Homework Assignment 9
“Brave and adept from this day on”

At least half of the problems here are excellent practice for your final exam.

Practice Problems

These problems are graded on effort only.
Problem 1: Reproduce the derivation of generalized Ehrenfest theorem (Eq. 3.73).
Griffiths: P3.18, P3.19, P3.25, P3.26

Additional Problems

Problem 2: Suppose that the state vector is:

$$|\Psi\rangle = \frac{i}{\sqrt{3}}\,|\lambda_1\rangle + k\,|\lambda_2\rangle$$

where $|\lambda_n\rangle$ is a properly normalized eigenvector of an observable $\hat{Q}$ with eigenvalue λ_n. Assume $\hat{Q}$ has a non-degenerate spectrum. Hint: I am not trying to trick you here (or anywhere in this course, really!) If any of the answers below seem easy to you, then good for you!

(A) What is the probability of a measurement yielding the value λ_1?

(B) What is the probability of a measurement yielding the value λ_2? (Give a numerical answer, such as 1/10, not something that depends on k).

(C) What is expectation value of the observable $\hat{Q}$?

(D) What is $\langle\lambda_1|\lambda_1\rangle$?

(E) What is $\langle\lambda_1|\lambda_2\rangle$?

(F) What is $\langle\lambda_1|\Psi\rangle$?

(G) What is $\langle\Psi|\lambda_1\rangle$?

(H) What is $|\langle\lambda_2|\Psi\rangle|$? (Notice the norm symbol $|\ldots|$ and give a numerical answer.)

Problem 3: Suppose that the properly-normalized state vector at $t = 0$ is:

$$|\Psi\rangle = C\left(|a_1\rangle + i\,|a_2\rangle\right)$$

where C is a positive real constant and $|a_n\rangle$ is a properly normalized eigenvector of an observable $\hat{A}$ with eigenvalue a_n. Assume $\hat{A}$ has a non-degenerate discrete spectrum (with $n = 1, 2, 3, \ldots$). Further suppose the spectrum of the Hamiltonian is discrete and non-degenerate with eigenvalues $\{E_i\}$ and properly normalized eigenvectors $|E_i\rangle$. Suppose we can write:

$$|E_1\rangle = \frac{1}{\sqrt{2}}\,|a_1\rangle + \frac{1}{\sqrt{2}}\,|a_2\rangle$$

$$|E_2\rangle = -\frac{i}{\sqrt{2}}\,|a_1\rangle + \frac{i}{\sqrt{2}}\,|a_2\rangle$$

Do not leave an explicit dependence on C in any your answers below (i.e. solve for C).

(A) Deduce the inner products $\langle E_1|a_1\rangle$ and $\langle E_1|a_2\rangle$.

(B) Deduce the inner products $\langle E_2|a_1\rangle$ and $\langle E_2|a_2\rangle$.

(C) Deduce the inner product $\langle E_1|a_m\rangle$ for $m > 2$.

(D) Deduce the inner product $\langle E_2|a_m\rangle$ for $m > 2$.

(E) Rewrite the state vector $|\Psi\rangle$ in terms of the stationary states $|E_i\rangle$ by inserting a complete set:

$$1 = \sum_i |E_i\rangle\,\langle E_i|$$

and using your results from A-D.

(F) What is the expectation value for the observable $\hat{A}$ at $t = 0$?

(G) What is the expectation value for the total energy?

(H) Write down the time-dependent state vector $|\Psi(t)\rangle$.

(I) Deterimine the time-dependent wave function $\Psi(x,t)$ in terms of the wave functions of the stationary states:

$$\Psi_i(x) \equiv \langle x|E_i\rangle$$

You must start from your answer in (H) and explicitly use the Dirac notation to determine the wave function $\Psi(x,t)$.

Problem 4: The Parseval-Plancherel identity:

$$\int_{-\infty}^{+\infty} |\Psi(x)|^2 \, dx \;=\; \int_{-\infty}^{+\infty} |\widetilde{\Psi}(p)|^2 \, dp$$

shows that if a position-space wave function $\Psi(x)$ is normalized, then so is it's momentum-space wave function $\widetilde{\Psi}(p)$.

(A) Derive the Parseval-Plancherel identity by explicit integration. **Hint:** start with the LHS and write out $\Psi(x)$ and $\Psi^*(x)$ in terms of their Fourier transforms. Make sure you use two different dummy variables for the integral over momentum (e.g. dq and dp) . At this point, you should have **three** integrals: dx,dp, and dq. Look carefully and note that only two factors involve x. Move the integral dx to include just those factors, and use the orthogonality of the complex exponentials:

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \exp(i(q-p)x/\hbar) \, dx = \delta(q-p)$$

This kills the integral over dx, then use the resulting δ-function to kill the integral of dq.

(B) Derive Parseval-Plancherel identity using the Dirac notation. Start with:

$$\langle \Psi | \Psi \rangle$$

then insert the identity:

$$\int dx \, |x\rangle \langle x| = 1$$

and identify:

$$\Psi(x) = \langle x | \Psi \rangle$$

Next, do the same thing but using the identity:

$$\int dp \, |p\rangle \langle p| = 1$$

and identify:

$$\widetilde{\Psi}(p) = \langle p | \Psi \rangle$$