

Homework Assignment 3

Infinite Square Well Potential

Practice Problems

These problems are graded on effort only.

Griffiths: P1.17, P2.1bc, P2.3

Additional Problems

Problem 1: Suppose that $\psi_n(x)$ is a solution to the TISE:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_n}{\partial x^2} + V(x) \psi_n(x) = E_n \psi_n(x)$$

for

$$n = 1, 2, 3, \dots$$

Show that:

$$\Psi(x, t) \equiv \sum_{n=1}^{\infty} c_n \exp\left(-\frac{i E_n t}{\hbar}\right) \psi_n(x)$$

is a solution to the SE:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi$$

Problem 2: We found the complete set of orthonormal solutions to the TISE for the infinite square well potential to be:

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases} \quad n = 1, 2, 3, \dots \quad (1)$$

Show that if the particle is in any stationary state $\psi_n(x)$ then the expectation value of the position is:

$$\langle x \rangle = \frac{a}{2}$$

You may use the definite integral:

$$\int_0^{n\pi} x \sin^2 x \, dx = \frac{n^2 \pi^2}{4}$$

Problem 3: Suppose at time $t=0$, a particle is in the state

$$\Psi(x, 0) = \frac{1}{\sqrt{2}}\psi_\alpha(x) + \frac{1}{\sqrt{2}}\psi_\beta(x)$$

where $\psi_\alpha(x)$ and $\psi_\beta(x)$ are two specific $\psi_n(x)$ from Problem 2 with definite energies E_α and E_β and $\alpha \neq \beta$. Remember that $\psi_\alpha(x)$ and $\psi_\beta(x)$ are real functions.

(A) Write down the time-dependent wave function $\Psi(x, t)$ (You verified in Problem 1 that this will be a solution to the SE.)

(B) Calculate $|\Psi|^2$. You should find:

$$|\Psi|^2 = \frac{|\Psi_\alpha|^2 + |\Psi_\beta|^2}{2} + \dots$$

where the remaining bit is due to interference.

(C) Calculate the expectation value of the position $\langle x \rangle$ at any time t . Make sure you use the results from Problem 2 so you should just have one not-so-bad integral to compute. You may use:

$$\sin a \sin b = \frac{\cos(a - b) - \cos(a + b)}{2}$$

to help compute the integral.

Problem 4: The axiomatic definition of a vector space V and inner product space H over the real numbers \mathbb{R} is detailed in Table 1 on the next page.

(A) Show that D1 follows from A1-5 and M1-5.

Hint: we already know for the scalars that $0 + 0 = 0$

(B) Show that D2 follows from A1-5, M1-5, and D1.

Hint: you need to show $x + (-1)x = 0$. And we already know that $1 + (-1) = 0$.

(C) Show that D3 follows from I1 and I2.

(D) Show that D4 follows from I1 and I3.

Table 1: Here we define the properties of a vector space V and an inner product space H . Note that no complex conjugation appears in these definitions as the scalar field is the real numbers.

Useful Math Symbols:

| | |
|---------------------------------|--|
| $\forall x \in V$ | for all x in V (for any vector x) |
| $\forall \alpha \in \mathbb{R}$ | for all α in \mathbb{R} (for any real number α) |
| $\exists! y$ | there exists unique y |
| s.t. | such that |

Properties of Addition:

| | | | |
|-----------|--------------------|--|-----------------------------|
| A1 | Closure | $\forall x, y \in V$ | $(x + y) \in V$ |
| A2 | Commutative | $\forall x, y \in V$ | $x + y = y + x$ |
| A3 | Associative | $\forall x, y, z \in V$ | $(x + y) + z = x + (y + z)$ |
| A4 | Zero | $\exists! 0$ s.t. $\forall x \in V$ | $x + 0 = x$ |
| A5 | Inverse | $\forall x \in V \exists! (-x) \in V$ s.t. | $x + (-x) = 0$ |

Properties of Scalar Multiplication:

| | | | |
|-----------|---------------------|--|--|
| M1 | Closure | $\forall x \in V$ and $\forall \alpha \in \mathbb{R}$ | $\alpha x \in V$ |
| M2 | Identity | $\forall x \in V$ | $1x = x$ |
| M3 | Associative | $\forall x \in V$ and $\forall \alpha, \beta \in \mathbb{R}$ | $\alpha(\beta x) = (\alpha\beta)x$ |
| M4 | Distributive | $\forall x, y \in V$ and $\forall \alpha \in \mathbb{R}$ | $\alpha(x + y) = \alpha x + \alpha y$ |
| M5 | Distributive | $\forall x \in V$ and $\forall \alpha, \beta \in \mathbb{R}$ | $(\alpha + \beta)x = \alpha x + \beta x$ |

Deducible Properties:

| | | |
|-----------|-------------------|----------------|
| D1 | $\forall x \in V$ | $0x = 0$ |
| D2 | $\forall x \in V$ | $(-1)x = (-x)$ |

Properties of Inner Products:

| | | |
|-----------|---|---|
| I1 | $\forall x, y \in H$ | $\langle x y \rangle = \langle y x \rangle$ |
| I2 | $\forall x, y, z \in H$ and $\forall \alpha \in \mathbb{R}$ | $\langle x \alpha y \rangle = \alpha \langle x y \rangle$ |
| I3 | $\forall x, y, z \in H$ | $\langle x + y z \rangle = \langle x z \rangle + \langle y z \rangle$ |
| I4 | $\forall x \in H$ | $\langle x x \rangle \geq 0$ |
| I5 | $\forall x \in H$ | $\langle x x \rangle = 0$ if and only if $x = 0$ |

Deducible Properties:

| | | |
|-----------|--|---|
| D3 | $\forall x, y \in H$ and $\forall \alpha \in \mathbb{R}$ | $\langle \alpha x y \rangle = \alpha \langle x y \rangle$ |
| D4 | $\forall x, y, z \in H$ | $\langle x y + z \rangle = \langle x y \rangle + \langle x z \rangle$ |