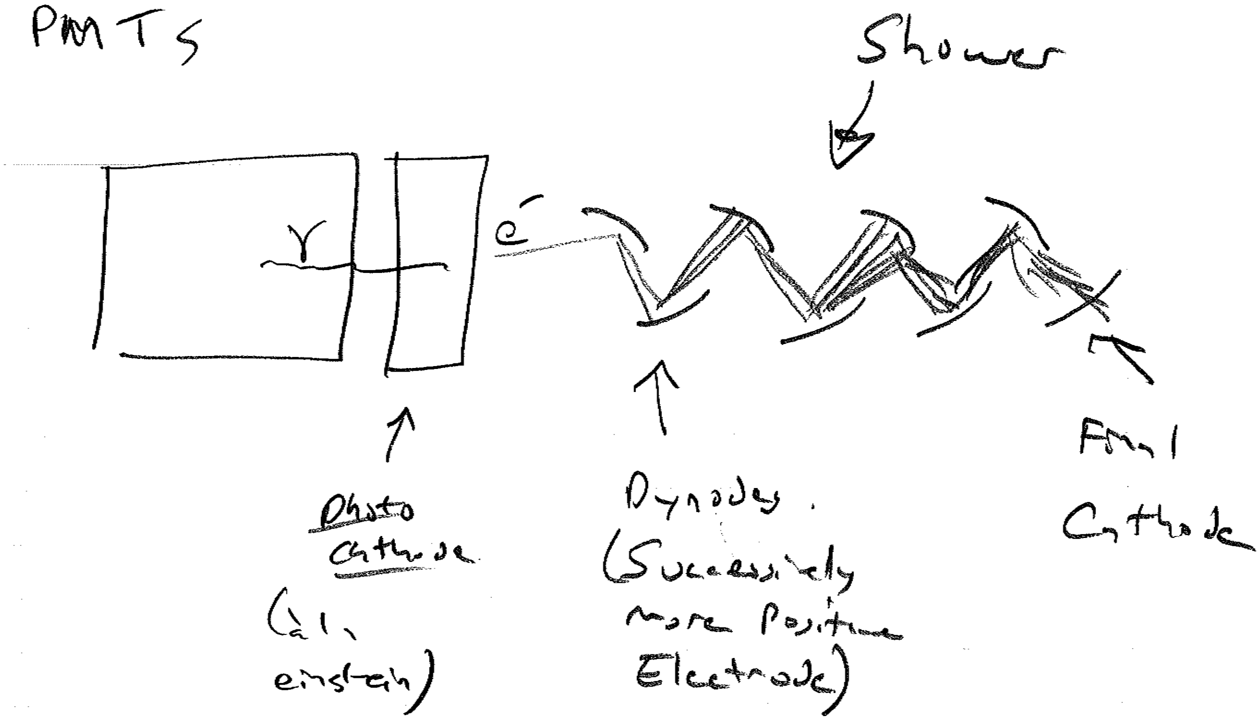
(prob of single decay! $N_0 = 1$)

Scintillator



Only Fraction of Energy of

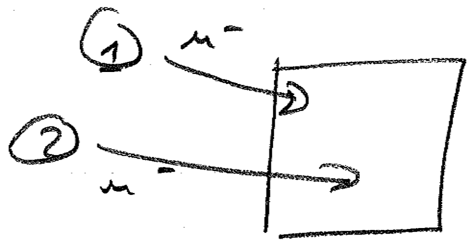
PMTs



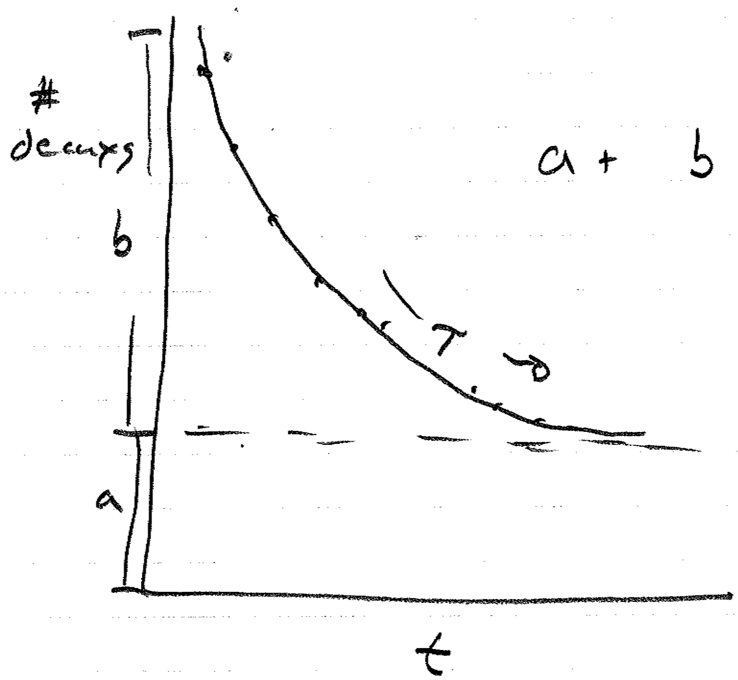
Gain is voltage dependent, but typically 100,000 to 1 million.

(Tuned to application)

Background :

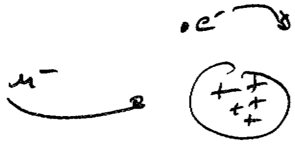


Key Point: An additional muon is equally likely to arrive in any given time interval... **FLAT**



$$a + b \exp(-t/\tau)$$

μ^+ / μ^- differences



μ^- can occupy same orbital as an electron, Then decay via

$$\mu^- + p \rightarrow n + \nu_\mu$$

This provides additional decay for $\underline{\mu^-}$ not possible by $\underline{\mu^+}$.

So two decay rates:

λ^+ and λ^-

We measure a <u>weighted</u> <u>average</u>.

$$\langle \lambda \rangle = \frac{N^+ \lambda^+ + N^- \lambda^-}{N^+ + N^-}$$

Define $\quad \rho = N^+ / N^- \qquad \gamma = \frac{1}{\lambda}$

$$\langle \tfrac{1}{\gamma} \rangle = \frac{\rho \frac{1}{\gamma^+} + \frac{1}{\gamma^-}}{\rho + 1}$$

$$\boxed{\langle \gamma \rangle = (\rho + 1) \frac{\gamma^+ \gamma^-}{\gamma^+ + \rho \gamma^-} = \gamma_{obs}}$$

Solve for ρ

$$(\rho + 1)\ \tau^+ \tau = \tau_{obs} \cdot (\tau^+ + \rho \tau^-)$$

$$\rho(\tau^+ \tau^- - \tau_{obs} \tau^-) = \tau_{obs}\, \tau^+ - \tau^+ \tau^-$$

$$\rho = \frac{\tau_{obs}\, \tau^+ - \tau^+ \tau^-}{\tau^+ \tau^- - \tau_{obs}\, \tau^-}$$

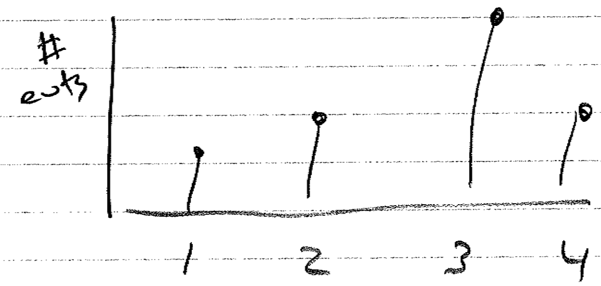
$$= - \frac{\tau^+}{\tau^-} \cdot \left(\frac{\tau^- - \tau_{obs}}{\tau^+ - \tau_{obs}} \right)$$

Histograms + Uncertainties

Histograms

Have students pick number from 1 to 4.

Tally the results and build a histogram

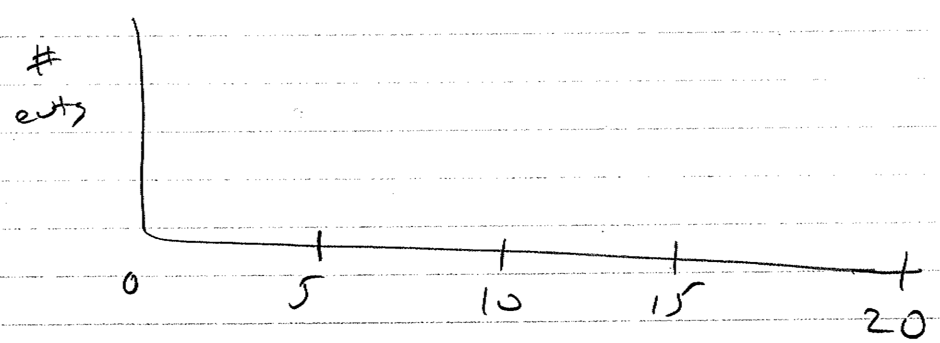


(Probably will be biased toward 3!)

Aside: Can mention (as motivation) statistical uncertainty of $\sqrt{N}$ to check for bias.

Q: How to handle data like:

15.4, 11.3, 12.1, 6.5, 10.1, 17.3



Note: Now each bin has a width and a number of entries.

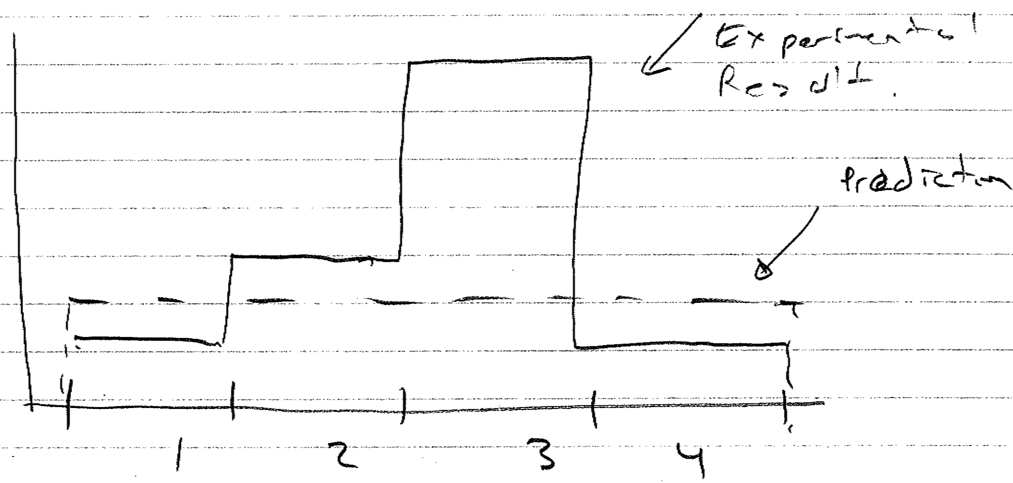
(2 things to keep track of)

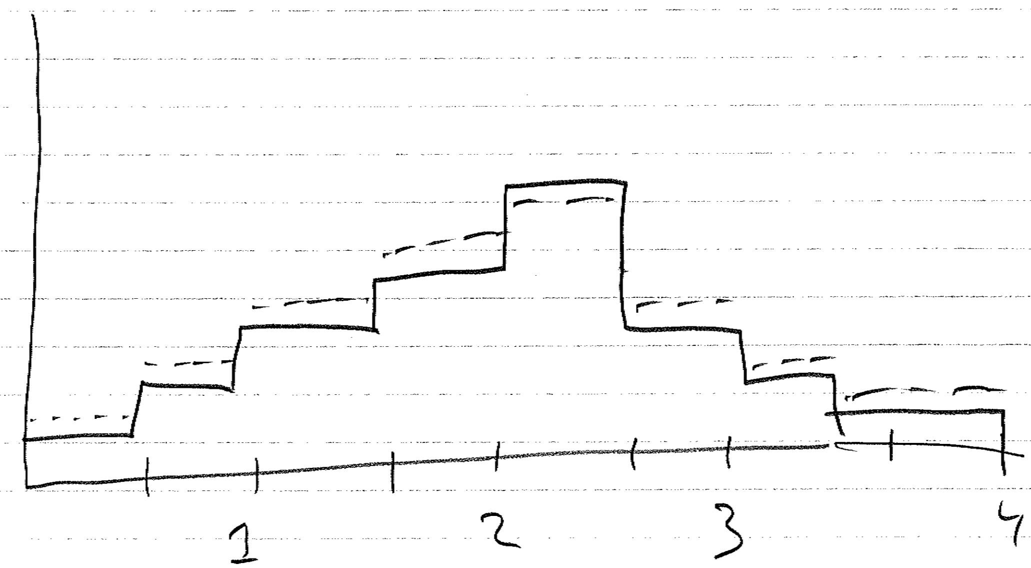
We'll assume (for now) all have same width (evenly spaced).

Distributions:

Science has one and only one purpose:
predicting the results of experiments!

If most experiments are this way (even **without** quantum mechanics!) with similar but non-identical results in well-performed experiments ... what does a prediction look like.





Prediction: # of events in a particular bin
Result: # of event in a particular bin.

This would tend to mean our theory (which predicts outcomes of experiments) depends explicitly on our choice of bins. Also, it would depend on number of measurements. NOT PRETTY! We get a lot from fixing this.

Our theory would better tell us something like:

$$p(x) = \text{probability to measure } x.$$

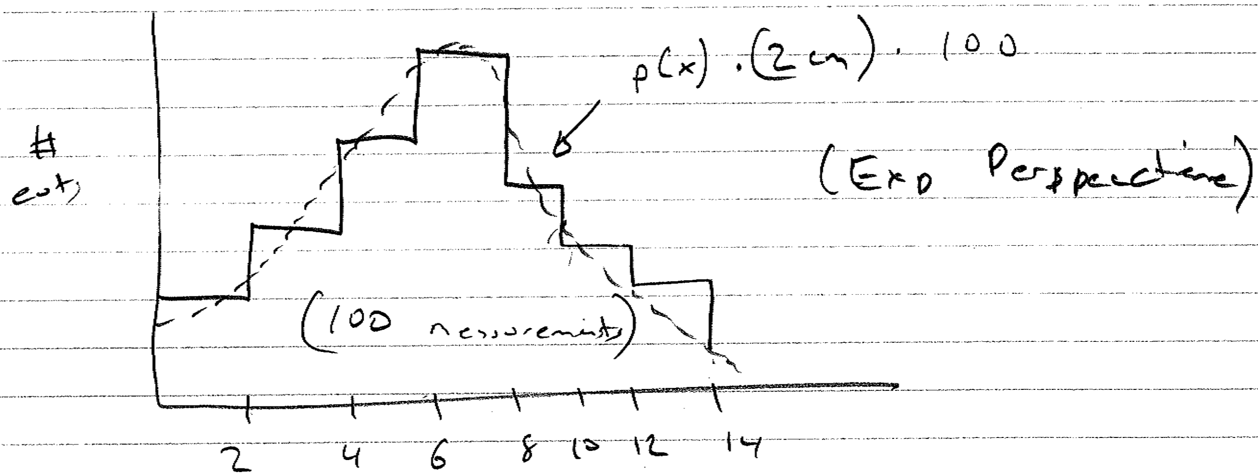
Except that doesn't work!

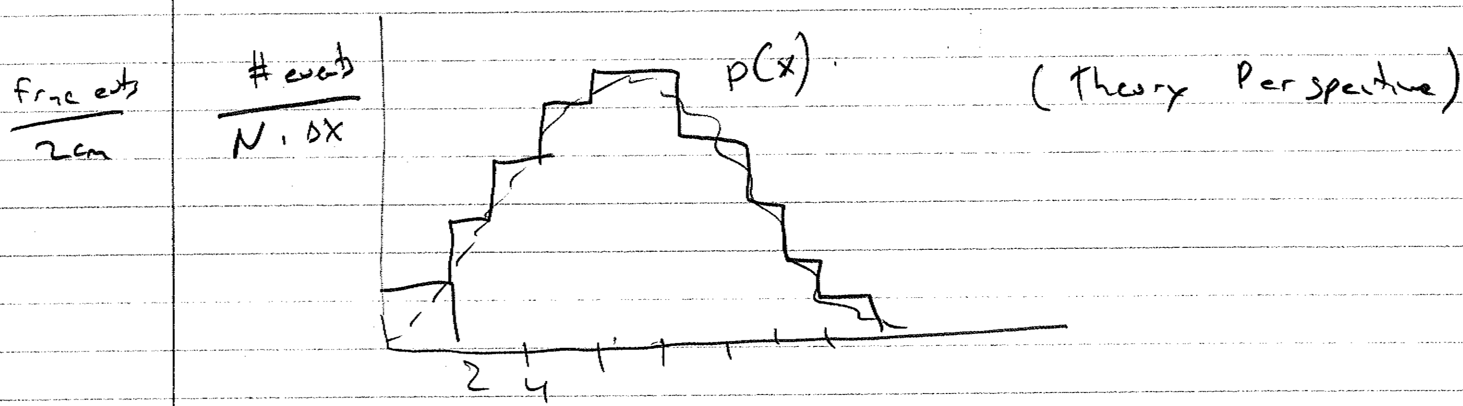
Q: What is probability to measure exactly the value "1.2357612532"?

A: 0.

Probabilities are only non-zero over non-infinitesimal over some range...

Whichever way you think about it, you must remember to account for normalization of prediction to experiment!





There are lots of ways to think about what comes next...

1) Jump to answer:

$$\underbrace{N_{pred}}_{\text{predicted \# for particular experiment}} = \underbrace{p(x)}_{\text{theory prediction}} \cdot \underbrace{\Delta X \cdot N}_{\text{bin size and number of measurements made (exp dependent)}}$$

Q: Suppose x is a measurement of "cm". What are dimensions of p(x)?

A) $\frac{1}{cm}$ in general $[p(x)] = \left[\frac{1}{x}\right]$.

2) $$P(x, x_0) = \int_{x_0}^{x} p(x)\, dx$$

$$\rightarrow p(x) = \frac{\text{prob } (x \text{ and } x + dx)}{dx}$$

$$\rightarrow p(x) = \frac{d P(x, x_0)}{dx}$$

3) Book writes: $p(x)\, dx = \text{prob } (x \text{ to } x+dx)$

We can do a lot with our prediction $p(x)$ which is called a probability distribution function (PDF).

I) $$\int_{-\infty}^{\infty} p(x)\, dx = ?$$

II) $$\bar{x} = \int x\, p(x)\, dx$$

is theory prediction for <u>average</u> <u>value</u>.

III) RMS:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2\, f(x)\, dx$$

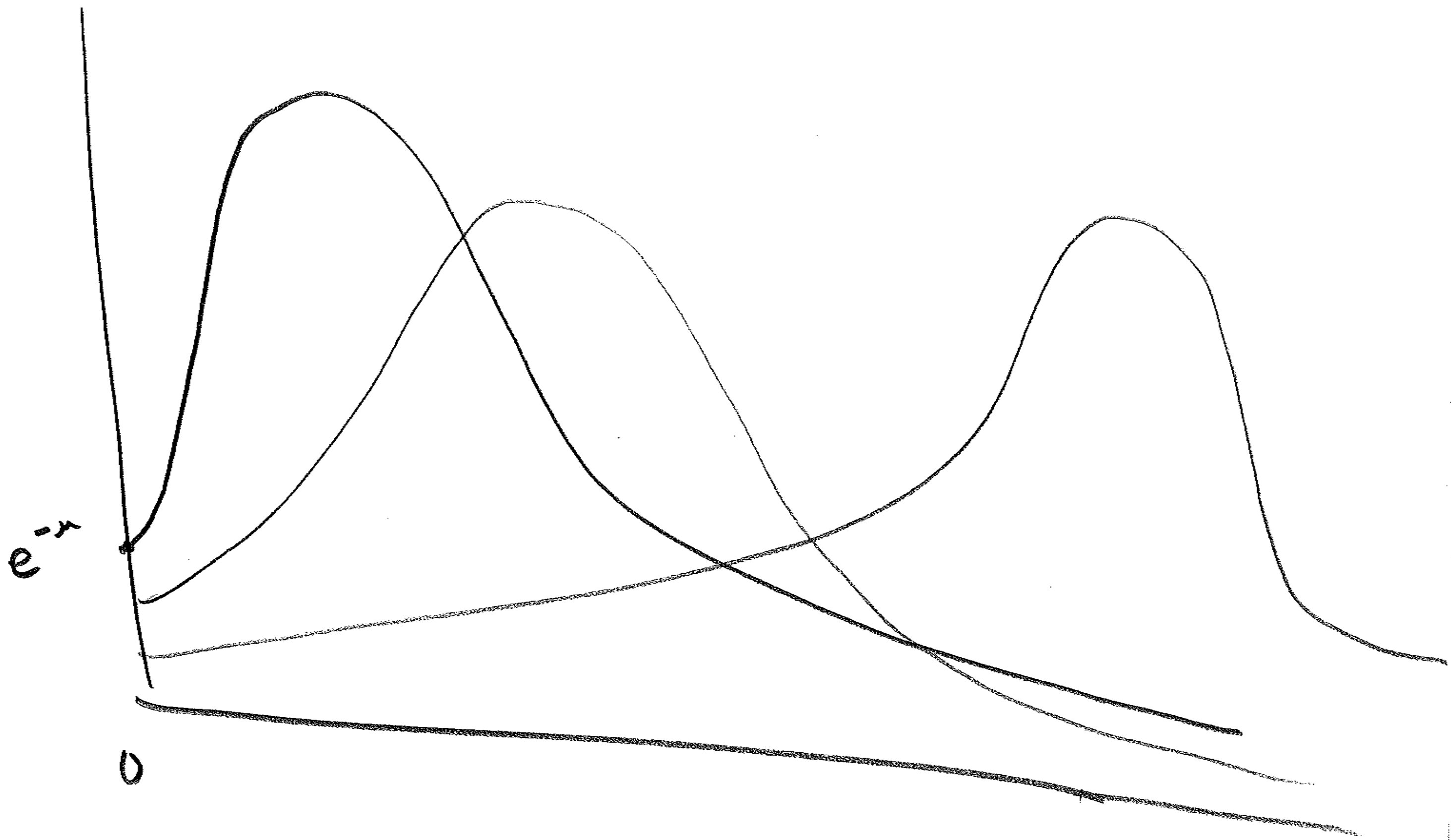
(It will turn out σ_x is related to <u>uncertainty</u>...)

$\Rightarrow$ Now some specific <u>distributions</u>.

INTRODUCE, will <u>prove</u> later

Intro to Poisson Distribution:

$$P_\mu(n) = e^{-\mu}\,\frac{\mu^n}{n!}$$



Std Dev of Poisson Distribution

$$\overline{v^2} = \sum v^2 e^{-\lambda} \frac{\lambda^v}{v!}$$

$$= \sum_{v=1}^{\infty} v \; e^{-\lambda} \frac{\lambda^v}{(v-1)!}$$

$$= \lambda \cdot \sum_{v=1}^{\infty} \left[(v-1) + 1\right] \frac{\lambda^{v-1}}{(v-1)!}$$

$$= \lambda (\lambda + 1) = \lambda^2 + \lambda$$

$$\overline{v^2} - \lambda^2 = \lambda$$

$$\overline{v^2} - \overline{v}^2 = \lambda$$

$$\sigma_v^2 = \lambda$$

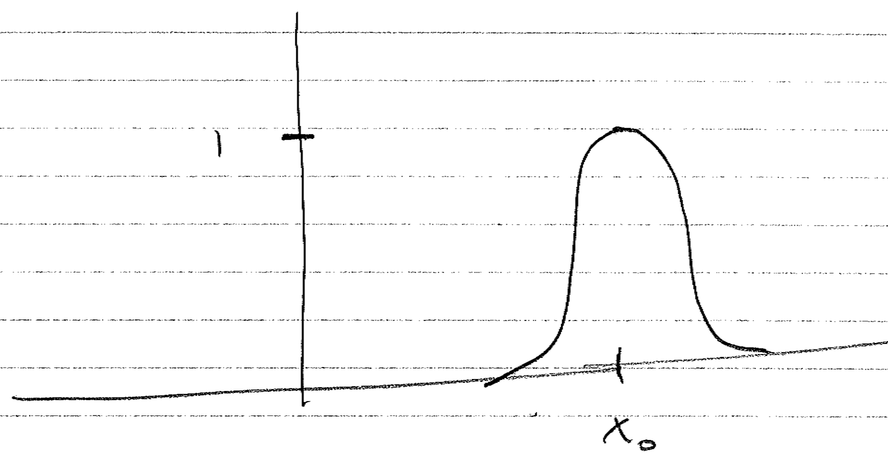
$$\boxed{\sigma_v = \sqrt{\lambda}}$$

Normal Distribution

(From central limit theorem, later)

Limit of many measurements:

$$\exp\left(-(x-x_0)/2\sigma^2\right)$$



- symmetric around x_0
- drops quick to zero as x differs from x_0

Normalization:

$$p(x) \sim \exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right)$$

$$p(x) = N \cdot \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$

$$1 = \int_{-\infty}^{\infty} p(x)\, dx = N \cdot \int \exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right) dx$$

$$\Rightarrow \frac{1}{N} = \int_{-\infty}^{\infty} \exp\left(\frac{-(x-x_0)^2}{2\sigma^2}\right) dx$$

$$x' = \frac{x-x_0}{\sqrt{2}\sigma} \qquad dx' = \frac{dx}{\sqrt{2}\sigma}$$

$$\frac{1}{\sqrt{2}N\sigma} = \int_{-\infty}^{\infty} \exp\left(-x'^2\right) dx'$$

(See trick)

$$\frac{1}{\sqrt{2}\, N\, \sigma} = \sqrt{\pi}$$

$$\Rightarrow N = \frac{1}{\sigma\sqrt{2\pi}}$$

$$\boxed{p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-X)^2}{2\sigma^2} \right)}$$

Trick For:

$$I = \int_{-\infty}^{\infty} dx \; \exp(-x^2)$$

$$I^2 = \int_{-\infty}^{\infty} dx \; \exp(-x^2) \cdot \int_{-\infty}^{\infty} dy \; \exp(-y^2)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy \; \exp(-(x^2+y^2))$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} r\, d\Theta\, dr \; \exp(-r^2)$$

$$= 2\pi \int_{0}^{\infty} dr \; r \exp(-r^2)$$

$$x = r^2, \qquad dx = 2r\, dr$$

$$= \pi \int_{0}^{\infty} dx \; \exp(-x)$$

$$= \pi \; -\exp(-x)\Big|_{0}^{\infty}$$

$$= \pi$$

$$\Rightarrow I = \sqrt{\pi}$$

$$\bar{x} = \int_{-\infty}^{\infty} dx \; x \cdot p(x) = \int_{-\infty}^{\infty} dx \; x \; \frac{\exp\left(-(x-x_0)^2/2\sigma^2\right)}{\sigma\sqrt{2\pi}}$$

$$y = x - x_0 \qquad dy = dx \qquad x = y + x_0$$

$$\bar{x} = \int_{-\infty}^{\infty} dy \; (y + x_0) \; \exp\left(\frac{-y^2}{2\sigma^2}\right) / \sigma\sqrt{2\pi}$$

$$= x_0 \underbrace{\int_{-\infty}^{\infty} dy \; \frac{\exp\left(\frac{-y^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}}_{1} + \underbrace{\int_{-\infty}^{\infty} dy \cdot y \cdot \frac{\exp\left(\frac{-y^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}}_{0}$$

$$\boxed{\bar{x} = x_0}$$

$$RMS^2 = \sigma_x^2 = \int (x - \bar{x})^2 \; p(x) \; dx$$

$$= \int_{-\infty}^{\infty} (x - x_0)^2 \; \exp\left(\frac{-(x-x_0)^2}{2\sigma^2} \right) dx \; \frac{1}{\sqrt{2\pi}\,\sigma}$$

$y = x - x_0$

$$= \int_{-\infty}^{\infty} y^2 \exp\left(\frac{-y^2}{2\sigma^2} \right) dy \cdot \frac{1}{\sqrt{2\pi}\,\sigma}$$

$z = y / \sigma \quad \Rightarrow \quad dy = dz \cdot \sigma$

$y^2 = z^2 \cdot \sigma^2$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sigma^2 \int_{-\infty}^{\infty} y^2 \exp\left(-\frac{y^2}{2} \right) dy$$

(See Next)

$$= \sigma^2$$

Aside:

$$dv = y \exp\left(\frac{-y^2}{2}\right) \qquad v = y$$

$$v = - \exp\left(\frac{-y^2}{2}\right) \qquad du = dy$$

$$\int_{-\infty}^{\infty} y^2 \exp\left(-\frac{y^2}{2}\right) = \int v\,du = uv - \int u\,dv$$

$$= -y \exp\left(-\frac{y^2}{2}\right)\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\exp\left(-\frac{y^2}{2}\right) dy$$

$$= 0 + \sqrt{2\pi}$$

Recap:

mean value

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(\frac{-(x-\bar{x})^2}{2\sigma_x^2}\right)$$

$\left(\sigma^2 = \text{variance}\right)$ ~~RMS~~ (std dev)

$$\left[p(x)\right] = \left[\frac{1}{x}\right]$$ as required

Binomial Coefficients

$(a+b)^n = ?$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = (a+b) \times (a^2 + 2ab + b^2)$

$= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

$$(a+b)^n = \sum_{v=0}^{n} \binom{n}{v} a^v b^{n-v}$$

$\binom{n}{v} = \frac{n!}{v!(n-v)!}$ = # ways to choose v items from n.

1st choice: n
2nd choice $(n-1)$
3rd choice $(n-2)$
...
v choice $(n-(v-1))$

Total Possibilities: $n \cdot (n-1) \cdot (n-2) \ldots (n-(v-1))$

$= \frac{n!}{(n-v)!}$

But, order doesn't matter, v items, $v!$ orderings.

$\Rightarrow \boxed{\binom{n}{v} = \frac{n!}{(n-v)!\,v!}}$

$$\binom{n}{v} = \frac{n!}{v!\,(n-v)!}$$

$$\binom{n+1}{v} = \binom{1}{1}\binom{n}{v-1} + \binom{1}{0}\binom{n}{v}$$

$$= \frac{n!}{(v-1)!\,(n-v+1)!} + \frac{n!}{v!\,(n-v)!}$$

$$= \frac{n!\left[(v) + (n-v+1)\right]}{v!\,(n+1-v)!}$$

$$= \frac{(n+1)\;n!}{v!\;((n+1)-v)!}$$

$$= \frac{(n+1)!}{v!\;((n+1)-v)!}$$

Assume $\binom{n}{v} = \frac{n!}{v!\,(n-v)!}$ for all $v < n$.

Shown previously $\binom{n+1}{v} = \frac{n!}{v!\,(n-v)!}$ for $v < n$

Need only show $\binom{n+1}{n+1} = \frac{n+1!}{(n+1)!\ 0!} = 1$

which is obvious!

$$+ \binom{n}{v} = \binom{1+n}{v}$$

$$\frac{!(n-v)\ !n}{!v} = \binom{n}{v}$$

Binomial Distribution

Suppose you have efficiency ε of succeeding at a task?

Q: What is probability of failure?

A: $(1-\varepsilon)$

Binomial Distribution is

P(r successes in n trials)

$= (\#\text{ combination of } r \text{ success}) \times \left(\begin{array}{c}\text{Prob of 1}\\ \text{such comb}\end{array}\right)$

$$= \binom{n}{r} \times \varepsilon^{r} (1-\varepsilon)^{n-r}$$

$$= \frac{n!}{(n-r)!\, n!} \varepsilon^{r} (1-\varepsilon)^{n-r}$$

Now we have this distribution we want to calculate $\bar{v}$ and σ_v ...

$$\bar{v} = \sum_{v=0}^{n} v \cdot \binom{n}{v} \varepsilon^v \cdot (1-\varepsilon)^{n-v} = ?$$

We use a <u>trick</u>:

$$(p+q)^n = \sum_{v=0}^{n} \binom{n}{v} p^v q^{n-v}$$

This is true for <u>all</u> p and q. So we can fix q and differentiate wrt p:

$$n(p+q)^{n-1} = \sum_{v=0}^{n} \binom{n}{v} v p^{v-1} q^{n-v}$$

then multiply by p:

$$np(p+q)^{n-1} = \sum_{v=0}^{n} \binom{n}{v} v p^v q^{n-v}$$

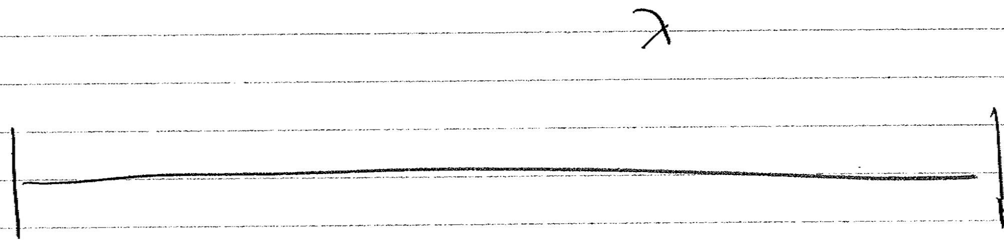
The RHS looks much like what we want <u>if</u> $p \to \varepsilon$ $q \to 1-\varepsilon$

Miracle: $p+q = \varepsilon + 1 - \varepsilon = 1$

$$\bar{v} = \sum_{v=0}^{n} \binom{n}{v} v \varepsilon^v (1-\varepsilon)^{n-v} = np$$

$$\boxed{\bar{v} = np}$$

Poisson Distribution from Binomial



Imagine over some time interval, we expect λ events.

We can divide this interval into n blocks of time, with probability of having an event

$$p = \frac{\lambda}{n}$$

We take λ constant,

$n \to \infty$

(and so $p \to 0$).

$$p(\nu) = \binom{n}{\nu} \varepsilon^{\nu} (1-\varepsilon)^{n-\nu}$$

$\varepsilon \to \frac{\lambda}{n}$ and $n \to \infty$

$$p(\nu) = \frac{n!}{\nu!\,(n-\nu)!} \left(\frac{\lambda}{n}\right)^{\nu} \left(1 - \frac{\lambda}{n}\right)^{n-\nu}$$

$$= \left(\frac{\lambda^{\nu}}{\nu!}\right) \left[\frac{n!}{(n-\nu)!} \cdot \frac{1}{n^{\nu}}\right] \left(1 - \frac{\lambda}{n}\right)^{n} \left[\left(1 - \frac{\lambda}{n}\right)^{-\nu}\right]$$

In the limit $n \to \infty$ the terms in brackets go to 1. (See following).

Since $\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n} = e^{-\lambda}$

We are left with:

$$p(\nu) \to \frac{\lambda^{\nu} e^{-\lambda}}{\nu!} \equiv \boxed{\text{Poisson Distribution}}$$

Proof:

$$\frac{n!}{(n-r)!} \cdot \frac{1}{n^r} = \frac{\overbrace{n\,(n-1)\,\cdots\,(n-r+1)}^{r\text{ terms}}}{n^r}$$

$$= \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-r+1}{n}\right)$$

which all goes to 1 in lim $n \to \infty$ ...

Mean of Poisson Distribution

$$\bar{v} = \sum_{v=0}^{\infty} v \, e^{-\lambda} \frac{\lambda^v}{v!}$$

Since first term is zero $(v=0)$

$$\bar{v} = \sum_{v=1}^{\infty} v \, e^{-\lambda} \frac{\lambda^v}{v!}$$

$$= \sum_{v=1}^{\infty} e^{-\lambda} \frac{\lambda^v}{(v-1)!}$$

$$= \lambda e^{-\lambda} \sum_{v=1}^{\infty} \frac{\lambda^{v-1}}{(v-1)!}$$

$$= \lambda e^{-\lambda} \sum_{v=0}^{\infty} \frac{\lambda^v}{v!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{\bar{v} = \lambda}$$

Central Limit Theorem:

1) We derived an example:
   Poisson Distribution $\longrightarrow$ Gaussian Distribution
   for large N.

2) In scipy, we saw that
   many random numbers in $[0, 1]$ lead
   to Gaussian distribution

Generally CLT says arithmetic mean of
independent random variables with finite
variance converges to a Gaussian distribution
in limit of large N.

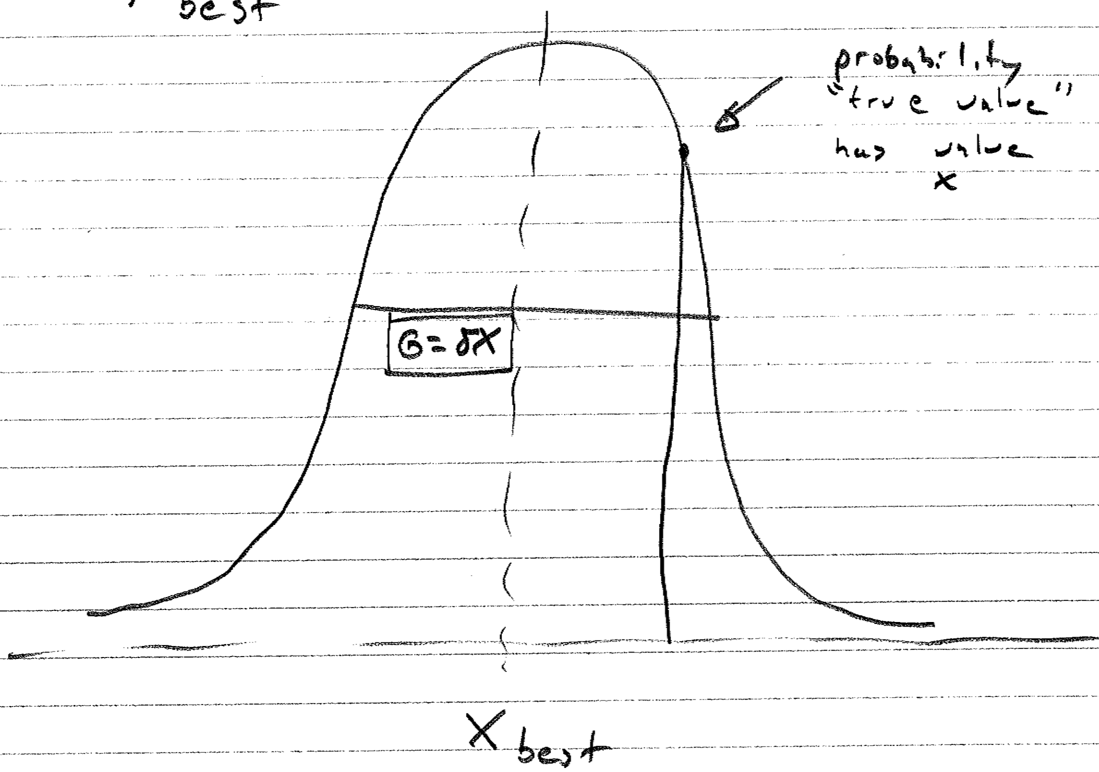
In case of Poisson, we saw this convergence
can be very fast. ($N \sim 10$).

In practice: this means the most
likely distribution we will encounter
is ... the Gaussian distribution.

Therefore; when we know no-better
we assume our PDF is Gaussian...

Gaussian Uncertainties:

$$X_{best} \pm \delta X$$



Conventions

$X_{best} \pm \delta X$ means PDF is Gaussian with

Why?

1) CLT
2) Simple: 2 parameters! mean and sigma
3) Math is easy... as we will see, extracting best values of x and σ is just
   - mean
   - rms
4) almost any measurement

Error Function

Suppose we measure the speed of light to be:

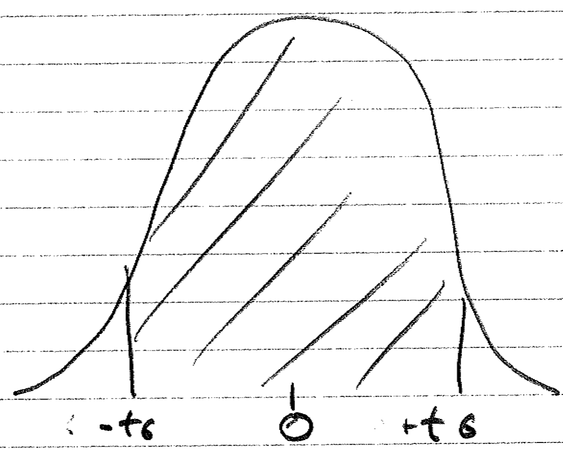
$$3.3 \pm 0.2 \quad \times 10^8 \text{ m/s}$$

How consistent is this with the generally accepted value

$$3.0 \quad \times 10^8 \text{ m/s}$$

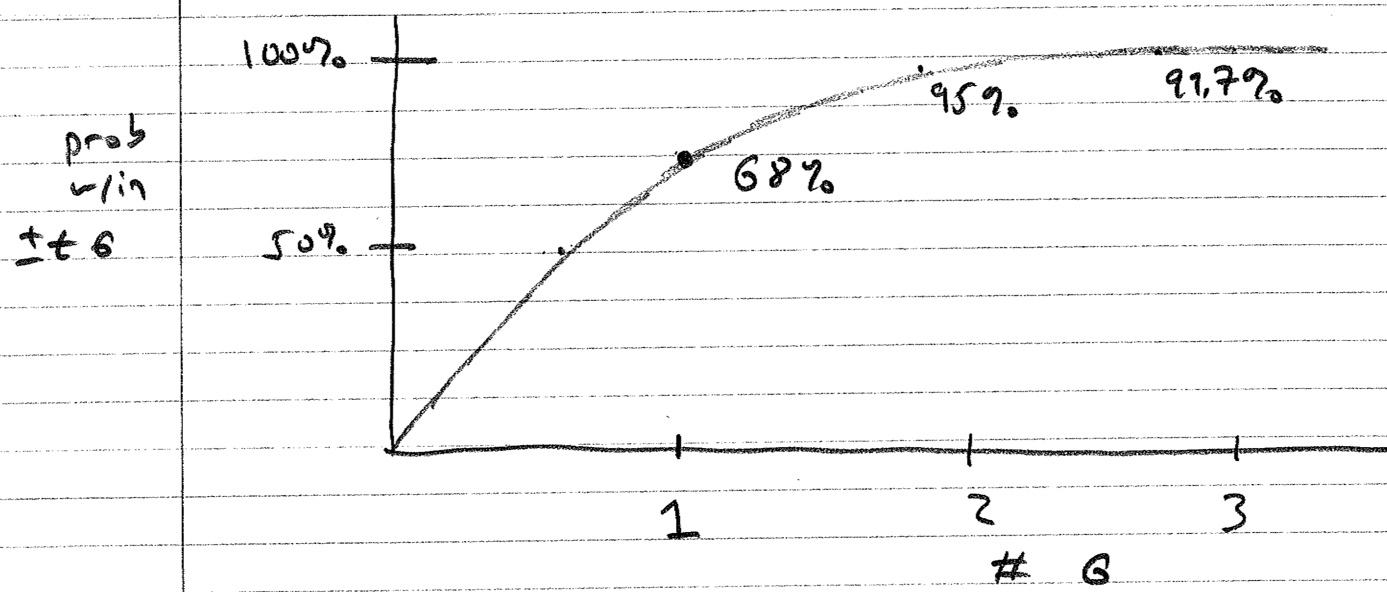
First answer: $\frac{3.3 - 3.0}{0.2} = \boxed{1.5 \text{ sigma}}$

But what if we want to know probability enclosed within $\pm t\sigma$.



$$\boxed{\text{erf}(t) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} \exp\left(-\frac{x^2}{2}\right)}$$

No analytic solution, so we just define this function and tabulate it...



HEP uses 5σ as discovery threshold

$\boxed{99.99994\ \%}$

Our choice to report errors as 1σ for Gaussian means

"true value within uncertainty 68% of time"

Why Do Errors Add in Quadrature?

$$P(x) = \frac{1}{\sqrt{2\pi}\, a} \exp\left(-\frac{x^2}{2a^2}\right) \qquad \left(\text{assume } x_0 = 0\right)$$

$$P(y) = \frac{1}{\sqrt{2\pi}\, b} \exp\left(-\frac{y^2}{2b^2}\right) \qquad \left(\text{assume } y_0 = 0\right)$$

$$P(x,y) = P(x) \cdot P(y) = \frac{1}{2\pi ab} \exp\left(-\left(\frac{x^2}{2a^2} + \frac{y^2}{2b^2}\right)\right)$$

Suppose we want to know probability that "$x+y$" adds to some value "u" $P(u)$?

$$P(u) = \int dx\, dy\; P(x,y)\, \delta(x+y-u) \qquad (\text{optional})$$

$$= \int dx\; P(x, u-x)$$

$$= \frac{1}{2\pi ab} \int dx\; \exp\left(-\frac{1}{2}\left[\frac{x^2}{a^2} + \frac{(u-x)^2}{b^2}\right]\right)$$

$$= \frac{1}{2\pi ab} \int dx\; \exp\left(-\frac{1}{2}\frac{1}{a^2 b^2}\left\{b^2 x^2 + a^2 (u-x)^2\right\}\right)$$

$$\{\} = (a^2+b^2)\,x^2 - 2a^2 u x + a^2 u^2$$

$$= (a^2+b^2)\left[x^2 - \frac{2a^2}{a^2+b^2}\,u x\right] + a^2 u^2$$

$$= (a^2+b^2)\left[x - \cancel{\frac{a^2}{(a^2+b^2)}}\,u\right]^2 - \frac{a^4}{a^2+b^2}\,u^2 + a^2 u^2$$

integrate from $-\infty \to \infty$

$$\frac{a^2 b^2}{a^2+b^2}$$

$$P(u) = \frac{1}{2\pi ab}\left\{\int dx\, \exp\left(-\frac{1}{2}\,\frac{a^2+b^2}{a^2 b^2}\,x^2\right)\right\} \exp\left(-\frac{1}{2}\,\frac{u^2}{a^2+b^2}\right)$$

$$= \frac{1}{2\pi ab}\left(\frac{\sqrt{2\pi}\;ab}{a^2+b^2}\right) \exp\left(-\frac{1}{2}\,\frac{u^2}{a^2+b^2}\right)$$

$$\boxed{P(u) = \frac{1}{\sqrt{2\pi}\,(a^2+b^2)} \exp\left(-\frac{1}{2}\,\frac{u^2}{a^2+b^2}\right)}$$

(OR)

$$[\ "\ ] = \frac{x^2}{a^2} + \frac{(u-x)^2}{b^2}$$

$$= \frac{b^2x^2 + a^2(u^2 - 2ux + x^2)}{a^2b^2}$$

$$= \frac{(a^2+b^2)x^2 - 2a^2ux + a^2u^2}{a^2b^2}$$

Define: $c^2 = a^2 + b^2$

$$[\ ] = \frac{x^2 - 2\frac{a^2}{c^2}ux + \frac{a^2}{c^2}u^2}{a^2b^2/c^2}$$

Complete Square:

$$(x + Au)^2 = x^2 + 2Aux + A^2u^2$$

$$[\ ] = \frac{\left(x - \frac{a^2}{c^2}u\right)^2 - \frac{a^4}{c^4}u^2 + \frac{a^2}{c^2}u^2}{a^2b^2/c^2}$$

$$\frac{-\frac{a^4}{c^4}u^2 + \frac{a^2}{c^2}u^2}{a^2b^2/c^2} = \frac{\frac{a^2}{c^2}\left(1 - \frac{a^2}{c^2}\right)u^2}{a^2b^2/c^2} = \boxed{\frac{u^2}{c^2}}$$

$$\left[\left(1 - \frac{a^2}{c^2}\right) = \left(\frac{c^2 - a^2}{c^2}\right) = \left(\frac{a^2 + b^2 - a^2}{c^2}\right) = \frac{b^2}{c^2} \quad !!\right]$$

$$P(u) = \exp\left(-\frac{1}{2}\frac{u^2}{c^2}\right) \cdot \frac{1}{2\pi ab} \cdot \int_{-\infty}^{\infty} dx \; \exp\left(-\frac{1}{2}\frac{\left(x-\frac{a^2}{c^2}u\right)^2}{(ab/c)^2}\right)$$

The integral is just a Gaussian (unnormalized!)

$$\int \exp\left(-\frac{1}{2}\frac{(x-x_0)^2}{\sigma^2}\right)$$

with $x_0 = \frac{a^2}{c^2} u$

and $\sigma = ab/c$

It's integral is therefore $\sqrt{2\pi} \cdot \sigma = \sqrt{2\pi}\, ab/c$

Hence

$$P(u) = \frac{1}{2\pi ab} \cdot \sqrt{2\pi}\, ab/c \cdot \exp\left(-\frac{1}{2}\frac{u^2}{c^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\, c} \exp\left(-\frac{1}{2}\frac{u^2}{c^2}\right)$$

i.e.,

$$P(x+y) = \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left(-\frac{1}{2}\frac{(x+y)^2}{\sigma_x^2+\sigma_y^2}\right)$$

(See * below)

Fourier Series:

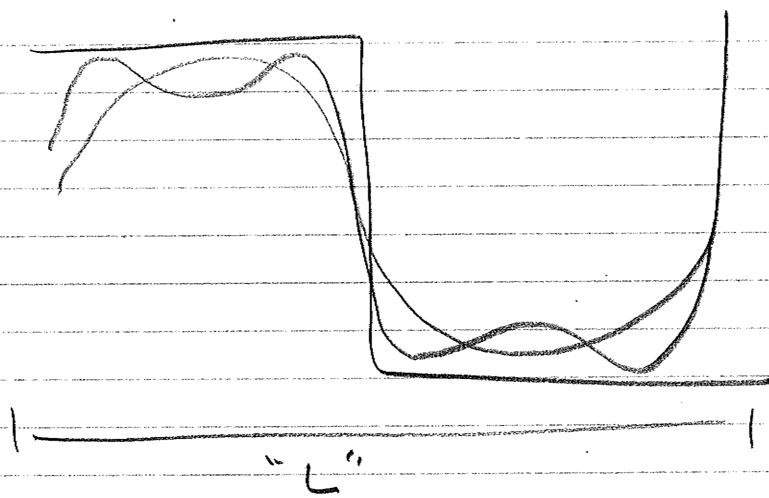
$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n x}{L} + b_n \sin\frac{2\pi n x}{L}\right)$$

Sin + cos are **complete**: means

any periodic function can be expressed

as infinite series of sines and cosines

(no proof)



. . .

(we'll see an example shortly)

[*] Preface:

(1) Good Simple Idea w/

powerful results → many different versions of same idea

(2) Show vector analogy for

→ orthogonality . . .

→ completeness . . .

sin + cos are orthogonal Functions too!

$$\int_{-L/2}^{L/2} \sin\frac{2n\pi x}{L} \sin\frac{2m\pi x}{L}\,dx = \delta_{mn}\frac{L}{2}$$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\frac{2n\pi x}{L} \cos\frac{2m\pi x}{L} = 0$$

$$\int_{-L/2}^{L/2} \cos\frac{2n\pi x}{L} \cos\frac{2m\pi x}{L} = \delta_{mn}\frac{L}{2}$$

(No Proof)

(ie, if we think of each $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ as a vector, each vector is orthogonal to every other vector)

(Normalization Factor)

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin^2\left(\frac{2\pi n x}{L}\right) dx$$

$$u = \frac{2\pi n x}{L} \qquad du = \frac{2\pi n}{L}\, dx$$

$$I = \frac{L}{2\pi n} \int_{-n\pi}^{n\pi} \sin^2(u)$$

$$= \frac{L}{2\pi n} \cdot n \int_{-\pi}^{\pi} \sin^2(u)$$

$$= \frac{L}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \cancelto{0}{\cos 2u}}{2}$$

$$= \frac{L}{2}$$

$$s(x) = \sum_{m=1}^{\infty} a_m \cos \frac{2\pi m x}{L}$$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos \frac{2\pi n x}{L} \cdot s(x) \, dx$$

$$= \int \cos \frac{2\pi n x}{L} \sum_{m=1}^{\infty} a_m \cos \frac{2\pi m x}{L} \, dx$$

$$= \sum_{m=1}^{\infty} a_m \int \cos \frac{2\pi n x}{L} \cos \frac{2\pi m x}{L}$$

$$= \sum_{m=1}^{\infty} a_m \, \delta_{mn} \cdot \frac{L}{2}$$

$$= a_n \frac{L}{2}$$

ie.

$$\boxed{\begin{aligned} a_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos \frac{2\pi n x}{L} \, s(x) \\ b_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin \frac{2\pi n x}{L} \, s(x) \end{aligned}}$$

Continuous Case

If function not periodic, we take $L \to \infty$

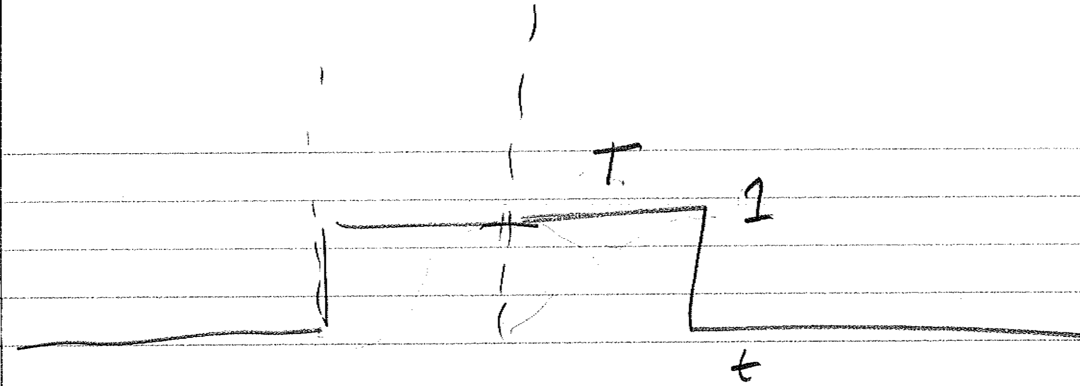
$$F(t) = \int d\nu \; F_c(\nu)\cos(2\pi\nu t) + F_s(\nu)\sin(2\pi\nu t)$$

$$\begin{pmatrix} F_c(\nu) \\ F_s(\nu) \end{pmatrix} = 2\int_{-\infty}^{\infty} dt \begin{pmatrix} \cos(2\pi\nu t) \\ \sin(2\pi\nu t) \end{pmatrix} F(t)$$

A calculation would show

$$\int |F(t)|^2\, dt = \int \underbrace{\left(|F_c(\nu)|^2 + |F_s(\nu)|^2\right)}_{\text{(Power at frequency ``}\nu\text{'')}} d\nu$$

Fourier Transform Example



$$f(\nu) = 2 \int_{-T/2}^{T/2} d\nu \ \cos 2\pi\nu t \ .$$

$$= 2 \ \frac{\sin 2\pi\nu t}{2\pi\nu} \Bigg|_{t=T/2}^{t=-T/2}$$

$$= 4 \ \frac{\sin 2\pi\nu T/2}{2\pi\nu}$$

$$\boxed{f(\nu) = (2T) \ \frac{\sin(\pi\nu T)}{\pi\nu T}}$$

Connection of Sine / Cosine to Fourier Transforms,

<u>Conventional Definition</u>:

$$F(\nu) = \int e^{-2\pi i \nu t} f(t) \, dt$$

$$F(t) = \int e^{2\pi i \nu t} F(\nu) \, d\nu$$

$$= \int \left[\cos(2\pi i \nu t) + i \sin(2\pi i \nu t)\right] f(t)$$

$$= \frac{1}{2} \mathcal{F}_c(\nu) + i\frac{1}{2} \mathcal{F}_s(\nu)$$

<u>Why do we prefer this?</u>

→ Having a sine and cosine transform was really about <u>phase</u>. (Sine and Cosine are same function shifted by a phase)

→ Likewise $F(\nu)$ has two parts: $(re^{i\phi})$

r: $|F(\nu)|$ → amplitude $(\mathcal{F}_c^2 + \mathcal{F}_s^2)$

ϕ: → phase

Note that $|\mathcal{F}(\nu)|^2 = \mathcal{F}_c^2(\nu) + \mathcal{F}_s^2(\nu)$

Completeness of F.T. . . .

$$F(\nu) \equiv \int_{-\infty}^{\infty} dt\, e^{-i2\pi\nu t} f(t)$$

$$f(t) = e^{i2\pi\nu' t}$$

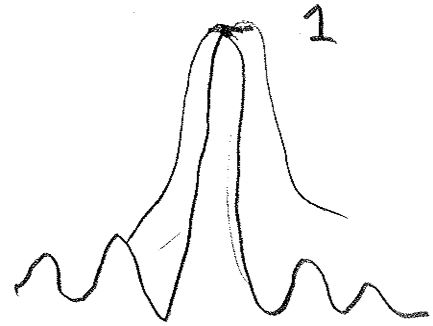
$$F(\nu) = \int_{-\infty}^{\infty} dt \quad e^{2\pi i(\nu'-\nu)t} \quad dt$$

$$F(\nu) = \lim_{L\to\infty} \int_{-L}^{L} dt \quad e^{2\pi i(\nu'-\nu)t} \quad dt$$

$$= \left. \frac{e^{2\pi i(\nu'-\nu)t}}{2\pi i(\nu'-\nu)} \right|_{-L}^{L}$$

$$= \frac{e^{2\pi i(\nu'-\nu)L} - e^{-2\pi i(\nu'-\nu)L}}{2\pi i(\nu'-\nu)}$$

$$= T\,\frac{\sin(2\pi(\nu'-\nu)T)}{2\pi i(\nu'-\nu)T}$$



$$\underbrace{\sim \quad \delta(\nu'-\nu)}_{\text{orthogonality}}$$

Similar $\delta(t'-t)$
(completeness)

Best Estimates for X:

Suppose we make n measurements
"drawn from a Gaussian distribution"

$$x_1, x_2, x_3, \ldots, x_n$$

What is our best estimate for the "true value" of x?

For a "true value" of X and an uncertainty of each measurement σ, the probability of measuring within Δx of x_i is

$$P(x_1) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x_1 - X)^2}{2\sigma^2}\right) \Delta x$$

The combined probability of our whole series is

$$P = P(x_1) \cdot P(x_2) \cdot P(x_3) \cdots P(x_n)$$

$$= \left(\frac{\Delta x}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left(-\frac{(x_1 - X)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{(x_2 - X)^2}{2\sigma^2}\right) \cdots$$

$$= \left(\frac{\Delta x}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left(-\frac{\sum_i (x_i - X)^2}{2\sigma^2}\right)$$

The main technique (not "trick" because we will use it many times) is to maximize probability by setting

$$\boxed{\frac{dP}{dX} = 0}$$

$$\frac{dP}{dX} = P \cdot \frac{d}{dX}\left(-\frac{\sum_i (x_i - X)^2}{2\sigma^2}\right)$$

$$= \frac{P}{2\sigma^2} \sum_i (x_i - X)$$

Only way $\frac{dP}{dX} = 0$ is

$$\sum_i (x_i - X) = 0$$

$$\sum_i x_i - NX = 0$$

$$\boxed{X = \frac{1}{N}\sum_i x_i}$$

Best Estimate for σ

$$P \sim \frac{1}{\sigma^N} \exp\left(-\frac{\sum_i (x_i - X)^2}{2\sigma^2}\right)$$

Since $\frac{d}{d\sigma}\frac{1}{\sigma^N} = \frac{-N}{\sigma^{N+1}} = \frac{1}{\sigma^N}\left(\frac{-N}{\sigma}\right)$

and $\frac{d}{d\sigma}\frac{1}{\sigma^2} = -\frac{2}{\sigma^3}$

$$\frac{dP}{d\sigma} = P\left(-\frac{N}{\sigma} + \frac{\sum_i (x_i - X)^2}{\sigma^3}\right)$$

$$= 0$$

$$\Rightarrow \boxed{\sigma^2 = \frac{\sum_i (x_i - X)^2}{N}}$$

*** To describe a distribution by a Gaussian, need only calculate

mean $\rightarrow X$

RMS $\rightarrow \sigma$

Standard Deviation of the mean

Suppose we make N measurements of a quantity x each with uncertainty σ.

We calculate the mean:

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{N}$$

Note that as each x_i is centered on the true value X, our $\bar{x}$ will also be centered on X, but with **better** uncertainty!

Just use error propagation

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{\partial \bar{x}}{\partial x_1} \cdot \sigma\right)^2 + \dots + \left(\frac{\partial \bar{x}}{\partial x_n} \cdot \sigma\right)^2}$$

But each $\frac{\partial x}{\partial x_i} = \frac{1}{N}$, so we have

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{1}{N}\sigma\right)^2 + \dots + \left(\frac{1}{N}\sigma\right)^2}$$

$$= \sigma \cdot \sqrt{N \cdot \frac{1}{N^2}} = \boxed{\frac{\sigma}{\sqrt{N}}}$$

Repeating Measurement N times reduces uncertainty by a factor $\frac{1}{\sqrt{N}}$!!!

χ^2 Intro Lecture

χ^2 Tests

Want to understand how likely a set of data:

$$\{x_1 \pm \sigma_1, \; x_2 \pm \sigma_2, \; x_3 \pm \sigma_3, \ldots\} \equiv x_i \pm \sigma_i$$

is the result of a corresponding theoretical prediction:

$$\{y_1, \; y_2, \; \ldots, \; y_n\} \equiv y_i$$

Assume Gaussian uncertainties on x, then probability of one measurement is just a Gaussian PDF

$$P_i = \frac{1}{\sqrt{2\pi}\,\sigma_i} \exp\left(-\frac{1}{2}\frac{(y_i - x_i)^2}{\sigma_i^2}\right)$$

The probability for the complete set of n measurements is called the Likelihood and is just the probability

$$= \prod_i \frac{1}{\sqrt{2\pi}\,\sigma_i} \exp\left(-\frac{1}{2}\frac{(y_i - x_i)^2}{\sigma_i^2}\right)$$

elihood (and what follows) is
more general, we could have
in some circumstances, and
Likelihood function)

- Since sums are easier than products, and log is monotonically increasing

$$\log \mathcal{L}$$

- Since we physicists think about minimization (as opposed to maximization)

$$-\log \mathcal{L}$$

- Since there is an annoying factor of $\frac{1}{2}$ in exponent, scale

$$-2\log \mathcal{L}$$

Let's calculate it:

$$-2\log \mathcal{L} = -2\sum_i \log\left(\frac{1}{2\pi\sigma_i}\right) + \sum_i \frac{(y_i - x_i)^2}{\sigma_i^2}$$

Does not depend on x_i or y_i, only a constant depending on precision of experiment

$\equiv \chi^2$

$$\chi^2 = \sum_i \frac{(y_i - x_i)^2}{\sigma_i^2} \qquad \left(= -2\log\mathcal{L} + \text{const}\right)$$

SMALL $\chi^2 \Rightarrow$ Large Probability data and prediction agree

$\boxed{*}$ For counting exp

$x_i \to N_i \quad \sigma_i^2 \to N_i$

χ^2 fit

$$\chi^2 \equiv \sum_i \frac{\left(x^i_{pred} - x^i_{meas}\right)^2}{\sigma_i^2}$$

Suppose our prediction is a relationship:

$$y_i = a x_i + b \qquad \text{(Measure "y")}$$

$$\chi^2 = \sum_i \frac{\left(a x_i + b - y_i\right)^2}{\sigma_i^2}$$

$$\frac{\partial \chi^2}{\partial a} = \sum_i \frac{2\left(a x_i + b - y_i\right) \cdot x_i}{\sigma_i^2} = 0$$

$$\frac{\partial \chi^2}{\partial b} = \sum_i \frac{2\left(a x_i + b - y_i\right)}{\sigma_i^2} = 0$$

$$\langle ? \rangle \equiv \sum_i \frac{?}{\sigma_i^2} \Big/ \sum_i \frac{1}{\sigma_i^2}$$

$$a\langle x^2 \rangle + b\langle x \rangle - \langle xy \rangle = 0$$

$$a\langle x \rangle + bN - \langle y \rangle = 0$$

$$N a \langle x^2 \rangle + b \langle x \rangle N - \langle xy \rangle N = 0$$

$$a \langle x \rangle^2 + b \langle x \rangle N - \langle x \rangle \langle y \rangle = 0$$

$$a\left(\langle x^2 \rangle - \langle x \rangle^2 \right) = \langle xy \rangle - \langle x \rangle \langle y \rangle$$

$$\boxed{a = \frac{N\langle xy \rangle - \langle x \rangle \langle y \rangle}{N\langle x^2 \rangle - \langle x \rangle^2}}$$

$$a \langle x^2 \rangle \langle x \rangle + b \langle x \rangle^2 - \langle xy \rangle \langle x \rangle = 0$$

$$a \langle x \rangle \langle x^2 \rangle + bN \langle x^2 \rangle - \langle y \rangle \langle x^2 \rangle = 0$$

$$b\left(N\langle x^2 \rangle - \langle x \rangle^2 \right) = \langle x^2 \rangle \langle x \rangle - \langle xy \rangle \langle x \rangle$$

$$\boxed{b = \frac{\langle y \rangle \langle x^2 \rangle - \langle xy \rangle \langle x \rangle}{N \langle x^2 \rangle - \langle x^2 \rangle}}$$