Exploring the theta term in QM

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Abstract

The objective of this exploration is to determine whether the theta term that comes out from the QCD theory of heavy particle collisions system is an an angle, as the naming of the parameter suggests. We will first explore a 1D system. The theta term we explore appears in the classical Lagrangian as follows:

$$\mathscr{L} = \frac{1}{2}m\dot{x}^2 - \frac{\theta}{2\pi}\dot{x} \tag{1}$$

The θ term does not affect classical mechanics as you could deduce from the equation. However, after transforming the term to QM system, it is evident that the theta term plays an important role in QM with regards to the energy spectrum of a given particle. We are interested in exploring circular motion of particles under some magnetic field, to begin with the most simple version of this problem consider a circular 1D problem where for a given length L where if a particle reaches L it will start off at zero, and vice versa. More precisely, $\Psi(0) = \Psi(L)$. With that boundary condition in mind, we found the momentum eigen state of the system to be $e^{ik_n x}$ where $K_n = \frac{2*pi}{L} * n$, and n is some integer. Then, we found that the momentum operator commutes with the energy operator (the Hamiltonian), which allows us to use the eigen vector of the momentum solution in the energy equation. Solving for the energy of the system we get the following:

$$E = \frac{1}{2m}(\hbar K_n - \frac{\theta}{2 * \pi})^2 \tag{2}$$

However, this result is not final, the glaring problem is that the units within the parenthesis does not work out. The left side of the parenthesis has units of momentum and the left side is unit less. In order to fix this, we can multiply the term $\frac{\hbar * 2 * \pi}{L}$ to the right side to correct for this and get:

$$E = \frac{1}{2m}(\hbar K_n - \frac{\hbar * 2 * \pi}{L}\frac{\theta}{2 * \pi})^2 = \frac{1}{2m}(\hbar K_n - \frac{\hbar}{L}\theta)^2 \tag{3}$$

Since we are trying to observe how does the theta term affects QM, we observe the change of our energy spectrum as a change of theta. On its own, the energy function for a given wavelength is a parabolic function, however, if we graph the ground state energy for a given theta, we get a periodic result over 2π, which is a strong argument for the angular nature of the theta term.

In retrospect, the Lagrangian we used did not have the correct dimensions either. One way we could have approached it is to fix the Lagrangian is to multiply the theta term with a constant of units $\frac{J}{m/s} = \frac{Js}{m}$ to ensure the right hand side had units of energy. The term would be $\frac{\hbar}{L}$; the same answer we got eventually. Note that multiplying this term in the Lagrangian instead of the Hamiltonian will not suffer alterations from the Hamiltonian transformation, this is left as an exercise for the reader. Alternatively, we could use dimensional analysis to arrive to the same conclusion. If we use mass dimensions, notice how the left hand side of the Lagrangian has units of 1, and the left hand side is unit less to fix that we need to multiply the right hand side of the Lagrangian with a constant of unit 1, which would be $\frac{1}{L}$. Notice how $\frac{1}{L}$ is the same as using $\frac{\hbar}{L}$ with natural units.

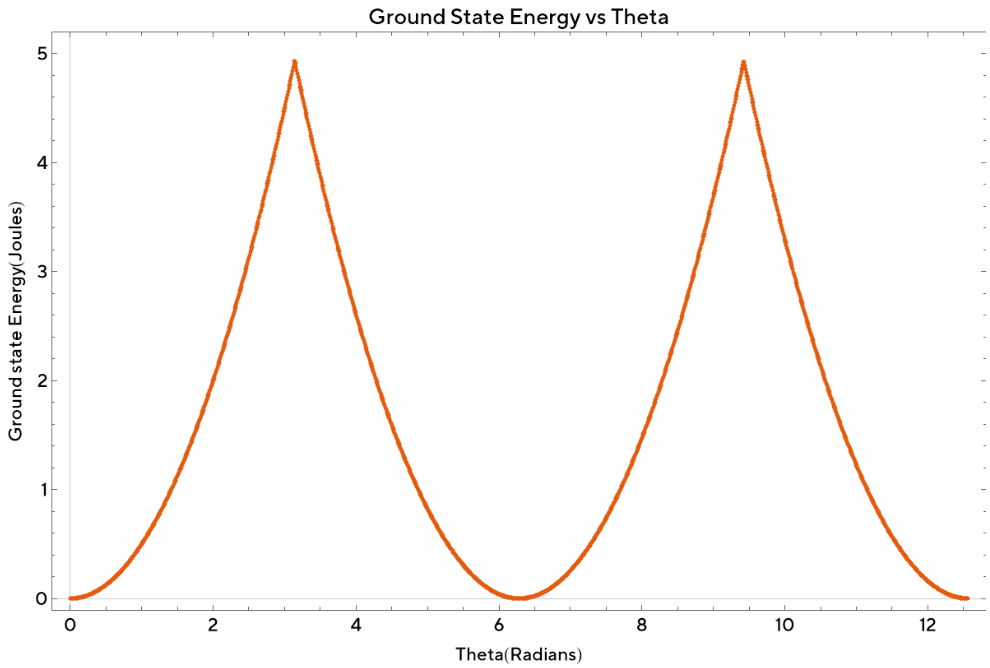

Figure 1: Ground State Energy vs Theta graph. This is the result we got after adding the term to fix the dimensional of our expression. Notice how the graph is periodic over $2 * \pi$, a result we confirmed analytically as well.

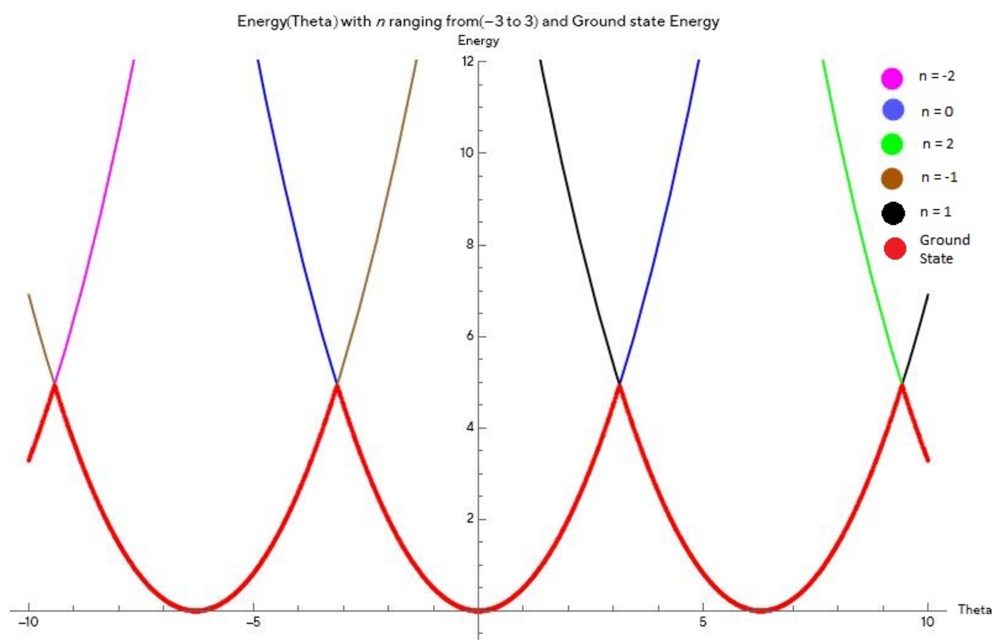

Figure 2: In red, you can see the graph from figure 1. In other colours you can see the energy spectrum for a given n value for varying theta values. Notice how the lowest possible value for the energy at every value of theta results in the red graph.

For our 2D problem we know that the Lagrangian takes the following form:

$$\mathscr{L} = \frac{1}{2}m(\dot{q}_x^2 + \dot{q}_y^2) - \frac{\theta_x}{2\pi}\dot{q}_x * \frac{1}{L_x} - \frac{\theta_y}{2\pi}\dot{q}_y * \frac{1}{L_y} \tag{4}$$

Which in essence has the form of our 1D Lagrangian but with both dimensions, including the additional term using natural units. It is more evident if we present the equation as follows:

$$\mathscr{L} = (\frac{1}{2}m\dot{q}_x^2 - \frac{1}{L_x}\frac{\theta_x}{2\pi}\dot{q}_x) + (\frac{1}{2}m\dot{q}_y^2 - \frac{1}{L_y}\frac{\theta_y}{2\pi}\dot{q}_y) \tag{5}$$

From equation 5, it is evident that the problem is completely separable which would lend us to think that the energy spectrum would have two terms that are similar to our 1D problem for each dimension. This turns out to be the case:

$$E = \frac{1}{2m}(K_n - \frac{1}{L_x}\theta_x)^2 + \frac{1}{2m}(K_n - \frac{1}{L_y}\theta_y)^2 \tag{6}$$