Collaborative estimation over the collision channel

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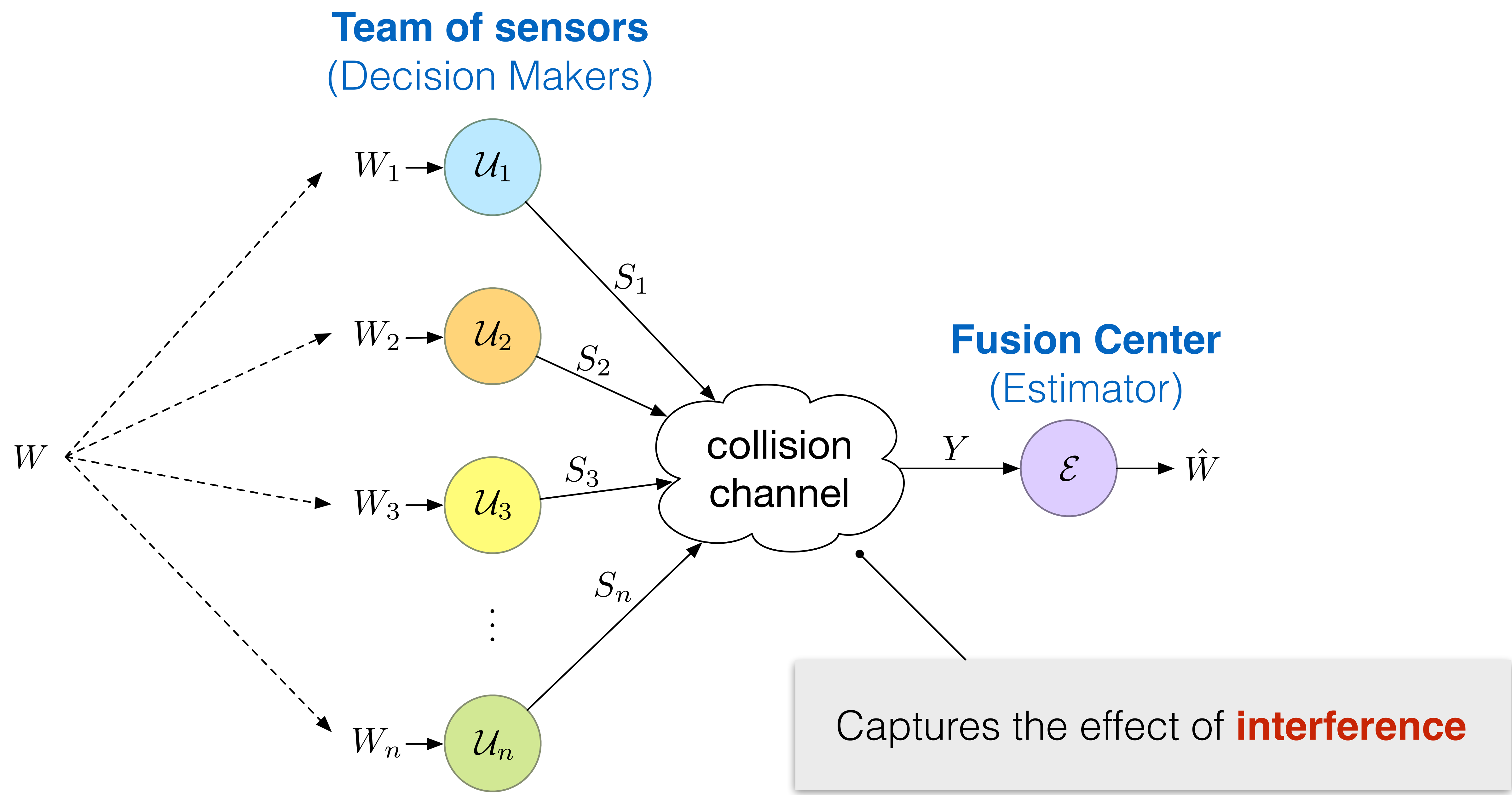
12-11-16

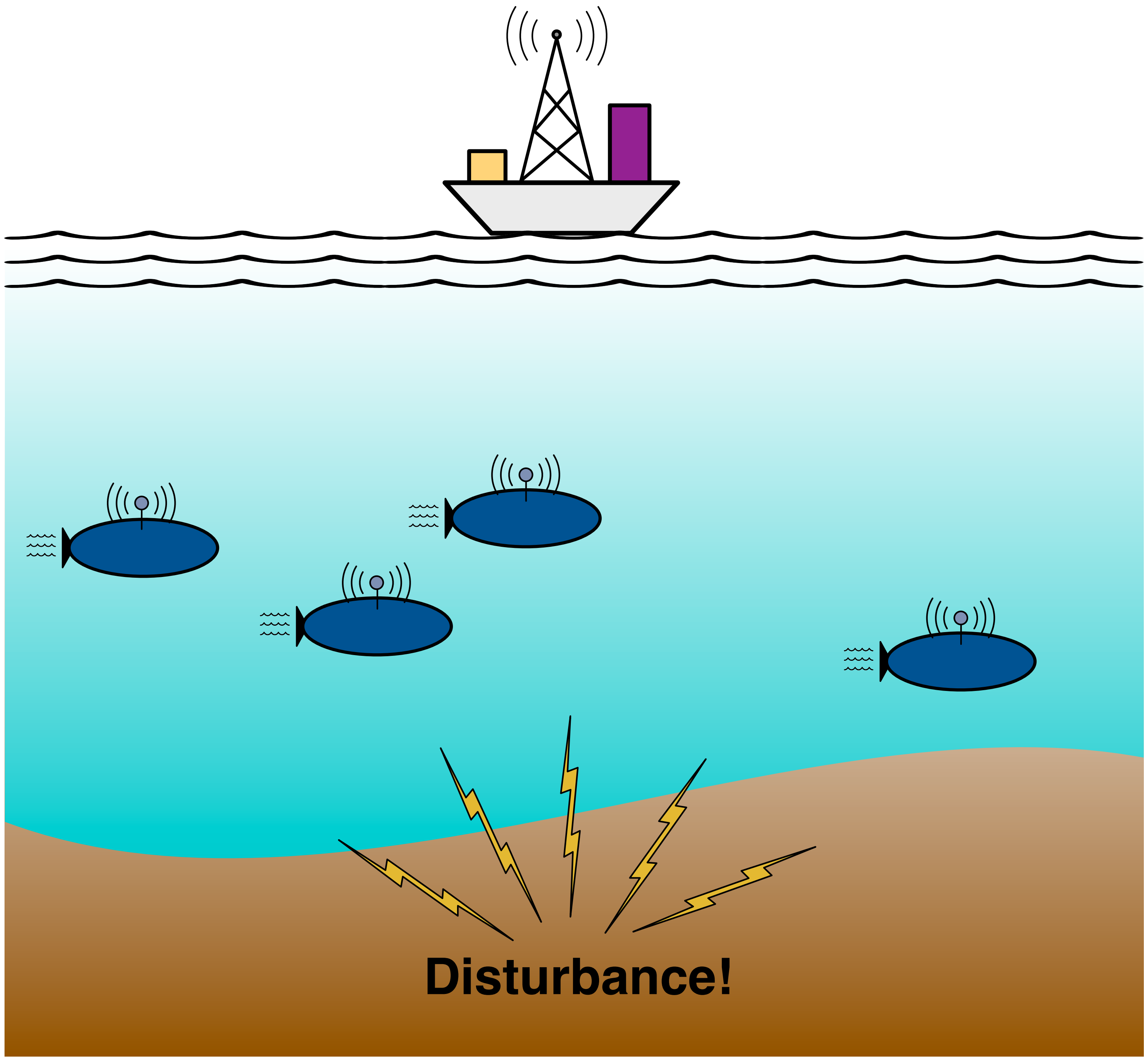
Basic framework



Design jointly optimal communication and estimation policies

Application: Underwater acoustic sensor networks

Environmental monitoring - **quickly** detect a random event or disturbance



Features

- Teams of sensors
- Cooperation
- Decentralized system

Challenges[1,2]

- Collisions (interference)
- Long delays
- Lack of feedback

No coordination protocols

1. Bullo, Cortés and Martínez, *Distributed Control of Robotic Networks*, 2009.
2. Climent et al., “Underwater Acoustic Wireless Sensor Networks,” *IEEE Sensors* 2014.

Collision channel

Model of **interference**:

- **Widely used in wireless communications**[1,2]
- >1 transmission results in a **collision**
- Sensors decide whether to transmit or not

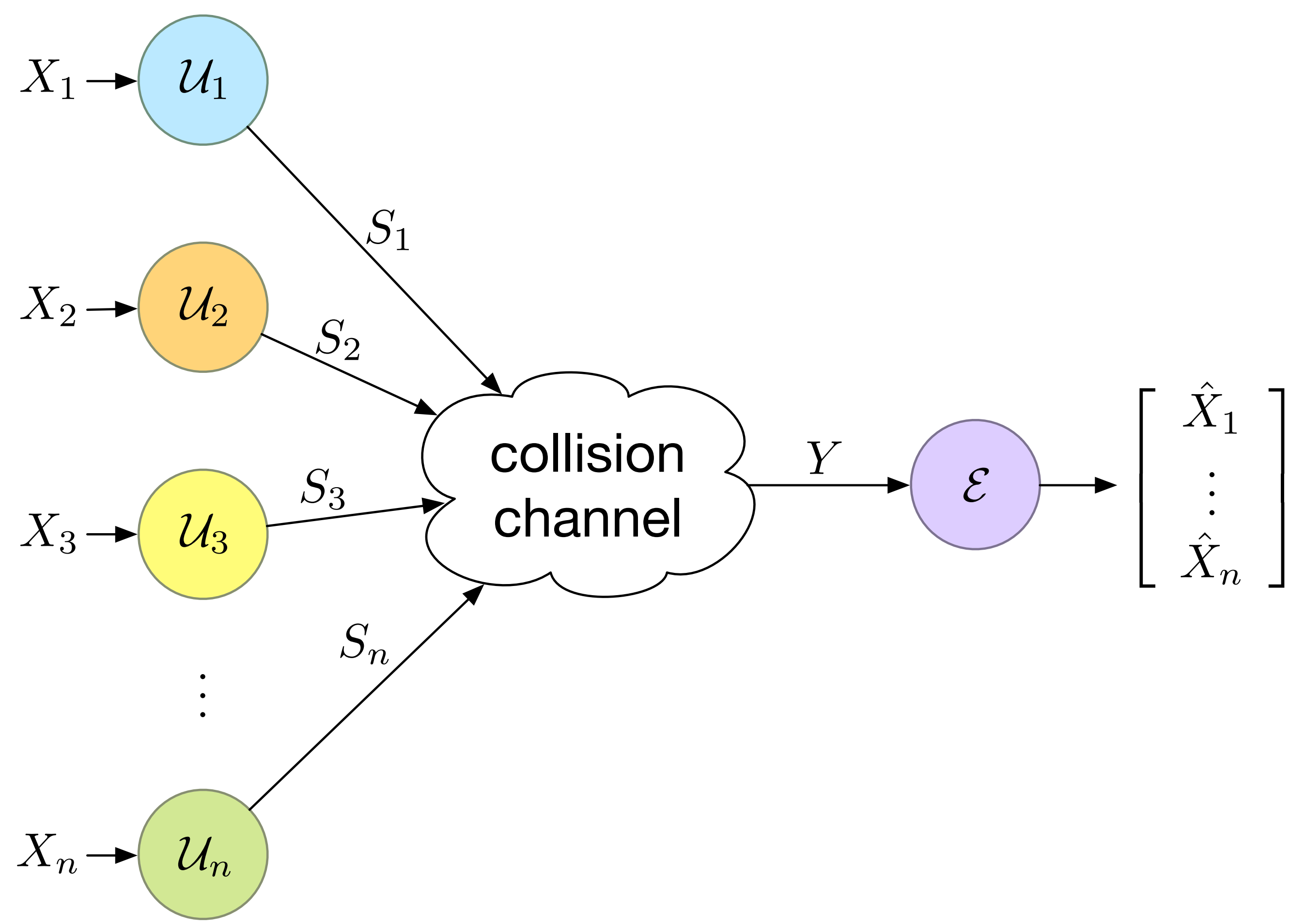
Decision variables: U_i

$$U_i = 1 \quad \Longrightarrow \quad S_i = (i, X_i)$$

(**transmit**)

$$U_i = 0 \quad \Longrightarrow \quad S_i = \varnothing$$

(**stay silent**)



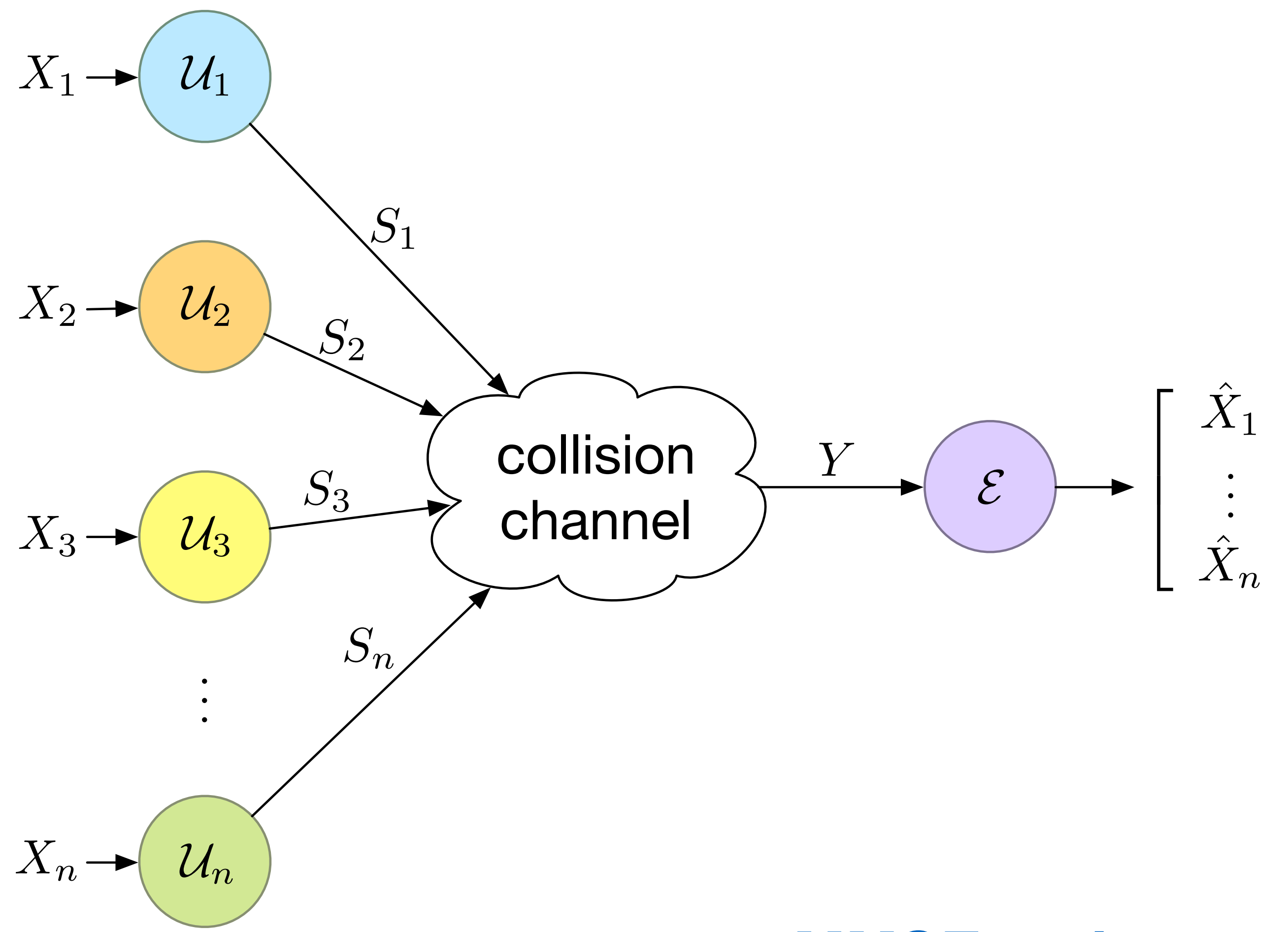
1. Goldsmith, *Wireless Communications*, 2005.
2. Bertsekas and Gallager, *Data Networks*, 1992.

Estimation over the collision channel

$$W = [X_1, \cdots, X_n]$$

$$X_i, \quad i \in \{1, \cdots, n\}$$

- mutually **independent**
- **continuous** rvs
- supported on the real line
- **any distribution**



Stochastic policies

$$\mathbf{prob}(U_i = 1 | X_i = x_i) = \mathcal{U}_i(x_i)$$

MMSE estimator

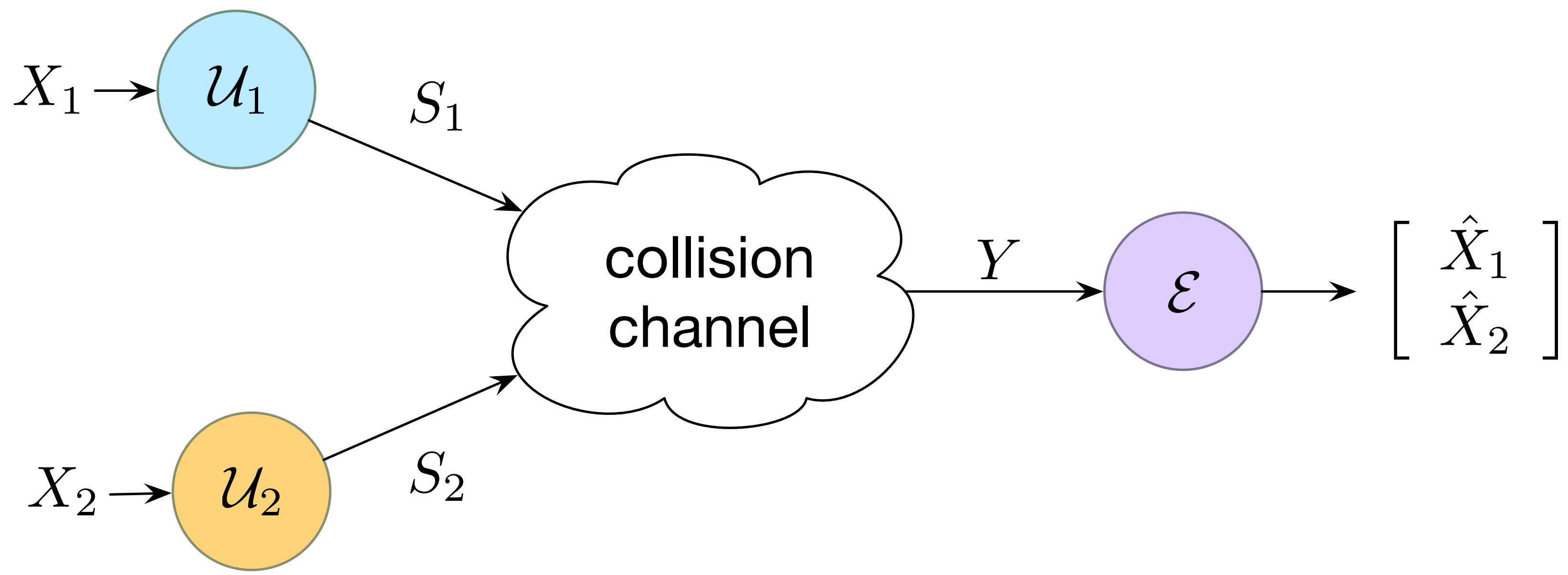
$$\mathcal{E}(y) = \mathbf{E}\big[W | Y = y\big]$$

Problem

$$\text{minimize} \qquad \mathcal{J}(\mathcal{U}_1, \ldots, \mathcal{U}_n) = \mathbf{E}\left[\sum_{i=1}^{n} (X_i - \hat{X}_i)^2\right]$$

Mean Squared Error

Simplest case: two sensors



$$\mathbf{prob}(U_i = 1 | X_i = x_i) = \mathcal{U}_i(x_i)$$

$$\mathbb{U}_i = \{\mathcal{U} \mid \mathcal{U} : \mathbb{R} \to [0,1]\}\,, \quad i \in \{1,2\}$$

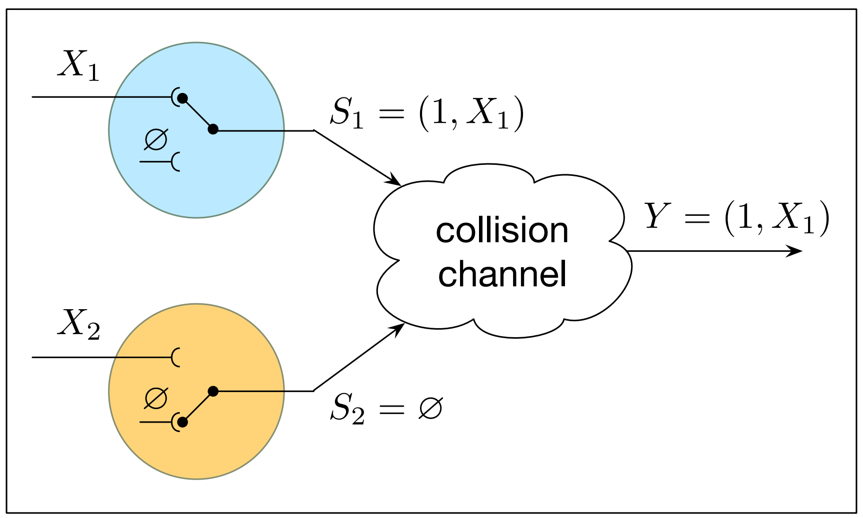
Problem 1

$$\text{minimize} \qquad \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\left[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\right]$$

Collision channel

single transmission

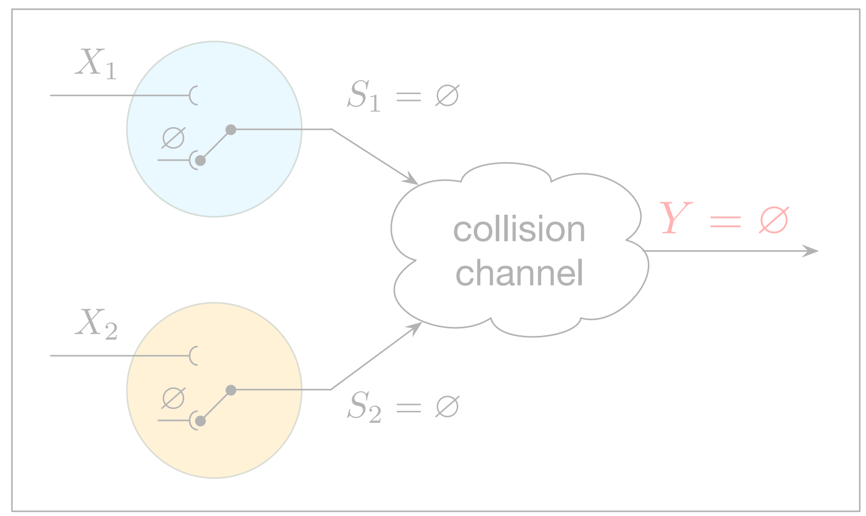
$U_1 = 1, U_2 = 0$



success!

no transmissions

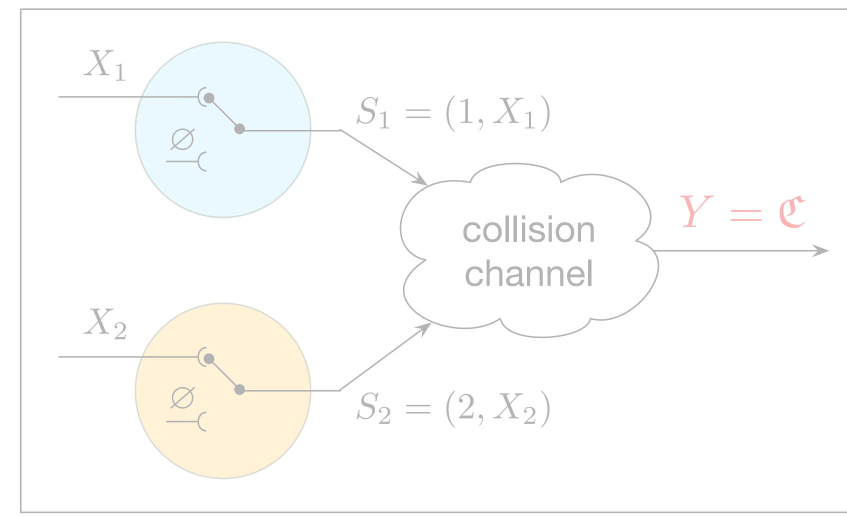
$U_1 = 0, U_2 = 0$



no transmission $\varnothing$

>1 transmissions

$U_1 = 1, U_2 = 1$



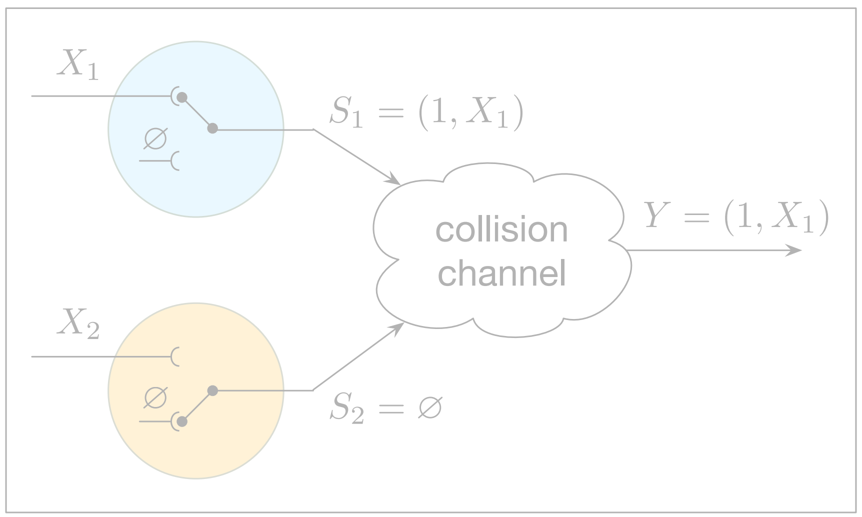
collision $\mathfrak{C}$

From the channel output we can always recover U_1 and U_2.

Collision channel

single transmission

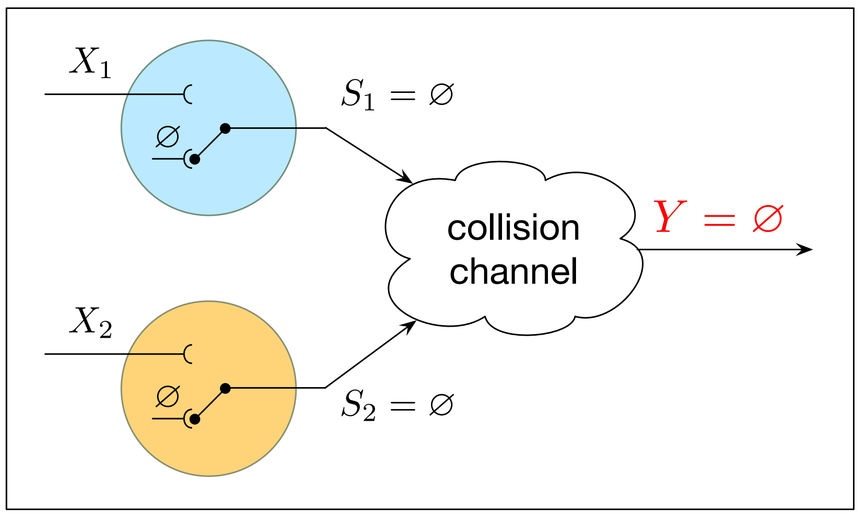
$U_1 = 1, U_2 = 0$



success!

no transmissions

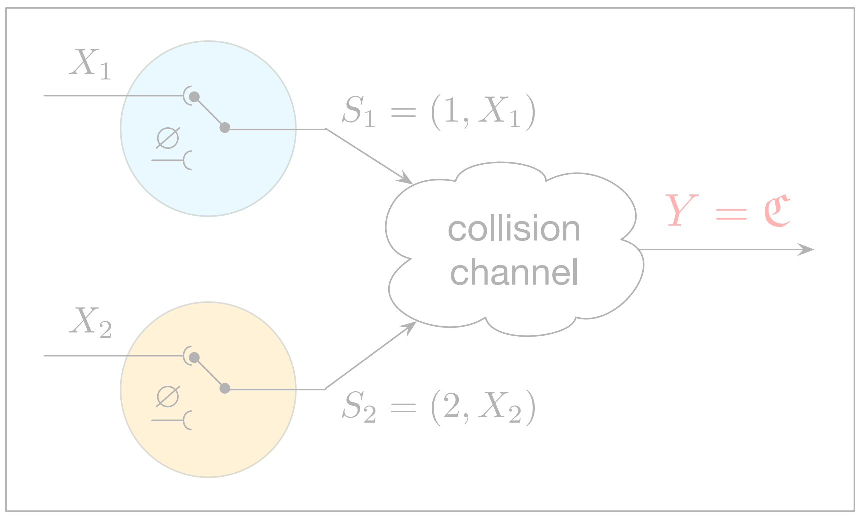
$U_1 = 0, U_2 = 0$



no transmission $\varnothing$

>1 transmissions

$U_1 = 1, U_2 = 1$



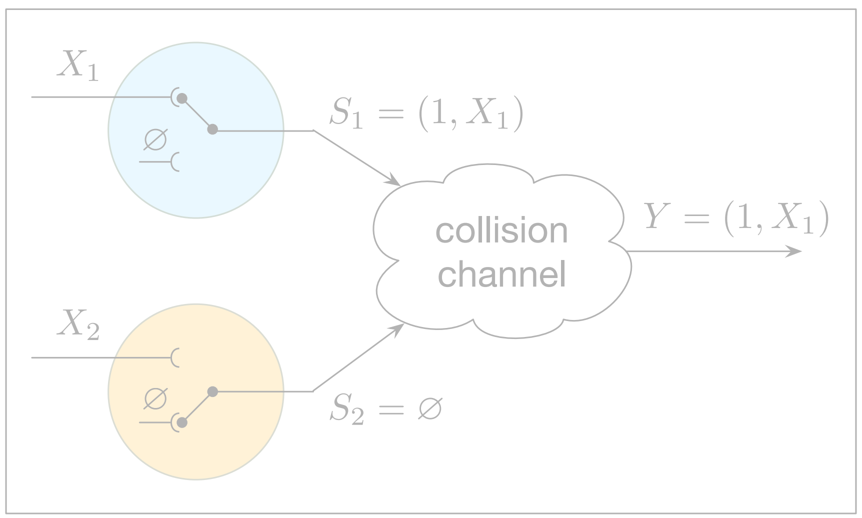
collision $\mathfrak{C}$

From the channel output we can always recover U_1 and U_2.

Collision channel

single transmission

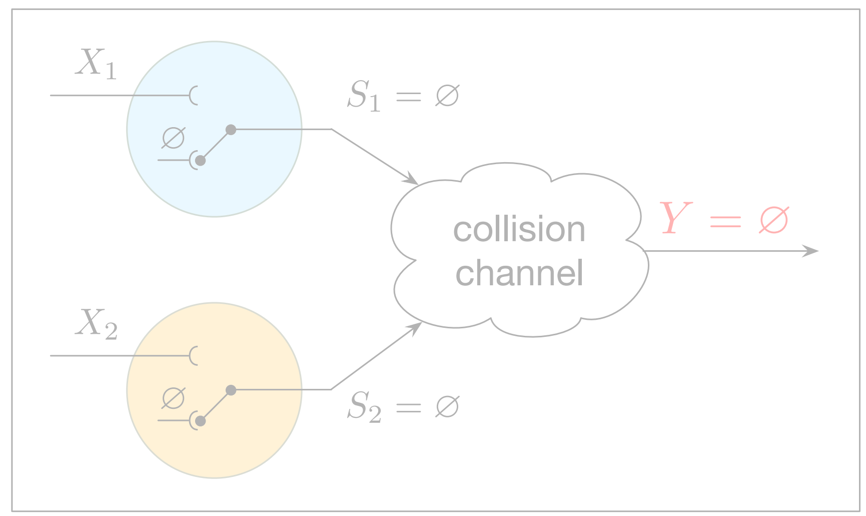
$U_1 = 1, U_2 = 0$



success!

no transmissions

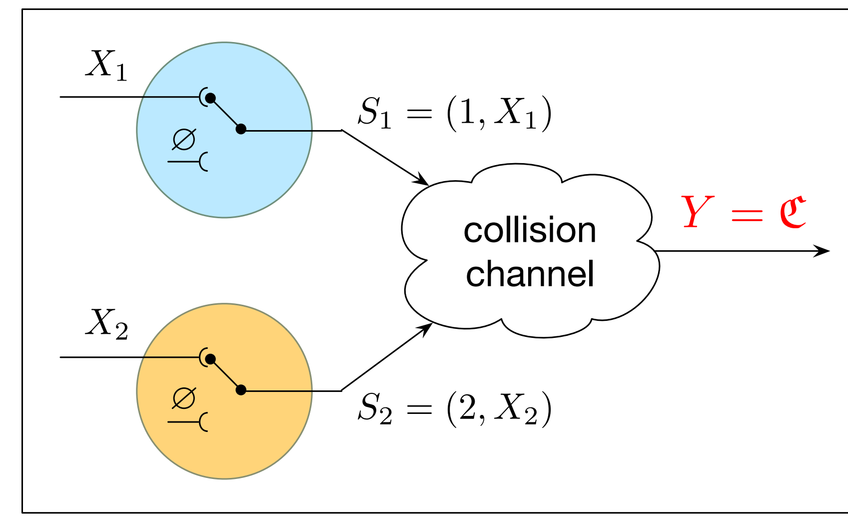
$U_1 = 0, U_2 = 0$



no transmission $\varnothing$

>1 transmissions

$U_1 = 1, U_2 = 1$



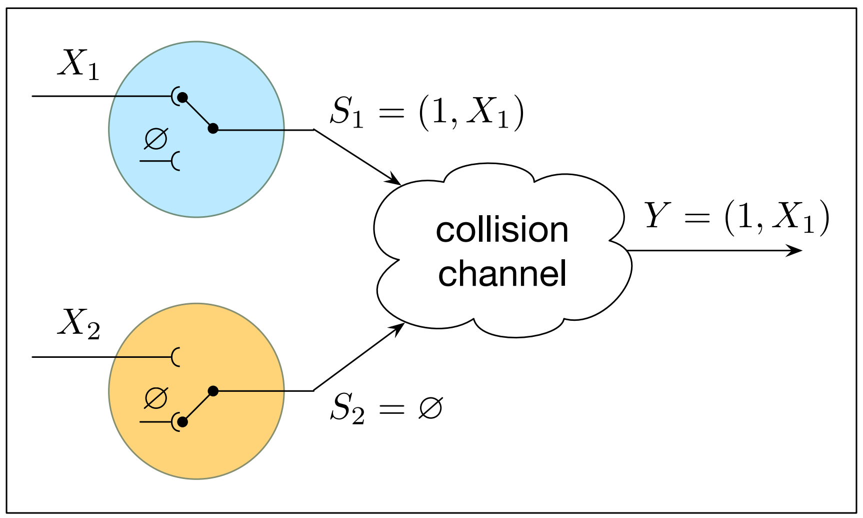
collision $\mathfrak{C}$

From the channel output we can always recover U_1 and U_2.

Collision channel

single transmission

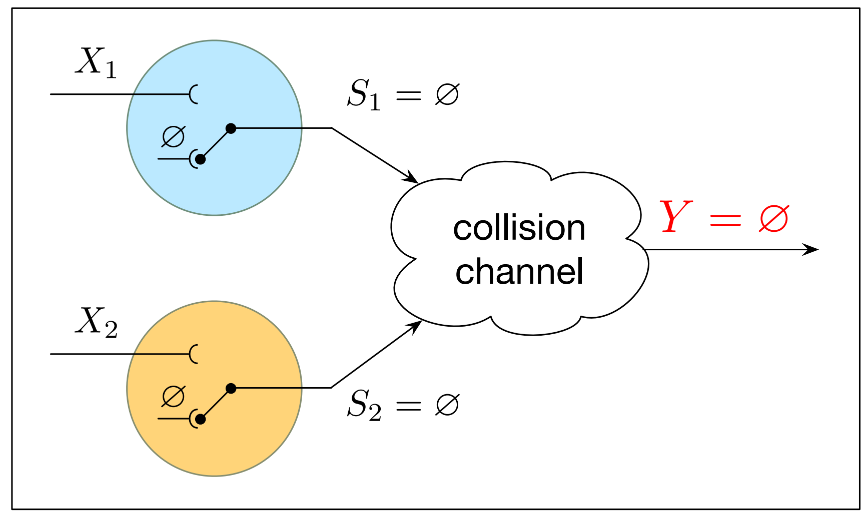
$U_1 = 1, U_2 = 0$



success!

no transmissions

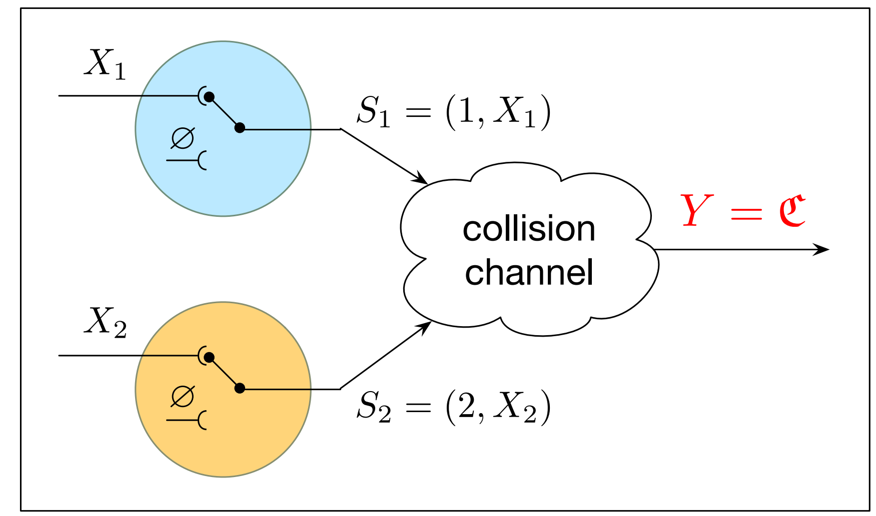
$U_1 = 0, U_2 = 0$



no transmission $\varnothing$

>1 transmissions

$U_1 = 1, U_2 = 1$

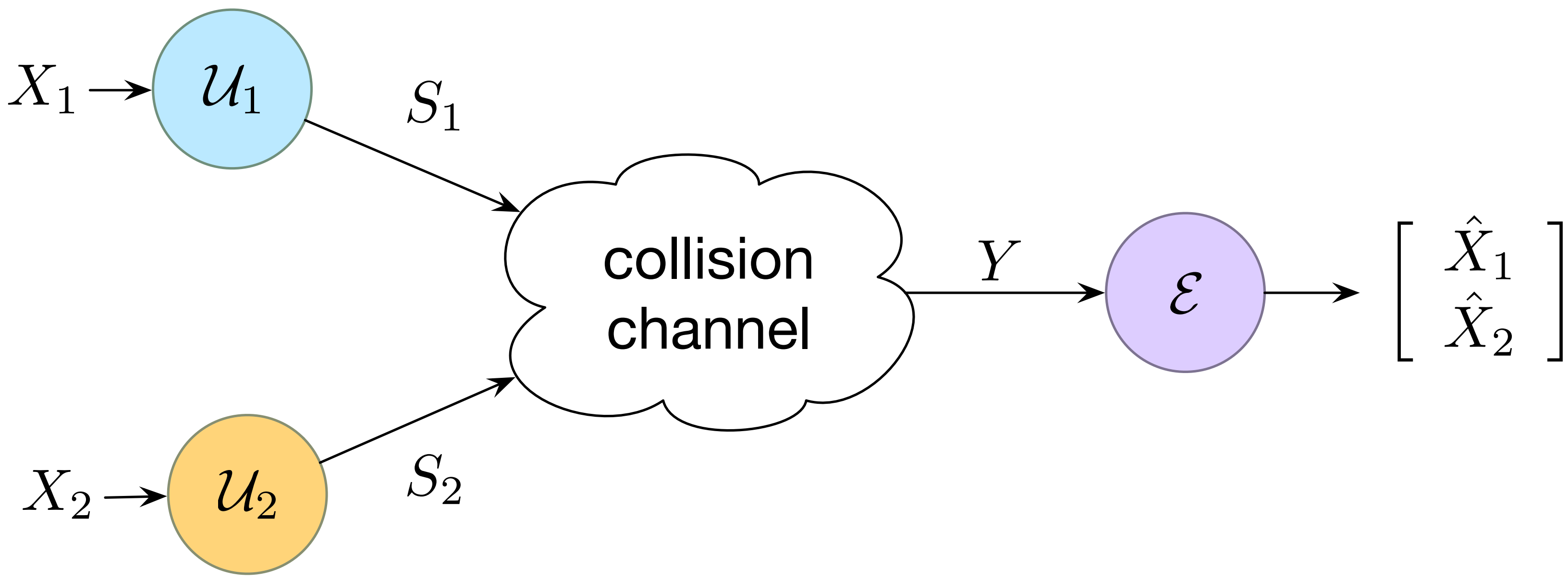


collision $\mathfrak{C}$

The collision channel is fundamentally different from the packet drop channel[1,2]

1. Sinopoli et al, "Kalman filtering with intermittent observations," *IEEE TAC* 2004.
2. Gupta et al, "Optimal LQG control across packet-dropping links," *Systems and Control Letters* 2007.

Why is this problem interesting?



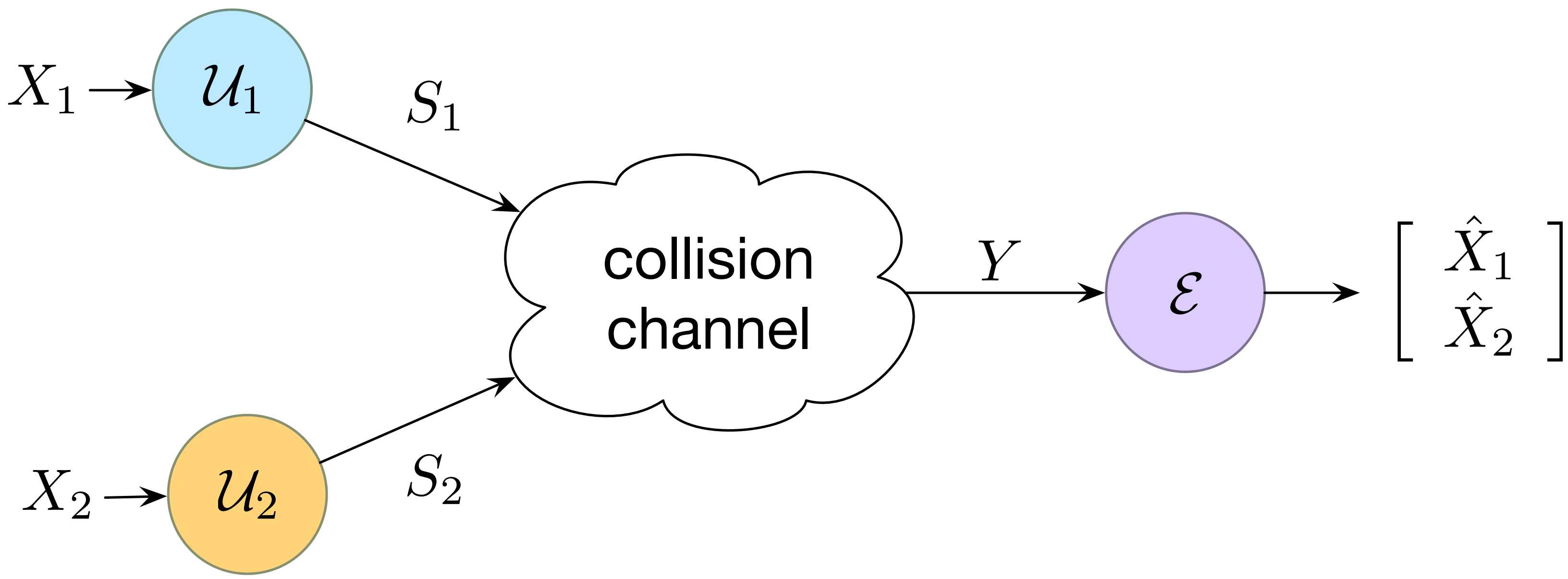
Problem 1

$$\text{minimize} \qquad \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\left[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\right]$$

Team-decision problem $\Longrightarrow$ **Non-convex** (in most cases) **intractable**[1,2]

1. Witsenhausen, “A counterexample in optimal stochastic control,” *SIAM J. Control* 1968.
2. Tsitsiklis & Athans, “On the complexity of decentralized decision making and detection problems,” *IEEE TAC* 1985.

Why is this problem interesting?



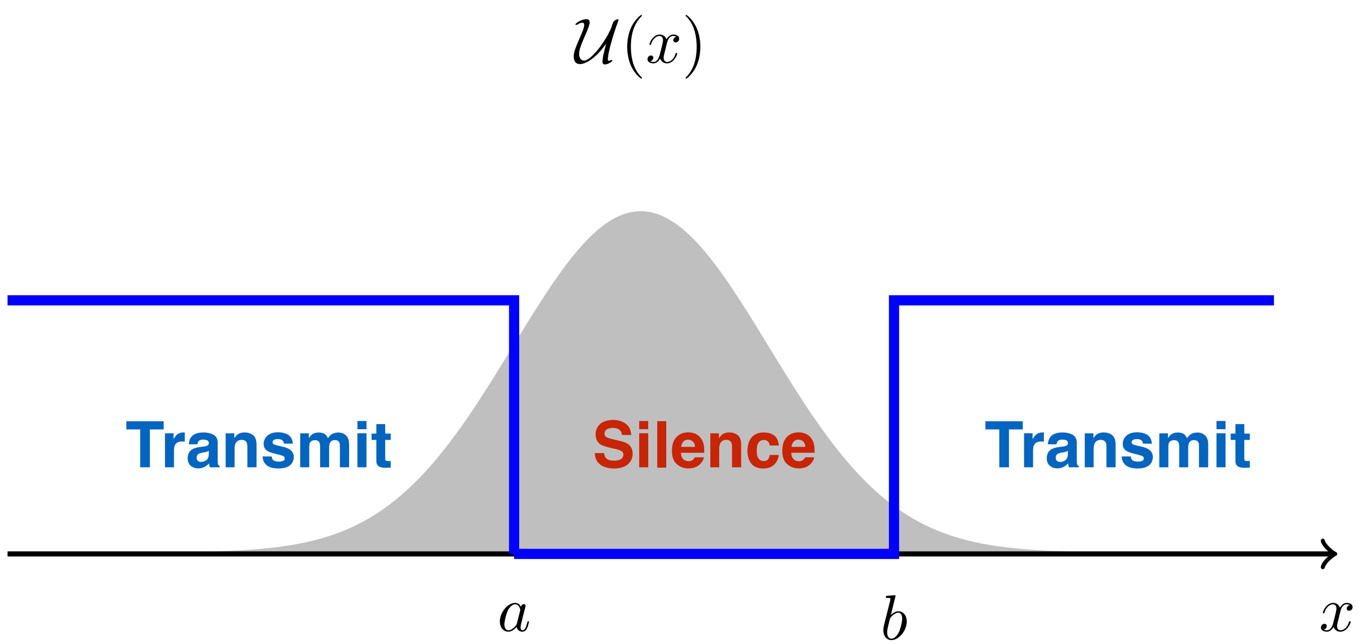
Problem 1

$$\text{minimize} \quad \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\left[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\right]$$

Look for a class parametrizable policies that contains an optimal strategy

1. Witsenhausen, "A counterexample in optimal stochastic control," *SIAM J. Control* 1968.
2. Tsitsiklis & Athans, "On the complexity of decentralized decision making and detection problems," *IEEE TAC* 1985.

Deterministic threshold policies



Threshold policy

$$\mathcal{U}(x) = \begin{cases} 0 & a \leq x \leq b \\ 1 & \text{otherwise} \end{cases}$$

1. Imer & Basar, "Optimal estimation with limited measurements", *Int. Journal of Syst., Cont. and Comm.* 2010.
2. Lipsa & Martins, "Remote state estimation with communication costs for first-order LTI systems". *IEEE TAC* 2011.

Main idea

Team-optimality

$$\mathcal{J}(\mathcal{U}_1^*, \mathcal{U}_2^*) \leq \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2), \quad (\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$$

$$\Longrightarrow$$

$$\not\Longleftarrow$$

Person-by-person optimality

$$\mathcal{J}(\mathcal{U}_1^*, \mathcal{U}_2^*) \leq \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2^*), \quad \mathcal{U}_1 \in \mathbb{U}_1$$

$$\mathcal{J}(\mathcal{U}_1^*, \mathcal{U}_2^*) \leq \mathcal{J}(\mathcal{U}_1^*, \mathcal{U}_2), \quad \mathcal{U}_2 \in \mathbb{U}_2$$

Given any pair of person-by-person optimal policies **construct a new pair** with **equal or better cost**, where each policy is **threshold**

$$(\mathcal{U}_1^*, \mathcal{U}_2^*) \in \mathbb{U}_1 \times \mathbb{U}_2 \quad \longrightarrow \quad (\breve{\mathcal{U}}_1^*, \breve{\mathcal{U}}_2^*) \in \mathbb{U}_1 \times \mathbb{U}_2$$

$$\mathcal{J}(\mathcal{U}_1^*, \mathcal{U}_2^*) \geq \mathcal{J}(\breve{\mathcal{U}}_1^*, \breve{\mathcal{U}}_2^*)$$

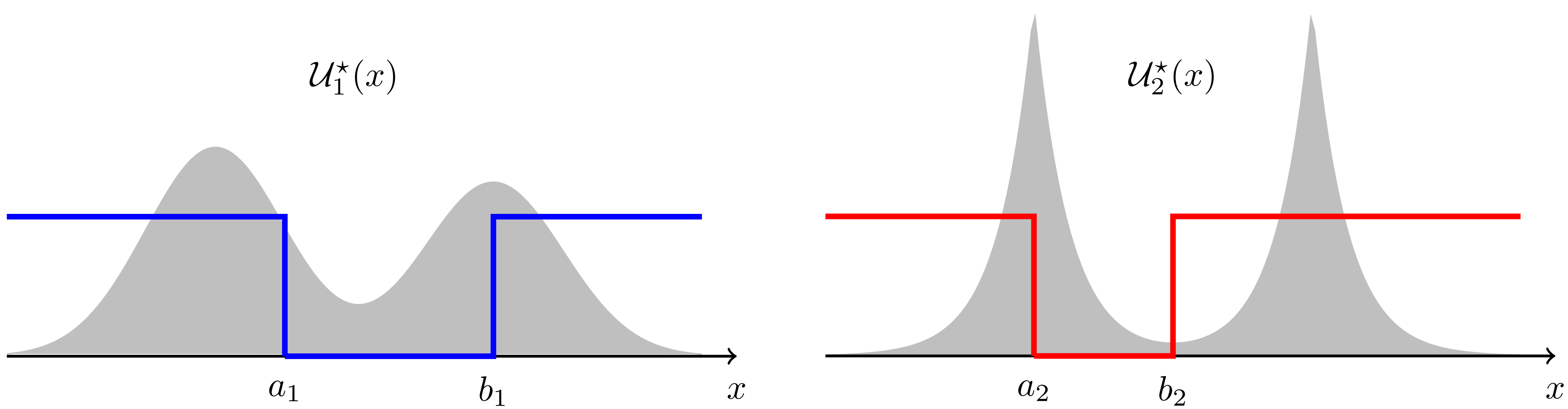
threshold policies

1. Yuksel & Basar, *Stochastic networked control systems,* Birkhauser 2013.
2. Mahajan et al, "Information structures in optimal decentralized control," CDC 2012.

Characterization of team-optimal policies

Theorem

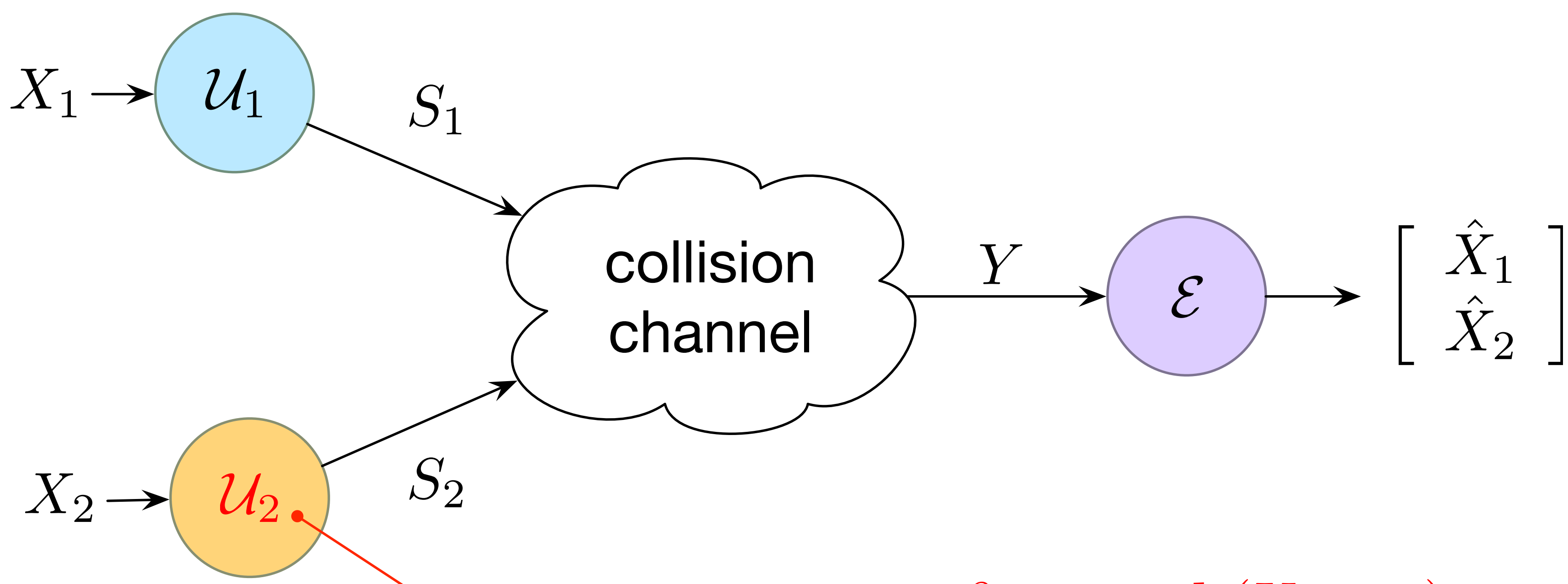
There exists an team-optimal pair of **threshold policies** for Problem 1.



Sketch of Proof:

- Step 1: Equivalent single DM problem
- Step 2: Lagrange duality for infinite dimensional LPs

Remote estimation with communication costs



arbitrarily fixed $\Rightarrow$ $\beta_2 = \mathbf{prob}(U_2 = 1)$ **is constant**

Original cost:

$$\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\Big[(X_1 - \hat{X}_1)^2 + (X_2 - \hat{X}_2)^2\Big]$$

Cost from the perspective of DM$_1$:

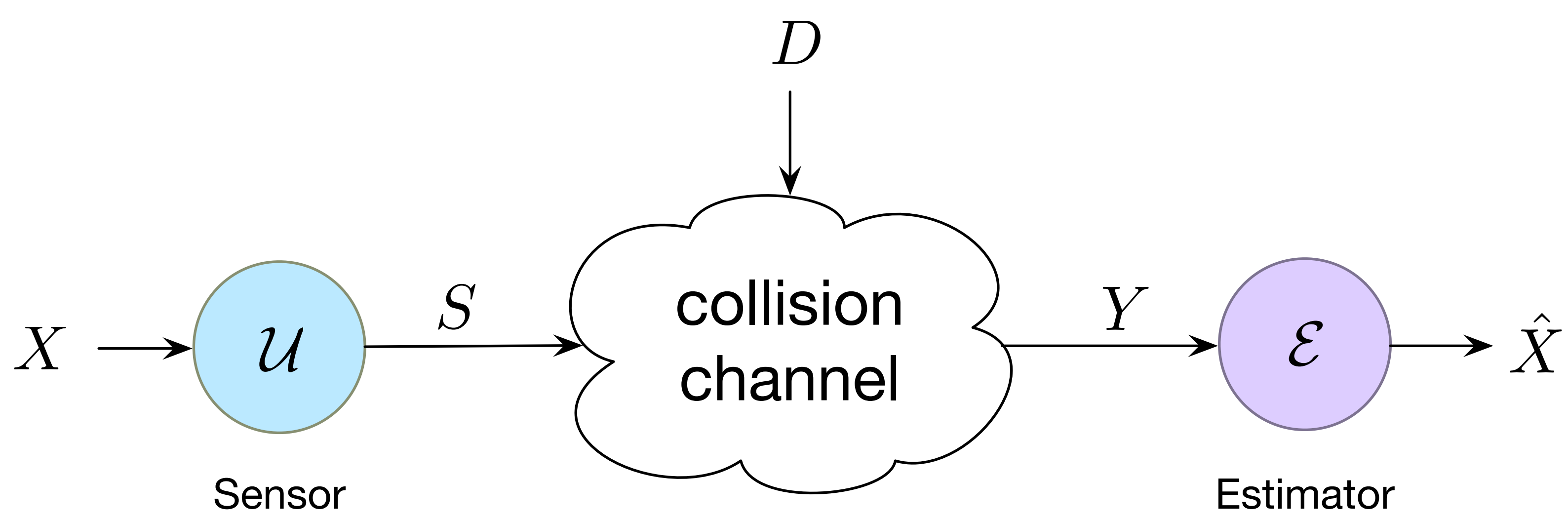
$$\mathcal{J}_1(\mathcal{U}_1) = \mathbf{E}\Big[(X_1 - \hat{X}_1)^2\Big] + \rho_2 \cdot \mathbf{prob}(U_1 = 1) + \theta_2$$

do not depend on $\mathcal{U}_1$

Communication cost:

$$\rho_2 = \mathbf{E}\Big[(X_2 - \hat{X}_2)^2 | U_1 = 1\Big] - \mathbf{E}\Big[(X_2 - \hat{X}_2)^2 | U_1 = 0\Big] \geq 0$$

Single DM subproblem



$D \sim \mathcal{B}(\beta)$

Determines if the channel is occupied or not

$X \perp\!\!\!\perp D$

Problem 2

$$\text{minimize} \qquad \mathcal{J}(\mathcal{U}) = \mathbf{E}\Big[(X - \hat{X})^2\Big] + \rho \cdot \mathbf{prob}(U = 1)$$

$$\mathbf{prob}(U = 1 | X = x) = \mathcal{U}(x) \qquad \mathbb{U} = \{\mathcal{U} \mid \mathcal{U} : \mathbb{R} \to [0,1]\}$$

Lemma[1,2]

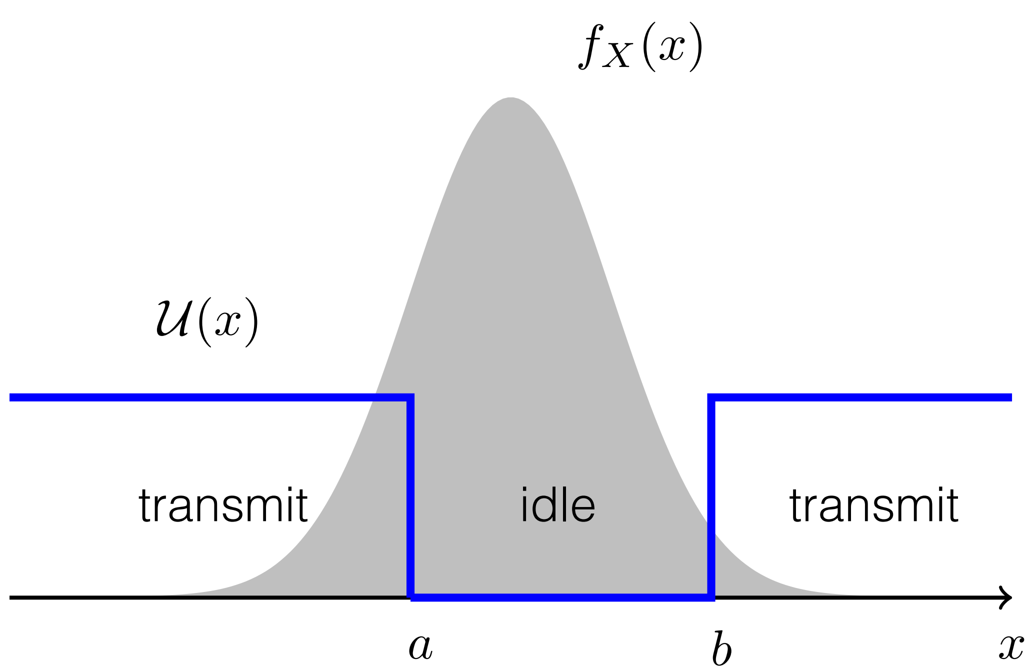
There exists an optimal **threshold policy** for Problem 2.

1. Byrnes & Lindquist, "A convex optimization approach to generalized moment problems," Springer 2003.
2. Borwein & Lewis, "Partially finite convex prog., Part I: Quasi relative interiors and duality theory," *Math. Prog.* 1992.

Remarks

1. Valid for **any continuous probability distribution**
2. **Vector observations** and **any number of sensors**

Assumption:
Finite 1st and 2nd moments
(req. for strong duality)

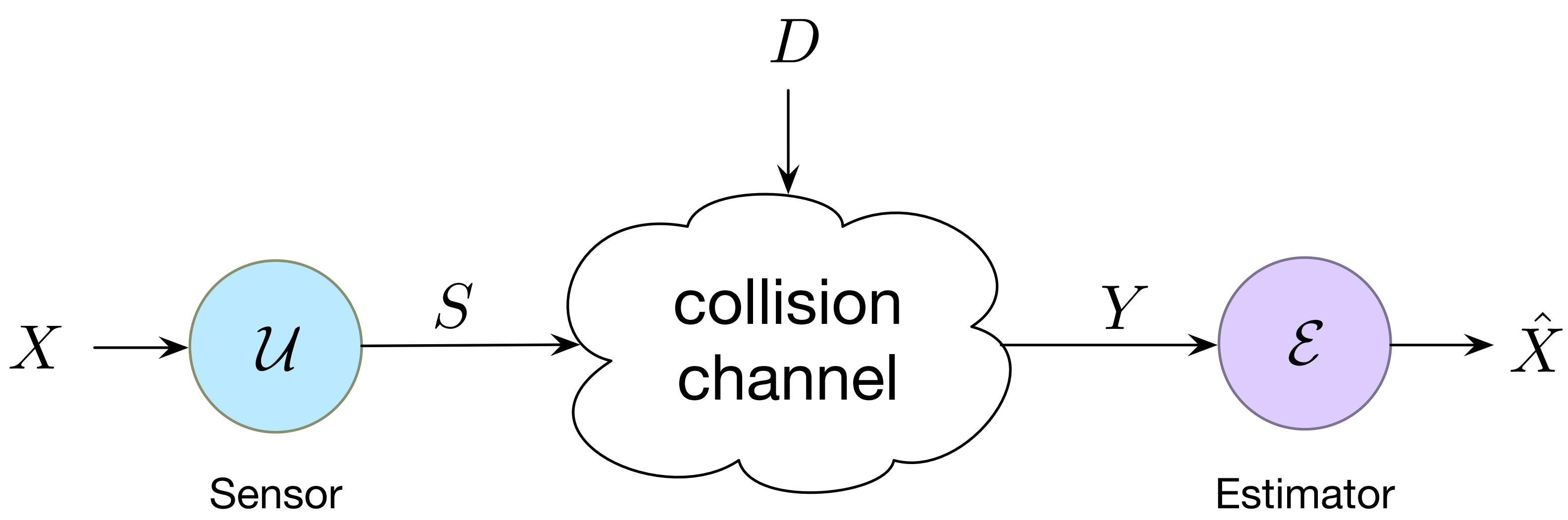


Additional assumption:
The fusion center can decode the indices of all sensors involved in a collision

How do we compute the optimal thresholds?

Vasconcelos & Martins, "Optimal estimation over the collision channel," to appear *IEEE TAC* 2017.

Computing optimal thresholds



$$\mathcal{U}(x) = \begin{cases} 0 & a \leq x \leq b \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{E}(y) = \begin{cases} x & y = x \\ \hat{x}_{\varnothing} & y = \varnothing \\ \hat{x}_{\mathfrak{C}} & y = \mathfrak{C} \end{cases}$$

$$\mathcal{J}(a, b, \hat{x}_{\varnothing}, \hat{x}_{\mathfrak{C}}) = \int_{[a,b]} (x - \hat{x}_{\varnothing})^2 f_X(x) dx + \int_{\bar{\mathbb{R}} \setminus [a,b]} [\beta(x - \hat{x}_{\mathfrak{C}})^2 + \rho] f_X(x) dx$$

binary quantization with **asymmetric** distortion

1. Lloyd, “Least squares quantization in PCM”, *IEEE IT* 1982.
2. Fleischer, “Sufficient conditions for achieving minimum distortion in a quantizer”, *IEEE Int. Conv. Rec.* 1964.

Binary quantization with asymmetric distortion

$$\begin{array}{ll} \text{minimize} & \mathcal{J}(a, b, \hat{x}_\varnothing, \hat{x}_\mathfrak{c}) \\ \text{subject to} & a \leq b \end{array}$$

$$x \in [a^*, b^*] \Leftrightarrow (x - \hat{x}_\varnothing)^2 \leq \beta(x - \hat{x}_\mathfrak{c})^2 + \rho$$

nearest neighbor condition

Let $\hat{x} = (\hat{x}_\varnothing, \hat{x}_\mathfrak{c})$ is the pair representation points for no-transmission and collision symbols

$$a(\hat{x}), b(\hat{x}) = \frac{1}{1-\beta}\left[(\hat{x}_\varnothing - \beta\hat{x}_\mathfrak{c}) \pm \sqrt{\beta(\hat{x}_\varnothing - \hat{x}_\mathfrak{c})^2 + (1-\beta)\rho}\right]$$

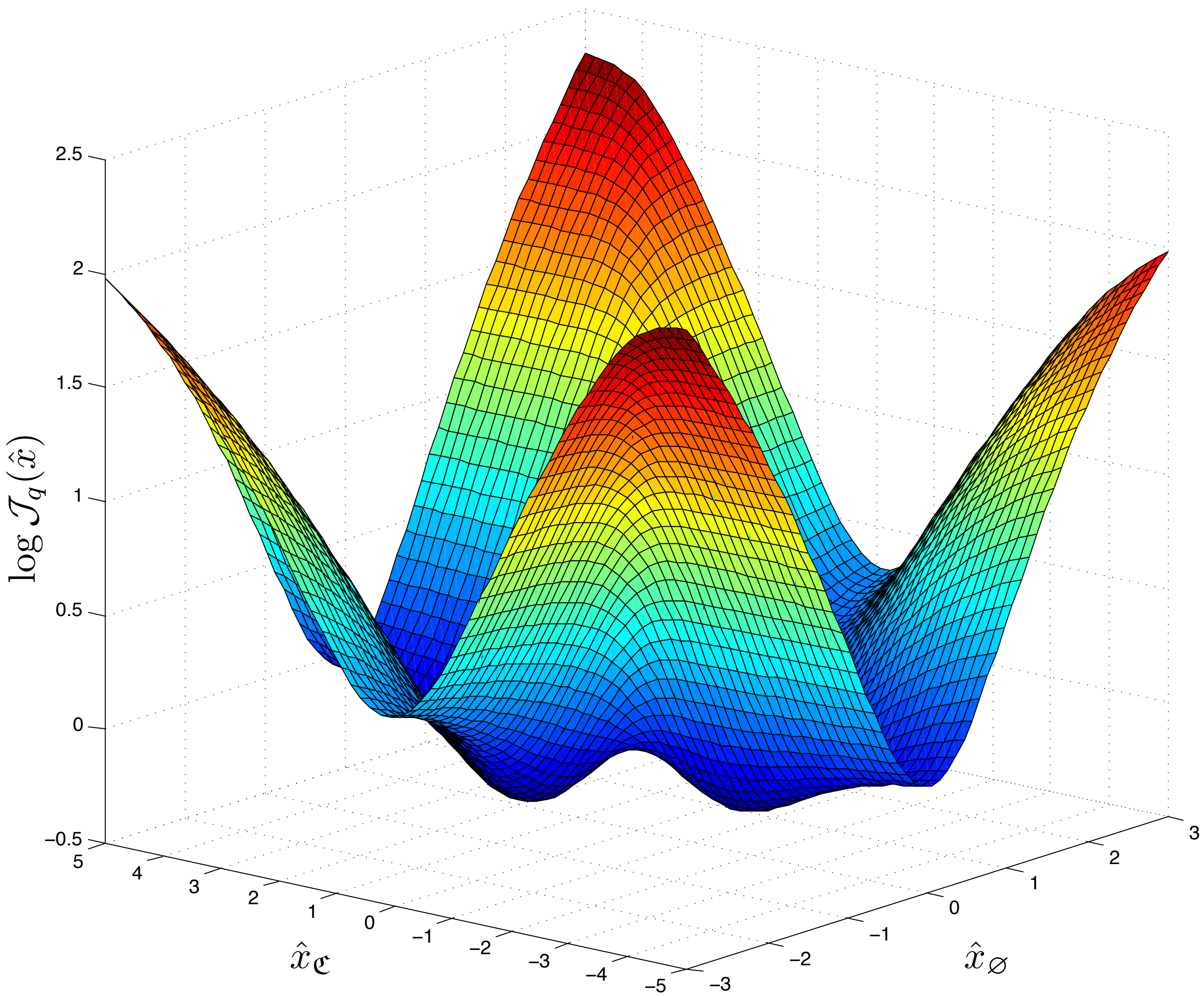
Define a new cost: $\mathcal{J}_q(\hat{x}) = \mathcal{J}(a(\hat{x}), b(\hat{x}), \hat{x}_\varnothing, \hat{x}_\mathfrak{c})$

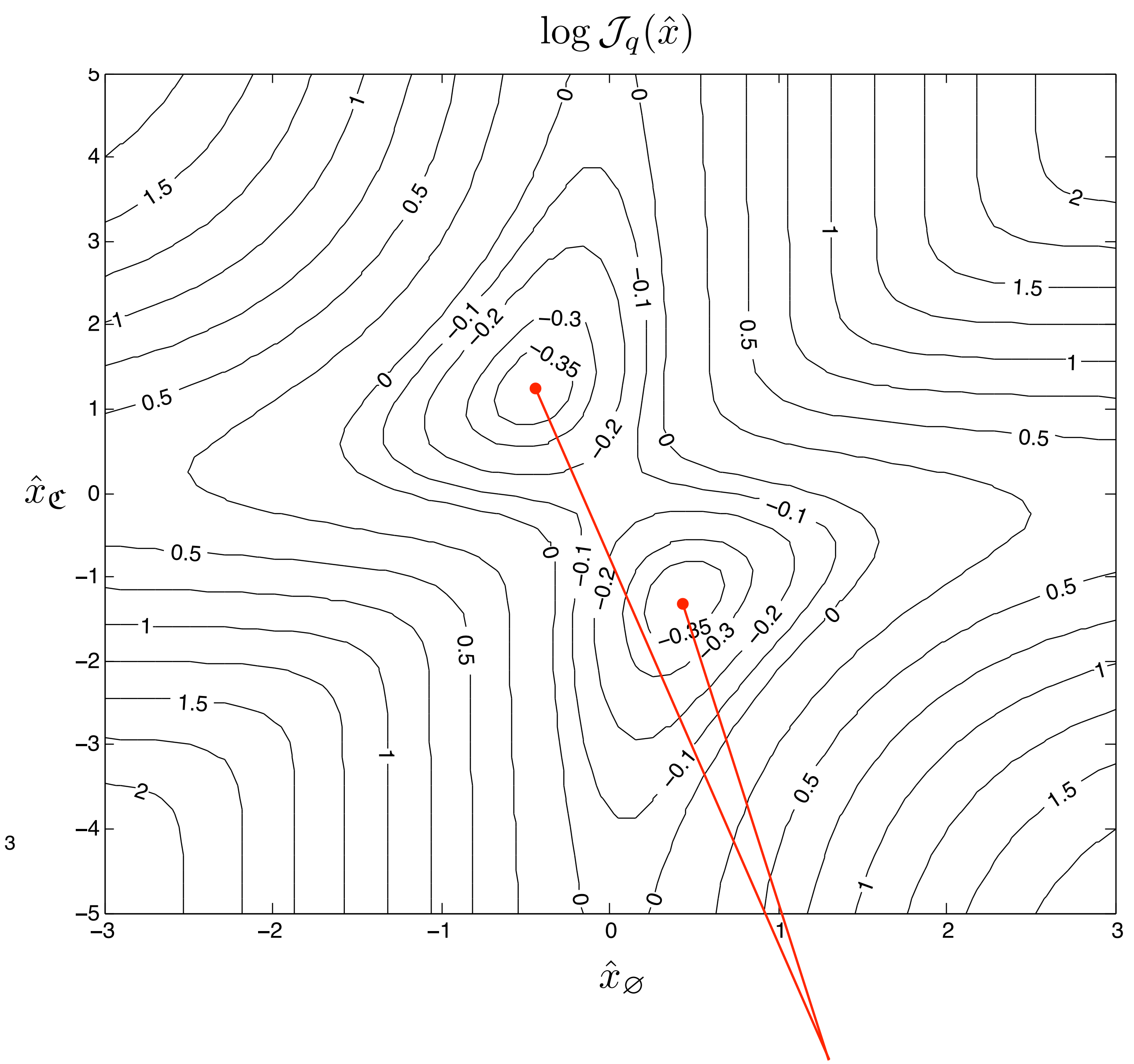
$$\text{minimize} \quad \mathcal{J}_q(\hat{x})$$

unconstrained problem

Quantizer distortion function

$X \sim \mathcal{N}(0,1)$ $\beta = 0.5$ $\rho = 1$





Neither convex nor quasi-convex

Continuously differentiable

Modified Lloyd-Max Algorithm

$$\underset{\hat{x} \in \mathbb{R}^2}{\text{minimize}} \quad \mathcal{J}_q(\hat{x})$$

$$\nabla \mathcal{J}_q(\hat{x}) = 0 \qquad \longleftrightarrow \qquad \hat{x} = \mathcal{F}(\hat{x})$$

Lloyd's Map

$$\mathcal{F}(\hat{x}) = \begin{bmatrix} \mathbf{E}\Big[X \,|\, X \in [a(\hat{x}), b(\hat{x})]\Big] \\ \mathbf{E}\Big[X \,|\, X \notin [a(\hat{x}), b(\hat{x})]\Big] \end{bmatrix}$$

Modified Lloyd-Max

$$\hat{x}^{(0)} \neq (0,0)$$

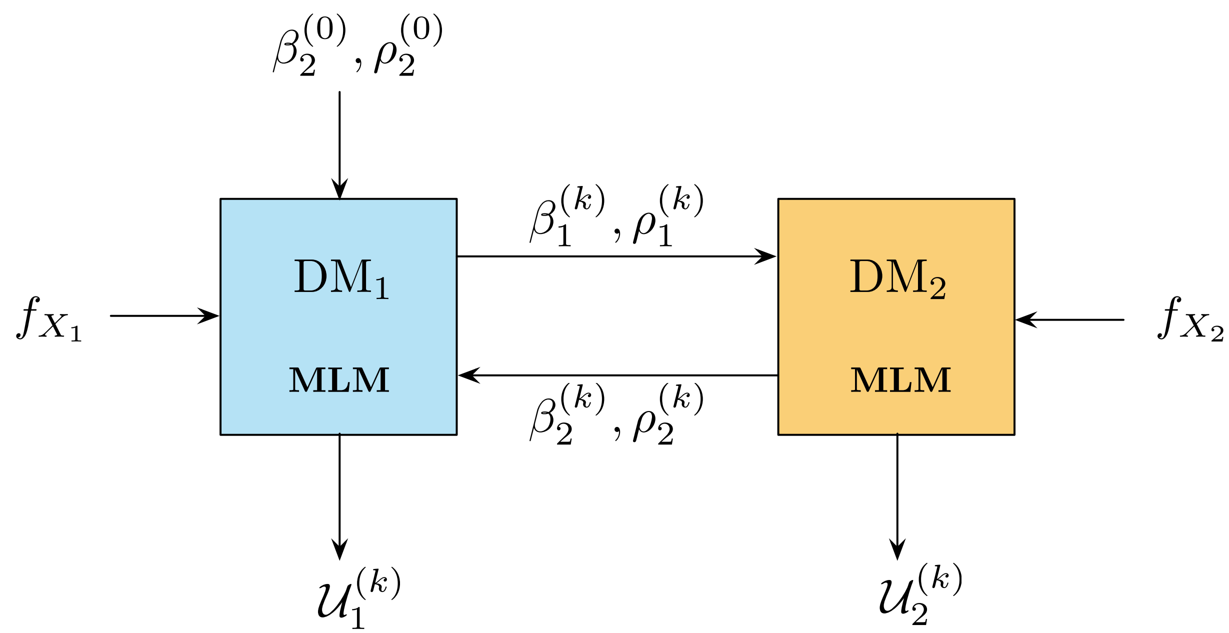
$$\hat{x}^{(k+1)} = \mathcal{F}(\hat{x}^{(k)}), \quad k = 0, 1, \ldots$$

Step 1 From $\hat{x}^{(k)}$ update the thresholds $a(\hat{x}^{(k)})$ and $b(\hat{x}^{(k)})$

Step 2 Compute the centroids of the new quantization regions

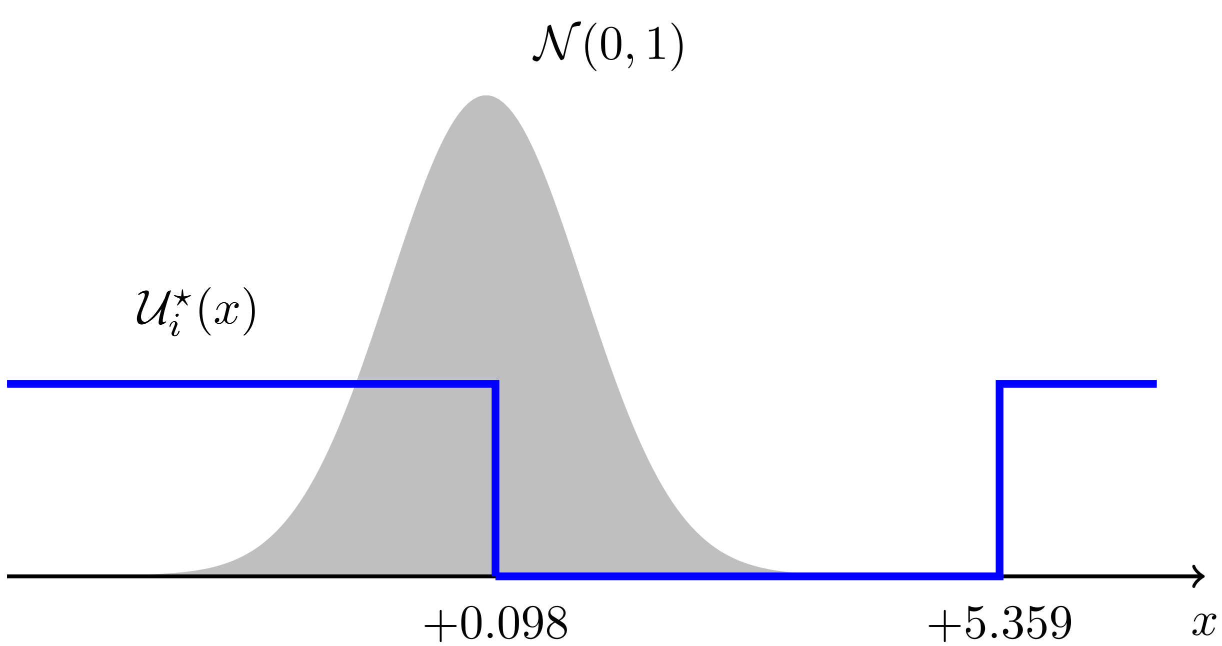
Vasconcelos & Martins, "Optimal thresholds for remote estimation over the collision channel," CDC 2015.

Numerical procedure



Repeat until the cost cannot be further reduced

Example $X_1, X_2 \sim \mathcal{N}(0,1)$



i.i.d. observations, symmetric pdf[1]

asymmetric thresholds[2]

$$\mathcal{J}(\mathcal{U}_1^*, \mathcal{U}_2^*) = 0.54$$

Gain of 46% over open loop scheduling policies

1. Lipsa & Martins, "Remote state estimation with communication costs for first-order LTI systems," *IEEE TAC* 2011.
2. Vasconcelos & Martins, "Optimal estimation over the collision channel," to appear *IEEE TAC* 2017.

Collision channel with common and private observations

$$W = \begin{bmatrix} X_1 \\ Z \\ X_2 \end{bmatrix}$$

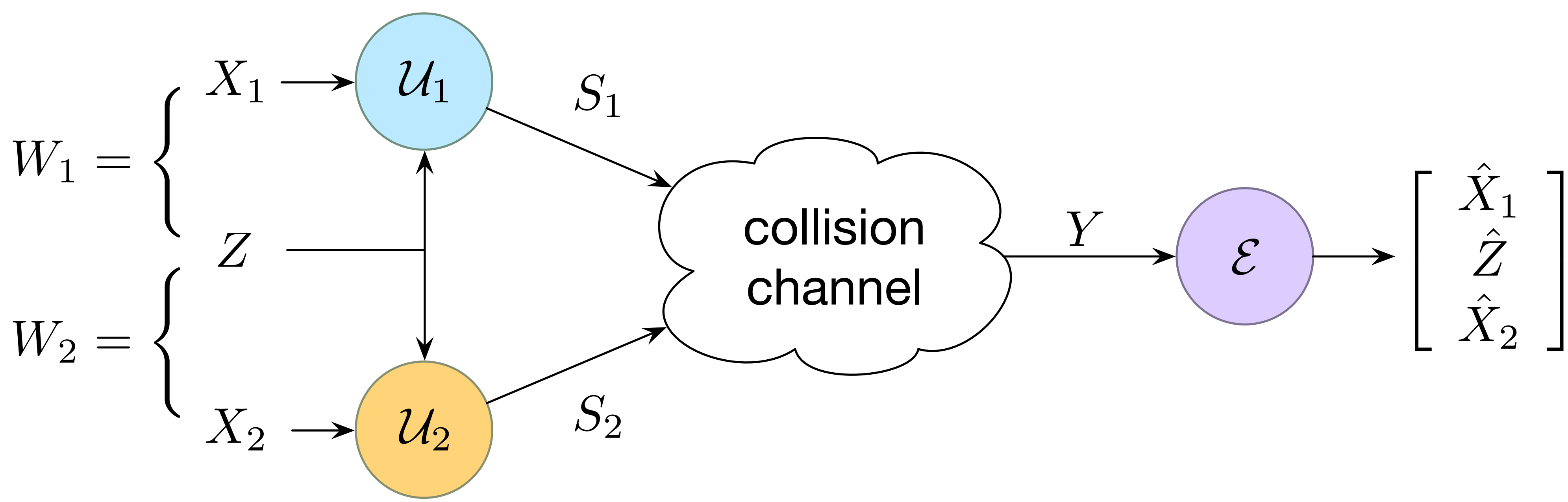
$$W_i = \begin{bmatrix} \color{blue}{X_i} \\ \color{red}{Z} \end{bmatrix}$$

private observation

common observation

$$f_W = f_Z \cdot f_{X_1|Z} \cdot f_{X_2|Z}$$

$$X_1 \leftrightarrow Z \leftrightarrow X_2$$



Problem 3

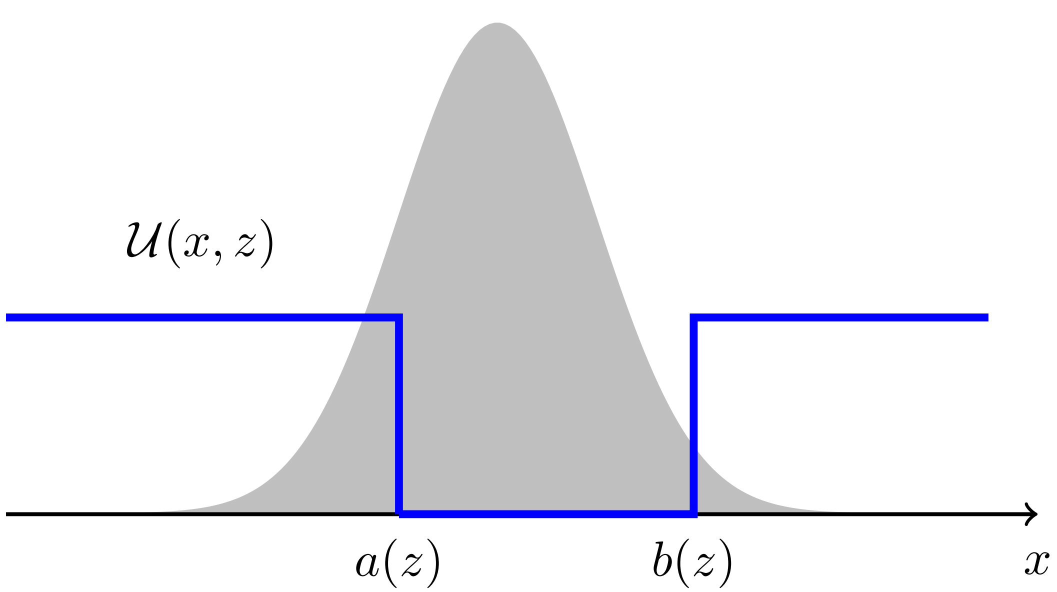
$$\text{minimize} \quad \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2) = \mathbf{E}\Big[(X_1 - \hat{X}_1)^2 + (Z - \hat{Z})^2 + (X_2 - \hat{X}_2)^2\Big]$$

Threshold policy on private information

Theorem:
There exists a team-optimal pair of **threshold policies on private information** for Problem 3.

Threshold policy on private information

$$\mathcal{U}(x,z) = \begin{cases} 0 & a(z) \leq x \leq b(z) \\ 1 & \text{otherwise} \end{cases}$$



Vasconcelos & Martins, "The structure of optimal communication policies for remote estimation over the collision channel with private and common observations," CDC 2016.

Summary

1. **New class of problems** in remote sensing
2. **Optimality of event-based policies** (threshold policies)
3. **Algorithm** to compute locally optimal thresholds
4. Extension: **dependent observations** (talk tomorrow)

Future work

1. Extend to **arbitrarily correlated observations**
2. **Scalability** results
3. **Hypothesis testing** and **parameter estimation**
4. Performance trade-offs in **sensor placement**