DEGENERACY PRESSURE IN COLD NEUTRON STARS

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ABSTRACT

Neutron stars, composed of highly dense neutron-rich matter, represent one of the most extreme states of matter in the observable universe, shortly behind black holes. Their unique properties present significant challenges for theoretical physics. In this paper, we approach neutron stars in the low-temperature limit, a simplification that allows for more accessible calculations of the Equation of State (EOS) and degeneracy pressure. The theoretical predictions derived from this model can then be compared to observational data to validate the accuracy and utility of the low-temperature approximation.

1. THEORETICAL BACKGROUND

1.1. The Birth of Neutron Stars

Stars, vast astronomical entities, sustain nuclear fusion within their cores, a process facilitated by the immense gravitational pressure they endure. Stars exist in hydrostatic equilibrium, where forces balance perfectly across all spatial units, resulting in a net sum of zero. The equilibrium occurs between two opposing forces: the star's own gravity and the pressure created by the heat from nuclear fusion.

However, this balanced state only persists until iron is generated from the fusion process. Beyond this point, fusion transitions to an endothermic process, a reaction that necessitates its input instead of releasing energy. Hence, once a star exhausts the material suitable for nuclear fusion, hydrostatic equilibrium is lost, and the star undergoes a typically violent contraction, a supernova event. Such a cosmic event can spawn a variety of celestial objects.

The resulting object from a supernova depends on the remnant star's mass. For reasons not fully explored in this paper, if the remnant mass, M, is within the range of $1.4M_{\odot} < M < 2.9M_{\odot}$, where $M_{\odot}$ denotes a solar mass, equivalent to the mass of our sun, the star will turn into a neutron star. This leaves the question: what force prevents the further collapse of a neutron star under its own gravity?

1.2. Assumptions and the Fermi Gas

Fermions are a class of particles characterized by half-odd-integer spins, such as $\frac{1}{2}$ or $\frac{3}{2}$. This unique feature connects them to a fundamental principle of Quantum Mechanics known as the **Pauli Exclusion Principle**. This principle asserts that two fermions cannot simultaneously share the same quantum state. The relevance of this principle to our analysis lies in the fact that neutrons themselves are fermions.

Since neutron stars are predominantly composed of neutrons, the Pauli Exclusion Principle plays a crucial role in understanding their structure. Specifically, it is this principle that prevents the neutron star from collapsing further under its own gravity. In other words, it holds the key to deciphering the origin of the pressure that balances the gravitational force within the neutron star: The immense gravitational pressure tries to squeeze the neutrons into the same place and the same state. But the Pauli Exclusion Principle prevents this from happening: it stops any two neutrons from occupying the same state.

Therefore, the key assumptions we are going to make are:

1) A neutron star is composed of "Fermi Gas" - matter composed of non-interacting fermions. In our case, this means we regard a neutron star to be composed solely of neutrons.

2) $T = 0$, Under this conditions, the pressure exerted by the neutrons (the degeneracy pressure) arises not from thermal motion (as would be the case in a classical gas), but from the Pauli exclusion principle. This pressure is what supports the neutron star against gravitational collapse.

2. DISCUSSION

2.1. Degenerate Fermi Gas in a Cold Neutron Star

Having established our working framework, it's time to delve into the behavior of the Fermi gas. As is customary, we'll start by calculating the grand partition function for a single particle. By then applying this understanding to every particle within the system, we'll have the ability to extract the physics that captures our interest. A similar discussion was covered in *Tong's Lecture Notes* in section 3.6 [1].

In the context of a neutron star, we assume the grand canonical ensemble not because the number of particles is continuously changing, but rather because the exact number of particles is not a fixed, known quantity. This is a typical situation in many-particle quantum systems, especially those where particles can be created and destroyed, and it's often more practical to work in the grand canonical ensemble for such systems.

First, let us consider a single fermion of the state r with corresponding energy E_r; therefore, the partition function is:

$$\mathcal{Z}_r = \sum_{n=0,1} e^{-\beta n(E_r-\mu)} \tag{1}$$

This is not the partition function over all possible energies, rather, the partition function of a single energy state E_r, and thus, it is either occupied or not. Naturally, the next step is to find the partition function of all states, which would be done by multiplying the grand partition functions corresponding to all energy states, more compactly:

$$\mathcal{Z} = \Pi_r \mathcal{Z}_r \tag{2}$$

Now, in order to find the average number of particles in state E_r, denoted n_r, we use the definition of average and get:

$$n_r = \frac{1}{Z_r}\sum_{N_r} N_r * P(N_r) \tag{3}$$

$$= \frac{0\cdot e^{-\beta(E_r-\mu\cdot 0)} + 1\cdot e^{-\beta(E_r-\mu\cdot 1)}}{1+e^{-\beta(E_r-\mu)}} \tag{4}$$

$$n_r = \frac{1}{e^{\beta(E_r-\mu)}+1} \tag{5}$$

Where we get $P(N_r) = e^{-\beta(E_r-\mu * N_r)}$ from the Boltzmann distribution, and $N_r = 0, 1$ due to the Poli-Exclusion principle, so a state is either occupied, or empty.

In the limit which we are interested, $T \to 0$, we have:

$$n_r = \frac{1}{e^{\beta(E_r-\mu)}+1} \to \begin{cases} 1 & \text{for } E_r < \mu \\ 0 & \text{for } E_r > \mu \end{cases} \tag{6}$$

This is an exciting result, while slightly confusing at first. an excellent analogy to understand what is going on is as follows: Imagine you're at a movie theater where the seats represent the different energy states. Each seat can be occupied by one person only (just as the Pauli exclusion principle dictates, each energy state can be occupied by one fermion only). Now, if you're the first person to walk into the theater, you would probably choose the best seat available, let's say, the one in the middle with the perfect view - it's the lowest energy state, it requires the least effort, and gives you the best experience.

As more people (or fermions) enter the theater (or the system), they take up the next best available seats. This continues until all the best seats (or lowest energy states) are taken up. The next person that walks into the theater has no choice but to take a less ideal seat that requires a bit more effort to get to and maybe doesn't have as good a view. This is equivalent to a higher energy state.

At low temperatures, fermions behave just like people entering an empty theater. They want to occupy the lowest energy states first. So, they "fill up" these states one by one, starting from the lowest, until no more particles are left. At $T = 0$, the fermions simply do not have enough energy to fill up the higher energy states; this is a central reason for the simplicity of our analysis.

The energy of the last filled state is denoted as E_F, and at the low-temperature-limit, we get:

$$\mu(T=0) = E_F \tag{7}$$

In order to find E_F, we approximate that the Fermi energy in n-space is similar to the square well energy:

$$E_F = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2+n_y^2+n_z^2) = \frac{\pi^2\hbar^2}{2mL^2}R_n^2 \tag{8}$$

Where n_x, n_y, n_z are the principal quantum number in three-dimensional n-space, and R_n is the magnitude. Since all states lower than E_F are filled, we can perform the following sum to find the total energy of all fermions in a sphere:

$$E_{tot} = \sum_{n_x}\sum_{n_y}\sum_{n_z}\frac{\pi^2\hbar^2}{2mL^2}(n_x^2+n_y^2+n_z^2) \tag{9}$$

As it turns out, assuming continuous n-space and performing the integral counterpart to the sum above yields a perfectly good approximation and thus:

$$E_{tot} = \int\int\int \frac{\pi^2\hbar^2}{2mL^2}(n_x^2+n_y^2+n_z^2)dn_y dn_y dn_z \tag{10}$$

However, notice that $\int\int\int dn_y dn_y dn_z = \int(4\pi r_n^2)dR_n$, and since we are interested in only positive values of n_i, we notice that the positive values of n_i in n-space correspond to an eighth of a quadrant in R_n space, and thus, we get a factor of $\frac{1}{8}$. Thus, our integral becomes:

$$E_{tot} = 2\frac{1}{8}\int_0^{R_n}\frac{\pi^2\hbar^2}{2mL^2}R_n^2(4\pi r_n^2)dR_n \tag{11}$$

where the two corresponds to the two possible spin states for each fermion.

by computing the integral we get:

$$E_{tot} = \frac{\pi^3\hbar^2}{10mL^2}R_n^5 \tag{12}$$

If we assume there are N states, which are all filled, we would get:

$$N = 2\frac{1}{8}\frac{4}{3}\pi R_n^3 \tag{13}$$

Where we are considering $\frac{1}{8}$ of a sphere and two spin states for each position. Solving for R_n, and substituting for E_{tot} and E_F, and substituting $L^3 = V$ we get:

$$R_n = (\frac{3N}{\pi}))^{\frac{1}{3}} \tag{14}$$

$$E_{tot} = \frac{\pi^3\hbar^2}{10mV^{\frac{2}{3}}}(\frac{3N}{\pi})^{\frac{5}{3}} \tag{15}$$

$$E_F = \frac{\pi^2\hbar^2}{2m}(\frac{3N}{\pi V})^{\frac{2}{3}} \tag{16}$$

2.2. Deriving Thermodynamic Quantities: Degeneracy Pressure

At last, we can use thermodynamic relations to find the degeneracy pressure from the total energy. Namly, we shall apply Eq(15) to the following relation:

$$P_{tot} = P_d = -(\frac{\partial E_{tot}}{\partial V}) = \frac{\pi^3\hbar^2}{15m}(\frac{3N}{\pi V})^{\frac{5}{3}} \tag{17}$$

Where P_d is degeneracy pressure. The reason $P_{tot} = P_d$ is due to our assumption for which $T = 0$; thus, the force opposing gravity is solely due to the pressure of the degenerate matter.

Finally, This is the degeneracy pressure! More particularly, it is the degeneracy pressure of a uniform star composed of fermi gas. However, a similar derivation can be made to calculate the degeneracy pressure of a white dwarf, for which the degeneracy pressure is solely due to electrons, which are also fermions. Thus, the particular degeneracy pressure of a white dwarf and a neutron star are respectively as follows:

$$P_e = \frac{\pi^3\hbar^2}{15m_e}(\frac{3N_e}{\pi V})^{\frac{5}{3}} \tag{18}$$

$$P_n = \frac{\pi^3\hbar^2}{15m_n}(\frac{3N_n}{\pi V})^{\frac{5}{3}} \tag{19}$$

Where e and n subscripts refer to electrons and neutrons, respectively.

2.3. Implication of Hydrostatic Equilibrium

As discussed, a neutron star is an example of a hydrostatic equilibrium, where external forces balance, in our case, the balance happens between the degeneracy pressure and the gravitational pressure. Using a similar technique to what we used in this paper, the gravitational pressure, P_g, is as follows:

$$P_g = -\frac{1}{5}G(Nm_n)^2(\frac{4\pi}{3})^{\frac{1}{3}}V^{-\frac{4}{3}} \tag{20}$$

Thus, the balance of the pressures implies $P_n = -P_g$:

$$\frac{\pi^3\hbar^2}{15m_n}(\frac{3N_n}{\pi V})^{\frac{5}{3}} = \frac{1}{5}G(Nm_n)^2(\frac{r\pi}{3}^{\frac{1}{3}}V^{-\frac{4}{3}}) \tag{21}$$

Solving for R by using the substitution $V = \frac{4}{3}\pi R^3$ and $m_n N = M$ we get:

$$R = \frac{\hbar^2}{Gm_n^{\frac{8}{3}}}(\frac{9\pi}{4})^{\frac{2}{3}}M^{-\frac{1}{3}} \tag{22}$$

3. CONCLUSION

Using a similar insight as was used in Timlin's article [2], we can observe that Eq(22) states that the radius of a neutron star is inversely proportional to the cube root of its mass. Employing the approximation of $M = 2M_\odot$ and incorporating it into our equation with the appropriate parameters yields an outcome of $R \sim 10km$. This estimate is reassuringly consistent with current research and prevalent literature on neutron star radii, which generally propose a range of $R \sim 10 - 15km$. Consequently, our derivation and the assumptions underlying it gain credibility, indicating that we're on the right track even though precise measurement of neutron star radii continues to be an active area of scientific inquiry.

Rearranging equation (18), and using $V = \frac{4}{3}\pi R^3$ and $N_n = \frac{M}{m_n}$, eventually get:

$$P_n = \frac{3^{\frac{10}{3}}\hbar^2}{15\pi^{\frac{1}{3}}m_n^{\frac{8}{3}}R^5}(\frac{M}{4})^{\frac{5}{3}} \tag{23}$$

Which is much neater expression of the degeneracy pressure, since it only depends on two observables; M, the mass of the neutron star, and R, its radius.

after plugging $M = 2M_\odot$, $R = 10000m$, and the other appropriate constants, we get that the degeneracy pressure is $P_n \sim 5.5 * 10^{33}kg/ms^2$

Using quantum mechanics and statistical physics principles, we were able to derive an expression for the degeneracy pressure of a neutron star and a white dwarf. However, it is to note this is just a theoretical framework that we used to find a rough approximation. However, it turns out that our approximation was not far from reality as the observables we derived were not far off values accepted in empiric values of the scientific literature. This perhaps stems from the possibility that if the temperature was not zero, the additional pressure stemming from the temperature is insignificant compared to the degeneracy pressure.

4. APPENDIX A: PROBLEM SET - GRAVITATIONAL PRESSURE APPROXIMATION

1. In our initial derivation, we briefly touched on the method used to derive the gravitational pressure exerted on neutron stars. Now, to reinforce our understanding and proficiency in the techniques utilized within the study, we will delve deeper and undertake the task of calculating this pressure. This, relies on the approximation where the density of the neutron star, ρ, is constant.

$a)$ By considering a spherical shell of mass $m_s = 4\pi r^2 \rho$ and a volume element of mass $m_i = \frac{4}{3}\pi r^3 \rho$, write down and solve the gravitational energy in integral form. [Hint: $M_{tot} = \frac{4}{3}\pi R^3 \rho$]

$b)$ Rewrite your result in terms of the volume, V.

$c)$ Using the approximation you found and equation (16), find the pressure, P_g.

2. [Bonus Problem] Explore the intriguing implications of a universe where the neutron-proton mass difference is less than the electron mass, namely: $M_n - M_p < m_e$, illustrating the delicate balance of forces and mass relationships that give rise to the neutron stars we observe. Would neutron star be possible if $M_n - M_p < m_e$ was not true? Explain.

5. APPENDIX B - SOLUTION TO APPENDIX A

$a)$

$$E_g = -G \int_0^R \frac{m_s m_i}{r} dr$$

$$= -G \int_0^R \frac{(4\pi r^2 \rho)(\frac{4}{3}\pi r^3 \rho)}{r} dr$$

$$-\frac{16 G \pi \rho^2 R^5}{15} = -\frac{3}{5}\frac{GM^2}{R}$$

$b)$

$$E_g = -\frac{3}{5} G(M)^2 (\frac{4\pi}{3V})^{\frac{1}{3}}$$

$c)$

$$P_g = -\frac{1}{5} G(N m_n)^2 (\frac{4\pi}{3})^{\frac{1}{3}} V^{-\frac{4}{3}}$$

2. In a universe where the neutron-proton mass difference is less than the electron mass, i.e., $M_n - M_p < m_e$, neutron stars may not exist as we know them. The key process in the creation of a neutron star is the beta decay of a neutron into a proton, an electron, and an electron antineutrino, and this process is only energetically favorable when $M_n - M_p > m_e$. If $M_n - M_p < m_e$ was the case, this fundamental process might not occur, implying that the conditions necessary for neutron star formation could be drastically altered or non-existent.

6. REFERENCES

[1] David Tong, "Statistical physics," http://www.damtp.cam.ac.uk/user/tong/statphys/sp.pdf, 2012.

[2] John Timlin, "Neutron degeneracy pressure," http://www.physics.drexel.edu/~bob/Term_Reports/John_Timlin.pdf, 2013.