Experiments on Komlós conjecture

Thoughts, Ideas and Experiments

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May 1, 2018

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Conjecture

Statement

Given n, let $K(n)$ denote the infimum over all real numbers such that: for all set of n vectors $u_1 \cdots u_n \in \mathbb{R}^n$ satisfying $||u_i||_2 \leq 1$, there exist signs $\epsilon_i = \pm 1$ such that

$$||\epsilon_1 u_1 + \cdots + \epsilon_n u_n||_\infty \leq K(n)$$

There exists a universal constant K such that $K(n) \leq K$ for all n.

What is it saying?

Statement

$\min \|A\epsilon\|_\infty \leq K(n) \leq K$

Consider this as a two player game, where Player 1 picks matrix A, and Player 2 picks vector ϵ in such a way as to minimize $\|A\epsilon\|_\infty$. Since Player 2's universe of game choices is known, $\pm 1^n$, we could simply take the position of Player 1 and ask ourselves how do we create a matrix A so as to maximize $A\epsilon$'s infinity norm?

$\mathbb{R}^2$

$$\left\| \begin{bmatrix} a & u \\ b & v \end{bmatrix} \begin{bmatrix} \pm 1 \\ \pm 1 \end{bmatrix} \right\|_\infty \leq K(2) \leq K;\ a^2 + b^2 \leq 1;\ u^2 + v^2 \leq 1$$

$\mathbb{R}^3$

$$\left\| \begin{bmatrix} a & p & u \\ b & q & v \\ c & r & w \end{bmatrix} \begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{bmatrix} \right\|_\infty \leq K(3) \leq K;$$

$$a^2 + b^2 + c^2 \leq 1;\ p^2 + q^2 + r^2 \leq 1;\ u^2 + v^2 + w^2 \leq 1$$

Propositions

Trivially, we can assert that $K(1) = 1$

$u_1 = [1]$

$\epsilon_1 = \pm 1$ yields $K(1) = 1$

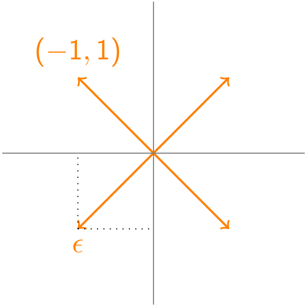
Working in $\mathbb{R}^2$

One could think of vectors in $\mathbb{R}^2$ that yield the maximal infinity norms. One might naturally attempt the orthogonal basis vectors e_1, e_2. That would yield us

$$\left\| 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|_\infty = 1$$

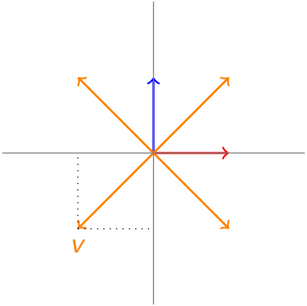
Can one do better?

Corners of a cube, ϵ



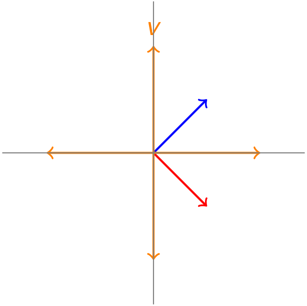
$\|.\|_\infty$ is only 1
(Matrix $A_n : \mathbb{R}^n \to \mathbb{R}^n$)

Plotting in $\mathbb{R}^2$



Rotation?

Rotated in $\mathbb{R}^2$



Rotation Matrix (by $\frac{\pi}{4}$): $\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}$

$\|.\|_\infty$ is now $\sqrt{2} \approx 1.414$

$K(2) = \sqrt{2}$

Lemma

For $n = 2$, Komlós constant $K(n) = \sqrt{2}$

Proof.

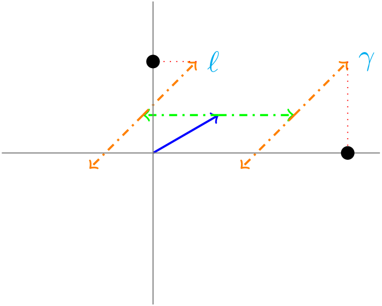
$\begin{bmatrix} \cos\theta & \cos\alpha \\ \sin\theta & \sin\alpha \end{bmatrix}$ *can be used to represent a 2D matrix with unit norm columns. By symmetry, there are only two unique vector assignments possible* $[1, 1]$ *and* $[-1, 1]$. *Assuming equal maximums for these assignments* $\cos\theta + \cos\alpha = \sin\theta - \sin\alpha$, *and setting the first order derivatives to* 0, *we arrive at* $\frac{d\alpha}{d\theta} = -\frac{\sin\theta}{\sin\alpha}$, *and* $\frac{d\theta}{d\alpha} = \frac{\cos\alpha}{\cos\theta}$. *Simplifying,* $\theta = -\alpha = \frac{\pi}{4}$. *We have intentionally ignored the case of vectors with norm strictly less than* 1, *because vectors with norm* $= 1$ *strictly dominate in* $\mathbb{R}^2$

$\square$

Turn based Analogy

Sequential Turns

Imagine each player takes turns. Player 1 selects a vector from A at each time step t. Player 2 picks the corresponding ϵ_t. Define the loss accrued by Player 2 as $\|\sum_t A_t \epsilon_t\|_\infty$. Obviously she wishes to minimize the loss.



Minimize $\|.\|_\infty$ at each step

Proposition

$K(n)$ is upper bounded by $O(\sqrt{n})$

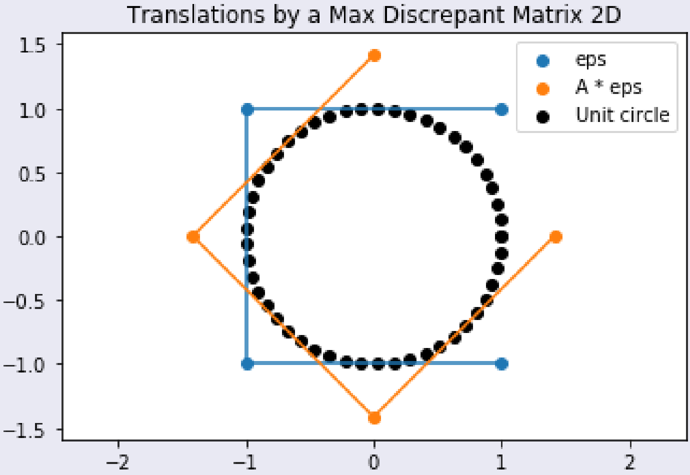
1. At each turn, Player 2 could choose a ± 1 assignment that greedily minimizes $\|.\|_\infty$ of the running sum $\sum_t A_t \epsilon_t$
2. In order to counter that, Player 1 starts by picking A_1 so that it is equally distant from all coordinate axes, thus projecting $\frac{1}{\sqrt{n}}$ on each
3. Player 2 is indifferent to ± 1 here
4. For the subsequent vectors, Player 1 picks a vector, A_t, orthogonal to all previously chosen vectors, $A_{1:t-1}$
5. Player 1 accrues, in a best case, $\frac{1}{\sqrt{n}}$ at each pick along at least one coordinate axis, thus summing to $O(\sqrt{n})$
6. *Note that this is a very loose upper bound*

Results

n	$K(n)$ estimation observations	Repetitions
3	1.5559	10 million
4	1.6206	400 million
5	1.4799	12 million
6	1.3885	12 million
7	n/a	n/a
8	1.1649	120 thousand
9	1.0828	120 thousand
10	0.9564	1200
11	0.9850	1200

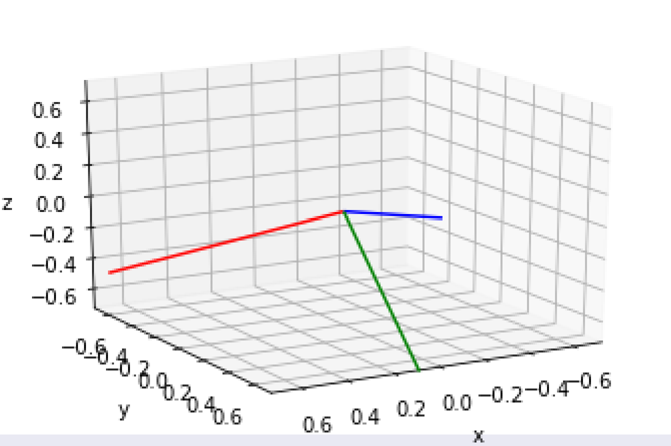
Reading the Results

Results from $\mathbb{R}^2$



Reading the Results

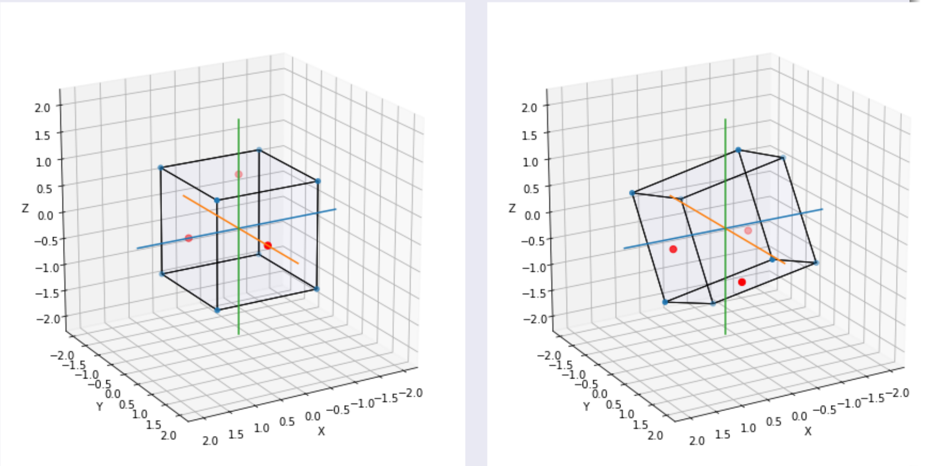
Empirical best in $\mathbb{R}^3$



Transformed unit vectors. Angular separations of: 86.96°, 102.72°, and 90.55°

Reading the Results

Empirical best in $\mathbb{R}^3$



Related Ideas

Pertinent for future work

- Ideas from the Lévy-Steinitz Theorem and Polygonal Confinement Theorem
- On-line learning and regret bounds in a bandit setting, with regrets defined as $\frac{1}{T}(\sum_{i=1}^{T} v_t^T A_t - \inf_{i_t} \sum_{t=1}^{T} v_{i_t}^T A_t)$
- Relax ϵ_i to be a vector in $\mathbb{R}^n$, along the lines of work by A. Nikolov on vector colorings

Thank you for your attention!

Miscellaneous

ML

Imagine each player takes turns. Player 1 select a vector of A at each time step t. Player 2 sees all the previous choices made and accordingly picks $\epsilon_t = \pm 1$. This is exactly the same as regret bounds in On-line Machine Learning and the bounds are $O(\sqrt{T})$ (in a generic setting), where T is the total rounds played. The regret to Player 2 in such a setting is typically defined as $\frac{1}{T}(\sum_{i=1}^{T} v_t^T A_t - \inf_{i_t} \sum_{t=1}^{T} v_{i_t}^T A_t)$. It is as if Player 2 has the opportunity to look at all possible combinations of v_{i_t} assignments and then choose the one with the minimum regret (loss). Here the relaxation is that $v_t \in \mathbb{R}^n$, so that $v_t^T A_t \in \mathbb{R}$