GROUNDWATER ANALYSIS USING FUNCTION.XLS

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Introduction

The following notes are designed to explain how Excel spreadsheets and a collection of user-defined functions entitled Function.xls can be used for analyzing some problems in groundwater resource analysis. It is assumed that you are already acquainted with at least some of the fundamentals of fully saturated groundwater flow and the use of Excel spreadsheets for numerical calculations. The text by Liengme (2000) is highly recommended for anyone who would like to review the use of Excel spreadsheets.

We will begin by explaining the use of the Theis solution for analyzing pumping test data for aquifers with no leakage. Then some of these principles will be used to analyze pumping tests in leaky aquifers with the Hantush solution before moving on to consider the use of the Boulton solution for delayed-yield pumping tests. This leads to an extension of the Hantush-Jacob leaky-aquifer idea in which the Boulton delayed-yield solution is used to predict the long-term behavior of a leaky aquifer with finite storage. Finally, we will consider the stream-depletion problem, in which pumping from a well depletes flow from a nearby stream. Additional topics covered by these notes are flow depletion from a spring, the rise of a groundwater mound beneath an irrigated area, flow to non-vertical wells, transport from contaminant sources in uniform flow and well recovery tests and other flows in which well abstraction rates, aquifer recharge rates and contaminant mass injection rates vary with time and flow to horizontal and vertical wells in Boulton-type semi-confined and Neuman-type unconfined aquifers.

The Theis Solution

The Theis (1935) solution describes unsteady flow to a well in an aquifer that is homogeneous and has an infinite horizontal extent with no vertical recharge or leakage. The aquifer is usually assumed to be isotropic, although it may be anisotropic if the principal directions of the permeability tensor are in the vertical and horizontal directions and if permeabilities are the same in all horizontal directions. Top and bottom aquifer boundaries are impermeable, but the top boundary can be a free surface if maximum free-surface drawdowns are not a significant portion of the saturated aquifer thickness. In this case the piezometric surface in the confined aquifer is replaced with the free surface of the unconfined aquifer. Sketches of the aquifer geology for a confined aquifer are shown in Fig. 1.

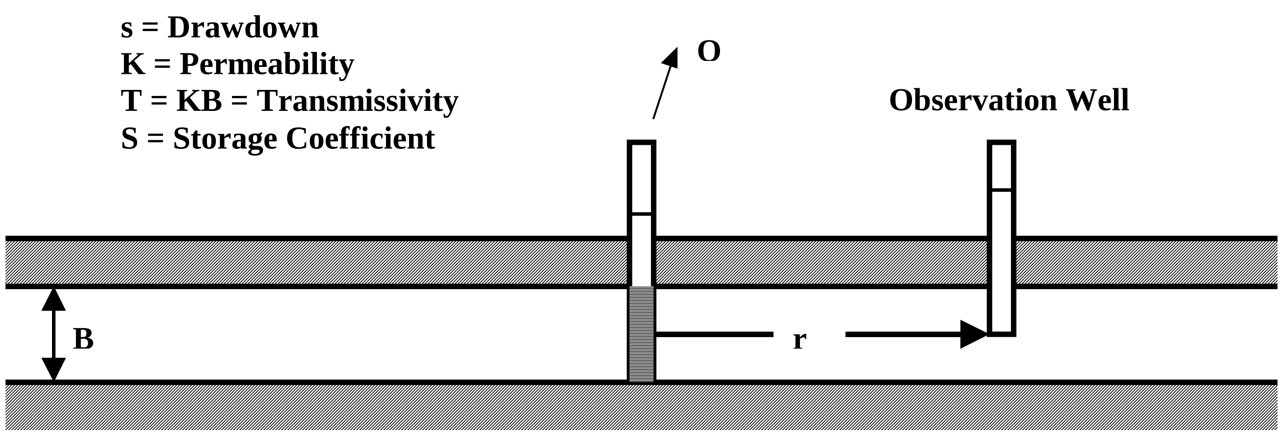

Fig. 1 Aquifer geology for the Theis solution.

The mathematical solution for this problem satisfies the following equations:

$$\frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)=S\frac{\partial s}{\partial t} \quad \left(0<r<\infty,\ 0<t<\infty\right) \tag{1}$$

$$s\left(r,0\right)=0 \quad \left(0<r<\infty\right) \tag{2}$$

$$s\left(\infty,t\right)=0 \quad \left(0<t<\infty\right) \tag{3}$$

$$Limit\left(r\frac{\partial s}{\partial r} \right)_{r\to 0} = -\frac{Q}{2\pi T} \quad \left(0 < t < \infty \right) \tag{4}$$

where s = drawdown of the piezometric head , r = radial coordinate, t = time, T = transmissivity = KB, K = coefficient of permeability, B = saturated thickness of the aquifer, S = storage coefficient and Q = flow rate to the well. The solution of Eqs. (1) - (4) was published in 1935 by Theis in the following form:

$$\frac{s4\pi T}{Q} = W\left(u \right) \quad \left(u = \frac{Sr^2}{4Tt} \right) \tag{5}$$

where W(u) is called the Theis well function. The well function is also known as the exponential integral, $E_1(u)$, and is given by the following definite integral:

$$W\left(u \right) = E_1\left(u \right) = \int_u^\infty e^{-x} \frac{dx}{x} \tag{6}$$

The numerical value of W(u) or $E_1(u)$ is fixed once a numerical value is chosen for the lower limit, u, in the definite integral in Eq. (6).

Numerical values of W(u) can be calculated for small to moderately large values of u from the following infinite series, which converges for all finite values of u:

$$W(u) = E_1(u) = -\ln(u) - \gamma - \sum_{n=1}^{\infty} \frac{(-1)^n u^n}{(n)(n!)} \tag{7}$$

where $\gamma =$ Euler's constant = 0.57721 56649… For larger values of u, W(u) can be calculated from the following asymptotic series:

$$W(u) = E_1(u) \sim \frac{e^{-u}}{u}\left(1 - \frac{1}{u} + \frac{2}{u^2} - \frac{6}{u^3} + \frac{24}{u^4} - \frac{120}{u^5} + ... \right) \tag{8}$$

You can compute the Theis solution more easily by opening the Function.xls software and typing in any spreadsheet cell

$$= W\_1\left(\frac{r}{L}, \frac{tT}{SL^2} \right) \tag{9}$$

where L is an arbitrarily chosen length. [Since L cancels out when $\mathrm{r/L}$ and $\mathrm{tT/\left(SL^2\right)}$ are used to compute u in Eq. (5). For example, if units of meter are used in a problem, then it is convenient to set $\mathrm{L = 1}$ meter.] The resulting number that will appear in the cell is a numerical value for the following dimensionless variable:

$$\frac{sT}{Q} \tag{10}$$

At this point a few words will be said about choosing suitable locations for observation wells when measuring drawdowns in pumping tests. Eq. (1) makes the assumption that streamlines in the aquifer are horizontal and, therefore, that piezometric heads do not change along vertical lines. (i.e. Piezometric levels in the observation well do not change with changes in observation well depth. This is known as the Dupuit approximation.) Since abstraction wells are seldom screened over the complete saturated thickness of the aquifer, this means that all observation wells should be at least one aquifer thickness, and preferably two or three aquifer thicknesses, away from the pumped well to ensure that they are located in zones of horizontal flow. However, observation wells that are placed too far away may require the test to continue for an unrealistically long time to obtain measurable drawdowns. Therefore, observation well locations for any pumping test should be chosen only after using Eq. (5) with estimated values for T and S to obtain estimates for the time variation of drawdown at each observation well location. Since an unconfined aquifer has an order of magnitude for S of about 0.1, while a fully confined aquifer has an order of magnitude for S of 10^{-4} or 10^{-5}, observation wells for an unconfined aquifer may be only 20 to 50 meters from the pumped well but may be several hundred meters from the pumped well in a confined aquifer. Well drilling and pumping tests can be expensive procedures, and a little advance planning can save worthwhile sums of money.

Values of T and S are usually obtained by comparing the solution given by Eq. (5) with measured drawdowns from a pumping test. In the past, T and S have usually been calculated by using either the Theis match-point method or the Jacob straight-line approximation. However, the speed of modern computers now makes it easier and at least as accurate to carry out this calculation with spreadsheets by using a trial and error procedure. (An optimist calls this a procedure of successive approximation!) A numerical example shown on page 5 was worked by opening Function.xls and entering data given by Hunt (1983) for Q, r and the measured drawdowns. Then the measured drawdowns were plotted as unfilled circles in a semi-log plot of s versus log(t). Next, guesses for T and S were entered in cells D2 and E2, and drawdowns over the experimental time span were calculated in column E by entering the following formula in cell F6:

=($A$2/$D$2)*W_1($B$2,E6*$D$2/$E$2) (11)

The use of absolute and relative addressing in Eq. (11) allowed this formula to be dragged down to cell F35. Then these calculated drawdowns were plotted as a solid curve, and values of T and S were adjusted to obtain good agreement between the calculated curve and measured points. Values of T and S were adjusted by noting that Eq. (7) gives, for $t \to \infty$ and $u \to 0$,

$$s \sim \frac{Q}{4\pi T}\ln(t) \qquad (t \to \infty) \tag{12}$$

Thus, T was adjusted to give the correct slope of the straight-line portion of the curve at larger values of t, and S was changed to move the calculated curve leftward or rightward. This curve fitting procedure makes use of the same basic principle that is used in the Jacob straight-line approximation.

Values of t in column E were computed so that n points were equally spaced on a logarithmic scale from $t = 10^a$ to $t = 10^b$ by using the formula $t = 10^{a+(b-a)(k-1)/(n-1)}$. (In this problem, a = 0, b = 3 and n = 30 were specified in cells F1:H2, and k is specified in column D). This is an efficient way of minimizing the number of computations required for a smooth semi-log plot, which can be important for solutions that require more computational time. Thus, the formula =10^($F$2+($G$2-$F$2)*(D6-1)/($H$2-1)) was entered in cell E6 and dragged downward. However, programs in Function.xls do not require that t be calculated in this way.

As suggested by the equations given herein, all input and output variables used in Function.xls are in dimensionless form and are defined at the beginning of each program. Access to these programs is obtained by clicking on **Tools**, **Macro** and **Visual Basic Editor**. Four different modules appear on the left side of your screen in the Visual Basic Editor, and you can view all of the programs referred to herein by double clicking on the module **Hydraulics**. (If these four modules do not appear on the left side of your screen, then click on **View** and **Project Explorer** on the toolbar at the top of the page in the Visual Basic Editor.)

	A	B	C	D	E	F	G	H
1	Q (m^3/min)	r (m)	L (m)	T (m^2/min)	S	a	b	n
2	2.295	296	1	1.65	0.00004	0	3	30
3	The Theis solution for flow to a well.							
4	Measured Values				Calculated Values			
5	t (min)	s (m)		k	t (min)	s (m)		
6	2	0.1		1	1	0.0579808		
7	5	0.2		2	1.268961	0.0744265		
8	10	0.27		3	1.610262	0.0926009		
9	15	0.31		4	2.0433597	0.1122653		
10	20	0.34		5	2.5929438	0.1331899		
11	30	0.39		6	3.2903446	0.1551646		
12	50	0.44		7	4.1753189	0.1780039		
13	75	0.48		8	5.2983169	0.2015487		
14	120	0.53		9	6.7233575	0.2256647		
15	195	0.58		10	8.5316785	0.2502407		
16				11	10.826367	0.2751855		
17				12	13.738238	0.3004247		
18				13	17.433288	0.3258984		
19				14	22.122163	0.3515585		
20				15	28.072162	0.3773663		
21				16	35.622479	0.4032913		
22				17	45.203537	0.4293089		
23				18	57.361525	0.4553998		
24				19	72.789538	0.4815485		
25				20	92.367086	0.5077429		
26				21	117.21023	0.5339734		
27				22	148.73521	0.5602324		
28				23	188.73918	0.5865137		
29				24	239.50266	0.6128128		
30				25	303.91954	0.6391259		
31				26	385.66204	0.6654499		
32				27	489.39009	0.6917826		
33				28	621.01694	0.7181222		
34				29	788.04628	0.7444671		
35				30	1000	0.7708163		
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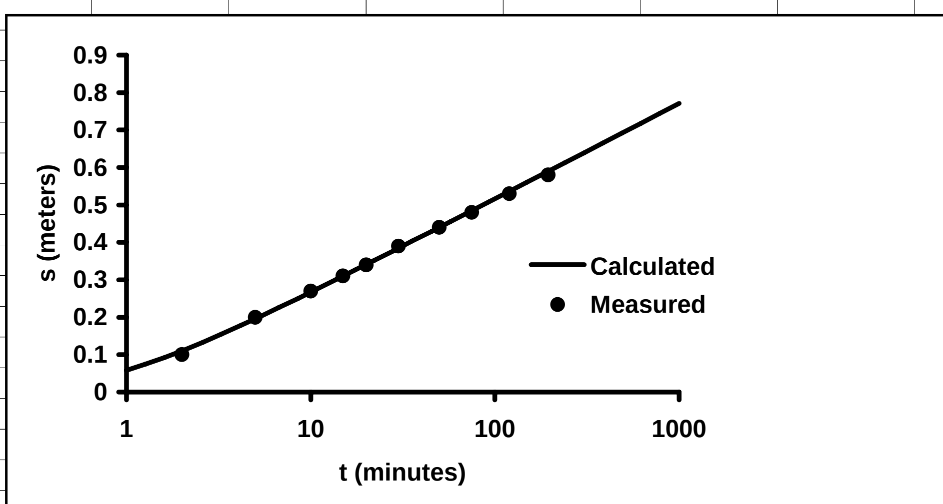


The Hantush Solution

The Hantush (1955) leaky-aquifer solution assumes that the pumped aquifer is bounded on top by a low permeability aquitard beneath a more permeable aquifer containing a standing water table, as shown in Fig. 2. Initially, piezometric levels in the bottom pumped aquifer coincide with the free surface elevation in the top aquifer. Pumping lowers piezometric levels in the bottom aquifer and causes both a vertical piezometric gradient and downward flow through the aquitard. The Hantush solution assumes that the free surface elevation in the top aquifer remains unchanged as the cone of depression in the pumped aquifer expands. Consequently, when total recharge flow through the aquitard equals flow extracted from the well, the cone of depression stops expanding and steady flow occurs.

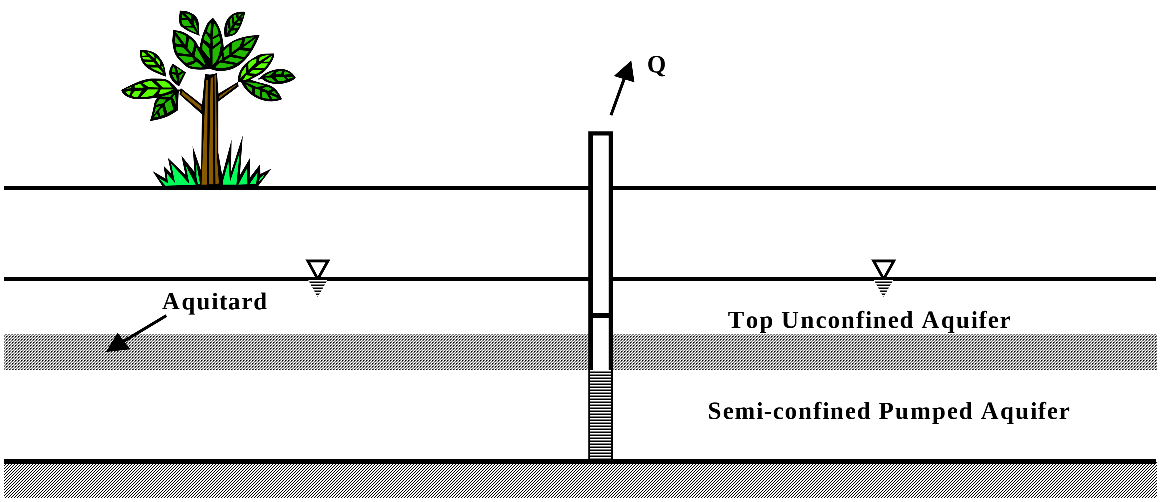


Fig. 2 Aquifer geology for the Hantush leaky-aquifer solution.

The following equations describe flow to a well in a leaky aquifer:

$$\frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)=S\frac{\partial s}{\partial t}+\left(\frac{K'}{B'}\right)s \qquad \left(0<r<\infty,\ 0<t<\infty\right) \tag{13}$$

$$s(r,0)=0 \quad \left(0<r<\infty\right) \tag{14}$$

$$s(\infty,t)=0 \quad \left(0<t<\infty\right) \tag{15}$$

$$Limit\left(r\frac{\partial s}{\partial r}\right)_{r\to 0}=-\frac{Q}{2\pi T} \qquad \left(0<t<\infty\right) \tag{16}$$

where K' and B' are the permeability and thickness, respectively, of the aquitard. The Hantush solution of these equations is

$$\frac{s4\pi T}{Q}=W\left(u,\ r\sqrt{\frac{K'/B'}{T}}\right) \tag{17}$$

where u is defined in Eq. (5). The leaky-aquifer function, $W(u,\ \alpha)$, is given by the value of the following definite integral:

$$W(u,\ \alpha)=\int_u^\infty e^{-x-\frac{\alpha^2}{4x}}\frac{dx}{x} \tag{18}$$

The steady-flow solution is found by letting $t\to\infty$ in this solution to obtain

$$s(r,\ \infty)=\frac{Q}{4\pi T}W\left(0,\ r\sqrt{\frac{K'/B'}{T}}\right) \tag{19}$$

where right side is a multiple of the zero-order, modified Bessel function of the second kind, $K_0(x)$.

$$W\left(0,\ r\sqrt{\frac{K'/B'}{T}}\right)=2K_0\left(r\sqrt{\frac{K'/B'}{T}}\right) \tag{20}$$

In most applications the dimensionless ratio $r\sqrt{\frac{K'/B'}{T}}$ is very small, and the steady-flow solution has the asymptotic behavior

$$s(r,\ \infty)\sim\frac{Q}{2\pi T}\ln\left(\frac{1}{r\sqrt{\frac{K'/B'}{T}}}\right)\qquad\left(r\sqrt{\frac{K'/B'}{T}}\to 0\right) \tag{21}$$

The leaky-aquifer solution has three aquifer parameters, T, S and K'/B', and this makes the calculation of these parameters from pumping test data relatively difficult. Numerous methods of analysis are given in the literature, but probably the most accurate and easiest method fits Eq. (19) to either measured or estimated steady-flow drawdowns to obtain values for T and K'/B'. This requires steady-flow drawdown estimates from at least two, and preferably more than two, observation wells at different values of r from the pumped well. The storage coefficient, S, is found by fitting Eq. (17) to the unsteady portion of drawdown curves after T and K'/B' have been found.

An example of a steady-flow spreadsheet analysis for a leaky aquifer is shown on page 8. The flow, Q, and measured drawdowns, s, were entered on the sheet, and guessed values for T and S were entered in cells M2 and N2. Function.xls computes the Hantush leaky-aquifer solution in the form

$$\frac{sT}{Q}=W\_2\left(\frac{r}{L},\frac{tT}{SL^2},\frac{(K'/B')L^2}{T}\right) \tag{22}$$

where L is an arbitrarily chosen length that is usually chosen as one metre when units of metre are used in a problem. Thus, the following formula was entered in cell F8 and dragged down to cell F37:

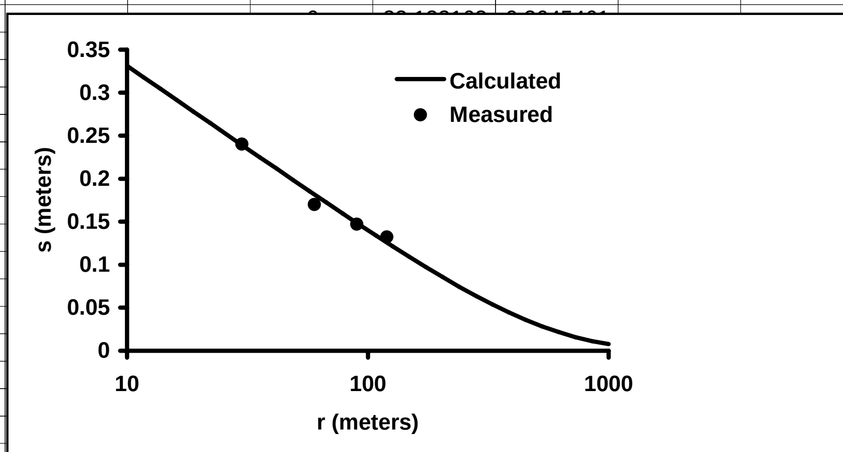
$$=(\$A\$2/\$C\$2)*W\_2(E8,\$F\$2*\$C\$2/\$D\$2,\$E\$2/\$C\$2) \tag{23}$$

Eq. (23) has used L = 1, a value for S in cell D2 that has been guessed and the value for t in cell F2 that has been increased until drawdowns stop changing. (In the end, it only matters that t / S is chosen to be a large enough number to give the steady-flow solution.)

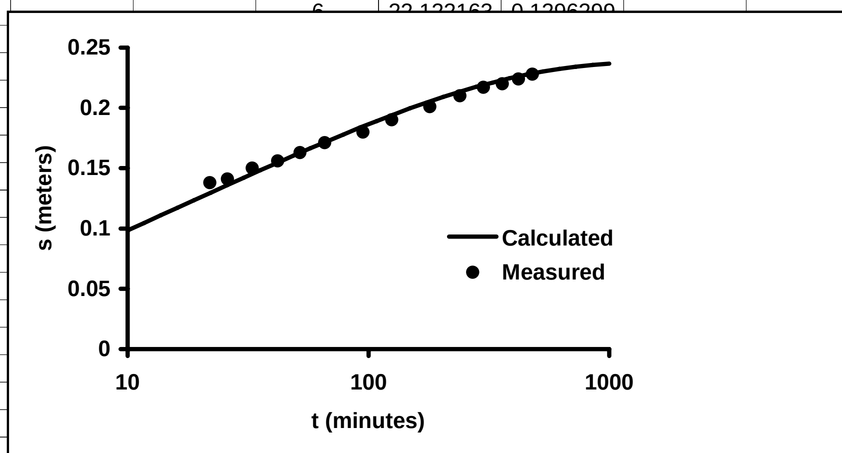
As suggested by Eq. (21), the slope of the straight-line portion of the calculated curve was adjusted by changing T, and the calculated curve was moved normal to itself by changing K'/B' until the calculated curve provided a satisfactory fit to the measured data.

Once T and L were calculated from the steady-flow analysis, then the unsteady analysis shown on page 9 was used to determine S. Data for this example was taken from Kruseman and DeRidder (1979).

	A	B	C	D	E	F	G	H	I
1	Q (m^3/min)	L (m)	T (m^2/min)	S	K'/B' (min^{-1})	t (min)	a	b	n
2	0.52848	1	1	0.0001	4.80E-06	10000	1	3	30
3									
4	Steady-flow drawdown analysis for a leaky aquifer								
5									
6	Measured Values				Calculated Values				
7	r (m)	s (m)		k	r (m)	s (m)			
8	30	0.24		1	10	0.3311744			
9	60	0.17		2	11.721023	0.3178342			
10	90	0.147		3	13.738238	0.3044993			
11	120	0.132		4	16.10262	0.2911714			
12				5	18.873918	0.2778528			
13				[illegible]	[illegible]	[illegible]			
14									
15									
16									
17									
18									
19									
20									
21									
22									
23									
24									
25									
26									
27									
28									
29				[illegible]	[illegible]	[illegible]			
30				23	329.03446	0.0537509			
31				24	385.66204	0.0444281			
32				25	452.03537	0.0359045			
33				26	529.83169	0.0282677			
34				27	621.01694	0.0215893			
35				28	727.89538	0.0159158			
36				29	853.16785	0.0112595			
37				30	1000	0.0075916			
38									



	A	B	C	D	E	F	G	H	I
1	Q (m^3/min)	r (m)	L (m)	T (m^2/min)	S	K'/B' (min^{-1})	a	b	n
2	0.52848	30	1	1	0.0025	4.80E-06	1	3	30
3									
4	Unsteady drawdown analysis for a leaky aquifer								
5									
6	Measured Values					Calculated Values			
7	t (min)	s (m)			k	t (min)	s (m)		
8	22	0.138			1	10	0.0984375		
9	26	0.141			2	11.721023	0.1046468		
10	33	0.15			3	13.738238	0.1108813		
11	42	0.156			4	16.10262	0.1171303		
12	52	0.163			5	18.873918	0.1233833		
13	66	0.171			6	22.122162	0.1296299		
14	95	0.18							
15	125	0.19							
16	180	0.201							
17	240	0.21							
18	300	0.217							
19	360	0.22							
20	420	0.224							
21	480	0.228							
22									
23									
24									
25									
26									
27									
28									
29									
30					23	329.03446	0.2211501		
31					24	385.66204	0.224514		
32					25	452.03537	0.2275052		
33					26	529.83169	0.2301118		
34					27	621.01694	0.2323299		
35					28	727.89538	0.2341658		
36					29	853.16785	0.2356367		
37					30	1000	0.2367713		
38									



The Boulton Solution

Boulton (1963) obtained a solution for a problem that has become known as delayed-yield flow to a well. In a later publication, Boulton (1973) showed that this type of aquifer response can occur when a pumped aquifer is bounded on top with an aquitard containing a shallow standing water table, as shown in Fig. 3. Immediately after well abstraction begins the pumped aquifer behaves as a confined aquifer. At intermediate times, however, water starts to move downward through the aquitard to recharge the pumped aquifer, and during this period the aquifer response is described closely with the Hantush solution for flow to a well in a leaky aquifer. Unlike the Hantush solution, though, the Boulton solution allows the free surface in the aquitard to move downward as water is drained from the aquitard into the pumped aquifer. Thus, at larger times piezometric levels in the aquitard and in the pumped aquifer approach each other, and the aquifer response becomes similar to the response predicted with the Theis solution for an unconfined aquifer with a storage coefficient equal to the effective porosity of the aquitard.

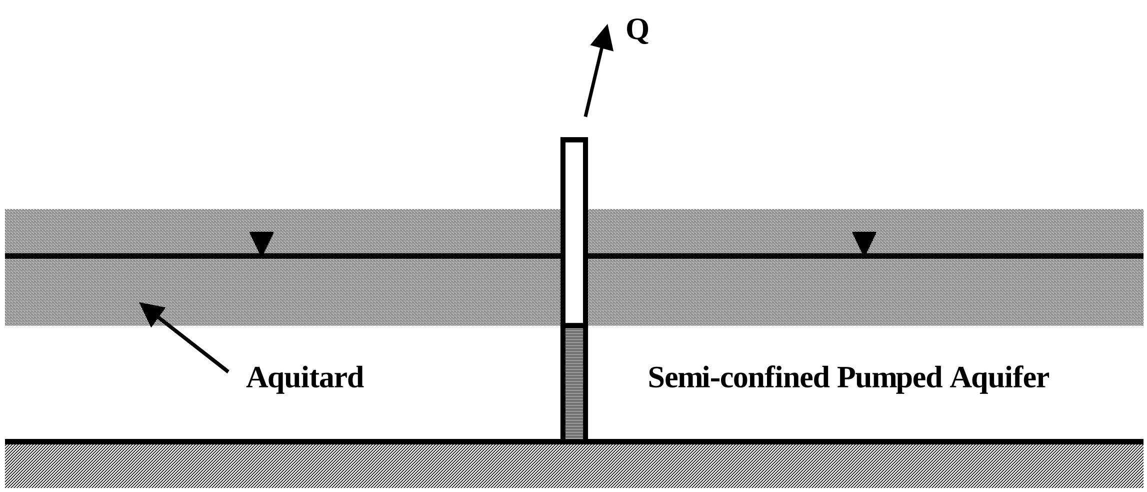


Fig.3 Aquifer geology for delayed-yield flow to a well.

Hunt (2003a) has shown that drawdowns in the pumped aquifer and aquitard are described by the following equations:

$$\frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)=S\frac{\partial s}{\partial t}+\left(\frac{K'}{B'}\right)(s-\eta) \quad (0<r<\infty,\ 0<t<\infty) \tag{24}$$

$$\sigma\frac{\partial \eta}{\partial t}+\left(\frac{K'}{B'}\right)(\eta-s)=0 \quad (0<r<\infty,\ 0<t<\infty) \tag{25}$$

$$s(r,\ 0)=\eta(r,\ 0)=0 \quad (0<r<\infty) \tag{26}$$

$$s(\infty, t) = 0 \qquad (0 < t < \infty) \tag{27}$$

$$Limit\left(r\frac{\partial s}{\partial r} \right)_{r \to 0} = -\frac{Q}{2\pi T} \qquad (0 < t < \infty) \tag{28}$$

where s and η = drawdowns in the pumped aquifer and aquitard, respectively, K' and B' = aquitard permeability and saturated thickness, respectively, S = pumped aquifer storage coefficient (storativity) and σ = aquitard effective porosity (specific yield). Solutions for s and η are calculated by Function.xls with user-defined functions written in the following dimensionless variables:

$$\frac{sT}{Q} = W\_3\left(\frac{r}{L}, \frac{tT}{SL^2}, \frac{S}{\sigma}, \frac{(K'/B')L^2}{T} \right) \tag{29}$$

$$\frac{\eta T}{Q} = Eta\_3\left(\frac{r}{L}, \frac{tT}{SL^2}, \frac{S}{\sigma}, \frac{(K'/B')L^2}{T} \right) \tag{30}$$

where L is a length that may be chosen arbitrarily. For example, L may be set equal to one meter if units of meter are used in the problem.

Hunt and Scott (2005) have shown that the Boulton solution also applies when the pumped aquifer is bounded on the top and bottom by any number of aquifer and aquitard layers provided that the bottom layer has an impermeable bottom boundary, the top layer contains a standing water table, no unpumped layer has a transmisssivity that exceeds five percent of the pumped layer transmissivity and the elastic storage coefficients of the unpumped layers are much smaller than the porosity or specific yield of the top unconfined layer. An example is shown in Fig. 4. In this case the value of K'/B' becomes an "effective" value that is given by

$$(K'/B')_{effective} = \frac{1}{\sum_{i=1}^{n} (B'/K')_i} \tag{31}$$

where $(B'/K')_i$ is the ratio of thickness to permeability for each of the n layers above the pumped aquifer. In practice, the right side of Eq.(31) is fixed by the layer or layers with the smallest value of B'/K'.

When a large number of layers exist above the pumped aquifer, it is probably better to consider this entire region as an anisotropic aquifer with a ratio of vertical to horizontal permeability that is very much less than one. Eqs.(24)-(25) still apply in this case, provided that K'/B' is replaced with its effective value given by Eq.(31). An effective value of the vertical permeability, which isn't needed for use in Eqs.(29)-(30), can be calculated from

$$K_{effective} = (K'/B')_{effective} \sum_{i=1}^{n} B'_i \tag{32}$$

In applications it may not be possible to recognize any distinct layering from well logs.

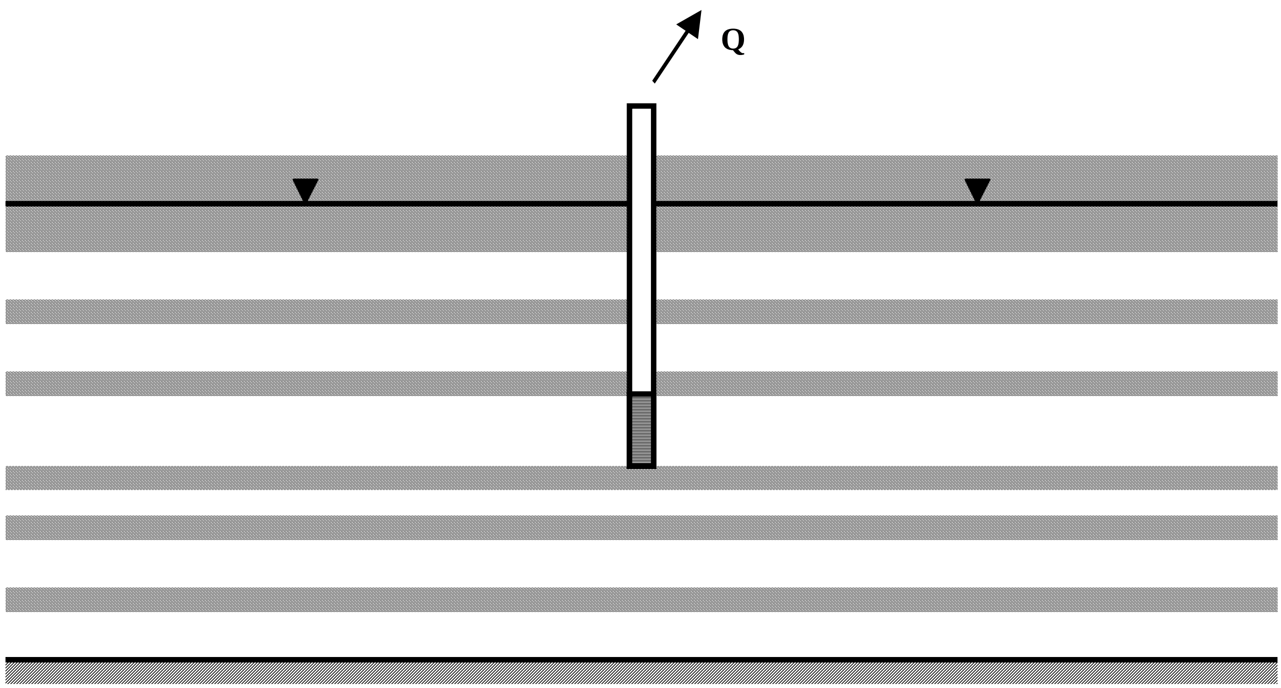


Fig. 4 An extension of the Boulton solution to describe flow to a well in a multi-layer sytem.

This extension suggests that the Boulton solution is a generalization of the Hantush leaky-aquifer concept in which leakage into the pumped aquifer is limited by free surface drawdowns in the top layer. Thus, steady flow that is reached in the Hantush solution is in reality an illusion that may appear to occur at intermediate pumping times but will disappear at larger times. Many, if not most, pumping tests in leaky aquifers probably last an insufficient time to show a departure from this pseudo steady-flow condition. An example of this is shown in Fig. 5, where drawdowns for the pumped aquifer and the free surface in the top layer are plotted for the example shown on page 9 by using an assumed value for the effective porosity of $\sigma = 0.1$ in the top layer. The value used for σ has a significant influence upon the time at which drawdowns start to depart from the pseudo steady-flow value predicted with the Hantush solution. In this case, drawdowns start to depart from pseudo steady-flow after almost seven days from the start of pumping, but the test lasted only eight hours. Drawdowns in the pumped aquifer and the top layer approach similar values after about 69 days from the start of pumping.

Some pumping tests do last long enough to show the departure from pseudo steady flow. For an example, the spreadsheet on page 13 plots and analyzes some of the data used by Boulton (1963) in his original paper on delayed-yield flow to a well. This pumping test was carried out by W.C. Walton in the U.S.A. and lasted two days. Calculations were carried out by using trial and error to adjust values for T, S, K'/B' and σ in the program W_3 until the calculated curve provided a good fit for the measured data. Generally, T controls the slope of the straight-line asymptotes at both small and large times (both of these slopes are identical in the Boulton solution), changing S translates the straight-line asymptote at small times normal to itself, changing σ translates the straight-line asymptote at large times normal to itself and changing K'/B' moves the pseudo steady-flow portion of the drawdown curve in either the upward or downward direction.

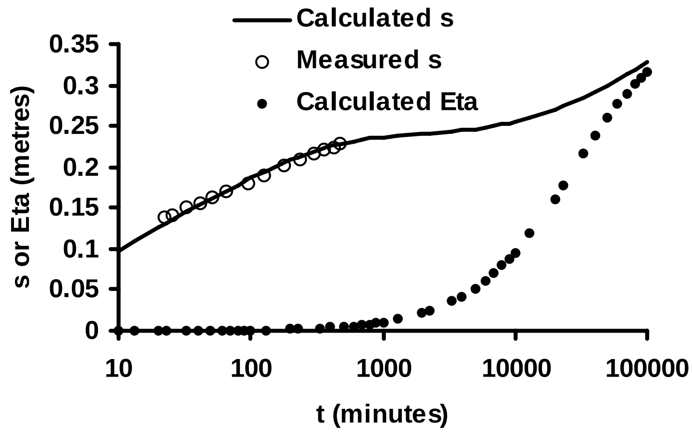
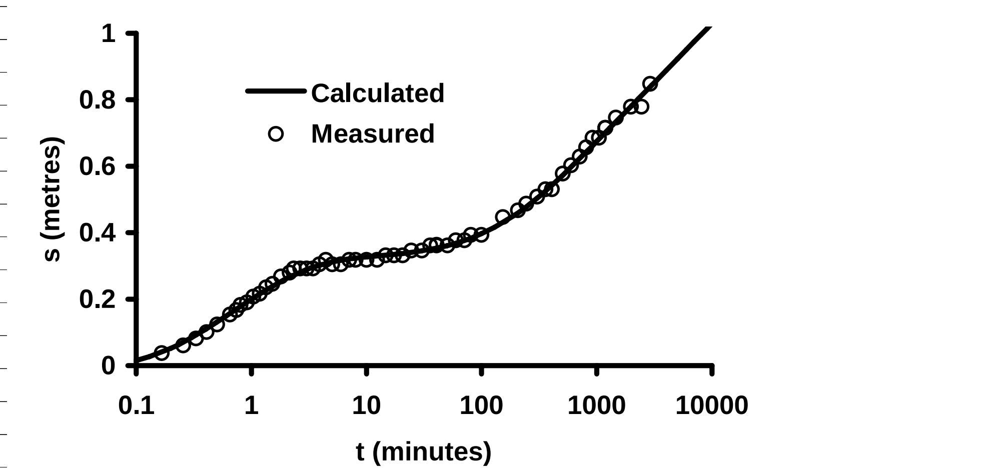


Fig.5 Calculated and measured drawdowns for delayed-yield flow to a well for the data shown on page 9 with σ = 0.1.

Q (m^3/min)	r (m)	T (m^2/min)	S	K'/B' (min^{-1})	Sigma	L (m)	
4.083	22.25	2.1	0.0025	0.0008	0.12	1	
Data used by Boulton for a delayed-yield analysis.							
Measured Values			Calculated Values				
t (min)	s (m)		t (min)	s (m)			
0.2	0.038		0.1	0.0157304			
0.3	0.060		0.13	0.02648765			
0.3	0.081		0.2	0.05182895			
0.4	0.101		0.23	0.06198371			
0.5	0.124		0.3	0.08352839			
0.7	0.154		0.33	0.09187619			
0.7	0.168		0.4	0.10953072			
0.8	0.183		0.5	0.13103999			
7.1	0.318		9	0.32550737			
8.1	0.318		10	0.32709385			
10.0	0.318		13	0.33065505			
12.4	0.318		20	0.3369582			



The Stream-depletion Problem

Abstracting water from a well beside a stream also depletes water from the stream. In fact, if pumping continues for a long enough time, and if the stream continues to flow, then flow depleted from the stream ultimately equals flow abstracted by the well. However, at smaller values of time well abstraction exceeds stream depletion. Thus, an ability to predict stream depletion as a function of time for any given well abstraction can allow pumping schedules to be devised that will control the amount of environmental damage done to the stream.

Hunt (2003a) has obtained a mathematical solution for the geology that is pictured in Fig. 6. The pumped well is a distance L from a long straight stream that extends to plus and minus infinity. The stream width is allowed to approach zero in the solution, which suggests that the pumped well and all observation wells should be at least ten multiples of the stream width, b, from the nearest stream edge. The aquifer is a delayed-yield aquifer in which the stream partially penetrates the top aquitard. Thus, the solution can also be expected to hold for a more general multi-layer aquifer like the one discussed in the previous section.

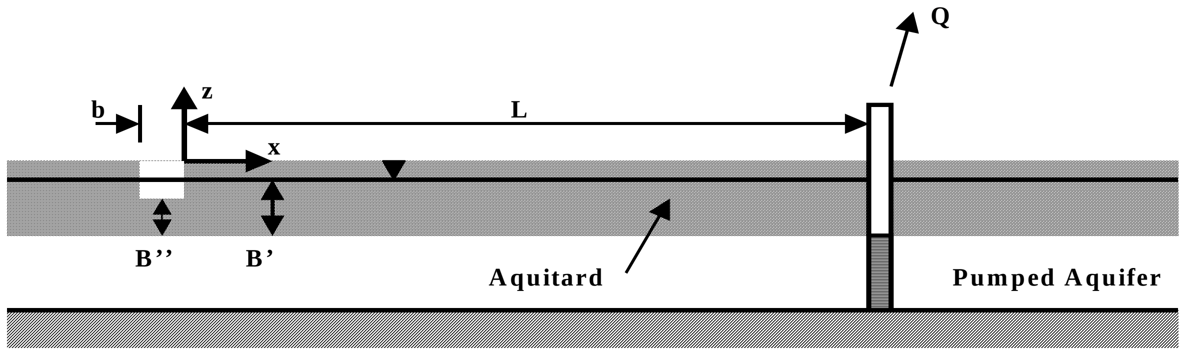


Fig. 6 Geology for the stream depletion problem.

The problem shown in Fig. 6 is described by the solution of the following problem:

$$T\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}\right)=S\frac{\partial s}{\partial t}+\left(\frac{K'}{B'}\right)(s-\eta) \qquad (-\infty<x<\infty,\ -\infty<y<\infty,\ 0<t<\infty) \tag{33}$$

$$\sigma\frac{\partial \eta}{\partial t}+\left(\frac{K'}{B'}\right)(\eta-s)=0 \qquad (-\infty<x<\infty,\ -\infty<y<\infty,\ 0<t<\infty) \tag{34}$$

$$Limit\left(R\frac{\partial s}{\partial R}\right)_{R\to 0}=-\frac{Q}{2\pi T} \qquad \left(R=\sqrt{(x-L)^2+y^2},\ 0<t<\infty\right) \tag{35}$$

$$\left(T\frac{\partial s}{\partial x}\right)_{x=0+}-\left(T\frac{\partial s}{\partial x}\right)_{x=0-}=\lambda s(0,\ y,\ t) \qquad (-\infty<y<\infty,\ 0<t<\infty) \tag{36}$$

$$(\mathrm{s})_{\mathrm{x}=0+}-(\mathrm{s})_{\mathrm{x}=0-}=0 \tag{37}$$

$$s(x,\ y,\ 0)=\eta(x,\ y,\ 0)=0 \qquad (-\infty<x<\infty,\ -\infty<y<\infty) \tag{38}$$

$$Limit(s)_{R\to\infty}=0 \qquad (0<t<\infty) \tag{39}$$

where s = drawdown in the pumped aquifer, η = drawdown of the free surface in the aquitard and λ is a streambed leakage parameter defined by

$$\lambda=K''\frac{b}{B''} \tag{40}$$

where B'' denotes the aquitard thickness beneath the streambed, which may differ from the aquitard saturated thickness, B', at points away from the stream. Eqs. (33) - (34) are the partial differential equations that describe flow in a delayed-yield aquifer, where K'/B' is replaced with an equivalent value given by Eq. (31) if more than one layer exists above the pumped aquifer. Eq. (35) requires that flow seeping into the well equal the well abstraction, and Eqs. (36) and (40) require that the difference in aquifer flow (per unit stream length) along the two stream sides equal the flow (per unit stream length) that has seeped vertically downward through the aquitard from the stream. Eq. (37) requires that drawdowns in the pumped aquifer be continuous under the stream, Eq. (38) requires that drawdowns vanish at t = 0 in both the aquitard and pumped aquifer and Eq. (39) requires that drawdowns in the pumped aquifer vanish at infinity.

If Eqs. (33) - (39) are applied to the more general layered aquifer discussed in the previous section, then it is important to note that not only must K'/B' be replaced with the equivalent value given by Eq. (31) but λ must also be replaced with an effective value given by

$$\lambda_{\mathrm{effective}} = \frac{b}{\sum_{i=1}^{n-1}\left(B'/K'\right) + B''/K'} \tag{41}$$

Thus, increasing the number of layers above the pumped aquifer reduces the values of both K'/B' and λ. In this case, the top layer **must** be an aquitard that contains both the free surface and the partially penetrating stream. If this is not the case, then lateral piezometric gradients in the top layer may cause significant horizontal outflow from the stream, and Eq.(36) neglects this horizontal outflow.

A user-defined function in Function.xls computes drawdowns in the aquifer with

$$\frac{sT}{Q} = W\_4\left(\frac{x}{L}, \frac{y}{L}, \frac{tT}{SL^2}, \frac{\lambda L}{T}, \frac{\left(K'/B'\right)L^2}{T}, \frac{S}{\sigma}\right) \tag{42}$$

Free-surface drawdowns in the aquitard are computed with

$$\frac{\eta T}{Q} = Eta\_4\left(\frac{x}{L}, \frac{y}{L}, \frac{tT}{SL^2}, \frac{\lambda L}{T}, \frac{\left(K'/B'\right)L^2}{T}, \frac{S}{\sigma}\right) \tag{43}$$

Flow depleted from the stream, ΔQ, can be computed by using

$$\frac{\Delta Q}{Q} = Q\_4\left(\frac{tT}{SL^2}, \frac{\left(K'/B'\right)L^2}{T}, \frac{S}{\sigma}, \frac{\lambda L}{T}\right) \tag{44}$$

Typical dimensionless plots computed from Eqs. (42) and (44) are shown in Figs. 7 and 8. Points to note include the reverse in curvature that occurs for all curves that describe a delayed-yield aquifer response and the horizontal asymptote, indicating steady flow, that occurs for both drawdown and stream depletion. The asymptotic value of one for stream depletion indicates that any well pumped for a sufficiently long period of time eventually obtains all of its flow from the adjacent stream, and a horizontal asymptote for the drawdown curves occurs because the stream is assumed in the mathematical solution to contain an infinite amount of water that can be recharged to the aquifer. A horizontal asymptote in the drawdown solution distinguishes the stream-depletion solution from the Boulton solution for flow to a well in an aquifer of infinite horizontal extent, which does not have a similar steady-flow asymptote. Fig. 9 shows the result of fitting these mathematical solutions to some field data obtained by Weir (1999).

Strangely enough, this solution for a stream of zero width can also be used to model flow to a well beside a stream whose width extends from $x = 0$ to $x = -\infty$ simply by increasing $\lambda L / T$ until the solution stops changing. This can be thought of as the result of letting $b \to \infty$ in Eq.(40), and it may require setting $\lambda L / T = 10,000$ or more. This follows from the fact that all of the equations that describe flow for negative values of x become homogeneous in the limit as $\lambda \to \infty$. Thus, taking this limit gives the solution $s = \eta = 0$ when $-\infty < x < 0$ and $-\infty < y < \infty$.

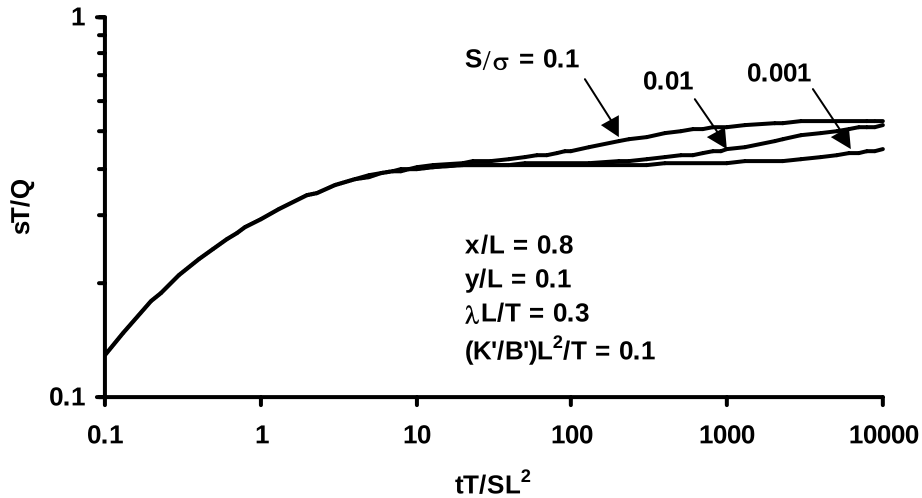


Fig. 7 Typical drawdowns calculated from Eq.(42)**.**

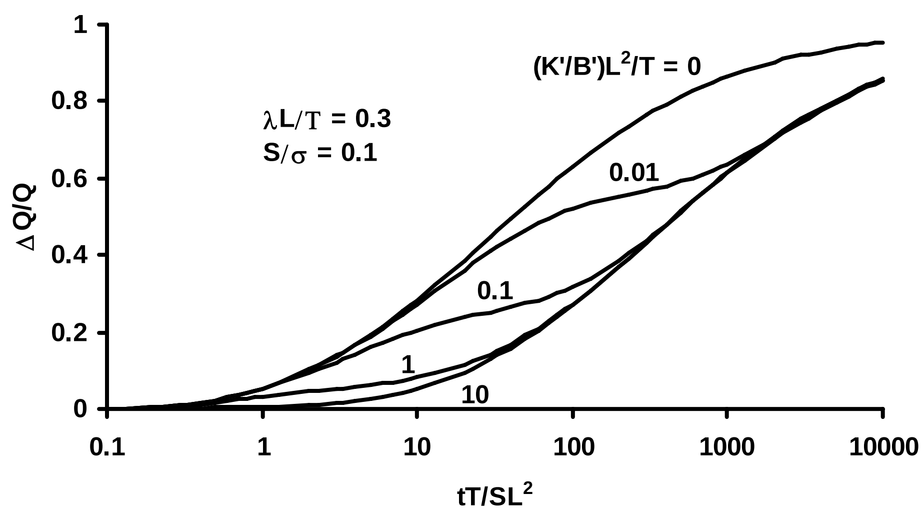


Fig. 8 Typical stream-depletions calculated from Eq. (44)**.**

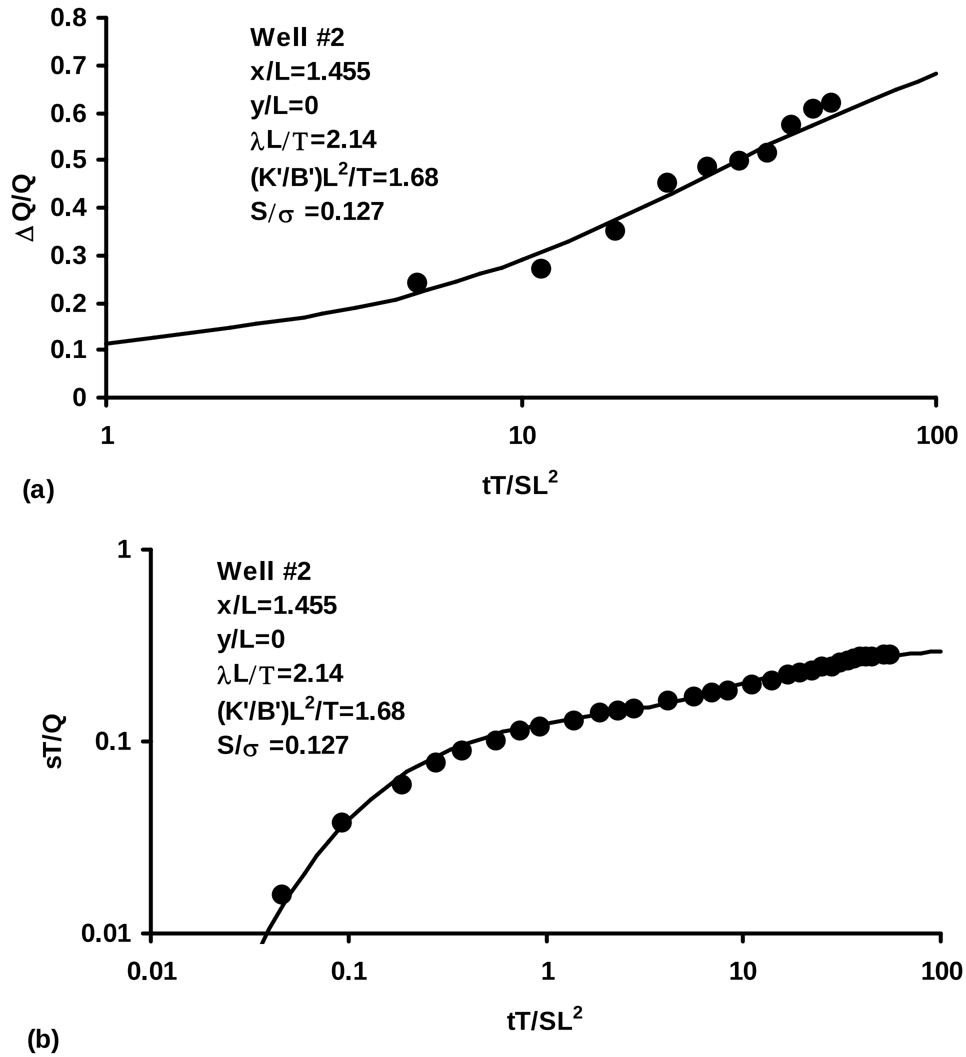


Fig. 9 Dimensionless plots of (a) stream depletion and (b) drawdown fitted to field data obtained by Weir (1999).

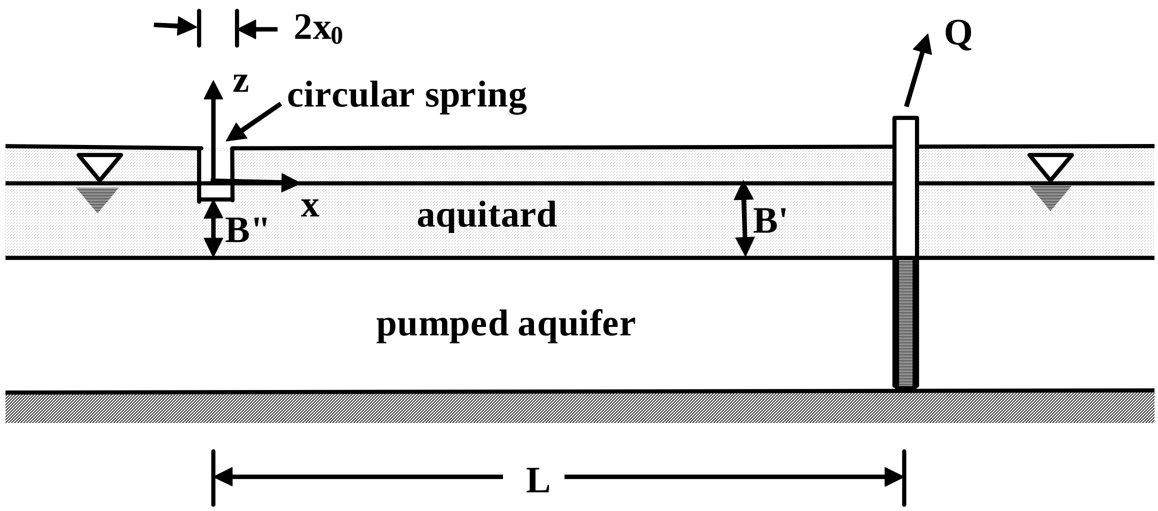


Fig. 10 Pumping from a well beside a spring causing spring depletion.

Spring Depletion

Pumping from a well beside a spring decreases flow to the spring in a process known as spring depletion. A definition sketch for this problem is shown above in Fig. 10. The first solution for this problem was given by Hunt (2004). An improved solution and an analysis of some field data have been published by Hunt and Smith (2008).

The pumped aquifer has a thickness B, a transmissivity T and an elastic storage coefficient S, and the abstraction well has the coordinates $(x, y) = (L, 0)$ in the horizontal (x, y) plane. An overlying aquitard contains a free surface and has a saturated thickness B' and a vertical hydraulic conductivity K'. This is the same aquifer-aquitard system that was described by the Boulton (1954, 1963) solution for delayed flow to a well, as shown by Boulton (1973), Cooley and Case (1973) and Hunt (2003). Hunt and Scott (2005) showed that the equations that describe this aquifer system can also be used when the overlying aquitard is replaced with any number of aquitard layers provided that the transmissivity of each layer does not exceed about 5 % of the pumped aquifer transmissivity and the elastic storage coefficient of the pumped aquifer is much less than the specific yield (effective porosity) near the free surface in the aquitard containing the free surface. The spring is circular with a radius x_0, and it will be assumed that $x_0 << L$. The ratio of vertical hydraulic conductivity to aquitard thickness directly beneath the spring is K''/B'', and this ratio can be expected to be much larger than the corresponding ratio for the aquitard at points away from the spring $\left(\text{i.e. } K''/B'' >> K'/B'\right)$. If flow in the pumped aquifer is assumed to be entirely horizontal (the Dupuit approximation), and if flow in the aquitard is assumed to be entirely vertical $\left(\text{i.e. } K' << K, \text{where } K = \text{horizontal hydraulic conductivity of the pumped aquifer}\right)$, then this problem is described by the following set of equations:

$$T\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}\right)=S\frac{\partial s}{\partial t}+\left(\frac{K'}{B'}\right)(s-\eta)+Q_d(t)\delta(x)\delta(y)-Q\delta(x-L)\delta(y) \quad (45)$$

$$\sigma\frac{\partial \eta}{\partial t}+\left(\frac{K'}{B'}\right)(\eta-s)=0 \quad (46)$$

$$\lim_{r\to 0} s(x,y,t)=0 \qquad \left(r=\sqrt{x^2+y^2}\right) \quad (47)$$

$$s(x,y,0)=\eta(x,y,0)=0 \quad (48)$$

where s = drawdown in the pumped aquifer, η = free surface drawdown in the aquitard, x and y = horizontal coordinates, t = time, T = pumped aquifer transmissivity, S = elastic storage coefficient of the pumped aquifer, $K'/B' =$ ratio of vertical conductivity to saturated thickness for the aquitard, $Q_d(t) =$ flow depleted from the spring, Q = constant flow abstracted from the pumped well and $\delta(x) =$ Dirac's delta function. Values of x, y and t lie within the ranges $-\infty < x < \infty$, $-\infty < y < \infty$ and $0 < t < \infty$, respectively.

Eq. (45) results from combining Darcy's law with an equation of continuity. The first term on the right side of Eq. (45) allows water to be released from elastic storage within the pumped aquifer, the second term allows water from the overlying aquitard to recharge the pumped aquifer, the third term models the spring as a vertical line source with a time varying recharge rate and the last term models the abstraction well as a vertical line sink with a constant abstraction rate. Eq. (46) equates the vertical velocity of a point on the free surface in the aquitard with the vertical velocity of flow within the aquitard. [The volume of water released from elastic storage within the aquitard is regarded as small compared with the volume of water released from storage near the free surface as the free surface draws down, which means that a linear variation of drawdown exists within the aquitard. Therefore, the vertical drawdown gradient within the aquitard is independent of the vertical coordinate and is given by the second term on the left side of Eq.(46).] Eq. (47) requires that drawdown within the pumped aquifer vanish at large distances from the spring, and Eq. (48) requires that free surface drawdowns and pumped aquifer drawdowns vanish everywhere at t = 0. The time derivative of η in Eq. (46) allows an initial condition to be prescribed for η, but no boundary conditions can be prescribed for η because spatial derivatives of η do not appear in any of the governing equations.

It will be assumed that spring depletion is related to drawdown in the pumped aquifer by the following equation:

$$Q_d(t)=\alpha\ T\ s(x_0,0,t) \quad (49)$$

where α = dimensionless spring discharge coefficient and $x_0 =$ effective spring radius. Physical considerations and Darcy's law suggest that α is given by

$$\alpha = \frac{\left(K''/B''\right)\pi x_0^{\;2}}{T} \tag{50}$$

which is the reciprocal of the definition of α given previously by Hunt (2004). However, vertical flow through a spring fissure is not laminar and is not described by Darcy's law, and the effective spring radius never has an obvious value. Therefore, α must be regarded as an experimental constant to be determined by comparisons of calculated values of $s(x,y,t)$ and $Q_d(t)$ with corresponding experimental values rather than by an application of Eq. (50).

Calculations show that solutions of Eqs. (45)- (49) become independent of α for sufficiently large values of α. Inspection of Eq. (49) suggests that this is because $s(x_0,0,t)\to 0$ as $\alpha\to\infty$. In this case Eq. (50) does not give a valid physical interpretation of α when α becomes relatively large.

It has become apparent to the writer during the past several years that the full generality of the equations describing this type of aquifer is not well understood by some research workers and practitioners. In particular, if $K'/B'=0$, Eqs. (45) and (46) reduce to the equations that describe fully confined flow in the pumped aquifer. If $K'/B'=0$ and $S=\sigma$, these equations reduce to the linearized form of the equations that describe unconfined flow in the pumped aquifer. If the limit $\sigma\to\infty$ is taken, these equations reduce to the equations that describe flow in a Hantush-Jacob leaky aquifer. In this case, drawdown in the aquitard is zero for all time, and the aquitard provides an infinite volume of stored water. Then maximum steady flow spring depletion values can be expected to die off exponentially with distance between the spring and the pumped well. It is the authors' opinion, however, that use of this latter limit would amount to professional malpractice since no geologic formation has an infinite amount of storage, and the resulting spring depletion values are likely to be grossly underestimated.

Function computes pumped aquifer drawdowns, spring depletion values and free surface drawdowns within the aquitard by using the following call statements:

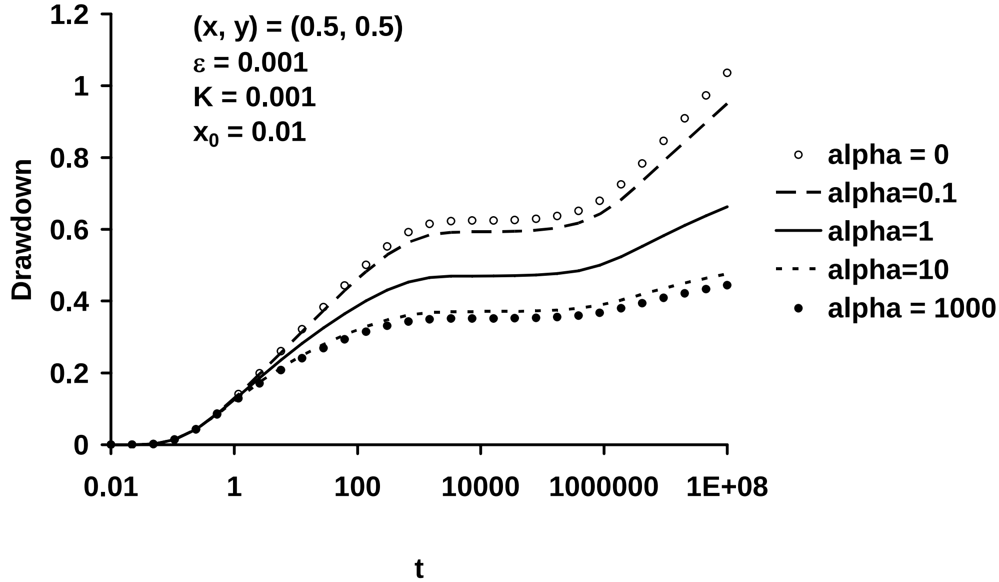
$$\frac{sT}{Q} = W\_5\left(\frac{x}{L},\frac{y}{L},\frac{tT}{SL^2},\frac{S}{\sigma},\frac{\left(K'/B'\right)L^2}{T},\frac{x_0}{L},\alpha\right) \tag{51}$$

$$\frac{Q_d}{Q} = Q\_5\left(\frac{tT}{SL^2},\frac{S}{\sigma},\frac{\left(K'/B'\right)L^2}{T},\frac{x_0}{L},\alpha\right) \tag{52}$$

$$\frac{\eta T}{Q} = Eta\_5\left(\frac{x}{L},\frac{y}{L},\frac{tT}{SL^2},\frac{S}{\sigma},\frac{\left(K'/B'\right)L^2}{T},\frac{x_0}{L},\alpha\right) \tag{53}$$

where terms on the left of the equal sign are omitted when inserting these statements in a spreadsheet cell.

Fig. 11 shows the effect of changing α on pumped aquifer drawdowns and spring depletion values. Since increasing α is seen from Eq. (49) to increase $Q_d(t)$, and since increasing the pumped aquifer recharge can be expected to decrease aquifer drawdowns, we see that drawdown values decrease and spring depletion values increase as α increases. A value of α = 0 gives the Boulton (1963, 1973) drawdown solution for delayed yield flow to a well at (x, y) = (1, 0) and a spring depletion of zero for all time. At the other extreme, a value of $\alpha = \infty$ gives a solution in which $s(x_0, 0, t) = 0$ for $0 \le t < \infty$, as pointed out earlier. It is interesting to notice the very large effect that α has on these solutions as it increases from 0 to about 10. After α exceeds 10, however, there is very little effect upon either drawdown or spring depletion values.



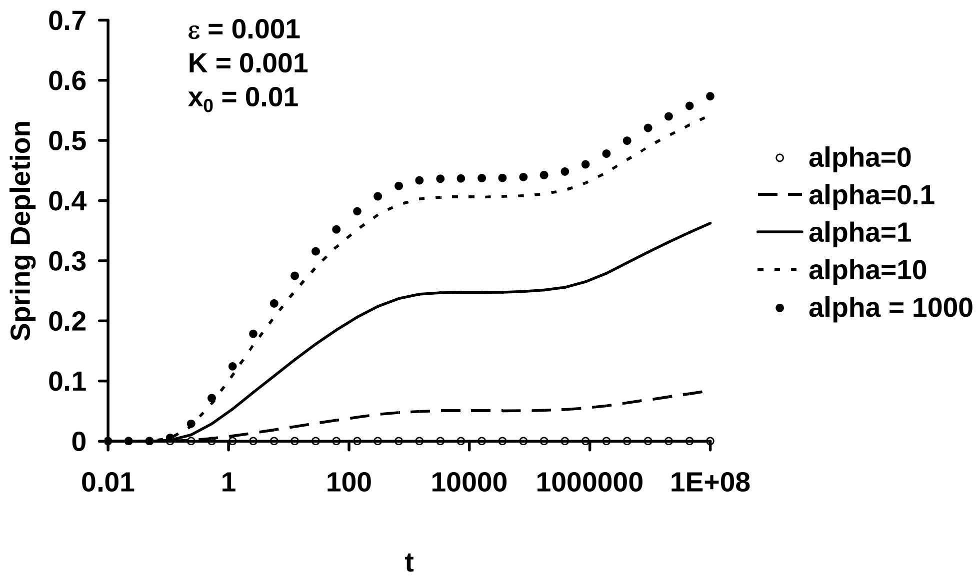


Fig. 11 The effect of changing a upon drawdown and spring depletion values.

The Rise of a Groundwater Mound

Downward seeping water, perhaps as the result of irrigation, enters the free surface of a shallow unconfined aquitard that overlies a more permeable aquifer. I sometimes refer to this type of aquifer as a delayed-yield aquifer since it is the same aquitard-aquifer system that is described by the Boulton solution for delayed-yield flow to a well. The recharge causes a rise in the free surface in the aquitard that is usually described as a groundwater mound. Figure 12 shows a sketch for the start of a groundwater mound. The problem for recharge over a rectangular area with dimensions 2a and 2b in the x and y directions, respectively, is described by the solution of the following equations:

$$T\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)=S\frac{\partial h}{\partial t}+\left(\frac{K'}{B'}\right)(h-\eta) \quad (-\infty<x<\infty,\ -\infty<y<\infty,\ 0<t<\infty) \tag{54}$$

$$\sigma\frac{\partial \eta}{\partial t}+\left(\frac{K'}{B'}\right)(\eta-h)=w \tag{55}$$

$$h(x,y,0)=0 \quad (-\infty<x<\infty,\ -\infty<y<\infty) \tag{56}$$

$$\lim_{(x^2+y^2)\to\infty} h(x,y,t)=0 \quad (0<t<\infty) \tag{57}$$

where h = piezometric head rises in the aquifer, $\eta =$ free surface rise in the aquitard and the vertical recharge per unit area, w, has dimensions of a velocity and is given by

$$\begin{aligned} w(x,y,t)&=R \quad (-a<x<a,\ -b<y<b,\ 0<t<\infty) \\ &=0 \quad (\text{all other values of } x \text{ and } y \text{ for } 0<t<\infty) \end{aligned} \tag{58}$$

where R is a constant. Definitions for the other variables are given in the section describing the Boulton solution. The solution for h is given by the following integral:

$$\begin{aligned} \frac{Sh}{R}=\frac{t}{2}\int_0^1 &\left\{\operatorname{erf}\left[\frac{(a-x)}{2\tau}\sqrt{\frac{S}{Tt}}\right]+\operatorname{erf}\left[\frac{(a+x)}{2\tau}\sqrt{\frac{S}{Tt}}\right]\right\} \\ &\left\{\operatorname{erf}\left[\frac{(b-y)}{2\tau}\sqrt{\frac{S}{Tt}}\right]+\operatorname{erf}\left[\frac{(b+y)}{2\tau}\sqrt{\frac{S}{Tt}}\right]\right\}\tau\, F(\alpha,\beta)\, d\tau \end{aligned} \tag{59}$$

where erf = error function, $\alpha=\varepsilon Kt\left(1-\tau^2\right)$, $\beta=\tau\sqrt{Kt}$ and the function $F(\alpha,\beta)$ is computed from the following formulae:

$$\begin{aligned} F(\alpha,\beta)&=e^{-\left(\sqrt{\alpha}-\beta\right)^2}\sum_{n=1}^{\infty}e^{-2\beta\sqrt{\alpha}}I_n\left(2\beta\sqrt{\alpha}\right)\left(\frac{\sqrt{\alpha}}{\beta}\right)^n \quad \left(\frac{\sqrt{\alpha}}{\beta}\le 1\right) \\ &=\frac{1}{2}\left[1-e^{-2\alpha}I_0(2\alpha)\right] \quad \left(\frac{\sqrt{\alpha}}{\beta}=1\right) \\ &=1-e^{-\left(\sqrt{\alpha}-\beta\right)^2}\sum_{n=0}^{\infty}e^{-2\beta\sqrt{\alpha}}I_n\left(2\beta\sqrt{\alpha}\right)\left(\frac{\beta}{\sqrt{\alpha}}\right)^{-n} \quad \left(\frac{\sqrt{\alpha}}{\beta}\ge 1\right) \end{aligned} \tag{60}$$

where $I_n(B)=$ modified Bessel function of the first kind of order n.

Function.xls computes the piezometric head rise in the in the aquifer in the following dimensionless form:

$$\frac{hT}{RL^2} = W\_6\left(\frac{x}{L}, \frac{y}{L}, \frac{tT}{SL^2}, \frac{S}{\sigma}, \frac{(K'/B')L^2}{T}, \frac{a}{L}, \frac{b}{L}\right) \tag{61}$$

where L is a length that may be chosen in any way. Water table rises in the aquitard are computed with the following routine:

$$\frac{\eta T}{RL^2} = Eta\_6\left(\frac{x}{L}, \frac{y}{L}, \frac{tT}{SL^2}, \frac{S}{\sigma}, \frac{(K'/B')L^2}{T}, \frac{a}{L}, \frac{b}{L}\right) \tag{62}$$

An example of the use of Eq. (61) is shown in Fig. 13.

The growth of a groundwater mound when an overlying aquitard is absent can be computed from Eqs. (61) by setting S/σ equal to a very large number. Start by setting $S/\sigma = 10$, and then increase this ratio to 100, 1000 etc. until the solution stops changing. At this point, the solution will no longer depend upon K'/B', and S in the dimensionless term tT/SL^2 must be interpreted as the aquifer porosity, σ. Also, the principles of time translation and superposition can be used with all of the solutions discussed in this section to obtain solutions in which w in Eq. (58) changes discontinuously with time but is constant over the time intervals between these discontinuities.

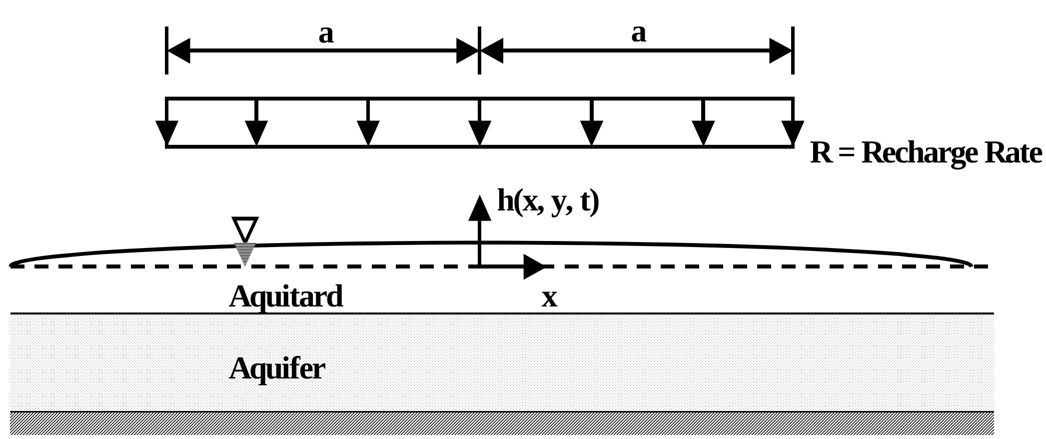


Fig. 12 Vertical recharge to an unconfined aquifer.

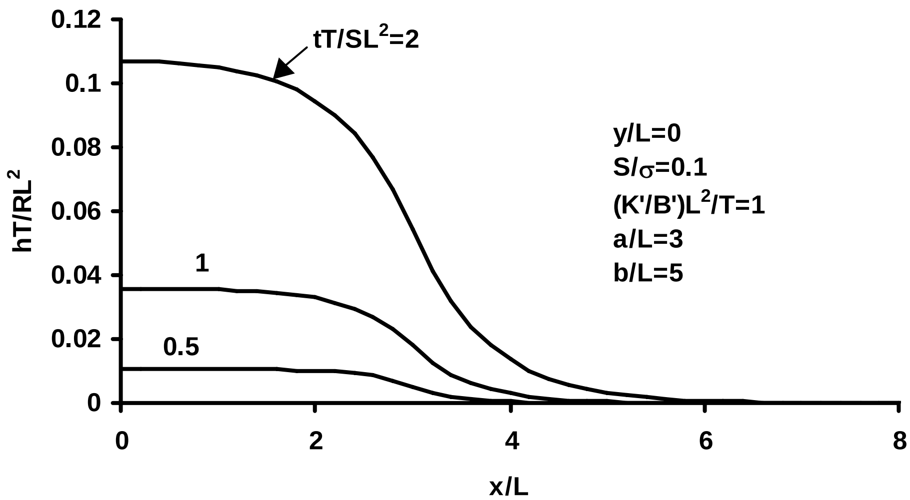


Fig. 13 Head rises in an aquifer that result from recharge to an overlying aquitard.

The aquifer head rise for recharge over a circular area with an overlying aquitard present has the following solution:

$$\frac{T\ h(r,t)}{R\ L^2} = 2t\int_0^1 \tau\ F(\alpha_1,\beta_1)\ F(\alpha,\beta)\ d\tau \tag{63}$$

where r = radial coordinate, a = radius of the recharge area, $\alpha_1 = a^2/4t\tau^2$, $\beta_1 = r/2\tau\sqrt{t}$ and α and β are as defined for Eq. (59). Function.xls computes aquifer head rises from Eq. (63) as follows:

$$\frac{hT}{RL^2} = W\_7\left(\frac{r}{L},\frac{tT}{SL^2},\frac{S}{\sigma},\frac{(K'/B')L^2}{T},\frac{a}{L}\right) \tag{64}$$

Figure 14 shows aquifer heads plotted from Eq. (64). The growth of a groundwater mound when an overlying aquitard is absent can also be computed from Eqs. (64) by setting S/σ equal to a very large number.

The free surface rise, η, for recharge over a circular area can be computed with Function.xls using the following routine:

$$\frac{\eta T}{RL^2} = Eta\_7\left(\frac{r}{L},\frac{tT}{SL^2},\frac{S}{\sigma},\frac{(K'/B')L^2}{T},\frac{a}{L}\right) \tag{65}$$

Figure 15 shows water table rises computed with Eq. (65) for the same example considered in Fig. 14. The discontinuity at the edge of the recharge area is the result of neglecting horizontal derivatives in the partial differential equations for the aquitard. Including these derivatives would lead to a solution, if it could be calculated, in which η changes in a very rapid but continuous manner across a boundary-layer region. This boundary-layer thickness is small enough to disregard for values of t that satisfy

$$\sqrt{\frac{tT'}{\sigma a^2}} \leq 0.1 \tag{66}$$

where T' = K'B' = aquitard transmissivity and a = recharge zone radius or minimum half width.

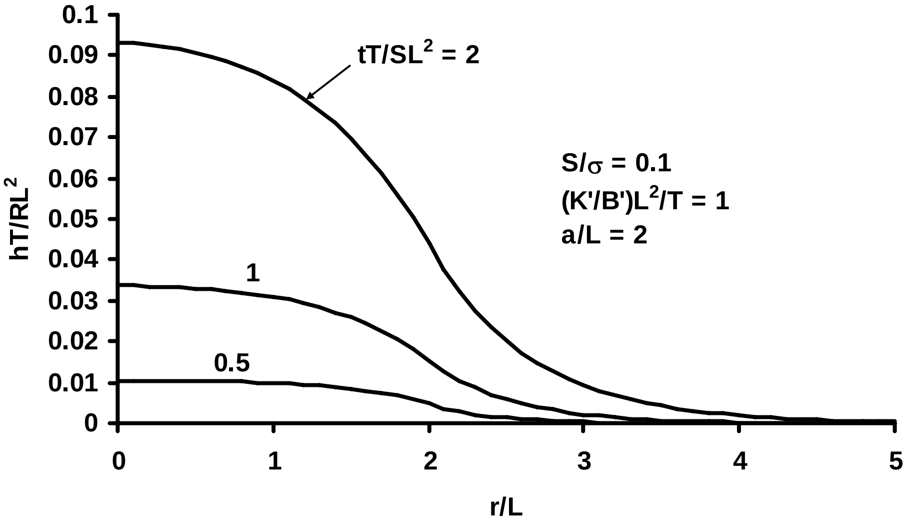


Fig. 14 Aquifer piezometric head rises computed for recharge over a circular area.

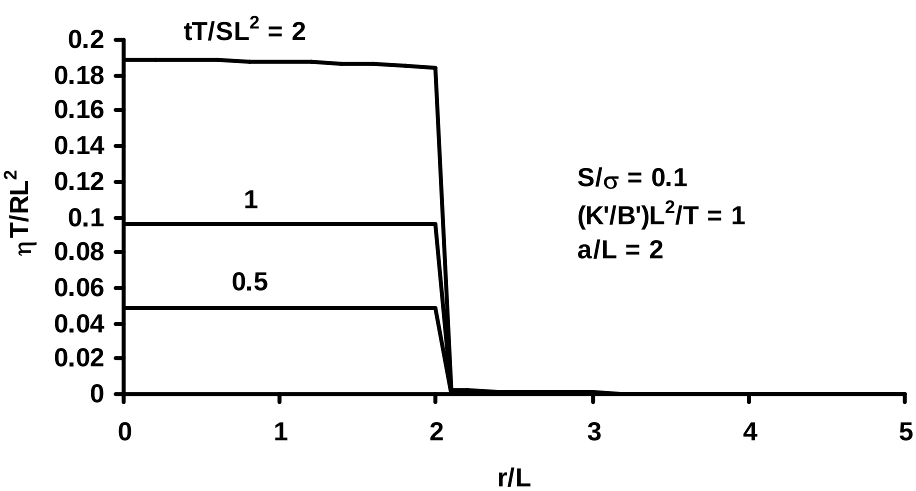


Fig. 15 Water table rises in an aquitard that result from recharge over a circular area.

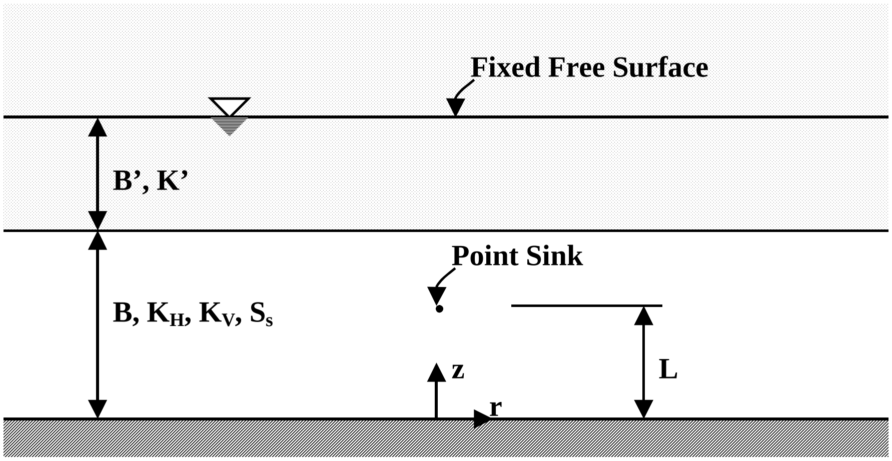


Fig. 16 Flow to a point sink in an anisotropic leaky aquifer.

Flow to Vertical and Non-vertical Wells

Sometimes it is necessary to model flow to either partially penetrating vertical wells or horizontal or slanted wells with finite length. These flows can all be modeled by first considering flow to the continuous point sink shown in Fig. 16. The aquifer is assumed to be homogeneous, anisotropic and elastic, and free surface drawdowns in the top layer, an aquitard, are negligible. [Consequently, this solution applies only under two sets of circumstances: (1) when the free surface occurs in a reservoir of water underlain by an aquitard, in which case the reservoir serves as an infinite recharge source for the pumped aquifer, or, if the free surface occurs within an overlying aquitard, as shown in Fig.16, (2) during only the first portion of a longer term period of well abstraction before significant free surface drawdowns start to occur. For problems in which significant free surface drawdowns occur, the reader should consult a later section entitled "Flow to Vertical, Inclined and Horizontal Wells in Unconfined and Semi-Confined Aquifers".]Hunt (2005) describes this problem with the following equations:

$$\frac{K_H}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)+K_V\frac{\partial^2 s}{\partial z^2}=S_s\frac{\partial s}{\partial t} \qquad \left(0<r<\infty,\ 0<z<B,\ 0<t<\infty\right) \tag{67}$$

$$\lim_{r\to 0}\left(r\frac{\partial s}{\partial r}\right)=-\frac{Q}{2\pi K_H}\delta\left(z-L\right) \qquad \left(0<z<B,\ 0<t<\infty\right) \tag{68}$$

$$s\left(\infty,z,t\right)=0 \qquad \left(0<z<B,\ 0<t<\infty\right) \tag{69}$$

$$K_V\frac{\partial s\left(r,B,t\right)}{\partial z}+\left(K'/B'\right)s\left(r,B,t\right)=0 \qquad \left(0<r<\infty,\ 0<t<\infty\right) \tag{70}$$

$$\frac{\partial s\left(r,0,t\right)}{\partial z}=0 \qquad \left(0<r<\infty,\ 0<t<\infty\right) \tag{71}$$

$$s(r,z,0)=0 \qquad (0<r<\infty,\ 0<z<B) \tag{72}$$

where s = drawdown, K_H and K_V = horizontal and vertical hydraulic conductivities, respectively, r = radial coordinate, z = vertical coordinate, t = time, S_s = specific storage, Q = flow to the sink, δ = Dirac's delta function, B = aquifer thickness, K' = vertical aquitard permeability and B' = aquitard saturated thickness. The same problem formulation can also be obtained by placing either an aquifer or a reservoir with a standing water table on top of the aquitard, as shown in Fig. 2. In this case B' becomes the aquitard thickness. Eq.(67) is the partial differential equation that describes flow with both horizontal and vertical velocity components in an aquifer that is compressible and anisotropic with principal directions for the permeability tensor in the horizontal and vertical directions. Eq.(68) requires a constant flow to the point sink, and Eq.(69) requires drawdowns to vanish at large distances from the sink Eqs.(70) and (71) require the top and bottom aquifer boundaries to be leaky and impermeable, respectively, and Eq.(72) requires that drawdowns be zero when pumping first starts. The linearity of these equations has allowed the problem to be formulated in terms of the drawdown or change in piezometric head from any existing head distribution, and the calculated drawdown can, in principle, always be superimposed upon head changes created in the future by other influences, such as abstractions by other wells.

The problem statement and solution can be simplified by introducing the following dimensionless variables:

$$(s^*,r^*,z^*,t^*,L^*,\delta^*,K^*,\lambda)=\left(\frac{sK_HB}{Q},\frac{r}{B},\frac{z}{B},\frac{tK_H}{S_sB^2},\frac{L}{B},B\delta,\frac{K_V}{K_H},\frac{K'/B'}{K_V/B}\right) \tag{73}$$

Then the solution for the point sink can be written as follows:

$$s(r,z,t)=\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\cos(\alpha_n L)\cos(\alpha_n z)}{1+\frac{\sin(2\alpha_n)}{2\alpha_n}}\,W\left(\frac{r^2}{4t},\alpha_n r\sqrt{K}\right) \tag{74}$$

where the asterisk superscript has been omitted for notational convenience, W(x,y) = the Hantush leaky-aquifer well function and α_n = the nth root of the following equation:

$$\alpha_n \tan(\alpha_n)=\lambda \tag{75}$$

When no leakage occurs, $\lambda=0$ and $\alpha_n=(n-1)\pi$.

Flow to a well that starts at (x_0,y_0,z_0) and terminates at (x_M,y_M,z_M) can be approximated by dividing the well up into M segments of equal length and placing a sink at the midpoint of each segment. If Q is now interpreted as the total flow to the well, and if each segment abstracts the same flow, then this solution is obtained from Eq.(74) in the following form:

$$s(x,y,z,t)=\frac{1}{2\pi M}\sum_{i=1}^{M}\sum_{n=1}^{\infty}\frac{\cos(\alpha_n z_{i/2})\cos(\alpha_n z)}{1+\frac{\sin(2\alpha_n)}{2\alpha_n}}\,W\left(\frac{r_{i/2}^{\ 2}}{4t},\alpha_n r_{i/2}\sqrt{K}\right) \tag{76}$$

where $r_{i/2}$ is distance between the point (x,y,z) and the midpoint of the ith element.

Function.xls computes drawdowns from Eq.(76) by using the following user-defined function:

$$\frac{sK_H B}{Q} = W\_8\left(\frac{x}{B},\frac{y}{B},\frac{z}{B},\frac{tK_H}{S_s B^2},\frac{K_V}{K_H},\frac{K'/B'}{K_V/B},\frac{x_0}{B},\frac{y_0}{B},\frac{z_0}{B},\frac{x_M}{B},\frac{y_M}{B},\frac{z_M}{B},M,\text{nterms}\right) \quad (77)$$

where nterms = the number of terms retained in the infinite series in Eq.(76). Thus, a user can increase both M and nterms until the drawdown, s, stops changing. In practice, a value of 10 or so is usually sufficient for M, and nterms doesn't usually exceed 10 or 20.

Flow to a number of wells can be modeled by using a spreadsheet to add drawdowns from each well. If Q in Eq.(77) is the total flow to all wells, then the drawdown contribution from each well must be weighted by the portion of total flow that is contributed by that well. If the flow per unit screen length is identical for all wells, then the weighting factor for each well is the ratio of the screen length to the sum of all screen lengths. Figs.17 and 18 show dimensionless plots of drawdowns for flow to a horizontal well, and Fig.19 shows drawdown contours for two horizontal wells crossed at right angles.

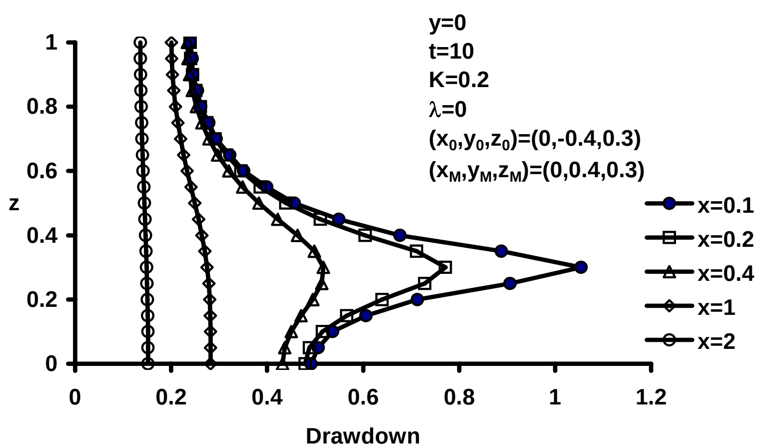


Fig.17 Drawdowns in the vertical plane y = 0 for flow to a horizontal well at z = 0.3 in an anisotropic aquifer.

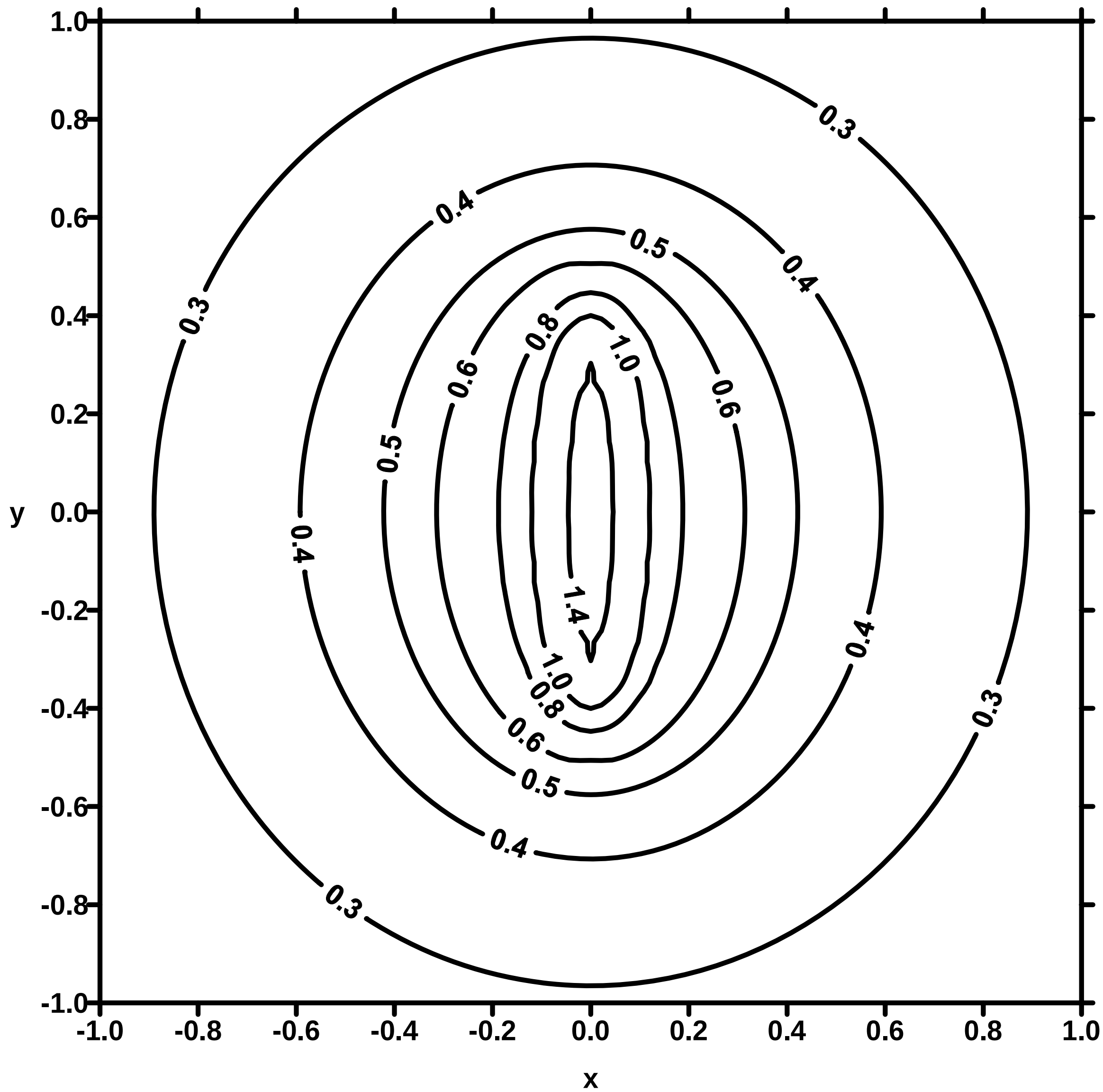


Fig.18 Drawdown contours in the horizontal plane of the well, z = 0.3, for the problem considered in Fig.17.

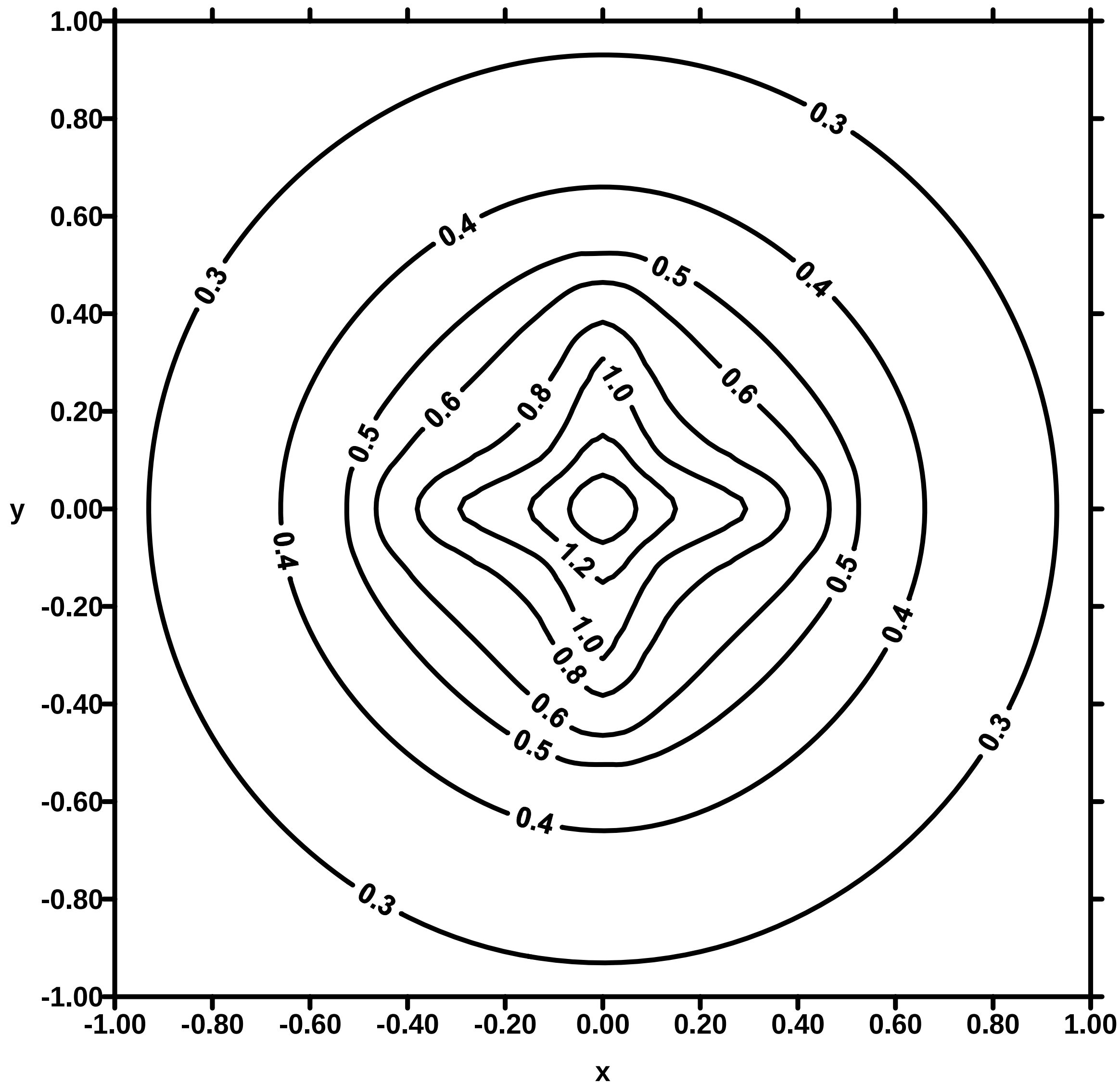


Fig.19 Drawdown contours in the plane z = 0.3 for two horizontal wells crossed at right angles with the same set of parameters shown in Fig.17.

In environmental applications it is often important to track the position of fluid particles as a function of time. This is done by integrating the following equation:

$$\sigma \frac{d\bar{r}}{dt} = K_H \left(\hat{i} \frac{\partial s}{\partial x} + \hat{j} \frac{\partial s}{\partial y} \right) + K_V \hat{k} \frac{\partial s}{\partial z} \tag{78}$$

where $\sigma =$ porosity, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} =$ position vector and $\hat{i}$, $\hat{j}$ and $\hat{k} =$ unit base vectors in the x, y and z directions, respectively. Eq.(73) can be used to rewrite Eq.(78) in the following dimensionless form:

$$\frac{d\bar{r}}{dt} = Q\left(\hat{i}\frac{\partial s}{\partial x} + \hat{j}\frac{\partial s}{\partial y} + \hat{k}\,K\frac{\partial s}{\partial z} \right) \tag{79}$$

where Q = dimensionless flow rate defined by

$$Q^* = \frac{QS_s}{\sigma K_H B} \tag{80}$$

Function.xls computes dx/dt with the following user-defined function:

$$\frac{dx/dt}{K_H/(S_s B)} = \text{dxdt}\left(\frac{x}{B}, \frac{y}{B}, \frac{z}{B}, \frac{tK_H}{S_s B^2}, \frac{K_V}{K_H}, \frac{QS_s}{\sigma K_H B}, \frac{K'/B'}{K_V/B}, \frac{x_0}{B}, \frac{y_0}{B}, \frac{z_0}{B}, \frac{x_M}{B}, \frac{y_M}{B}, \frac{z_M}{B}, M, \text{nterms} \right) \tag{81}$$

in which all dimensionless variables have been rewritten in dimensional form. Similar programs exist for computing dy/dt and dz/dt.

The three scalar differential equations given in Eq.(79) must be integrated simultaneously. A second-order Runge-Kutta method is recommended, in which a first approximation, x^1, for the particle location at the end of the time step is given by

$$x^1(t + \Delta t) = x(t) + \frac{dx(t)}{dt}\,\Delta t \tag{82}$$

Then x^1 is used to obtain an approximation for dx/dt at the end of the time step, and the trapezoidal rule gives the final value for x at the end of the time step:

$$x(t+\Delta t) = x(t) + \frac{\Delta t}{2}\left(\frac{dx(t)}{dt} + \frac{dx(t+\Delta t)}{dt} \right) \tag{83}$$

Care must be taken in this calculation to ensure that Δt is small enough to give a stable solution.

The spreadsheet shown below was used to track fluid particles for the problem considered in Figs.17 and 18 for an isotropic aquifer (K=1).

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	K	Q	λ	M	nterms	time step	x_0	y_0	z_0	x_M	y_M	z_M	
2	1	0.1	0	10	20	0.1	0	-0.4	0.3	0	0.4	0.3	
3													
4	t	x	y	z	dx/dt	dy/dt	dz/dt	x	y	z	dx/dt	dy/dt	dz/dt
5	0	1	0	0.1	0	0	0	1	0	0.1	-0.001819	-4.88E-20	-6.25E-05
6	0.1	0.999909	-2.44E-21	0.0999969	-0.00182	-1.17E-20	-6.25E-05	0.999727	-3.61E-21	0.0999906	-0.005493	-3.8E-20	-0.000284
7	0.2	0.9995433	-4.92E-21	0.0999796	-0.005497	-3.58E-20	-0.000284	0.9989936	-8.5E-21	0.0999512	-0.007373	-1.07E-20	-0.000395
8	0.3	0.9988998	-7.25E-21	0.0999456	-0.007374	-1.19E-19	-0.000395	0.9981624	-1.91E-20	0.0999061	-0.008477	-2E-19	-0.000437

Cells B5:D5 contain the particle initial coordinates, columns E-G contain derivatives calculated from values in columns A-D, columns H-J contain the first approximations for x, y and z at the end of the time step, and columns K-M contain derivatives at the end of the time step calculated from columns A and H-J. Columns B-D below row 5 contain final values for x, y and z that have been calculated with the trapezoidal rule by using values for x, y and z and their derivatives in the previous row. The entire process can be expedited by making careful use of absolute and relative addressing so that formulas can be duplicated and modified in both horizontal and vertical directions. An example showing the position of two fluid particles as a function of time is shown in Fig.20 for this particular problem.

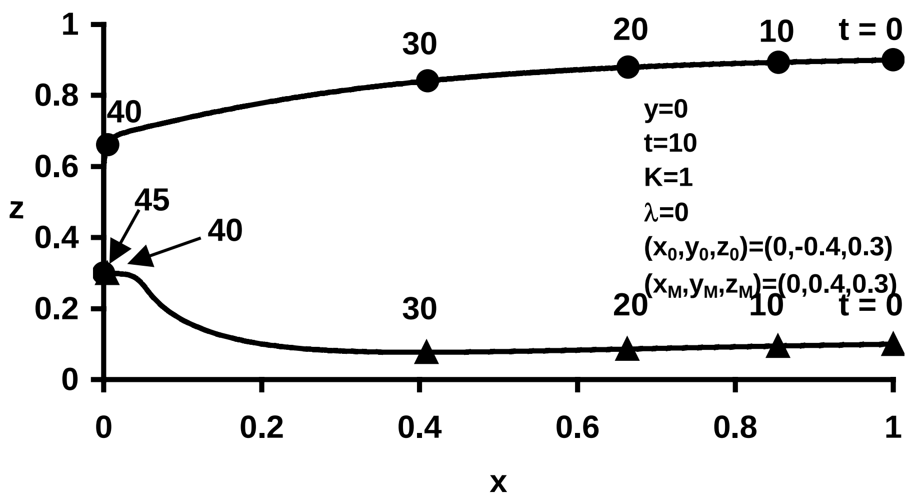


Fig.20 Particle positions in the vertical plane y = 0 for the problem considered in Figs.17 and 18. (Q = 0.1, K = 1)

Contaminant Transport Solutions

The module **Dispersion** contains programs for computing contaminant concentrations in uniform groundwater flow from various types of contaminant sources when the dispersion coefficients are constant. For example, dispersion from an instantaneous three-dimensional point source in an infinite aquifer is described by the following set of equations:

$$D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} + D_z \frac{\partial^2 c}{\partial z^2} = u \frac{\partial c}{\partial x} + R \frac{\partial c}{\partial t} + R\lambda c \quad (-\infty < x < \infty,\ -\infty < y < \infty,\ -\infty < z < \infty,\ 0 < t < \infty) \tag{84}$$

$$c(x,y,z,0) = \frac{M_3}{\sigma} \delta(x)\delta(y)\delta(z) \quad \left(-\infty < x < \infty,\ -\infty < y < \infty,\ -\infty < z < \infty\right) \tag{85}$$

$$\lim_{r\to\infty} c(x,y,z,t) = 0 \quad \left(r = \sqrt{x^2 + y^2 + z^2}\right) \tag{86}$$

where c = concentration in units of mass per unit volume of water, D_x, D_y, D_z = dispersion coefficients in the x, y and z directions, respectively, u = pore velocity in the x direction (the specific discharge divided by the aquifer porosity, σ), R = retardation factor (dimensionless), $\lambda =$ decay constant and $M_3 =$ mass of contaminant released instantaneously at t = 0. Dirac's delta function is denoted by $\delta(x)$.

It is convenient to introduce the following dimensionless variables:

$$\left(x', y', z', t', D_x', D_y', D_z', \lambda'\right) = \left(\frac{x}{L}, \frac{y}{L}, \frac{z}{L}, \frac{tu}{RL}, \frac{D_x}{uL}, \frac{D_y}{uL}, \frac{D_z}{uL}, \frac{\lambda RL}{u}\right) \tag{87}$$

where a prime denotes a dimensionless variable and L = a length scale that may be chosen arbitrarily. For example, L may be chosen to be one metre if metres are used for the length unit. The prime will be omitted for notational convenience throughout the remainder of this section.

The solution for the problem described by Eqs.(84)-(87) is given by

$$c = \frac{e^{-\left[\lambda t + \frac{(x-t)^2}{4D_x t} + \frac{y^2}{4D_y t} + \frac{z^2}{4D_z t}\right]}}{\sqrt{(4\pi t)^3 D_x D_y D_z}} \qquad \left(c' = \frac{c\sigma L^3}{M_3}\right) \tag{88}$$

This solution the value of c is computed in a cell with the call

$$= c\_3\left(x, y, z, t, D_x, D_y, D_z, \lambda\right) \tag{89}$$

The solution for an instantaneous source in two dimensions [i.e. c=c(x,y,t)]can be found by distributing three-dimensional sources along the entire z axis. This solution is given by

$$c=\frac{e^{-\left[\lambda t+\frac{(x-t)^2}{4D_x t}+\frac{y^2}{4D_y t}\right]}}{\sqrt{(4\pi t)^2 D_x D_y}} \qquad \left(c'=\frac{c\sigma L^2}{M_2}\right) \tag{90}$$

where $M_2 =$ mass per unit distance along the z axis. The call to this program follows:

$$=c\_2(x,y,t,D_x,D_y,\lambda) \tag{91}$$

A one-dimensional solution is obtained by distributing instantaneous sources over the entire y-z plane. This gives

$$c=\frac{e^{-\left[\lambda t+\frac{(x-t)^2}{4D_x t}\right]}}{\sqrt{(4\pi t)D_x}} \qquad \left(c'=\frac{c\sigma L}{M_1}\right) \tag{92}$$

and is computed with Function.xls with the call

$$=c\_1(x,t,D_x,\lambda) \tag{93}$$

where $M_1 =$ mass per unit area in the y-z plane.

Solutions for continuous sources can be obtained by distributing instantaneous sources over time. In other words, a constant mass flow rate is injected from t = 0 to the value of t at which the solution is calculated. This gives the following results for one-, two- and three-dimensional continuous sources:

$$\begin{gathered} c=\frac{e^{\frac{x}{2D_x}}}{4b\sqrt{D_x}}\left[e^{-ab}\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}-b\sqrt{t}\right)-e^{ab}\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}+b\sqrt{t}\right)\right] \\ \left(c'=\frac{c\sigma u}{RdM_1/dt}, a=\frac{|x|}{\sqrt{D_x}}, b=\sqrt{\lambda+\frac{1}{4D_x}}\right) \end{gathered} \tag{94}$$

$$c=\frac{e^{\frac{x}{2D_x}}}{4\pi\sqrt{D_x D_y}}W\left(\frac{a^2}{4t},ab\right) \qquad \left(c'=\frac{c\sigma uL}{RdM_2/dt}, a=\sqrt{\frac{x^2}{D_x}+\frac{y^2}{D_y}}, b=\sqrt{\lambda+\frac{1}{4D_x}}\right) \tag{95}$$

$$\begin{gathered} c=\frac{e^{\frac{x}{2D_x}}}{8\pi a\sqrt{D_x D_y D_z}}\left[e^{-ab}\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}-b\sqrt{t}\right)+e^{ab}\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}+b\sqrt{t}\right)\right] \\ \left(c'=\frac{c\sigma uL^2}{RdM_3/dt}, a=\sqrt{\frac{x^2}{D_x}+\frac{y^2}{D_y}+\frac{z^2}{D_z}}, b=\sqrt{\lambda+\frac{1}{4D_x}}\right) \end{gathered} \tag{96}$$

where erfc = complementary error function and W(x,y) = Hantush leaky-aquifer well function. The time derivative dM_i/dt is the mass flow rate of the contaminant M_i. These solutions are computed with the following calls:

$$=c\_4(x,t,D_x,\lambda) \tag{97}$$

$$= c\_6\left(x, y, t, D_x, D_y, \lambda\right) \tag{98}$$

$$= c\_7\left(x, y, z, t, D_x, D_y, D_z, \lambda\right) \tag{99}$$

There is another one-dimensional solution for a continuous source that is often used in laboratory experiments in which the concentration is kept constant at one end of a semi-infinite solution domain. This solution is described by the following equations:

$$D_x \frac{\partial^2 c}{\partial x^2} = u\frac{\partial c}{\partial x} + R\frac{\partial c}{\partial t} + \lambda Rc \qquad \left(0 < x < \infty, 0 < t < \infty\right) \tag{100}$$

$$c\left(0, t\right) = c_0 \qquad \left(0 < t < \infty\right) \tag{101}$$

$$c\left(\infty, t\right) = 0 \qquad \left(0 < t < \infty\right) \tag{102}$$

$$c\left(x, 0\right) = 0 \qquad \left(0 < x < \infty\right) \tag{103}$$

The solution of this problem can be written in the following dimensionless variables:

$$\left(c', x', t', \lambda'\right) = \left(\frac{c}{c_0}, \frac{xu}{D_x}, \frac{tu^2}{RD_x}, \frac{\lambda RD_x}{u^2}\right) \tag{104}$$

This solution follows:

$$c = \frac{e^{x/2}}{2}\left[e^{x\sqrt{\lambda+1/4}}\operatorname{erfc}\left(\frac{x}{2\sqrt{t}} + \sqrt{t\left(\lambda+1/4\right)}\right) + e^{-x\sqrt{\lambda+1/4}}\operatorname{erfc}\left(\frac{x}{2\sqrt{t}} - \sqrt{t\left(\lambda+1/4\right)}\right)\right] \tag{105}$$

where all primes have been omitted for notational convenience. The call to this program is

$$= c\_5\left(x, t, \lambda\right) \tag{106}$$

Superposition and time translation can always be used with this solution to obtain the solution for a contaminant pulse.

The solutions for sources can also be distributed with respect to their spatial coordinates. Therefore, routines have been added to Function.xls that distribute both instantaneous and continuous sources along straight lines with any specified orientation and length. Calls for line distributions of instantaneous sources follow:

$$= c\_1\_Line\left(x, t, D_x, \lambda, x_0, x_M, M\right) \tag{107}$$

$$= c\_2\_Line\left(x, y, t, D_x, D_y, \lambda, x_0, y_0, x_M, y_M, M\right) \tag{108}$$

$$= c\_3\_Line\left(x, y, z, t, D_x, D_y, D_z, \lambda, x_0, y_0, z_0, x_M, y_M, z_M, M\right) \tag{109}$$

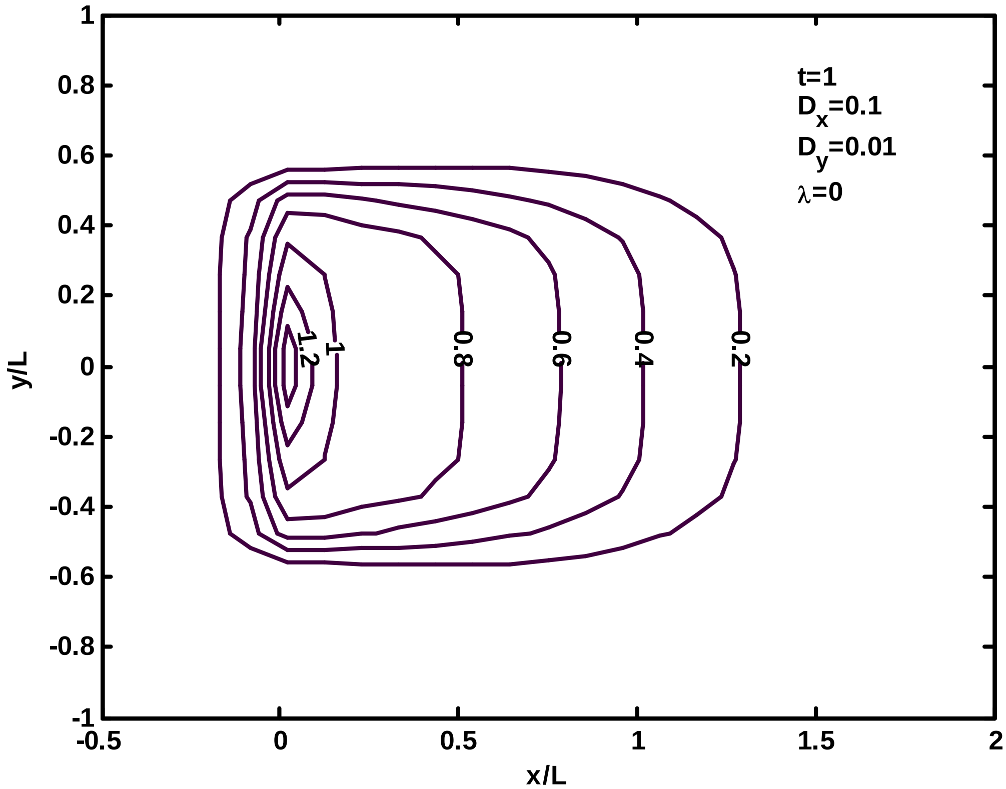
Calls for line distributions of continuous sources follow:

$$= c\_4\_Line\left(x, t, D_x, \lambda, x_0, x_M, M\right) \tag{110}$$

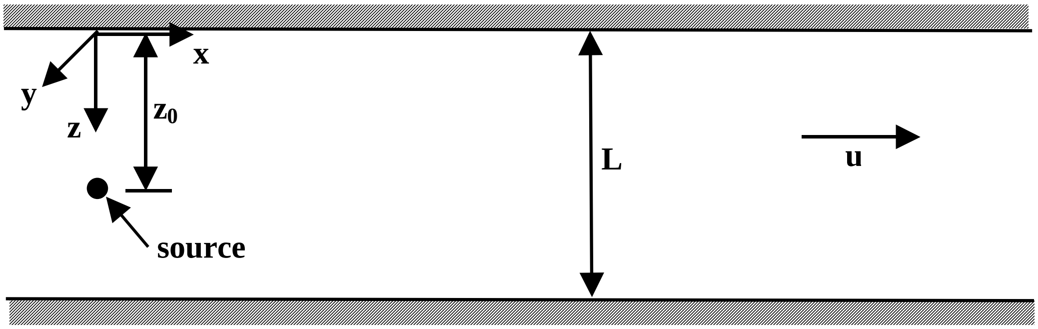
$$= c\_6\_Line\left(x, y, t, D_x, D_y, \lambda, x_0, y_0, x_M, y_M, M\right) \tag{111}$$

$$= c\_7\_Line\left(x, y, z, t, D_x, D_y, D_z, \lambda, x_0, y_0, z_0, x_M, y_M, z_M, M\right) \tag{112}$$

where the line source, in three dimensions, extends from $\left(x_0, y_0, z_0\right)$ to $\left(x_M, y_M, z_M\right)$. The line has been broken into M segments of equal length, and a point source has been placed at the mid-point of each segment. An example that uses Eq.(111) to distribute ten continuous sources over a line extending from $(x_0, y_0) = (0, -0.5)$ to $(x_M, y_M) = (0, 0.5)$ is shown in the following contour plot:



In many applications contaminant sources need to be modeled in an aquifer that has a finite depth. For example, consider the three-dimensional problem in which a contaminant source is placed in a uniform flow between two impermeable boundaries, as shown in the following sketch:



The concentration for an instantaneous source inserted at the point (x, y, z) = (0, 0, z_0) is a solution of the following problem:

$$D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} + D_z \frac{\partial^2 c}{\partial z^2} = u \frac{\partial c}{\partial x} + R \frac{\partial c}{\partial t} + R\lambda c \qquad (-\infty < x < \infty, -\infty < y < \infty, 0 < z < L, 0 < t < \infty) \tag{113}$$

$$c(x, y, z, 0) = \frac{M_3}{\sigma} \delta(x)\delta(y)\delta(z - z_0) \tag{114}$$

$$\lim_{r\to\infty} c(x, y, z, t) = 0 \qquad \left(r = \sqrt{x^2 + y^2}\right) \tag{115}$$

$$\frac{\partial c(x, y, 0, t)}{\partial z} = 0 \tag{116}$$

$$\frac{\partial c(x, y, L, t)}{\partial z} = 0 \tag{117}$$

The solution of this problem is given by

$$c(x, y, z, t) = \frac{e^{-\frac{(x-t)^2}{4tD_x} - \frac{y^2}{4tD_y} - \lambda t}}{4\pi t\sqrt{D_x D_y}} \left[1 + 2\sum_{n=1}^{\infty} e^{-tD_z\alpha_n^2} \cos(\alpha_n z_0) \cos(\alpha_n z)\right] \qquad (\alpha_n = n\pi) \tag{118}$$

in which all variables are dimensionless and are defined as follows:

$$\left(c', x', y', z', t', D_x^{'}, D_y^{'}, D_z^{'}, \lambda'\right) = \left(\frac{c\sigma L^3}{M_3}, \frac{x}{L}, \frac{y}{L}, \frac{z}{L}, \frac{tu}{RL}, \frac{D_x}{uL}, \frac{D_y}{uL}, \frac{D_z}{uL}, \frac{\lambda RL}{u}\right) \tag{119}$$

This solution can be calculated with Function by using the following call:

$$\frac{c\sigma L^3}{M_3} = c\_8\left(\frac{x}{L}, \frac{y}{L}, \frac{z}{L}, \frac{tu}{RL}, \frac{D_x}{uL}, \frac{D_y}{uL}, \frac{D_z}{uL}, \frac{\lambda RL}{u}, \frac{z_0}{L}\right) \tag{120}$$

The solution for a two-dimensional instantaneous source can be obtained by distributing three-dimensional sources along an infinitely long straight line parallel to the y axis. This gives the following result:

$$c(x,z,t)=\frac{e^{-\frac{(x-t)^2}{4tD_x}-\lambda t}}{\sqrt{4\pi tD_x}}\left[1+2\sum_{n=1}^{\infty}e^{-tD_z\alpha_n^2}\cos(\alpha_n z_0)\cos(\alpha_n z)\right]\qquad\left(c'=\frac{c\sigma L^2}{M_2},\ \alpha_n=n\pi\right)\text{(121)}$$

where $M_2 =$ mass per unit distance along the infinitely straight line. Function computes this solution with the following call:

$$\frac{c\sigma L^2}{M_2}=c\_9\left(\frac{x}{L},\frac{z}{L},\frac{tu}{RL},\frac{D_x}{uL},\frac{D_z}{uL},\frac{\lambda RL}{u},\frac{z_0}{L}\right)\qquad(122)$$

The instantaneous source solutions can also be distributed with respect to time to obtain solutions for continuous sources. The result for a three-dimensional continuous source located at the point (x, y, z) = (0, 0, z_0) is given by

$$c(x,y,z,t)=\frac{e^{\frac{x}{2D_x}}}{4\pi\sqrt{D_xD_y}}\left[W\left(\frac{a^2}{4t},ab_0\right)+2\sum_{n=1}^{\infty}W\left(\frac{a^2}{4t},ab_n\right)\cos(\alpha_n z_0)\cos(\alpha_n z)\right]\text{(123)}$$

where W(x, y) = Hantush-Jacob leaky aquifer solution and

$$\begin{aligned}
a&=\sqrt{\frac{x^2}{D_x}+\frac{y^2}{D_y}}\\
b_n&=\sqrt{\frac{1}{4D_x}+\lambda+D_z\alpha_n^2}\\
\alpha_n&=n\pi\\
c'&=\frac{c\sigma uL^2}{R\ dM_3/dt}
\end{aligned}\qquad(124)$$

Function computes this solution with the following call:

$$\frac{c\sigma uL^2}{R\ dM_3/dt}=c\_10\left(\frac{x}{L},\frac{y}{L},\frac{z}{L},\frac{tu}{RL},\frac{D_x}{uL},\frac{D_y}{uL},\frac{D_z}{uL},\frac{\lambda RL}{u},\frac{z_0}{L}\right)\qquad(125)$$

where $dM_3/dt =$ mass flow injected by the continuous source. The equivalent solution for a two-dimensional continuous source is given by

$$c(x,z,t)=f_0(x,z,t)+2\sum_{n=1}^{\infty}f_n(x,z,t)\cos(\alpha_n z_0)\cos(\alpha_n z)\qquad\left(c'=\frac{c\sigma uL}{R\ dM_2/dt}\right)\text{(126)}$$

where

$$f_n(x,z,t)=\frac{e^{\frac{x}{2D_x}}}{4b_n\sqrt{D_x}}\left[e^{-ab_n}\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}-b_n\sqrt{t}\right)-e^{ab_n}\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}+b_n\sqrt{t}\right)\right]\qquad(127)$$

in which

$$a = \frac{|x|}{\sqrt{D_x}}$$

$$b_n = \sqrt{\frac{1}{4D_x} + \lambda + D_z \alpha_n^2} \qquad (128)$$

$$\alpha_n = n\pi$$

Function computes this solution for a two-dimensional continuous source with the call

$$\frac{c\sigma uL}{R\ dM_2/dt} = c\_11\left(\frac{x}{L}, \frac{z}{L}, \frac{tu}{RL}, \frac{D_x}{uL}, \frac{D_z}{uL}, \frac{\lambda RL}{u}, \frac{z_0}{L}\right) \qquad (129)$$

Function also computes solutions by distributing sources along straight lines with finite length. For example, an instantaneous line source in three dimensions that extends from (x_0, y_0, z_0) to (x_M, y_M, z_M) is approximated numerically by dividing the line into M equal-length segments and placing a source at the midpoint of each segment. Function computes solutions for line sources with the following calls for instantaneous sources in three and two dimensions, respectively:

$$\frac{c\sigma L^3}{M_3} = c\_8\_Line\left(\frac{x}{L}, \frac{y}{L}, \frac{z}{L}, \frac{tu}{RL}, \frac{D_x}{uL}, \frac{D_y}{uL}, \frac{D_z}{uL}, \frac{\lambda RL}{u}, \frac{x_0}{L}, \frac{y_0}{L}, \frac{z_0}{L}, \frac{x_M}{L}, \frac{y_M}{L}, \frac{z_M}{L}, M\right) \qquad (130)$$

$$\frac{c\sigma L^2}{M_2} = c\_9\_Line\left(\frac{x}{L}, \frac{z}{L}, \frac{tu}{RL}, \frac{D_x}{uL}, \frac{D_z}{uL}, \frac{\lambda RL}{u}, \frac{x_0}{L}, \frac{z_0}{L}, \frac{x_M}{L}, \frac{z_M}{L}, M\right) \qquad (131)$$

Continuous line sources in three and two dimensions are computed by using the following two calls

$$\frac{c\sigma uL^2}{R\ dM_3/dt} = c\_10\_Line\left(\frac{x}{L}, \frac{y}{L}, \frac{z}{L}, \frac{tu}{RL}, \frac{D_x}{uL}, \frac{D_y}{uL}, \frac{D_z}{uL}, \frac{\lambda RL}{u}, \frac{x_0}{L}, \frac{y_0}{L}, \frac{z_0}{L}, \frac{x_M}{L}, \frac{y_M}{L}, \frac{z_M}{L}, M\right) \qquad (132)$$

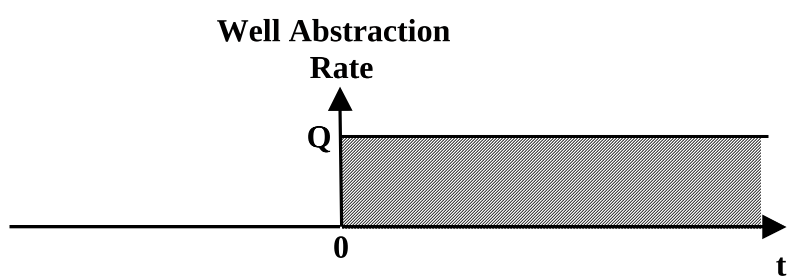
$$\frac{c\sigma uL}{R\ dM_2/dt} = c\_11\_Line\left(\frac{x}{L}, \frac{z}{L}, \frac{tu}{RL}, \frac{D_x}{uL}, \frac{D_z}{uL}, \frac{\lambda RL}{u}, \frac{x_0}{L}, \frac{z_0}{L}, \frac{x_M}{L}, \frac{z_M}{L}, M\right) \qquad (133)$$

The module **ScaleDepDispersion** contains some equivalent solutions for contaminant sources when dispersion coefficients increase with the first power of distance downstream. These solutions were obtained by Hunt (1998).

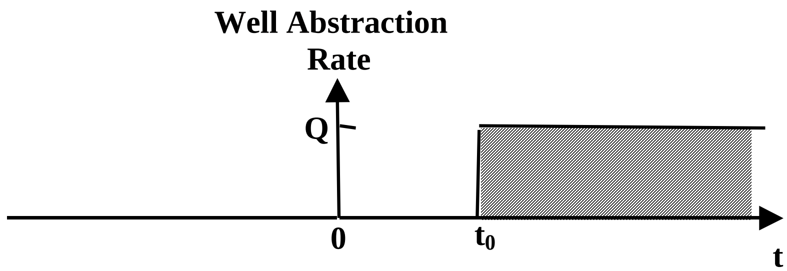
Well Recovery Tests and Flows in which Well Abstraction, Aquifer Recharge and Contaminant Injection Rates Vary with Time

Most of the solutions and corresponding programs discussed herein are directly proportional to a well abstraction rate, an aquifer recharge rate or a contaminant injection rate that vanishes for $-\infty < t < 0$ and is a non-zero constant for $0 < t < \infty$. However, it is sometimes necessary to model flows in which well abstraction rates, aquifer recharge rates or contaminant injection rates vary with time. For example, pumping tests ordinarily use a constant well abstraction rate, but it is often very useful to continue taking drawdown measurements after turning off the pump. These measurements are referred to as well recovery measurements.

All of the equations that we have considered are linear with coefficients that do not depend upon time, which means that superposition and time translation can always be used with solutions of these equations. For example, the Theis solution applies when the well abstraction rate is zero for $-\infty < t < 0$ and is a constant (that we called Q) for $0 < t < \infty$. The time translation principle means that replacing t with $t - t_0$ in this solution, where t_0 is a constant, gives a second solution in which the abstraction rate is zero for $-\infty < t < t_0$ and equals a constant, Q, for $t_0 < t < \infty$. The following sketches and equations illustrate this principle:

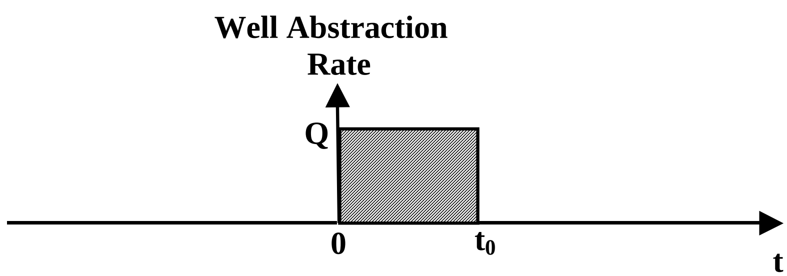


$$\frac{sT}{Q} = W\_1\left(\frac{r}{L}, \frac{tT}{SL^2}\right) \qquad (134)$$



$$\frac{sT}{Q} = W\_1\left(\frac{r}{L}, \frac{(t-t_0)T}{SL^2}\right) \tag{135}$$

The superposition principle allows these two solutions to be subtracted to obtain a solution that models both abstraction and recovery phases of a pumping test.



$$\begin{aligned}\frac{sT}{Q} &= 0 \qquad (-\infty < t \le 0) \\ &= W\_1\left(\frac{r}{L}, \frac{tT}{SL^2}\right) \qquad (0 \le t \le t_0) \\ &= W\_1\left(\frac{r}{L}, \frac{tT}{SL^2}\right) - W\_1\left(\frac{r}{L}, \frac{(t-t_0)T}{SL^2}\right) \qquad (t_0 \le t < \infty)\end{aligned} \tag{136}$$

Programming for this type of application is simplified considerably by using the fact that all programs in the **Hydraulics** and **Dispersion** modules return a value of zero when $t \le 0$. This means that the following single statement covers all of the three possibilities given in Eq. (136):

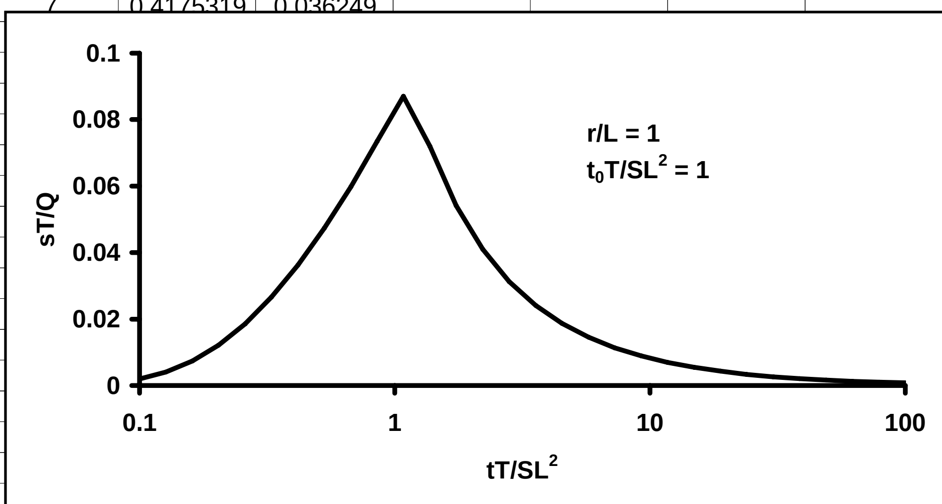
$$\frac{sT}{Q} = W\_1\left(\frac{r}{L}, \frac{tT}{SL^2}\right) - W\_1\left(\frac{r}{L}, \frac{(t-t_0)T}{SL^2}\right) \qquad (-\infty < t < \infty) \tag{137}$$

An example of the use of Eq. (137) is shown in the following spreadsheet, in which the formula

$$=W\_1(\$A\$2,B5)-W\_1(\$A\$2,B5-\$B\$2) \tag{138}$$

was entered in cell C5 and dragged downward:

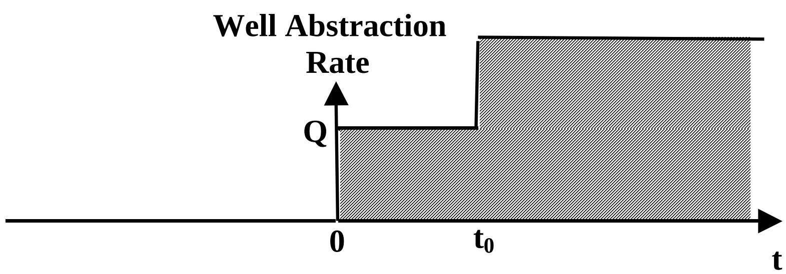
	A	B	C	D	E	F	G	H
1	r/L	t_0T/SL^2	a	b	n			
2	1	1	-1	2	30			
3								
4	k	tT/SL^2	sT/Q					
5	1	0.1	0.0019827					
6	2	0.1268961	0.004056					
7	3	0.1610262	0.0073637					
8	4	0.204336	0.0121468					
9	5	0.2592944	0.0185461					
10	6	0.3290345	0.0265979					
11	7	0.4175319	0.036249					
12								
13								
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25								
26								
27								
28	24	23.950266	0.003358					
29	25	30.391954	0.0026402					
30	26	38.566204	0.0020769					
31	27	48.939009	0.0016344					
32	28	62.101694	0.0012866					
33	29	78.804628	0.001013					
34	30	100	0.0007978					
35								



Likewise, adding the right sides of Eqs. (134) and (135) gives

$$\frac{sT}{Q} = W\_1\left(\frac{r}{L}, \frac{tT}{SL^2}\right) + W\_1\left(\frac{r}{L}, \frac{(t-t_0)T}{SL^2}\right) \qquad (-\infty < t < \infty) \qquad (139)$$

which models the following pumping schedule:

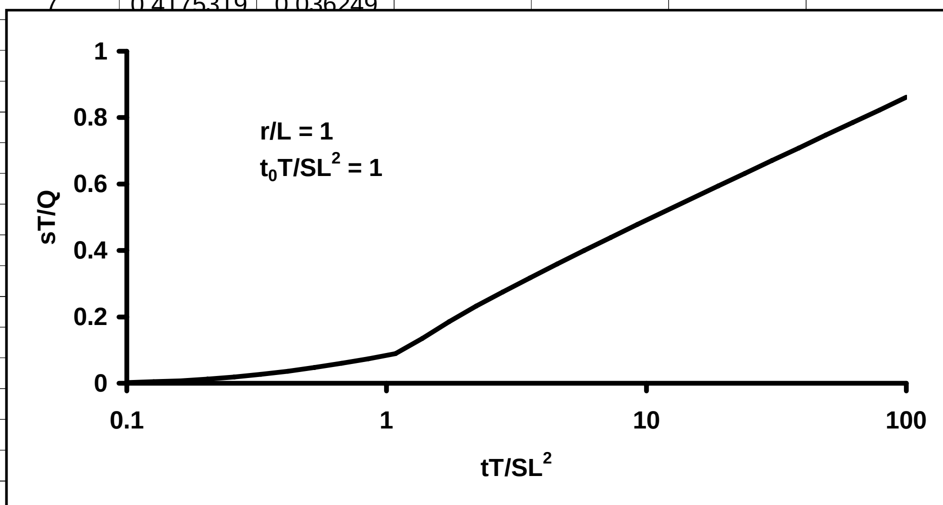


An example of the use of Eq. (139) is shown in the following spreadsheet, in which the formula

=W_1($A$2,B5)+W_1($A$2,B5-$B$2) (140)

was entered in cell C5 and dragged downward:

	A	B	C	D	E	F	G	H
1	r/L	t_0T/SL^2	a	b	n			
2	1	1	-1	2	30			
3								
4	k	tT/SL^2	sT/Q					
5	1	0.1	0.0019827					
6	2	0.1268961	0.004056					
7	3	0.1610262	0.0073637					
8	4	0.204336	0.0121468					
9	5	0.2592944	0.0185461					
10	6	0.3290345	0.0265979					
11	7	0.4175319	0.036249					
12								
13								
14								
15								
16								
17								
18								
19								
20								
21								
22								
23								
24								
25								
26								
27								
28	24	23.950266	0.6325407					
29	25	30.391954	0.6708184					
30	26	38.566204	0.7090157					
31	27	48.939009	0.7471506					
32	28	62.101694	0.785237					
33	29	78.804628	0.8232855					
34	30	100	0.8613043					
35								



It should be obvious from these two examples that time translation and superposition can be used with many of the programs in Function.xls to construct solutions for different variations in well abstraction rates, aquifer recharge rates and contaminant injection rates.

Flow to Vertical, Inclined and Horizontal Wells in Unconfined and Semi-confined Aquifers

Two different types of aquifer geology are used to model the delayed-yield type of behavior often observed when pumping from a well in an unconfined aquifer. The first type of aquifer, which will be referred to as a Boulton-type semi-confined aquifer, has been described earlier when discussing the Boulton solution for flow to a well in a delayed- yield aquifer. The Boulton-type aquifer consists of a pumped aquifer overlain by either a single aquitard or a series of aquitards containing a standing water table. For small values of time after turning on the pump, the semi-confined pumped aquifer behaves as a fully confined (Theis-type) aquifer. At intermediate values of time water seeps downward through the overlying aquitard (or aquitards) to recharge the pumped aquifer, which appears during this second period to respond as a Hantush leaky aquifer. Finally, at still larger values of time, free surface drawdowns in the top aquitard begin to drop and, ultimately, approach the same values that occur in the pumped aquifer. During this third period vertical leakage ceases and drawdowns in the pumped aquifer approach values predicted from the Theis solution when the elastic storage coefficient (storativity) is replaced with the effective porosity (specific yield) at the free surface in the top aquitard.

A Neuman-type unconfined aquifer is a single, unconfined, homogeneous anisotropic aquifer. The anisotropy may be caused either by a preferred orientation of solid particles in the aquifer matrix or by very fine layering. However, vertical and horizontal bulk permeabilities do not change with either depth or horizontal position within the aquifer.

Function.xls uses a solution obtained by Zhan and Zlotnik (2002) to model flow to a well in either of these two different types of aquifers by computing drawdowns created by flow to a point sink. Then programs W_9 and W_10 distribute these point sinks over a straight-line segment to model flow to a well screen with a finite length but zero radius. As in the program for W_8, this straight-line segment extends in any direction from $\left(x_0, y_0, z_0\right)$ to $\left(x_M, y_M, z_M\right)$ and contains one sink at the midpoint of each of M sub-segments. This makes it relatively easy to model both radial flow to partially penetrating vertical wells and three-dimensional flow to horizontal or inclined wells.

Flow to a point sink in a Neuman-type unconfined aquifer is modeled with the following equations:

$$\frac{K_H}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)+K_V\frac{\partial^2 s}{\partial z^2}=S_s\frac{\partial s}{\partial t} \qquad \left(0<r<\infty,\ 0<z<B,\ 0<t<\infty\right) \tag{141}$$

$$\lim_{r\to\infty}\left(r\frac{\partial s}{\partial r}\right)=-\frac{Q}{2\pi K_H}\delta\left(z-L\right) \qquad \left(0<z<B,\ 0<t<\infty\right) \tag{142}$$

$$s\left(\infty,z,t\right)=0 \qquad \left(0<z<B,\ 0<t<\pi\right) \tag{143}$$

$$\frac{\partial s\left(r,0,t\right)}{\partial z}=0 \qquad \left(0<r<\infty,\ 0<t<\infty\right) \tag{144}$$

$$K_V \frac{\partial s(r,B,t)}{\partial z} + \sigma \frac{\partial s(r,B,t)}{\partial t} = 0 \qquad (0 < r < \infty,\ 0 < t < \infty) \tag{145}$$

$$s(r,z,0) = 0 \qquad (0 < r < \infty,\ 0 < z \le B) \tag{146}$$

where s = drawdown, r = radial coordinate, z = vertical coordinate, t = time, B = saturated aquifer thickness at t = 0, $K_H =$ horizontal permeability, $K_V =$ vertical permeability, $\sigma =$ effective porosity (specific yield), $S_s =$ specific storage, Q = well flow rate, L = z coordinate of the sink (the impermeable bottom aquifer boundary has the coordinate z = 0) and $\delta(x) =$ Dirac's delta function. Eq.(141) is the governing partial differential equation, Eq.(142) requires a constant flow rate to the sink, Eq.(143) requires that drawdowns vanish at infinity, Eq.(144) requires an impermeable bottom boundary, Eq.(145) is a linearized form of the free-surface boundary condition and Eq.(146) is an initial condition that requires the drawdown to vanish everywhere at t = 0.

The Laplace transform can be used to remove all time derivatives and initial conditions from Eqs.(141)-(146). Then the resulting equations can be solved for the Laplace transform of the drawdown and inverted numerically with an approximate algorithm devised by Stehfest. This has been done in a Function.xls program entitled sink_9, which has the following call statement:

$$\frac{sK_H B}{Q} = \sin k\_9\left(\frac{r}{B}, \frac{z}{B}, \frac{tK_H}{S_s B^2}, \frac{L}{B}, \frac{K_V}{K_H}, \frac{K_V}{K_H}\frac{S_s B}{\sigma}\right) \tag{147}$$

The program sink_9 has been incorporated in a program entitled W_9 that distributes sinks over a straight-line segment that starts at (x_0, y_0, z_0) and terminates at (x_M, y_M, z_M). The program places one sink at the midpoint of each of M equal-length sub-segments. The call statement for W_9 follows:

$$\frac{s B K_H}{Q} = W\_9\left(\frac{x}{B}, \frac{y}{B}, \frac{z}{B}, \frac{t K_H}{S_s B^2}, \frac{K_V}{K_H}, \frac{K_V}{K_H}\frac{S_s B}{\sigma}, \frac{x_0}{B}, \frac{y_0}{B}, \frac{z_0}{B}, \frac{x_M}{B}, \frac{y_M}{B}, \frac{z_M}{B}, M\right) \tag{148}$$

A comparison is shown in Fig. 21 between approximate drawdowns calculated from W_9, shown with unfilled circles, and drawdowns calculated from an exact solution obtained by Neuman (1972, 1974), which is available in the commercial software AqteSolv. The comparison is very close, but it should be stressed that the solution is approximate and is sensitive to the number of terms used in the Stehfest algorithm. Therefore, if an application is encountered in which plotted points fail to follow a smooth curve, try decreasing or increasing the number of terms used in the algorithm by changing the parameter ninv in the program Neuman_inv.

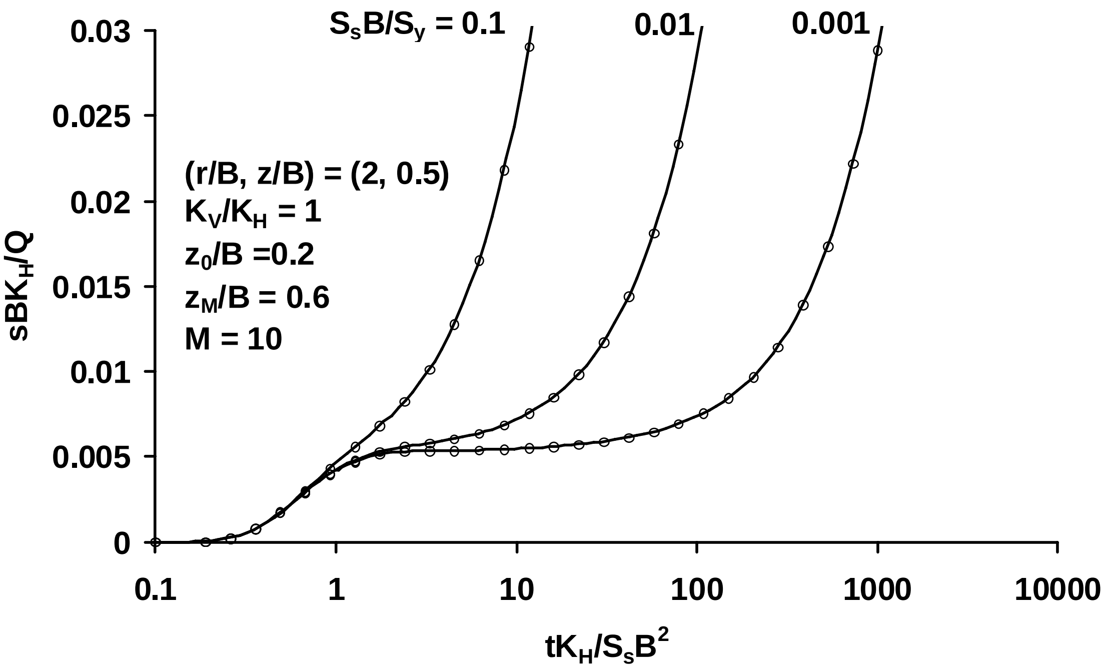


Fig. 21 Drawdowns created by pumping from a vertical well calculated using W_9 and AqteSolv.

Flow to a sink in a Boulton-type semi-confined aquifer is described by the following equations:

$$\frac{K_H}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)+K_V\frac{\partial^2 s}{\partial z^2}=S_s\frac{\partial s}{\partial t} \qquad \left(0<r<\infty,\ 0<z<B,\ 0<t<\infty\right) \tag{149}$$

$$\sigma\frac{\partial\eta(r,t)}{\partial t}+\left(\frac{K_V'}{B'}\right)\left[\eta(r,t)-s(B,r,t)\right]=0 \qquad \left(-<r<\infty,\ 0<t<\infty\right) \tag{150}$$

$$\lim_{r\to\infty}\left(r\frac{\partial s}{\partial r}\right)=-\frac{Q}{2\pi K_H}\delta(z-L) \qquad \left(0<z<B,\ 0<t<\infty\right) \tag{151}$$

$$s(\infty,z,t)=0 \qquad \left(0<z<B,\ 0<t<\pi\right) \tag{152}$$

$$\frac{\partial s(r,0,t)}{\partial z}=0 \qquad \left(0<r<\infty,\ 0<t<\infty\right) \tag{153}$$

$$K_V\frac{\partial s(r,B,t)}{\partial z}+\left(\frac{K_V'}{B'}\right)\left[s(r,B,t)-\eta(r,t)\right]=0 \qquad \left(0<r<\infty,\ 0<t<\infty\right) \tag{154}$$

$$s(r,z,0)=0 \qquad \left(0<r<\infty,\ 0<z<B\right) \tag{155}$$

$$\eta(r,0)=0 \qquad \left(0<r<\infty\right) \tag{156}$$

where s(r,z,t) = drawdown in the semi-confined pumped aquifer, $\eta(r,t)=$ free surface drawdown in the overlying aquitards, K_V' and $\sigma=$ the vertical permeability and effective porosity, respectively, at the free surface in the overlying aquitard, B' = aquitard thickness and all other variables are as defined above for flow to a sink in a Neuman-type unconfined aquifer. Eq.(149) is the governing partial differential equation for the pumped aquifer, and Eq.(150) is the governing partial differential equation for the overlying aquitard. [Eq.(150) neglects elastic effects and assumes that flow occurs only in the vertical direction. This leads directly to the result that vertical head distributions are linear across the aquitard, and Eq.(150) equates the constant vertical velocity at any point in the aquitard to the vertical velocity component of the free surface.] Eq.(151) requires a constant flow rate to the sink, Eq.(152) requires that drawdowns vanish at infinity, Eq.(153) requires that the pumped aquifer bottom boundary be impermeable, Eq.(154) equates vertical velocity components across the common boundary between the pumped aquifer and aquitard and Eqs.(155)-(156) are initial conditions that require zero drawdowns in both the pumped aquifer and the overlying aquitard at t = 0.

Unlike the Boulton solution calculated from W_3, the solution of Eqs.(149)-(156) allows flow in the pumped aquifer to have both vertical and horizontal components, and the aquifer itself can be anisotropic. Comparison with the equations that describe flow to a point sink in a Neuman-type unconfined aquifer shows that this solution will have one additional parameter, the ratio of the vertical aquitard permeability, K_V', to the aquitard thickness, B'. This solution for flow to a point sink, which is also calculated in Function.xls with the Stehfest algorithm for inversion of a Laplace transform, has the following call statement:

$$\frac{sK_H B}{Q} = \sin k\_10\left(\frac{r}{B},\frac{z}{B},\frac{tK_H}{S_s B^2},\frac{L}{B},\frac{K_V}{K_H},\frac{K_V/B}{K_V'/B'},\frac{K_V'/B'}{K_H/B}\frac{S_s B}{\sigma}\right) \tag{157}$$

The call statement for the program that distributes these sinks over a line segment with finite length follows:

$$\frac{sBK_H}{Q} = W\_10\left(\frac{x}{B},\frac{y}{B},\frac{z}{B},\frac{tK_H}{S_s B^2},\frac{K_V}{K_H},\frac{K_V/B}{K_V'/B'},\frac{K_V'/B'}{K_H/B}\frac{S_s B}{\sigma},\frac{x_0}{B},\frac{y_0}{B},\frac{z_0}{B},\frac{x_M}{B},\frac{y_M}{B},\frac{z_M}{B},M\right) \tag{158}$$

Since the solution calculated from W_10 can be expected to duplicate the Boulton solution calculated from W_3 when the line of distributed sinks is vertical and completely penetrates an isotropic pumped aquifer, a check can be obtained by comparing drawdowns calculated from W_10 with corresponding drawdowns calculated from W_3. This is done in Fig. 22, where it is seen that the comparison is very close. Once again, if an application is encountered in which plotted points fail to follow a smooth curve, try decreasing or increasing the number of terms used in the Stehfest algorithm by changing the parameter ninv in the program Boulton_inv.

The free surface drawdown in the overlying aquitard can be computed by using the following call statement:

$$\frac{\eta K_H B}{Q} = Eta\_10\left(\frac{x}{B},\frac{y}{B},\frac{z}{B},\frac{tK_H}{S_s B^2},\frac{K_V}{K_H},\frac{K_V/B}{K_V'/B'},\frac{K_V'/B'}{K_H/B}\frac{S_s B}{\sigma},\frac{x_0}{B},\frac{y_0}{B},\frac{z_0}{B},\frac{x_M}{B},\frac{y_M}{B},\frac{z_M}{B},M\right) \tag{159}$$

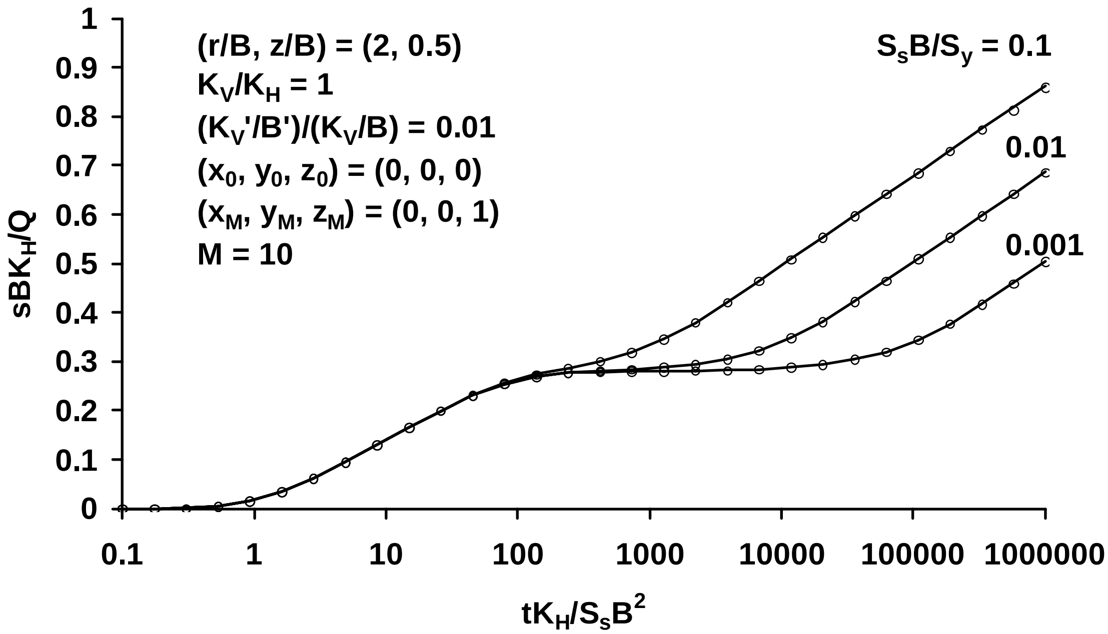


Fig. 22 Comparison of drawdowns calculated from W_3 and W_10 for a vertical well that completely penetrates an isotropic aquifer.

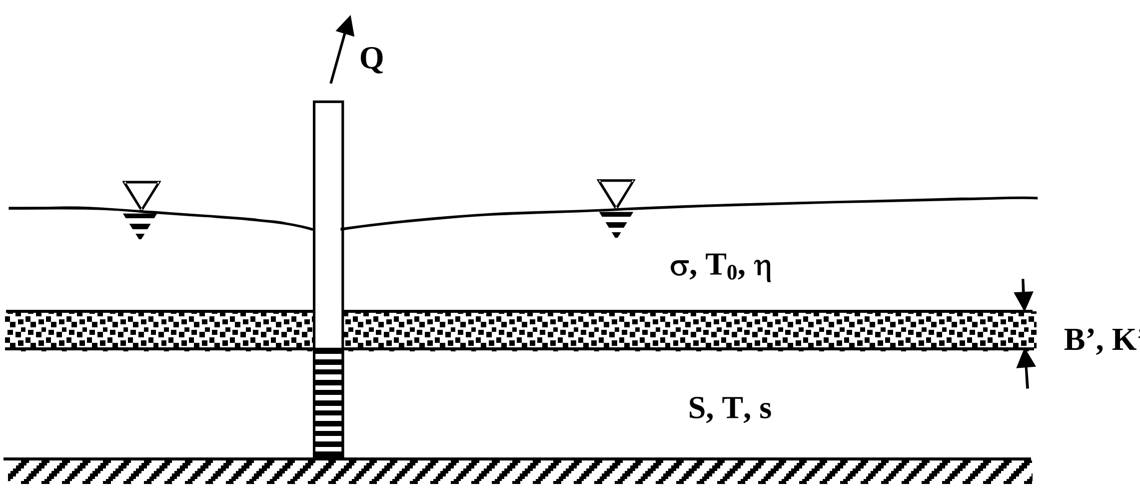


Fig. 23 Definition sketch for flow to a well in a two-aquifer system.

Flow to a Well in a Two-Aquifer System

This section concerns a generalization of the Boulton solution in which a well is screened in an aquifer that is overlain by both an aquitard and a second aquifer containing a free surface, as shown in Fig. 23. Flows in the top and bottom aquifers are assumed to be horizontal, and flow in the aquitard is assumed to be both vertical and incompressible. The vertical flow condition will be satisfied if the permeability of the aquitard is at least one, and preferably two or more, orders of magnitude less than the permeability of the bottom aquifer. The governing equations for this problem follow:

$$\frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)=S\frac{\partial s}{\partial t}+\left(K'/B'\right)\left(s-\eta\right) \qquad \left(0<r<\infty, 0<t<\infty\right) \tag{160}$$

$$\frac{T_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \eta}{\partial r}\right)=\sigma\frac{\partial \eta}{\partial t}+\left(K'/B'\right)\left(\eta-s\right) \qquad \left(0<r<\infty, 0<t<\infty\right) \tag{161}$$

$$\lim_{r\to 0}\left(r\frac{\partial s}{\partial r}\right)=-\frac{Q}{2\pi T} \qquad \left(0<t<\infty\right) \tag{162}$$

$$\lim_{r\to 0}\left(r\frac{\partial \eta}{\partial r}\right)=0 \qquad \left(0<t<\infty\right) \tag{163}$$

$$s\left(\infty,t\right)=\eta\left(\infty,t\right)=0 \qquad \left(0<t<\infty\right) \tag{164}$$

$$s\left(r,0\right)=\eta\left(r,0\right)=0 \qquad \left(0<r<\infty\right) \tag{165}$$

where s = drawdown in the pumped aquifer, $\eta =$ free surface drawdown in the top aquifer, r = radial distance from the pumped well, t = time, T = transmissivity of the pumped aquifer, T_0 = transmissivity of the top aquifer, S = elastic storage coefficient for the pumped aquifer, $\sigma =$ effective porosity or specific yield of the top aquifer, $K'/B' =$ permeability to thickness ratio for the aquitard and Q = well abstraction rate. Eqs.(160) and (161) are the partial differential equations that describe horizontal flow in the bottom and top aquifers, respectively; Eq.(162) specifies the flow abstracted through the well screen; Eq.(163) requires that no well screen exist in the top aquifer at r = 0; Eq.(164) requires that drawdowns vanish at infinity in both aquifers; and Eq.(165) requires zero drawdowns everywhere at the start of pumping.

Function calculates the numerical inversion of two Laplace transforms with the Stehfest algorithm to compute drawdowns in both aquifers with the following two calls:

$$\frac{sT}{Q}=W\_11\left(\frac{r}{L},\frac{tT}{SL^2},\frac{S}{\sigma},\frac{\left(K'/B'\right)L^2}{T},\frac{T_0}{T}\right) \tag{166}$$

$$\frac{\eta T}{Q}=Eta\_11\left(\frac{r}{L},\frac{tT}{SL^2},\frac{S}{\sigma},\frac{\left(K'/B'\right)L^2}{T},\frac{T_0}{T}\right) \tag{167}$$

where L = an arbitrarily chosen length.

The governing equations reduce to the equations that describe the Boulton solution when $T_0 \to 0$. This result is used to test the accuracy of the Stehfest inversion process in

Fig.24, where the Boulton solution is shown with solid and dashed lines and the solutions given by Eqs.(166) and (167) are shown with filled and unfilled circles, respectively.

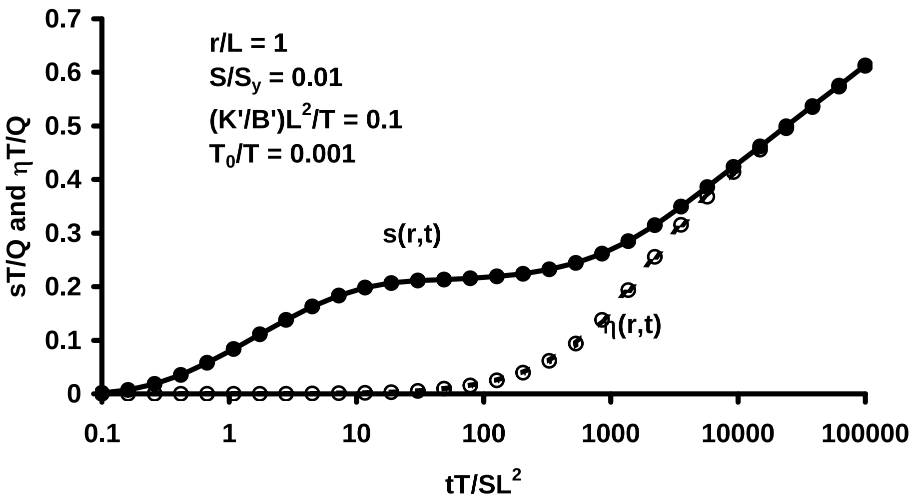


Fig. 24 The Boulton solution compared with solutions obtained from Eqs. (166)**-**(167)**.**

Fig. 25 shows the result of increasing the transmissivity ratio in Fig. 24 to $T_0/T = 0.5$. There are two important effects. First, the asymptotic slopes of the final portions of the drawdown curves are decreased. Second, the final asymptotes for s and η are parallel and are separated by a finite distance to maintain a constant drawdown gradient across the aquitard. This must happen since horizontal flow in the top aquifer is not abstracted directly by a well but instead leaks downward to recharge the lower pumped aquifer.

Fig. 26 shows the result of systematically increasing the transmissivity ratio, with the curve for $T_0/T = 0$ showing the Boulton solution and the curve for $T_0/T = 10$ approaching the Hantush-Jacob solution for a leaky aquifer. These curves show that the Boulton solution gives a sufficiently accurate solution for most purposes when $T_0/T \le 0.05$ and that increasing T_0/T beyond this value decreases the asymptotic slope of the last part of the curve. Therefore, T_0 can be found from pumping test data by adjusting its value until the final asymptotic slopes of the measured and calculated drawdown curves agree. Furthermore, since the asymptotic slopes of the first and final segments of the drawdown curve are identical in the Boulton solution, this gives a quick way to judge from experimental field data whether the top aquifer has a large enough transmissivity to carry a significant portion of flow abstracted by the well.

Programs have also been written to calculate the accumulated volumes of water abstracted up to any time t from each of the aquifers in a two-aquifer system. The volume of water abstracted from the top unconfined aquifer is calculated from

$$\frac{\text{Volume}}{\sigma(Q/T)L^2} = \text{Eta\_Volume\_11}\left(\frac{tT}{SL^2}, \frac{S}{\sigma}, \frac{(K'/B')L^2}{T}, \frac{T_0}{T}\right) \qquad (168)$$

The volume of water abstracted from the bottom semi-confined aquifer is calculated from

$$\frac{\text{Volume}}{\sigma(Q/T)L^2} = W\_Volume\_11\left(\frac{tT}{SL^2}, \frac{S}{\sigma}, \frac{(K'/B')L^2}{T}, \frac{T_0}{T}\right) \tag{169}$$

The volume of water abstracted from the top unconfined aquifer is very much larger than the volume of water abstracted from the bottom semi-confined aquifer because a specific yield is about three orders of magnitude larger than an elastic storativity.

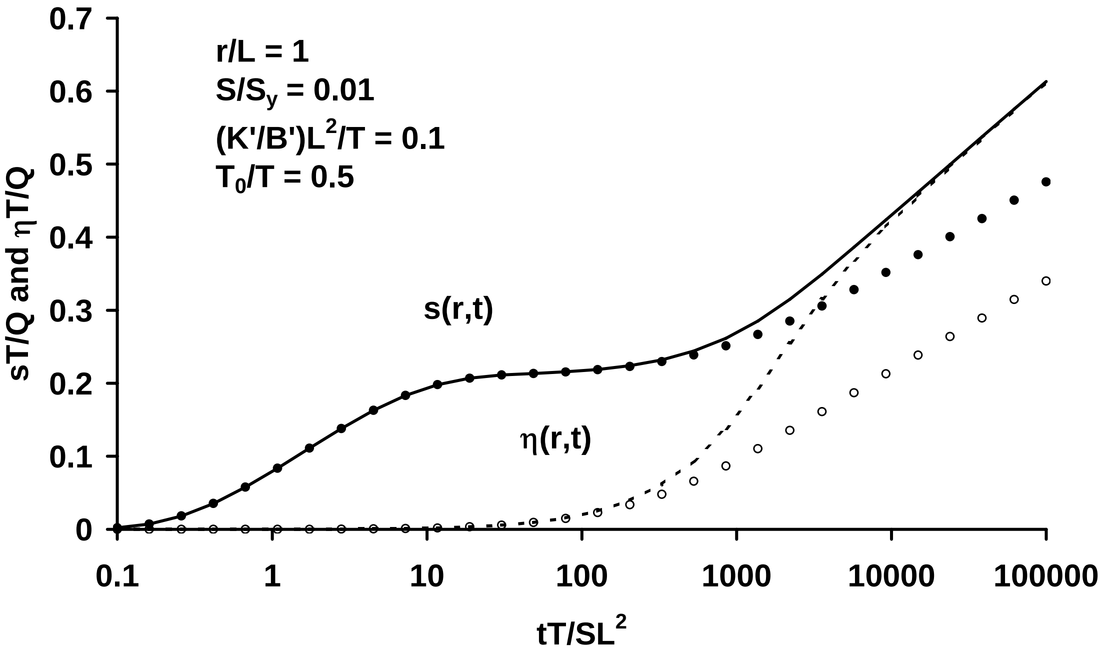


Fig. 25. The Boulton solution, shown with solid and dashed curves, compared with the two-layer aquifer solution, shown with filled and unfilled circles.

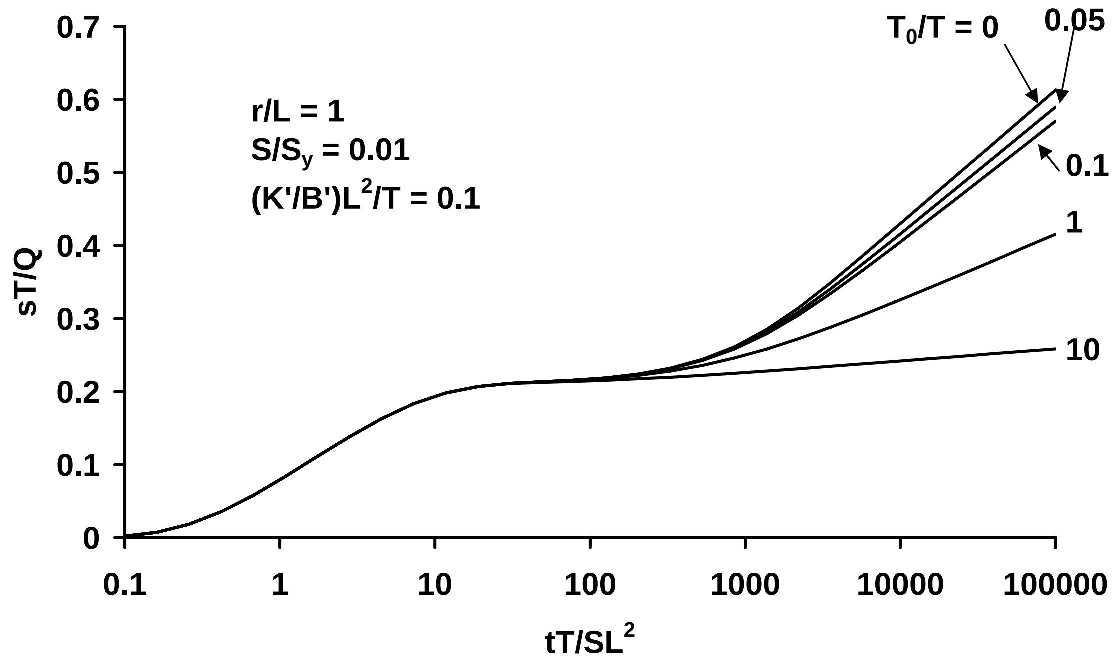


Fig. 26. The result of changing the transmissivity ratio in the two-layer aquifer solution.

Neuman and Witherspoon (1969) extended solution obtained by Hantush (1967) by considering compressible flow in the aquitard. This was done by substituting the following equations for Eqs.(160) and (161):

$$\frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)=S\frac{\partial s}{\partial t}-K'\frac{\partial s'}{\partial z}\Big|_{z=0} \qquad \left(0<r<\infty, 0<t<\infty\right) \tag{170}$$

$$\frac{T_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \eta}{\partial r}\right)=\sigma\frac{\partial \eta}{\partial t}+K'\frac{\partial s'}{\partial z}\Big|_{z=B'} \qquad \left(0<r<\infty, 0<t<\infty\right) \tag{171}$$

where z = vertical coordinate with the bottom and top aquitard boundaries at z = 0 and z = B’, respectively, and $s' =$ drawdown within the aquitard. Flow within the aquitard was considered vertical and compressible by solving the following equation:

$$K'\frac{\partial^2 s'}{\partial z^2}=S_s'\frac{\partial s'}{\partial t} \qquad \left(0<r<\infty,\ 0<z<B',\ 0<t<\infty\right) \tag{172}$$

where $S_s' =$ aquitard specific storage. Eqs.(162)-(165) remained unchanged, and conditions were also imposed that required continuity of drawdown and specific discharge across the top and bottom aquitard boundaries. Function calculates the numerical inversion of two Laplace transforms with the Stehfest algorithm to compute drawdowns in both aquifers for the Neuman-Witherspoon (1969) problem with the following two calls:

$$\frac{sT}{Q}=W\_12\left(\frac{r}{L},\frac{tT}{SL^2},\frac{S}{\sigma},\frac{\left(K'/B'\right)L^2}{T},\frac{T_0}{T},\frac{B'S_s'}{S}\right) \tag{173}$$

$$\frac{\eta T}{Q}=\mathrm{Eta\_12}\left(\frac{r}{L},\frac{tT}{SL^2},\frac{S}{\sigma},\frac{(K'/B')L^2}{T},\frac{T_0}{T},\frac{B'S_s'}{S}\right) \tag{174}$$

Fig. 27 shows the effect of aquitard compressibility in this problem. Hunt and Scott (2007) give a much more detailed presentation of material considered in this section.

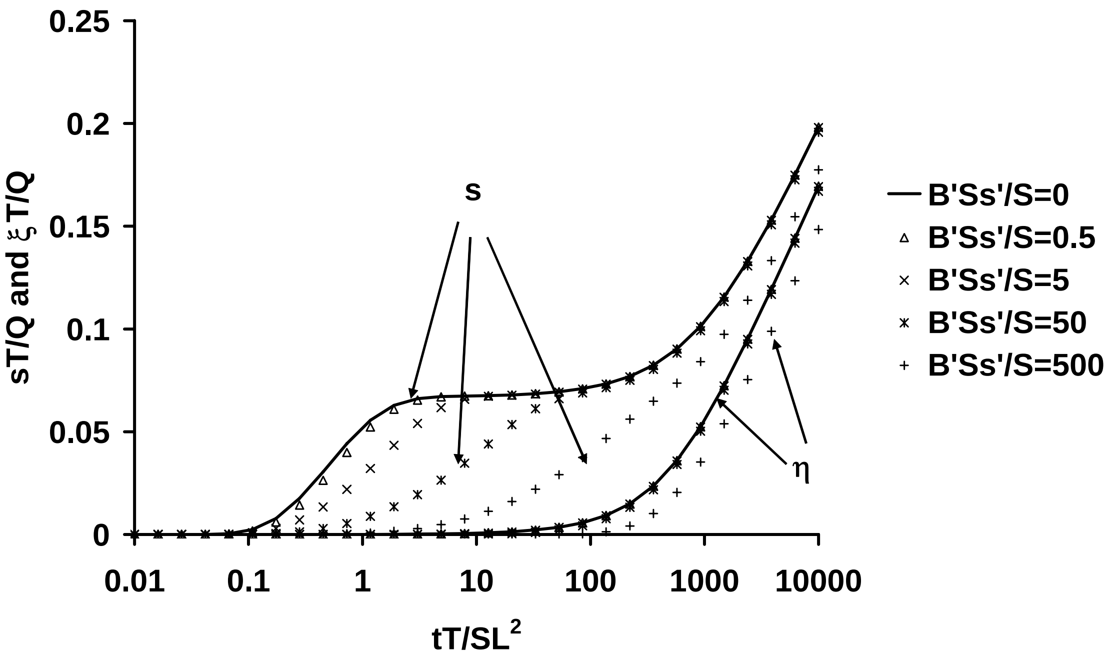


Fig.27 The effect of aquitard compressibility for flow to a well in a two-aquifer system.

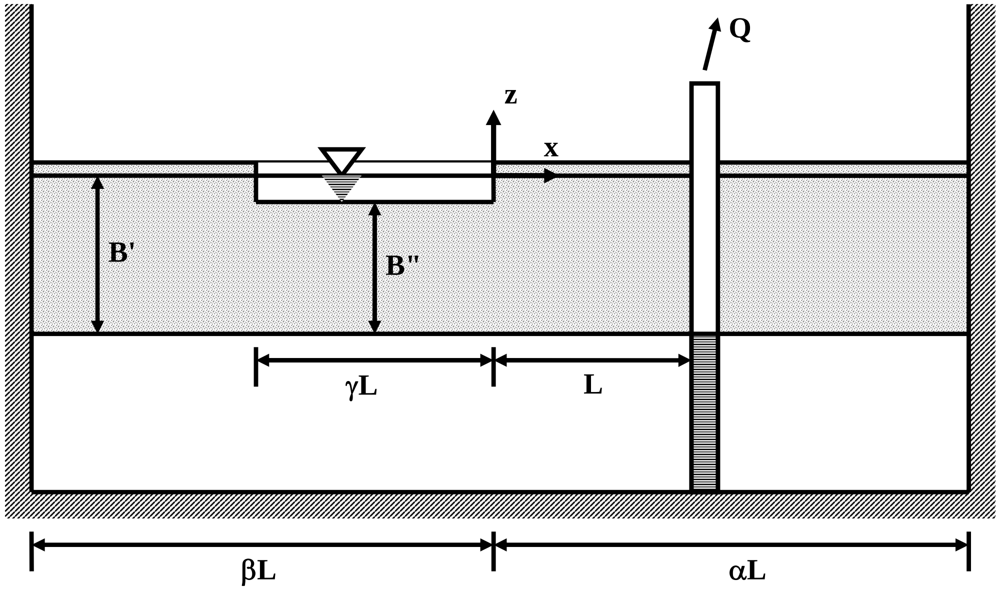


Fig. 28. Stream depletion when the stream and aquifer have finite widths.

Stream Depletion 2

Fig. 28 shows flow to a well beside a stream when the stream and aquifer have finite widths. The pumped aquifer is overlain by an aquitard, or a series of aquitard layers, containing a free surface. The geologic cross section shown in Fig. 28 remains unchanged for $-\infty < y < \infty$, flow in the aquitard is assumed to be vertical and flow in the pumped aquifer is assumed to be horizontal. This means that the aquifer is identical with the aquifer used in the Boulton delayed-yield solution for flow to a well in an infinite aquifer. However, this solution can also be used to describe horizontal unconfined flow in a homogeneous aquifer by (1) setting the vertical hydraulic conductivity of the aquitard equal to zero and (2) replacing the storativity (elastic storage coefficient) of the pumped aquifer with its specific yield (effective porosity). The following equations describe this problem:

$$T\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}\right)=S\frac{\partial s}{\partial t}+\left(\frac{K'}{B'}\right)(s-\eta)-Q\delta(x-L)\delta(y) \quad (-\beta L < x < -\gamma L,\ 0 < x < \alpha L) \tag{175}$$

$$\sigma\frac{\partial \eta}{\partial t}+\left(\frac{K'}{B'}\right)(\eta-s)=0 \qquad (-\beta L < x < -\gamma L,\ 0 < x < \alpha L) \tag{176}$$

$$T\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}\right)=S\frac{\partial s}{\partial t}+\left(\frac{K''}{B''}\right)s \qquad (-\gamma L < x < 0) \tag{177}$$

$$\frac{\partial s(\alpha L, y, t)}{\partial x} = 0 \tag{178}$$

$$\frac{\partial s(-\beta L, y, t)}{\partial x} = 0 \tag{179}$$

$$s(x, \pm\infty, t) = 0 \qquad (-\beta L < x < \alpha L) \tag{180}$$

$$s(x, y, 0) = \eta(x, y, 0) = 0 \qquad (-\beta L < x < \alpha L) \tag{181}$$

where s = drawdown in the pumped aquifer, $\eta =$ free surface drawdown in the aquitard, t = time, T = aquifer transmissivity, S = aquifer storativity, $K'/B' =$ hydraulic conductivity $\div$ saturated thickness of the aquitard at points not beneath the stream, $K''/B'' =$ hydraulic conductivity $\div$ thickness of the aquitard beneath the stream, Q = well abstraction rate, $\delta(x) =$ Dirac's delta function and $\sigma =$ aquitard specific yield. The drawdown of the stream free surface is assumed to be zero for all time.

Eqs. (175)and (177) are the result of combining Darcy's law with an equation of continuity, and Eq. (176) is a simplification that requires that the vertical specific discharge component in the aquitard equal the vertical velocity of the free surface. Hunt and Scott (2005) show that elimination of η from Eqs. (175) and (176) gives the same integral-differential equation that was solved by Boulton (1963, 1973) to obtain his solution for delayed-yield flow to a well. As just pointed out, however, these two equations also reduce to the equations that describe flow in a homogeneous unconfined aquifer if $K' \to 0$ and $S \to \sigma$. Eqs. (178) and (179) require that the boundaries at $x = \alpha L$ and $x = -\beta L$ be impermeable, Eq. (180) requires that drawdowns at infinity be unaffected by pumping from the well and Eq. (181) requires that drawdowns be zero everywhere in both the pumped aquifer and the aquitard at t = 0. After $s(x, y, t)$ has been found by solving Eqs. (175) -(181), the flow, ΔQ, depleted from the stream is computed from

$$\Delta Q(t) = \left(\frac{K''}{B''}\right) \int_{-\gamma L}^{0} \int_{-\infty}^{\infty} s(x, y, t)\, dy\, dx \tag{182}$$

Although not stated explicitly in these equations, the independent variables y and t vary throughout the ranges $-\infty < y < \infty$ and $0 < t < \infty$, respectively.

The solution of these equations for s, η and ΔQ have been coded as user-defined functions entitled W_13, Eta_13 and Q_13, respectively. The argument lists for these programs follow:

$$\frac{sT}{Q} = W\_13\left(\frac{x}{L}, \frac{y}{L}, \frac{tT}{SL^2}, \frac{(K'/B')L^2}{T}, \frac{(K''/B'')L^2}{T}, \frac{S}{\sigma}, \alpha, \beta, \gamma\right) \tag{183}$$

$$\frac{\eta T}{Q} = Eta\_13\left(\frac{x}{L}, \frac{y}{L}, \frac{tT}{SL^2}, \frac{(K'/B')L^2}{T}, \frac{(K''/B'')L^2}{T}, \frac{S}{\sigma}, \alpha, \beta, \gamma\right) \tag{184}$$

$$\frac{\Delta Q}{Q} = Q\_13\left(\frac{tT}{SL^2}, \frac{(K'/B')L^2}{T}, \frac{(K''/B'')L^2}{T}, \frac{S}{\sigma}, \alpha, \beta, \gamma\right) \tag{185}$$

The programs for W_13 and Eta_13 are believed to give reliable results for dimensionless values of time in the range 0.001 < tT/SL^2 < 600,000. There are no similar restrictions known at this time for the program Q_13. Hunt (2007) gives a much more detailed presentation of material considered in this section.

Boulton's Solution for a Finite-Width Aquifer

Programs have also been written to calculate drawdowns for the problem shown in Fig. 28 when no stream is present. The following equations describe this problem:

$$T\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}\right)=S\frac{\partial s}{\partial t}+\left(\frac{K'}{B'}\right)(s-\eta)-Q\delta(x)\delta(y)\quad(-\beta L<x<\alpha L)\tag{186}$$

$$\sigma\frac{\partial \eta}{\partial t}+\left(\frac{K'}{B'}\right)(\eta-s)=0\qquad(-\beta L<x<\alpha L)\tag{187}$$

$$\frac{\partial s(\alpha L,y,t)}{\partial x}=0\tag{188}$$

$$\frac{\partial s(-\beta L,y,t)}{\partial x}=0\tag{189}$$

$$s(x,\pm\infty,t)=0\qquad(-\beta L<x<\alpha L)\tag{190}$$

$$s(x,y,0)=\eta(x,y,0)=0\qquad(-\beta L<x<\alpha L)\tag{191}$$

Note that Eq. (186) shows that the coordinate origin in this problem is at the location of the well, whereas the coordinate origin in Fig. 28 is at a stream edge that does not exist in this problem. The solution for the previous problem can be used to obtain the solution for this problem by setting $\gamma=0$, but only if values of α,β and x are adjusted for the difference in location of the coordinate origin. It is easier, however, to make use of the following programs:

$$\frac{sT}{Q}=W\_14\left(\frac{x}{L},\frac{y}{L},\frac{tT}{SL^2},\frac{(K'/B')L^2}{T},\frac{S}{\sigma},\alpha,\beta,n_{max}\right)\tag{192}$$

$$\frac{\eta T}{Q}=Eta\_14\left(\frac{x}{L},\frac{y}{L},\frac{tT}{SL^2},\frac{(K'/B')L^2}{T},\frac{S}{\sigma},\alpha,\beta,n_{max}\right)\tag{193}$$

These programs distribute image wells to calculate a solution, and n_{max} controls the number of image wells used in the solution. (In particular, each additional value of n adds four image wells to the system.) It is recommended that users start with $n_{max}=5$ and then keep incrementing n_{max} by 5 until the solution stops changing. A value of $n_{max}=5$ will probably be sufficient for most applications.

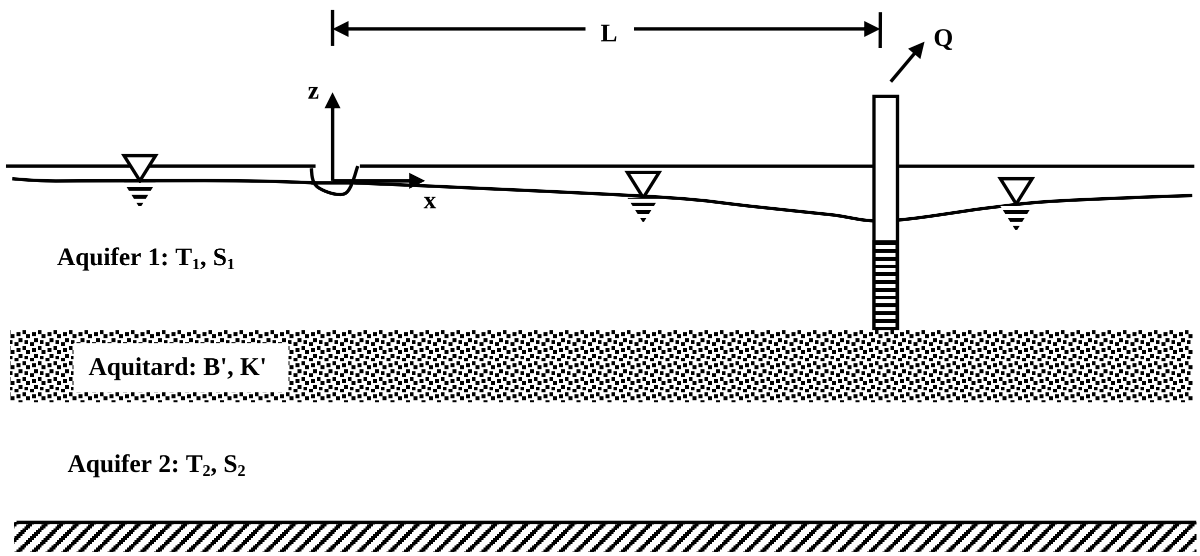


Fig. 29. Stream depletion in a leaky two-layer aquifer system.

Stream Depletion 3

Stream depletion problems often occur in the context of leaking systems of aquifers and aquitards. Fig. 29 shows stream depletion occurring in a two-layer aquifer in which water is pumped from a well in an unconfined aquifer underlain by an aquitard and a second aquifer. The solution domain consists of the entire horizontal (x, y) plane, and the stream, is assumed to have zero width and coincide with the y axis. The problem is modeled with the following equations:

$$T_1\left(\frac{\partial^2 s_1}{\partial x^2}+\frac{\partial^2 s_1}{\partial y^2}\right)=S_1\frac{\partial s_1}{\partial t}+\frac{K'}{B'}\left(s_1-s_2\right)-Q\delta\left(x-L\right)\delta\left(y\right)+\lambda\delta\left(x\right)s_1 \tag{194}$$

$$T_2\left(\frac{\partial^2 s_2}{\partial x^2}+\frac{\partial^2 s_2}{\partial y^2}\right)=S_2\frac{\partial s_2}{\partial t}+\frac{K'}{B'}\left(s_2-s_1\right) \tag{195}$$

$$\lim_{r\to\infty} s_1\left(x,y,t\right)=\lim_{r\to\infty} s_2\left(x,y,t\right)=0 \qquad \left(r\equiv\sqrt{x^2+y^2}\right) \tag{196}$$

$$s_1\left(x,y,0\right)=s_2\left(x,y.0\right)=0 \tag{197}$$

where the solution domain is the region $\left(-\infty<x<\infty,-\infty<y<\infty\right)$ for $0<t<\infty$. Drawdowns in aquifers one and two are denoted by s_1 and s_2, respectively, and the stream depletion factor, λ, is defined by Hunt et al. (2001) and Hunt (2003) to be the product of the stream width and the hydraulic conductivity of the aquitard layer beneath the stream divided by the aquitard layer thickness. In practice, λ is determined by comparing field measurements of drawdown and stream depletion with an analytic solution for the problem being considered. [Lough and Hunt (2006) give an example of this type of calculation.] Once a solution has been obtained for $s_1(x,y,t)$, then the stream depletion, ΔQ, can be calculated from the following equation:

$$\Delta Q(t) = \lambda \int_{-\infty}^{\infty} s_1(0,y,t)\, dy \tag{198}$$

Eq. (194) uses the Dupuit approximation of horizontal flow to describe slightly compressible flow in aquifer one. The second term on the right side accounts for leakage from aquifer two, the third term models a well in aquifer one with a vertical line sink and the last term requires that the difference in specific discharge in the x direction along the y axis equal λ times the drawdown along the y axis, which models the stream with a horizontal line sink. Eq. (195) uses the Dupuit approximation to describe slightly compressible flow in aquifer two, with the last term accounting for leakage from aquifer one. The well and streambed do not penetrate aquifer two. Eq. (196) requires that drawdowns vanish at infinity for all time in both aquifers, and Eq. (197) requires zero drawdown in both aquifers at $t=0$ throughout the entire (x, y) plane. Eq. (198) calculates the total stream depletion by integrating the x component of specific discharge in aquifer one along the entire stream at y = 0. These equations reduce to the equations for a Hantush-Jacob leaky aquifer when S_2 becomes infinite. (In this case $\partial s_2/\partial t = 0$ for all values of x, y and t and $s_2 = 0$ everywhere for all time when the initial condition given by Eq. (197) is used.) However, it would be incorrect to do so in an applied problem since aquifer two will always have finite drawdown values, and this would lead to seriously large underestimates of stream depletion at large values of time.

This problem has been solved by the writer and evaluated with the following programs:

$$\frac{\Delta Q}{Q} = Q\_15\left(\frac{tT_1}{S_1L^2}, \frac{T_2}{T_1}, \frac{S_2}{S_1}, \frac{(K'/B')L^2}{T_1}, \frac{\lambda L}{T_1}\right) \tag{199}$$

$$\frac{s_1T_1}{Q} = W\_15\left(\frac{x}{L}, \frac{y}{L}, \frac{tT_1}{S_1L^2}, \frac{T_2}{T_1}, \frac{S_2}{S_1}, \frac{(K'/B')L^2}{T_1}, \frac{\lambda L}{T_1}\right) \tag{200}$$

$$\frac{s_2T_1}{Q} = Eta\_15\left(\frac{x}{L}, \frac{y}{L}, \frac{tT_1}{S_1L^2}, \frac{T_2}{T_1}, \frac{S_2}{S_1}, \frac{(K'/B')L^2}{T_1}, \frac{\lambda L}{T_1}\right) \tag{201}$$

Fig. 30 shows a comparison of stream depletion calculated with Eq. (199) and an approximation obtained by using the sum of the two aquifer transmissivities to replace the transmissivity in the single aquifer solution obtained by Hunt(1999). The approximation is close enough to suggest that replacing the aquifer transmissivity in the single-aquifer solution Q_4 [Eq. (44)] with a sum of aquifer transmissivities should give a useful approximate solution for a stream depletion problem with a layered system of aquifers and aquitards.

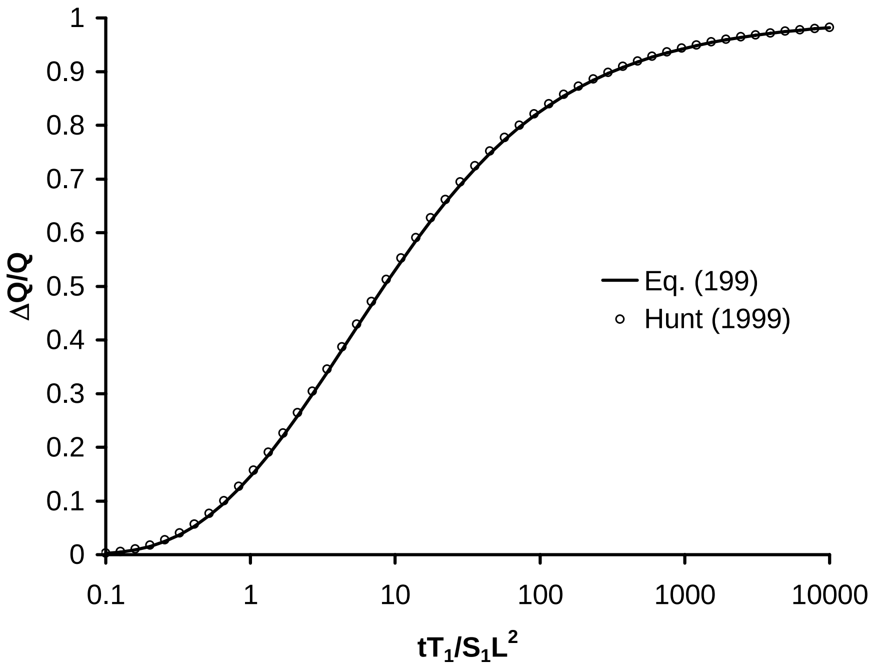


Fig. 30. Use of the Hunt (1999) solution for a single aquifer to approximate stream depletion in a two-aquifer problem. ($T_2/T_1 = 0.2$, $S_2/S = 0.001$, $(K'/B')L^2/T_1 = 1$ and $\lambda L/T_1 = 1$)

Stream Depletion 4

Ward and Lough (2011) have published a solution for the hydrogeology shown in Fig. 29 when a well pumps water from the bottom aquifer while the top aquifer remains unpumped. This problem is described by the solution of the following equations:

$$T_1\left(\frac{\partial^2 s_1}{\partial x^2}+\frac{\partial^2 s_1}{\partial y^2}\right)=S_1\frac{\partial s_1}{\partial t}+\frac{K'}{B'}(s_1-s_2)+\lambda\delta(x)s_1 \tag{202}$$

$$T_2\left(\frac{\partial^2 s_2}{\partial x^2}+\frac{\partial^2 s_2}{\partial y^2}\right)=S_2\frac{\partial s_2}{\partial t}+\frac{K'}{B'}(s_2-s_1)-Q\delta(x-L)\delta(y) \tag{203}$$

$$\lim_{r\to\infty} s_1(x,y,t)=\lim_{r\to\infty} s_2(x,y,t)=0 \qquad \left(r\equiv\sqrt{x^2+y^2}\right) \tag{204}$$

$$s_1(x,y,0)=s_2(x,y.0)=0 \tag{205}$$

$$\Delta Q(t)=\lambda\int_{-\infty}^{\infty} s_1(0,y,t)\,dy \tag{206}$$

The solutions can be calculated with the following programs:

$$\frac{\Delta Q}{Q}=Q\_16\left(\frac{tT_2}{S_2L^2},\frac{T_1}{T_2},\frac{S_1}{S_2},\frac{(K'/B')L^2}{T_2},\frac{\lambda L}{T_2}\right) \tag{207}$$

$$\frac{s_1 T_2}{Q} = W\_16\left(\frac{x}{L},\frac{y}{L},\frac{tT_2}{S_2 L^2},\frac{T_1}{T_2},\frac{S_1}{S_2},\frac{(K'/B')L^2}{T_2},\frac{\lambda L}{T_2}\right) \tag{208}$$

$$\frac{s_2 T_2}{Q} = Eta\_16\left(\frac{x}{L},\frac{y}{L},\frac{tT_2}{S_2 L^2},\frac{T_1}{T_2},\frac{S_1}{S_2},\frac{(K'/B')L^2}{T_2},\frac{\lambda L}{T_2}\right) \tag{209}$$

Figs. 31 and 32 show stream depletion and drawdowns calculated from Eqs. (207) and (208)-(209), respectively. Note that the three changes in curvature that characterize both stream depletion and drawdown in a Boulton type aquifer (i.e. When the free surface is contained within an overlying aquitard rather than a more permeable aquifer) are missing in both the stream depletion plot and the drawdown curve for the top aquifer. However, three changes in curvature are present in the drawdown curve for the bottom aquifer. This suggests that delayed-yield behaviour, characterized by three changes of curvature for a drawdown curve in a pumped aquifer, can be caused either by having a free surface contained within an aquitard overlying a pumped aquifer or by having a free surface within an overlying aquifer provided that an aquitard lies between the free surface and the underlying pumped aquifer.

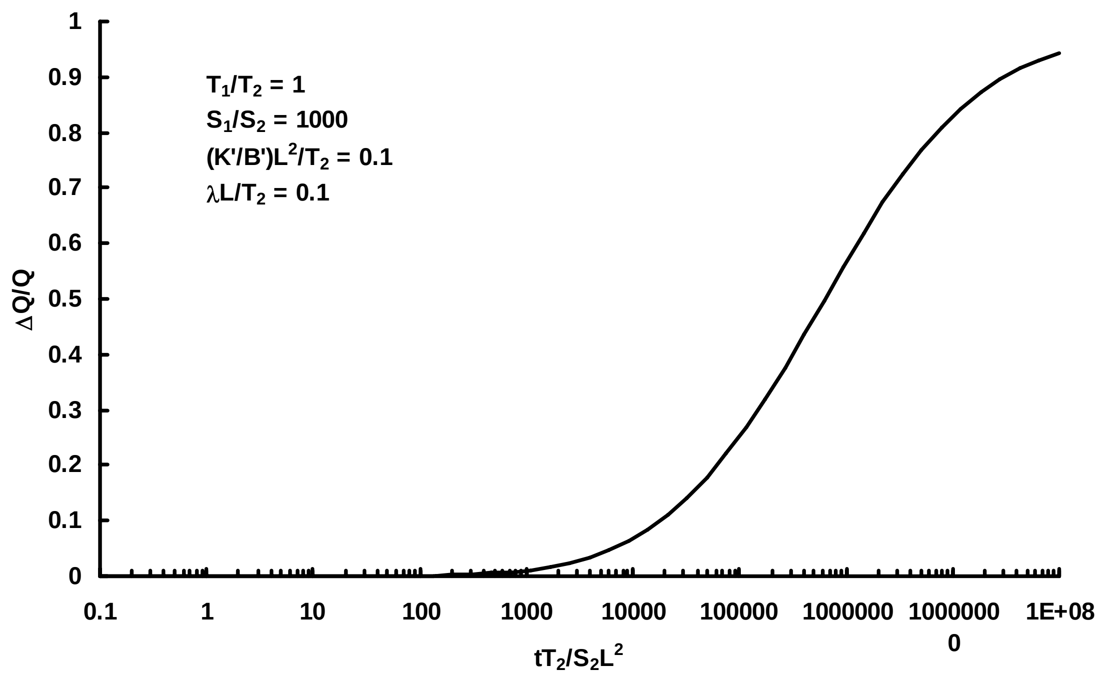


Fig. 31. Stream depletion calculated from Eq. (207).

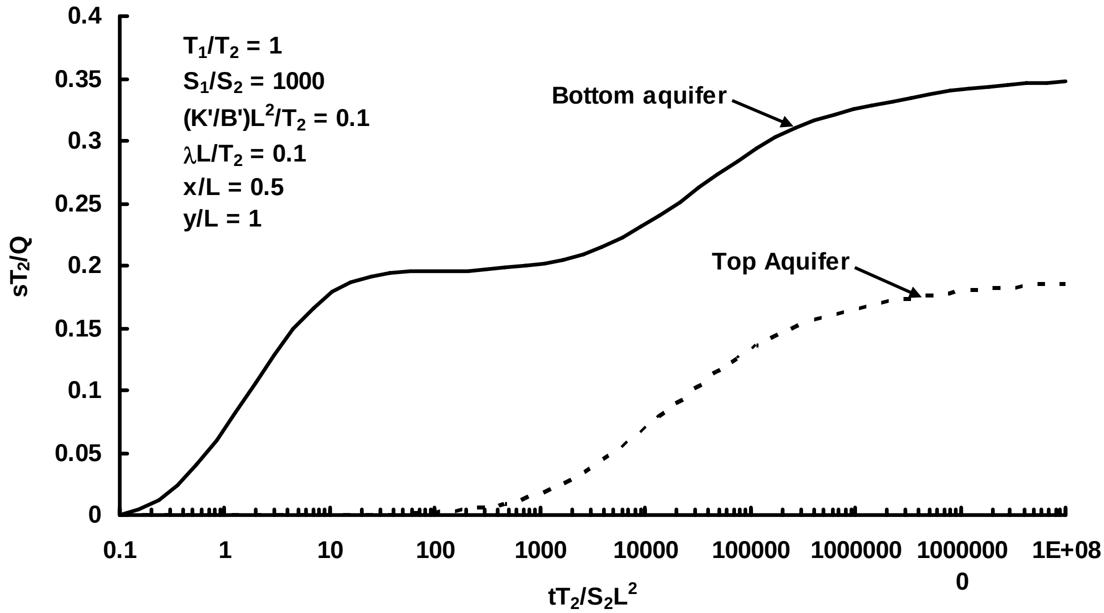


Fig. 32. Drawdowns for the top and bottom aquifers calculated from Eqs. (208)-(209).

References

Boulton, N.S. 1963. Analysis of data from non-equilibrium pumping tests allowing for delayed yield from storage. *Proc. Inst. Civil Eng.*, **26**, Paper No. 6693: 469-482.

Boulton, N.S. 1973. The influence of delayed drainage on data from pumping tests in unconfined aquifers. *J. Hydro*.19, no.2: 157-169.

Cooley, R. L., and Case, C. M. (1973). "Effect of a water table aquitard on drawdown in an underlying pumped aquifer." *Water Resour. Res.*, 9(2), 434-447.

Hantush, M.S. and C.E. Jacob. 1955. Non-steady radial flow in an infinite leaky aquifer. *Trans. Am. Geophys. Union*, 36, no.1: 95-100.

Hantush, M.S. 1967. Flow to wells in aquifers separated by a semipervious layer. *Journal of Geophysical Research*, 72(6), 1709-1720.

Hunt, B. 1983. *Mathematical analysis of groundwater resources*. Butterworths, London. 122-123, 217-221.

Hunt, B. (1998) Contaminant source solutions with scale-dependent dispersivities, ASCE Jnl. Hydrologic Engrg., Vol.3, No.4, 268-275.

Hunt, B. 2003a. Unsteady stream depletion when pumping from semi-confined aquifer. *ASCE Journal of Hydrologic Engineering*, Vol. 8, No.1, 12-19.

Hunt, B. 2004. A spring-depletion solution. ASCE *Journal of Hydrologic Engineering*, Vol.9, No.2: 144-149.

Hunt, B. 2005. Flow to vertical and non-vertical wells in leaky aquifers. *ASCE Journal of Hydrologic Engineering*, Vol. 10, No. 6: 477-484.

Hunt, B. 2008. Stream depletion for streams and aquifers with finite widths. ASCE *Journal of Hydrologic Engineering*, Vol. 13, No. 2, 80-89.

Hunt, B. and D. Scott 2005. An extension of the Hantush and Boulton solutions. ASCE Journal of Hydrologic Engineering, Vol. 10, No. 3, 223-236.

Hunt, B. and D. Scott 2007. Flow to a well in a two-aquifer system. ASCE *Journal of Hydrologic Engineering*, Vol. 12, No. 2: 146-155.

Hunt, B. and M. Smith 2008. Improved spring depletion solution and analysis. ASCE *Journal of Hydrologic Engineering*, Vol. 13, No.. 2, 80-89.

Kruseman, G.P. and N.A. DeRidder 1979. *Analysis and evaluation of pumping test data*. International Institute for Land Reclamation and Improvement, P.O. Box 45, 6700 AA Wageningen , The Netherlands. 69-72 and 80-81.

Liengme, B.V. 2000. *A guide to Microsoft Excel for scientists and engineers*. 2nd edition. Butterworth/Heinemann, Auckland.

Neuman, S.P. 1972. Theory of flow in unconfined aquifers considering delayed response of the water table. *Water Resources Research*, 8, no. 4: 1031-1045.

Neuman, S.P. 1974. Effect of partial penetration on flow in unconfined aquifers considering delayed gravity response. *Water Resources Research*, 10, no. 2: 303-312.

Neuman, S.P. and P.A. Witherspoon. 1969. “Theory of flow in a confined two aquifer system. *Water Resour. Res.* 5(4), 803-816.

Theis, C.V. 1935. The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage. *Trans. Am. Geophys. Union*, 16: 519-524.

Ward, N.D. and H. Lough. 2011. Stream depletion from pumping a semiconfined aquifer in a two-layer leaky aquifer system. *Journal of Hydrologic Engineering*, Vol. 16, No. 11, 955-959.

Weir, J. 1999. Stream depletion from groundwater pumping. M.E. thesis, Department of Civil Engineering, University of Canterbury, Christchurch, New Zealand.

Zhan, H. and V. A. Zlotnik. 2002. Groundwater flow to a horizontal or slanted well in an unconfined aquifer. *Water Resour. Res*. 38 (7).