# Homework Assignment 6
# Scatter There Thy Cheerful Beams!

## Practice Problems

These problems are graded on effort only.

**Griffiths: P2.23, P2.25, P2.58**
Hint for 2.23b: use integration by parts to move the derivative off of $\theta(x)$.

## Additional Problems

**Problem 1:** Consider the vector space of 2x1 column vectors, for example the vectors $x$ and $y$:

$$x = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad\qquad y = \begin{pmatrix} c \\ d \end{pmatrix}$$

with transpose:

$$x^T = \begin{pmatrix} a & b \end{pmatrix}$$

We can create an inner product space by defining the inner product as:

$$\langle x|y\rangle \;\equiv\; (x^*)^T\, y \;=\; a^*\, c + b^*\, d$$

(A) Verify that this definition does satisfy the properties of an inner product:

| | | |
|---|---|---|
| **I1** | $\forall x, y \in H$ | $\langle x\|y\rangle = \langle y\|x\rangle^*$ |
| **I2** | $\forall x, y, z \in H$and $\forall \alpha \in \mathbb{C}$ | $\langle x\|\alpha y\rangle = \alpha\,\langle x\|y\rangle$ |
| **I3** | $\forall x, y, z \in H$ | $\langle x + y\|z\rangle = \langle x\|z\rangle + \langle y\|z\rangle$ |
| **I4** | $\forall x \in H$ | $\langle x\|x\rangle \geq 0$ |
| **I5** | $\forall x \in H$ | $\langle x\|x\rangle = 0$ if and only if $x = 0$ |

By definition, operators return a new vector for a given vector. In this finite-dimensional vector space, any linear operator $O$ can be represented as a 2x2 matrix:

$$O\, x = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} O_{11}\, a + O_{12}\, b \\ O_{21}\, a + O_{22}\, b \end{pmatrix}$$

(B) Show that the hermetian adjoint of $O$ is given by:

$$O^\dagger = (O^*)^T = \begin{pmatrix} O^*_{11} & O^*_{21} \\ O^*_{12} & O^*_{22} \end{pmatrix}$$

from our definition:

$$\langle x|O^\dagger y\rangle = \langle Ox|y\rangle$$

Hint: calculate:

$$\langle Ox|y\rangle$$

then do whatever it takes to bring the action over onto the $y$ instead, and then read off $O^\dagger$. You may use the property of matrices $A$ and $B$ that:

$$(AB)^T = B^T A^T$$

An operator $U$ is unitary if

$$U^\dagger U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv I$$

Note that:

$$Ix = x$$

for any vector $x$.

(C) Show that the rotation matrix:

$$R \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is unitary.

(D) Show that for a unitary matrix $U$:

$$\langle Ux|Uy\rangle = \langle x|y\rangle$$

**Problem 2:** In lecture we studied the scattering states (with $E > 0$) of the delta-function potential:

$$V(x) = -\alpha\delta(x)$$

We found that the general solution:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x \leq 0 \\ Fe^{ikx} + Ge^{-ikx} & x \geq 0 \end{cases}$$

has boundary conditions:

$$F + G = A + B$$

and

$$F - G = A(1 + 2i\beta) - B(1 - 2i\beta)$$

where:

$$\beta = \frac{m\alpha}{\hbar^2 k}$$

Notice that the waves (with coefficients) $A$ and $G$ are incoming, while the waves $B$ and $F$ are outgoing. They are connected by the scattering matrix $S$:

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$$

(A) Calculate the $S$-matrix from the boundary conditions.

Hint: Put $B$ and $F$ on the LHS of the boundary conditions and $A$ and $G$ on the RHS. Eliminate $F$ and solve for $B$ in terms of $A$ and $G$. Then read off:

$$B = S_{11}\,A + S_{12}\,G$$

(B) Show that the $S$-matrix you calculated in (A) is unitary.

(C) Calculate the probability current:

$$J(x) = \frac{i\hbar}{2m}\left(\psi\frac{d\psi^*}{dx} - \psi^*\frac{d\psi}{dx}\right)$$

for $x < 0$.
Hint: this explodes a bit but you should get nice cancellation leaving you just two terms in your answer.

(D) Also calculate the probability current $J(x)$ for $x > 0$.

Hint: do not calculate this again from scratch! Use your results from (C) and substitution!

(E) For normalizable solutions, to the SE, we showed that:

$$J(a) = J(-a)$$

in the limit $a \to \infty$. These are not normalizable solutions, but let's assume still that the probability current you calculated in (C) equals the probability current you calculated in (D). Calculate a condition on $|A|^2$, $|B|^2$, $|F|^2$, and $|G|^2$ that results from this.

(F) Define the outgoing waves $O$ and the incoming waves $I$ as:

$$O = \begin{pmatrix} B \\ F \end{pmatrix}, \qquad\qquad I = \begin{pmatrix} A \\ G \end{pmatrix}$$

Calculate a condition on $|A|^2$, $|B|^2$, $|F|^2$, and $|G|^2$ from:

$$\langle O|O\rangle = \langle I|I\rangle$$

Compare to your condition in (E).

(G) Show that:

$$\langle O|O\rangle = \langle I|I\rangle$$

implies that $S$ is unitary.

Hint: use $O = SI$.

**Problem 3:** Consider the potential:

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$$

and assume $E > V_0$.

(A) Show that the general solutions to the SE can be written as:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x \leq 0 \\ Fe^{i\eta kx} + Ge^{-i\eta kx} & x \geq 0 \end{cases}$$

where:

$$\eta \equiv \sqrt{\frac{E - V_0}{E}}$$

(B) Determine the boundary conditions at $x = 0$ and use these conditions to determine the scattering matrix $S$ defined by:

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$$

(C) Is $S$ unitary? Is there anything unphysical about this potential which might explain this?

(D) Calculate the reflection from the left:

$$R = \frac{|B|^2}{|A|^2}\bigg|_{G=0}$$

(E) And transmission from the left:

$$T = \frac{|F|^2}{|A|^2}\bigg|_{G=0}$$