Best Estimates for X:

Suppose we make n measurements
"drawn from a Gaussian distribution"

$x_1, x_2, x_3, \ldots, x_n$

What is our best estimate for the "true value" of x?

For a "true value" of X and an uncertainty of each measurement σ, the probability of measuring within Δx of x_i is

$$P(x_1) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x_1 - X)^2}{2\sigma^2}\right) \Delta x$$

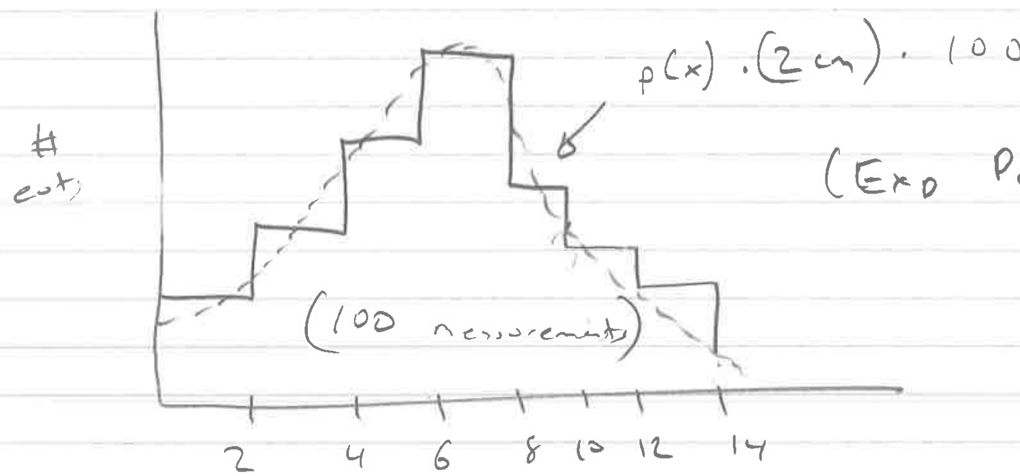
The combined probability of our whole series is

$$P = P(x_1) \cdot P(x_2) \cdot P(x_3) \ldots P(x_n)$$

$$= \left(\frac{\Delta x}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left(-\frac{(x_1-X)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{(x_2-X)^2}{2\sigma^2}\right) \cdots$$

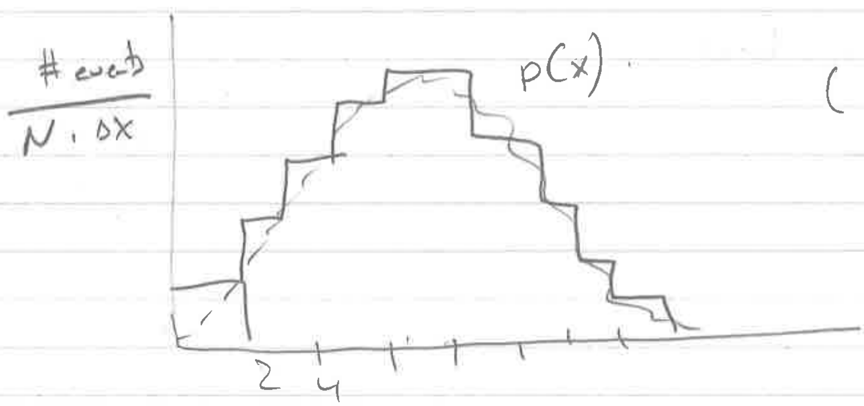
$$= \left(\frac{\Delta x}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left(-\frac{\sum_i (x_i - X)^2}{2\sigma^2}\right)$$

Whichever way you think about it, you must remember to account for normalization of prediction to experiment!



(Exp Perspective)

$\frac{\text{frac evts}}{2\text{cm}}$



(Theory Perspective)

$\boxed{\text{Covariance:}}$

$$q_i = q(x_i, y_i) \qquad \begin{array}{l} x_i \longrightarrow \bar{x}\,,\ \sigma_x \\ y_i \longrightarrow \bar{y}\,,\ \sigma_y \end{array}$$

Near $x = \bar{x}$, $y = \bar{y}$:

$$q_i = q(x_i, y_i) \approx q(\bar{x}, \bar{y}) + \frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y})$$

$$\bar{q} = \frac{1}{N}\sum_{i=1}^{N} q_i \approx \frac{1}{N}\sum_{i=1}^{N}\left[q(\bar{x},\bar{y}) + \frac{\partial q}{\partial x}\underbrace{\cancel{(x_i - \bar{x})}}_{0} + \frac{\partial q}{\partial y}\underbrace{\cancel{(y_i - \bar{y})}}_{0}\right.$$

$$= q(\bar{x}, \bar{y})$$

$$\boxed{\bar{q} = q(\bar{x}, \bar{y})}$$

$$\sigma_q^2 = \frac{1}{N}\sum_i (q_i - \bar{q})^2$$

$$= \frac{1}{N}\sum_i \left(\frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y})\right)^2$$

$$= \left(\frac{\partial q}{\partial x}\right)^2 \frac{1}{N}\sum_i (x_i - x)^2 + \left(\frac{\partial q}{\partial y}\right)^2 \frac{1}{N}\sum_i (y_i - y)^2 + 2\left(\frac{\partial q}{\partial x}\right)\left(\frac{\partial q}{\partial y}\right)$$

$$\sum_i (x - x_i)(y - y_i)$$

$$= \left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2 \sigma_y^2 + 2\frac{\partial q}{\partial x}\frac{\partial q}{\partial y}\,\sigma_{xy}$$

$$\sigma_{xy} \equiv \frac{1}{N}\sum_i (\bar{x} - x_i)(\bar{y} - y_i)$$

σ_{xy} is the <u>covariance</u> of x and y.

Corresponding PDF:

Multivariate Normal

$$\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\frac{x - x_0}{\sigma^2}\right)$$

$$\longrightarrow \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{x}_0)\cdot \Sigma^{-1} \cdot (\vec{x}-\vec{x}_0)\right)$$

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \\ \sigma_{xy} & \sigma_y^2 & \\ & & \ddots \end{bmatrix}$$

In 2-D

$$\rho \equiv \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$ (correlation between x and y)

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix} \frac{1}{\sigma_x^2\sigma_y^2 - \rho^2\sigma_x^2\sigma_y^2}$$

$$= \frac{1}{1-\rho^2} \begin{bmatrix} \frac{1}{\sigma_x^2} & -\frac{\rho}{\sigma_x\sigma_y} \\ -\frac{\rho}{\sigma_x\sigma_y} & \frac{1}{\sigma_y^2} \end{bmatrix}$$

$$\chi^2 = \underbrace{\chi^2(a^*,b^*)}_{\text{"Const"}} + \underbrace{\cancel{\frac{\partial \chi^2}{\partial a}\bigg|_{a^*}}}_{0}(a-a^*) + \underbrace{\cancel{\frac{\partial \chi^2}{\partial b}\bigg|_{b^*}}}_{0}(b-b^*)$$

$$+ \frac{1}{2}\left[\frac{\partial^2 \chi^2}{\partial a^2}(a-a^*)^2 + \frac{\partial^2 \chi^2}{\partial b^2}(b-b^*)^2 + 2\frac{\partial^2 \chi}{\partial a\,\partial b}(a-a^*)(b-b^*) \right]$$

$$\chi^2 \sim \chi^2(a^*,b^*) + (a-a^*,\ b-b^*)\,\frac{1}{2}\begin{bmatrix} \frac{\partial^2\chi^2}{\partial a^2} & \frac{\partial^2\chi^2}{\partial a\,\partial b} \\ \frac{\partial^2\chi^2}{\partial a\,\partial b} & \frac{\partial^2\chi^2}{\partial b^2} \end{bmatrix}\begin{pmatrix} a-a^* \\ b-b^* \end{pmatrix}$$

Compare to multivariate Gaussian

$$Q = N \cdot \exp\left(-\frac{1}{2}\ \vec{\Delta}\ \Sigma^{-1}\ \vec{\Delta}\right)$$

$$-2\log Q = \text{const} + \vec{\Delta}\ \Sigma^{-1}\ \vec{\Delta}$$

$$\Rightarrow \quad \frac{1}{2} D = \Sigma^{-1}$$

$$\Sigma = 2\,D^{-1}$$

$$\begin{pmatrix} \sigma_a^2 & \sigma_{ab} & \sigma_{ac} & \\ \sigma_{ab} & \sigma_b^2 & \sigma_{bc} & \\ \sigma_{ac} & \sigma_{bc} & \sigma_c^2 & \\ & & & \ddots \end{pmatrix} = 2 \cdot \begin{bmatrix} \frac{\partial\chi^2}{\partial a^2} & \frac{\partial\chi^2}{\partial a\,\partial b} & \\ \frac{\partial\chi^2}{\partial a\,\partial b} & \frac{\partial\chi^2}{\partial b^2} & \\ & & \ddots \end{bmatrix}^{-1}$$

So $$G(x, y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}}$$

$$\cdot \exp\left(-\frac{1}{2} \frac{1}{(1-\rho^2)} \left[\frac{(x-x_0)^2}{\sigma_x^2} + \frac{(y-y_0)^2}{\sigma_y^2} - \frac{2\rho (x-x_0)(y-y_0)}{\sigma_x \sigma_y} \right]\right)$$