The main <u>technique</u> (not "trick" because we will use it many times) is to maximize probability by setting

$$\boxed{\frac{dP}{dX} = 0}$$

$$\frac{dP}{dX} = P \cdot \frac{d}{dX}\left(-\frac{\sum_i (x_i - X)^2}{2\sigma^2}\right)$$

$$= -\frac{P}{2\sigma^2} \sum_i (x_i - X)$$

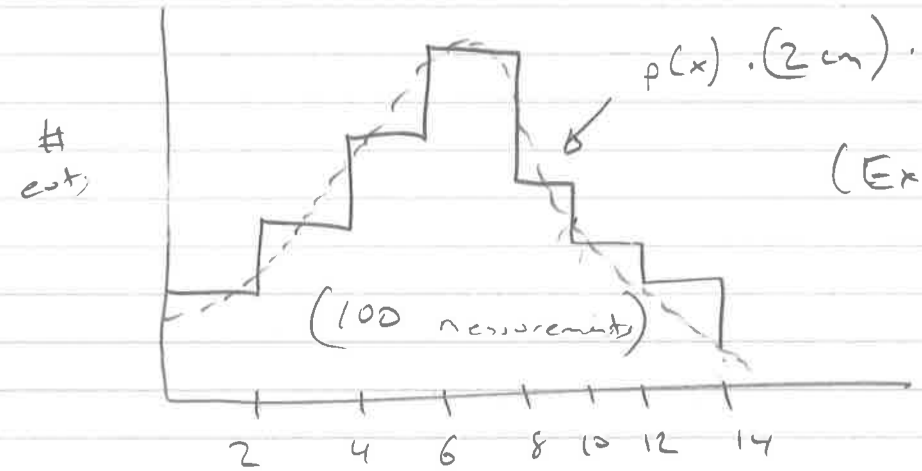
Only way $\frac{dP}{dX} = 0$ is

$$\sum_i (x_i - X) = 0$$

$$\sum_i x_i - NX = 0$$

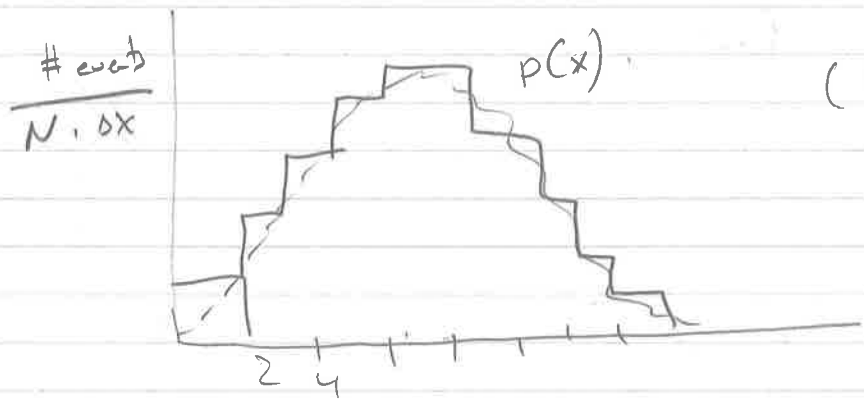
$$\boxed{X = \frac{1}{N}\sum_i x_i}$$

Whichever way you think about it, you <u>must</u> remember to account for normalization of prediction to experiment!



(Exp Perspective)

$\frac{\text{frac evts}}{2cm}$

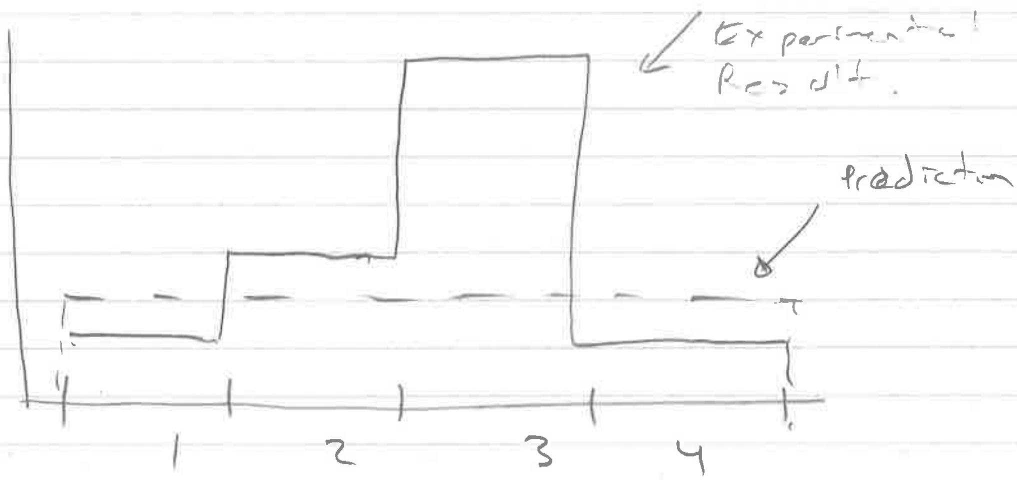


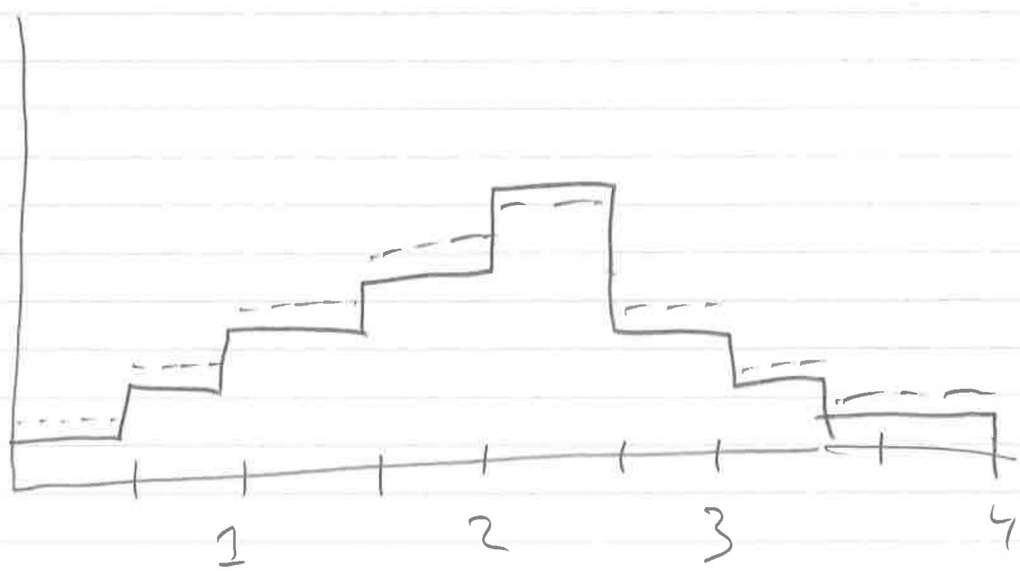
(Theory Perspective)

Distributions:

Science has one and only one purpose: predicting the results of experiments!

If most experiments are this way (even *without* quantum mechanics!) with similar but non-identical results in well-performed experiments ... what does a prediction look like.





Prediction: # of events in a particular bin
Result: # of event in a particular bin.

This would tend to mean our <u>theory</u> (which predicts outcomes of experiments) depends explicitly on our choice of bins. Also, it would depend on number of measurements. NOT PRETTY! We get a lot from <u>fixing this</u>.

Our theory would better tell us something like:

$$p(x) = \text{probability to measure } x.$$

Except that doesn't work!

Q: What is probability to measure exactly the value "1.2357612531 2"?

A: <u>0</u>.

Probabilities are only non-zero over non-infinitesimal ~~over some range~~

We can do a lot with our prediction $p(x)$ which is called a probability distribution function (PDF).

I) $\int_{-\infty}^{\infty} p(x)\, dx = \, ?$

II) $\bar{x} = \int x\, p(x)\, dx$

is theory prediction for <u>average</u> <u>value</u>.

III) RMS:

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 \, f(x)\, dx$$

(It will turn out σ_x is related to <u>uncertainty...</u>)

$\Rightarrow$ Now some specific <u>distributions</u>.

INTRODUCE, will <u>prove</u> later

There are <u>lots</u> of ways to think about what comes next...

1) Jump to answer:

$$\underbrace{N_{pred}}_{\substack{\text{predicted}\\ \text{for particular}\\ \text{experiment}}} = \underbrace{p(x)}_{\substack{\text{theory}\\ \text{prediction}}} \cdot \underbrace{\Delta X \cdot N}_{\substack{\text{bin size}\\ \text{and number}\\ \text{of measurements}\\ \text{made (exp dependant)}}}$$

Q: Suppose x is a measurement of "cm"

What are dimensions of $p(x)$?

A) $\frac{1}{cm}$ in general $[p(x)] = \left[\frac{1}{x}\right]$.

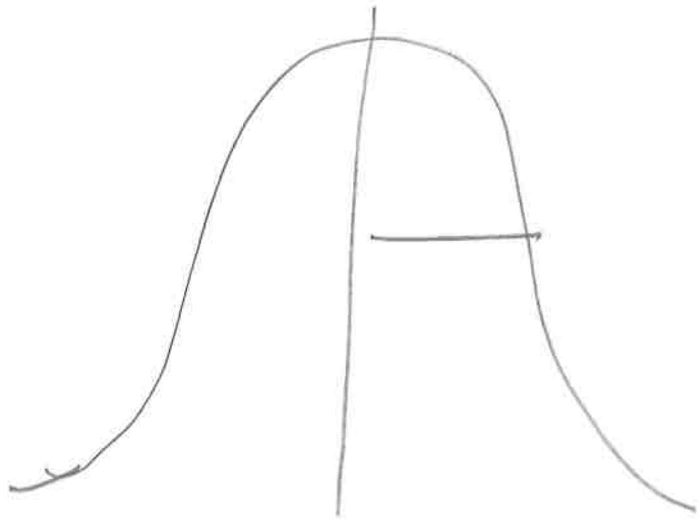
2) $$P(x, x_0) = \int_{x_0}^{x} p(x)\, dx$$

$$\rightarrow \quad p(x) = \frac{\text{prob } (x \text{ and } x + dx)}{dx}$$

$$\rightarrow \quad p(x) = \frac{d P(x, x_0)}{dx}$$

3) Book writes: $p(x)\, dx = \text{prob } (x \text{ to } x + dx)$

Standard Interpretation of Errors



We know how to handle $\int_{-\infty}^{\infty} G \, dx$

What of, say $\pm \sigma$

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\sigma}^{\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \qquad ?$$

Not analytically solvable

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$

(Interpretation $\sigma^2 = \frac{1}{2}$, $P[-x, x]$)

$$P = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} \exp(-x^2) \, dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \exp(-x^2) \, dx \equiv \operatorname{erf}(x)$$

$1\sigma \quad \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) = 68.3\%$

$2\sigma \quad \mathrm{erf}\left(2/\sqrt{2}\right) = 95.4\%$

$3\sigma \quad \mathrm{erf}\left(3/\sqrt{2}\right) = 98.5\%$

$4\sigma \quad \mathrm{erf}\left(4/\sqrt{2}\right) = 99.5\%$

$5\sigma \quad \mathrm{erf}\left(5/\sqrt{2}\right) = 99.8\%$

Standard Deviation of the mean

Suppose we make N measurements of a quantity x each with uncertainty σ.

We calculate the mean:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{N}$$

Note that as each x_i is centered on the true value X, our $\bar{x}$ will also be centered on X, but with better uncertainty!

Just use error propagation

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{\partial \bar{x}}{\partial x_1} \cdot \sigma\right)^2 + \cdots + \left(\frac{\partial \bar{x}}{\partial x_n} \cdot \sigma\right)^2}$$

But each $\frac{\partial x}{\partial x_i} = \frac{1}{N}$, so we have

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{1}{N}\sigma\right)^2 + \cdots + \left(\frac{1}{N}\sigma\right)^2}$$

$$= \sigma \cdot \sqrt{N \cdot \frac{1}{N^2}} = \boxed{\frac{\sigma}{\sqrt{N}}}$$

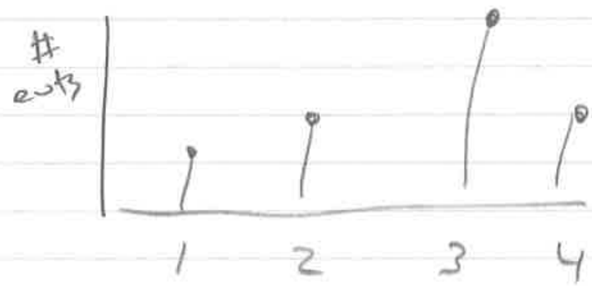
Repeating Measurement N times reduces uncertainty by a factor $\frac{1}{\sqrt{N}}$!!!

Histograms + Uncertainties

Histograms

Have students pick number from 1 to 4.

Tally the results and build a histogram

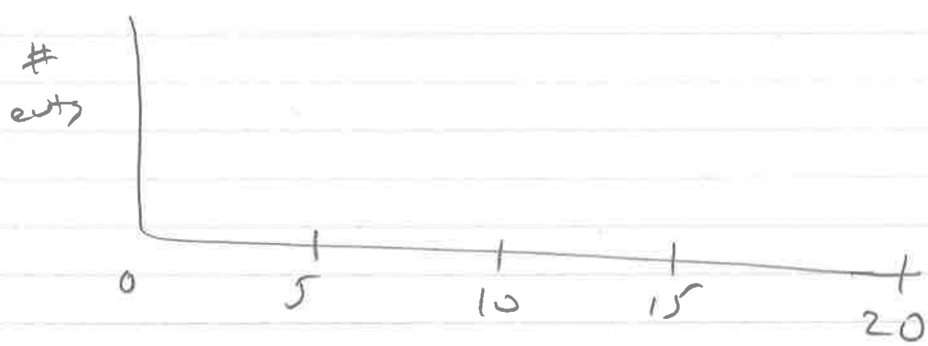


(Probably will be biased toward 3!)

Aside: Can mention (as motivation) statistical uncertainty of $\sqrt{N}$ to check for bias.

Q: How to handle data like:

15.4, 11.3, 12.1, 6.5, 10.1, 17.3



Note: Now each bin has a width and a number of entries.

(2 things to keep track of)

We'll assume (for now) all have same width (evenly spaced).

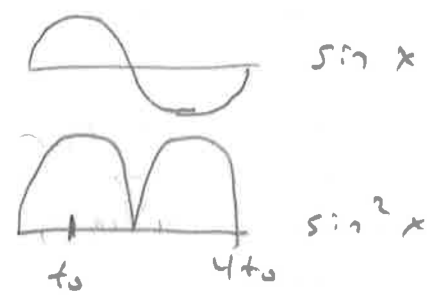
Maybe?

Johnson Noise (Continued)

Last time, showed: (actually had "fudged" as [illegible])

$$\langle V^2 \rangle_{t_0} = RkT \frac{1}{t_0}$$

We need to convert this to frequency domain... think of which frequencies are contributing to this RMS...



If we calculate RMS of a function with frequency $\nu = \frac{1}{4t_0}$ for time interval

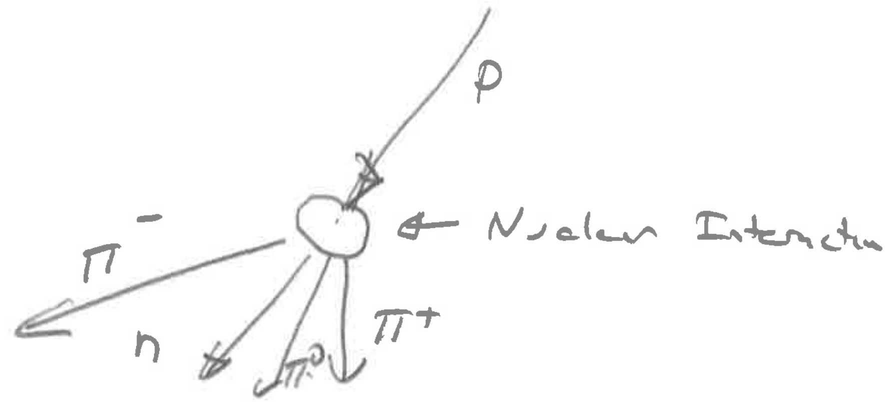
$$\langle V^2 \rangle_{\nu > \nu_0} = 4RkT\nu_0$$

If we want to find RMS for frequencies in range ν_1 to ν_2

$$\langle V^2 \rangle_{\nu > \nu_2} - \langle V^2 \rangle_{\nu > \nu_1} = 4RkT(\nu_2 - \nu_1)$$

$$\boxed{\langle V^2 \rangle_{d\nu} = 4RkT\, d\nu}$$

Muon Production + Decay



$\pi^{\pm} \rightarrow \mu^{\pm} \nu_{\mu}$ $\qquad$ $\pi^0 \rightarrow \gamma\gamma$

$\mu^{\pm} \rightarrow e^{\pm}$

$\boxed{1/\text{minute} \cdot \text{cm}^2 \quad \text{at} \quad E \sim 4\,\text{GeV}}$

↳ Prof of relativity

Muon Lab Lecture

Decay Rate:

μ is fundamental particle
→ discussed radioactive decay, it has no "age" and therefore decays at a fixed rate

$$dN = -N(t)\,\lambda\,dt$$

$$\Rightarrow \quad N(t) = N_0 \exp(-t/\tau)$$

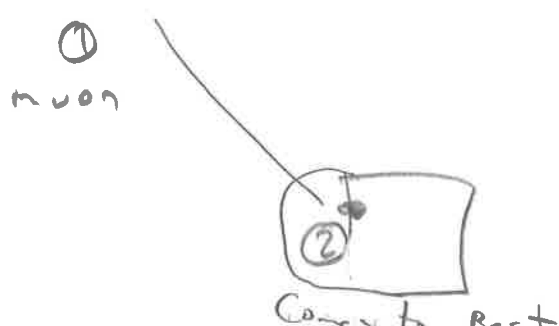
$$\tau \equiv \text{lifetime} = \frac{1}{\lambda}$$

(Relate to half-life):

$$N(t_{1/2}) = \frac{1}{2} N_0 = N_0 \exp(-t_{1/2}/\tau)$$

$$\boxed{\tau \cdot \log(2) = t_{1/2}}$$

Plan:



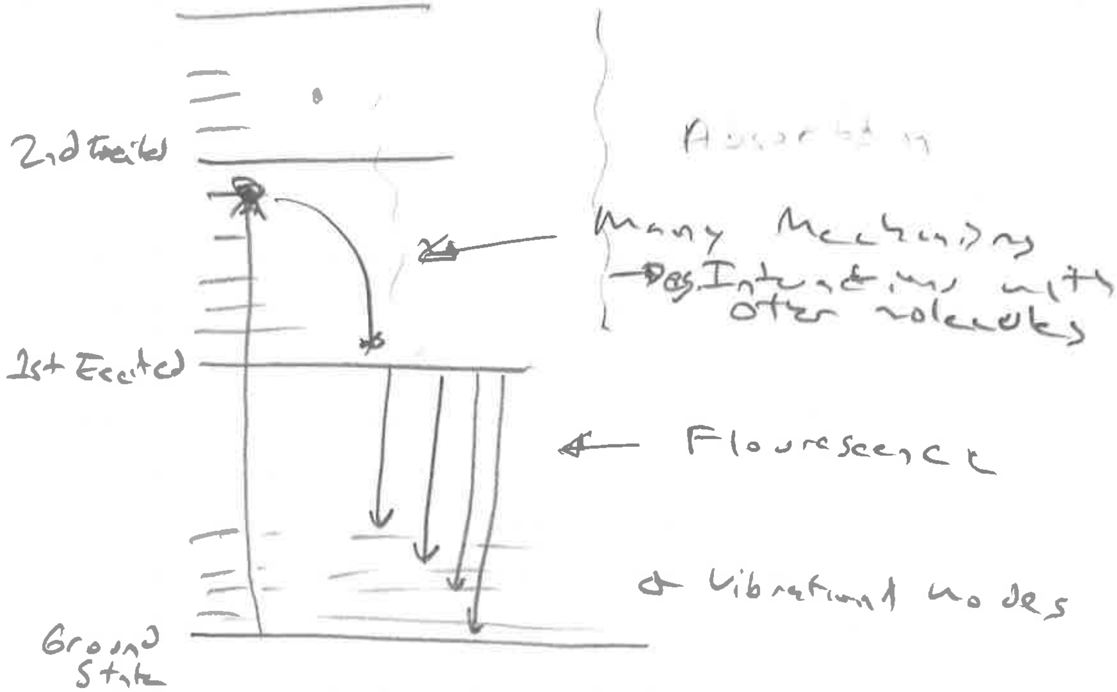
Q: Don't know how long muon took to there

Does it matter?

$$dN = -\lambda N_0 \exp(-\lambda t)$$

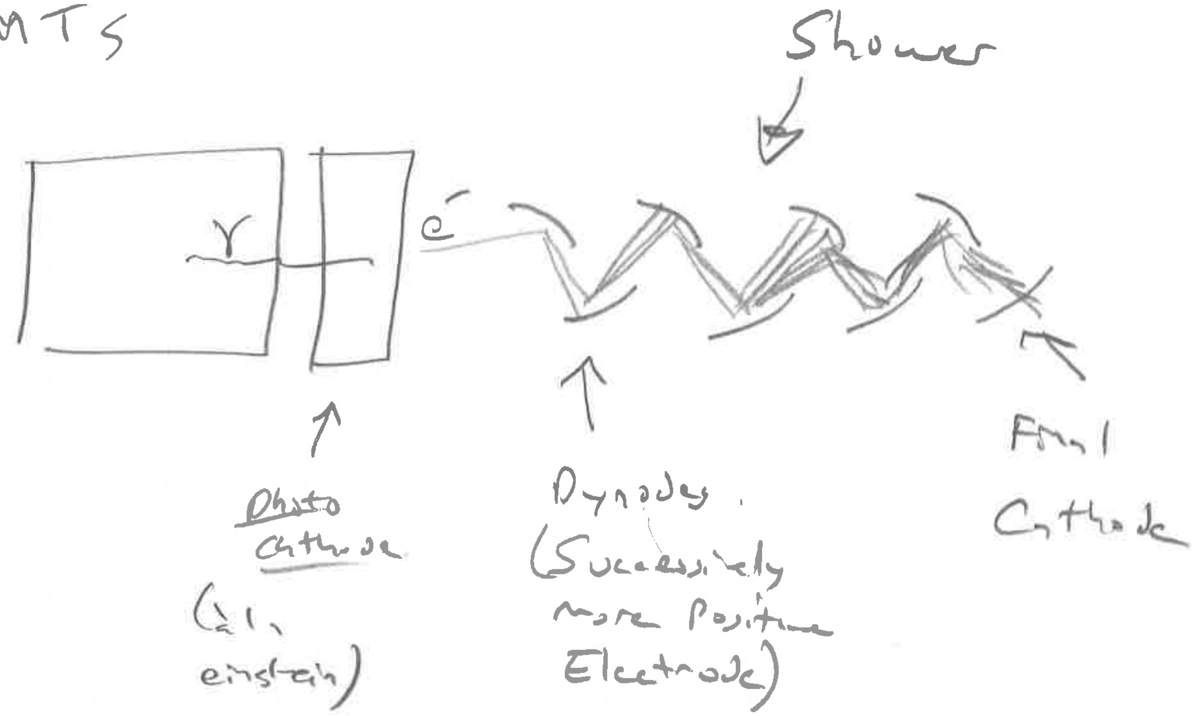
$$\boxed{dW = -\lambda \exp(-\lambda t)}$$ (prob of single decay, $N_0 = 1$)

Scintillator



Only Fraction of Energy of

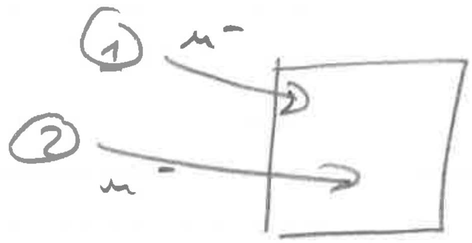
PMTs



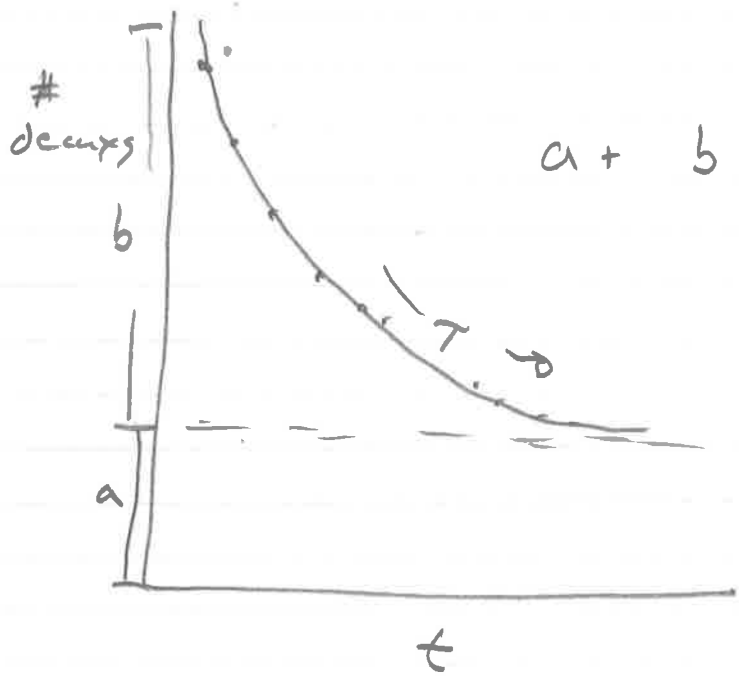
Gain is voltage dependent, but typically 100,000 to 1 million.

(Tuned to application)

Background:

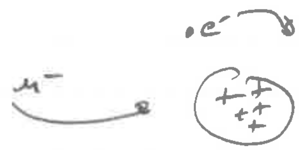


Key Point: An additional muon is equally likely to arrive in any given time interval... FLAT



$$a + b \exp(-t/\tau)$$

μ^+ / μ^- differences



μ^- can occupy same orbital as an electron. Then decay via

$$\mu^- + p \rightarrow n + \nu_\mu$$

This provides additional decay for $\underline{\mu^-}$ not possible by $\underline{\mu^+}$.

So two decay rates:

λ^+ and λ^-

We measure a <u>weighted</u> <u>average</u>.

$$\langle \lambda \rangle = \frac{N^+ \lambda^+ + N^- \lambda^-}{N^+ + N^-}$$

Define $\rho = N^+ / N^-$ $\qquad \tau = \frac{1}{\lambda}$

$$\langle \frac{1}{\tau} \rangle = \frac{\rho \frac{1}{\tau^+} + \frac{1}{\tau^-}}{\rho + 1}$$

$$\boxed{\langle \tau \rangle = (\rho+1) \frac{\tau^+ \tau^-}{\tau^+ + \rho \tau^-} = \tau_{obs}}$$

Solve for ρ

$$(\rho+1)\,\tau^+\tau^- = \tau_{obs}\cdot(\tau^+ + \rho\tau^-)$$

$$\rho(\tau^+\tau^- - \tau_{obs}\tau^-) = \tau_{obs}\tau^+ - \tau^+\tau^-$$

$$\rho = \frac{\tau_{obs}\tau^+ - \tau^+\tau^-}{\tau^+\tau^- - \tau_{obs}\tau^-}$$

$$= -\frac{\tau^+}{\tau^-}\cdot\left(\frac{\tau^- - \tau_{obs}}{\tau^+ - \tau_{obs}}\right)$$

$\boxed{\text{Covariance:}}$

$$q_i = q(x_i, y_i) \qquad x_i \longrightarrow \bar{x},\ \sigma_x \qquad y_i \longrightarrow \bar{y},\ \sigma_y$$

Near $x = \bar{x},\ y = \bar{y}$:

$$q_i = q(x_i, y_i) \approx q(\bar{x}, \bar{y}) + \frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y})$$

$$\bar{q} = \frac{1}{N}\sum_{i=1}^{N} q_i \approx \frac{1}{N}\sum_{i=1}^{N}\left[q(\bar{x},\bar{y}) + \frac{\partial q}{\partial x}\underbrace{\cancel{(x_i - \bar{x})}}_{0} + \frac{\partial q}{\partial y}\underbrace{\cancel{(y_i - \bar{y})}}_{0}\right]$$

$$= q(\bar{x}, \bar{y})$$

$$\boxed{\bar{q} = q(\bar{x}, \bar{y})}$$

$$\sigma_q^2 = \frac{1}{N}\sum_i (q_i - \bar{q})^2$$

$$= \frac{1}{N}\sum_i \left(\frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y})\right)^2$$

$$= \left(\frac{\partial q}{\partial x}\right)^2 \frac{1}{N}\sum_i (x_i - x)^2 + \left(\frac{\partial q}{\partial y}\right)^2 \frac{1}{N}\sum_i (y_i - y)^2 + 2\left(\frac{\partial q}{\partial x}\right)\left(\frac{\partial q}{\partial y}\right) \sum_i (x - x_i)(y - y_i)$$

$$= \left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2 \sigma_y^2 + 2\frac{\partial q}{\partial x}\frac{\partial q}{\partial y}\sigma_{xy}$$

$$\sigma_{xy} \equiv \frac{1}{N}\sum_i (\bar{x} - x_i)(\bar{y} - y_i)$$

σ_{xy} is the <u>covariance</u> of x and y.

~~Corresponding PDF:~~

<u>Multivariate Normal</u>

$$\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\frac{x-x_0}{\sigma^2}\right)$$

$$\longrightarrow \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{x}_0)\cdot \Sigma^{-1}(\vec{x}-\vec{x}_0)\right)$$

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \\ \sigma_{xy} & \sigma_y^2 & \\ & & \ddots \end{bmatrix}$$

In 2-D

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$ (correlation between x and y)

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix} \frac{1}{\sigma_x^2\sigma_y^2 - \rho^2\sigma_x^2\sigma_y^2}$$

$$= \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma_x^2} & -\frac{\rho}{\sigma_x\sigma_y} \\ -\frac{\rho}{\sigma_x\sigma_y} & \frac{1}{\sigma_y^2} \end{bmatrix}$$

Interpretation of $\Delta\chi^2$?

$$\mathcal{L} = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\,\frac{x^2}{\sigma_x^2}\right)$$

$$-2 \log \mathcal{L} = \text{const} + \underbrace{\frac{x^2}{\sigma_x^2}}_{\chi^2}$$

$$\Delta x = \sigma_x \Rightarrow \Delta\chi^2 = 1$$

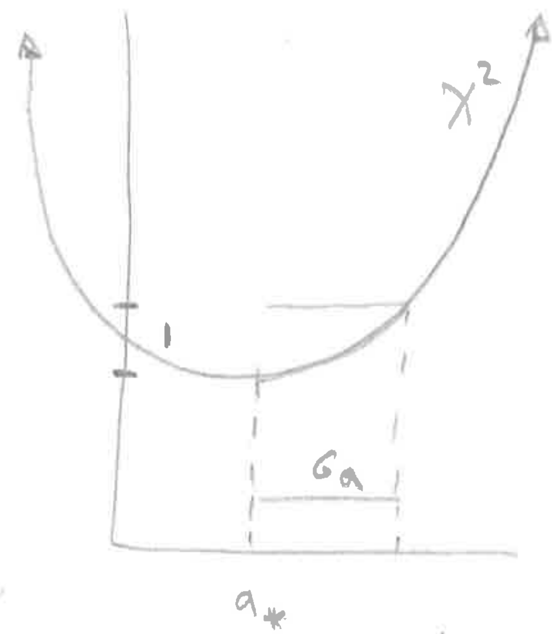
$$\left(\Delta\chi^2 = \chi^2(\Delta x) - \chi^2(0)\right)$$

One approach to determining parameter uncertainties

$$\boxed{\Delta\chi^2 = 1}$$

We can use this to determine parameter uncertainties:

Picture:



$$\chi^2 = \chi^2(a_*) + \cancel{\left.\frac{\partial \chi^2}{\partial a}\right|_{a_*}}(a-a_*) + \frac{1}{2}\left.\frac{\partial^2 \chi}{\partial a^2}\right|_{a^*}(a-a^*)^2$$

0 at a_*!

$$\chi^2 = \chi^2(a_*) + \frac{1}{2}\left.\frac{\partial^2 \chi}{\partial a^2}\right|_{a^*}(a-a^*)^2 + \ldots$$

(Non Gaussian HOTs)

$\Delta \chi^2 = 1$ when $a - a^* = \sigma_a$

$$1 = \frac{1}{2}\frac{\partial^2 \chi}{\partial a^2}\sigma_a^2$$

$$\Rightarrow \sigma_a^2 = \frac{2}{\left.\frac{\partial^2 \chi^2}{\partial a^2}\right|_{a^*}}$$

So $$G(x, y) = \frac{1}{2\pi \, \sigma_x \, \sigma_y \sqrt{1-\rho^2}}$$

$$\cdot \exp\left(-\frac{1}{2} \frac{1}{(1-\rho^2)} \left[\frac{(x-x_0)^2}{\sigma_x^2} + \frac{(y-y_0)^2}{\sigma_y^2} - \frac{2\rho \, (x-x_0)(y-y_0)}{\sigma_x \, \sigma_y} \right] \right)$$

$$\chi^2 = \underbrace{\chi^2(a^*,b^*)}_{\text{"Const"}} + \underbrace{\cancel{\left.\frac{\partial\chi^2}{\partial a}\right|_{a^*}}}_{0}(a-a^*) + \underbrace{\cancel{\left.\frac{\partial\chi^2}{\partial b}\right|_{b^*}}}_{0}(b-b^*)$$

$$+ \frac{1}{2}\left[\frac{\partial^2\chi^2}{\partial a^2}(a-a^*)^2 + \frac{\partial^2\chi^2}{\partial b^2}(b-b^*)^2 + 2\frac{\partial^2\chi}{\partial a\,\partial b}(a-a^*)(b-b^*)\right]$$

$$\chi^2 \sim \chi^2(a^*,b^*) + (a-a^*,\ b-b^*)\,\frac{1}{2}\begin{bmatrix}\frac{\partial^2\chi^2}{\partial a^2} & \frac{\partial^2\chi^2}{\partial a\,\partial b}\\ \frac{\partial^2\chi^2}{\partial a\,\partial b} & \frac{\partial^2\chi^2}{\partial b^2}\end{bmatrix}\begin{pmatrix}a-a^*\\ b-b^*\end{pmatrix}$$

Compare to multivariate Gaussian

$$Q = N \cdot \exp\left(-\frac{1}{2}\,(\vec{\Delta}, \Sigma^{-1}\vec{\Delta})\right)$$

$$-2\log Q = \text{const} + \vec{\Delta}\,\Sigma^{-1}\vec{\Delta}$$

$$\Rightarrow \quad \frac{1}{2}D = \Sigma^{-1}$$

$$\Sigma = 2D^{-1}$$

$$\begin{pmatrix}\sigma_a^2 & \sigma_{ab} & \sigma_{ac} & \\ \sigma_{ab} & \sigma_b^2 & \sigma_{bc} & \\ \sigma_{ac} & \sigma_{bc} & \sigma_c^2 & \\ & & & \ddots\end{pmatrix} = 2\cdot\begin{bmatrix}\frac{\partial\chi^2}{\partial a^2} & \frac{\partial\chi^2}{\partial a\,\partial b} & \\ \frac{\partial\chi^2}{\partial a\,\partial b} & \frac{\partial\chi^2}{\partial b^2} & \\ & & \ddots\end{bmatrix}^{-1}$$