# Lab 4- Fourier analysis using Python

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## Aims

The purpose of this investigation was to use Fourier analysis for both periodic and aperiodic signals. Simpson's rule was also investigated and compared with analytical solutions, along with the use of Fourier Transform and Fourier series, used to compute aperiodic signals.

## Introduction

Simpson's rule is a numerical method used to approximate definite integrals, that is, area underneath graph functions. It is much more accurate than other methods such as the Trapezoid Rule, primarily because it utilises parabolas rather than straight lines. We divide the area into n segments (where n is an even integer) of equal width, this width is equal to $\frac{b-a}{n}$. Using this, we find that for a function f(x), which is defined between the limits a and b, the Simpson's rule approximation for the area between these limits is as follows:

$$\int_a^b f(x) \approx \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j} - 1) + f(x_n)\right]$$

Exercise 1 primarily involved coding a script to execute Simpson's rule for an arbitrary function, which is this case was $e^x$ integrated over the interval [0,1]. An analytical solution was also coded, and the results were compared to test the validity of the Simpson's method. The next part of exercise 1 involved the modification of the script in order to display the result given by the Fourier Series expansion of a function. This function was also plotted, along with the Fourier Coefficients resulting from this method.

The Fourier series is used for the reconstruction periodic signals- it is an infinite sum of the sine and cosine components, which each have a frequency that is a multiple of 1/T- these components gradually make up the signal, and so allows it to be analysed. The Fourier Series makes use of integral identities and is thus defined as:

$$f(t) = a_0 + \sum(a_n cos(n\omega t) + b_n sin(n\omega t))$$

Where $a_n, b_n$ and $a_0$ are known as the Fourier coefficients. These can be determined by the integral identities below:

$$a_0 = \int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,dt$$

There are times where all the $a_n$ coefficients (including $a_0$) may be set to zero, and other times where all the $b_n$ coefficients may be set to zero. This is applicable for even and odd functions respectively. This is the case because sine is an even function whereas cosine is an odd function.

Exercise 2 involved interpreting a square wave using the Fourier Series. The square wave function used is shown below:

$$f(\theta) = \begin{cases} 1, & 0 \leq x \leq \pi \\ -1, & \pi \leq x \leq 2\pi \end{cases}$$

This is an odd function with period, T, equal to $\frac{2\pi}{\omega}$. The Fourier coefficients were then determined and compared to those given by the analytical solution. This square wave was also plotted.

Continuing with exercise 2, the code was then modified, and the steps listed previously were then repeated. These processes demonstrate how, upon adding more terms to the summation of the Fourier series, the graph plot represented by this sum because more accurate- i.e. increasingly similar to the original function. The more terms there are included, the closer the reconstructed square wave will be to the exact square wave. Furthermore, for square waves, due to the presence of the jump discontinuities, evident within the function definition, there often evolves "overshoots" on either side of the resultant function limits. This is known as Gibbs phenomenon, and is evident for both the square and rectangular waves. However, for most real-life applications of the Fourier series, where f(t) can be very accurately approximated, wherever the function is continuous.

In the case of functions which are aperiodic, a more suitable method known as Fourier transform is used. In the previous case, the integer k represents how the integral will be executed for discrete values of the frequency of the wave given. In this case, however, the k is no longer present in the analysis, and the expansion is carried out over a continuous range of frequencies. This is subsequent from the fact that for a non-periodic function, the period can essentially be treated as infinite, and so the angular fundamental frequency becomes infinitely small. This means that when summing the integral over discrete values, the result will be a sum over a continuous range of fundamental frequency values. Here, the summation is redefined as an integral:

$$f(t) = \int_0^x \{a(\omega)cos\omega t + b(\omega)sin\omega t\,\}d\omega$$

And so, the coefficients are now:

$$a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)cos\omega t\ \ dt$$

$$b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)sin\omega t\ dt$$

From this we can define the Fourier transform using complex notation, from Euler's formula $e^{i\omega t}$=$cos\omega t + isin\omega t$, we find that:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}$$

Which is the Fourier transform of $f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} F(\omega)e^{i\omega t}$

Knowing this information, we can see how this leads on to Discrete Fourier Transform, which is the equivalent of the Fourier Transform of a function where only specific points of the signal are known at specific points in time. A Fourier transform for a periodic signal can be analysed over a finite interval, rather than from $-\infty$ to $+\infty$. A similar method can be used for DFT as this method treats the values as periodic. The main application for this method is where data of amplitude or intensity as a function of time is analysed in this way, as in many cases the nature of the period is unknown. The period of the function is assumed to be $\tau = Nh$ where N is the number of samples taken, at interval values, h.

Therefore, the equation for such a situation is as follows:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

This was investigated in exercise 3, whereby the sampling rate and fundamental frequency were first determined involving both real and imaginary terms. N was set to 128 and h to 0.1 for the function $f(t) = \sin(0.45\pi t)$. The value of h was then altered in order to find an optimum value of the sampling time, which would ideally be an integer multiple of the period, T. The sampling rate for various values of h were then compared to the Nyquist frequency, which is the lowest value of h for which a signal can be analysed without any error occurrence.

**Results**

Exercise 1- Upon comparison of the analytical result of $\int_0^1 e^x$, and that determined by the Simpson's method, it was found that they were very similar. The analytical result- i.e. e-1 was found to be 1.71828182846. This was very similar, to an accuracy of 10^-9 degrees, to the numerical solution, which was 1.71828182855. This was executed using 50 steps in the integration- the higher the number of steps/smaller interval widths, the more accurate the result.

Figure 1.1

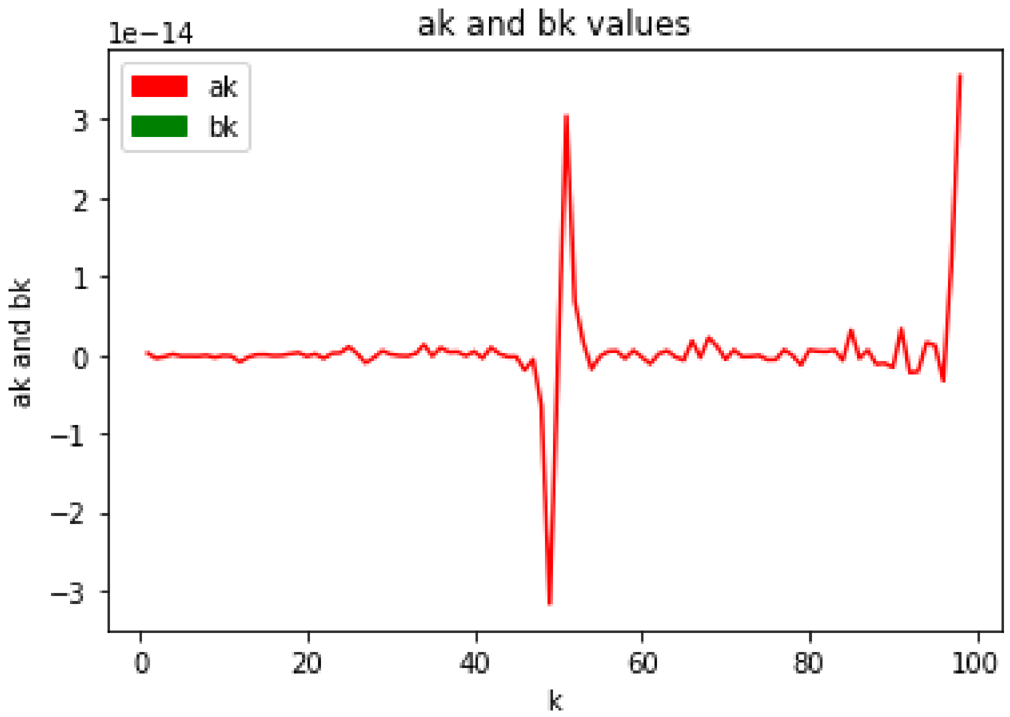
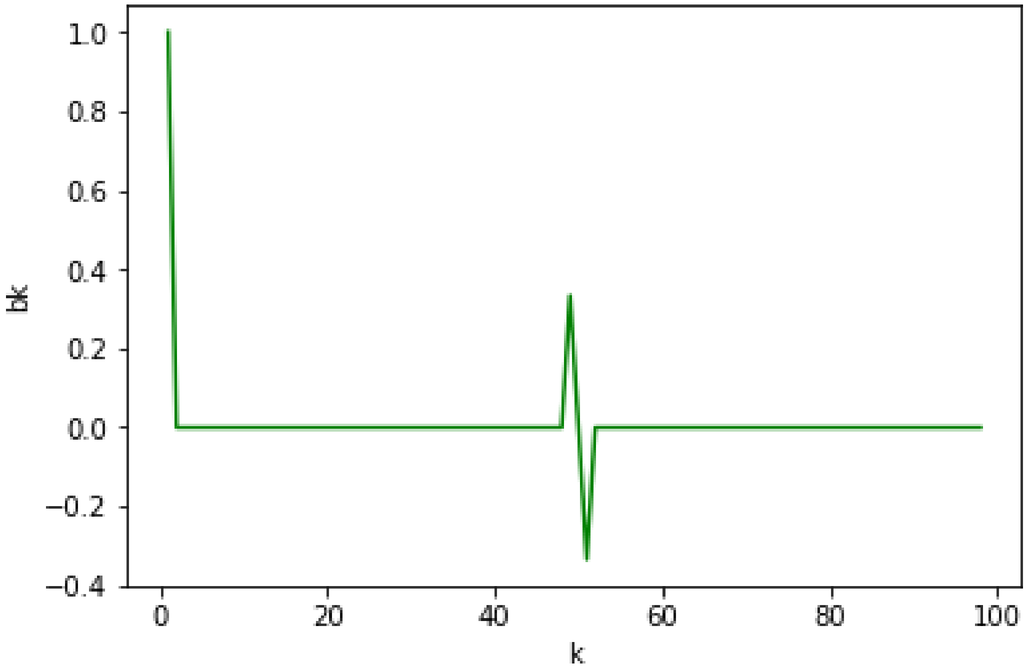



Figure 1.1 displays the result for, solely, the 'a' coefficient values of the Fourier series. Figure 1.2, below, shows the second part of this figure, although poorly labelled.

Figure 1.2

This graph displays, solely, values for the ‘b’ coefficients of the Fourier series.




Figures 1.1 and 1.2 show the expected trends, whereby for increasingly high values of k, the Fourier Coefficients a and b are zero. However, clearly there are sudden changes to non- zero values of both the ak and bk coefficients.

Figure 1.3

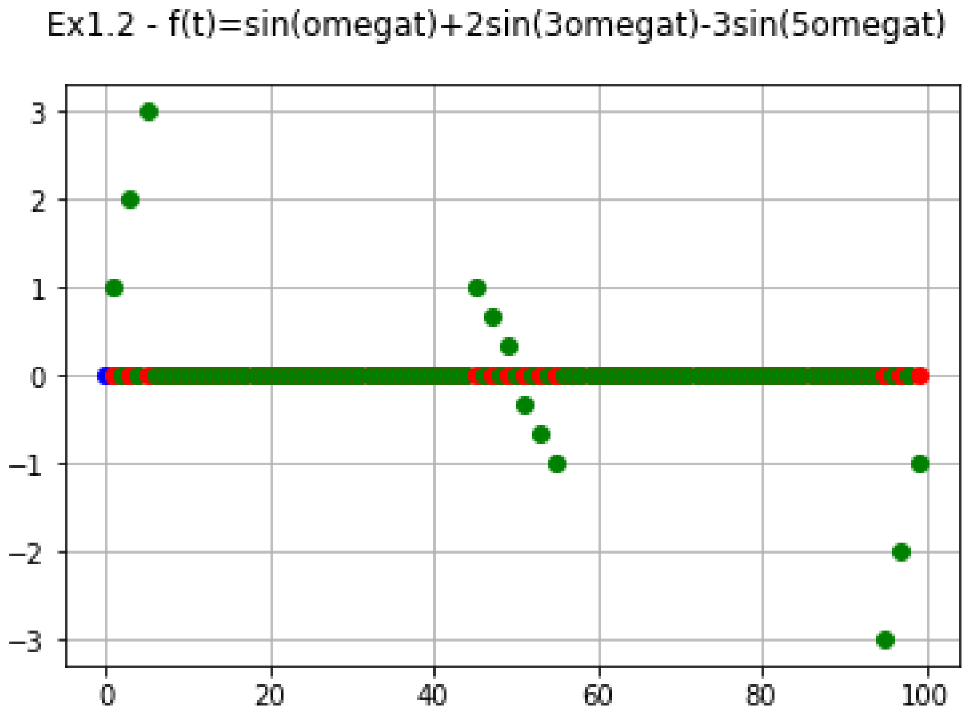



Figure 1.3 displays the function f(t) versus time. Clearly it is a sinusoidal wave but with many values centred at the origin. It verifies the validity of the Fourier series formula within the code.

Figure 2.1

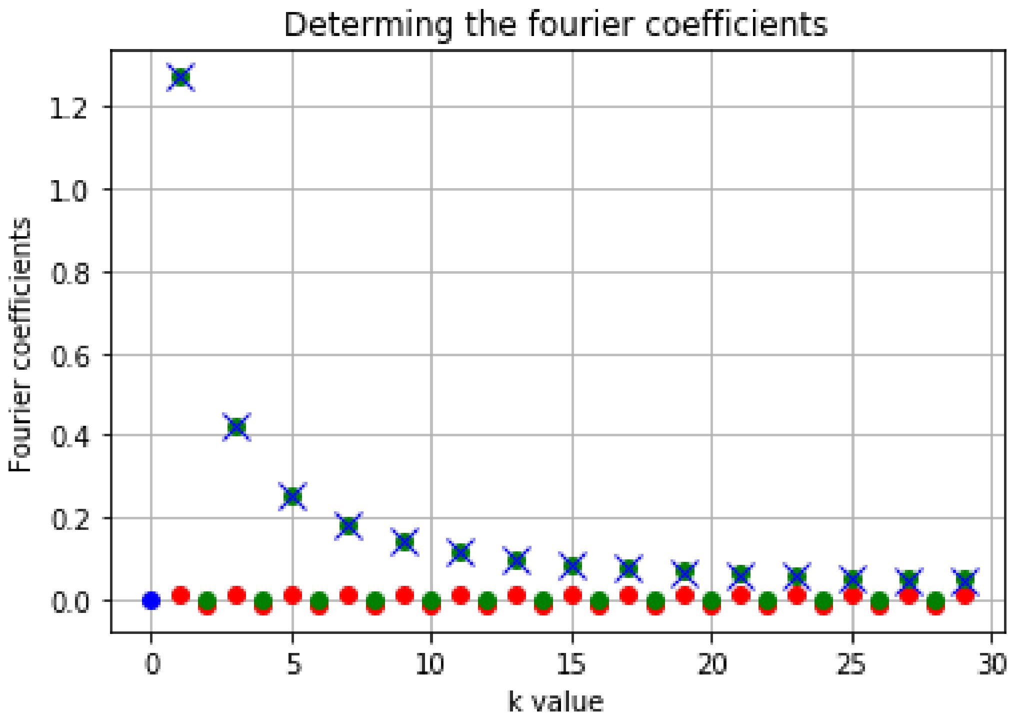



Figure 2.1 demonstrates how the Fourier coefficient will gradually tend to zero as k increases. These are the ‘b’ coefficients, whereas the ‘a’ coefficient values are already zero for all values of k. Here we can see that the values of b and a closely match those found through the analytical method, and so also verifies the validity of the code.

Figure 2.1

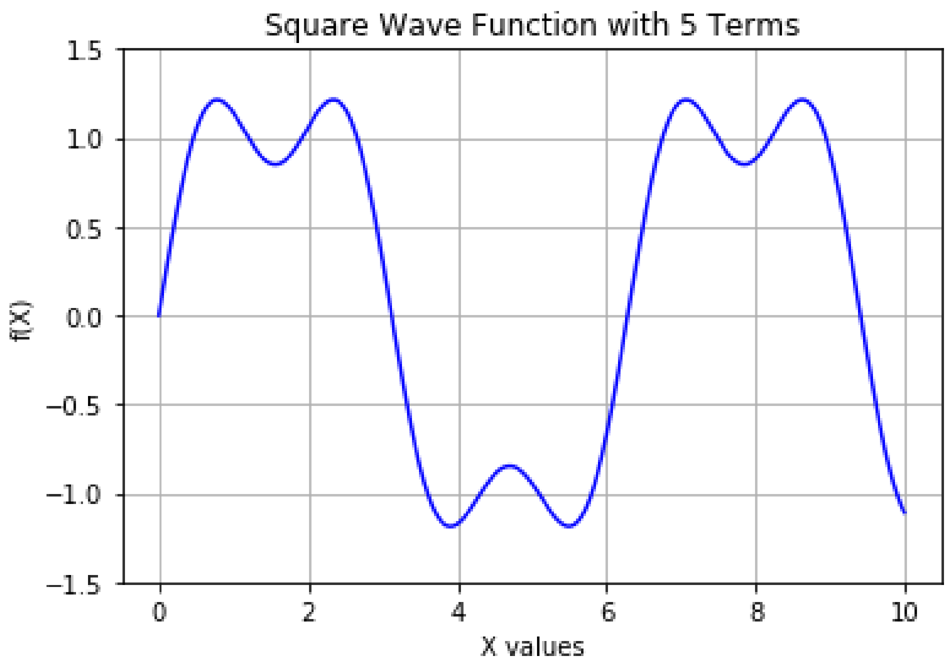



Figure 2.2

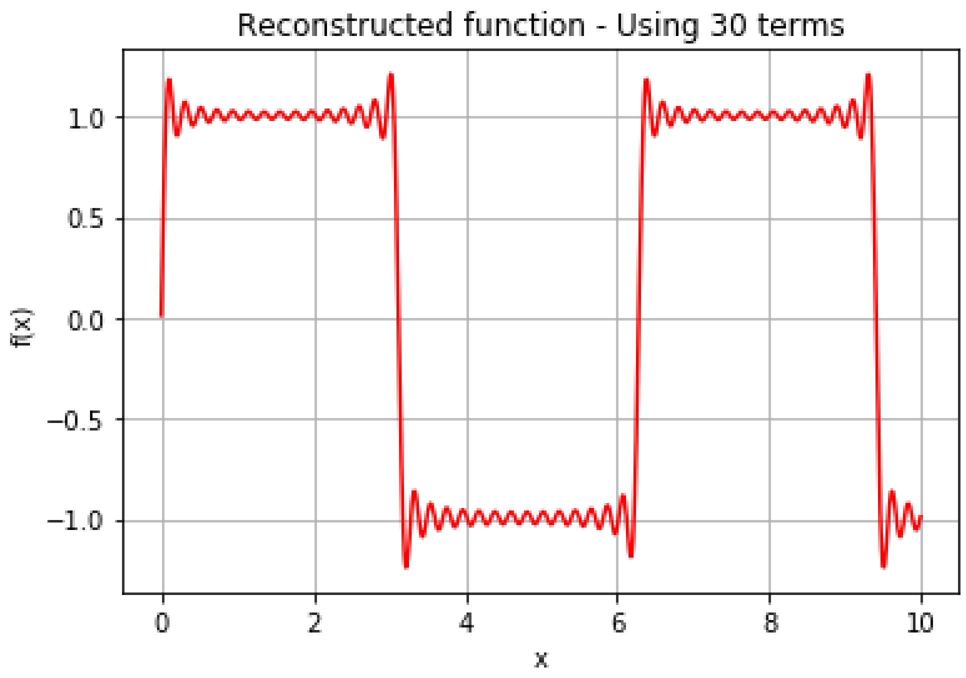



Figure 2.1 and 2.2 are arbitrary graph selections to demonstrate how a greater number of terms include in the Fourier summation, results in a more accurate representation of the original function. For instance, Figure 2.1 shows the result when using just 5 terms in the summation. It is unclear just what function will be represented, but upon using more terms, as shown in figure 2.2, the result in clear. Here 30 terms were used, and the function is now known to have originally been a square wave. Gibbs phenomenon is also evident, as expected when working with this method.

## Conclusion

This investigation was a success in that it demonstrates the many applications of Fourier analysis and Simpson's rule. Exercise 1 clearly shows the accuracy of Simpson's rule in analysing the area under the graph of e^x between x=0 and x=1. The accuracy will be further increased as the width of the interval is made smaller. Exercise 1 also resulted in the expected trends for the Fourier coefficient values.

For the piecewise function, the results were as expected, showing that increasing the number of terms included in the summation of the Fourier Series, there is a significantly more accurate depiction of the original function. The Fourier coefficient values were as expected, and Gibbs phenomenon was very much evident as the number of terms increased.

The Fourier Series and transform was shown to accurately construct periodic and aperiodic functions using sine and cosine terms. It was also shown that if h was altered to an integer multiple of the period, the term closest to the fundamental frequency will certainly dominate.

Overall, this investigation was successful in demonstrating the real-life applications of Fourier analysis, and allows us to accurately analyse signals in a different domain.