Computational lab 2- The pendulum

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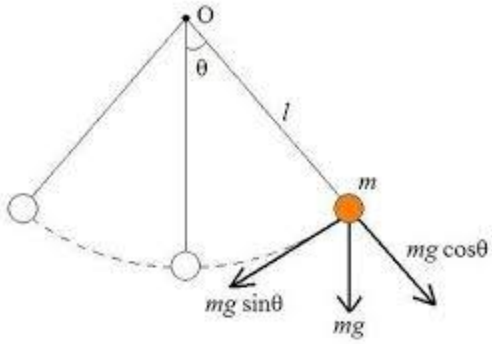
Aims

The aim of this investigation was to analyse the motion of both linear and non-linear pendulum systems using a python script, and further analysing damped systems as well as damped and driven systems. These have a higher application value to real-life situations. This investigation also involved the comparison of the Runge-Kutta and Trapezoid rule numerical methods for systems of this kind.

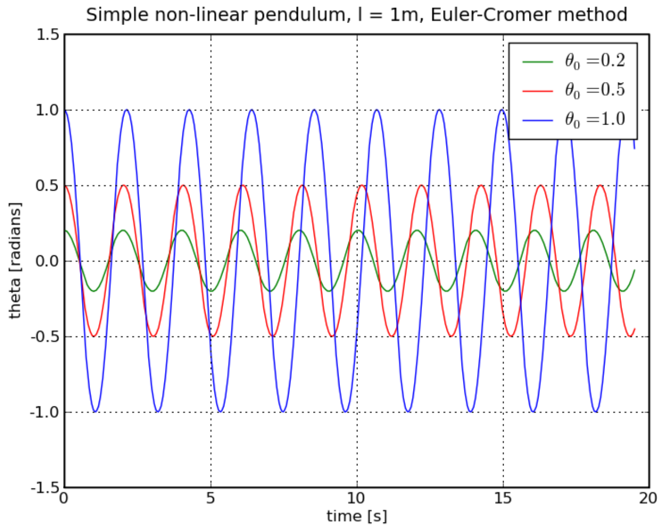
Introduction

This experiment involved the use of Python in order to replicate the results of linear pendulum and non-linear pendulum systems. Whereby the latter may involve a driving force and/or damping which greatly affect the pendulum motion- as seen later from the graphs of theta and omega versus time in exercise 4 and 5.

For linear pendulum system, the motion is periodic as it is both regular and repeated. External resistive or driving forces are neglected when describing this motion and so the only forces considered are the tension in the string to which the bob is attached, and the gravitational force acting downwards throughout the pendulum's motion. The restoring force is one of the resolved components of gravity, which is tangent to the pendulum's circular motion, as shown below. This will always act towards the equilibrium position.



The motion displayed by this system is sinusoidal, with the equation $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$. However, it is acceptable in this case to equate $\sin\theta$ to θ for small angle values. This is the case because the arc length lθ is of a very similar value to the length sinθ. The equation previously mentioned is therefore explicable and gives the periodic result $\theta = A\sin(\omega t + \delta)$ where A is the amplitude, ω is the natural frequency of the system and δ is the initial phase.



The example graph above displays the sinusoidal behaviour of a linear pendulum, with theta repeating regularly with time. Theta in this case is the angular velocity, however.

For a non-linear pendulum, however, the above theory is not exactly applicable. For the purposes of real-life applications, a demonstration of a non-linear pendulum system is often useful, as there can be a frictional force hindering the pendulum's continuous repeating motion, as shown in exercises 4 and 5. This frictional force is often in the form of air resistance and/or tension between the string and the bob. For the case of a linear pendulum, as mentioned previously, $\sin\theta$ was approximately equal to θ, but for a non-linear pendulum, $\sin\theta$ remains in the equation $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$ and is an essential function upon θ in the equation $f(\theta, \omega, t) = -\frac{g}{L}\sin\theta - k\omega + A\cos\varphi t$, which is inputted into python to compare the linear with the non-linear pendulum system.

The difference in the equations inputted is shown below, the first is for exercise 1- a linear pendulum, and the second for a non-linear pendulum system as executed in exercise 2.

```
def f(theta, omega, t):     result = -(g/L) * theta - k * omega + A *
math.cos(phi * t)
    return result

def f(theta, omega, t):     result = -(g/L) * math.sin(theta) - k*omega
+ A * math.cos(phi * t)
    return result
```

Within this investigation, a numerical method known as the trapezoid rule was used in solving differential equations describing the motion of the pendulum. This works by assigning regions under a graph with a trapezoid and calculating its area in order to approximate the actual area under the graph. As evident throughout the graphs involved in this investigation, the trapezoid rule will converge very quickly for periodic functions- such as those in exercise 1. In summary, the equations derived from the trapezoid rule which were subsequently used in the code are

$$\theta n + 1 = \theta n + \omega n \frac{\Delta t}{2} + (\omega n + f(\theta n, \omega n, t)\Delta t)\frac{\Delta t}{2}$$

$$\omega n + 1 = \omega n + f(\theta n, \omega n, t)\, \Delta t\, /\, 2 + f(\theta n + 1, \omega n + f(\theta n, \omega n, t)\, \Delta t), t + \Delta t)\, \Delta t\, /\, 2$$

The Runge-Kutta method of the fourth order is also very useful for this type of investigation involving non-linear damped, driven pendulums. This method is often found to be more accurate as it utilises the Taylor series expansion at the midpoint of each time step, contrast to the Trapezoid rule where it is carried out at the beginning. The equations used to solve for motion of this kind are given below:

$$\theta(t + h) = \theta + \frac{k1a + 2k2a + 2k3a + k4a}{6}$$

$$\omega(t + h) = \omega(t) + \frac{k1b + 2k2b + 2k3b + k4b}{2}$$

These equations were used in the code as such, simulating a non-linear pendulum's motion using a 'for' loop (in which the fourth order Runge-Kutta equations were utilised).

```
for nsteps in range(2000):
…     theta = theta + (k1a + 2 * k2a + 2 * k3a + k4a) / 6
      omega = omega + (k1b + 2 * k2b + 2 * k3b + k4b) / 6
…     nsteps += 1print(nsteps)
```

The quantity k, which can be varied within the code, represents damping. The driving force is represented by A.

While a simple harmonic oscillator such as a linear pendulum does not lose energy, a non-linear pendulum will. The magnitude of the damping will greatly affect the amount dissipated and results in the pendulum coming to a gradual stop.

The driving force will preform work on a damped system, this work is equal to the sum of the total energy of the system and the work done by the damping force.

Method

The purpose of exercise 1 was to solve for linear equations of motion of a pendulum. The function for a linear pendulum was defined within the python code and values for A, k and phi were initialised to 0.0, 0.0 and 0.66667 respectively. This is the case because had A or k been non-zero values, the function would not be describing simple linear harmonic motion, but rather motion which is undergoing a driving or frictional force. Values for t and omega were also set to 0.0 and theta was set to 0.2. The equations derived from the trapezoid method mentioned previously were inserted into the code. This found the change in theta and omega during the motion, depending on their initial conditions. Once the sinusoidal graphs of theta and omega versus 'nsteps' were plotted, these conditions were varied several times to illustrate how these sinusoidal functions do depend on the initial variable values, namely the angular amplitude and angular velocity.

Exercise 2 was then carried out to solve non-linear equations of a pendulum, this was executed by making a simple change of theta to sin(θ) in the equation for the motion used in exercise 1. The graphs of theta and omega versus time were then plotted under the order of this new function.

The third exercise involved the use of the Runge-Kutta Integration method to solve the second order differentials involved in the non-linear pendulum equations. It continuously updates theta and omega within the range set for the number of steps, with the initial values for theta and omega being 3.0 and 0.0 respectively. Again, graphs of theta versus time and omega versus time were plotted.

Utilising this numerical method further, exercise 4 involved the plotting of graphs of theta and omega versus time again, but with the initial value of k being set to 0.5- therefore the effects of damping were evident in these plots.

Exercise 4 kept the initial value of A to be 0.0, however, for exercise 5, the magnitude of the driving force was continuously increased per each run of the script. This illustrated the motion of a damped, driven non-linear pendulum. This driving force is sinusoidal and is initially set to 0.9. A was then increased to set values and the results were observed- these values were 1.07, 1.35, 1.47, 1.5.

Results

Exercise 1

Figure 1.1

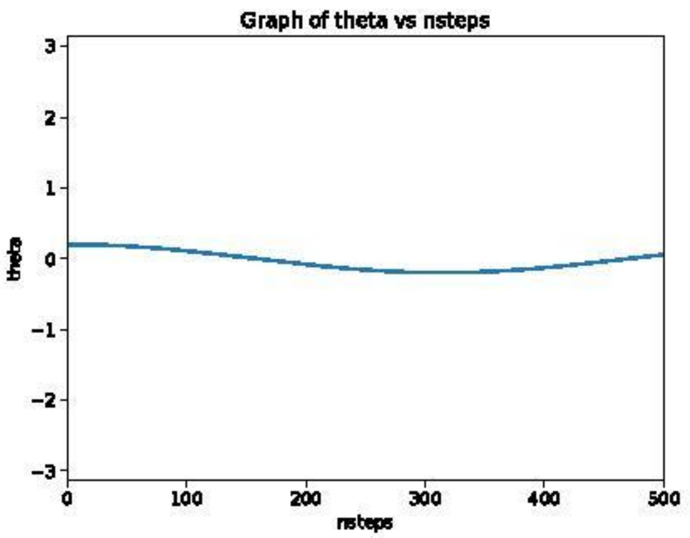


Figure 1.2

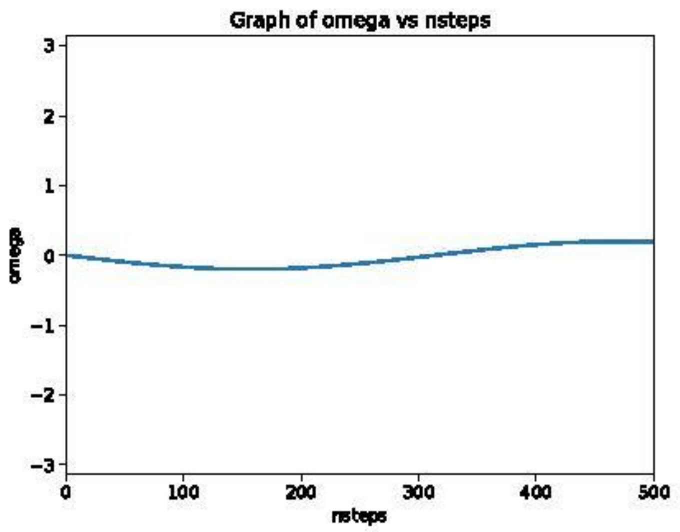


Figure 1.3

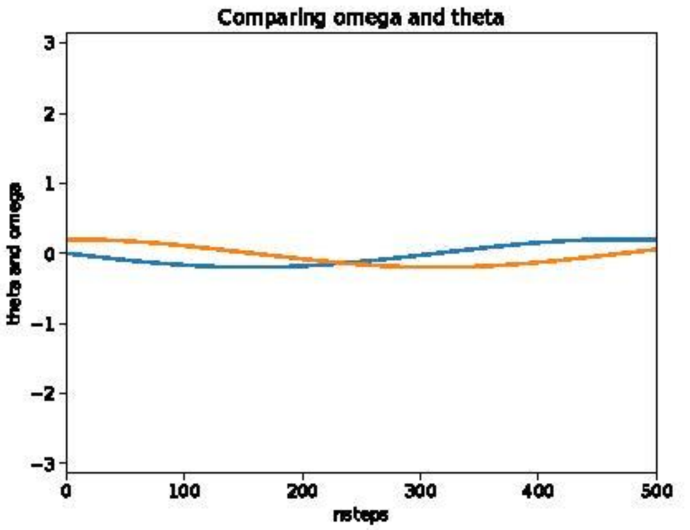


Figure 1.3 compares theta and omega as shown in figure 1.1 and 1.2. Omega is plotted as a blue line while theta is plotted as an orange line. Both verify that the graphs of theta and omega versus 'nsteps' are sinusoidal.

Figure 1.4

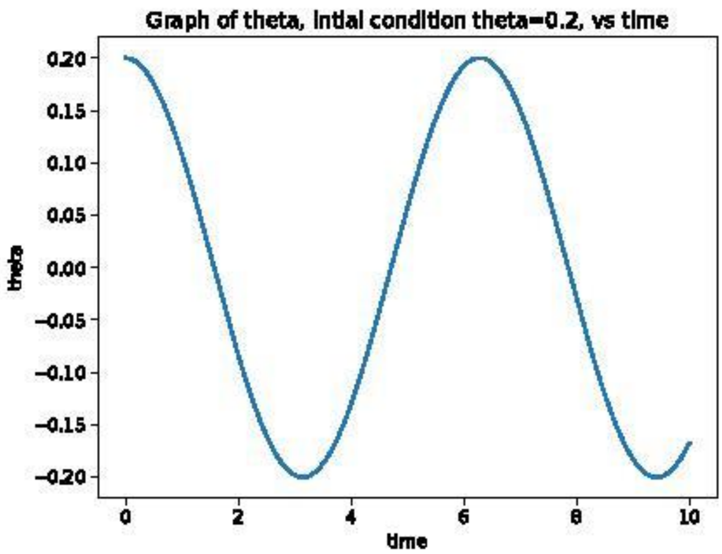


Figure 1.4, above, shows theta versus time, initialising theta to 0.2 with omega, t and 'nsteps' initialised to 0.0. The increment dt was initialised to 0.01. The angular amplitude is 0.2. As expected for such a system, the angular amplitude will remain constant, undergoing no forces which could affect this.

Figure 1.5a

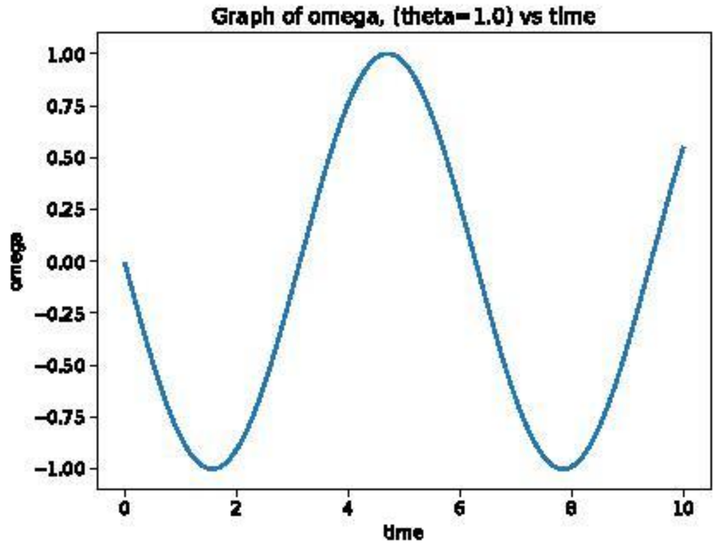
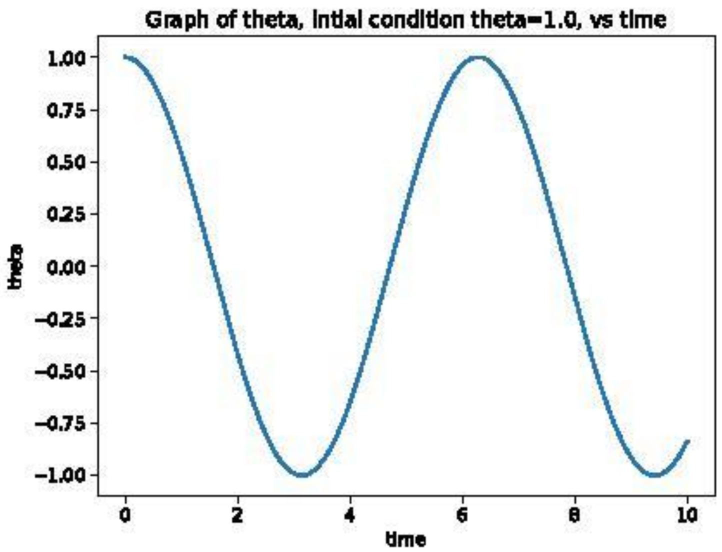


Figure 1.5a and b show plots of omega and theta versus time respectively, with the initial condition of omega= 0.0 and theta=1.0. With theta initialised to 1 radian, it is expected that the initial value of the angular velocity, omega, would be zero. Again, it demonstrates perfect simple harmonic motion. The period shown here is approximately 6 seconds. This is the case for graphs 1.4, 1.5 and 1.6, as expected, as the periodic time is independent of the amplitude of the motion.

Figure 1.5b



Again, the graph shows a period of around 6 seconds and indicates no decay in the angular amplitude. The initialised value of theta is evident on the graph as 1 radian.

Figure 1.6a

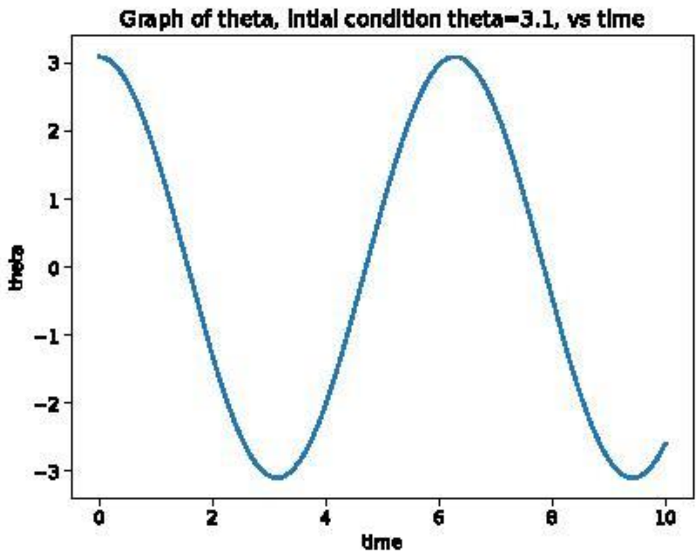
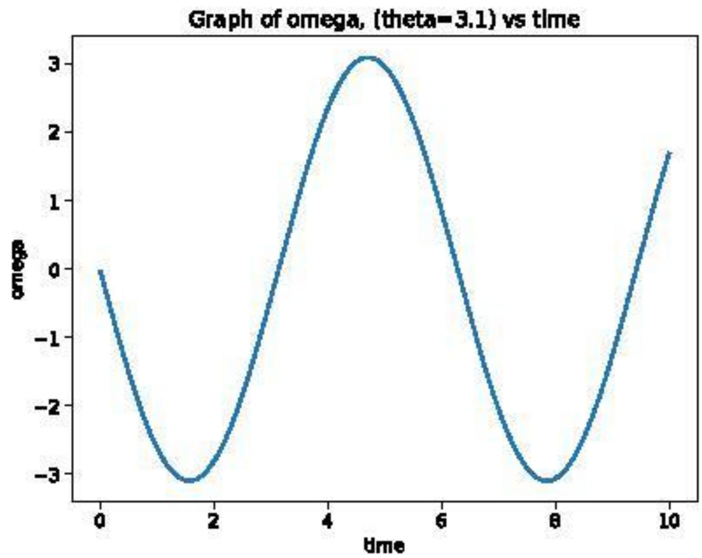
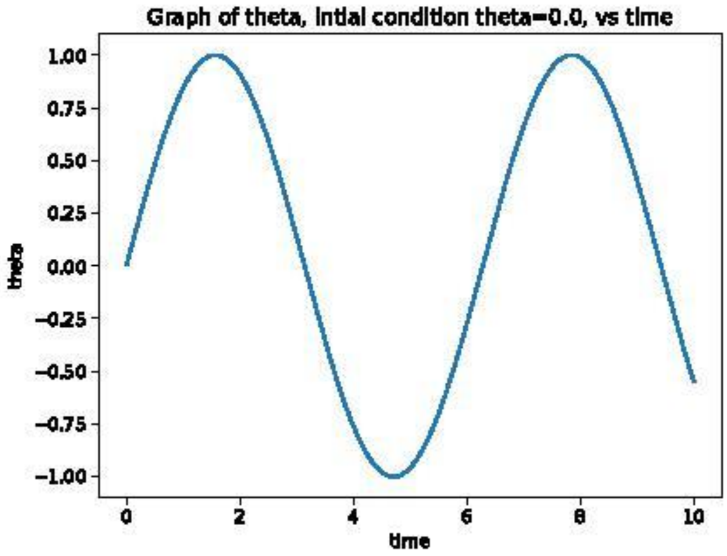


Figure 1.6b



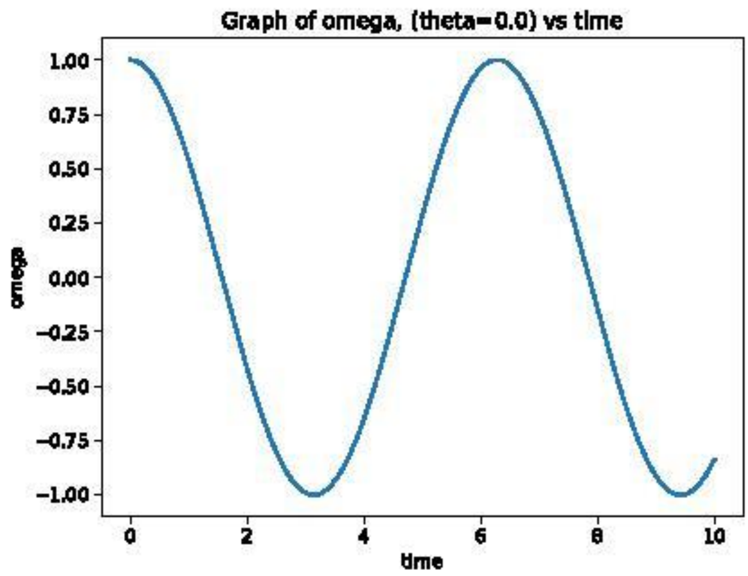
Figures 1.6a and b again show a same period of 6 seconds and constant amplitude of 3.1 radians, which has no effect on periodic time, as mentioned previously. Omega is initially zero, as would be expected, as at the extremities of a pendulums motion i.e. where it has reached its maximum displacement, the angular velocity will be zero and the acceleration will be at a maximum towards the equilibrium position.

Figure 1.7a



For theta initialised to zero, as shown in the figures 1.7a and b, it is not quite the same situation as discussed before. At theta=0, the pendulum bob will be located at the equilibrium position and so the angular velocity will be at a maximum. The graph of theta is very much sinusoidal.

Figure 1.7b



Here, omega has been initialised to 1.0, since if it were zero, as in the previous graphs, the pendulum would either be at a non-zero displacement, or the system would be stationary altogether. However, as with the previous cases, the angular amplitude will be constant (in this case it is 1 radian), and the motion will be continuously periodic without any decay. The period is also still the same value as in the previous scenarios, being a value of 6 seconds.

Exercise 2

Figure 2.1a

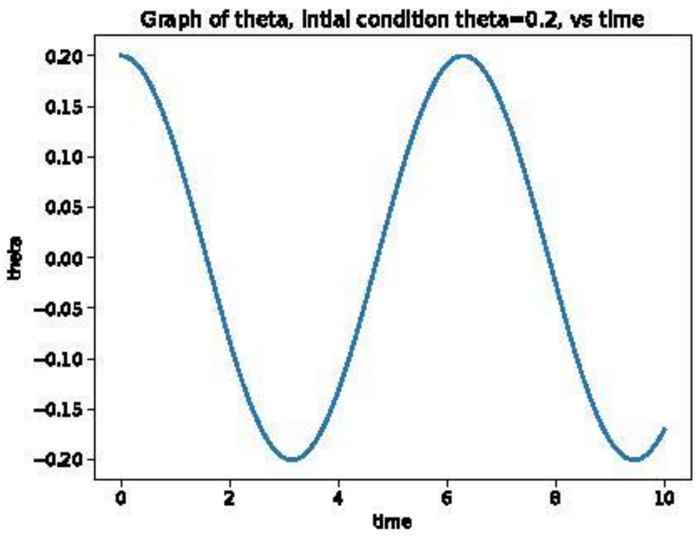
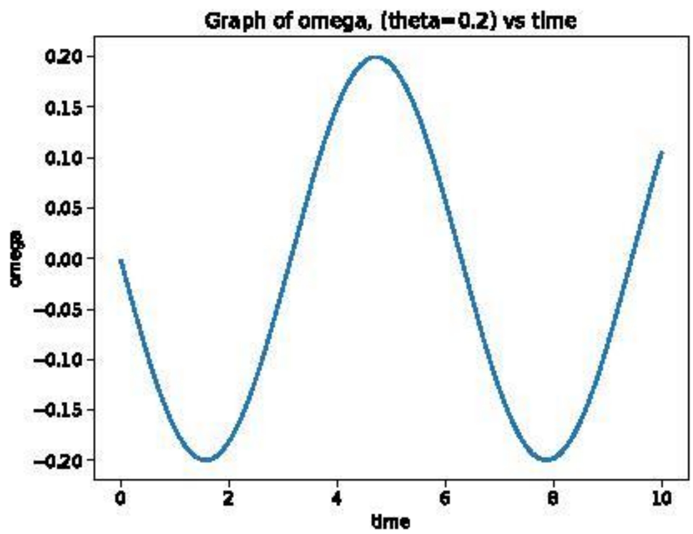


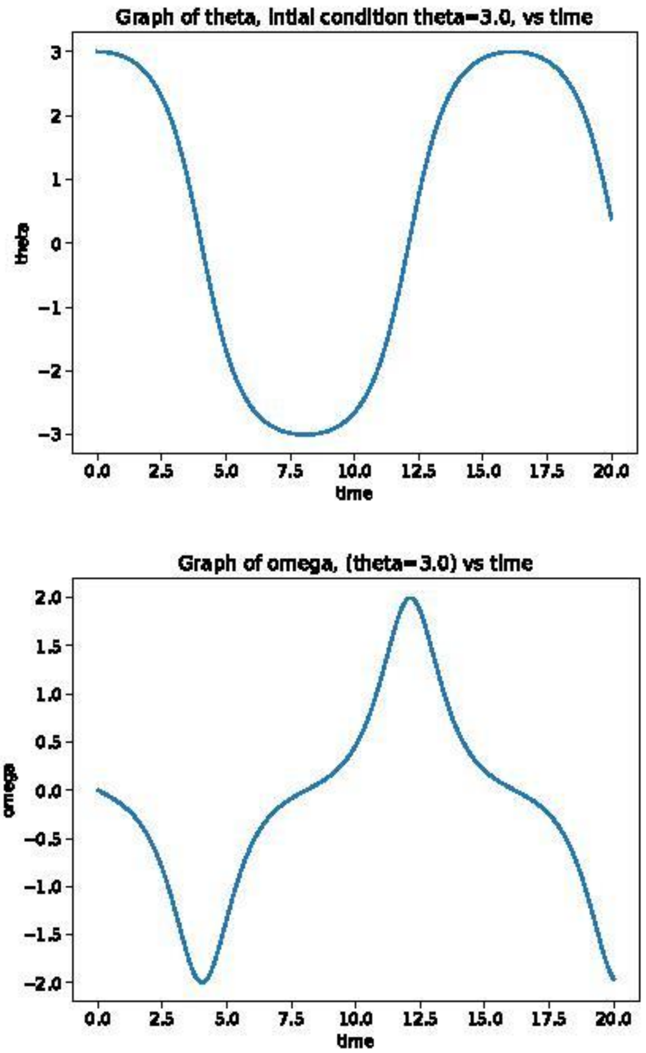
Figure 2.1b



The graphs for linear and non-linear were shown to be identical in every way and so overlaid each other in the comparison plot. This is as expected as $\sin\theta$ is very similar in value to θ for small angles, such as theta= 0.2 as shown above.
However, for larger initial values of theta, the non-linear graph tends to “lag”, in other words, the period of the non-linear system is slightly larger than that of the linear system. Therefore, it is seen that the period of a non-linear pendulum is affected by the amplitude, contrast to the case with a linear pendulum, as the period will increase with the angular amplitude, theta.

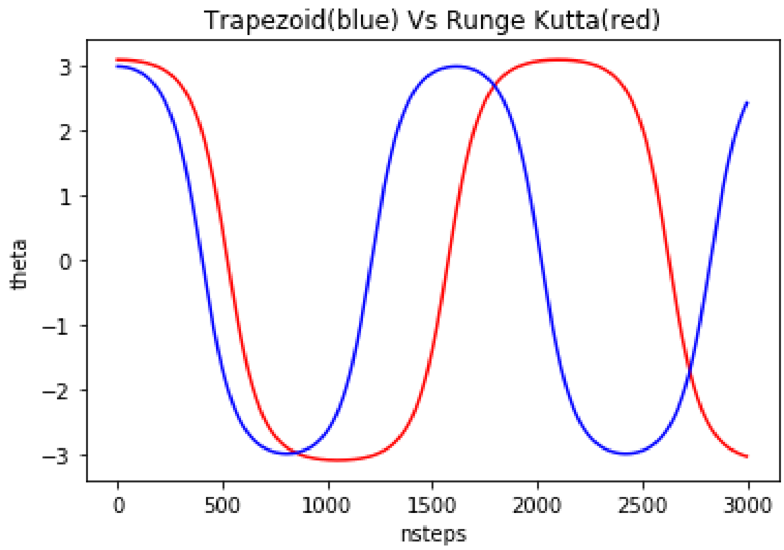
This is shown in figure 2.1c, for the initial value of theta being 3.1 radians, the period of the motion is almost increased threefold to the period of a linear pendulum.

Figure 2.1c



Figures 3.1a and b, as stated before, show the pattern of theta and omega versus time for a non-linear pendulum. For theta= 3.1, the period is much larger than for a linear pendulum. The graph for omega versus time is very different to the equivalent linear graph. The non-linear plot does not resemble a sinusoidal wave, but rather a jagged pattern.

Figure 3.1a



This graph compares the trapezoid method and the Runge-Kutta method for theta versus time for a non-linear pendulum. Here it is evident that the Runge-Kutta method is significantly more accurate than the trapezoid method as it greater represents the motion.

Figure 3.1b

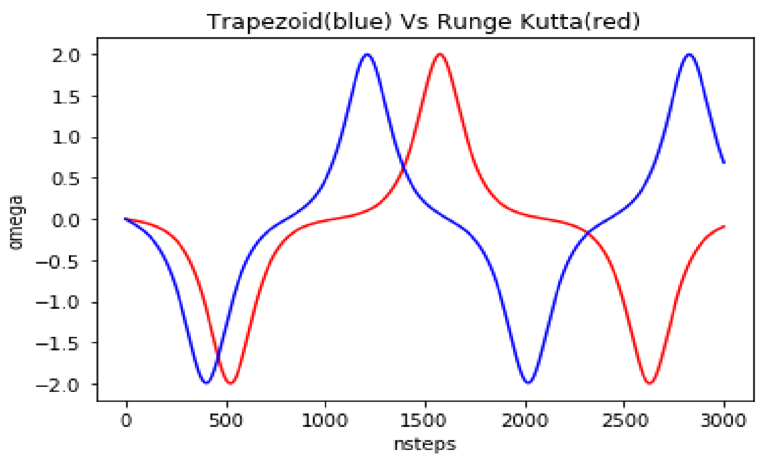
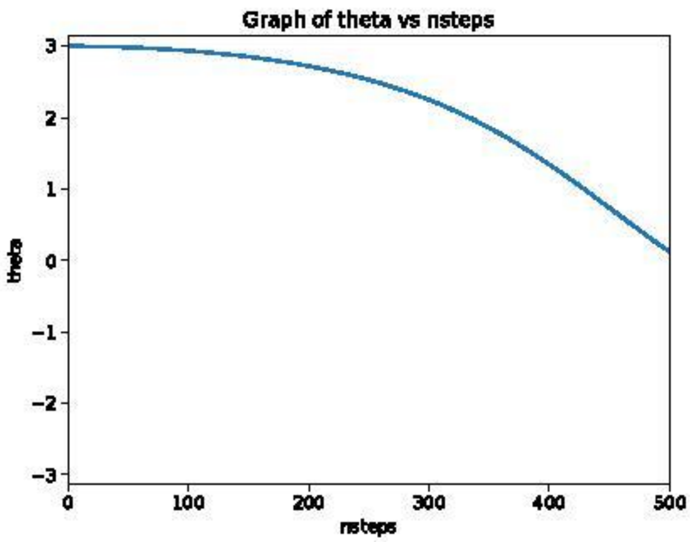


Figure 3.2b compares the Runge-Kutta method with the Trapezoid method for omega versus 'nsteps' for a non-linear, undamped pendulum. Again, the Runge-kutta method is clearly more representative of the correct motion.

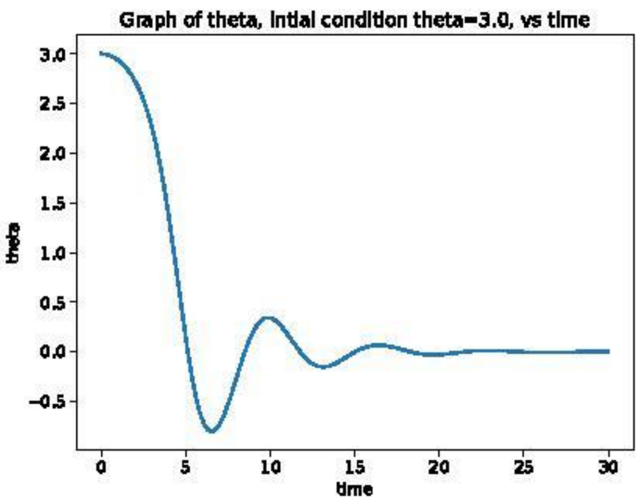
Exercise 4

Figure 4.1a



This figure would be improved had there been a greater range for 'nsteps' set. Nevertheless, it shows how theta, the angular amplitude will decrease with time for a damped, non-linear pendulum. The value of k in this instance was set to 0.5, therefore implementing a damping factor. This will subsequently cause the loss of energy and so the pendulum will not swing freely nor with constant amplitude as the motion continues.
A greater range was used for time in figure 4.1b.

Figure 4.1b



This graph shows the gradual decay of theta from its maximum initial value of 3.0 to a value of 0.0 in the time frame of 30 seconds. The damping effect is clearly seen here as within the first oscillation, the angular amplitude has decreased to a magnitude of less than 1.0 radians. The value of k therefore has a significant influence on the motion of the pendulum as its amplitude has been decreased more than threefold of its initial value when it is carrying out its second swing. The pendulum comes to a rest very quickly due to this damping effect, as the energy within the system has been dissipated.

Figure 4.2a

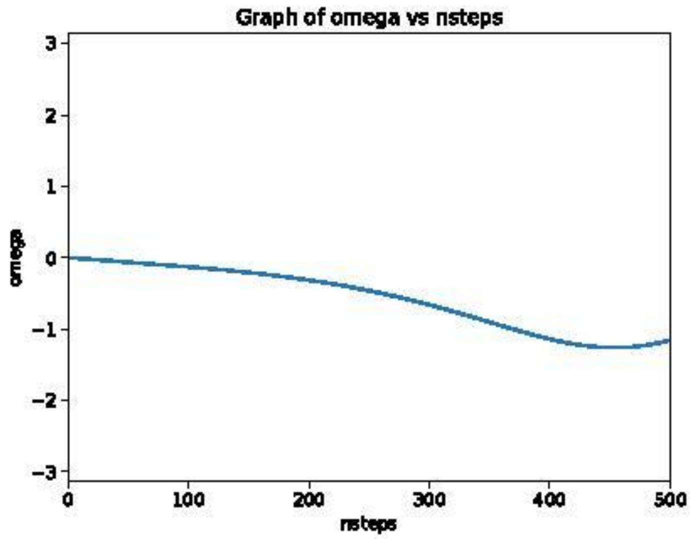
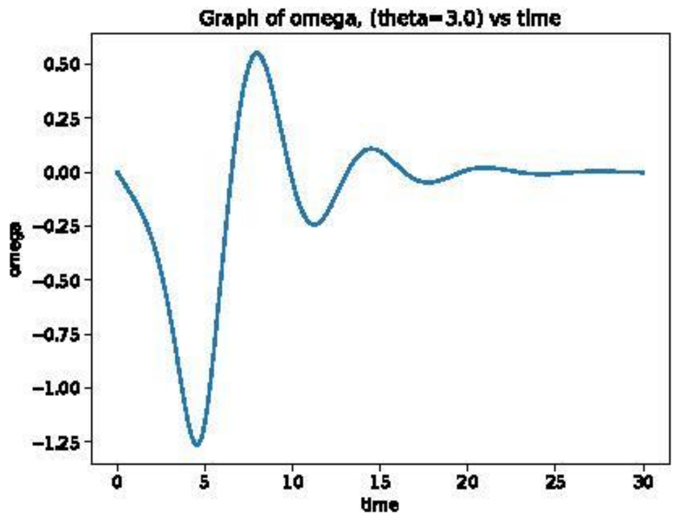


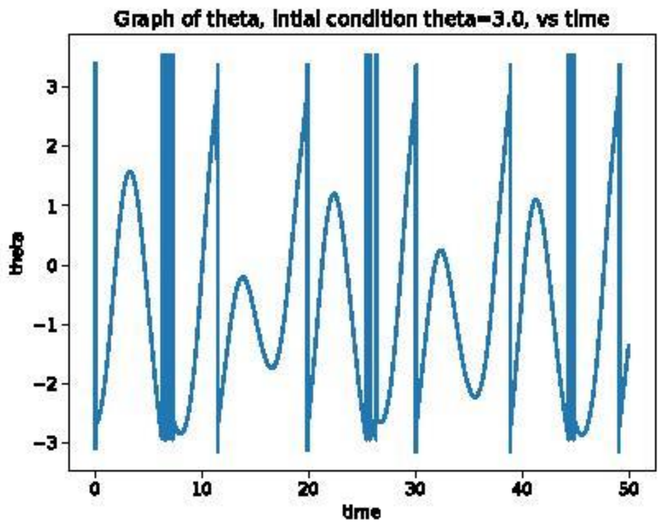
Figure 4.2b



The same situation applies to omega as shown in figures 4.2a and 4.2b. Omega begins at a value of zero radians per second at the initial position of theta (which is the amplitude of the motion) and increases in magnitude to a value around 1.25 as it passes through the equilibrium position. This is the maximum value it reaches- subsequent angular velocity values as the bob passes through equilibrium progressively diminish.

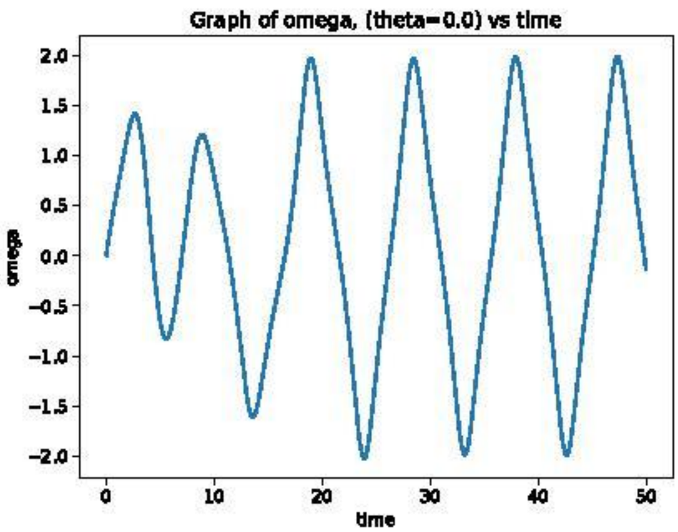
Exercise 5

Figure 5.1a



In figure 5.1a, we can see how the motion is gradually increasing in amplitude as the motion continues, due to the driving force. The motion then becomes periodic at around 20 seconds with a period of roughly 10 seconds. Initial conditions greatly affect the transient behaviour of a damped, driven pendulum. Applying a driving force to a damped system can result in the pattern shown above, whereby the amplitude increases into periodic motion where it is then restricted by the damping component of the system.

Figure 5.1b



(This graph has been mislabelled as the initial condition should state theta to be 3.0)
This transient process is relatively short, which is due to a large frictional component. The initial values for the angular velocity and angular amplitude only have an effect during the transient phase of the motion. It is expected that the transient phase occurs at the beginning of the motion, as it occurs due to a sudden change of state i.e. where the initial conditions are set and the damping and driving forces are first applied. A so-called steady state is reached once the transient component of the equation of motion vanishes.

Figure 5.2

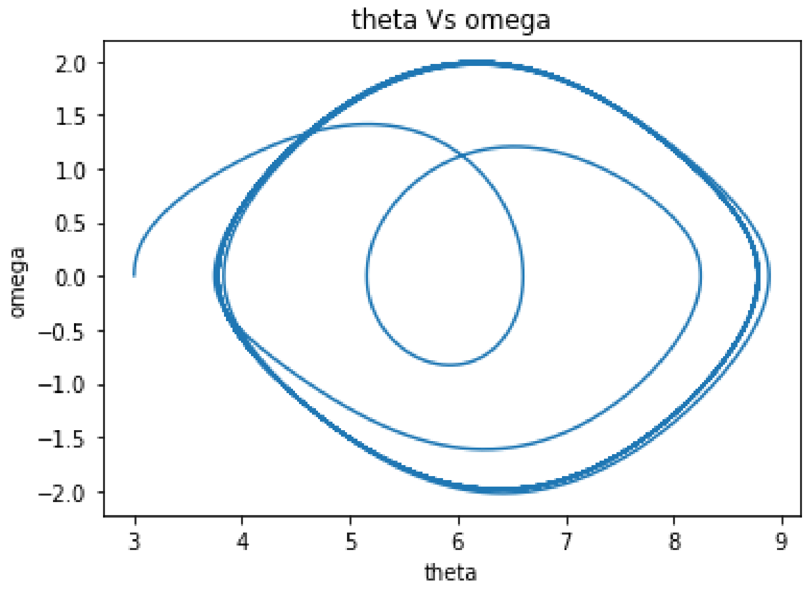


Figure 5.2 shows the phase diagram for A=0.9, the transient phase of the motion is evident until a steady state is reached. This is as expected for a small driving force and for theta initialised to 3.0 radians. This steady state involves the pendulum oscillating and doubling of the period occurs.

Figure 5.3

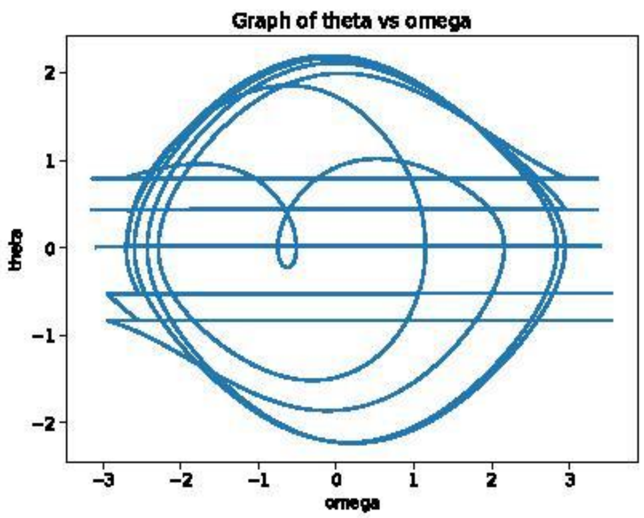
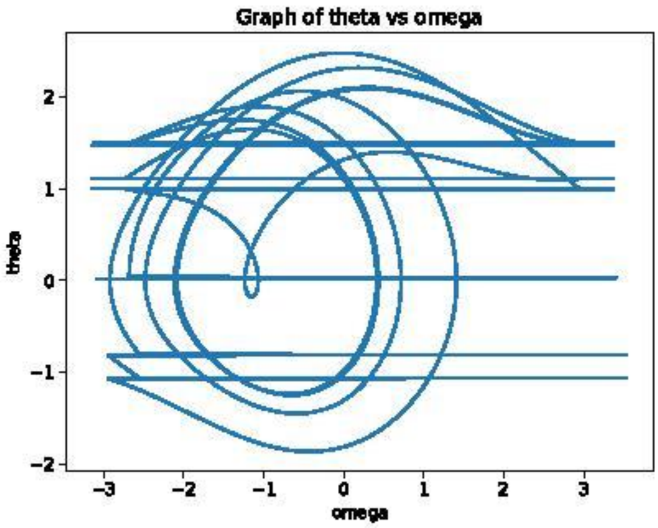


Figure 5.3 shows the phase diagram for a slightly increased value of A to 1.07. Horizontal lines are evident in this figure, as well as figures 5.4, 5.5 and 5.6d. These straight lines are present due to the 'if' statement within the 'for' loop of the python code. As shown below:

```
if (math.fabs(theta) > math.pi - 0.2):
          theta -= 2 * math.pi * math.fabs(theta)/theta - 0.2
```

This states that any values which are greater than pi are reduced by 2(pi) and so are represented on the phase diagram.

Figure 5.4



For the driving force set to 1.35 as shown in figure 5.5, the periodicity is observed to break down significantly, to the point where chaotic motion is observed.

Figure 5.5

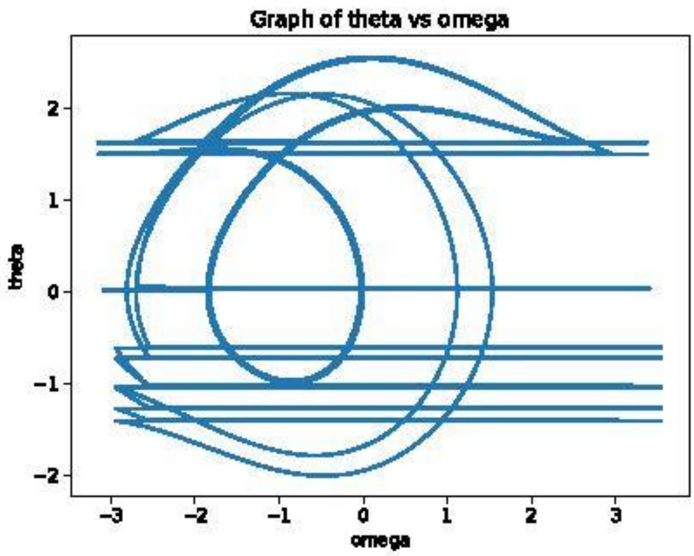
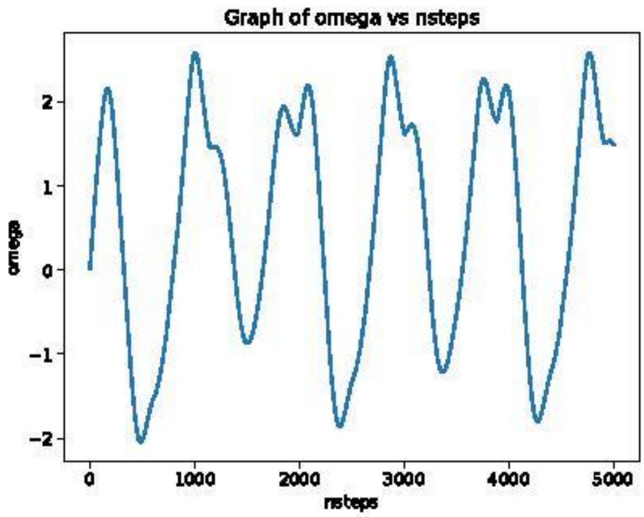


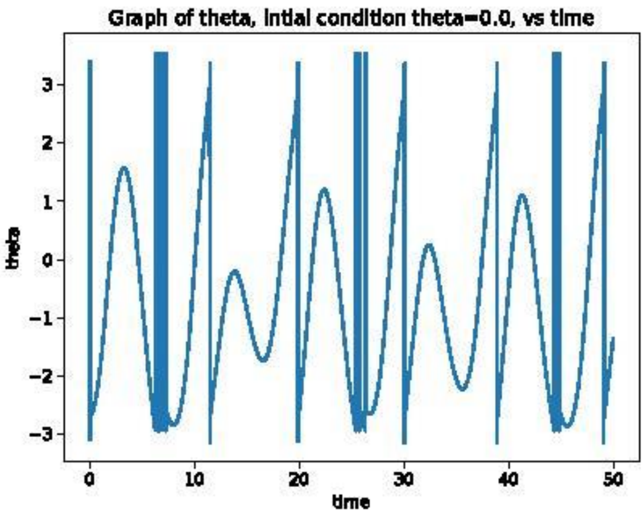
Figure 5.5 shows the phase diagram for the driving force amplitude being set to 1.47. Small adjustments to the driving force cause unpredictable and highly reliant behaviours on the initial conditions. This motion is therefore described as chaotic.

Figure 5.6a



The chaotic motion of the pendulum is now very clear for the figure 5.6a, where the angular velocity is somewhat periodic but neither sinusoidal nor smooth.

Figure 5.6b



Again, the figure above has been mislabelled, but for A=1.5, theta versus time is clearly chaotic.

Figure 5.6c

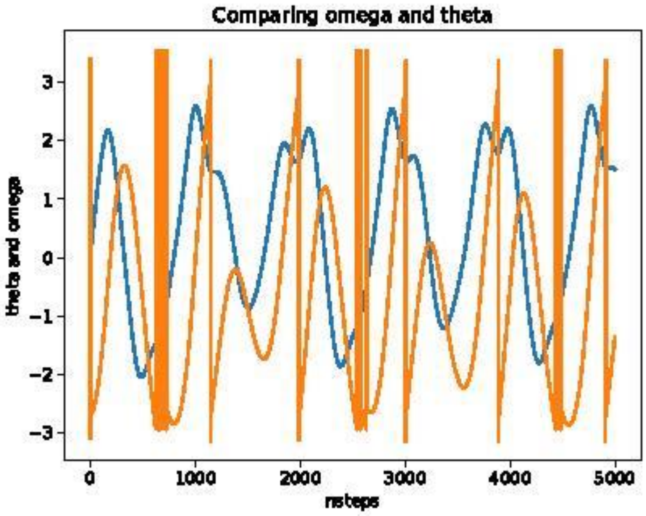
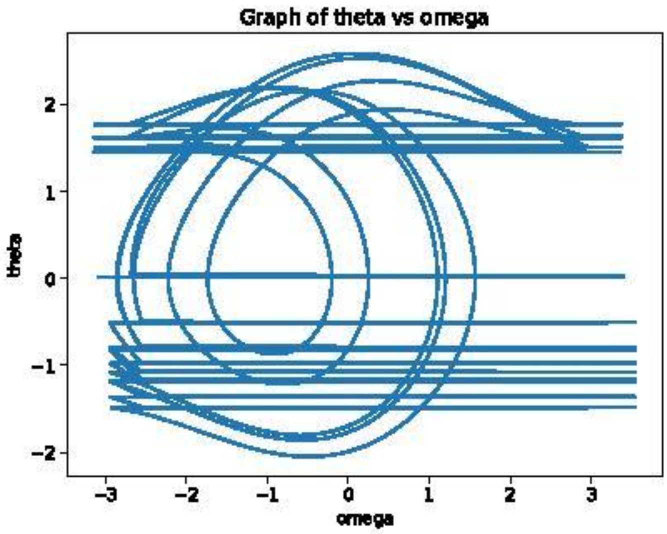


Figure 5.6c compares theta (orange) and omega (blue) versus time. Again, the motion is clearly not sinusoidal. The damping force does not limit the driving force for the system to reach a steady state in any way.

Figure 5.6d



Finally, the phase diagram 5.6d is shown above for A-1.5. The motion is clearly chaotic with no symmetry.

Conclusion

As is the case with all the graphs in exercise 1, the angular amplitude, theta, remains constant throughout the pendulum's motion. This is the situation for all conservative systems of this kind- no energy is lost and so the system will continue to move in a periodic, constant motion without deterioration. The graphs plotted show very similar expected trends for both initial conditions whereby the bob begins at its maximum displacement and at the equilibrium position of its motion. Of course, this is not a very accurate representation of real-life scenarios, and so the following exercises were proven to be more applicable.

For exercise 2, again, the expected trends were shown. The linear and non-linear systems only match for very small values of theta, specifically those less than 1 radian. In any other case, the non-linear pendulum system demonstrates a sinusoidal system but with a longer period.

For part 3 of the investigation, the comparison of the Runge-Kutta and Trapezoidal method showed the expected results. For small values of angular amplitude, both methods were of equal accuracy, but the difference is clear once values of theta were initialised to 3.1 radians, the Runge-Kutta method is much more accurate for a non-linear system.

Experiment 4 investigated the behaviour of a non-linear, damped system. The reduction of the angular amplitude is clear and shows that the code has effectively executed the behaviour of such a system.

For exercise 5, the graphs plotted for non-integer values of A demonstrate how the motion is chaotic. There is a very sensitive change to the behaviour of the system even when A is varied by small values such as 0.3 (i.e. from 1.47 to 1.5). This chaotic effect is certainly not periodic but is not random- the result of any driving force shown in the graphs is completely deterministic and extremely dependent on the initial conditions set within the code. The chaotic behaviour clearly increases with the driving force. As the number of iterations were increased, the lack of pattern becomes more evident- more lines were visible showing the phase was never repeated, verifying that this behaviour is chaotic.