



Optimal threshold strategies for estimation over the collision channel with communication costs

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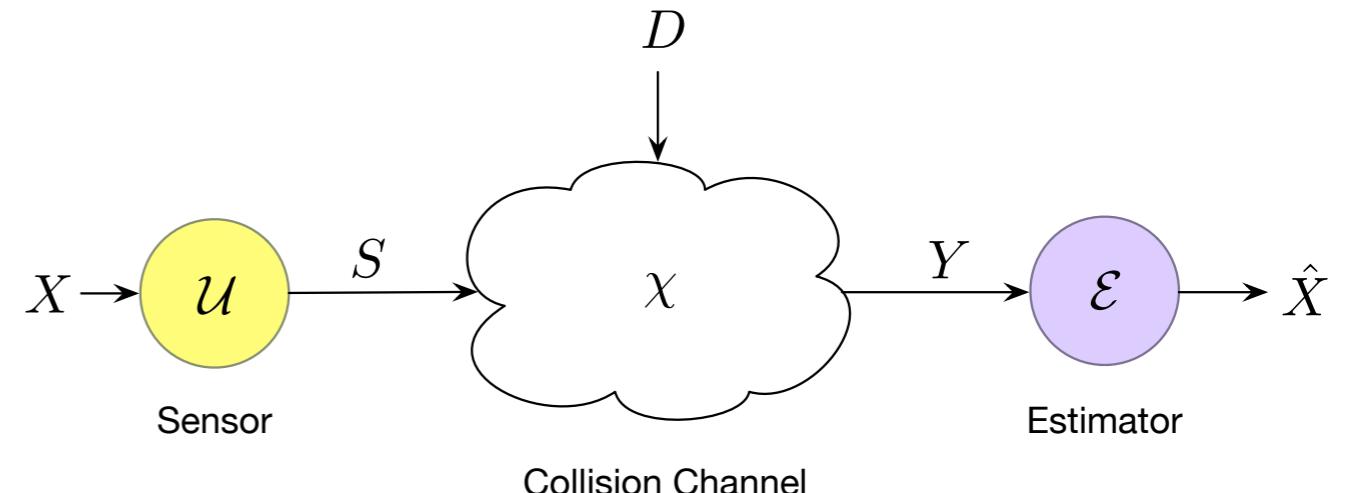
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ECEGSA academic seminar

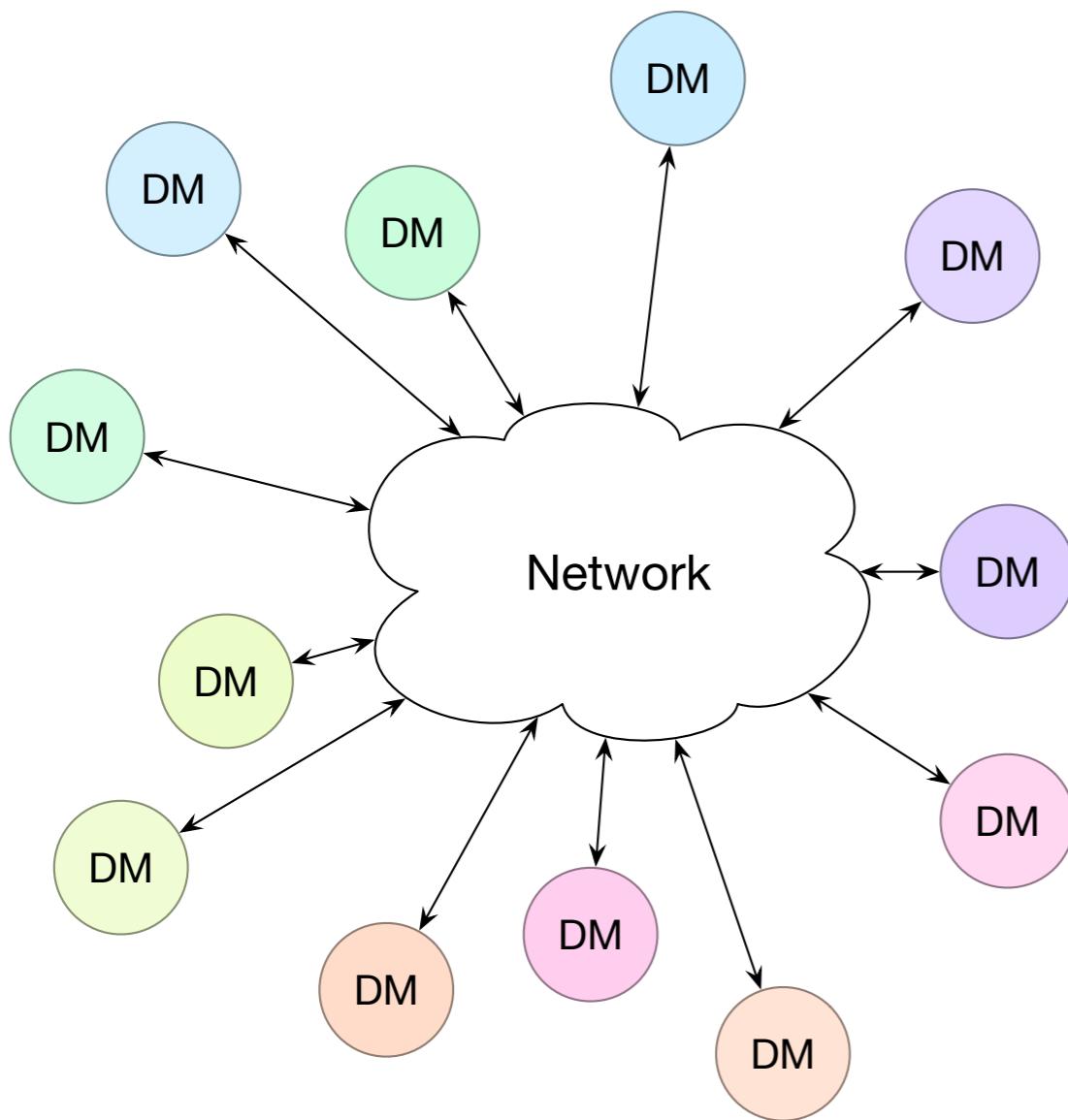
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Outline

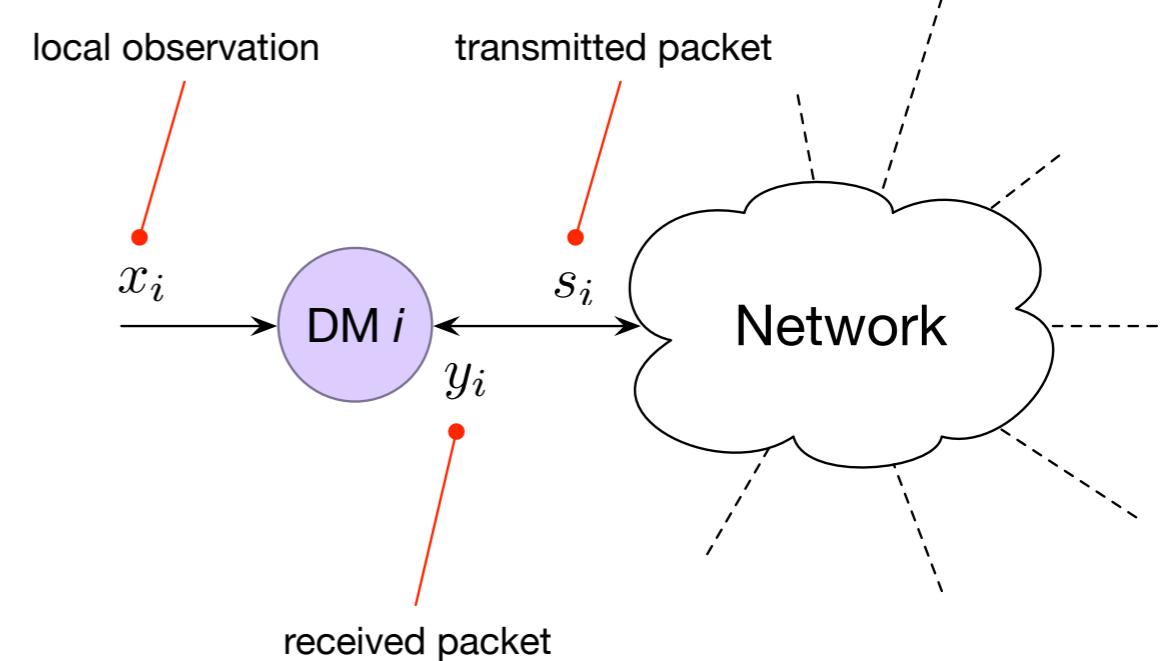
- Context and motivation
- Problem statement
- Our main results:
 1. Existence of optimal thresholds
 2. Symmetry of optimal thresholds
 3. Iterative method and its convergence
- Conclusion and future work



Motivation



Networked decision system

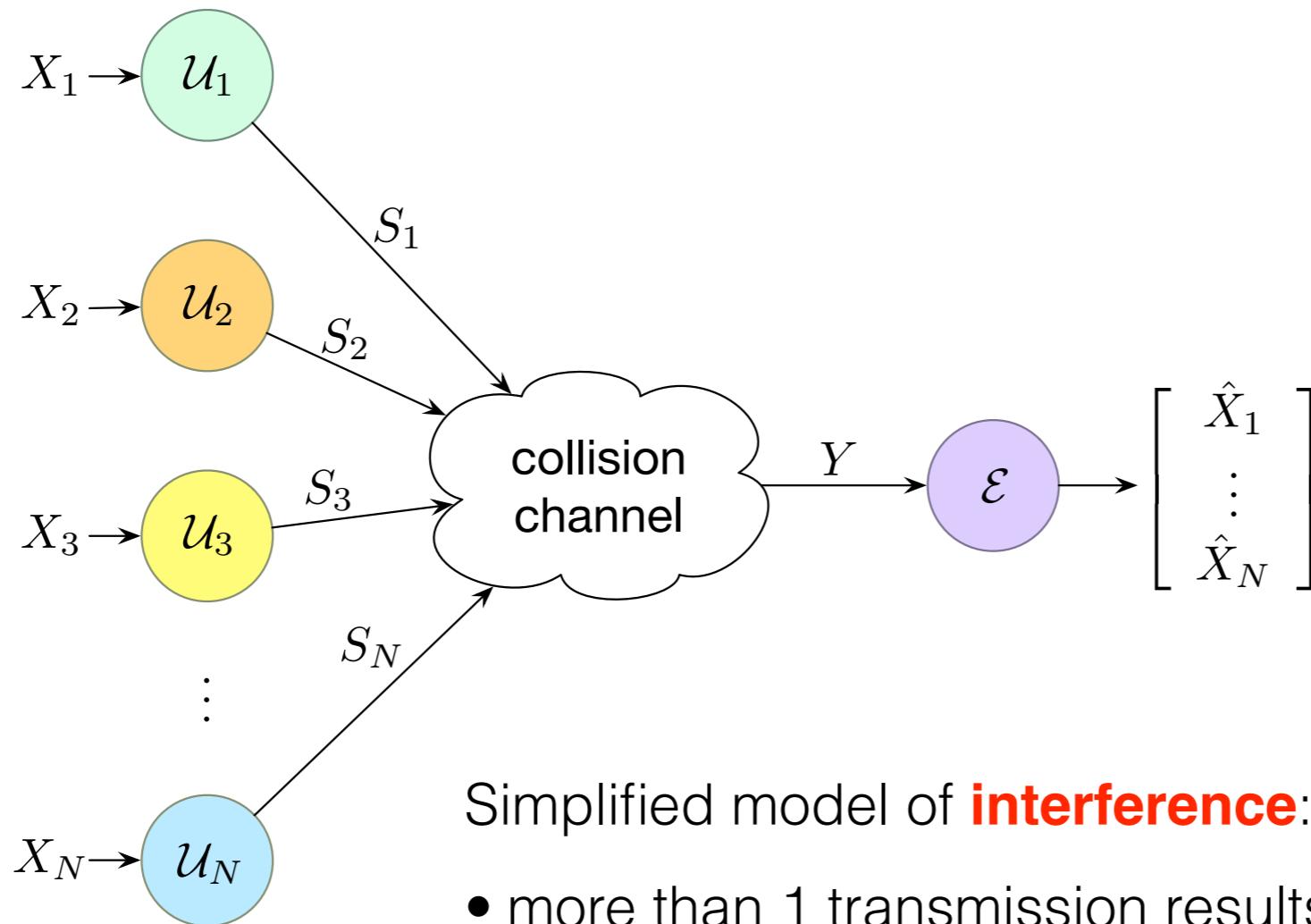


Capabilities of each node

Applications: distributed sensing, detection, computation, control, ...

Goal: study the effect of **interference** in **networked estimation**

The collision channel

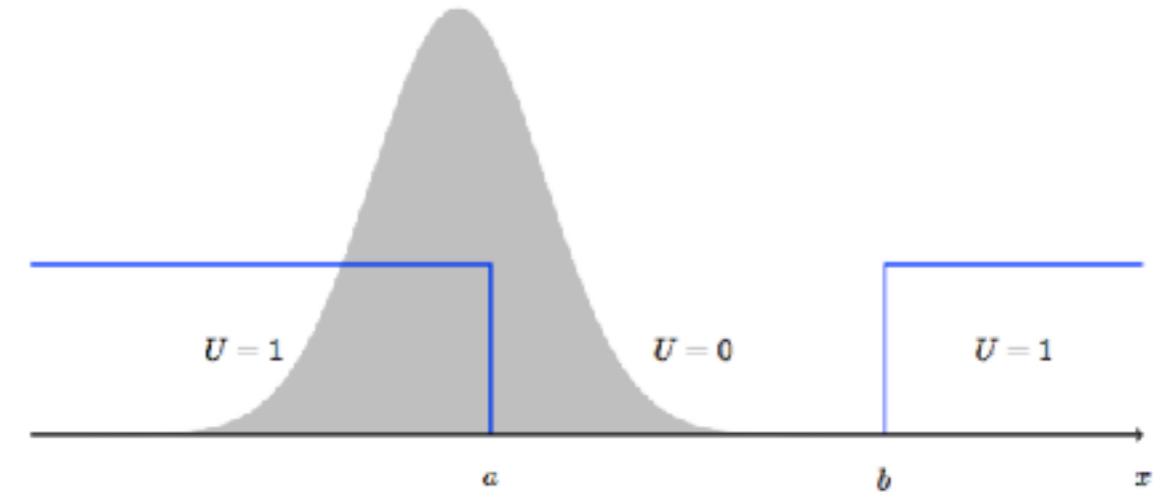
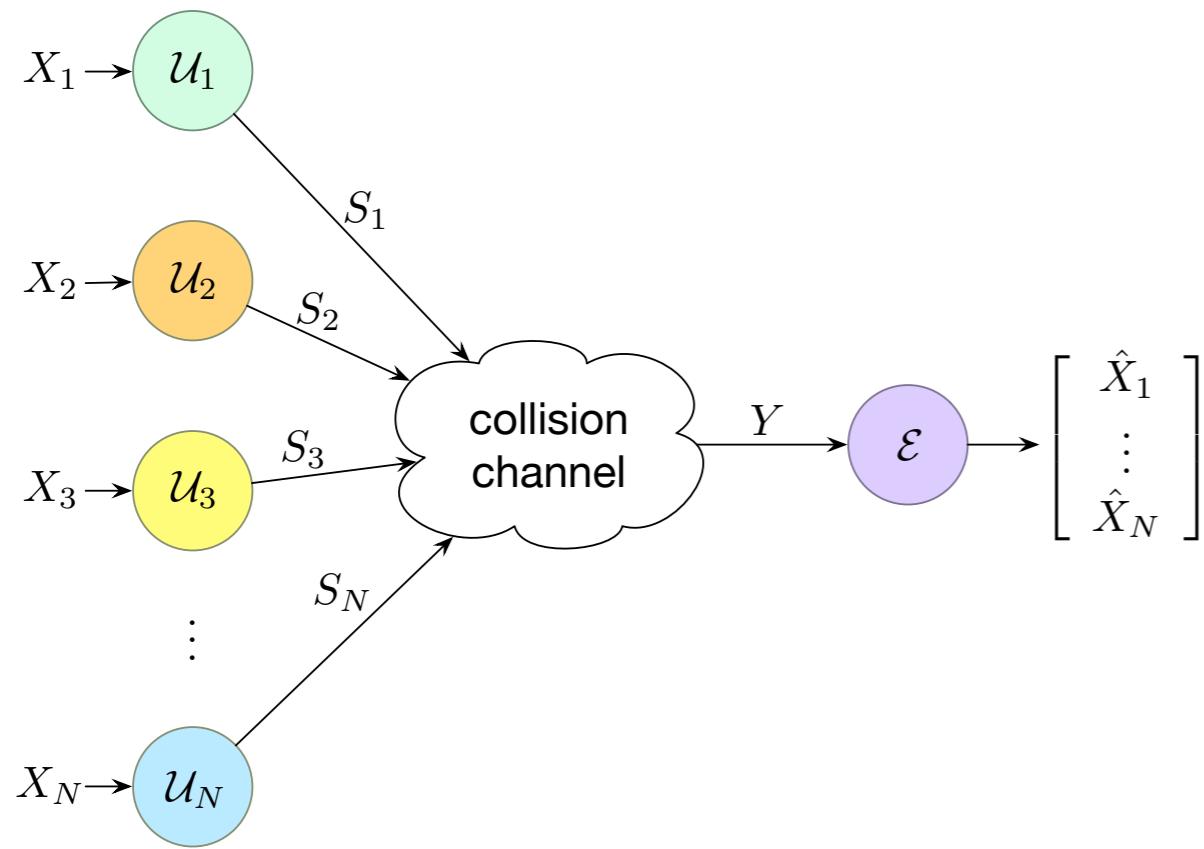


Simplified model of **interference**:

- more than 1 transmission results in a **collision**
- no transmissions results in an **erasure**

$$\mathcal{J}(\mathcal{U}_1, \dots, \mathcal{U}_N, \mathcal{E}) = \mathbb{E} \left[\sum_{i=1}^N (X_i - \hat{X}_i)^2 \right]$$

Our previous work on the collision channel



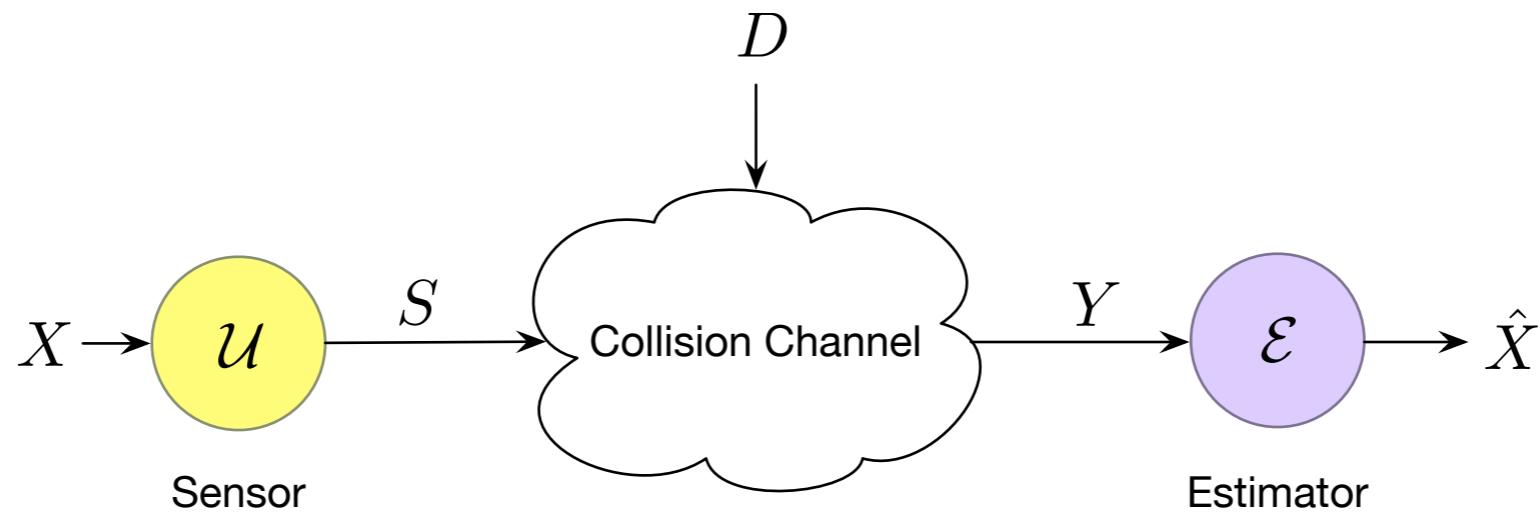
threshold structure

$$\mathcal{U}(x) = \begin{cases} 0 & a \leq x \leq b \\ 1 & \text{otherwise} \end{cases}$$

1. Optimal policies have a **deterministic threshold structure**
2. Structure holds for any set of independent continuous random variables

[Vasconcelos and Martins - Allerton '13]

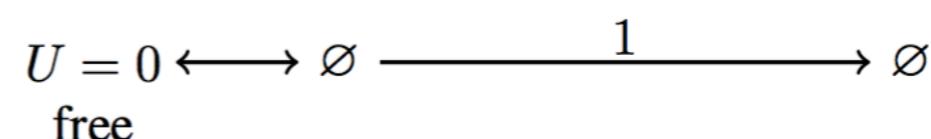
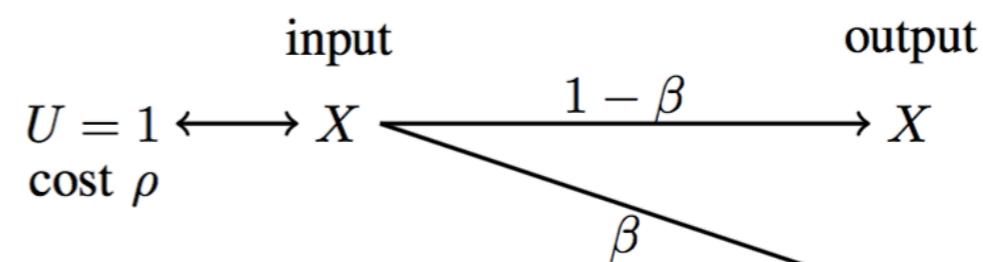
Problem formulation



$$X \sim \mathcal{N}(0, \sigma^2)$$

$$D \sim \mathcal{B}(\beta)$$

$$X \perp\!\!\!\perp D$$

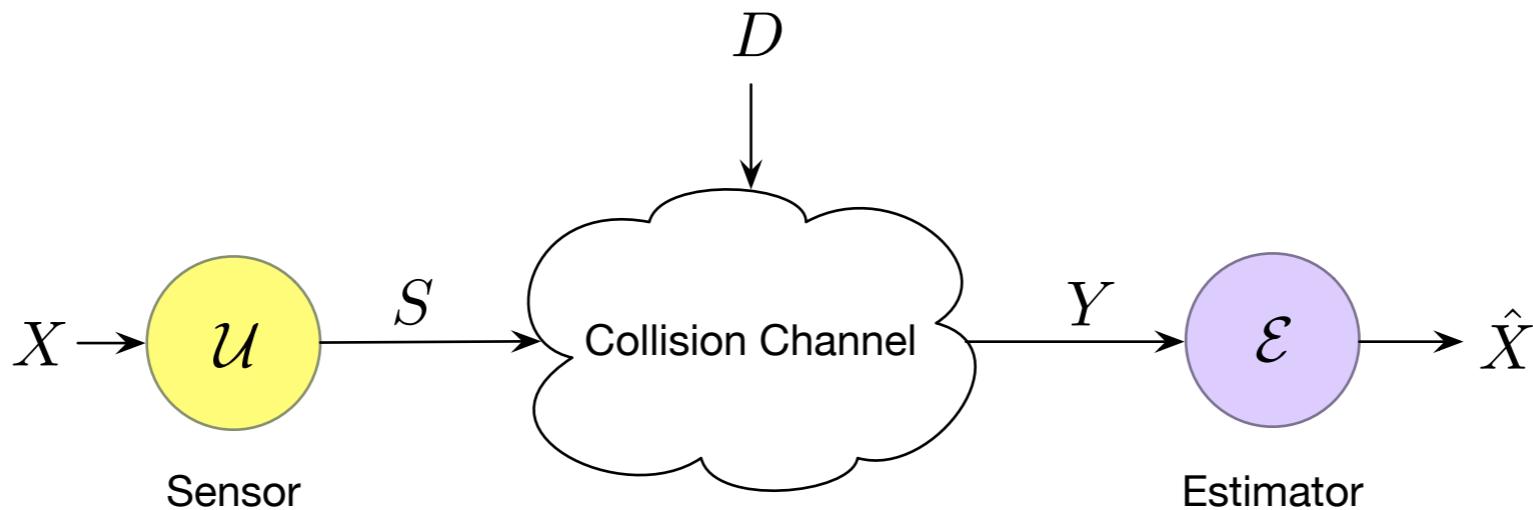


Collision channel

$$\mathcal{J}(\mathcal{U}, \mathcal{E}) = \mathbb{E}[(X - \hat{X})^2] + \rho \Pr(U = 1)$$

communication cost

Problem formulation



$$\mathcal{U}(x) = \begin{cases} 0 & a \leq x \leq b \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{E}(y) = \begin{cases} x & y = x \\ \hat{x}_\emptyset & y = \emptyset \\ \hat{x}_{\mathfrak{C}} & y = \mathfrak{C} \end{cases}$$

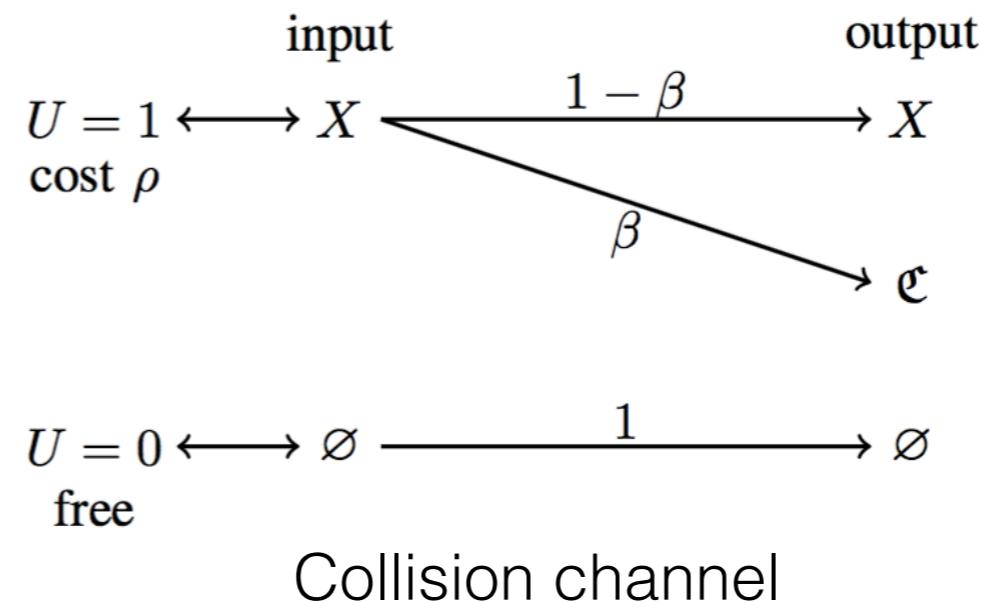
Minimize:

$$\mathcal{J}(a, b, \hat{x}_\emptyset, \hat{x}_{\mathfrak{C}}) = \int_{[a,b]} (x - \hat{x}_\emptyset)^2 f_X(x) dx + \int_{\bar{\mathbb{R}} \setminus [a,b]} [\beta(x - \hat{x}_{\mathfrak{C}})^2 + \rho] f_X(x) dx$$

binary quantization problem

Why is this problem relevant?

1. **New channel model** for networked control
2. Presence of collisions cause the **asymmetry** of optimal thresholds
3. **Building block** for sequential decentralized estimation problems



1. The **packet-drop channel** is a special case when $\emptyset = C$
2. The **collision channel** is widely used in wireless communications

Existence of optimal thresholds

$$\mathcal{J}(a, b, \hat{x}_\emptyset, \hat{x}_{\mathfrak{C}}) = \int_{[a,b]} (x - \hat{x}_\emptyset)^2 f_X(x) dx + \int_{\bar{\mathbb{R}} \setminus [a,b]} [\beta(x - \hat{x}_{\mathfrak{C}})^2 + \rho] f_X(x) dx$$

minimize $\mathcal{J}(a, b, \hat{x}_\emptyset, \hat{x}_{\mathfrak{C}})$
subject to $a \leq b$

$$x \in [a^*, b^*] \Leftrightarrow (x - \hat{x}_\emptyset)^2 \leq \beta(x - \hat{x}_{\mathfrak{C}})^2 + \rho$$

necessary optimality condition

$$a(\hat{x}), b(\hat{x}) \triangleq \frac{1}{1-\beta} \left[(\hat{x}_\emptyset - \beta \hat{x}_{\mathfrak{C}}) \pm \sqrt{\beta(\hat{x}_\emptyset - \hat{x}_{\mathfrak{C}})^2 + (1-\beta)\rho} \right] \quad \hat{x} \triangleq (\hat{x}_\emptyset, \hat{x}_{\mathfrak{C}})$$

Define a new cost: $\mathcal{J}_q(\hat{x}) \triangleq \mathcal{J}(a(\hat{x}), b(\hat{x}), \hat{x}_\emptyset, \hat{x}_{\mathfrak{C}})$

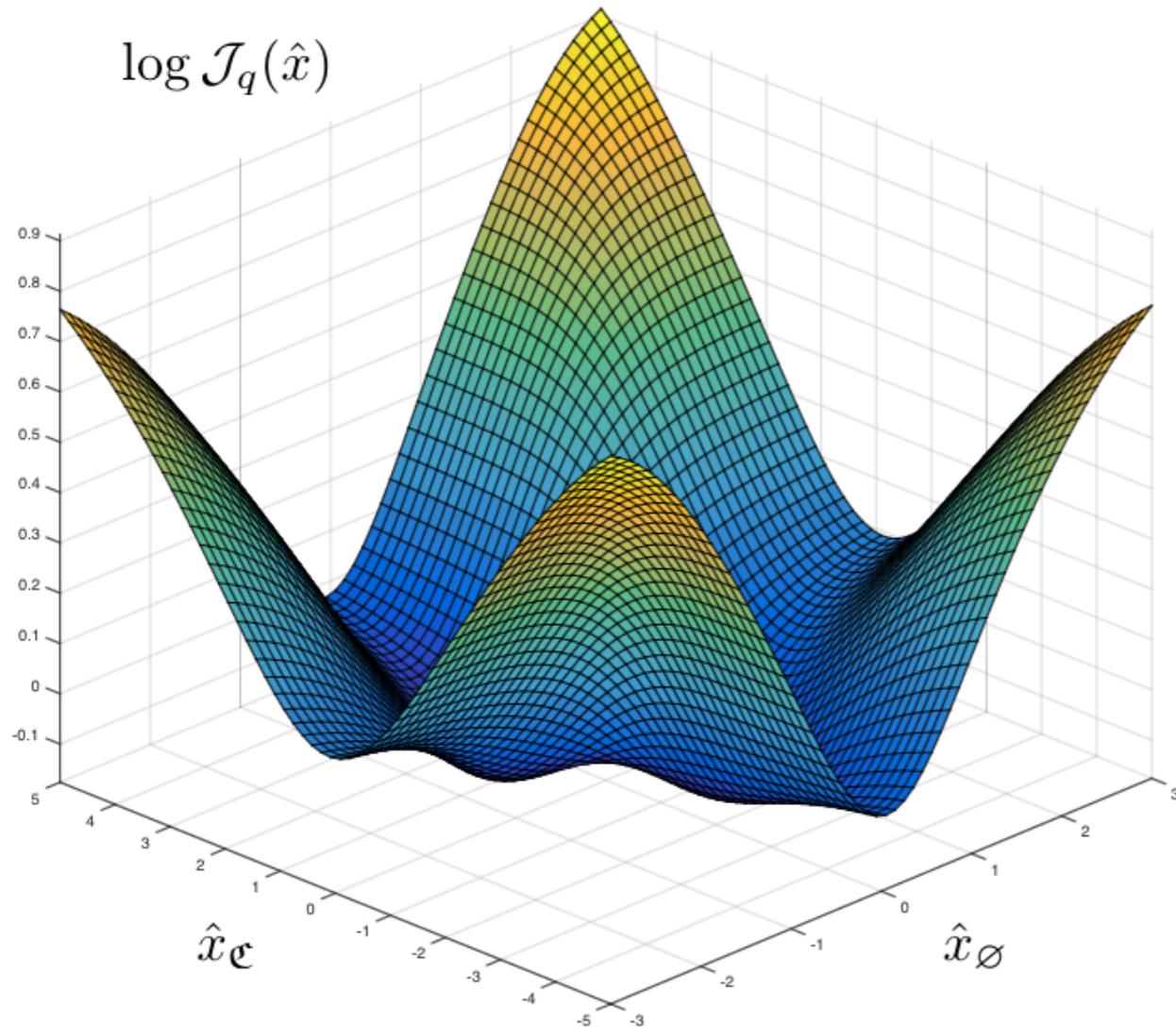
minimize $\mathcal{J}_q(\hat{x})$

main mathematical object
of this talk

Existence of optimal thresholds

Theorem 1:

Provided that $0 < \rho < +\infty$, the global minimizer of $\mathcal{J}_q(\hat{x})$ exists.



Sketch of Proof:

1. The cost function is continuous on \mathbb{R}^2
2. There exists a point \hat{x}^* such that:

$$\mathcal{J}_q(\hat{x}^*) \leq \mathcal{J}_q(\hat{x}) \text{ as } \|\hat{x}\| \rightarrow +\infty$$

Property similar to **coercivity** where:

$$\mathcal{J}_q(\hat{x}) \rightarrow +\infty \text{ as } \|\hat{x}\| \rightarrow +\infty$$

Existence of optimal thresholds

Step 1:

There exists a symmetric policy that outperforms the ***always-transmit*** and ***never-transmit*** degenerate policies.

$$\mathcal{U}(x) \equiv 1 \quad \mathcal{J}(-\infty, +\infty, \hat{x}_\emptyset, \hat{x}_{\mathfrak{C}}) = \beta(\sigma^2 + \hat{x}_{\mathfrak{C}}^2) + \rho \geq \beta\sigma^2 + \rho$$

$$\mathcal{U}(x) \equiv 0 \quad \mathcal{J}(a, a, \hat{x}_\emptyset, \hat{x}_{\mathfrak{C}}) = \sigma^2 + \hat{x}_\emptyset^2 \geq \sigma^2$$

$$0 < \rho < +\infty \quad \text{implies} \quad \mathcal{J}\left(-\sqrt{\frac{\rho}{1-\beta}}, \sqrt{\frac{\rho}{1-\beta}}, 0, 0\right) < \min\{\sigma^2, \beta\sigma^2 + \rho\}$$

Step 2:

$$\mathcal{J}_q(\hat{x}) \rightarrow \begin{cases} +\infty \\ \beta(\sigma^2 + x_{\mathfrak{C}}^2) + \rho \\ \sigma^2 + x_\emptyset^2 \end{cases} \quad \text{as} \quad \|\hat{x}\| \rightarrow +\infty$$

cost of the ***always*** and ***never-transmit*** policies

The global minimum must exist

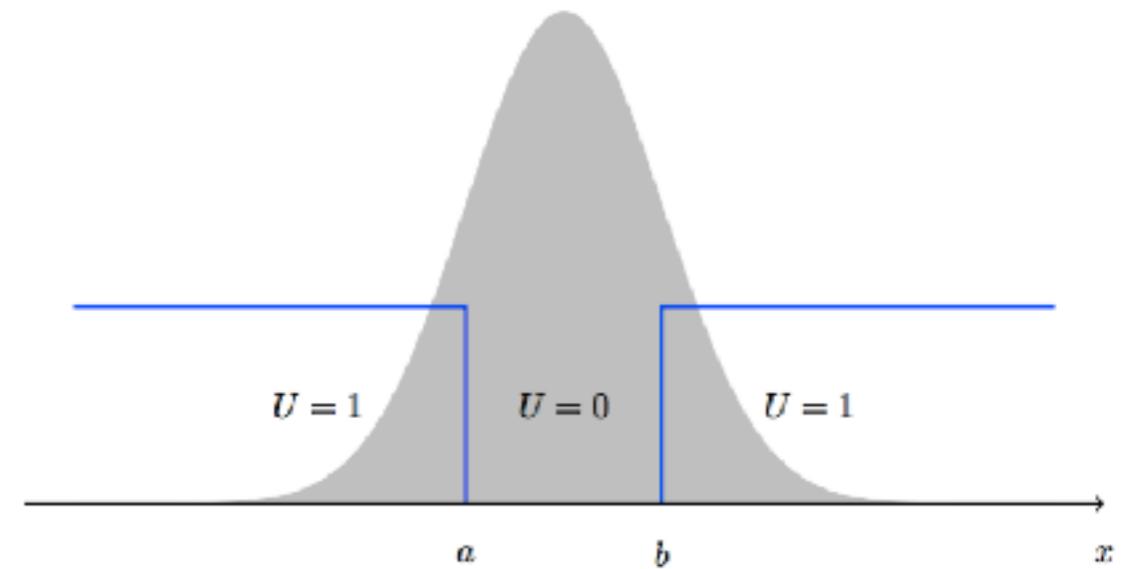
Asymmetry of optimal thresholds

First order optimality condition: $\nabla \mathcal{J}_q(\hat{x}^*) = 0$

$$\hat{x}_{\emptyset}^* = \mathbb{E}[X | X \in [a(\hat{x}^*), b(\hat{x}^*)]]$$

$$\hat{x}_{\mathfrak{C}}^* = \mathbb{E}[X | X \notin [a(\hat{x}^*), b(\hat{x}^*)]]$$

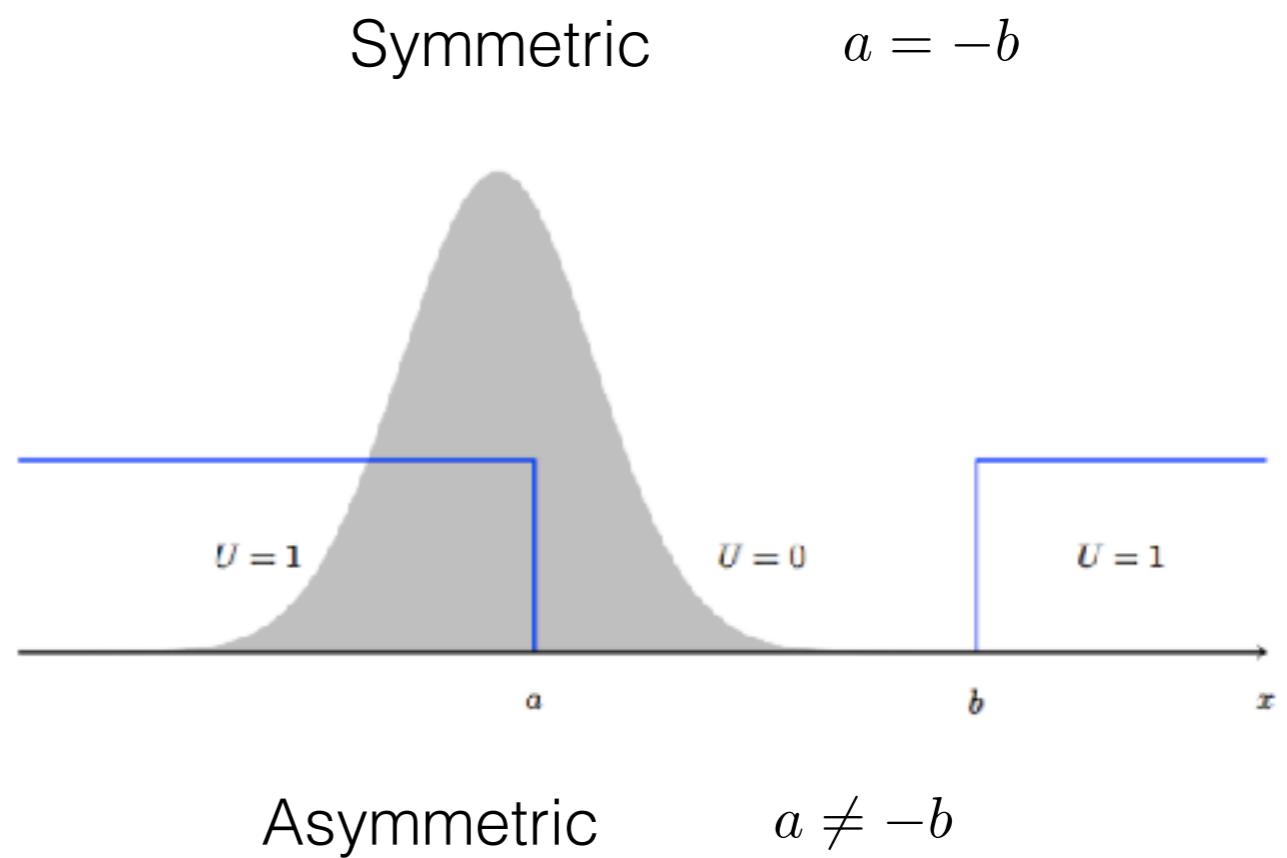
Centroid condition



$\hat{x} = (0, 0)$ always satisfies this condition

corresponds to the best
symmetric threshold policy

**When are the optimal thresholds
asymmetric?**



Asymmetric $a \neq -b$

Asymmetry of optimal thresholds

Theorem 2:

If $\mathcal{G}(\beta, \rho) > \frac{1}{2}$ the optimal thresholds are asymmetric.

$$\mathcal{G}(\beta, \rho) \triangleq \frac{\mathcal{M}(\beta, \rho)}{2} + (1 - (1 - \beta)\mathcal{M}(\beta, \rho)) \frac{\partial}{\partial \beta} \log \mathcal{M}(\beta, \rho)$$

related to $\det(\nabla^2 \mathcal{J}_q(0, 0))$

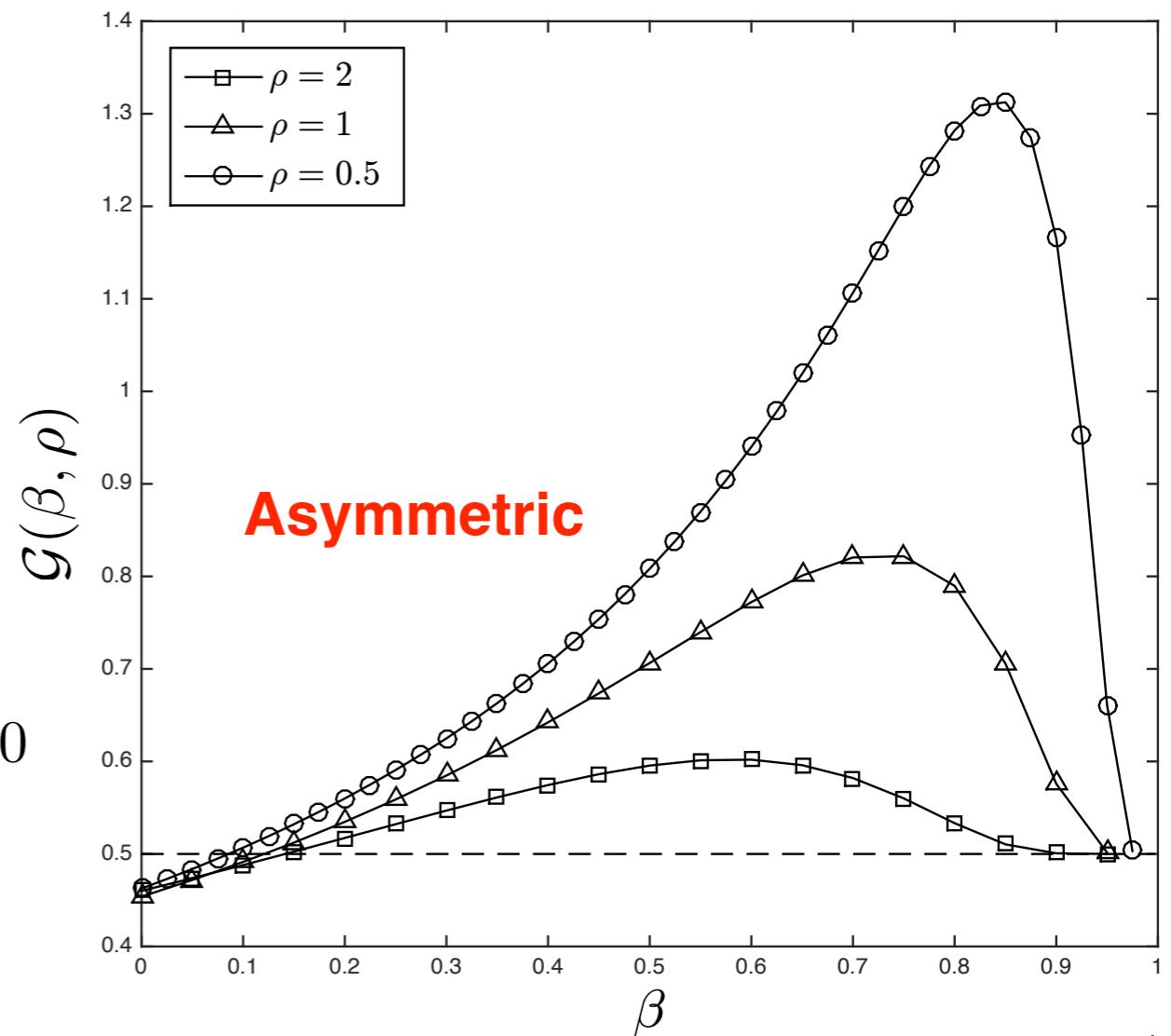
$$\mathcal{M}(\beta, \rho) \triangleq \int_{-\sqrt{\frac{\rho}{1-\beta}}}^{\sqrt{\frac{\rho}{1-\beta}}} f_X(x) dx$$

Sketch of Proof:

$\hat{x} = (0, 0)$ is a local minimum then $\nabla^2 \mathcal{J}_q(0, 0) \succeq 0$

$$\nabla^2 \mathcal{J}_q(0, 0) \succeq 0 \Leftrightarrow \mathcal{G}(\beta, \rho) \leq \frac{1}{2}$$

■



Modified Lloyd-Max and its convergence

$$\underset{\hat{x} \in \mathbb{R}^2}{\text{minimize}} \quad \mathcal{J}_q(\hat{x})$$

$$\nabla \mathcal{J}_q(\hat{x}) = 0 \quad \longleftrightarrow \quad \hat{x} = \mathcal{F}(\hat{x})$$

Lloyd's Map

$$\mathcal{F}(\hat{x}) \triangleq \begin{bmatrix} \mathbb{E}[X | X \in [a(\hat{x}), b(\hat{x})]] \\ \mathbb{E}[X | X \notin [a(\hat{x}), b(\hat{x})]] \end{bmatrix}$$

Modified Lloyd Max

$$\hat{x}^{(0)} \neq (0, 0)$$

$$\hat{x}^{(k+1)} = \mathcal{F}(\hat{x}^{(k)}), \quad k = 0, 1, \dots$$

Step 1 From $\hat{x}^{(k)}$ update the thresholds $a(\hat{x}^{(k)})$ and $b(\hat{x}^{(k)})$

Step 2 Compute the centroids of the new quantization regions

Modified Lloyd-Max and its convergence

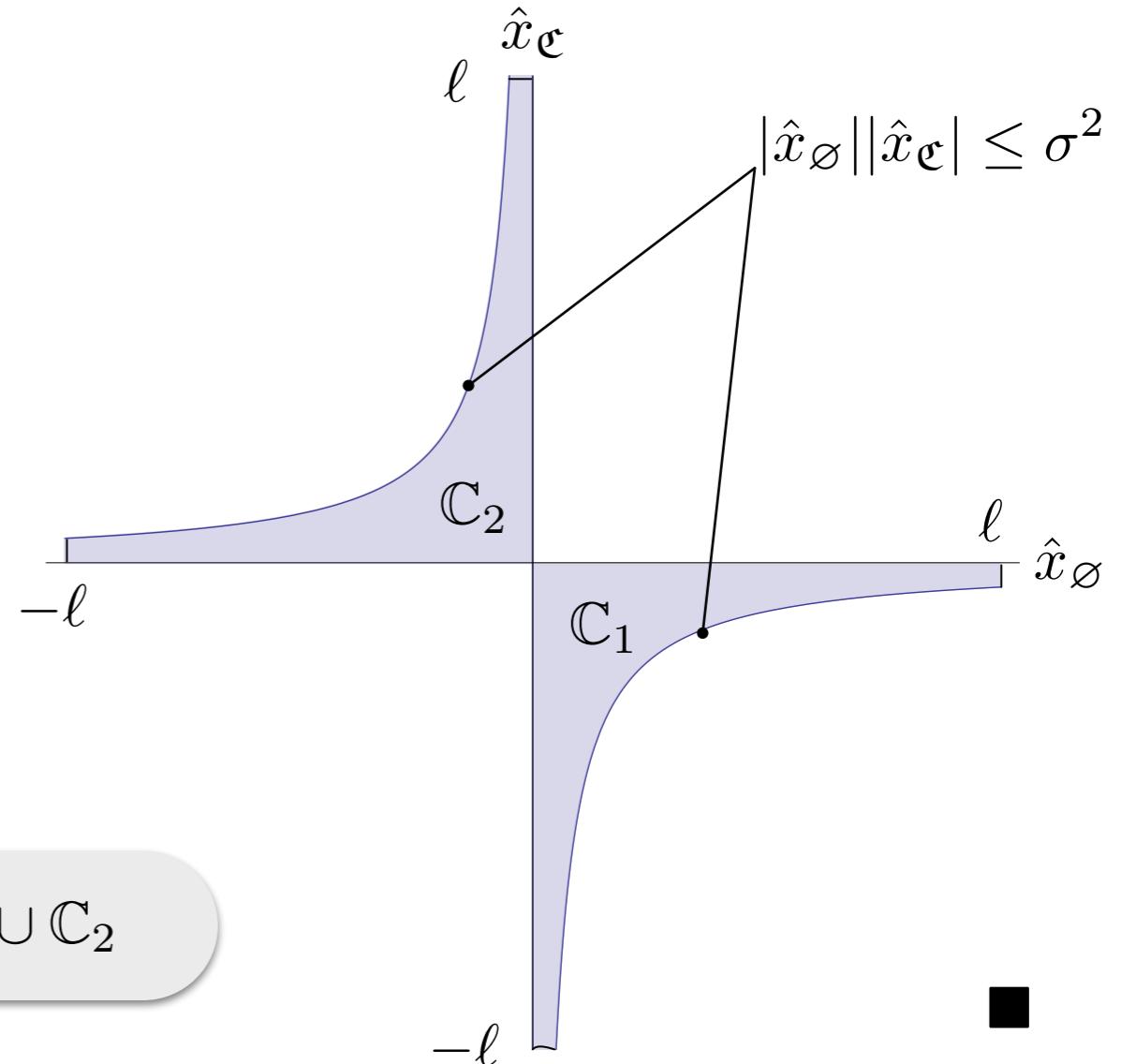
Theorem 3:

The modified Lloyd-Max algorithm is globally convergent to a critical point of $\mathcal{J}_q(\hat{x})$.

Sketch of Proof:

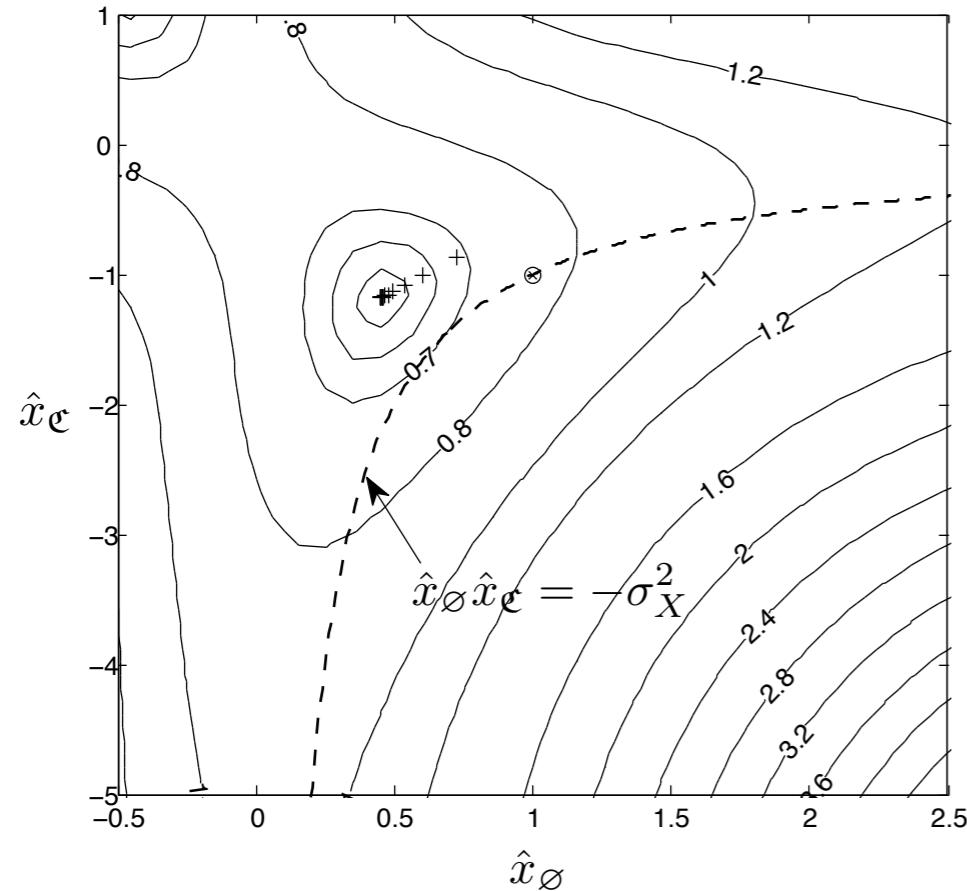
- Based on a result by
Du et al. - SIAM J. Num. Analysis '06
- Find a compact set \mathbb{C} that contains all the critical points of $\mathcal{J}_q(\hat{x})$
- Show that $\mathcal{F}(\mathbb{C}) \subset \mathbb{C}$

$$\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$$

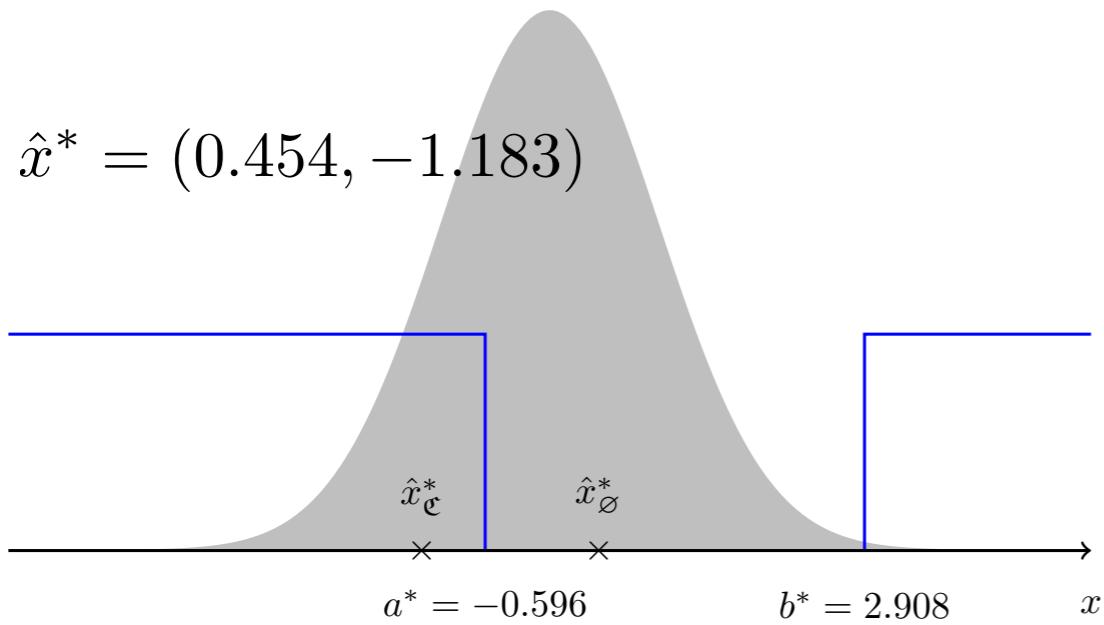


Examples

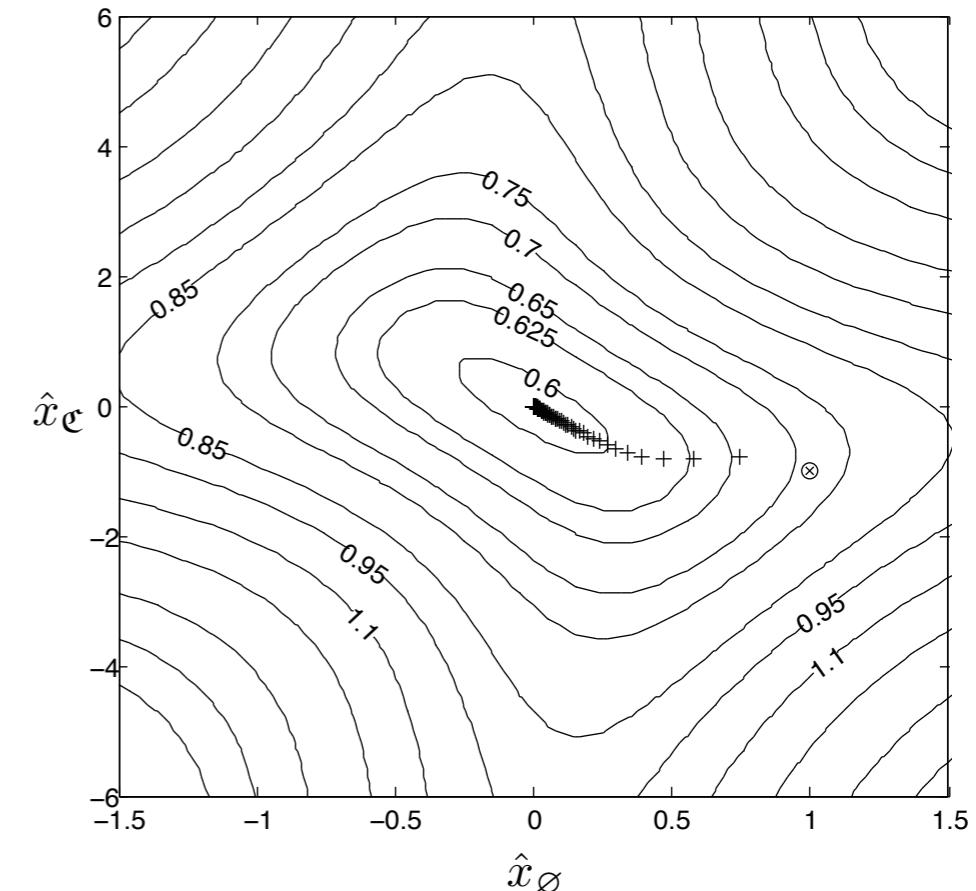
$X \sim \mathcal{N}(0, 1)$, $\beta = 0.3$ and $\rho = 1$



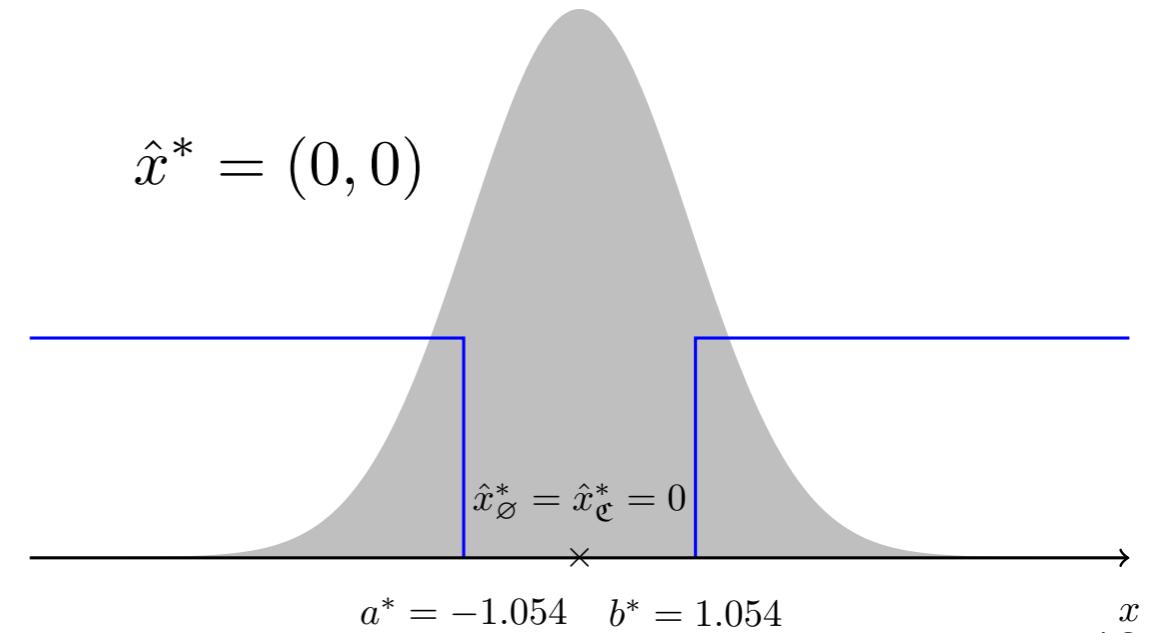
$$\hat{x}^* = (0.454, -1.183)$$



$X \sim \mathcal{N}(0, 1)$, $\beta = 0.1$ and $\rho = 1$



$$\hat{x}^* = (0, 0)$$



Conclusion and future work

- **Existence** of optimal thresholds
- **Sufficient condition for the asymmetry of optimal thresholds**
- **Global convergence** of the Modified Lloyd-Max algorithm

- Solving the **sequential case** with channel output feedback
- Proving the convergence to a **global minimum**
- **Control** over the collision channel

