Homework Assignment 7
"Just when I thought I was out..."

Practice Problems

These problems are graded on effort only.

Griffiths: P2.44

Hint: evaluate

$$\psi_2 \frac{d\psi_1}{dx} - \psi_1 \frac{d\psi_2}{dx}$$

at $x \to +\infty$ to determine it's constant value.

Additional Problems

The first two problems are from the midterm exam. If you are confident of your solution there, you may just indicate "done" as your solution and we will substitute your exam solution.

Problem 1: From the midterm exam:

A particle is in a certain potential $V(x)$ with corresponding Hamiltonian operator $\hat{H}$. Suppose that the properly normalized stationary states $\psi_1(x)$, $\psi_2(x)$, and $\psi_3(x)$ have definite energies:

$$E_1 = 2\,\epsilon, \qquad E_2 = 3\,\epsilon, \qquad E_3 = 5\,\epsilon$$

for some positive real constant ϵ. Using positive real constants when possible, construct a state $\Psi(x,t)$ with the properties:

$$\langle \Psi | \psi_2 \rangle = 0$$

and:

$$\langle \hat{H} \rangle = 3\epsilon$$

Problem 2: From the midterm exam:

Consider a particle in a harmonic oscillator potential with allowed energies:

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) \tag{1}$$

for

$$n = 0, 1, 2, 3, \ldots$$

Consider a general solution:

$$\Psi(x,t) = \sum_n c_n \, \psi_n(x) \, e^{-i\,n\,\omega\,t}$$

Using the ladder operators $\hat{a}_+$ and $\hat{a}_-$, with the properties recalled in Problem 2, verify that Ehrenfest's Theorem holds in this case:

$$\frac{d}{dt}\langle p\rangle = -\langle\frac{dV}{dx}\rangle = -m\omega^2\langle x\rangle$$

Suggested approach:
(A) Calculate $\langle\psi_m|\hat{a}_+\psi_n\rangle$ and $\langle\psi_m|\hat{a}_-\psi_n\rangle$ and express the answer in terms of δ_{ij}.
(B) Calculate $\langle\Psi|\hat{a}_+\Psi\rangle$ and $\langle\Psi|\hat{a}_-\Psi\rangle$ and leave your answer as an infinite series.
(C) Calculate $\langle p\rangle$ and $\langle x\rangle$ from the (B).
(D) Calculate:

$$\frac{d}{dt}\langle p\rangle$$

and hope that it all works out!

Additional Note: You may have noticed that as we have defined the c_n above, they have absorbed a time-dependent factor $e^{-i\omega t/2}$. But then note also that $c_n^* c_m$ is still time-independent for any values of n and m.

Problem 3: Recall that the probability current for a wave function $\psi(x)$ is defined as:

$$J(x) \equiv \frac{i\hbar}{2m}\left(\psi\frac{d\psi^*}{dx} - \psi^*\frac{d\psi}{dx}\right)$$

In Problem 3 of the previous homework assignent (HW06) we considered the potential

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$$

and assumed that $E > V_0$. You showed that the general solution is:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x \leq 0 \\ Fe^{i\eta kx} + Ge^{-i\eta kx} & x \geq 0 \end{cases}$$

where:

$$\eta \equiv \sqrt{\frac{E - V_0}{E}}.$$

In this problem, we will assume scattering from the left, i.e. $G = 0$.

(A) Calculate the probability current J_i for just the incoming wave:

$$\Psi_i = Ae^{ikx}$$

(B) Calculate the probability current J_r for just the reflected wave:

$$\Psi_r = Be^{-ikx}$$

(C) Define the coefficient of reflection as:

$$R \equiv \left|\frac{J_r}{J_i}\right|$$

and compare to your answer for Problem 3D of the previous homework.
(D) Calculate the probability current J_t for just the transmitted wave:

$$\Psi_t = Fe^{i\eta kx}$$

(E) Define the coefficient of transmission as:

$$T \equiv \left| \frac{J_t}{J_i} \right|$$

and compare to your answer for Problem 3E of the previous homework.

(F) Calculate the current density J_L for the complete solution for $x < 0$.

(G) Calculate the current density J_R for the complete solution for $x > 0$. (We still have $G = 0$)

(H) Set $J_L = J_R$ and show that:

$$T + R = 1$$