Optimal remote estimation over the collision channel

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9-7-16

Applications

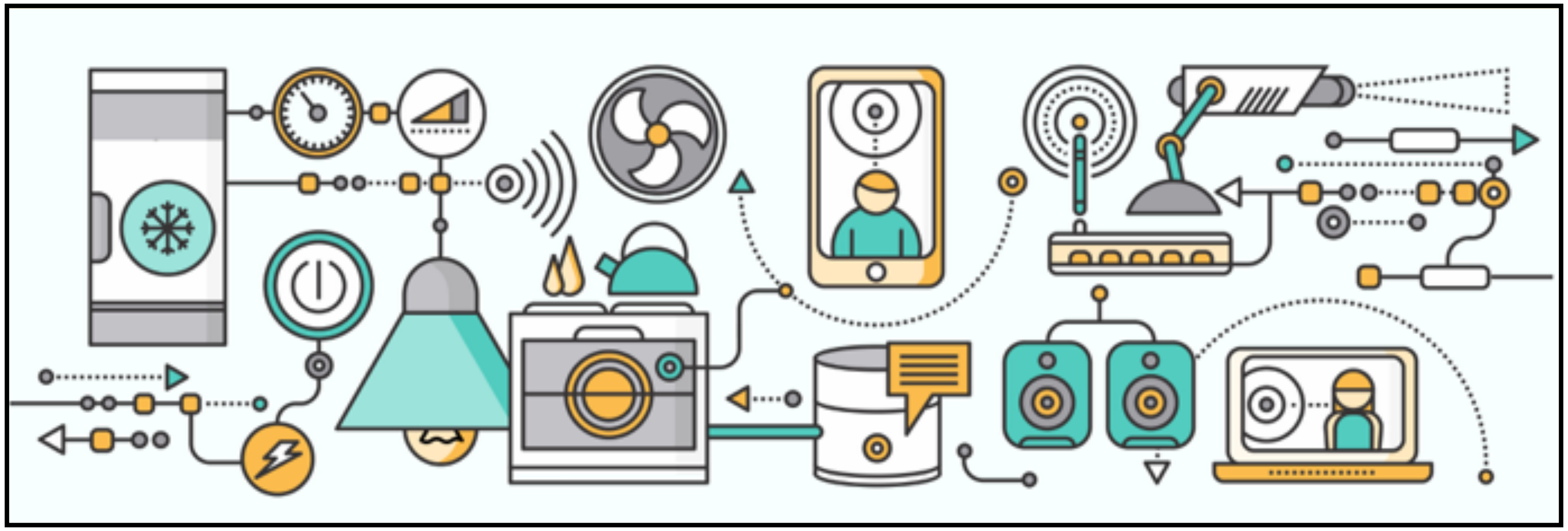
- Sensor networks - **one-time catastrophic events**

Oil & gas pipelines

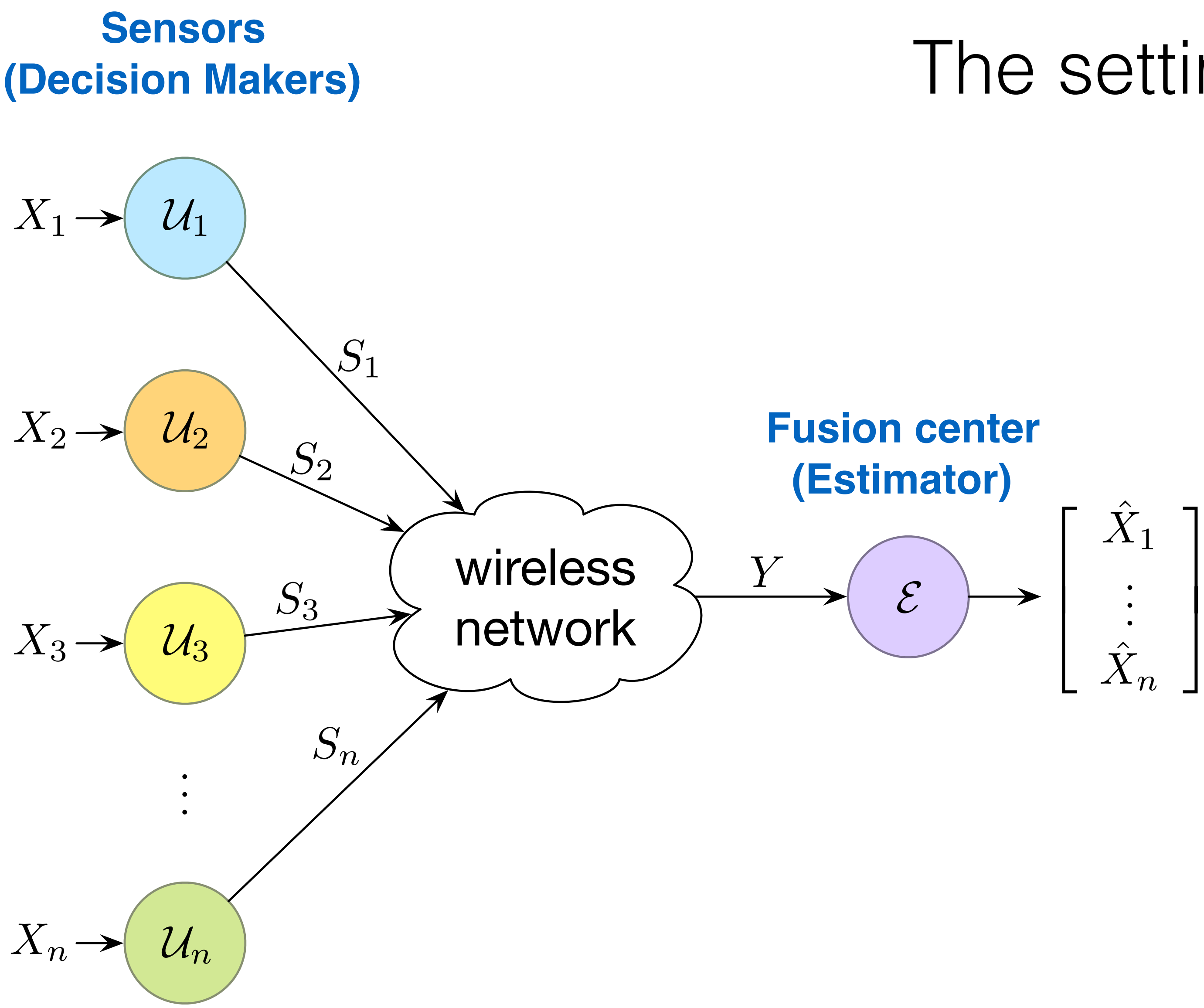
Powergrid

Bridges

- Internet-of-things - **real time wireless networking**



The setting



Channel models used in networked control

- Rate-limited channel[1]
- Packet drop channel[2]
- **Collision channel**[new]

- **One-shot estimation**
- **Decentralized** = **no coordination** between sensors

1. Nair et al, “Feedback control under data rate constraints: An overview,” *Proceedings IEEE* 2007.
2. Schenato et al, “Foundations of control and estimation over lossy networks,” *Proceedings IEEE* 2007.

Collision channel

Model of **interference**:

- **Widely used in wireless communications**[1,2]
- >1 transmission results in a **collision**
- Sensors decide whether to transmit or not

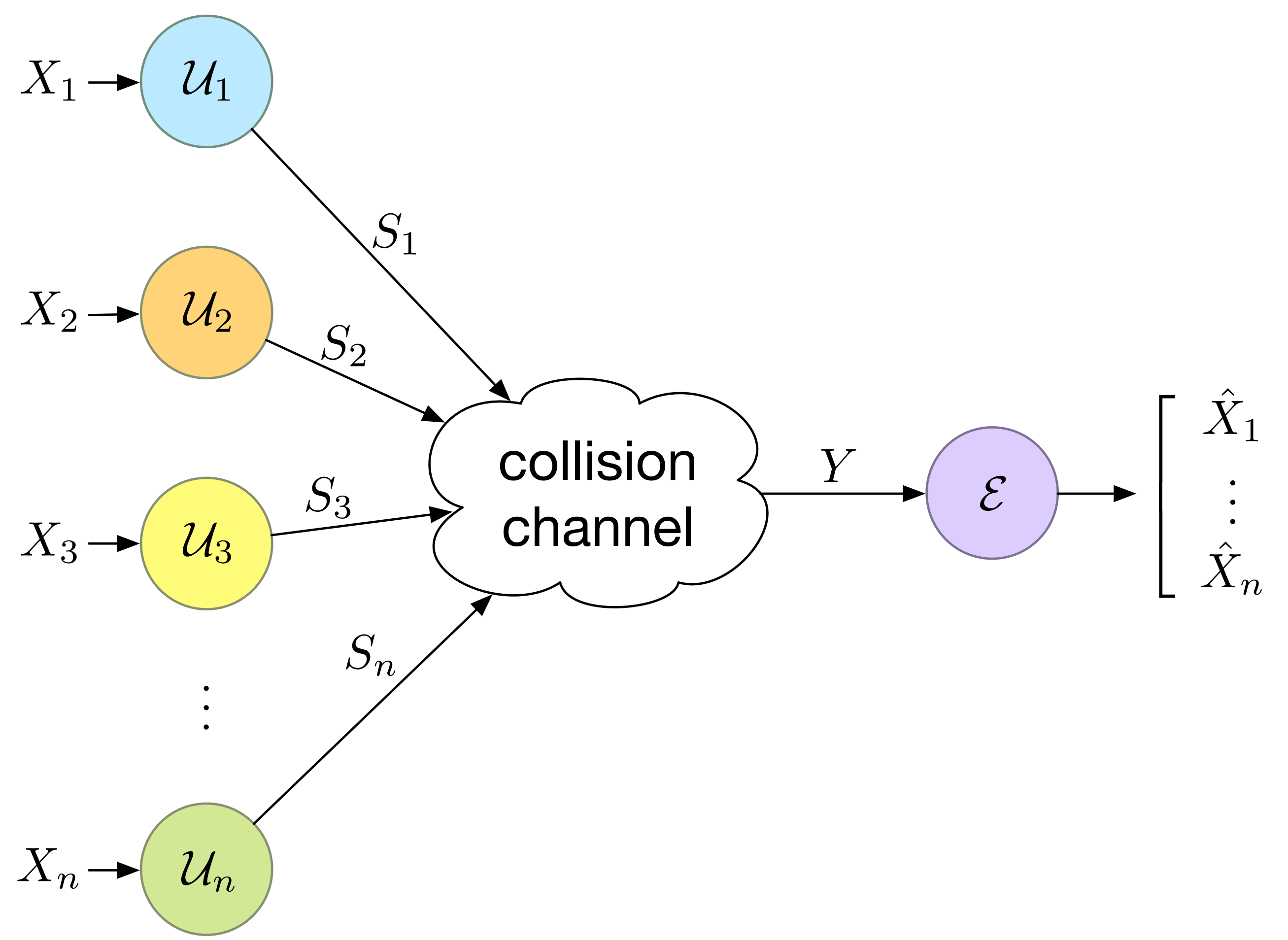
Decision variables: U_i

$$U_i = 1 \implies S_i = (i, X_i)$$

(**transmit**)

$$U_i = 0 \implies S_i = \varnothing$$

(**stay silent**)



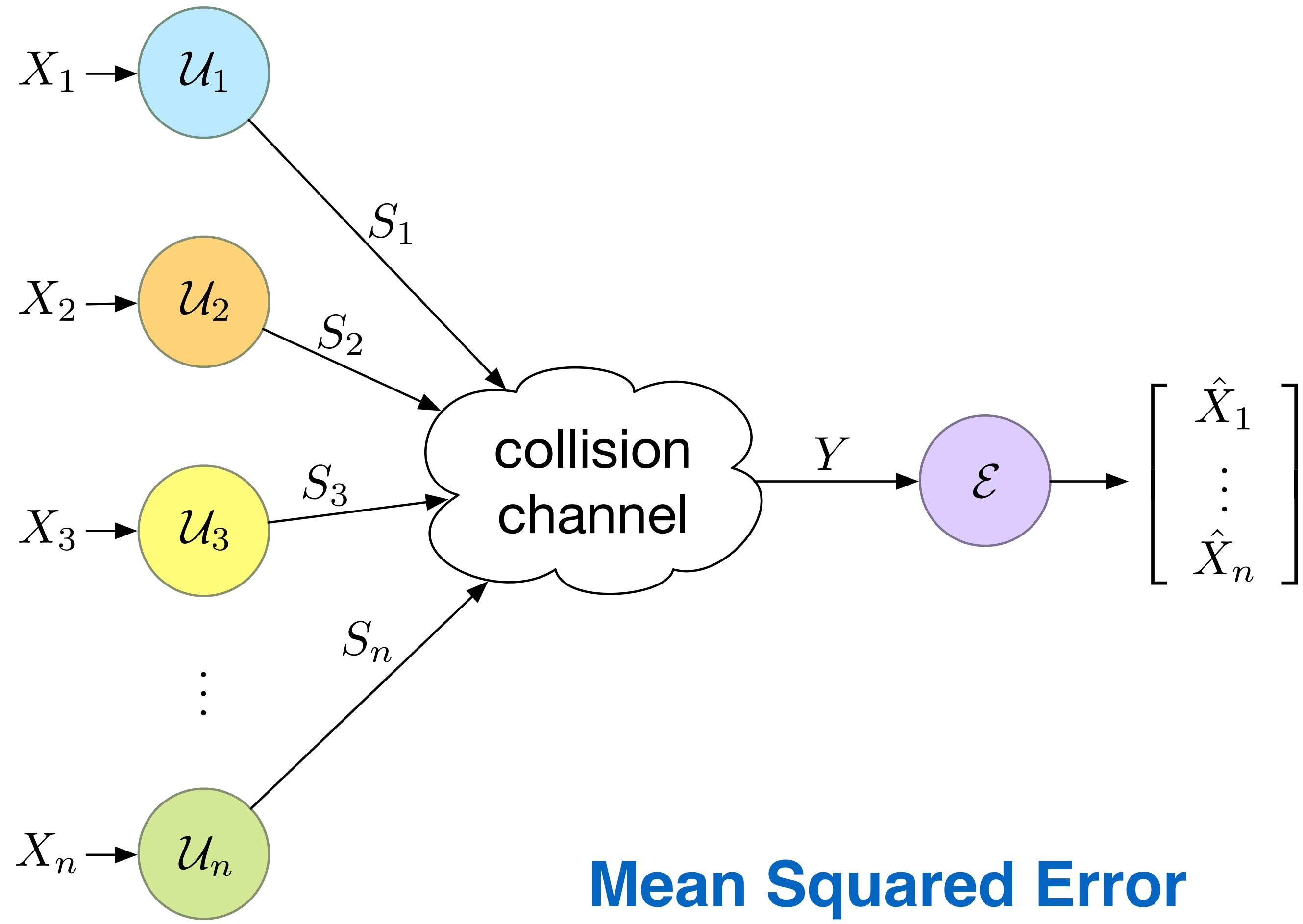
1. Goldsmith, *Wireless Communications*, 2005.
2. Bertsekas and Gallager, *Data Networks*, 1992.

Part 1: MMSE estimation over the collision channel

$$W = [X_1, \cdots, X_n]$$

$$X_i, \quad i \in \{1, \cdots, n\}$$

- mutually independent
- **continuous** rvs
- supported on the real line
- arbitrarily distributed



Mean Squared Error

Problem

$$\text{minimize} \qquad \mathcal{J}(\mathcal{U}_1, \ldots, \mathcal{U}_n) = \mathbf{E}\left[\sum_{i=1}^{n} (X_i - \hat{X}_i)^2\right]$$

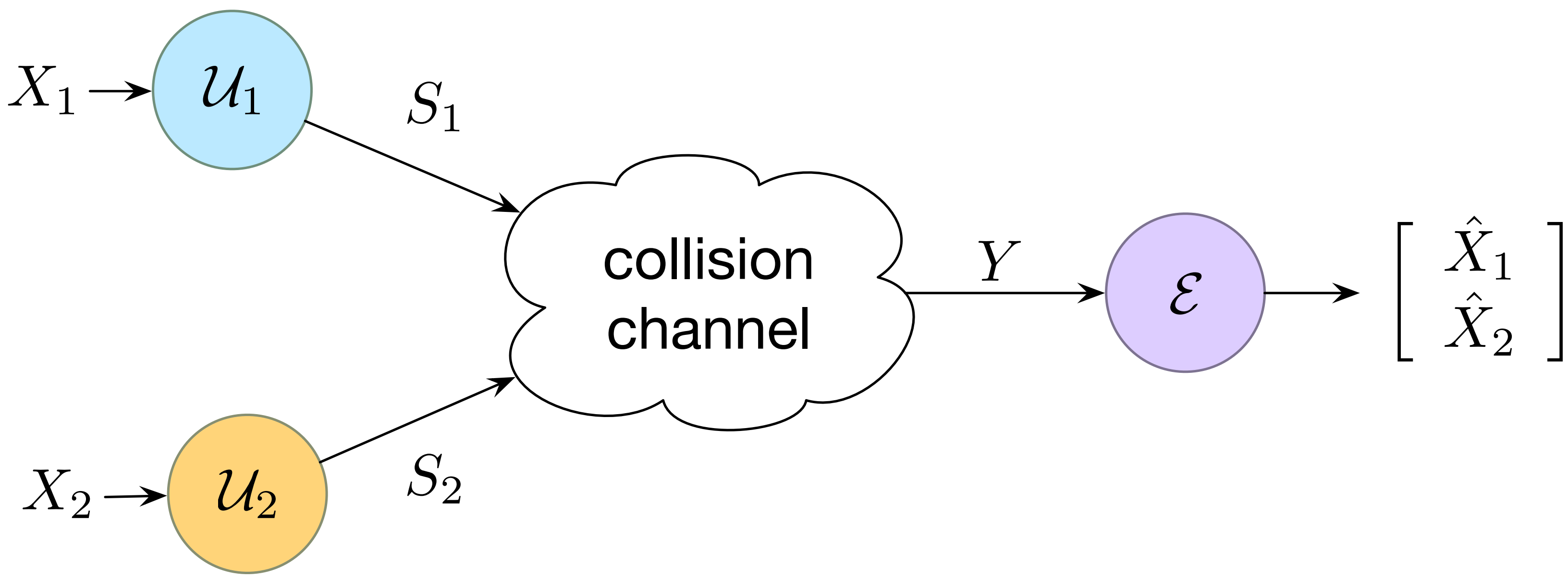
Stochastic policies

$$\mathbf{prob}(U_i = 1 | X_i = x_i) = \mathcal{U}_i(x_i)$$

MMSE estimator

$$\mathcal{E}(y) = \mathbf{E}\big[W | Y = y\big]$$

Simplest case: two sensors



$$\mathbf{prob}(U_i = 1 | X_i = x_i) = \mathcal{U}_i(x_i)$$

Problem 1

minimize $\quad \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\left[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\right]$

$$\mathbb{U}_i = \{\mathcal{U} \mid \mathcal{U} : \mathbb{R} \to [0,1]\}, \quad i \in \{1, 2\}$$

Collision channel

single transmission

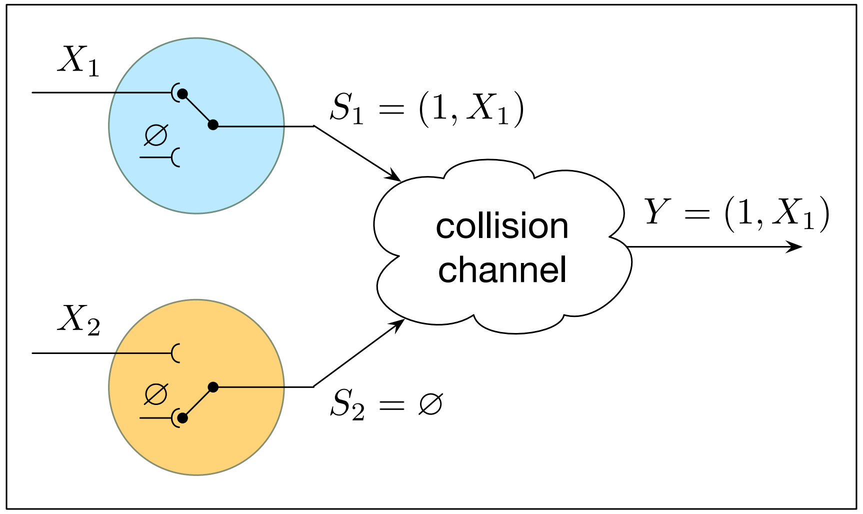
$U_1 = 1, U_2 = 0$

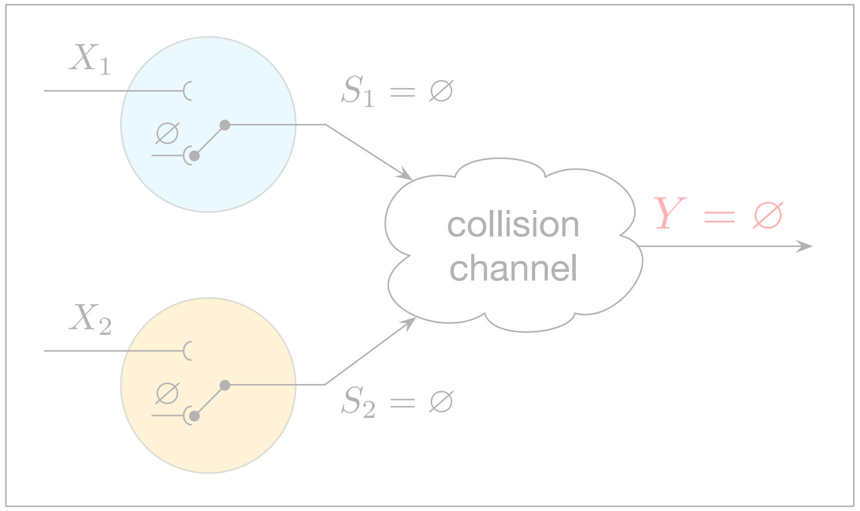
no transmissions

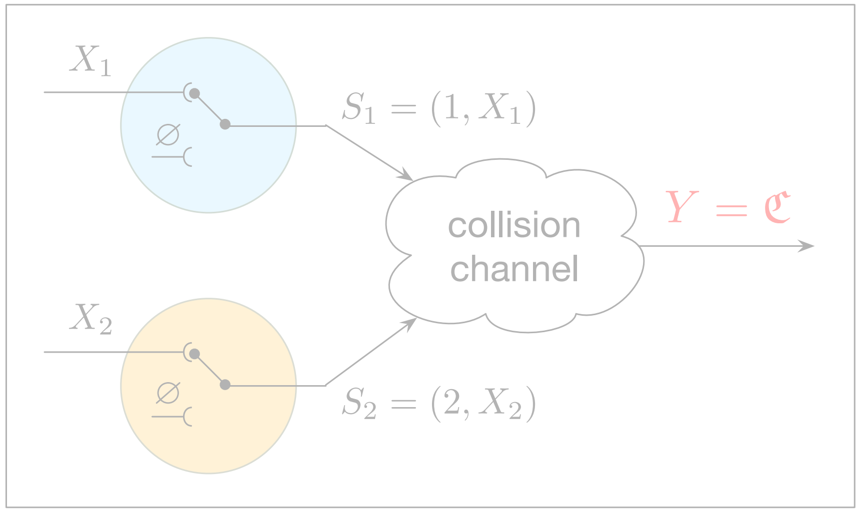
$U_1 = 0, U_2 = 0$

>1 transmissions

$U_1 = 1, U_2 = 1$







success!

no transmission $\varnothing$

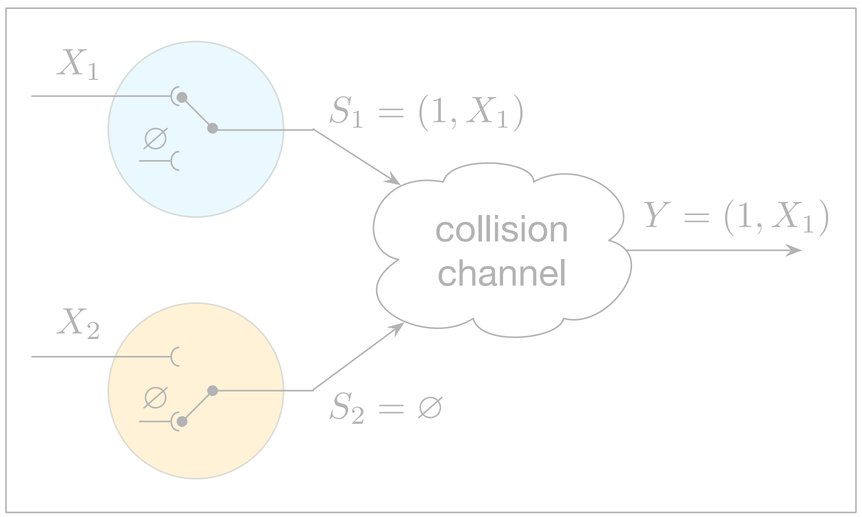
collision $\mathfrak{C}$

From the channel output we can always recover U_1 and U_2.

Collision channel

single transmission

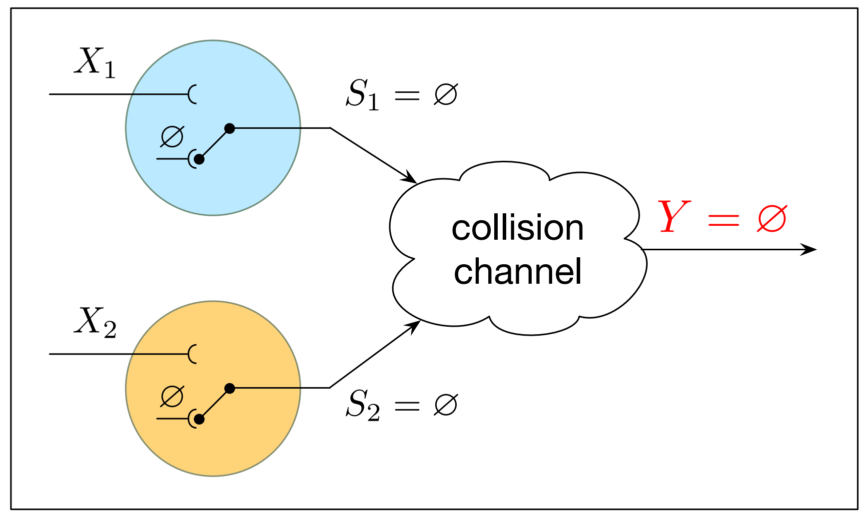
$U_1 = 1, U_2 = 0$



success!

no transmissions

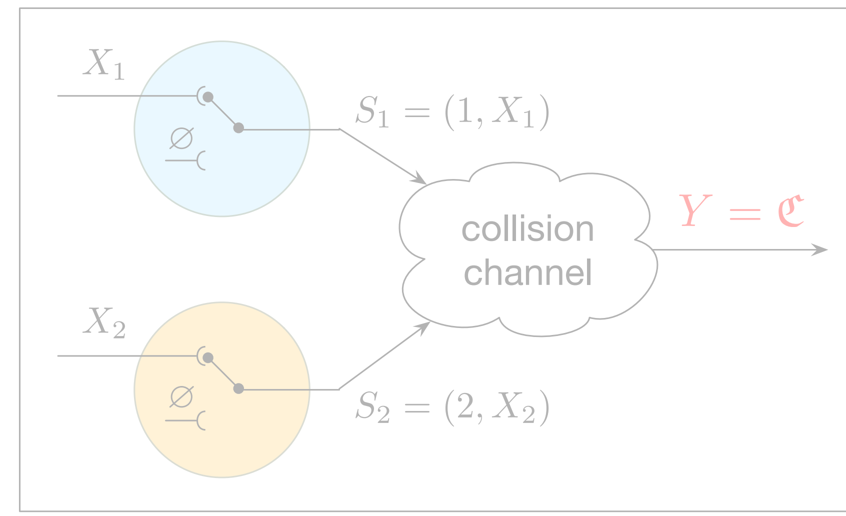
$U_1 = 0, U_2 = 0$



no transmission $\varnothing$

>1 transmissions

$U_1 = 1, U_2 = 1$



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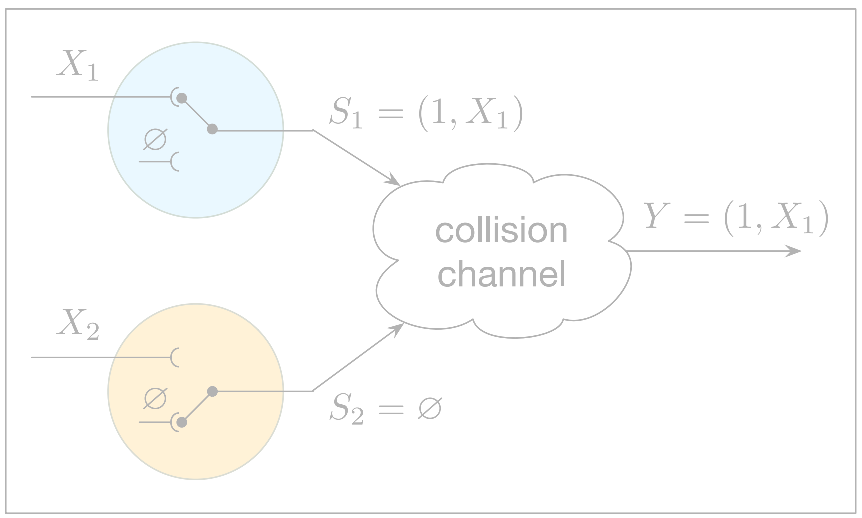
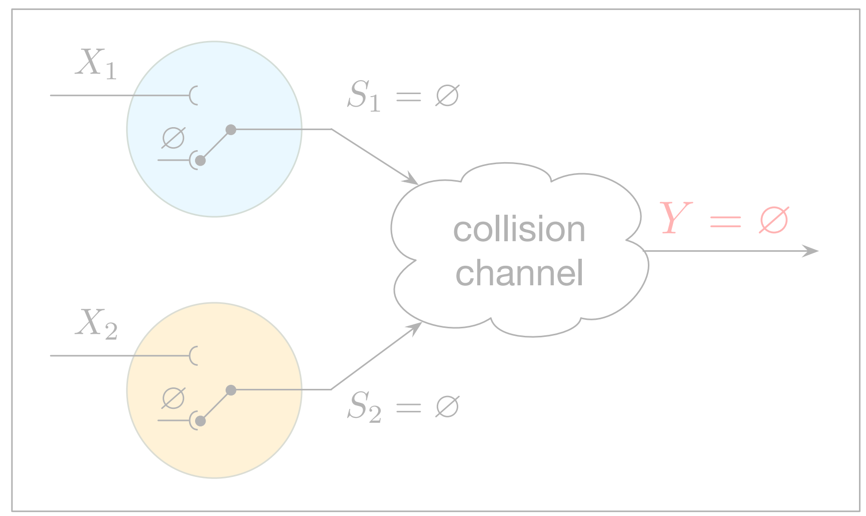


success!

no transmissions

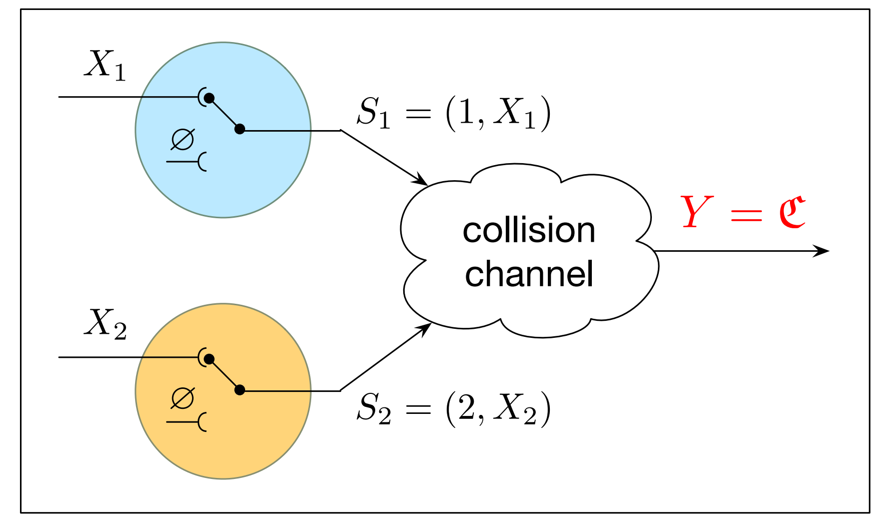
$U_1 = 0, U_2 = 0$



no transmission $\varnothing$

>1 transmissions

$U_1 = 1, U_2 = 1$

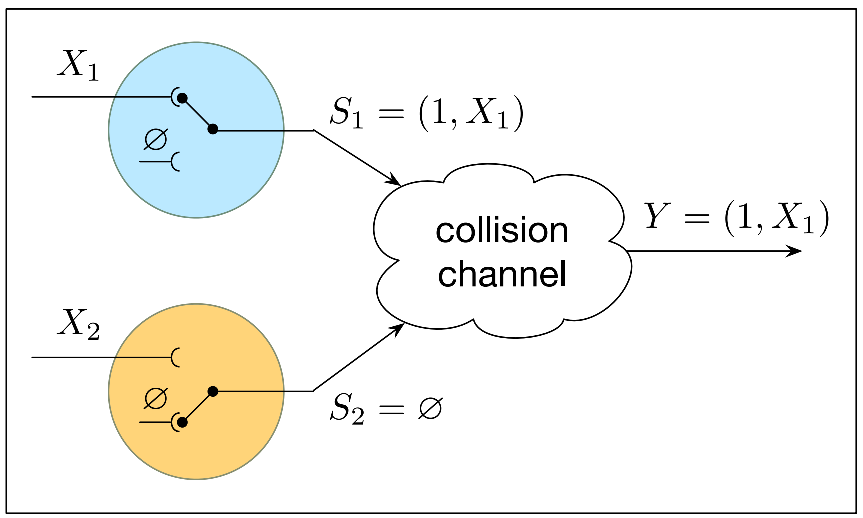


collision $\mathfrak{C}$

From the channel output we can always recover U_1 and U_2.

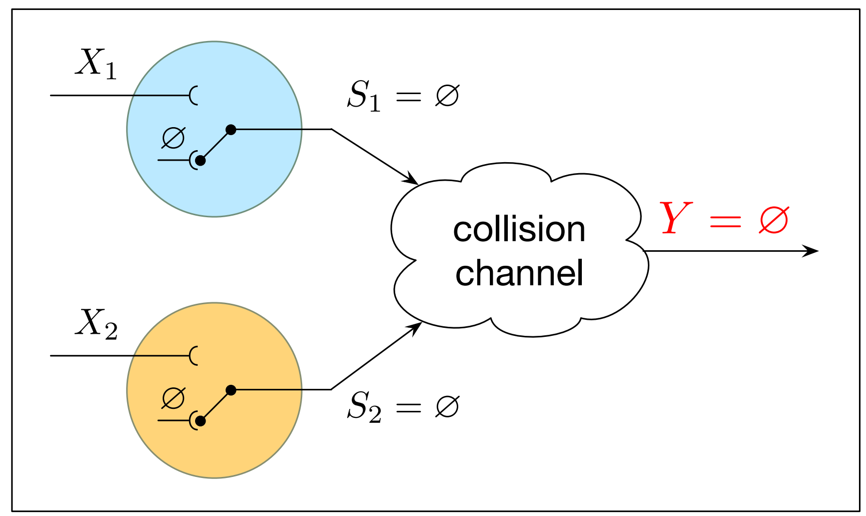
Collision channel

single transmission
$U_1 = 1, U_2 = 0$



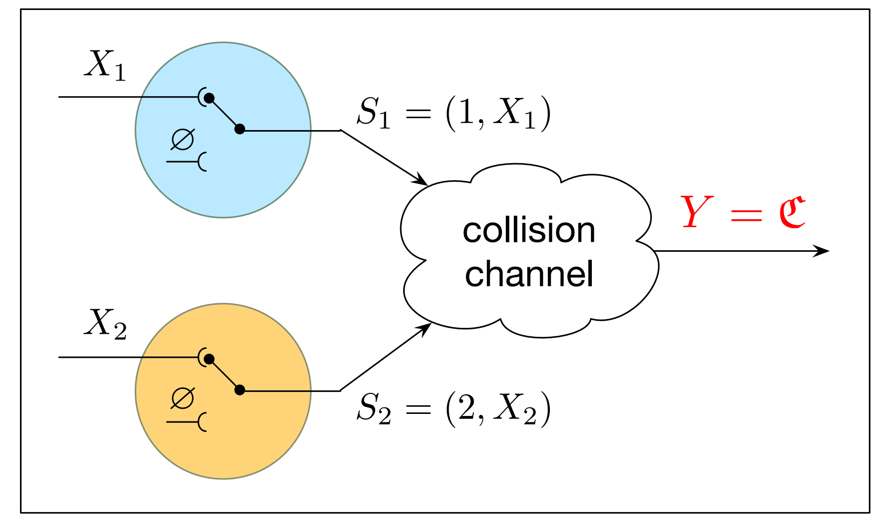
success!

no transmissions
$U_1 = 0, U_2 = 0$



no transmission $\varnothing$

>1 transmissions
$U_1 = 1, U_2 = 1$

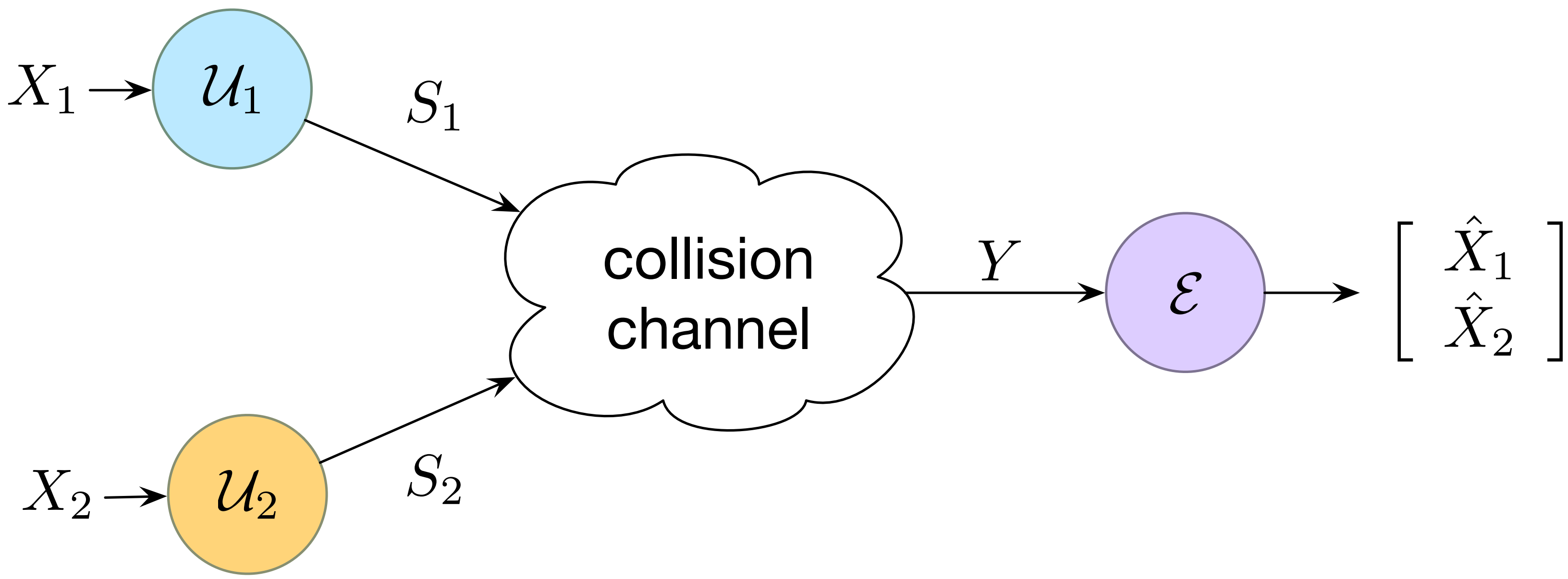


collision $\mathfrak{C}$

The collision channel is fundamentally different from the packet drop channel[1,2]

1. Sinopoli et al, "Kalman filtering with intermittent observations," *IEEE TAC* 2004.
2. Gupta et al, "Optimal LQG control across packet-dropping links," *Systems and Control Letters* 2007.

Why is this problem interesting?



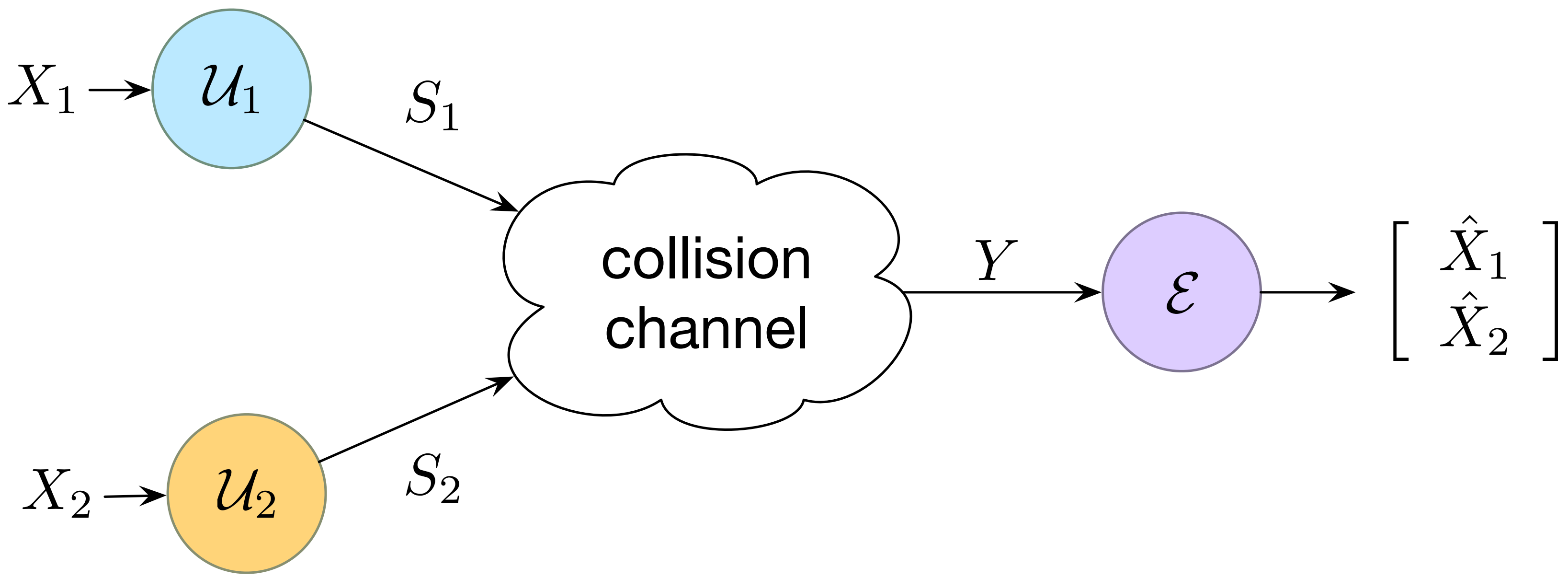
Problem 1

$$\text{minimize} \qquad \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\left[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\right]$$

Static **team-decision problem** with **non-classical information** structure $\Longrightarrow$ **Non-convex** (in most cases) **intractable**[1,2]

1. Witsenhausen, "A counterexample in optimal stochastic control," *SIAM J. Control* 1968.
2. Tsitsiklis & Athans, "On the complexity of decentralized decision making and detection problems," *IEEE TAC* 1985.

Why is this problem interesting?



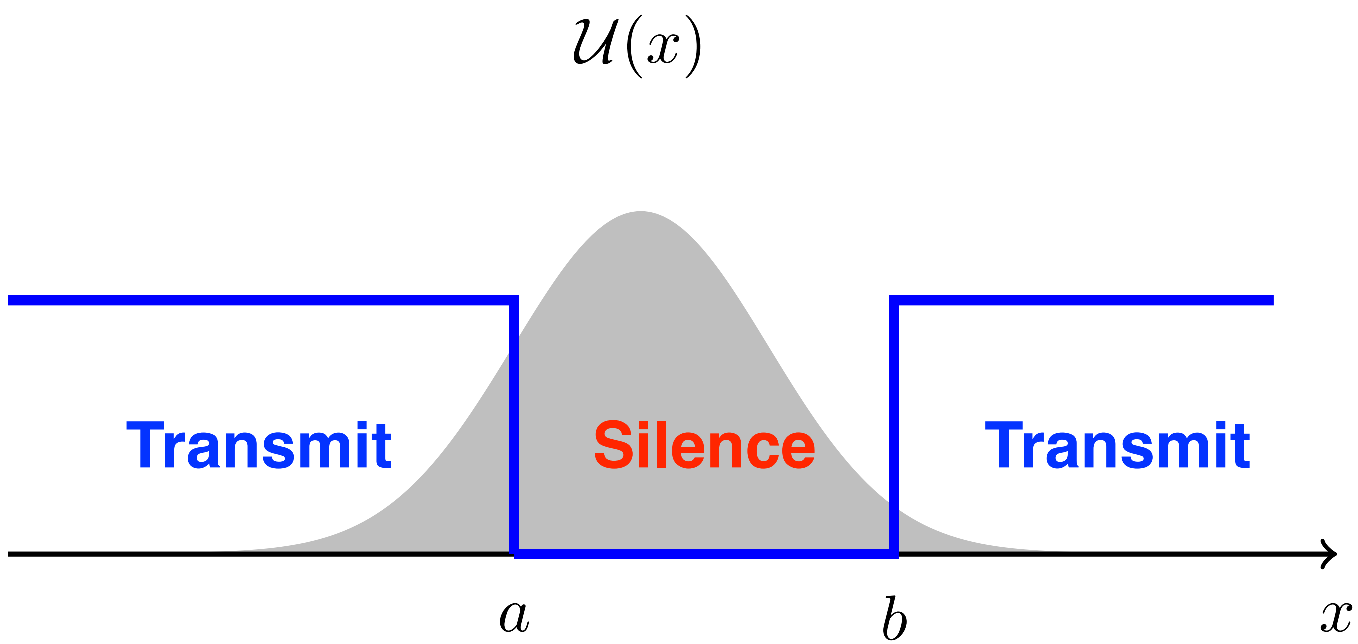
Problem 1

$$\text{minimize} \qquad \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\Big[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\Big]$$

Look for a class parametrizable policies that contains an optimal strategy

1. Witsenhausen, "A counterexample in optimal stochastic control," *SIAM J. Control* 1968.
2. Tsitsiklis & Athans, "On the complexity of decentralized decision making and detection problems," *IEEE TAC* 1985.

Deterministic threshold policies



Threshold policy

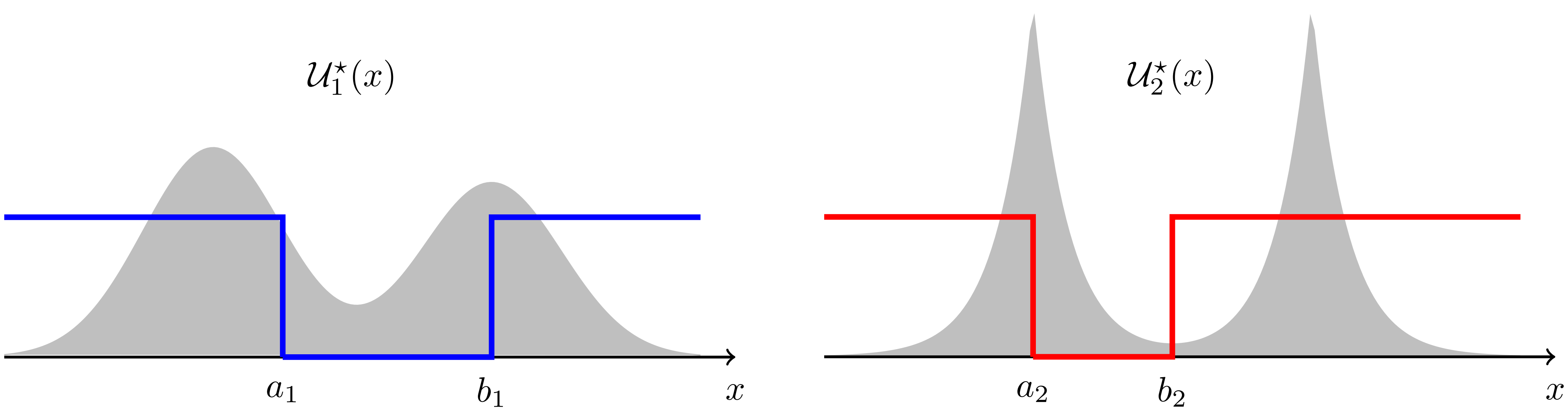
$$\mathcal{U}(x) = \begin{cases} 0 & a \leq x \leq b \\ 1 & \text{otherwise} \end{cases}$$

1. Imer & Basar, "Optimal estimation with limited measurements", *Int. Journal of Syst., Cont. and Comm.* 2010.
2. Lipsa & Martins, "Remote state estimation with communication costs for first-order LTI systems". *IEEE TAC* 2011.

Characterization of team-optimal policies

Theorem

There exists an team-optimal pair of **threshold policies** for Problem 1.



Sketch of Proof:

- Step 1: Equivalent single DM problem
- Step 2: Lagrange duality for infinite dimensional LPs

Main idea

Team-optimality

$$\mathcal{J}(\mathcal{U}_1^*,\mathcal{U}_2^*) \leq \mathcal{J}(\mathcal{U}_1,\mathcal{U}_2), \quad (\mathcal{U}_1,\mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$$

$$\Longrightarrow$$

$$\not\Longleftarrow$$

Person-by-person optimality

$$\mathcal{J}(\mathcal{U}_1^*,\mathcal{U}_2^*) \leq \mathcal{J}(\mathcal{U}_1,\mathcal{U}_2^*), \quad \mathcal{U}_1 \in \mathbb{U}_1$$

$$\mathcal{J}(\mathcal{U}_1^*,\mathcal{U}_2^*) \leq \mathcal{J}(\mathcal{U}_1^*,\mathcal{U}_2), \quad \mathcal{U}_2 \in \mathbb{U}_2$$

$$(\mathcal{U}_1^*,\mathcal{U}_2^*) \in \mathbb{U}_1 \times \mathbb{U}_2 \quad \longrightarrow \quad (\breve{\mathcal{U}}_1^*,\breve{\mathcal{U}}_2^*) \in \mathbb{U}_1 \times \mathbb{U}_2$$

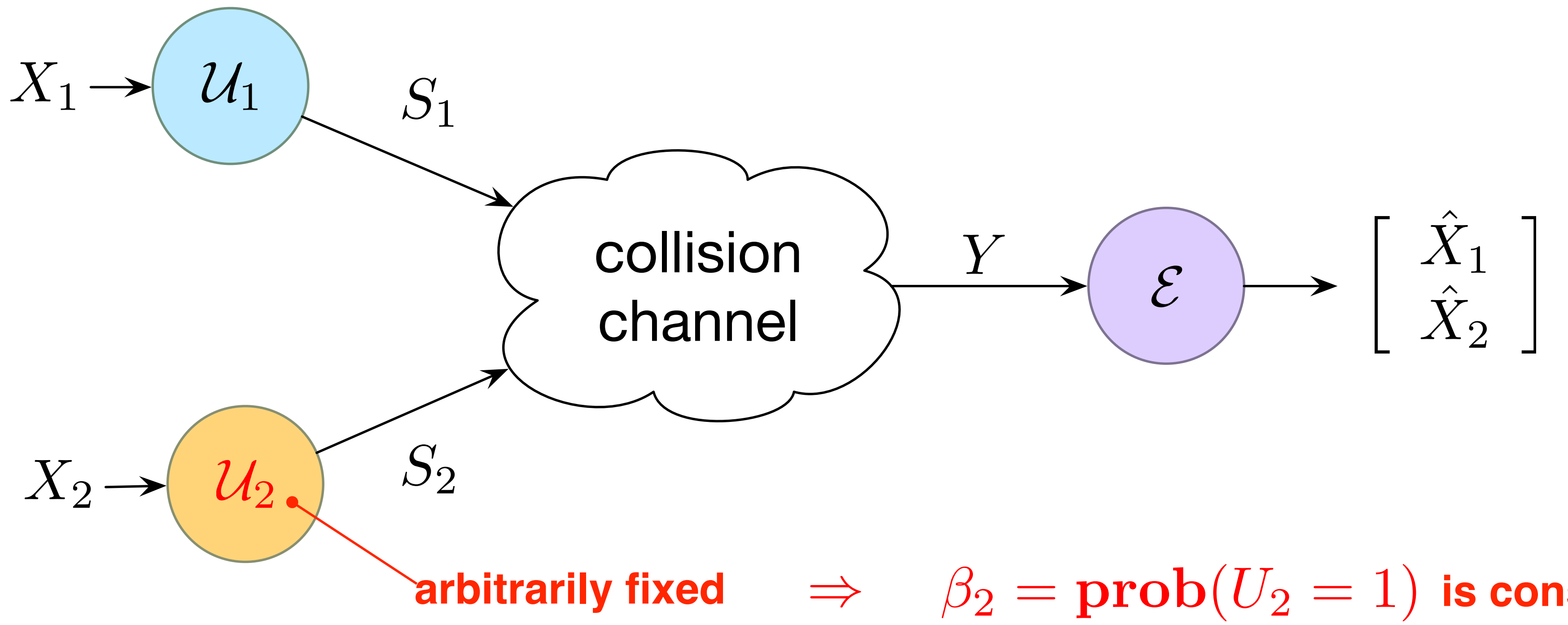
$$\mathcal{J}(\mathcal{U}_1^*,\mathcal{U}_2^*) \geq \mathcal{J}(\breve{\mathcal{U}}_1^*,\breve{\mathcal{U}}_2^*)$$

threshold policies

Given any pair of person-by-person optimal policies **construct a new pair** with **equal or better cost**, where each policy is **threshold**

1. Yuksel & Basar, *Stochastic networked control systems,* Birkhauser 2013.
2. Mahajan et al, "Information structures in optimal decentralized control," CDC 2012.

Remote estimation with communication costs



$\Rightarrow$ $\beta_2 = \mathbf{prob}(U_2 = 1)$ **is constant**

Original cost:

$$\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\Big[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\Big]$$

Cost from the perspective of DM$_1$:

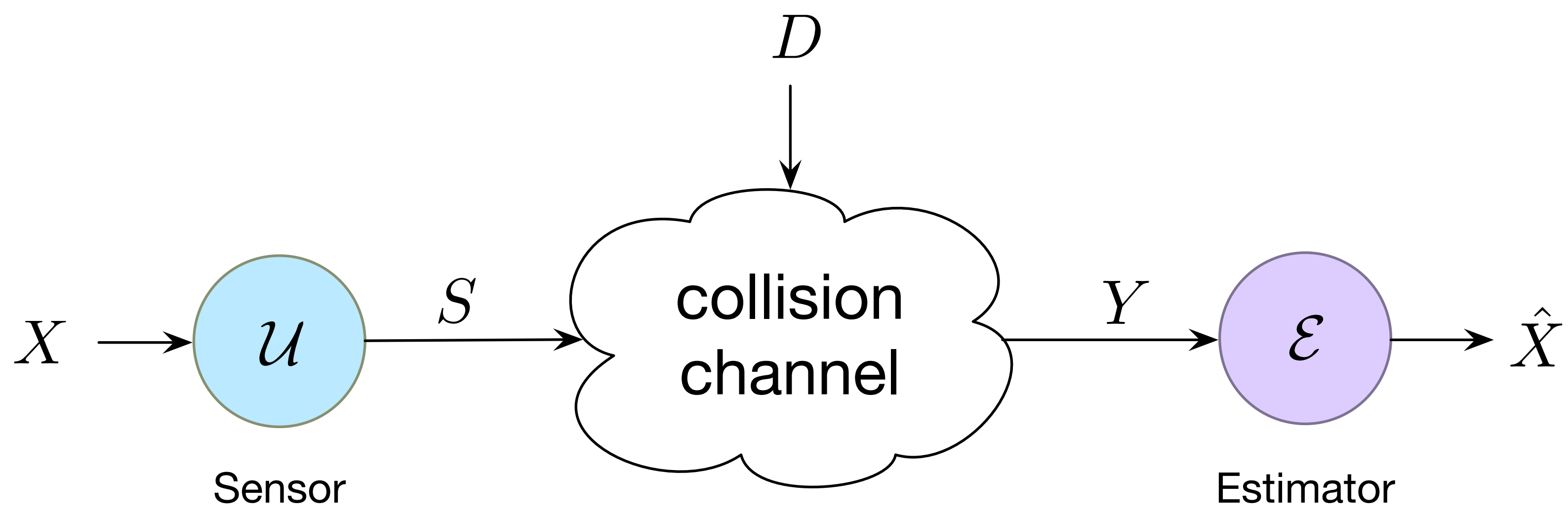
$$\mathcal{J}_1(\mathcal{U}_1) = \mathbf{E}\Big[(X_1 - \hat{X}_1)^2\Big] + \rho_2 \cdot \mathbf{prob}(U_1 = 1) + \theta_2$$

do not depend on $\mathcal{U}_1$

Communication cost:

$$\rho_2 = \mathbf{E}\Big[(X_2 - \hat{X}_2)^2 | U_1 = 1\Big] - \mathbf{E}\Big[(X_2 - \hat{X}_2)^2 | U_1 = 0\Big] \geq 0$$

Single DM subproblem



$D \sim \mathcal{B}(\beta)$

Determines if the channel is occupied or not

$X \perp\!\!\!\perp D$

Problem 2

$$\text{minimize} \qquad \mathcal{J}(\mathcal{U}) = \mathbf{E}\Big[(X - \hat{X})^2\Big] + \rho \cdot \mathbf{prob}(U = 1)$$

$$\mathbf{prob}(U = 1 | X = x) = \mathcal{U}(x) \qquad \mathbb{U} = \{\mathcal{U} \mid \mathcal{U} : \mathbb{R} \to [0, 1]\}$$

Lemma

There exists an optimal **threshold policy** for Problem 2.

Sketch of Proof

1. Express the cost as

$$\mathcal{J}(\mathcal{U}) = \mathbf{E}\Big[\beta(X - \hat{x}_{\mathfrak{C}})^2 + \rho \mid U = 1\Big] \cdot \mathbf{prob}(U = 1) + \mathbf{E}\Big[(X - \hat{x}_{\varnothing})^2 \mid U = 0\Big] \cdot \mathbf{prob}(U = 0)$$

$$\hat{x}_{\mathfrak{C}} = \mathbf{E}[X|U = 1] \qquad \hat{x}_{\varnothing} = \mathbf{E}[X|U = 0]$$

2. After **introducing two linear constraints** and a **change of variables**, we have:

$$\mathbf{prob}(U = 1) = \alpha \qquad \mathcal{G}(x) = \frac{1 - \mathcal{U}(x)}{1 - \alpha}$$

$$\mathbf{E}[X|U = 0] = \gamma$$

$$\begin{array}{ll} \underset{\mathcal{G} \in L^2_\mu(\mathbb{R})}{\text{minimize}} & \mathbf{E}\big[X^2 \mathcal{G}(X)\big] \\ \text{subject to} & \mathbf{E}\big[X \mathcal{G}(X)\big] = \gamma \\ & \mathbf{E}[\mathcal{G}(X)] = 1 \\ & 0 \leq \mathcal{G}(x) \leq \dfrac{1}{1 - \alpha} \end{array}$$

moment optimization problem with variable bounds (convex)

1. Akhiezer, *The Classical Moment Problem,*1965.
2. Byrnes & Lindquist, "A convex optimization approach to generalized moment problems," Springer 2003.

Sketch of Proof

3. The Lagrange dual function is

$$\mathcal{C}^*(\nu) = -\nu_1 - \nu_0\gamma - \frac{1}{1-\alpha}\mathbf{E}\left[[X^2 + \nu_0 X + \nu_1]^-\right]$$

strong duality holds[1,2]

4. The solution to the primal problem is

$$\mathcal{G}_{\nu^*}(x) = \begin{cases} \frac{1}{1-\alpha} & \text{if } x^2 + \nu_0^* x + \nu_1^* \le 0 \\ 0 & \text{otherwise} \end{cases}$$

5. In the original optimization variable:

$$\mathcal{U}_{\nu^*}(x) = \begin{cases} 0 & \text{if } x^2 + \nu_0^* x + \nu_1^* \le 0 \\ 1 & \text{otherwise} \end{cases} \implies \boxed{\mathcal{U}^*(x) = \begin{cases} 0 & \text{if } a \le x \le b \\ 1 & \text{otherwise} \end{cases}}$$

$\blacksquare$

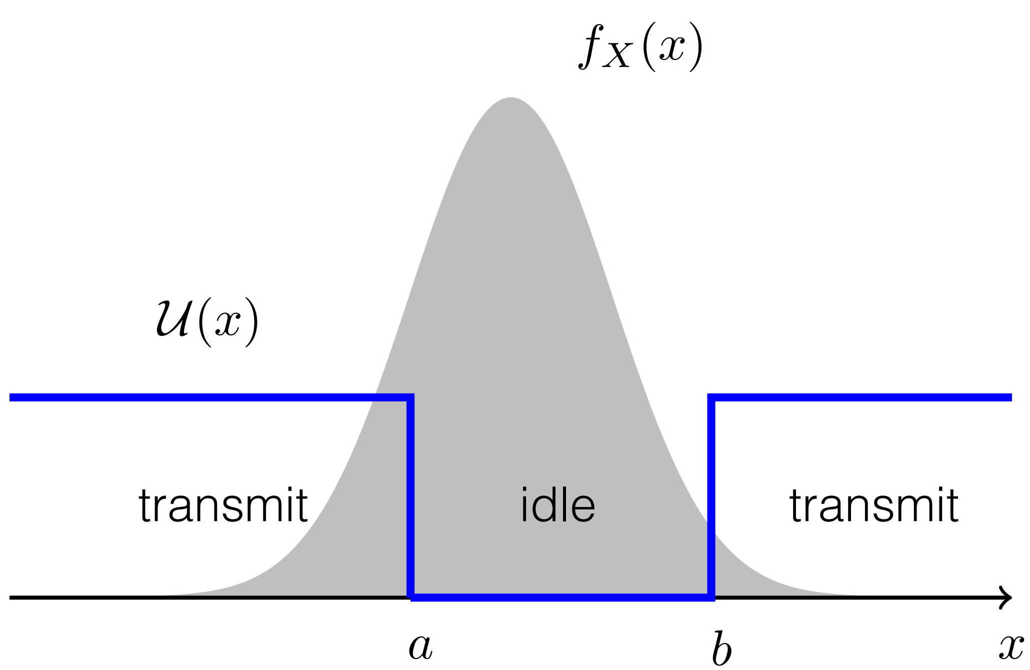
1. Borwein & Lewis, “Partially finite convex programming, Part I: Quasi relative interiors and duality theory,” Math. Prog. 1992.
2. Limber & Goodrich, “Quasi interiors, Lagrange multipliers, and Lp spectral estimation with lattice bounds,” JOTA 1993.

Remarks

1. Valid for **any continuous probability distribution**
2. **Vector observations** and **any number of sensors**

Assumption:
Finite 1st and 2nd moments
(req. for strong duality)

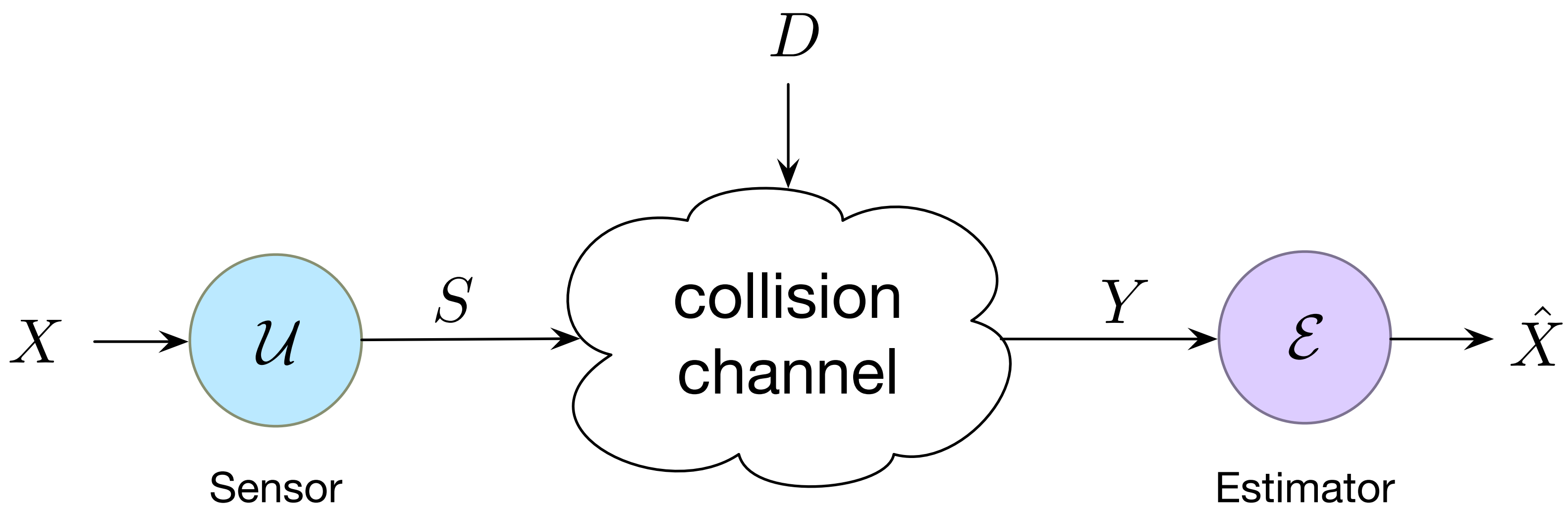
Additional assumption:
The fusion center can decode the indices of all sensors involved in a collision



How do we compute the optimal thresholds?

Vasconcelos & Martins, "Optimal estimation over the collision channel," to appear *IEEE TAC* 2017.

Part 2. Computing optimal thresholds



$$\mathcal{U}(x) = \begin{cases} 0 & a \leq x \leq b \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{E}(y) = \begin{cases} x & y = x \\ \hat{x}_{\varnothing} & y = \varnothing \\ \hat{x}_{\mathfrak{C}} & y = \mathfrak{C} \end{cases}$$

$$\mathcal{J}(a, b, \hat{x}_{\varnothing}, \hat{x}_{\mathfrak{C}}) = \int_{[a,b]} (x - \hat{x}_{\varnothing})^2 f_X(x) dx + \int_{\bar{\mathbb{R}} \setminus [a,b]} [\beta(x - \hat{x}_{\mathfrak{C}})^2 + \rho] f_X(x) dx$$

binary quantization with **asymmetric** distortion

1. Lloyd, "Least squares quantization in PCM", *IEEE IT* 1982.
2. Fleischer, "Sufficient conditions for achieving minimum distortion in a quantizer", *IEEE Int. Conv. Rec.*, 1964.

Binary quantization with asymmetric distortion

$$\begin{array}{ll} \text{minimize} & \mathcal{J}(a, b, \hat{x}_{\varnothing}, \hat{x}_{\mathfrak{c}}) \\ \text{subject to} & a \leq b \end{array}$$

$$x \in [a^*, b^*] \Leftrightarrow (x - \hat{x}_{\varnothing})^2 \leq \beta(x - \hat{x}_{\mathfrak{c}})^2 + \rho$$

necessary optimality condition

Let $\hat{x} = (\hat{x}_{\varnothing}, \hat{x}_{\mathfrak{c}})$ is the pair representation points for no-transmission and collision symbols

$$a(\hat{x}), b(\hat{x}) = \frac{1}{1-\beta}\left[(\hat{x}_{\varnothing} - \beta\hat{x}_{\mathfrak{c}}) \pm \sqrt{\beta(\hat{x}_{\varnothing} - \hat{x}_{\mathfrak{c}})^2 + (1-\beta)\rho}\right]$$

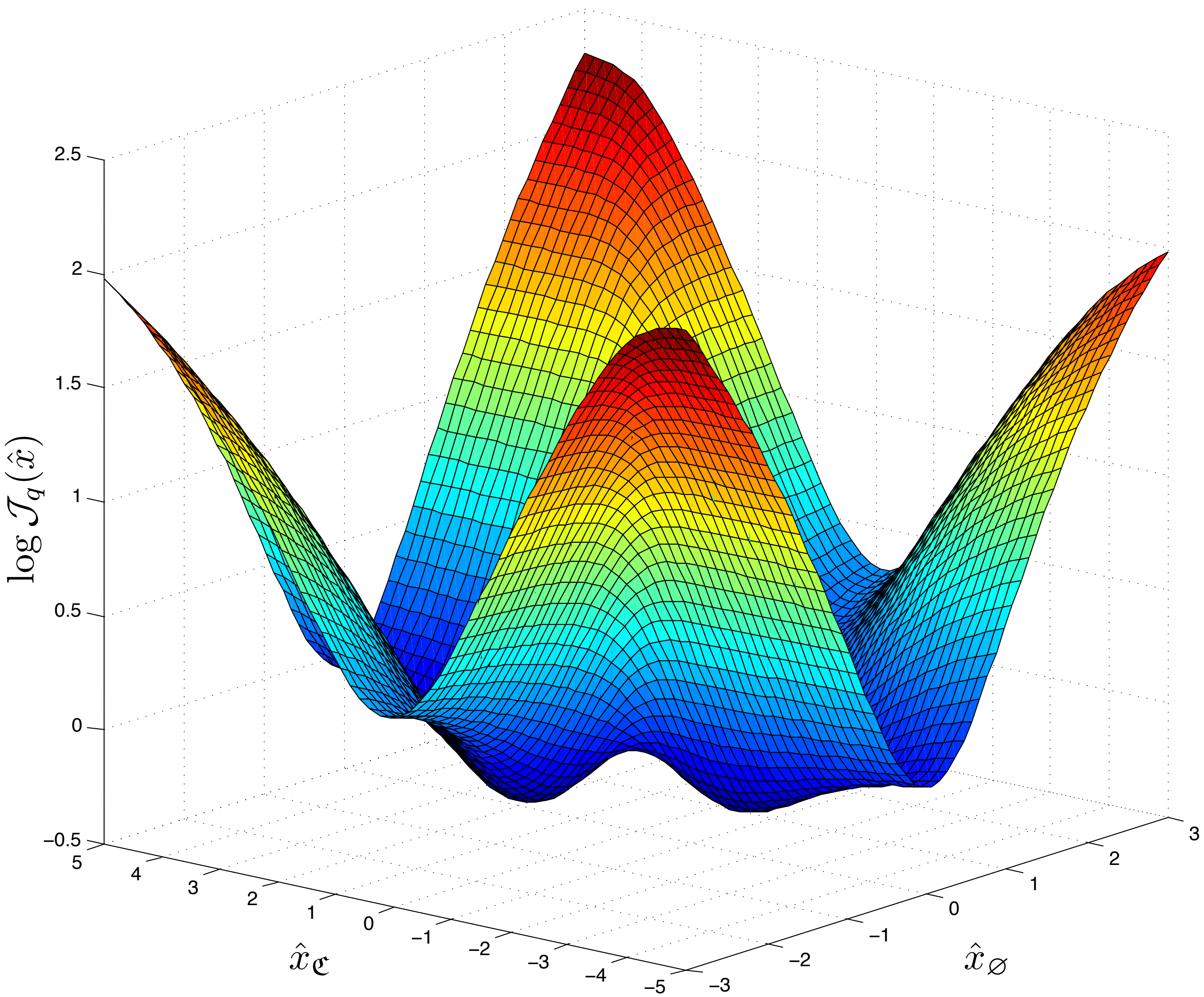
Define a new cost: $\mathcal{J}_q(\hat{x}) = \mathcal{J}(a(\hat{x}), b(\hat{x}), \hat{x}_{\varnothing}, \hat{x}_{\mathfrak{c}})$

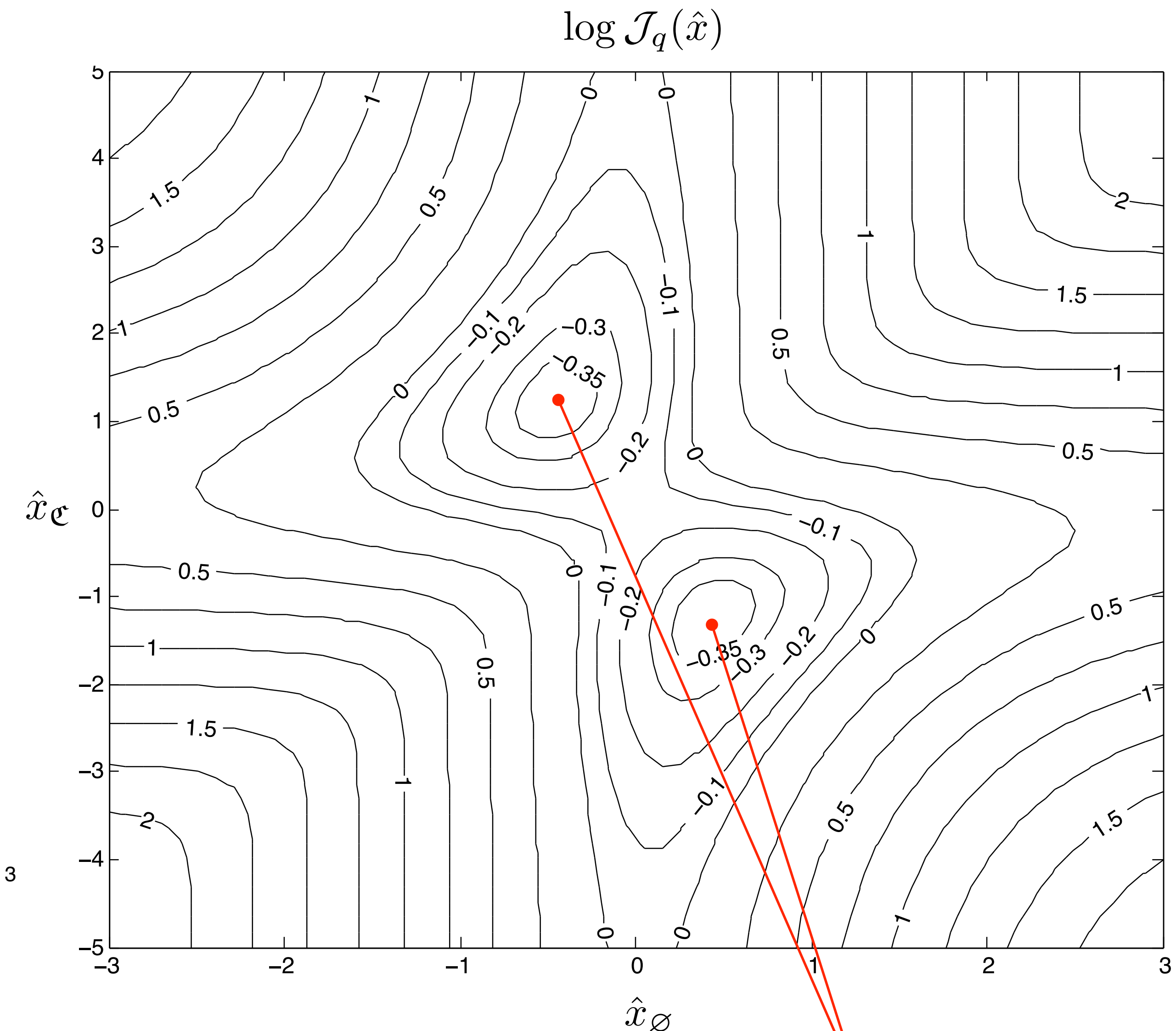
$$\text{minimize} \quad \mathcal{J}_q(\hat{x})$$

unconstrained problem

Quantizer distortion function

$X \sim \mathcal{N}(0,1)$ $\qquad$ $\beta = 0.5$ $\qquad$ $\rho = 1$





non-unique minima

Neither convex nor quasi-convex

Continuously differentiable

Modified Lloyd-Max Algorithm

$$\underset{\hat{x}\in\mathbb{R}^2}{\text{minimize}} \quad \mathcal{J}_q(\hat{x})$$

$$\nabla \mathcal{J}_q(\hat{x}) = 0 \qquad \longleftrightarrow \qquad \hat{x} = \mathcal{F}(\hat{x})$$

Lloyd's Map

$$\mathcal{F}(\hat{x}) = \begin{bmatrix} \mathbf{E}\Big[X|X \in [a(\hat{x}), b(\hat{x})]\Big] \\ \mathbf{E}\Big[X|X \notin [a(\hat{x}), b(\hat{x})]\Big] \end{bmatrix}$$

Modified Lloyd-Max

$$\hat{x}^{(0)} \neq (0,0)$$
$$\hat{x}^{(k+1)} = \mathcal{F}(\hat{x}^{(k)}), \quad k = 0, 1, \ldots$$

Step 1 From $\hat{x}^{(k)}$ update the thresholds $a(\hat{x}^{(k)})$ and $b(\hat{x}^{(k)})$

Step 2 Compute the centroids of the new quantization regions

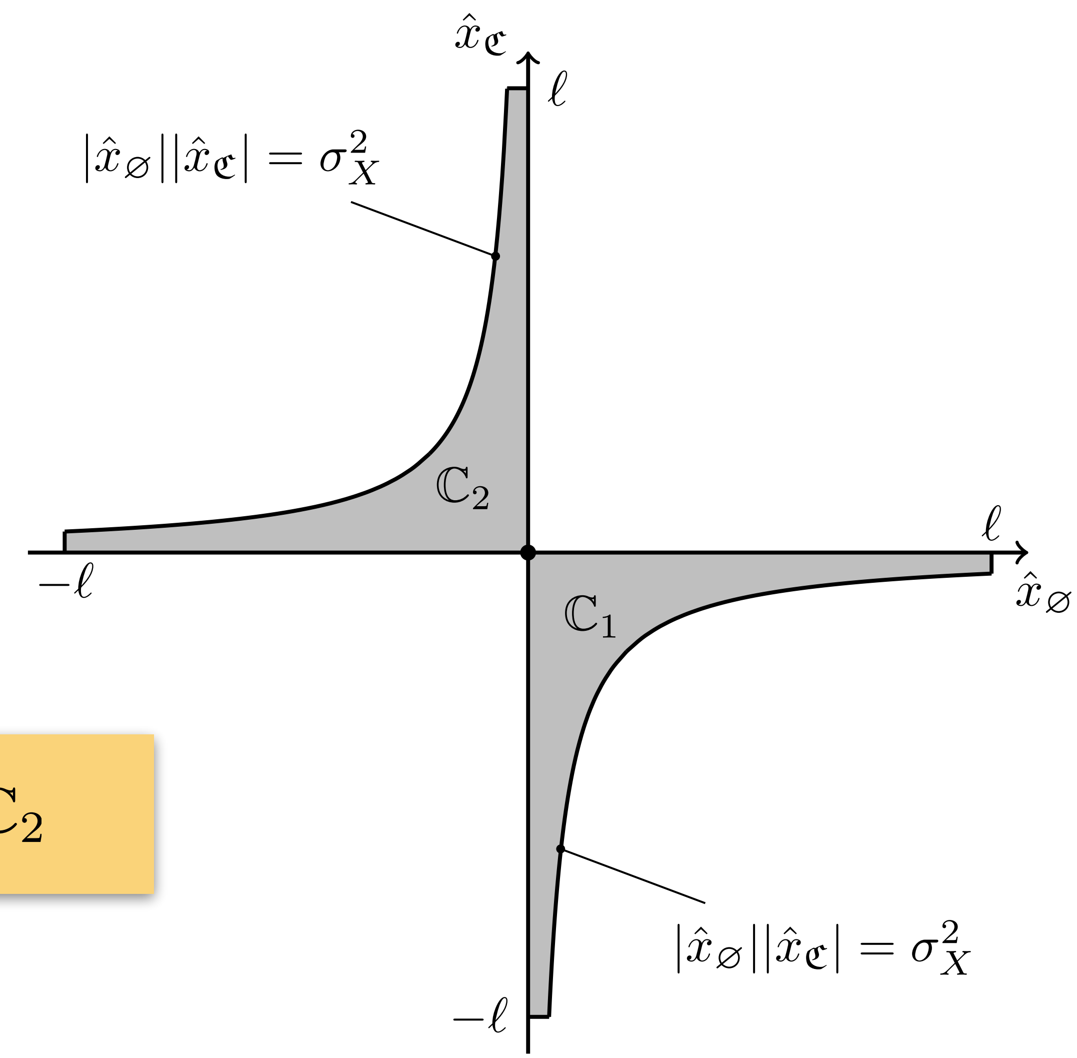
Convergence

Theorem:

For $X \sim \mathcal{N}(0, \sigma_X^2)$ the Modified Lloyd-Max algorithm is **globally convergent** to a local minimum of $\mathcal{J}_q(\hat{x})$.

Sketch of Proof[1,2]:

- Find a compact set $\mathbb{C}$ that contains all the critical points of $\mathcal{J}_q(\hat{x})$ such that $\mathcal{F}(\mathbb{C}) \subset \mathbb{C}$

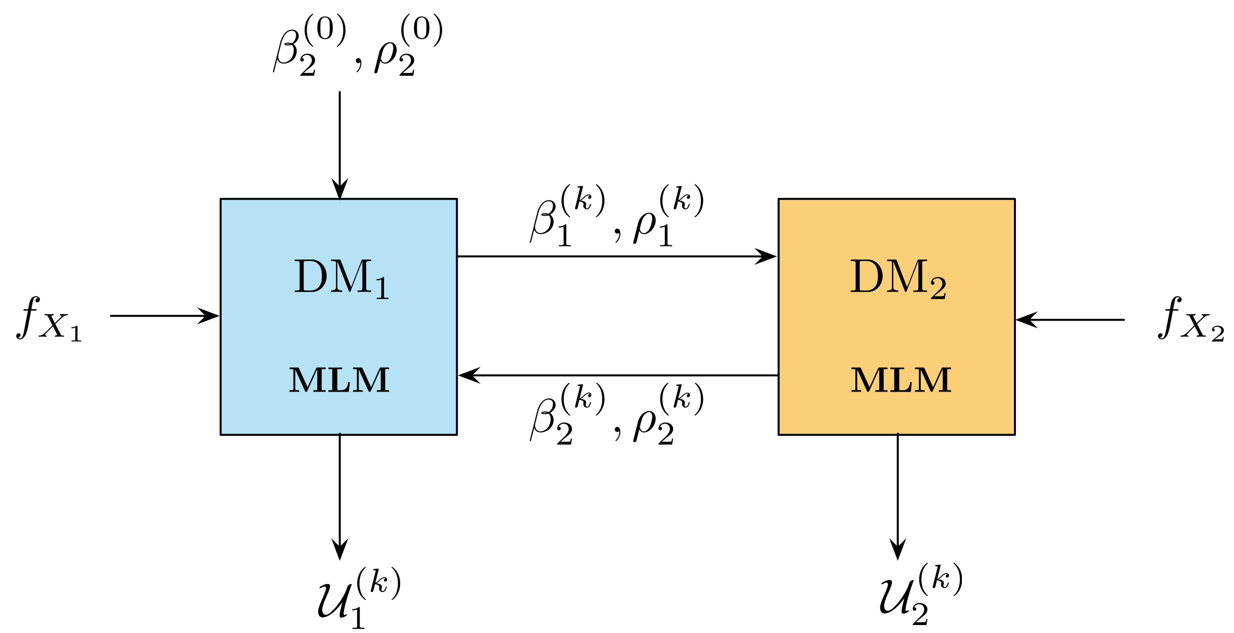


$$\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$$

$\blacksquare$

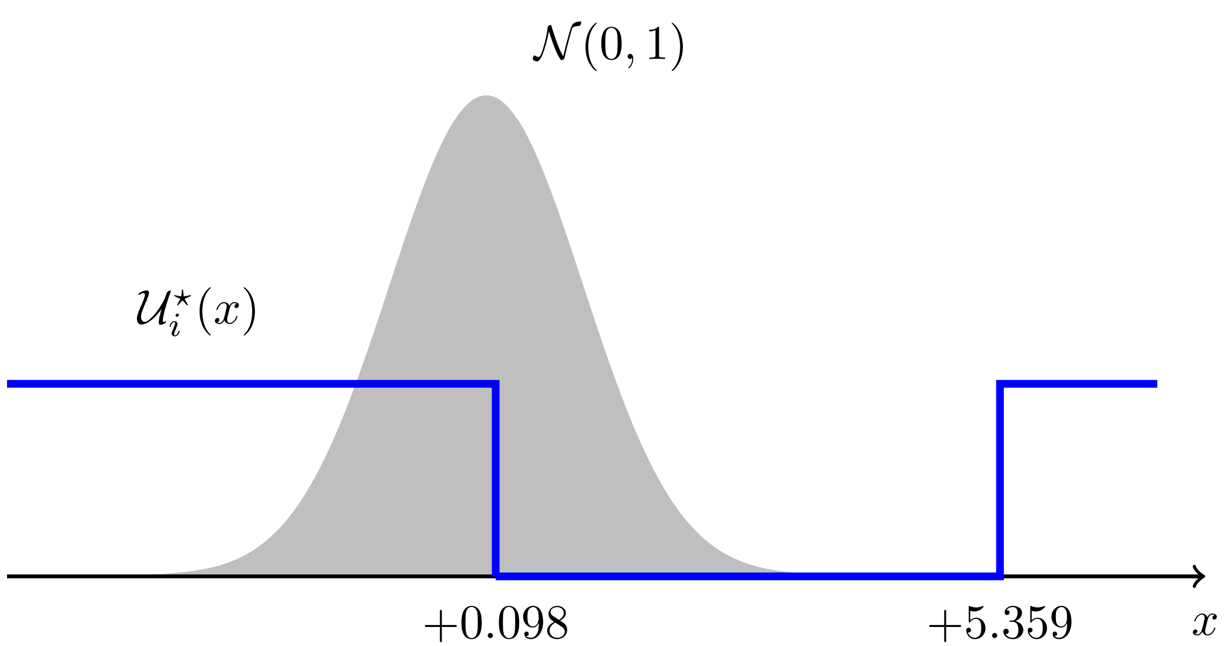
1. Du et. al., "Conv. of the Lloyd algorithm for computing Voronoi tessellations," *SIAM Num. Analysis* 2006.
2. Vasconcelos & Martins, "Optimal thresholds for remote estimation over the collision channel," CDC 2015.

Numerical procedure



Repeat until the cost cannot be further reduced

Example $X_1, X_2 \sim \mathcal{N}(0,1)$



i.i.d. observations, symmetric pdf

asymmetric thresholds

$$\mathcal{J}(\mathcal{U}_1^*, \mathcal{U}_2^*) = 0.54$$

Gain of 46% over scheduling policies

Lipsa & Martins, "Remote state estimation with communication costs for first-order LTI systems". *IEEE TAC* 2011.

Collision channel with common and private observations

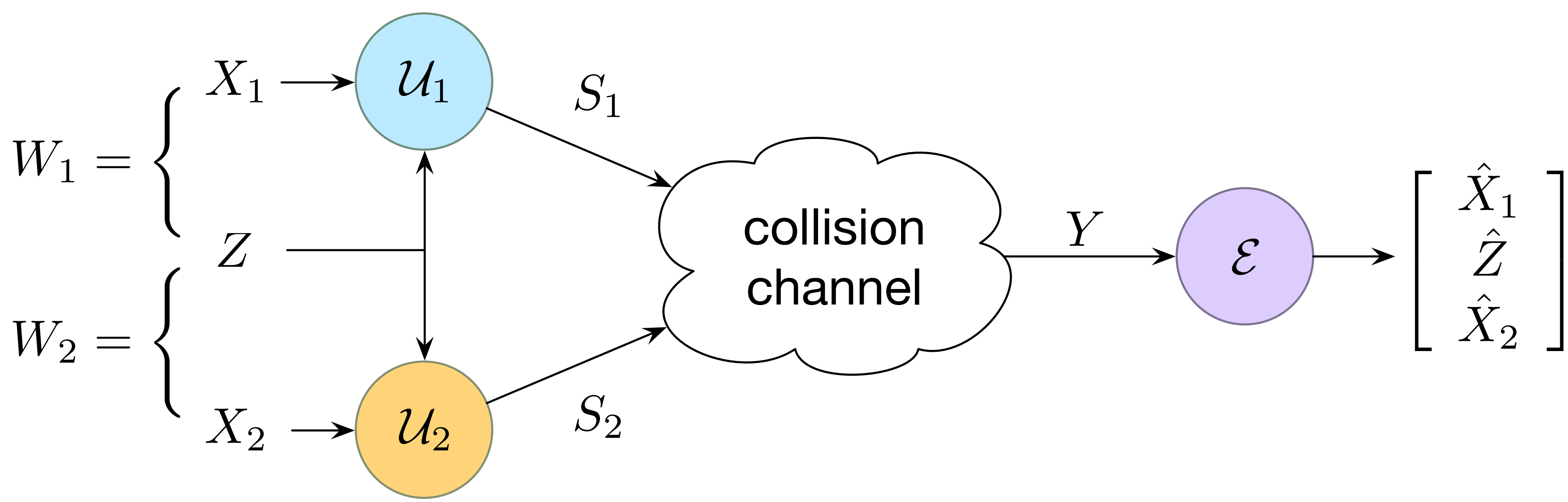
$$W = \begin{bmatrix} X_1 \\ Z \\ X_2 \end{bmatrix} \qquad W_i = \begin{bmatrix} X_i \\ Z \end{bmatrix}$$

private observation

common observation

$$f_W = f_Z \cdot f_{X_1|Z} \cdot f_{X_2|Z}$$

$$X_1 \leftrightarrow Z \leftrightarrow X_2$$

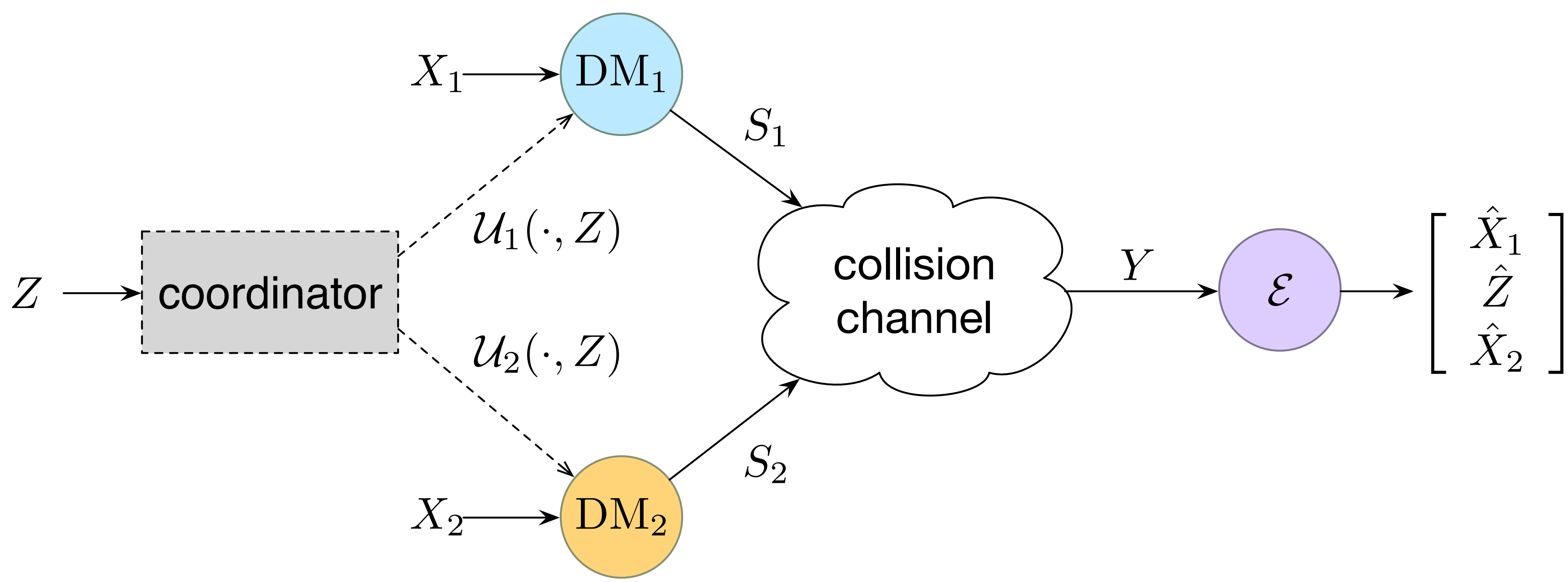


Problem 3

minimize $\quad \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\left[(X_1 - \hat{X}_1)^2 + (Z - \hat{Z})^2 + (X_2 - \hat{X}_2)^2\right]$

Common information approach

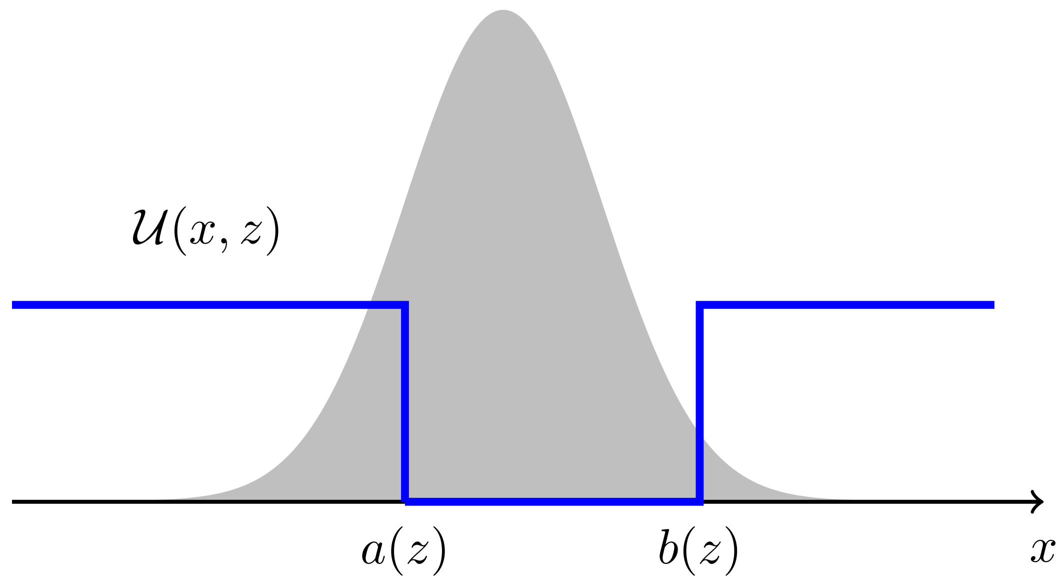
Common information[1] can be used to **simplify** and **characterize** optimal solutions of team problems.



$$\text{minimize} \qquad \mathcal{J}^z(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\left[(W - \hat{W})^T (W - \hat{W}) \mid Z = z\right]$$

1. Nayyar, Mahajan & Teneketzis, "Decentralized Stochastic Control with Partial History Sharing". *IEEE TAC* 2013.

Threshold policy on private information



Threshold policy on private information

$$\mathcal{U}(x,z) = \begin{cases} 0 & a(z) \leq x \leq b(z) \\ 1 & \text{otherwise} \end{cases}$$

Theorem:

There exists a team-optimal pair of **threshold policies on private information** for Problem 3.

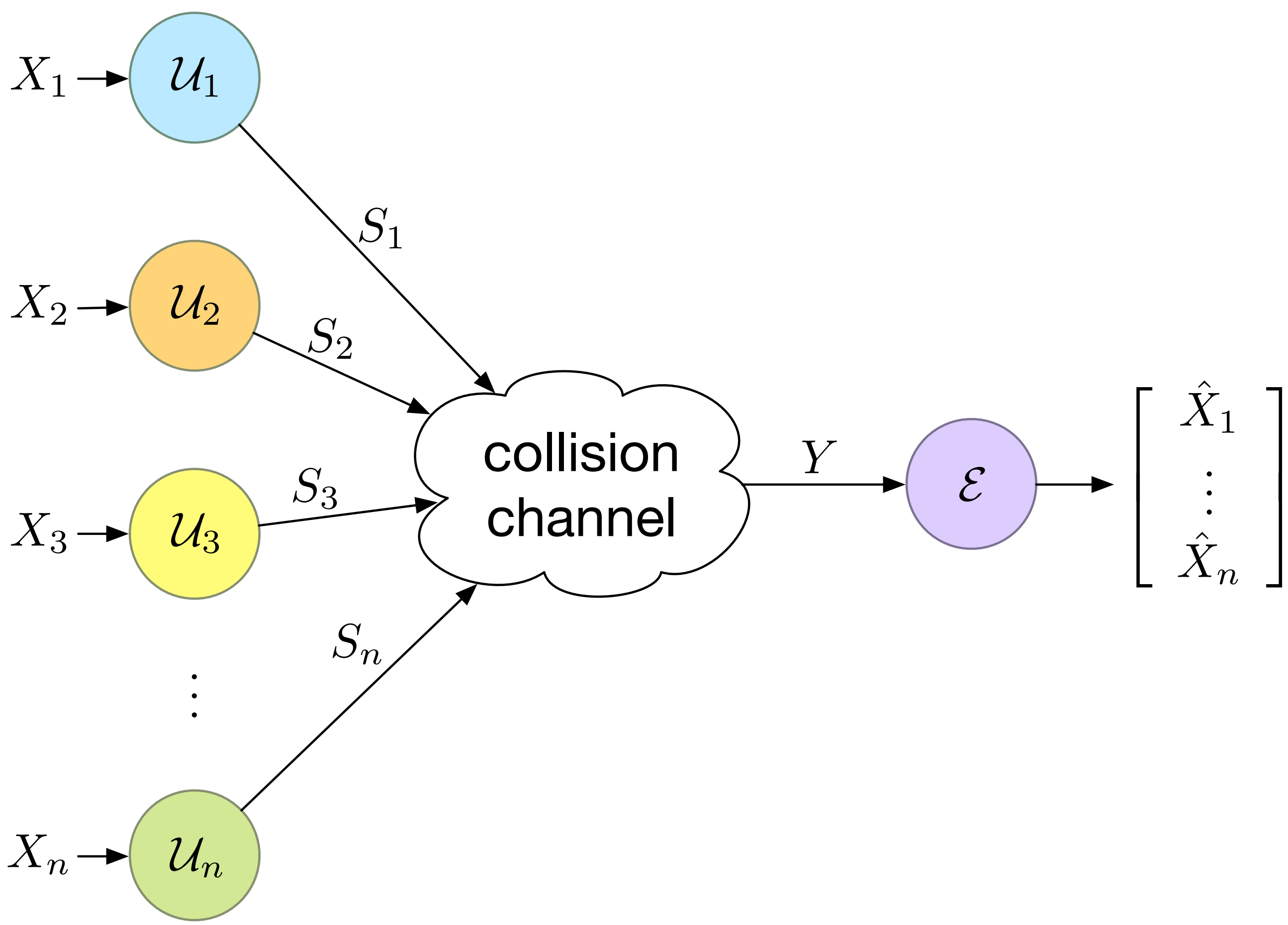
Vasconcelos & Martins, "The structure of optimal communication policies for remote estimation over the collision channel with private and common observations," accepted at CDC 2016.

Part 3. MAP estimation over the collision channel

$$W = [X_1, \cdots, X_n]$$

$$X_i, \quad i \in \{1, \cdots, n\}$$

- mutually independent
- **discrete** rvs
- supported on $\mathbb{X}_i$
- arbitrarily distributed



Probability of error

$$\text{minimize} \quad \mathcal{J}_B(\mathcal{U}_1, \cdots, \mathcal{U}_n) = \mathbf{prob}(W \neq \hat{W})$$

MAP estimator

$$\mathcal{E}(y) = \arg\max_{w \in \mathbb{W}} \ \mathbf{prob}(W = w | Y = y)$$

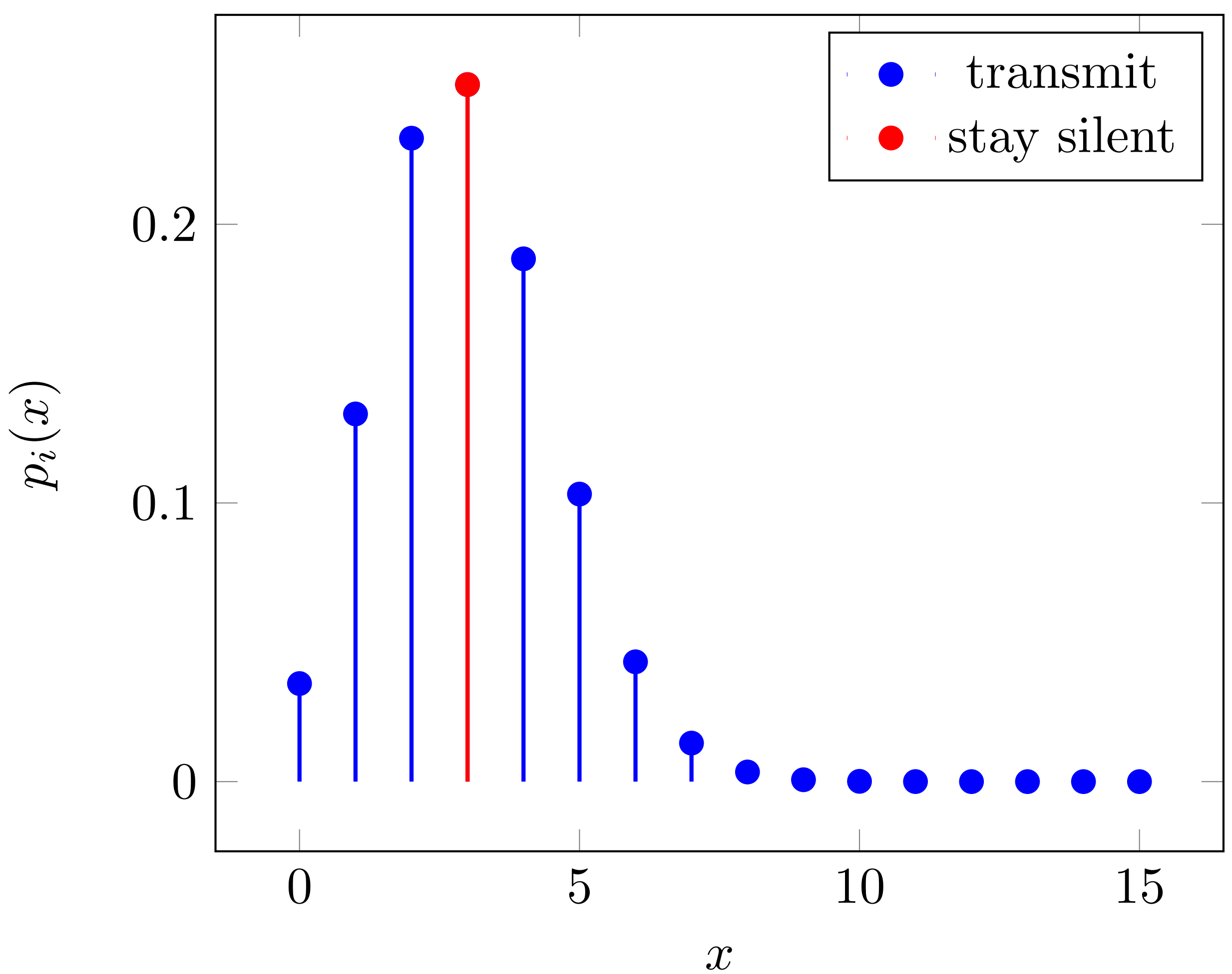
$$\mathbb{W} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_n$$

Structural result

Theorem

There exists a team-optimal strategy where each sensor transmits **all but the most likely** observation.

$\mathcal{U}_i^*(x)$



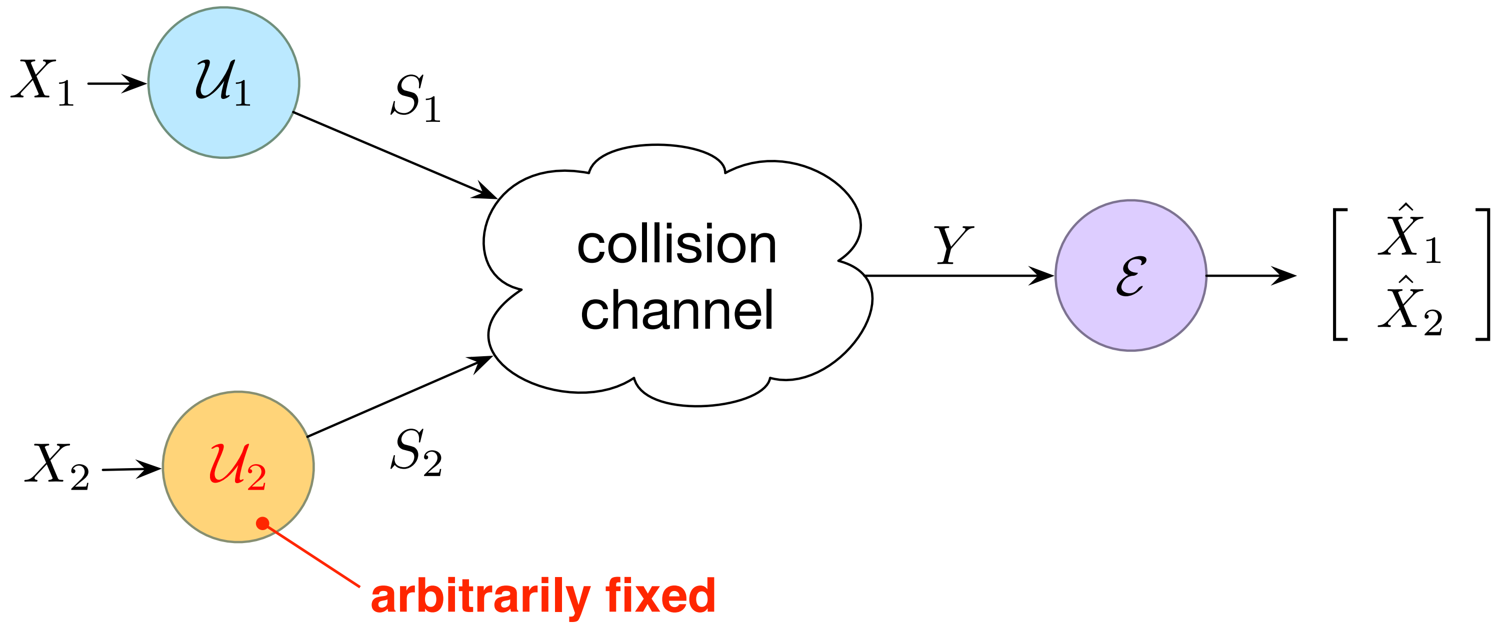
$$\mathcal{U}^* = (\mathcal{U}_1^*, \cdots, \mathcal{U}_n^*)$$

$$\mathcal{U}_i^*(x) = \begin{cases} 0 & \text{if } x = x_{i,[1]} \\ 1 & \text{otherwise.} \end{cases}$$

$$[\mathbb{X}_i] = \{x_{i,[1]}, x_{i,[2]}, x_{i,[3]}, \cdots\}$$

$$p_i(x_{i,[1]}) \geq p_i(x_{i,[2]}) \geq p_i(x_{i,[3]}) \geq \cdots$$

Sketch of proof



Original cost: $\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{prob}(\{X_1 \neq \hat{X}_1\} \cup \{X_2 \neq \hat{X}_2\})$

$$\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2) = 1 - \tau_2 \max_{x \in \mathbb{X}_1} \mathcal{U}_1(x) p_1(x) - \varrho_2 \sum_{x \in \mathbb{X}_1} \mathcal{U}_1(x) p_1(x) - (\beta_2 + \varrho_2) \max_{x \in \mathbb{X}_1} (1 - \mathcal{U}_1(x)) \, p_1(x)$$

where:

$$\tau_2 = \max_{x \in \mathbb{X}_2} \mathcal{U}_2(x) p_2(x)$$

$$\varrho_2 = \max_{x \in \mathbb{X}_2} (1 - \mathcal{U}_2(x)) p_2(x)$$

$$\beta_2 = \sum_{x \in \mathbb{X}_2} \mathcal{U}_2(x) p_2(x)$$

$$\beta_2, \varrho_2, \tau_2 \geq 0$$

$$\beta_2 \geq \tau_2$$

important!

Single DM subproblem

For $\beta, \varrho, \tau \in [0,1]$ such that $\beta \geq \tau$:

$$\begin{aligned} &\text{minimize} \quad && \tilde{\mathcal{J}}_B(\mathcal{U}) \\ &\text{subject to} \quad && 0 \leq \mathcal{U}(x) \leq 1, \quad x \in \mathbb{X} \end{aligned}$$

$$\tilde{\mathcal{J}}_B(\mathcal{U}) \stackrel{\text{def}}{=} 1 - \tau \max_{x \in \mathbb{X}} \mathcal{U}(x)p(x) - \varrho \sum_{x \in \mathbb{X}} \mathcal{U}(x)p(x) - (\varrho + \beta) \max_{x \in \mathbb{X}} (1 - \mathcal{U}(x))\, p(x)$$

concave function

1. Constrain to **deterministic policies**

$\mathbb{X} \to \mathbb{X}_0^{\mathcal{U}}$ — no transmission set

$\mathbb{X} \to \mathbb{X}_1^{\mathcal{U}}$ — transmission set

Sketch of Proof

The cost becomes:

$$\tilde{\mathcal{J}}_B(\mathcal{U}) \quad = \quad 1 - \tau \max_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) - \varrho \sum_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) - (\varrho + \beta) \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x)$$

1. "Converse" part:

$$\sum_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) \leq 1 - \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x) \quad \Longrightarrow \quad \tilde{\mathcal{J}}_B(\mathcal{U}) \geq 1 - \varrho - \tau \max_{x \in \mathbb{X}_1^{\mathcal{U}}} p(x) - \beta \max_{x \in \mathbb{X}_0^{\mathcal{U}}} p(x)$$

$$\boxed{\beta \geq \tau} \quad \Longrightarrow \quad \boxed{\tilde{\mathcal{J}}_B(\mathcal{U}) \geq 1 - \varrho - \tau p_{[2]} - \beta p_{[1]}}$$

2. "Achievability" part:

$$\mathcal{U}^*(x) = \begin{cases} 0 & \text{if } \ x = x_{[1]} \\ 1 & \text{otherwise} \end{cases} \qquad \boxed{\tilde{\mathcal{J}}_B(\mathcal{U}^*) = 1 - \varrho - \tau p_{[2]} - \beta p_{[1]}}$$

■

Example: the i.i.d. case

$$X_i \sim p(x), \quad x \in \mathbb{X}, \quad i \in \{1, \cdots, n\}$$

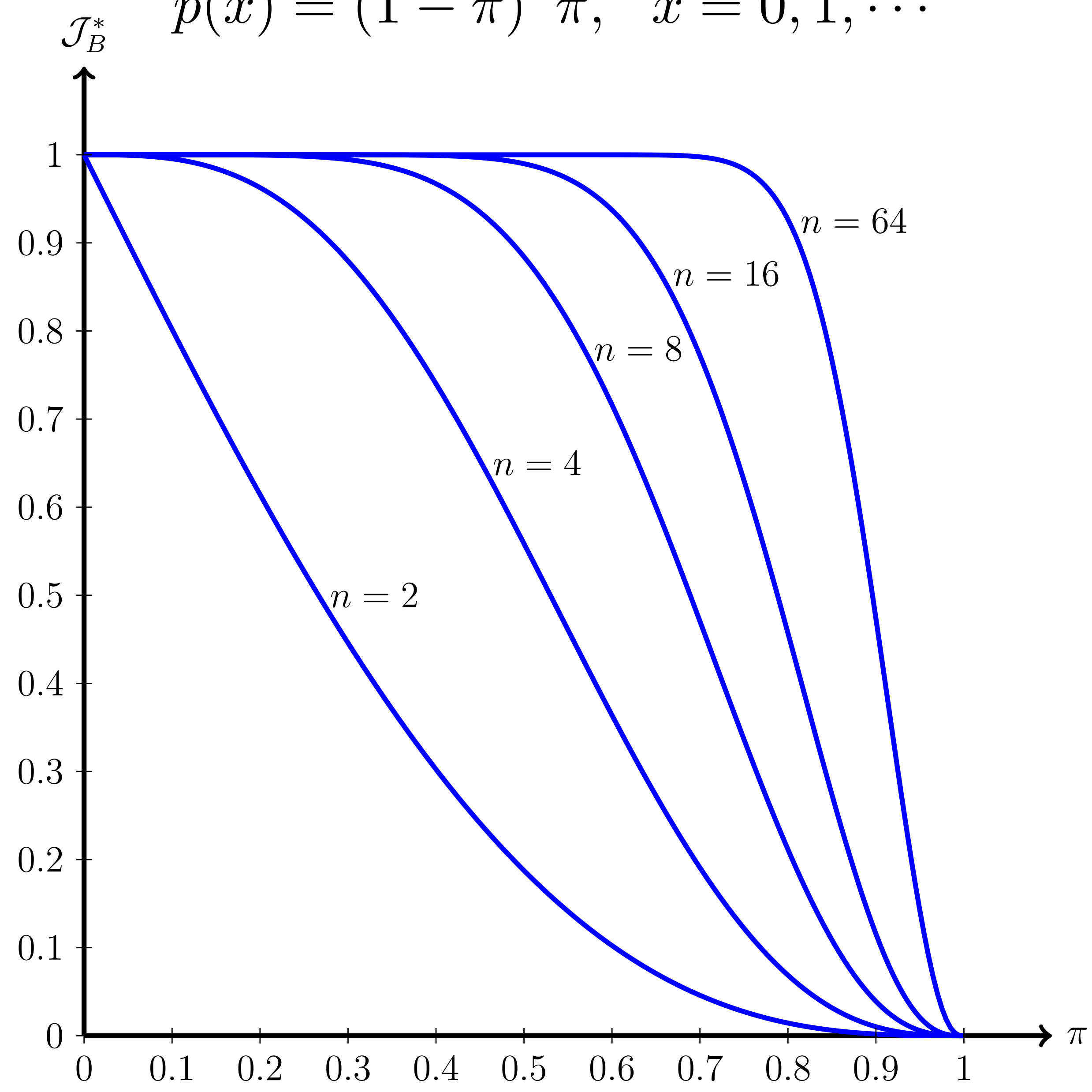
$$\mathcal{J}_B(\mathcal{U}^*) = 1 - np_{[1]}^{n-1}(1 - p_{[1]} - p_{[2]}) - (p_{[1]} + p_{[2]})^n$$

Two observations:

$$\mathcal{J}_B^{\text{bin}}(\mathcal{U}^*) = 0$$

$$\lim_{n\to\infty} \mathcal{J}_B(\mathcal{U}^*) = 1$$

$$p(x) = (1-\pi)^x \pi, \quad x = 0, 1, \cdots$$



Vasconcelos & Martins, "Estimation of discrete random variables over the collision channel with minimum probability of error," to be submitted.

Summary

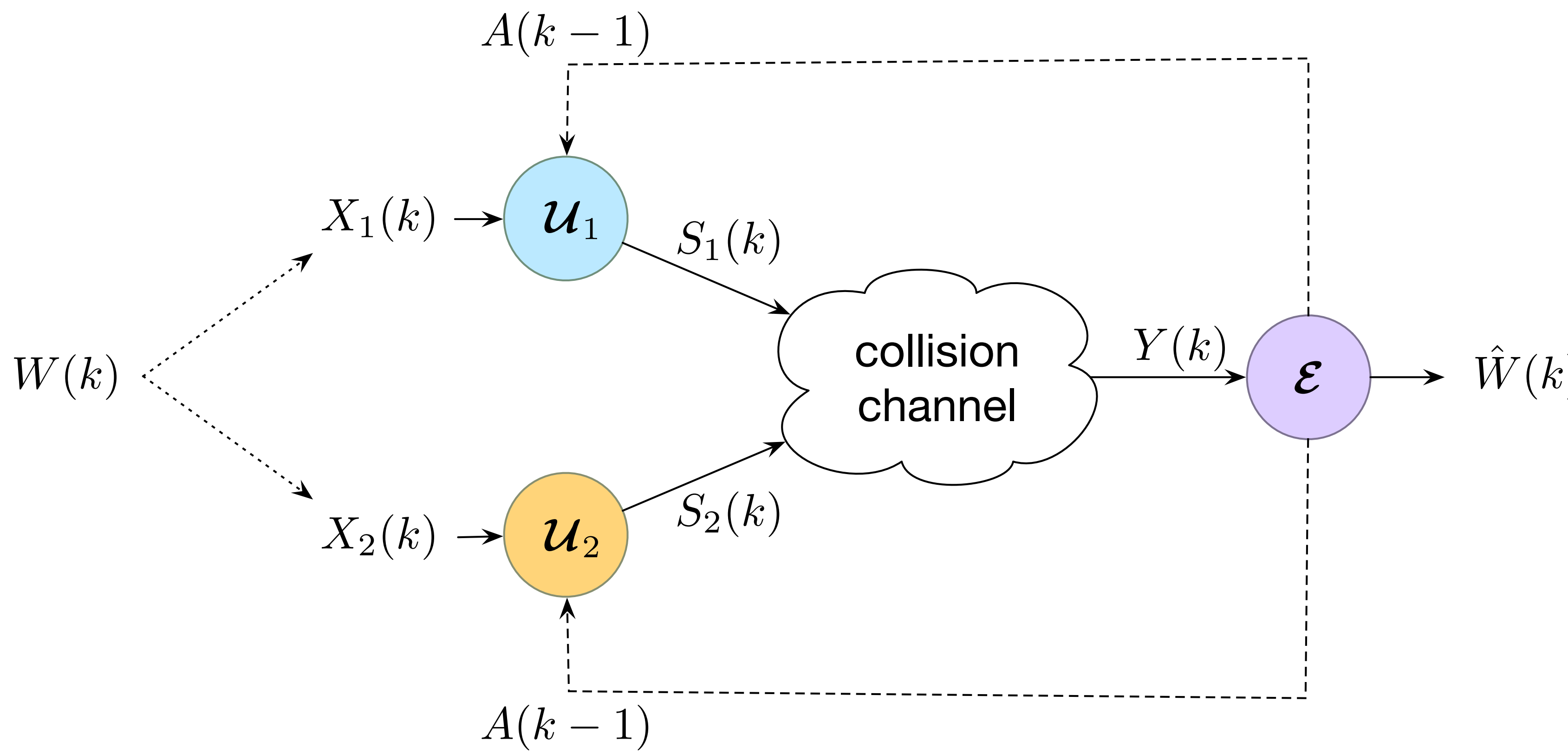
1. **New class of problems** in remote estimation
2. Structural results for the MSE & Prob. of Error cases
3. **Optimality of event-based policies**
4. Algorithm to compute optimal thresholds & its convergence
5. Extensions: dependent observations & n sensors

Future work

1. Extend to **arbitrarily correlated observations**
2. Different types of scalability results
3. **The sequential case** with acknowledgements
4. Lower and upper bounds on estimation error
5. Applications in biological communications

Appendix

The sequential case

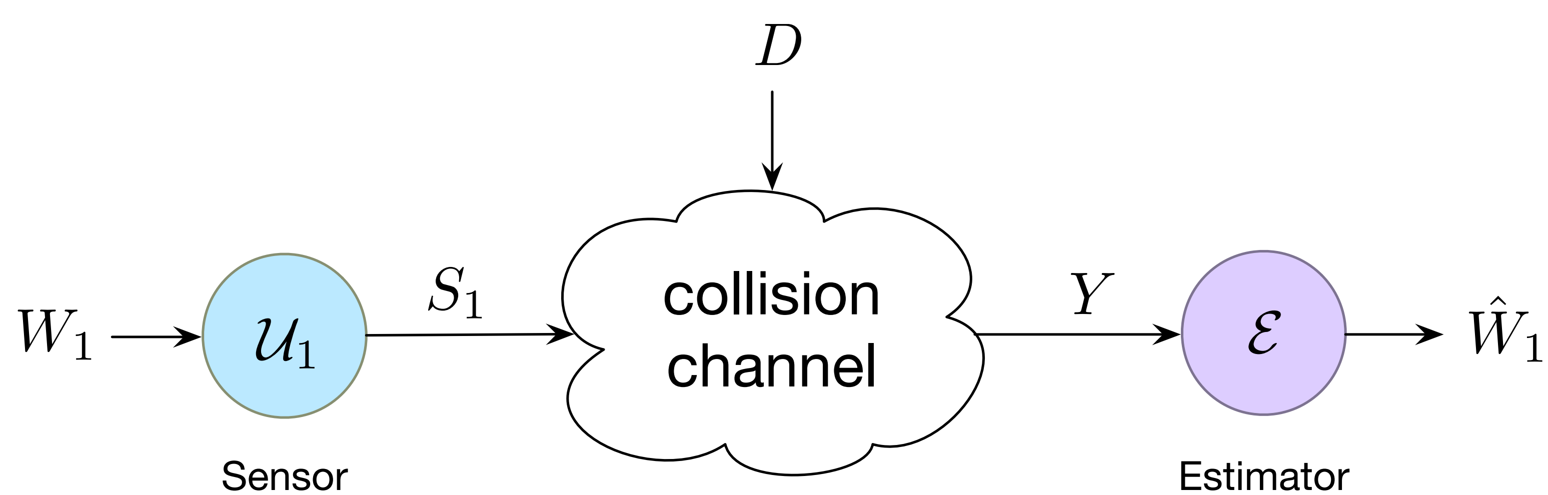


$$\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2, \mathcal{E}) = \sum_{k=0}^{T} \mathbf{E}\left[d\big(W(k), \hat{W}(k)\big)\right]$$

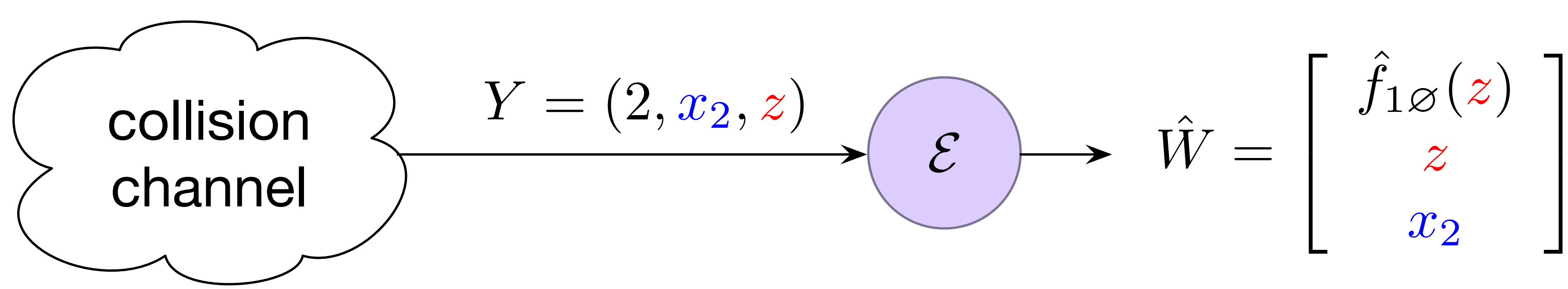
1. Mahajan, "Optimal decentralized control of coupled subsystems with control sharing", *IEEE TAC* 2013.
2. Bobrovsky & Zakai "A lower bound on the estimation error for Markov processes". *IEEE TAC* 1975.
3. Weiss & Weinstein. "A lower bound on the mean-square error in random parameter estimation". *IEEE IT* 1985.

Dependent observations: two difficulties

The event that there is a concurring transmission is **not independent** of W_1.

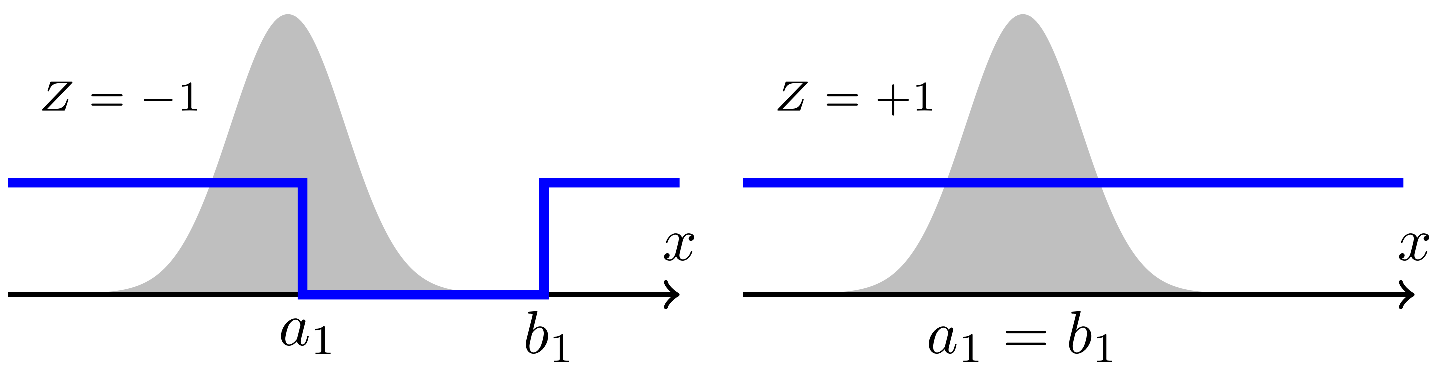


A successful transmission made by sensor 2 **contains side information** for the estimation of X_1.

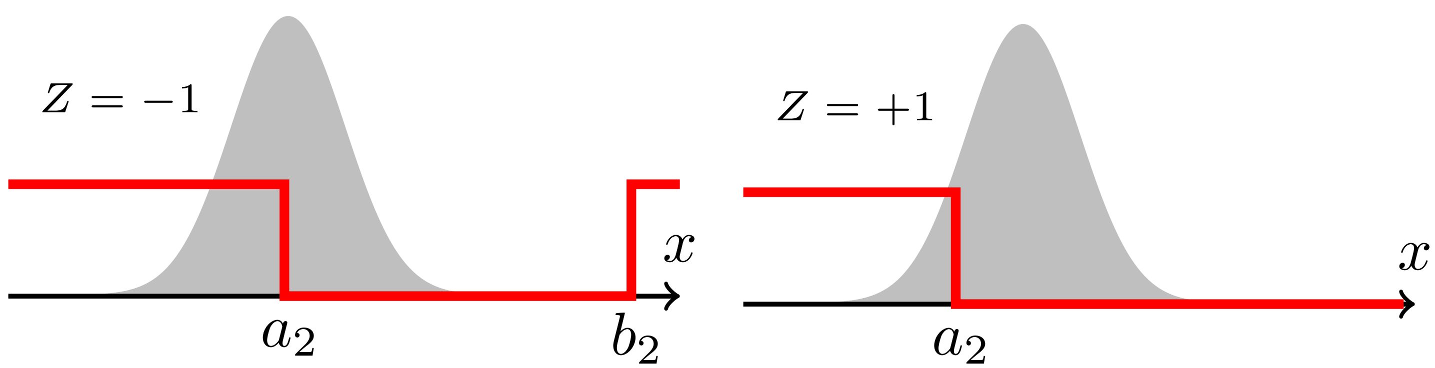


Scheduling vs. event-based policies

Example $X_1, X_2 \sim \mathcal{N}(0,1)$ $Z = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$



(a) Communication policy $\mathcal{U}_1$



(b) Communication policy $\mathcal{U}_2$

p	$\mathcal{J}^*$
0	0.54
0.1	0.59
0.2	0.63
0.3	0.68
0.4	0.73
0.5	0.78

Combination of **scheduling** and **event-based** policies.

$$p = 0.5 \implies \mathcal{J}(\mathcal{U}_1^*, \mathcal{U}_2^*) = 0.78$$

Gain of 22% over scheduling policies