AN OPTIMAL SYMMETRIC THRESHOLD STRATEGY FOR REMOTE ESTIMATION OVER THE COLLISION CHANNEL

Xu Zhang[⋆†] *Marcos M. Vasconcelos*[†] *Wei Cui*[⋆] *Urbashi Mitra*[†]

[⋆]School of Information and Electronics, Beijing Institute of Technology, China
[†] Ming Hsieh Department of Electrical Engineering, University of Southern California, USA

ABSTRACT

A wireless sensing system with n sensors, observing independent and identically distributed continuous random variables with a symmetric probability density function, and one non-collocated estimator acting as a fusion center is considered. The sensors transmit information to the fusion center via a limited capacity communication medium modeled by a collision channel. It is assumed that there is no communication among the sensors prior to transmission, and the collision channel allows at most $k < n$ simultaneous transmissions. Assuming that each sensor uses a symmetric threshold communication strategy, the problem of designing a threshold that minimizes a mean-squared error criterion is considered. Theoretical analysis shows the existence and uniqueness of the optimal threshold for this optimization problem.

Index Terms— Remote estimation, threshold strategies, collision channel, network control systems, optimization

1. INTRODUCTION

Modern large-scale distributed systems such as cyber-physical systems and the Internet of Things often consist of components (nodes) that communicate/interact over shared networks of limited bandwidth and operate with minimal delay [1]. One way to model this constraint is to assume that, at any time instant, a limited number of packets (smaller than total number of devices) can be reliably transmitted over the network to its destination at a time. If this limit is exceeded, a collision is declared and information is lost. In order to properly share the available communication resources, the nodes must decide to transmit or not their data to the destination.

There exists a rich literature on the optimal design of remote estimation systems under different technical assumptions. Remote sensing with a single sensor under a limited number of transmissions was considered in [2]. A similar system but with costly communication was studied in [3]. Systems with energy harvesting were considered in [4, 5]. Multi-sensor systems in the presence of collisions were introduced in [6, 7]. Systems with quality feedback allowing for collisions were considered in [8, 9]. Remote M-ary detection with single transmissions and dynamics was studied in [10, 11].

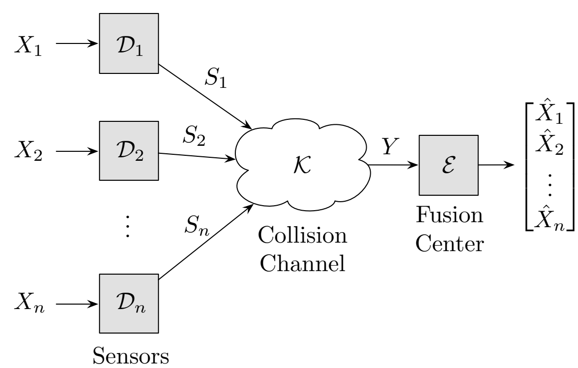

Fig. 1: System diagram for remote estimation over the collision channel.

We study a remote estimation system depicted in Fig. 1, where n distributed sensors observing identically distributed continuous random variables communicate with a fusion center over the collision channel. The channel can only support the reliable transmission of at most k packets, where $k < n$. If the number of simultaneous transmissions are larger than k, a collision symbol is observed at the fusion center. The goal is to design a transmission strategy at the sensors that minimizes the mean-squared error between the sensors observations and their estimates at the fusion center. Previous work [6] has shown that there are asymmetric threshold strategies that are optimal for this system in the case when $k = 1$. This work extends previous work by allowing the channel to support multiple ($k > 1$) simultaneous packets, and addresses the optimal design of a symmetric threshold strategy to be used by all the nodes in the network. Our theoretical analysis shows that the mean-squared error (MSE) is strictly quasi-convex in the threshold, which guarantees the existence and uniqueness of an optimal threshold.

The remainder of the paper is organized as follows. The problem formulation is presented in Section 2, where we describe the remote estimation system. The main results are derived in Sections 3 and 4. The performance of the optimal

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threshold strategy as a function of the collision channel capacity is provided in Section 5. Conclusions are provided in Section 6.

2. PROBLEM FORMULATION

In this section, we establish the problem setup for decentralized remote estimation system over the k-capacity collision channel. Consider the system diagram shown in Fig. 1. There are n sensors (or decision makers) $\mathcal{D}_1, \cdots, \mathcal{D}_n$ and a fusion center $\mathcal{E}$, which are connected by a collision channel $\mathcal{K}$. Each sensor $\mathcal{D}_i$ observes a zero-mean random variable X_i. The random variables $\{X_i\}_{i=1}^n$ are independent and identically distributed (i.i.d), and admit a symmetric probability density function (pdf) $f_X(x)$[1], where $f_X(x) > 0$ for $x \in \mathbb{R}$. Each sensor decides whether to transmit its observed measurement to the fusion center or to remain silent according to a symmetric threshold strategy, D_i, $i \in \{1, \cdots, n\}$, defined as follows.

Definition 1 (Symmetric threshold strategy)**.** *Let $D_i \in \{0, 1\}$ be the decision variable of the i-th sensor. Let $D_i = 1$ denote that the i-th sensor decides to transmit its observation, and $D_i = 0$ denote that the i-th sensor decides to remain silent. A symmetric threshold strategy for the i-th sensor is a function $\mathcal{D}_i : \mathbb{R} \to \{0, 1\}$ such that*

$$\mathcal{D}_i(x) \triangleq \mathbf{1}(|x| \geq T), \tag{1}$$

where $T \in [0, +\infty)$ denotes the threshold and $\mathbf{1}(\cdot)$ denotes the indicator function.

After making a decision, each sensor outputs a channel input, S_i, defined as follows:

$$S_i \triangleq \begin{cases} (i, X_i) & \text{if } D_i = 1 \\ \varnothing & \text{if } D_i = 0 \end{cases}, \quad i \in \{1, \ldots, n\}. \tag{2}$$

Remark 1. We assume that if a sensor decides to transmit, its unique identification number i is transmitted along with its measurement. This is done so that the receiver can identify the origin of the received communication packets without ambiguity.

The collection of n sensors share a k-capacity collision channel $\mathcal{K}$, which is defined as follows:

Definition 2 (k-capacity collision channel)**.** *The k-capacity collision channel allows at most k simultaneous transmitted packets. Let $\mathbb{D} \triangleq \{i \mid D_i = 1\}$ denote the set of indices of all transmitting sensors. The output of the collision channel Y is given by:*

$$Y \triangleq \begin{cases} \varnothing & \text{if } |\mathbb{D}| = 0 \\ \big\{(i, X_i) \mid i \in \mathbb{D}\big\} & \text{if } 1 \leq |\mathbb{D}| \leq k \\ (\mathfrak{C}, \mathbb{D}) & \text{otherwise.} \end{cases} \tag{3}$$

[1]A pdf f_X is symmetric if it satisfies:

$$f_X(-x) = f_X(x), \quad x \in \mathbb{R}.$$

The special symbol $\mathfrak{C}$ denotes a collision occurred and $\varnothing$ denotes that the channel is idle (silent).

Remark 2. When a collision occurs, we assume that the fusion center can decode the indices of the transmitting sensors.

Our purpose is to solve the following estimation problem over collision channel under the MSE criterion.

Problem 1. *Assume that each sensor uses a threshold strategy of the form given in Eq. (1). Given the number of sensors, n, the pdf of the sensors' observations, f_X, and the capacity of the collision channel, k; find a threshold T that minimizes*

$$\mathcal{J}_{n,k}(T) \triangleq \frac{1}{n}\mathbf{E}\left[\sum_{i=1}^{n}\left(X_i - \hat{X}_i\right)^2\right], \tag{4}$$

where the estimates $\hat{X}_i$ are given by:

$$\hat{X}_i = \mathbf{E}[X_i \mid Y], \quad i \in \{1, \cdots, n\}. \tag{5}$$

Remark 3. Due to the collision nature of the channel, any estimate $\hat{X}_i$ is determined by all the decision variables $\{D_i\}_{i=1}^n$, which will be derived in the following section.

3. MAIN RESULTS

In this section, we begin by providing alternative expressions for Eqs. (4) and (5). Then, we present a theoretical analysis for Problem 1, which shows the existence and uniqueness of a globally optimal threshold.

Lemma 1. *Given the decision variables $\{D_i\}_{i=1}^n$ and $\mathbb{D} = \{i \mid D_i = 1\}$, the output of the estimator can be rewritten as*

$$\hat{X}_i = \begin{cases} X_i & \text{if } |\mathbb{D}| \leq k \text{ and } i \in \mathbb{D}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

for $i \in \{1, \cdots, n\}$.

Proof. We compute the conditional expectation in Eq. (5) for every possible output of the collision channel.

When there is no collision and X_i was transmitted, i.e., $|\mathbb{D}| \leq k$ and $D_i = 1$, we have $(i, X_i) \in Y$, which implies that

$$\hat{X}_i = \mathbf{E}[X_i \mid Y] = X_i. \tag{7}$$

When a collision occurs and X_i was transmitted, i.e., $|\mathbb{D}| > k$ and $D_i = 1$, we have $Y = (\mathfrak{C}, \mathbb{D})$ and know $i \in \mathbb{D}$ from Remark 2, which implies that

$$\hat{X}_i \overset{(a)}{=} \mathbf{E}[X_i \mid D_i = 1] = \mathbf{E}[X_i \mid |X_i| \geq T] \overset{(b)}{=} 0, \tag{8}$$

where (a) is due to $\{X\}_{i=1}^n$ being a collection of independent random variables, and (b) is due to the symmetry of the pdf f_X.

When X_i is not transmitted, i.e., the index i does not appear in the channel output Y, which implies $D_i = 0$. In this case, we have

$$\hat{X}_i = \mathbf{E}[X_i \mid D_i = 0] = \mathbf{E}[X_i \mid |X_i| < T] \overset{(c)}{=} 0, \quad (9)$$

where (c) is due to the symmetry of the pdf f_X.

□

Lemma 2. *Let $\{X_i\}_{i=1}^n$ be an i.i.d. sequence distributed according to the pdf f_X. The objective function in Problem 1 can be rewritten as:*

$$\mathcal{J}_{n,k}(T) = \mathbf{E}\left[X^2\right] - \mathbf{E}\left[X^2\mathbf{1}(|X| \geq T)\right] F_{n-1,k}(T), \quad (10)$$

where

$$F_{n,k}(T) \triangleq \sum_{\ell=0}^{k-1} \binom{n}{\ell} (1-p(T))^{\ell} p(T)^{n-\ell}, \quad (11)$$

and

$$p(T) \triangleq \mathbf{P}(|X| < T). \quad (12)$$

Sketch of Proof. Define the following event:

$$\mathfrak{E}_{i,k} \triangleq \Big\{ \sum_{\ell \neq i} D_\ell \leq k-1 \Big\} \quad (13)$$

and let $\mathfrak{E}_{i,k}^c$ denote its complement. Using the law of total expectation and the independence of D_i and $\{D_\ell\}_{\ell \neq i}$, the objective function in Eq. (4) becomes

$$\begin{aligned}\mathcal{J}_{n,k}(T) = \frac{1}{n}\sum_{i=1}^{n}\Bigg[&\mathbf{E}\left[\left(X_i - \hat{X}_i\right)^2 \;\Big|\; D_i = 0\right]\mathbf{P}(D_i = 0)\\ &+ \mathbf{E}\left[\left(X_i - \hat{X}_i\right)^2 \;\Big|\; D_i = 1, \mathfrak{E}_{i,k}^c\right]\mathbf{P}(D_i = 1)\mathbf{P}(\mathfrak{E}_{i,k}^c)\\ &+ \mathbf{E}\left[\left(X_i - \hat{X}_i\right)^2 \;\Big|\; D_i = 1, \mathfrak{E}_{i,k}\right]\mathbf{P}(D_i = 1)\mathbf{P}(\mathfrak{E}_{i,k})\Bigg].\end{aligned} \quad (14)$$

Substituting Eq. (6) into Eq. (14), using the properties of the indicator function, and using the i.i.d. property of $\{X_i\}_{i=1}^n$ completes the proof. Several key algebraic manipulations which exploit the previously stated properties are omitted for brevity. □

Theorem 1. *There is a unique threshold such that $T^\star = \arg\min_{T\in\mathbb{R}} \mathcal{J}_{n,k}(T)$ for Problem 1.*

Sketch of Proof. We begin by calculating the derivative $\mathcal{J}'_{n,k}(T)$, which can be factored as

$$\mathcal{J}'_{n,k}(T) = F'_{n-1,k}(T) \underbrace{\left[\frac{2T^2 f_X(T) F_{n-1,k}(T)}{F'_{n-1,k}(T)} - \mathbf{E}\left[X^2\mathbf{1}(|X| \geq T)\right]\right]}_{\triangleq h(T)}, \quad (15)$$

for $T > 0$, where $F'_{n-1,k}(T)$ is the derivative of $F_{n-1,k}(T)$. It can be shown that $F'_{n-1,k}(T) > 0$ for all $T > 0$.

By substituting the definitions of $F_{n-1,k}(T)$ and $F'_{n-1,k}(T)$ [12], $h(T)$ becomes

$$h(T) = T^2 p(T) \sum_{j=0}^{k-1}\left[\frac{(k-1)!(n-1-k)!}{j!(n-1-j)!} \cdot \left(\frac{p(T)}{1-p(T)}\right)^{k-j-1}\right] - \mathbf{E}\left[X^2\mathbf{1}(|X| \geq T)\right],$$

Next, we can prove that $h(T)$ is a strictly increasing continuous function for $T > 0$ by using the following facts: 1) $p(T) \in (0,1)$ and $-\mathbf{E}\left[X^2\mathbf{1}(|X| \geq T)\right]$ are strictly increasing functions for $T > 0$; 2) the product of two positive strictly increasing functions is still a positive strictly increasing function; 3) the sum of two strictly increasing functions is still a strictly increasing function.

With $\lim_{T\to 0^+} h(T) < 0$ and $\lim_{T\to+\infty} h(T) > 0$, there exists only one $T^\star \in (0, +\infty)$ such that $h(T^\star) = 0$. Since $F'_{n-1,k}(T) > 0$ and $\mathcal{J}'_{n,k}(T) = F'_{n-1,k}(T)h(T)$, there is a unique $T^\star$ that minimizes $\mathcal{J}_{n,k}(T)$ for $T \in (0, +\infty)$. Finally, the continuity of $\mathcal{J}_{n,k}(T)$ implies that $T^\star$ minimizes $\mathcal{J}_{n,k}(T)$ for $T \in [0, +\infty)$. □

Remark 4. A closed-form expression to $T^\star$ is unlikely to exist, but we can compute it via iterative methods. Due to the continuity and unimodality of $\mathcal{J}_{n,k}(T)$ from Theorem 1, we can use one-dimensional search algorithms, such as the golden section search or Newton's method, to determine the optimal threshold [13].

4. APPROXIMATE LOWER BOUND

As mentioned in [14], when the goal is to minimize the MSE for zero-mean independent variables such as in Problem 1, a *centralized* scheme is to transmit the k largest measurements in magnitude. The performance of this strategy serves as a lower bound to *decentralized* communication schemes over the collision channel with capacity k. For the top-k strategy, the cost function becomes

$$\mathcal{J}_L = \frac{1}{n}\sum_{i=k+1}^{n} \mathbf{E}\left[Z_{(i)}^2\right], \quad (16)$$

where $Z_{(i)}$ is the i-th largest value in $\{|X_k|\}_{k=1}^n$.

From classical results on order statistics [15], we can approximate the mean and variance of $Z_{(i)}$, which leads to:

$$\mathcal{J}_L \approx \frac{1}{n}\sum_{i=k+1}^{n}\left\{\left[F_Z^{-1}\left(\frac{n+1-i}{n+1}\right)\right]^2 + \frac{i(n+1-i)}{(n+1)^2(n+2)}\frac{1}{\left(f_Z\left(F_Z^{-1}\left(\frac{n+1-i}{n+1}\right)\right)\right)^2}\right\}, \quad (17)$$

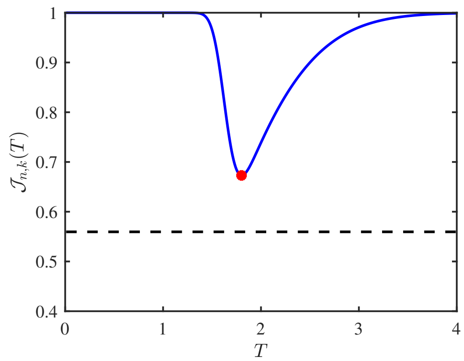

Fig. 2: Cost function $\mathcal{J}_{n,k}(T)$ as a function of the threshold T for standard Gaussian observations with with $n = 200$ sensors and a collision channel with capacity $k = 20$ packets. The red dot denotes the optimal point. The dotted horizontal line represents the centralized lower bound given in Eq. (17).

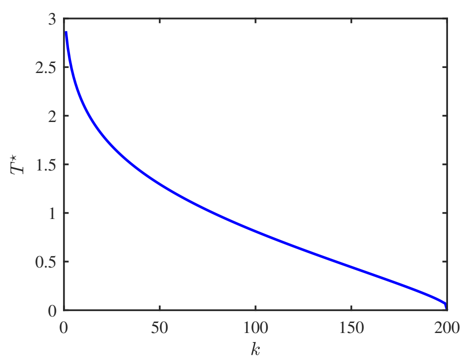

Fig. 3: Optimal threshold $T^\star$ as a function of the collision channel capacity, k, for standard Gaussian observations with $n = 200$ sensors.

where f_Z and F_Z are the pdf and cdf of $Z = |X|$, respectively. The approximations converge to the true values as n tends to infinity.

5. SIMULATIONS

In this section, we provide numerical results to examine the performance of the proposed method.

5.1. MSE as a function of the threshold

In this subsection, we present the MSE $\mathcal{J}$ as a function of the threshold T. We set the number of sensors to be $n = 200$ and the capacity of the collision channel to be $k = 20$ packets. The observations at the sensors are i.i.d. standard Gaussian random variables ($X_i \sim \mathcal{N}(0,1), i \in \{1,\ldots,n\}$). Figure 2 shows the cost and, as expected, the existence of a single globally optimal threshold, which can be numerically calculated using one-dimensional search algorithms. The lower bound is calculated using Eq. (17).

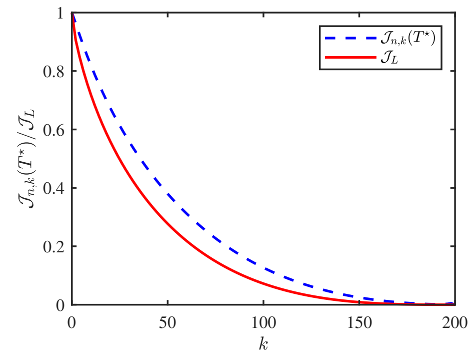

Fig. 4: Optimal cost $\mathcal{J}^\star_{n,k}$ and the lower bound $\mathcal{J}_L$ as functions of the capacity of the collision channel k (performance of the decentralized and centralized schemes) with $n = 200$. The observations are i.i.d. standard Gaussian variables.

5.2. Optimal threshold as a function of the capacity of the collision channel

We next show the relationship between the optimal threshold and the capacity of the collision channel for the decentralized scheme. We set $n = 200$ and increase k from 1 to 200. The observations are i.i.d. standard Gaussian random variables. Fig. 3 shows that the optimal threshold $T^\star$ decreases as the capacity k increases. As expected when $k = n$, i.e., the channel supports all packets, the optimal threshold is $T^\star = 0$.

5.3. Optimal MSE as a function of the capacity of the collision channel

Finally, we examine the relationship between the optimal MSE and the capacity of the collision channel k for the decentralized and centralized schemes. We set $n = 200$ and increase k from 0 to 200. As before, the observations are i.i.d. standard Gaussian random variables. The result is shown in Fig. 4. As expected as the capacity k increases, more measurements are successfully received at the fusion center, and the mean-squared error decreases. In addition, the decentralized scheme performs reasonably close to the centralized scheme.

6. CONCLUSIONS

We have studied the design of a symmetric threshold strategy for a decentralized remote estimation system over the collision channel. Assuming there is no communication among the sensors prior to transmission, and that the channel allows at most $k < n$ simultaneous transmissions. Our theoretical analysis showed the existence and uniqueness of an optimal threshold under the MSE criterion. Numerical results have been presented that shows that the decentralized scheme may have a performance reasonable close to the optimal centralized scheme.

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