# Optimal remote estimation of discrete random variables over the collision channel

Marcos M. Vasconcelos and Nuno C. Martins

**Abstract**

Consider a system comprising sensors that communicate with a remote estimator by way of a so-called collision channel. Each sensor observes a discrete random variable and must decide whether to transmit it to the remote estimator or to remain silent. The variables are independent across sensors. There is no communication among the sensors, which precludes the use of coordinated transmission policies. The collision channel functions as an ideal link when a single sensor transmits. If there are two or more simultaneous transmissions then a collision occurs and is detected at the remote estimator. The role of the remote estimator is to form estimates of all the observations at the sensors. Our goal is to design transmission policies that are globally optimal with respect to two criteria: the aggregate probability of error, which is a convex combination of the probabilities of error in estimating the individual observations; and the total probability of error. We show that, for the aggregate probability of error criterion, it suffices to sift through a finite set of candidate solutions to find a globally optimal one. In general, the cardinality of this set is exponential on the number of sensors but we discuss important cases in which it becomes quadratic or even one. For the total probability of error criterion, we prove that the solution in which each sensor transmits when it observes all but a preselected most probable value is globally optimal. So, no search is needed in this case. Our results hold irrespective of the probability mass functions of the observed random variables, regardless of their support.

**Index Terms**

Remote estimation, optimization, maximum a posteriori probability estimation, team decision theory, networked decision systems, concave minimization problems.

## I. Introduction

Cyber-physical systems have emerged as a framework of system design where multiple agents sense, communicate over a network and actuate on a physical system, operating as a team to achieve a common goal [1]. When the objective is (or requires) the optimization a certain cost, the system designer's task is to solve a problem of decentralized decision-making, where the agents have access to different information and choose an action that incurs in a cost that depends on the actions of all the decision makers. Furthermore, network constraints and stringent delay requirements on the flow of information between decision makers, forces them to make efficient use of potentially scarce communication resources.

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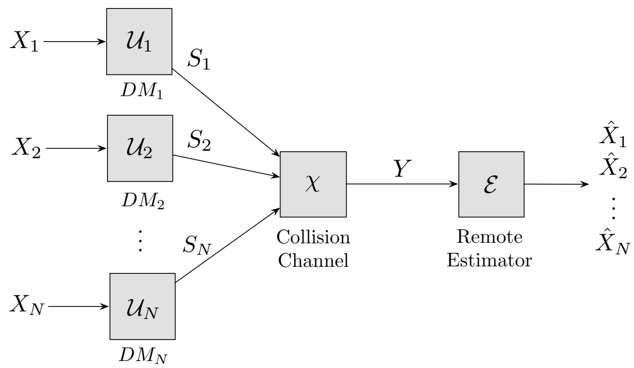


Fig. 1. Schematic representation of remote estimation over a collision channel.

We consider a Bayesian estimation problem illustrated by the block diagram of Fig. 1, where multiple sensors observing independent discrete random variables, decide whether to communicate their measurements to a remote estimator over a collision channel according to transmission policies. The communication constraint imposed by the collision channel is such that only one sensor can transmit its measurement perfectly; and if more than one sensor transmits simultaneously, a collision is declared. Upon observing the channel output, the estimator forms estimates of all the observations at the sensors. Our goal is to find the transmission policies that jointly optimize two performance criteria involving probabilities of estimation error.

### *A. Applications*

The collision channel captures data-transfer restrictions that may result, for instance, from the interference caused by wireless transmitters sharing the same frequency band and are not capable of executing scheduling or carrier-sense multiple access protocols. These constraints are present in large scale networks of simple devices, such as tiny low-power sensors. Potential applications include: nanoscale intra-body networks for health monitoring and drug delivery systems; and networks for environmental monitoring of air pollution, water quality and biodiversity control [2], [3]. Remote estimation systems of this type can also be applied in scenarios where the devices are heterogeneous and there is a strict requirement for real-time wireless networking. For example, ad-hoc networks that lack a coordination protocol among the devices such as the Internet of things [4]; data centers, which are subject to cascading power failures [5] or cyber-attacks [6] that must be detected in minimal time and as accurately as possible.

### *B. Related literature and prior work*

Many Cyber-physical systems are either discrete or hybrid (continuous and discrete) in nature. Although there exists a large body of work on remote estimation systems with a continuous state space, only a few papers deal with systems with discrete state spaces, e.g. [7], [8]. A class of problems in information theory known as real-time coding and decoding of discrete Markov sources is equivalent to remote estimation problems of plants with a discrete state space. Notably, many contributions in this area are derived using ideas from stochastic control such as identifying sufficient statistics for the optimal coding and decoding policies. Structural results of optimal policies for sequential problem formulations involving a single sensor and a remote estimator were obtained in [9]–[12]. The problem of estimating an independent and identically distributed (i.i.d.) discrete source observed by a single sensor with a limited number of measurements and a probability of error criterion was solved in [8]. A sequential, multi-sensor, real-time communication problem over parallel channels was investigated by [13].

Problems of distributed decision-making such as the one in this paper fall into the category of team decision problems with discrete observation and action spaces, and a nonclassical information structure [14], [15]. It is known that problems in this class are in general NP-complete [16]. One possible approach to team problems of this type is to use an approximation technique to obtain a suboptimal strategy within a fixed bound of the globally optimal solution [17]. Another set of results pertains to a class of problems for which the cost satisfies a property known as multimodularity, which allows the characterization of the set of person-by-person optimal solutions and efficient algorithms for searching for a globally optimal solution [18]. Despite the fact that discrete team decision problems are inherently difficult to solve, the remote estimation problem formulated in this paper admits a structural characterization of globally optimal solutions. Our results either significantly reduce the search space, allowing for the numerical search for globally optimal solutions or solve it completely, depending on which of the two criteria is used as the objective. The results reported here are exact and do not make use of any approximation techniques.

Our problem is an instance of a one-shot, remote estimation (real-time communication) problem over a multiple access channel with independent discrete observations. The problem of estimating independent continuous random variables over a collision channel while minimizing a mean squared error criterion was solved in [19], where it was shown the existence of globally optimal deterministic transmission policies with a threshold structure. The authors of [19] also show that this result is independent of the probability distributions of the observed random variables. Here, we solve two related problems where the objective is to optimize a probability of error criterion, which is a popular metric used in estimation of discrete random variables, statistical hypothesis testing and rate distortion theory [20]. A related team decision problem involving the optimization of a probability of error criterion similar to the one considered here was studied in [21], where the optimal strategy is based on ideas from coding theory. The main results of this paper relate to the discovery of a deterministic structure of certain globally optimal solutions, which hold for any probability mass functions that characterize the observed data. An interesting feature of our results is that the constraint imposed by the channel results in optimal policies for which only certain subsets of the measurements are transmitted and others are not. This phenomenon is known as data reduction via *censoring* and was explored for continuous observations with a mean squared error criterion in [22]. Although our problem is based on a framework introduced in [19], the techniques and results shown here do not follow from our previous work and have not appeared elsewhere in the literature.

### C. Summary of formulation and main results

We consider a system formed by a remote estimator and multiple sensors, in which each sensor makes an observation drawn from a discrete random variable. Observations are independent across sensors. Estimates of the observations are computed at the remote estimator based on information sent through a collision channel by the sensors, which have the authority to decide whether or not to transmit their measurements. The collision channel is a model of interference-limited communication according to which simultaneous transmissions cause a collision to occur and be detected by the remote estimator. A given collection of event-based transmission policies for the sensors is termed *solution*. The main goal of this article is to propose methods to compute globally optimal solutions. In Sections II and VI, we formulate this paradigm as a team decision problem with respect to two cost criteria: the aggregate probability of estimation error; and the total probability of estimation error. The former is a convex combination of the probabilities of estimation error for each observation. In order to simplify the introduction of concepts and methods, our results are derived first for the two-sensor case, followed by extensions for an arbitrary number of sensors. The results summarized below hold for any number of sensors and for any probability mass functions of the observed random variables, regardless of their support.

a) We show that there is a solution that globally minimizes the aggregate probability of estimation error. Our result can be stated precisely once we preselect a most probable value and a second most probable value of each random variable observed by a sensor. In particular, Theorem 1 and Theorem 3 show that it suffices to consider one of three strategies for each sensor, according to which it either (i) transmits when it observes all but the most probable value, (ii) it transmits when it observes the second most probable value (iii) it never transmits. These results imply that even when the support of the observations is infinite, it suffices to sift through a finite set of candidate solutions to find a globally optimal one. Using Theorem 3, in Section VI-D we show that, for any parameter selection, the search for an optimal solution is practicable for up to $N = 16$ sensors. In general, the cardinality of the set of candidate solutions is exponential in the number of sensors but, as we discuss below, there are important cases in which it becomes quadratic or even one.
    - The i.i.d. case: In Theorem 4, we use symmetry to show that if the observations are equally distributed then it suffices to consider a set of $\frac{1}{2}(N+1)(N+2)$ candidate solutions. This is a remarkable complexity reduction, when compared to the general case discussed above, which, in the worst case, has a set of $3^N$ candidate solutions.
    - The i.i.d. case with equal weights: If the observations are identically distributed and the weights of the convex combination defining the cost are equal then we provide a closed-form globally optimal solution. More specifically, Theorem 5 shows that there is a globally optimal solution in which each sensor follows one of two policies, according to which (i) it transmits when it observes all but the most probable value or (ii) it transmits when it observes the second most probable value. The number of sensors adopting either policy is given by a closed-form expression.

b) We obtain in closed-form a solution that globally minimizes the total probability of estimation error. Our result can be stated precisely once we preselect a most probable value of each random variable observed by a sensor. Notably, Theorem 2 and Theorem 6 prove that a solution in which every sensor transmits when it observes all but the most probable value is globally optimal.

### D. Paper organization

The paper is structured in seven sections, including the Introduction. In Section II, we describe the problem setup for teams of two sensors; define the two fidelity criteria used to obtain optimal policies: the aggregate probability of estimation error and the total probability of estimation error. In Section III, we state the main structural results for a system with two sensors. In section IV, we prove the structural result for the aggregate probability of error in the case of two sensors and, in Section V, we prove the result for the total probability of error in the case of two sensors. In Section VI, we present structural results for teams of more than two sensors. The paper ends in Section VII with conclusions and suggestions for future work.

### E. Notation

We adopt the following notation: Functions and functionals are denoted using calligraphic letters such as $\mathcal{F}$ or $\mathscr{F}$. Sets are represented in blackboard bold font, such as $\mathbb{A}$. The cardinality of a set $\mathbb{A}$ is denoted by $|\mathbb{A}|$. The set of real numbers is denoted by $\mathbb{R}$. If $\mathbb{A}$ is a subset of $\mathbb{B}$ then $\mathbb{B}\backslash\mathbb{A}$ represents the set of elements in $\mathbb{B}$ that are not in $\mathbb{A}$. Discrete random variables, vectors of discrete random variables and discrete general random elements are represented using upper case letters, such as $W$. Realizations of $W$ are represented by the corresponding lower case letter $w$. The probability of an event $\mathfrak{E}$ is denoted by $\mathbf{P}(\mathfrak{E})$. The probability mass function (pmf) of a Bernoulli random variable $W$, for which $\mathbf{P}(W=1)=\delta$ , is denoted as $\mathcal{B}(\delta)$.

We also adopt the following conventions:

- If $\mathfrak{A}$ and $\mathfrak{B}$ are two events for which $\mathbf{P}(\mathfrak{B})=0$ then we adopt the convention that $\mathbf{P}(\mathfrak{A}\mid\mathfrak{B})=0$.
- Consider that a subset $\mathbb{W}$ of $\mathbb{R}^n$ and a function $\mathcal{F}:\mathbb{W}\to\mathbb{R}$ are given. If $\overline{\mathbb{W}}$ is the subset of elements that maximize $\mathcal{F}$ then $\arg\max_{\alpha\in\mathbb{W}}\mathcal{F}(\alpha)$ is the greatest element in $\overline{\mathbb{W}}$ according to the lexicographical order.

## II. Problem setup: Two sensors

Consider two independent discrete random variables $X_1$ and $X_2$ taking values on finite or countably infinite alphabets $\mathbb{X}_1$ and $\mathbb{X}_2$, respectively. Each random variable $X_i$ is distributed according to a given probability mass function $p_{X_i}(x_i)$ on $\mathbb{X}_i$, $i\in\{1,2\}$. Without loss of generality, we assume that every element of $\mathbb{X}_i$ occurs with a strictly positive probability, for $i\in\{1,2\}$.

There are two sensors[1] denoted by $DM_1$ and $DM_2$ that measure $X_1$ and $X_2$, respectively. Each $DM_i$ observes a realization of $X_i$, and must decide whether to remain silent or attempt to transmit $x_i$ to the estimator. The decision to attempt a transmission or not is represented by a binary random variable $U_i\in\{0,1\}$, where $U_i=1$ denotes the decision to attempt a transmission and $U_i=0$ denotes the decision to remain silent. The decision by $DM_i$ on whether to transmit is based solely on its measurement $x_i$, according to a transmission policy $\mathcal{U}_i$ defined as follows:

*Definition 1 (Transmission policies):* The transmission policy for $DM_i$ is specified by a function $\mathcal{U}_i:\mathbb{X}_i\to[0,1]$ that governs a randomized strategy as follows:

$$\mathbf{P}(U_i=1\mid X_i=x_i)\stackrel{\text{def}}{=}\mathcal{U}_i(x_i),\quad i\in\{1,2\}. \tag{1}$$

[1]We use the terminology *sensor* and *decision maker* ($DM$) interchangeably throughout the paper.

The set of all transmission policies for $DM_i$ is denoted by $\mathbb{U}_i \overset{\text{def}}{=} [0,1]^{|\mathbb{X}_i|}$.

*Assumption 1:* We assume that the randomization, which generates $U_1$ and $U_2$, according to Eq. (1), is such that the pairs $(U_1, X_1)$ and $(U_2, X_2)$ are independent, i.e.,

$$\mathbf{P}(U_1 = \mu_1, U_2 = \mu_2, X_1 = \alpha_1, X_2 = \alpha_2) = \mathbf{P}(U_1 = \mu_1, X_1 = \alpha_1)\mathbf{P}(U_2 = \mu_2, X_2 = \alpha_2), \quad (2)$$

for all $(\mu_1, \alpha_1, \mu_2, \alpha_2) \in \{0,1\} \times \mathbb{X}_1 \times \{0,1\} \times \mathbb{X}_2$.

*Definition 2:* The measurement $X_i$ and the decision $U_i$ by $DM_i$ specify the random element $S_i$, which will be used as a channel input, as follows:

$$s_i \overset{\text{def}}{=} \begin{cases} (i, x_i) & \text{if } u_i = 1 \\ \varnothing & \text{if } u_i = 0 \end{cases}, \quad i \in \{1,2\}, \quad (3)$$

Each random element $S_i$ takes values in $\{\varnothing\} \cup \{(i, \alpha_i) \mid \alpha_i \in \mathbb{X}_i\}$, where the symbol $\varnothing$ denotes *no-transmission*.

*Remark 1:* Notice that $S_1$ and $S_2$ contain the identification number of its sender. This allows the estimator to determine unambiguously the origin of every successful transmission.

*Definition 3 (Collision channel):* The collision channel takes $S_1$ and $S_2$ as inputs. The output $Y$ of the collision channel is characterized by the following deterministic map:

$$y = \chi(s_1, s_2) \overset{\text{def}}{=} \begin{cases} s_1 & \text{if } s_1 \neq \varnothing, \ s_2 = \varnothing \\ s_2 & \text{if } s_1 = \varnothing, \ s_2 \neq \varnothing \\ \varnothing & \text{if } s_1 = \varnothing, \ s_2 = \varnothing \\ \mathfrak{C} & \text{if } s_1 \neq \varnothing, \ s_2 \neq \varnothing, \end{cases} \quad (4)$$

where the symbol $\mathfrak{C}$ denotes a *collision*.

*Remark 2:* The fact that the collision channel discerns between a collision $\mathfrak{C}$ and the absence of a transmission (indicated by $\varnothing$), makes it fundamentally different from the erasure link commonly found in the literature of remote control and estimation, such as in [23]. This creates an opportunity to improve estimation performance by implicitly encoding information in $\mathfrak{C}$ and $\varnothing$.

### A. The aggregate probability of estimation error criterion

For any given pair of transmission policies $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$, we start by considering the following fidelity criterion consisting of a convex combination of the individual probabilities of error of estimating $X_1$ and $X_2$. Consider the cost $\mathcal{J}_A : \mathbb{U}_1 \times \mathbb{U}_2 \to \mathbb{R}$ defined as:

$$\mathcal{J}_A(\mathcal{U}_1, \mathcal{U}_2) \overset{\text{def}}{=} \eta_1 \mathbf{P}(X_1 \neq \hat{X}_1) + \eta_2 \mathbf{P}(X_2 \neq \hat{X}_2), \quad (5)$$

where $\eta_1, \eta_2 > 0$ are given positive constants satisfying $\eta_1 + \eta_2 = 1$, and $\hat{X}_1, \hat{X}_2$ are the estimates formed by the remote estimator of the observations at the sensors $X_1, X_2$.

The designer can choose the weights $\eta_1$ and $\eta_2$ to set the relative priority of each of the random variables it is interested in. It is straightforward to show that for any fixed pair of transmission policies, the receiver that minimizes the cost in Eq. (5) forms a *maximum a posteriori probability* (MAP) estimate of the random variable $X_i$ given the observed channel output $Y$ as follows:

$$\hat{X}_i = \mathcal{E}_i(Y), \tag{6}$$

where the functions $\mathcal{E}_i : \mathbb{Y} \to \mathbb{X}_i$ are defined as:

$$\mathcal{E}_i(y) \stackrel{\text{def}}{=} \arg \max_{\alpha \in \mathbb{X}_i} \mathbf{P}(X_i = \alpha \mid Y = y), \ \ i \in \{1, 2\}. \tag{7}$$

*Problem 1:* Given a pair of probability mass functions $p_{X_1}$ and $p_{X_2}$, find a pair of policies $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$ that minimizes $\mathcal{J}_A(\mathcal{U}_1, \mathcal{U}_2)$ in Eq. (5), subject to the communication constraint imposed by the collision channel of Eq. (4) and that the estimator employs the MAP rule of Eq. (7).

*Remark 3:* Notice that $\mathcal{E}_1$ and $\mathcal{E}_2$ are implicit functions of the transmission policies $\mathcal{U}_1$ and $\mathcal{U}_2$. It is this coupling between transmission and estimation policies that makes this problem non-trivial.

### *B. The total probability of estimation error criterion*

Given a pair of transmission policies $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$, we also consider the cost $\mathcal{J}_B : \mathbb{U}_1 \times \mathbb{U}_2 \to \mathbb{R}$ defined as:

$$\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2) \stackrel{\text{def}}{=} \mathbf{P}(\{X_1 \neq \hat{X}_1\} \cup \{X_2 \neq \hat{X}_2\}) \tag{8}$$

which accounts for the probability that at least one estimate is incorrect.

In this case, for any fixed pair of transmission policies, the receiver that minimizes the cost in Eq. (8) forms a MAP estimate of the random variables $(X_1, X_2)$ given the observed channel output $Y$ as follows:

$$(\hat{X}_1, \hat{X}_2) = \mathcal{E}(Y), \tag{9}$$

where the function $\mathcal{E} : \mathbb{Y} \to \mathbb{X}_1 \times \mathbb{X}_2$ is defined as:

$$\mathcal{E}(y) \stackrel{\text{def}}{=} \arg \max_{(\alpha_1, \alpha_2) \in \mathbb{X}_1 \times \mathbb{X}_2} \mathbf{P}(X_1 = \alpha_1, X_2 = \alpha_2 \mid Y = y). \tag{10}$$

*Problem 2:* Given a pair of probability mass functions $p_{X_1}$ and $p_{X_2}$, find a pair of policies $(\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2$ that minimizes $\mathcal{J}_B(\mathcal{U}_1, \mathcal{U}_2)$ in Eq. (8), subject to the communication constraint imposed by the collision channel of Eq. (4) and that the estimator employs the MAP rule of Eq. (10).

*Definition 4 (Global optimality):* A pair of transimission policies $(\mathcal{U}_1^\star, \mathcal{U}_2^\star) \in \mathbb{U}_1 \times \mathbb{U}_2$ is *globally optimal* for the cost $\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2)$ if the following holds:

$$\mathcal{J}\left(\mathcal{U}_1^\star, \mathcal{U}_2^\star\right) \leq \mathcal{J}\left(\mathcal{U}_1, \mathcal{U}_2\right), \quad (\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{U}_1 \times \mathbb{U}_2. \tag{11}$$

### *C. A motivating example*

Sensor scheduling is one way to guarantee that collisions never occur. However, in general, collision avoidance through sensor scheduling is not optimal. In order to illustrate this, consider the simple scenario where $X_1$ and $X_2$ are independent Bernoulli random variables with nondegenerate probability mass functions $p_{X_1}$ and $p_{X_2}$, respectively.

Using a sensor scheduling policy where only one sensor is allowed to access the channel in Problem 1, the best possible performance is given by

$$\mathcal{J}_A^{\mathrm{sch}} = 1 - \max_{i\in\{1,2\}} \max_{x\in\{0,1\}} \eta_i p_{X_i}(x) > 0. \tag{12}$$

However, it is possible to achieve zero aggregate probability of error, for any independent Bernoulli $X_1$ and $X_2$, by using the following pair of deterministic policies $(\mathcal{U}_1^\star, \mathcal{U}_2^\star)$:

$$\mathcal{U}_i^\star(x_i) = x_i, \ i \in \{1,2\}. \tag{13}$$

The pair $(\mathcal{U}_1^\star, \mathcal{U}_2^\star)$ achieves zero aggregate probability of error because $U_1$ and $U_2$, or equivalently in this case $X_1$ and $X_2$, can be exactly recovered from the channel output $Y$. This globally optimal solution exploits the distinction that the channel in Eq. (4) makes between no-transmissions and collisions to convey information about the observations to the remote estimator.

Motivated by this example, we are interested in investigating whether we can find globally optimal solutions for any given $p_{X_1}$ and $p_{X_2}$.

### D. Person-by-person optimality

Problems with a nonclassical information pattern, such as the ones considered here, are nonconvex in general. Therefore, determining globally optimal solutions is often intractable. In Sections IV and V, we proceed to showing that there are globally optimal solutions for Problems 1 and 2 with a convenient structure, for which numerical optimization is possible. This is accomplished via the following concept of *person-by-person* optimality [15], [18].

*Definition 5 (Person-by-person optimality):* A policy pair $(\mathcal{U}_1^\star, \mathcal{U}_2^\star) \in \mathbb{U}_1 \times \mathbb{U}_2$ is said to satisfy the person-by-person necessary conditions of optimality for the cost $\mathcal{J}(\mathcal{U}_1, \mathcal{U}_2)$ if the following holds:

$$\begin{aligned} \mathcal{J}(\mathcal{U}_1^\star, \mathcal{U}_2^\star) &\leq \mathcal{J}(\mathcal{U}_1, \mathcal{U}_2^\star), \quad \mathcal{U}_1 \in \mathbb{U}_1 \\ \mathcal{J}(\mathcal{U}_1^\star, \mathcal{U}_2^\star) &\leq \mathcal{J}(\mathcal{U}_1^\star, \mathcal{U}_2), \quad \mathcal{U}_2 \in \mathbb{U}_2. \end{aligned} \tag{14}$$

A policy pair that satisfies Eq. (14) is also called a person-by-person optimal solution.

## III. Main results: Two sensors

The main results of this paper pertain to unveiling the structure of globally optimal solutions for the problems stated in Sections II-A and II-B. One important feature of the results below is that they are independent of the distributions of the observations, and are valid even when the alphabets are countably infinite.

Before we continue, we proceed to defining a few important policies that will be used to characterize certain globally optimal solutions.

*Definition 6 (A useful total order):* For every discrete random variable $W$, taking values in $\mathbb{W}$, we define a totally ordered set $[\mathbb{W}, p_W]$ in which, for any $w$ and $\tilde{w}$ in $[\mathbb{W}, p_W]$, $w \prec \tilde{w}$ holds when $p_W(w) < p_W(\tilde{w})$ or, for the case in which $p_W(w) = p_W(\tilde{w})$, when $w < \tilde{w}$. The elements of $[\mathbb{W}, p_W]$ are enumerated so that $w_{[1]}$ is the maximal element and $w_{[i+1]} \prec w_{[i]}$, for every integer $i$ greater than or equal to 1. We also adopt $q_W$ to denote the following pmf:

$$q_W(i) \overset{\mathrm{def}}{=} \mathbf{P}(W = w_{[i]}), \quad 1 \leq i \leq |\mathbb{W}| \tag{15}$$

*Definition 7 (Candidate optimal policies):* Given a discrete random variable $W$, we use the total order in $[\mathbb{W}, p_W]$ to define:

$$\mathcal{V}_W^0(\alpha) \stackrel{\text{def}}{=} 0, \quad \alpha \in \mathbb{W} \tag{16}$$

$$\mathcal{V}_W^1(\alpha) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \alpha = w_{[1]} \\ 1 & \text{otherwise} \end{cases}, \quad \alpha \in \mathbb{W} \tag{17}$$

$$\mathcal{V}_W^2(\alpha) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \alpha = w_{[2]} \\ 0 & \text{otherwise} \end{cases}, \quad \alpha \in \mathbb{W} \tag{18}$$

*Remark 4:* Using the policy $\mathcal{V}_W^0$ means that the sensor never transmits; $\mathcal{V}_W^1$ means that the sensor transmits when it observes all but a preselected most probable value; and policy $\mathcal{V}_W^2$ it transmits when it observes a preselected second most probable value. Notice that the preselected most probable symbol is never transmitted when these policies are used.

*Theorem 1 (Globally optimal solutions for Problem 1):* There exists a globally optimal solution $(\check{\mathcal{U}}_1, \check{\mathcal{U}}_2)$ for which each component $\check{\mathcal{U}}_i$ is one of the three policies in $\{\mathcal{V}_{X_i}^0, \mathcal{V}_{X_i}^1, \mathcal{V}_{X_i}^2\}$, for $i \in \{1, 2\}$.

*Remark 5:* Theorem 1 implies that, regardless of the cardinality of $\mathbb{X}_1$ and $\mathbb{X}_2$, one can determine a globally optimal solution by checking <u>at most nine candidate solutions</u>. In fact, Corollary 1 shows in Section IV-C that one can reduce the search to at most five candidate solutions.

*Theorem 2 (Globally optimal solution for Problem 2):* The policy pair $(\mathcal{V}_{X_1}^1, \mathcal{V}_{X_2}^1)$ is a globally optimal solution.

Using the person-by-person optimality approach, Sections IV-C and V-B present proofs for Theorems 1 and 2, respectively.

## IV. Globally optimal solutions for Problem 1

The person-by-person approach to proving Theorem 1 involves the analysis of the associated problem of optimizing the policy of a single decision maker, while keeping the policy of the other sensor fixed. This problem is precisely formulated and solved in Section IV-B, where we also show that it is a concave minimization problem. Such problems are, in general, intractable, but we were able to find solutions using a two-step approach. More specifically, we obtain a lower bound that holds for any feasible policy (the converse part) and then we provide a structured deterministic policy that achieves the lower bound (the achievability part).

We start by using Bayes' rule to rewrite the cost in Eq. (5) in a way that clarifies the effect of modifying the policy of a single decision maker. More specifically, from the perspective of $DM_i$, $i \in \{1, 2\}$, assuming that the policy used by $DM_j$ is fixed to $\tilde{\mathcal{U}}_j \in \mathbb{U}_j$, $j \neq i$, we have:

$$\mathcal{J}_A(\mathcal{U}_i, \tilde{\mathcal{U}}_j) = \eta_i \mathbf{P}(X_i \neq \hat{X}_i) + \eta_j (\rho_{\tilde{\mathcal{U}}_j} \mathbf{P}(U_i = 1) + \theta_{\tilde{\mathcal{U}}_j}), \tag{19}$$

where

$$\rho_{\tilde{\mathcal{U}}_j} \stackrel{\text{def}}{=} \mathbf{P}(X_j \neq \hat{X}_j \mid U_i = 1) - \mathbf{P}(X_j \neq \hat{X}_j \mid U_i = 0) \tag{20}$$

and

$$\theta_{\tilde{\mathcal{U}}_j} \stackrel{\text{def}}{=} \mathbf{P}(X_j \neq \hat{X}_j \mid U_i = 0). \tag{21}$$

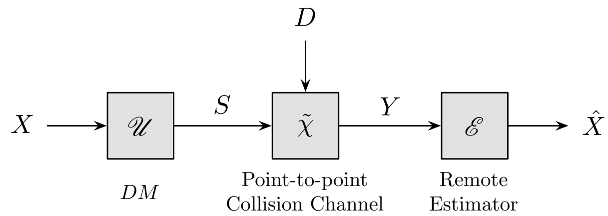


Fig. 2. An equivalent single $DM$ estimation problem over a collision channel.

The terms $\rho_{\tilde{\mathcal{U}}_j}$ and $\theta_{\tilde{\mathcal{U}}_j}$ are constant in $\mathcal{U}_i$. In particular, $\eta_j \rho_{\tilde{\mathcal{U}}_j}$ can be interpreted as a communication cost incurred by $DM_i$ when it attempts to transmit. A similar interpretation has been used in [19] and relates this problem to the multi-stage estimation case with limited actions solved in [8].

### A. The communication cost and offset terms

We proceed to characterizing the communication cost Eq. (19) and the offset terms in further detail.

*Proposition 1:* If $X_1$ and $X_2$ are independent and Assumption 1 holds, then for $i, j \in \{1, 2\}$ and $i \neq j$, the terms $\rho_{\tilde{\mathcal{U}}_j}$ and $\theta_{\tilde{\mathcal{U}}_j}$ are constant in $\mathcal{U}_i$, are non-negative, and upper bounded by $1$.

*Proof:* See Appendix A. ■

Proposition 1 will be used in the proof of Theorem 1.

### B. An equivalent single $DM$ subproblem

In order to use the person-by-person approach to proving Theorem 1, we need to consider the subproblem of optimizing the transmission policy of one decision maker, which we call $DM$, while assuming that the policy of $\widetilde{DM}$, representing the other sensor, is given and fixed. From the perspective of $DM$, the problem is depicted in Fig. 2. Here, $DM$ observes a random variable $X$ and must decide whether to attempt a transmission or to remain silent. A Bernoulli random variable $D$, which is independent of $X$, accounts for the effect that transmission attempts by $\widetilde{DM}$ have on the occurrence of collisions. The contribution of the policy of $\widetilde{DM}$ towards the cost is quantified in Eq. (19). Before we state the subproblem precisely in Problem 3, we proceed with a few definitions.

In order to emphasize the fact that $D$, which is determined by the fixed policy for $\widetilde{DM}$, is now a given Bernoulli random variable that can be viewed by $DM$ as a source of randomization inherent to the channel, we adopt the following definition:

*Definition 8 (Stochastic point-to-point collision channel):* Let $D$ be a given Bernoulli random variable with parameter $\beta$ for which $\mathbf{P}(D = 1) = \beta$. An associated point-to-point collision channel with input $S$ and output $Y = \tilde{\chi}(S, D)$ is specified by the following map:

$$\tilde{\chi}(s, d) \overset{\text{def}}{=} \begin{cases} \varnothing & \text{if } s = \varnothing \\ s & \text{if } s \neq \varnothing,\ d = 0 \\ \mathfrak{C} & \text{if } s \neq \varnothing,\ d = 1. \end{cases} \tag{22}$$

where $s$ is in the input alphabet $\mathbb{S} \overset{\text{def}}{=} \mathbb{X} \cup \{\varnothing\}$. The output alphabet of the channel is $\mathbb{Y} \overset{\text{def}}{=} \mathbb{X} \cup \{\mathfrak{C}\}$.

The input to the channel is governed by $DM$ according to:

$$s \overset{\text{def}}{=} \begin{cases} x & \text{if } u = 1 \\ \varnothing & \text{if } u = 0, \end{cases} \tag{23}$$

The following is the probability of attempting a transmission for a given measurement $\alpha$:

$$\mathbf{P}(U = 1 \mid X = \alpha) = \mathscr{U}(\alpha), \quad \mathscr{U} \in \mathbb{U}, \ \alpha \in \mathbb{X} \tag{24}$$

where $\mathscr{U}$ is the transmission policy used by $DM$.

*Assumption 2:* We assume that the randomization that generates $U$ according to Eq. (24), is such that $D$ is independent from the pair $(U, X)$.

Finally, based on Eq. (19), the cost to be minimized by $DM$ and the remote estimator, $\mathscr{J}_A : \mathbb{U} \to \mathbb{R}$, is defined as follows:

$$\mathscr{J}_A(\mathscr{U}) \overset{\text{def}}{=} \mathbf{P}(X \neq \hat{X}) + \varrho \mathbf{P}(U = 1). \tag{25}$$

where $\mathscr{U}$ represents a transmission policy for $DM$ and $\varrho$ can be viewed as a communication cost induced by $\widetilde{DM}$.

*Problem 3:* Consider that $\beta$ in $[0, 1]$, a non-negative $\varrho$ and a discrete random variable $X$ are given. Find a policy $\mathscr{U} \in \mathbb{U}$ that minimizes the cost $\mathscr{J}_A(\mathscr{U})$ in Eq. (25), subject to Eq. (22) with $\mathbf{P}(D = 1) = \beta$ and that the estimate $\hat{X} = \mathscr{E}(Y)$ is generated according to the following MAP rule:

$$\mathscr{E}(y) \overset{\text{def}}{=} \arg \max_{\alpha \in \mathbb{X}} \mathbf{P}(X = \alpha \mid Y = y), \quad y \in \mathbb{Y}. \tag{26}$$

We will provide a solution to Problem 3 using the following two lemmata.

*Lemma 1:* The cost $\mathscr{J}_A(\mathscr{U})$ is concave on $\mathbb{U}$.

*Proof:* Using the law of total probability, we rewrite the cost $\mathscr{J}_A(\mathscr{U})$ as:

$$\mathscr{J}_A(\mathscr{U}) = \Big(\beta \mathbf{P}(X \neq \hat{X} \mid U = 1, D = 1) + \varrho\Big) \mathbf{P}(U = 1) + \mathbf{P}(X \neq \hat{X} \mid U = 0)\mathbf{P}(U = 0). \tag{27}$$

Simplifying this expression using the relationships developed in Appendix A, we get:

$$\begin{aligned} \mathscr{J}_A(\mathscr{U}) = 1 + (\varrho + \beta - 1)\mathbf{P}(U = 1) \\ - \mathbf{P}(X = \mathscr{E}(\varnothing) \mid U = 0)\mathbf{P}(U = 0) \\ - \beta \mathbf{P}(X = \mathscr{E}(\mathfrak{C}) \mid U = 1)\mathbf{P}(U = 1). \end{aligned} \tag{28}$$

Using the definition of the MAP estimator, we can write the following probabilities in terms of $\mathscr{U}$:

$$\mathbf{P}(X = \mathcal{E}(\varnothing) \mid U = 0) = \max_{\alpha \in \mathbb{X}} \frac{(1 - \mathscr{U}(\alpha))p(\alpha)}{\mathbf{P}(U = 0)} \tag{29}$$

and

$$\mathbf{P}(X = \mathcal{E}(\mathfrak{C}) \mid U = 1) = \max_{\alpha \in \mathbb{X}} \frac{\mathscr{U}(\alpha)p(\alpha)}{\mathbf{P}(U = 1)}. \tag{30}$$

Finally, the cost can be rewritten as follows:

$$\mathscr{J}_A(\mathscr{U}) = 1 + (\varrho + \beta - 1)\sum_{\alpha \in \mathbb{X}} \mathscr{U}(\alpha)p(\alpha) - \max_{\alpha \in \mathbb{X}}(1 - \mathscr{U}(\alpha))p(\alpha) - \beta \max_{\alpha \in \mathbb{X}} \mathscr{U}(\alpha)p(\alpha). \tag{31}$$

The proof is concluded by using standard arguments found in [24, Ch. 3] to establish the concavity of Eq. (31). ■

*Lemma 2:* For $\beta \in [0,1]$ and $\varrho \geq 0$, the following policy minimizes $\mathscr{J}_A(\mathscr{U})$:

$$\mathscr{U}^{\star}_{\beta,\varrho} = \begin{cases} \mathcal{V}^1_X & \text{if } \ 0 \leq \varrho \leq 1-\beta \\ \mathcal{V}^2_X & \text{if } \ 1-\beta < \varrho \leq 1 \\ \mathcal{V}^0_X & \text{otherwise.} \end{cases} \tag{32}$$

*Proof:* Since $\mathscr{J}_A(\mathscr{U})$ is continuous and $\mathbb{U}$ is compact with respect to the weak$^*$ topology[2], a minimizer exists [25]. Due to the concavity of $\mathscr{J}_A(\mathscr{U})$ established in Lemma 1, the minimizer must lie on the boundary of the feasible set. Moreover, the search can be further constrained to the corners of the $|\mathbb{X}|$-dimensional hypercube that describes the feasible set and this implies that Problem 3 admits an optimal deterministic policy. Hence, it suffices to optimize with respect to policies $\mathscr{U}$ that take values in $\{0,1\}$. For each such policy, we use the alphabet partitions $\mathbb{X} = \mathbb{X}^{\mathscr{U},0} \cup \mathbb{X}^{\mathscr{U},1}$ defined as follows:

$$\mathbb{X}^{\mathscr{U},k} \overset{\text{def}}{=} \{\alpha \in \mathbb{X} \mid \mathscr{U}(\alpha) = k\}, \;\; k \in \{0,1\}. \tag{33}$$

We proceed to finding an optimal deterministic policy by solving the equivalent problem of searching for an optimal partition. In spite of the fact that the number of partitions grows exponentially with $|\mathbb{X}|$, as we show next, we can use the cost structure to render the search for an optimal partition tractable. We start by using the partitions to rewrite the cost as follows:

$$\mathscr{J}_A(\mathscr{U}) = \; 1 + (\varrho + \beta - 1) \sum_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha) - \max_{\alpha \in \mathbb{X}^{\mathscr{U},0}} p_X(\alpha) - \beta \max_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha). \tag{34}$$

We will obtain a lower bound that holds for every deterministic policy $\mathscr{U} \in \mathbb{U}$, and show that $\mathscr{U}^{\star}_{\beta,\varrho}$ always achieves it.

First, consider the case when $\varrho > 1 - \beta$. Using the inequality below

$$\sum_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha) \geq \max_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha) \tag{35}$$

we conclude that the cost satisfies the following lower bound:

$$\mathscr{J}_A(\mathscr{U}) \geq 1 - (1-\varrho) \max_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha) - \max_{\alpha \in \mathbb{X}^{\mathscr{U},0}} p_X(\alpha). \tag{36}$$

The right hand side of the inequality above can be minimized by assigning $x_{[1]}$ to the set $\mathbb{X}^{\mathscr{U},0}$. If $1 - \varrho \geq 0$, we assign $x_{[2]}$ to the set $\mathbb{X}^{\mathscr{U},1}$, otherwise we set $\mathbb{X}^{\mathscr{U},1} = \emptyset$. Therefore, we obtain the following lower bound for the cost:

$$\mathscr{J}_A(\mathscr{U}) \geq 1 - \max\{0, 1-\varrho\} q_X(2) - q_X(1). \tag{37}$$

When $1 - \beta \leq \varrho$, this lower bound is met with equality by the policy $\mathscr{U}^{\star}_{\beta,\varrho}$, for which the cost is given by:

$$\mathscr{J}_A(\mathscr{U}^{\star}_{\beta,\varrho}) = \begin{cases} 1 - q_X(1) & \text{if } \varrho > 1 \\ 1 - (1-\varrho) q_X(2) - q_X(1) & \text{otherwise.} \end{cases} \tag{38}$$

[2]This technical detail can be ignored when $|\mathbb{X}| < \infty$.

Similarly, when $0 \leq \varrho \leq 1-\beta$, we have:

$$\sum_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha) \leq 1 - \max_{\alpha \in \mathbb{X}^{\mathscr{U},0}} p_X(\alpha). \tag{39}$$

Therefore, for every $\mathscr{U}$ in $\mathbb{U}$, we establish the following lower bound on the cost:

$$\mathscr{J}_A(\mathscr{U}) \geq (\varrho+\beta)(1 - \max_{\alpha \in \mathbb{X}^{\mathscr{U},0}} p_X(\alpha)) - \beta \max_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha). \tag{40}$$

The right hand side of the inequality above can be minimized by assigning $x_{[1]}$ to the set $\mathbb{X}^{\mathscr{U},0}$. If $1-\varrho \geq 0$, we assign $x_{[2]}$ to the set $\mathbb{X}^{\mathscr{U},1}$. Therefore, we obtain the following lower bound for the cost:

$$\mathscr{J}_A(\mathscr{U}) \geq (\varrho+\beta)(1-q_X(1)) - \beta q_X(2). \tag{41}$$

The policy $\mathscr{U}^\star_{\beta,\varrho}$ achieves this lower bound, as the following calculation shows:

$$\begin{aligned}\mathscr{J}_A(\mathscr{U}^\star_{\beta,\varrho}) = 1 + (\varrho+\beta-1) &\sum_{\alpha \in \mathbb{X}\setminus\{x_{[1]}\}} p_X(\alpha) \\ &- q_X(1) - \beta \max_{\alpha \in \mathbb{X}\setminus\{x_{[1]}\}} p_X(\alpha) \\ &= (\varrho+\beta)(1-q_X(1)) - \beta q_X(2). \end{aligned} \tag{42}$$

■

*Remark 6:* Lemma 2 provides a solution to Problem 3 described only in terms of $\beta$, $\varrho$ and the two most probable outcomes of $X$. As a particular case, when $\beta$ is zero and $\varrho$ is in $[0,1]$, the optimal policy is

$$\mathscr{U}^\star_{0,\varrho}(\alpha) = \begin{cases} 0 & \text{if } \alpha = x_{[1]} \\ 1 & \text{otherwise.} \end{cases} \tag{43}$$

This result is related to a similar problem solved by Imer and Basar in [8].

## C. *Proof of Theorem 1*

We will now apply the results in Section IV-B to reduce the search space of possible optimal policies for each sensor in Problem 1. The strategy is to use a person-by-person optimality approach together with Lemma 2.

*Proof of Theorem 1:* Consider the cost $\mathcal{J}_A(\mathcal{U}_1,\mathcal{U}_2)$ in Problem 1. Arbitrarily fixing the policy $\tilde{\mathcal{U}}_2$ of $DM_2$, we have:

$$\mathcal{J}_A(\mathcal{U}_1,\tilde{\mathcal{U}}_2) \propto \mathbf{P}(X_1 \neq \hat{X}_1) + \frac{\eta_2}{\eta_1}(\rho_{\tilde{\mathcal{U}}_2}\mathbf{P}(U_1=1) + \theta_{\tilde{\mathcal{U}}_2}). \tag{44}$$

The problem of minimizing $\mathcal{J}_A(\mathcal{U}_1,\tilde{\mathcal{U}}_2)$ over $\mathcal{U}_1 \in \mathbb{U}_1$ is equivalent to solving an instance of Problem 3 with parameters $\varrho$ and $\beta$ selected as follows:

$$\varrho = \frac{\eta_2}{\eta_1}\rho_{\tilde{\mathcal{U}}_2} \quad \text{and} \quad \beta = \mathbf{P}(U_2=1). \tag{45}$$

Hence, from Lemmas 1 and 2, for each policy $\tilde{\mathcal{U}}_2$ in $\mathbb{U}_2$ there is at least one choice for $\mathcal{U}_1^\star$ in $\{\mathcal{V}^0_{X_1}, \mathcal{V}^1_{X_1}, \mathcal{V}^2_{X_1}\}$ for which $\mathcal{J}_A(\mathcal{U}_1^\star,\tilde{\mathcal{U}}_2) \leq \mathcal{J}_A(\mathcal{U}_1,\tilde{\mathcal{U}}_2)$ holds for any $\mathcal{U}_1$ in $\mathbb{U}_1$. Since this is true regardless of our choice of $\tilde{\mathcal{U}}_2$ and it also holds if we were to fix the policy of $DM_1$ and optimize $\mathcal{U}_2$, we conclude that given any person-by-person optimal pair $(\mathcal{U}_1^\star,\mathcal{U}_2^\star)$, there is a pair $(\check{\mathcal{U}}_1,\check{\mathcal{U}}_2)$ in $\{\mathcal{V}^0_{X_1}, \mathcal{V}^1_{X_1}, \mathcal{V}^2_{X_1}\} \times \{\mathcal{V}^0_{X_2}, \mathcal{V}^1_{X_2}, \mathcal{V}^2_{X_2}\}$, for which $\mathcal{J}_A(\check{\mathcal{U}}_1,\check{\mathcal{U}}_2) \leq \mathcal{J}_A(\mathcal{U}_1^\star,\mathcal{U}_2^\star)$

holds. The proof of Theorem 1 is complete once we recall that every globally optimal solution is also person-by-person optimal. ■

*Remark 7:* There may be other optimal solutions that do not have the same structure of the policies in Theorem 1. Note that the performance of an optimal remote estimation system is determined by the probabilities of the two most probable outcomes of $X_1$ and $X_2$. Also, the optimal performance of a system with binary observations is always zero, i.e., independent binary observations can be estimated perfectly from the output of the collision channel with two sensors. The globally optimal solution described in the motivating example of Section II-C also fits in the structure of the globally optimal solutions of Theorem 1.

We proceed to evaluating the performance of each of the nine candidate solutions listed in Theorem 1 using the expressions in Eqs. (20) and (21), and the following quantity

$$t_{X_i} \stackrel{\mathrm{def}}{=} 1 - q_{X_i}(1) - q_{X_i}(2),\ i \in \{1,2\}, \tag{46}$$

where $q_W$ is as defined in Eq. (15).

- If $\mathcal{U}_i = \mathcal{V}^1_{X_i}$, then

$$\mathbf{P}(U_i = 1) = 1 - q_{X_i}(1) \tag{47}$$

$$\rho_{\mathcal{U}_i} = t_{X_i} \quad \text{and} \quad \theta_{\mathcal{U}_i} = 0. \tag{48}$$

- If $\mathcal{U}_i = \mathcal{V}^2_{X_i}$, then

$$\mathbf{P}(U_i = 1) = q_{X_i}(2) \tag{49}$$

$$\rho_{\mathcal{U}_i} = 0 \quad \text{and} \quad \theta_{\mathcal{U}_i} = t_{X_i}. \tag{50}$$

- If $\mathcal{U}_i = \mathcal{V}^0_{X_i}$, then

$$\mathbf{P}(U_i = 1) = 0 \tag{51}$$

$$\rho_{\mathcal{U}_i} = 0 \quad \text{and} \quad \theta_{\mathcal{U}_i} = 1 - q_{X_i}(1). \tag{52}$$

We construct Table I, which lists the cost evaluated for all candidate solutions. It can be verified by inspection that the policy pairs for which $m$ is $6, 7, 8$ and $9$ are always outperformed by at least one of the others. This observation leads to the following corollary.

*Corollary 1:* The optimal cost obtained from solving Problem 1 is given by

$$\mathcal{J}^\star_A = \min_{1 \le m \le 5} \mathcal{J}^{(m)}_A \tag{53}$$

where $\mathcal{J}^m_A$, for $m$ in $\{1, \ldots, 9\}$, is specified in Table I.

### *D. Examples*

Corollary 1 further reduces the cardinality of the set of candidate solutions for Problem 1 to $5$. We will now explore the role that $p_{X_1}$ and $p_{X_2}$ have in determining which of the solutions ($m = 1$ through $5$ in Table I) is

TABLE I

VALUE OF THE COST FUNCTION $\mathcal{J}_A(\mathcal{U}_1,\mathcal{U}_2)$ AT EACH OF THE NINE CANDIDATE SOLUTIONS SPECIFIED IN THEOREM 1.

| $m$ | $(\mathcal{U}_1,\mathcal{U}_2)$ | $\mathcal{J}_A^{(m)} \overset{\text{def}}{=} \mathcal{J}_A(\mathcal{U}_1,\mathcal{U}_2)$ |
|---|---|---|
| 1 | $(\mathcal{V}^1_{X_1},\mathcal{V}^1_{X_2})$ | $\eta_1 t_{X_1}(1-q_{X_2}(1))+\eta_2 t_{X_2}(1-q_{X_1}(1))$ |
| 2 | $(\mathcal{V}^1_{X_1},\mathcal{V}^2_{X_2})$ | $\eta_1 t_{X_1} q_{X_2}(2)+\eta_2 t_{X_2}$ |
| 3 | $(\mathcal{V}^2_{X_1},\mathcal{V}^1_{X_2})$ | $\eta_1 t_{X_1}+\eta_2 t_{X_2} q_{X_1}(2)$ |
| 4 | $(\mathcal{V}^1_{X_1},\mathcal{V}^0_{X_2})$ | $\eta_2(1-q_{X_2}(1))$ |
| 5 | $(\mathcal{V}^0_{X_1},\mathcal{V}^1_{X_2})$ | $\eta_1(1-q_{X_1}(1))$ |
| 6 | $(\mathcal{V}^2_{X_1},\mathcal{V}^2_{X_2})$ | $\eta_1 t_{X_1}+\eta_2 t_{X_2}$ |
| 7 | $(\mathcal{V}^2_{X_1},\mathcal{V}^0_{X_2})$ | $\eta_1 t_{X_1}+\eta_2(1-q_{X_2}(1))$ |
| 8 | $(\mathcal{V}^0_{X_1},\mathcal{V}^2_{X_2})$ | $\eta_1(1-q_{X_1}(1))+\eta_2 t_{X_2}$ |
| 9 | $(\mathcal{V}^0_{X_1},\mathcal{V}^0_{X_2})$ | $\eta_1(1-q_{X_1}(1))+\eta_2(1-q_{X_2}(1))$ |

globally optimal. In the examples below, we assume[3] that $\eta_1=\eta_2$, which further reduces our search to policy pairs $m=1,2$ and 3. We will use the following quantities:

$$\mathcal{J}_A^{(2)}-\mathcal{J}_A^{(3)} = -t_{X_1}(1-q_{X_2}(2))+t_{X_2}(1-q_{X_1}(2)) \tag{54}$$

$$\mathcal{J}_A^{(2)}-\mathcal{J}_A^{(1)} = t_{X_2}(q_{X_1}(1)-t_{X_1}) \tag{55}$$

$$\mathcal{J}_A^{(3)}-\mathcal{J}_A^{(1)} = t_{X_1}(q_{X_2}(1)-t_{X_2}). \tag{56}$$

*Example 1 (Uniformly distributed observations):* For uniformly distributed observations, we have

$$p_{X_i}(x)=\frac{1}{N_i},\ x=1,2,\cdots,N_i. \tag{57}$$

Hence, the probabilities of the two most probable outcomes are

$$q_{X_i}(1)=q_{X_i}(2)=\frac{1}{N_i} \tag{58}$$

and the aggregate probability of all other outcomes is given by

$$t_{X_i}=1-\frac{2}{N_i},\ i\in\{1,2\}. \tag{59}$$

Without loss of generality, we assume that $N_1,N_2\geq 3$ and $N_1\leq N_2$. Since

$$\mathcal{J}_A^{(2)}-\mathcal{J}_A^{(3)} = \frac{1}{N_1}-\frac{1}{N_2}\geq 0 \tag{60}$$

$$\mathcal{J}_A^{(3)}-\mathcal{J}_A^{(1)} = \left(1-\frac{2}{N_1}\right)\times\left(\frac{3}{N_2}-1\right)\leq 0, \tag{61}$$

our assumptions imply that $\mathcal{J}_A^\star=\mathcal{J}_A^{(3)}$ and the pair of policies corresponding to $m=3$ is globally optimal.

[3]In this case, the weights $\eta_1$ and $\eta_2$ are irrelevant and we may assume that they are both equal to 1.

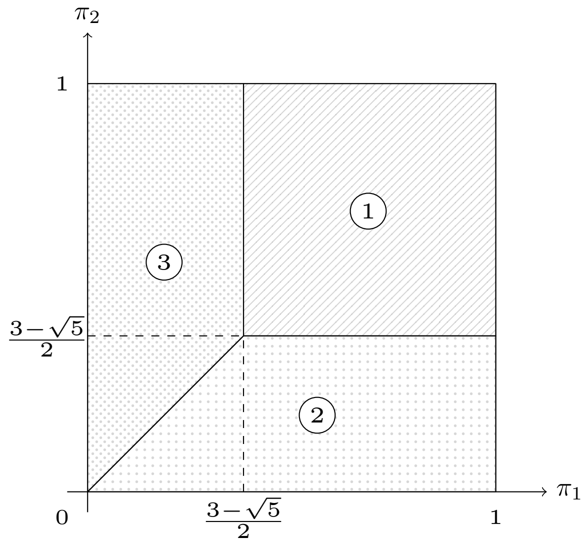


Fig. 3. Partition of the parameter space indicating where each policy pair is globally optimal for Example 2. The circled number corresponds to $m$ in Table I.

*Example 2 (Geometrically distributed observations):* For geometrically distributed observations with parameters $\pi_1$ and $\pi_2$, we have

$$p_{X_i}(x) = (1-\pi_i)^x \pi_i,\ x \geq 0,\ i \in \{1,2\} \tag{62}$$

The probabilities of the two most probable outcomes for each sensor are:

$$q_{X_i}(1) = \pi_i,\ i \in \{1,2\} \tag{63}$$

$$q_{X_i}(2) = (1-\pi_i)\pi_i,\ i \in \{1,2\} \tag{64}$$

and the aggregate probability of all other outcomes is:

$$t_{X_i} = (1-\pi_i)^2,\ i \in \{1,2\}. \tag{65}$$

Note that $\mathcal{J}_A^{(1)}$ is less than or equal to $\mathcal{J}_A^{(2)}$ and $\mathcal{J}_A^{(3)}$ if and only if $q_{X_i}(1) \geq t_{X_i}$, for $i \in \{1,2\}$, or equivalently, if the following holds:

$$-\pi_i^2 + 3\pi_i - 1 \geq 0,\ i \in \{1,2\}. \tag{66}$$

Also, $\mathcal{J}_A^{(2)}$ is less than or equal to $\mathcal{J}_A^{(3)}$ if and only if the following holds:

$$(1-\pi_2)^2\pi_1 \leq (1-\pi_1)^2\pi_2, \tag{67}$$

which is satisfied if $\pi_1 \leq \pi_2$. This yields the partitioning of the parameter space $(\pi_1, \pi_2) \in [0,1]^2$ into the three regions depicted in Fig. 3. Each region is labeled according to the policy pair that is optimal within it.

*Example 3 (Poisson distributed observations):* For Poisson distributed observations with parameters $\lambda_1$ and $\lambda_2$, which we assume here to be both greater than 1, the probabilities of the two most probable outcomes are

$$q_{X_i}(1) = q_{X_i}(2) = \frac{\lambda_i^{\lfloor \lambda_i \rfloor}}{\lfloor \lambda_i \rfloor !} e^{-\lambda_i} \tag{68}$$

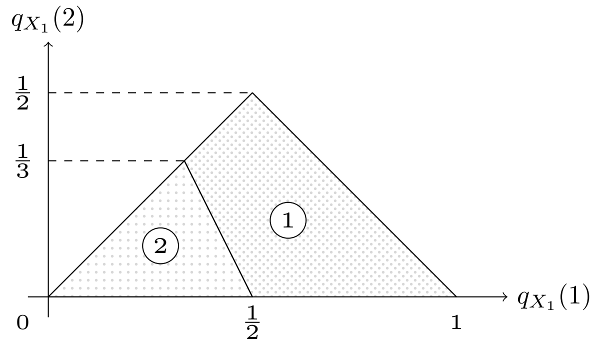


Fig. 4. Partition of the parameter space indicating where each policy pair is globally optimal for Example 4. The circled number corresponds to $m$ in Table I.

and the aggregate probability of all other outcomes is:

$$t_{X_i} = 1 - 2\frac{\lambda_i^{\lfloor \lambda_i \rfloor}}{\lfloor \lambda_i \rfloor !} e^{-\lambda_i},\ i \in \{1,2\}. \tag{69}$$

Using the same argument as in the previous example, we note that $\mathcal{J}_A^{(1)}$ is less than or equal to $\mathcal{J}_A^{(2)}$ and $\mathcal{J}_A^{(3)}$ if and only if $q_{X_i}(1) \geq t_{X_i}$ is satisfied, or equivalently, the following holds:

$$\frac{\lambda_i^{\lfloor \lambda_i \rfloor}}{\lfloor \lambda_i \rfloor !} e^{-\lambda_i} \geq \frac{1}{3},\ i \in \{1,2\}. \tag{70}$$

In order to check whether Eq. (70) holds, we define the following function:

$$\mathcal{F}(\lambda) \overset{\text{def}}{=} \frac{\lambda^{\lfloor \lambda \rfloor}}{\lfloor \lambda \rfloor !} e^{-\lambda} - \frac{1}{3}. \tag{71}$$

It can be shown that $\mathcal{F}(\lambda)$ is greater than or equal to zero if and only if $0 < \lambda \leq \bar{\lambda} \approx 1.5121$. Hence, we conclude that Eq. (70) holds if and only if $1 \leq \lambda_i \leq \bar{\lambda}$ for $i \in \{1,2\}$. Finally, $\mathcal{J}_A^{(2)} \leq \mathcal{J}_A^{(3)}$ holds if and only if

$$\frac{\lambda_2^{\lfloor \lambda_2 \rfloor}}{\lfloor \lambda_2 \rfloor !} e^{-\lambda_2} \geq \frac{\lambda_1^{\lfloor \lambda_1 \rfloor}}{\lfloor \lambda_1 \rfloor !} e^{-\lambda_1}, \tag{72}$$

which is satisfied when $\lambda_1 \geq \lambda_2$.

*Example 4 (Identically distributed observations):* When the observations are identically distributed, i.e., $p_{X_1} = p_{X_2}$, and $\mathbb{X}_1 = \mathbb{X}_2$, we have:

$$\mathcal{J}_A^{(2)} = \mathcal{J}_A^{(3)} = (1 - q_{X_1}(1) - q_{X_1}(2))(1 + q_{X_1}(2)) \tag{73}$$

and

$$\mathcal{J}_A^{(1)} = (1 - q_{X_1}(1) - q_{X_1}(2))(2 - 2q_{X_1}(1)) \tag{74}$$

Therefore, $\mathcal{J}_A^{(2)} \leq \mathcal{J}_A^{(1)}$ if and only if

$$2q_{X_1}(1) + q_{X_1}(2) \leq 1. \tag{75}$$

Recalling that $q_{X_1}(1) \geq q_{X_1}(2)$, we have the partitioning of the parameter space $[0,1]^2$ according to Fig. 4.

## V. Globally Optimal Solutions to Problem 2

In this section, we provide a proof for Theorem 2, which characterizes transmission policies for the sensors that minimize the probability that either $\hat{X}_1$ or $\hat{X}_2$, or both, give an incorrect estimate. The cost, which was initially defined as Eq. (8), is now rewritten as follows:

$$\mathcal{J}_B(\mathcal{U}_1,\mathcal{U}_2) = \mathbf{P}(W \neq \hat{W}). \tag{76}$$

where $W \overset{\text{def}}{=} (X_1, X_2)$ and the estimate $\hat{W} \overset{\text{def}}{=} \mathcal{E}(Y)$ is determined by the following MAP rule:

$$\mathcal{E}(y) = \arg\max_{\omega \in \mathbb{W}} \mathbf{P}(W = \omega \mid Y = y),\ \ y \in \mathbb{Y} \tag{77}$$

where $\mathbb{W} = \mathbb{X}_1 \times \mathbb{X}_2$.

The overall proof strategy is centered on the characterization of globally optimal solutions via the person-by-person optimality approach. This is possible in spite of the fact that the cost $\mathcal{J}_B(\mathcal{U}_1,\mathcal{U}_2)$ does not admit the additive decomposition used in Problem 1.

We start by stating two propositions that provide identities useful in Section V-A, where we use the total probability law to rewrite the cost in a convenient way.

*Proposition 2:* The following holds for Problem 2:

$$\begin{aligned}\mathbf{P}(W = \hat{W} \mid Y = \mathfrak{C}) = &\max_{\alpha_1 \in \mathbb{X}_1} \mathbf{P}(X_1 = \alpha_1 \mid U_1 = 1) \\ &\times \max_{\alpha_2 \in \mathbb{X}_2} \mathbf{P}(X_2 = \alpha_2 \mid U_2 = 1);\end{aligned} \tag{78}$$

and

$$\begin{aligned}\mathbf{P}(W = \hat{W} \mid Y = \varnothing) = &\max_{\alpha_1 \in \mathbb{X}_1} \mathbf{P}(X_1 = \alpha_1 \mid U_1 = 0) \\ &\times \max_{\alpha_2 \in \mathbb{X}_2} \mathbf{P}(X_2 = \alpha_2 \mid U_2 = 0).\end{aligned} \tag{79}$$

*Proposition 3:* The following holds for Problem 2:

$$\mathbf{P}(W = \hat{W} \mid Y = (i, X_i)) = \max_{\alpha_j \in \mathbb{X}_j} \mathbf{P}(X_j = \alpha_j \mid U_j = 0) \tag{80}$$

with $i, j \in \{1, 2\}$ and $i \neq j$.

### A. An equivalent single $DM$ subproblem

Here, we adopt an approach analogous to the one used in Section IV-B to prove Theorem 1. In particular, we proceed to providing preliminary results that we will be used in Section V-B to prove Theorem 2 via the person-by-person approach.

A key step is to characterize Problem 2 from the viewpoint of one decision maker, when the transmission policy of the other is given and fixed. Unlike Problem 1, in which the cost structure from the viewpoint of each decision maker allowed us to make an analogy with a remote estimation problem subject to communication costs, Problem 2 does not admit an insightful decomposition. Fortunately, we can still use the same techniques applied to this less convenient cost.

Similar to the approach adopted in Section IV-B, for a given discrete random variable $X$, we define $\mathbb{U}$ to be the class of all functions $\mathscr{U}$ with domain $\mathbb{X}$ taking values in $[0,1]$. Elements of $\mathbb{U}$ represent policies that govern transmission in the same manner described in Section IV-B, with the difference that we now consider the cost $\mathscr{J}_B : \mathbb{U} \to \mathbb{R}$ defined as follows:

$$\mathscr{J}_B(\mathscr{U}) \stackrel{\text{def}}{=} 1 - \tau \max_{\alpha \in \mathbb{X}} \mathscr{U}(\alpha) p_X(\alpha) - \varrho \sum_{\alpha \in \mathbb{X}} \mathscr{U}(\alpha) p_X(\alpha) - (\varrho + \beta) \max_{\alpha \in \mathbb{X}} \left(1 - \mathscr{U}(\alpha)\right) p_X(\alpha). \tag{81}$$

where $\varrho, \tau$ and $\beta$ are non-negative constants.

*Lemma 3:* The cost $\mathscr{J}_B$ is concave on $\mathbb{U}$.

*Proof:* The proof follows from standard arguments that can be found in [24, Ch. 3]. ∎

*Lemma 4:* Let $X$ be a given discrete random variable. If $\tau \leq \beta$ then $\mathcal{V}_X^1$ minimizes $\mathscr{J}_B$.

*Proof:* From Lemma 3, the cost is concave in $\mathscr{U}$. Therefore, without loss in optimality, we can constrain the optimization to the class of deterministic strategies. For any deterministic policy $\mathscr{U} \in \mathbb{U}$, define

$$\mathbb{X}^{\mathscr{U},k} \stackrel{\text{def}}{=} \{\alpha \in \mathbb{X} \mid \mathscr{U}(\alpha) = k\}, \quad k \in \{0,1\}. \tag{82}$$

Constraining the policies to be deterministic and using the notation defined above, the cost becomes

$$\begin{aligned} \mathscr{J}_B(\mathscr{U}) \quad = \quad & 1 - \tau \max_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha) - \varrho \sum_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha) \\ & - (\varrho + \beta) \max_{\alpha \in \mathbb{X}^{\mathscr{U},0}} p_X(\alpha). \end{aligned} \tag{83}$$

Since

$$\sum_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha) \leq 1 - \max_{\alpha \in \mathbb{X}^{\mathscr{U},0}} p_X(\alpha), \tag{84}$$

we obtain the following inequality, which holds for every deterministic policy $\mathscr{U} \in \mathbb{U}$:

$$\mathscr{J}_B(\mathscr{U}) \geq 1 - \varrho - \tau \max_{\alpha \in \mathbb{X}^{\mathscr{U},1}} p_X(\alpha) - \beta \max_{\alpha \in \mathbb{X}^{\mathscr{U},0}} p_X(\alpha). \tag{85}$$

If $\tau \leq \beta$ then the lower bound on the right hand side of the inequality above can be minimized by assigning the symbol $x_{[1]}$ to $\mathbb{X}^{\mathscr{U},0}$ and $x_{[2]}$ to $\mathbb{X}^{\mathscr{U},1}$, yielding:

$$\mathscr{J}_B(\mathscr{U}) \geq 1 - \varrho - \tau q_X(2) - \beta q_X(1). \tag{86}$$

The lower bound in Eq. (86) is achieved when $\mathscr{U} = \mathcal{V}_X^1$. ∎

## B. *Proof of Theorem 2*

We are now equipped to present a proof for Theorem 2.

*Proof of Theorem 2:* In order to use the person-by-person optimality approach, we will rewrite the cost from the perspective of a single decision maker. The law of total probability and the results in Propositions 2 and 3 allow us to re-express the cost as follows:

$$\begin{aligned}\mathcal{J}_B(\mathcal{U}_1,\mathcal{U}_2) = 1 - \max_{\alpha_1\in\mathbb{X}_1} \mathcal{U}_1(\alpha_1)p_{X_1}(\alpha_1) \max_{\alpha_2\in\mathbb{X}_2} \mathcal{U}_2(\alpha_2)p_{X_2}(\alpha_2) \\ - \max_{\alpha_1\in\mathbb{X}_1}(1-\mathcal{U}_1(\alpha_1))p_{X_1}(\alpha_1) \max_{\alpha_2\in\mathbb{X}_2}(1-\mathcal{U}_2(\alpha_2))p_{X_2}(\alpha_2) \\ - \sum_{\alpha_1\in\mathbb{X}_1} \mathcal{U}_1(\alpha_1)p_{X_1}(\alpha_1) \max_{\alpha_2\in\mathbb{X}_2}(1-\mathcal{U}_2(\alpha_2))p_{X_2}(\alpha_2) \\ - \sum_{\alpha_2\in\mathbb{X}_2} \mathcal{U}_2(\alpha_2)p_{X_2}(\alpha_2) \max_{\alpha_1\in\mathbb{X}_1}(1-\mathcal{U}_1(\alpha_1))p_{X_1}(\alpha_1).\end{aligned} \tag{87}$$

We start by fixing the transmission policy of $DM_2$ to an arbitrary choice $\tilde{\mathcal{U}}_2$. We use Eq. (87) to write the cost from the perspective of $DM_1$ as follows:

$$\begin{aligned}\mathcal{J}_B(\mathcal{U}_1,\tilde{\mathcal{U}}_2) = \ 1 - \tilde{\tau}_2 \max_{\alpha_1\in\mathbb{X}_1} \mathcal{U}_1(\alpha_1)p_{X_1}(\alpha_1) \\ - \tilde{\varrho}_2 \sum_{\alpha_1\in\mathbb{X}_1} \mathcal{U}_1(\alpha_1)p_{X_1}(\alpha_1) \\ - (\tilde{\varrho}_2 + \tilde{\beta}_2) \max_{\alpha_1\in\mathbb{X}_1}(1-\mathcal{U}_1(\alpha_1))p_{X_1}(\alpha_1),\end{aligned} \tag{88}$$

where

$$\tilde{\beta}_2 \overset{\text{def}}{=} \sum_{\alpha_2\in\mathbb{X}_2} \tilde{\mathcal{U}}_2(\alpha_2)p_{X_2}(\alpha_2), \tag{89}$$

$$\tilde{\varrho}_2 \overset{\text{def}}{=} \max_{\alpha_2\in\mathbb{X}_2}(1-\tilde{\mathcal{U}}_2(\alpha_2))p_{X_2}(\alpha_2) \tag{90}$$

and

$$\tilde{\tau}_2 \overset{\text{def}}{=} \max_{\alpha_2\in\mathbb{X}_2} \tilde{\mathcal{U}}_2(\alpha_j)p_{X_2}(\alpha_2). \tag{91}$$

Note that for any given $\tilde{\mathcal{U}}_2$ in $\mathbb{U}_2$, we have $\tilde{\beta}_2 \geq \tilde{\tau}_2$. Hence, from Lemma 4, $\mathcal{J}_B(\mathcal{V}^1_{X_1},\tilde{\mathcal{U}}_2) \leq \mathcal{J}_B(\mathcal{U}_1,\tilde{\mathcal{U}}_2)$ holds for any $\mathcal{U}_1$ in $\mathbb{U}_1$. Given the facts that the choice for $\tilde{\mathcal{U}}_2$ was arbitrary and that we could alternatively have chosen to fix the policy of sensor $DM_1$ to an arbitrary selection $\tilde{\mathcal{U}}_1$ and optimized with respect to $\mathcal{U}_2$, we conclude that $(\mathcal{V}^1_{X_1},\mathcal{V}^1_{X_2})$ is a globally optimal solution for Problem 2. ■

## VI. EXTENSIONS TO MORE THAN TWO SENSORS

We proceed to extending our results to allow for a team of $N$ sensors that access independent observations $\{X_1,\dots,X_N\}$ and communicate over a collision channel that can only support one transmission. Let $U = (U_1,\cdots,U_N)$ denote the $N$-tuple of transmission decision variables, and $U_{-i}$ denote the $N-1$ tuple obtained by excluding $U_i$ from $U$. Similarly, we use $\mathcal{U} = (\mathcal{U}_1,\cdots,\mathcal{U}_N)$ to represent the $N$-tuple of transmission policies and $\mathcal{U}_{-i}$ is obtained by omitting $\mathcal{U}_i$ from $\mathcal{U}$.

*Assumption 3:* The transmission decision $U_i$ of each $DM_i$ is generated as a function of $\mathcal{U}_i$ in the same manner as described in Definition 1, with the evident modification that we now consider that $i$ is in $\{1,\dots,N\}$. We also assume that the underlying randomization that generates $U$ from $\mathcal{U}$ is such that the pairs $\{(U_i, X_i)\}_{i=1}^{N}$ are mutually independent.

*Assumption 4:* The random elements $S_1$ through $S_N$ are generated in the same manner as described in Definition 2, with the evident modification that we now consider that $i$ is in $\{1, \ldots, N\}$.

The collision channel operates as follows:

*Definition 9:* The collision channel accepts inputs $S_1$ through $S_N$. The output of the collision channel is specified by the following map:

$$y = \chi(s_1, \ldots, s_N) \overset{\text{def}}{=} \begin{cases} s_i & \text{if } s_i \neq \varnothing \text{ and } s_j = \varnothing,\ j \neq i \\ \varnothing & \text{if } s_i = \varnothing,\ 1 \leq i \leq N \\ (\mathfrak{C}, u) & \text{otherwise.} \end{cases} \tag{92}$$

*Remark 8:* Notice that $\chi$ defined above is a natural extension of Eq. (4). The only difference is that, when there is a collision, $\chi$ also conveys the decision vector $U$. Notice that both channels are equivalent when $N$ is two because, in that case, a collision can only occur when $U_1$ and $U_2$ are both one. Hence, if the estimator is informed that a collision occurred and $N$ is two, the additional information on $U$ provided by $\chi$ becomes redundant. Notice that the recovery of $U$ at the receiver when there are collisions, as is assumed for $\chi$, has been demonstrated empirically in [26].

### A. Extension of Theorem 1 to more than two sensors

Consider a version of Problem 1 in which there are $N$ sensors, and the cost is as follows:

$$\mathcal{J}_A(\mathcal{U}) \overset{\text{def}}{=} \sum_{k=1}^{N} \eta_k \mathbf{P}(X_k \neq \hat{X}_k) \tag{93}$$

where $\eta_k$ are positive constants that sum up to 1. The following is the extended version of Theorem 1.

*Theorem 3:* There exists a globally optimal solution $(\check{\mathcal{U}}_1, \ldots, \check{\mathcal{U}}_N)$ for which $\check{\mathcal{U}}_i$ is in $\{\mathcal{V}^0_{X_i}, \mathcal{V}^1_{X_i}, \mathcal{V}^2_{X_i}\}$, for all $i$ in $\{1, \ldots, N\}$.

*Proof:* Select arbitrary $i$ in $\{1, \ldots, N\}$ and an $N-1$ tuple $\tilde{\mathcal{U}}_{-i}$, and write the cost from the perspective of $DM_i$ as follows:

$$\mathcal{J}_A(\mathcal{U}_i, \tilde{\mathcal{U}}_{-i}) = \eta_i \mathbf{P}(X_i \neq \hat{X}_i) + \rho_{\tilde{\mathcal{U}}_{-i}} \mathbf{P}(U_i = 1) + \theta_{\tilde{\mathcal{U}}_{-i}}, \tag{94}$$

where the "communication cost" and offset terms are given by

$$\rho_{\tilde{\mathcal{U}}_{-i}} \overset{\text{def}}{=} \sum_{j \neq i} \eta_j \big(\mathbf{P}(X_j \neq \hat{X}_j \mid U_i = 1) - \mathbf{P}(X_j \neq \hat{X}_j \mid U_i = 0)\big) \tag{95}$$

and

$$\theta_{\tilde{\mathcal{U}}_{-i}} \overset{\text{def}}{=} \sum_{j \neq i} \eta_j \mathbf{P}(X_j \neq \hat{X}_j \mid U_i = 0). \tag{96}$$

We can conclude by inspection that minimizing the cost above with respect to $DM_i$ is equivalent to Problem 3 with $\varrho$ and $\beta$ given by:

$$\varrho \overset{\text{def}}{=} \frac{\rho_{\tilde{\mathcal{U}}_{-i}}}{\eta_i}, \tag{97}$$

$$\beta \stackrel{\text{def}}{=} 1 - \prod_{j \neq i} \mathbf{P}(U_j = 0). \tag{98}$$

Here, $\beta$ is the probability that a transmission by $DM_i$ will collide with a transmission by at least one of the other $N-1$ sensors. At this point, the proof follows from Lemma 2. ■

### B. Further results to Problem 1 with more than two sensors

In the absence of a structural result, such as Theorem 3, the only systematic approach to Problem 1 is to exhaustively search over all possible deterministic solutions, which becomes impractical unless $N$ and the cardinality of $\mathbb{X}_1, \dots, \mathbb{X}_N$ are small. Fortunately, from Theorem 3, we conclude that even if there are observations with infinite support it suffices to sift through a set of $3^N$ candidate solutions. We now show that we can further leverage Theorem 3 to obtain remarkable complexity reductions for the two cases of practical significance discussed in Theorems 4 and 5.

Standard probability methods and algebraic manipulation lead to the following Proposition, which we state without proof. This result will be useful in the proof of Theorem 4.

*Proposition 4:* For every possible $N$-tuple of policies $\mathcal{U} = (\mathcal{U}_1, \cdots, \mathcal{U}_N)$ where each $\mathcal{U}_k$ is in $\{\mathcal{V}^0_{X_k}, \mathcal{V}^1_{X_k}, \mathcal{V}^2_{X_k}\}$, for $k \in \{1, \cdots, N\}$, the following holds:

$$\mathbf{P}(X_k \neq \hat{X}_k) = \begin{cases} 1 - q_{X_k}(1) & \text{if } \mathcal{U}_k = \mathcal{V}^0_{X_k} \\ t_{X_k}\big(1 - \prod_{j \neq k} \mathbf{P}(U_j = 0)\big) & \text{if } \mathcal{U}_k = \mathcal{V}^1_{X_k} \\ t_{X_k} & \text{if } \mathcal{U}_k = \mathcal{V}^2_{X_k} \end{cases} \tag{99}$$

and

$$\mathbf{P}(U_j = 0) = \begin{cases} 1 & \text{if } \mathcal{U}_j = \mathcal{V}^0_{X_j} \\ q_{X_j}(1) & \text{if } \mathcal{U}_j = \mathcal{V}^1_{X_j} \\ 1 - q_{X_j}(2) & \text{if } \mathcal{U}_j = \mathcal{V}^2_{X_j}. \end{cases} \tag{100}$$

*Theorem 4:* Consider that there are $N$ sensors and let them be labeled so that the weights defining the cost for Problem 1 are non-increasing ($\eta_1 \geq \eta_2 \geq \cdots \geq \eta_N$). Suppose that $N$ is greater than one and that the observations are identically distributed according to $X_k \sim p_X$, for $k \in \{1, \cdots, N\}$.

Under these conditions, there are nonnegative integers $n_1^\star$ and $n_2^\star$, satisfying $n_1^\star + n_2^\star \leq N$, for which the following is a globally optimal solution to Problem 1:

$$\mathcal{U}^\star = (\underbrace{\mathcal{V}^1_X, \cdots, \mathcal{V}^1_X}_{n_1^\star \text{ tuple}}, \underbrace{\mathcal{V}^2_X, \cdots, \mathcal{V}^2_X}_{n_2^\star \text{ tuple}}, \underbrace{\mathcal{V}^0_X, \cdots, \mathcal{V}^0_X}_{(N - n_1^\star - n_2^\star) \text{ tuple}}). \tag{101}$$

*Remark 9:* Theorem 4 implies that the search for an optimal solution is parametrized by two nonnegative integers $n_1$ and $n_2$ satisfying $n_1 + n_2 \leq N$. There are $\frac{1}{2}(N+1)(N+2)$ elements in this feasible set. Therefore, the complexity of searching for a globally optimal solution in this case is quadratic in the number of sensors.

*Proof of Theorem 4:* Consider an $N$-tuple of transmission policies $\mathcal{U}$ for Problem 1 where $n_1$ sensors use the policy $\mathcal{V}^1_X$, $n_2$ sensors use the policy $\mathcal{V}^2_X$ and $N - n_1 - n_2$ sensors use the policy $\mathcal{V}^0_X$. Let $\breve{\mathcal{U}}$ be an $N$-tuple of transmission policies constructed by rearranging the transmission policies of the individual sensors in $\mathcal{U}$ such that the $n_1$ sensors corresponding to the $n_1$ largest weights use $\mathcal{V}_X^1$; the following $n_2$ sensors corresponding to the $n_1+1$ through $n_1+n_2$ largest weights use $\mathcal{V}_X^2$; and the remaining $N-n_1-n_2$ sensors use $\mathcal{V}_X^0$.

From Proposition 4, the following chain of inequalities holds

$$\mathbf{P}(X_k \neq X_k)\Big|_{\mathcal{U}_k=\mathcal{V}_X^1} \leq \mathbf{P}(X_k \neq X_k)\Big|_{\mathcal{U}_k=\mathcal{V}_X^2} \leq \mathbf{P}(X_k \neq X_k)\Big|_{\mathcal{U}_k=\mathcal{V}_X^0}, \tag{102}$$

for $k \in \{1,\cdots,N\}$. The fact that the weights are nonincreasing and that Eq. (102) holds, imply that

$$\mathcal{J}_A(\breve{\mathcal{U}}) \leq \mathcal{J}_A(\mathcal{U}). \tag{103}$$

■

### C. Problem 1 with i.i.d. observations and uniform weights

We will show that, using the structural result of Theorem 3, we can solve Problem 1 exactly for an arbitrary number of sensors with identically distributed observations and uniform weights.

*Theorem 5:* Let $N$ greater than one represent the number of sensors for Problem 1. If $\eta_k = 1/N$ and the observations are identically distributed as $X_k \sim p_X$, for $k \in \{1,\cdots,N\}$, then the following solution is globally optimal for Problem 1:

$$\mathcal{U}^\star = \Big( \underbrace{\mathcal{V}_X^1,\cdots,\mathcal{V}_X^1}_{n^\star \text{ tuple}}, \underbrace{\mathcal{V}_X^2,\cdots,\mathcal{V}_X^2}_{(N-n^\star) \text{ tuple}} \Big), \tag{104}$$

where $n^\star$ is computed explicitly according to

$$n^\star = \min\left\{ \left\lfloor \frac{q_X(1)}{1-q_X(1)-q_X(2)} \right\rfloor + 1, N \right\}. \tag{105}$$

*Proof:* See Appendix C. ■

*Remark 10:* Theorem 5 is evidence that by using Theorem 3 and exploiting structure, one may arrive at exact solutions to Problem 1 for an arbitrary number of sensors. Using this result we may also study the asymptotic performance degradation of the optimal strategy as the number of sensors grows. Notice that, as the number of sensors tends to infinity, the proportion of sensors that use policy $\mathcal{V}_X^2$ tends to 1. Therefore, the optimal performance of the system tends to $t_X$, which, remarkably, is bounded away from 1.

### D. On the numerical search for a globally optimal solution: the general case

Using Proposition 4, it is possible to implement a numerical procedure to exhaustively search for a globally optimal solution to Problem 1 over the $3^N$ candidate solutions. On a standard laptop computer, the optimization can be carried out within reasonable times for up to $N=16$ sensors. Table II shows the average running time to perform this procedure. It is important to highlight that this numerical search is performed off-line and only once prior to deployment. The implementation of the policies $\mathcal{V}_{X_k}^0$, $\mathcal{V}_{X_k}^1$ and $\mathcal{V}_{X_k}^2$ by the sensors is trivial and do not require any computations. Therefore, the implementation of an optimal system with arbitrary pmfs and weights is practicable for $N \leq 16$ sensors.

TABLE II

TIME TO FIND A GLOBALLY OPTIMAL SOLUTION TO PROBLEM 1 FOR TEAMS OF $N$ SENSORS.

| $\boldsymbol{N}$ | $\boldsymbol{t}$ (sec) | $\boldsymbol{N}$ | $\boldsymbol{t}$ (sec) |
|---|---|---|---|
| 1 | - | 9 | 2.866 |
| 2 | 0.028 | 10 | 8.675 |
| 3 | 0.031 | 11 | 28.046 |
| 4 | 0.037 | 12 | 93.239 |
| 5 | 0.057 | 13 | 319.081 |
| 6 | 0.121 | 14 | 1047.081 |
| 7 | 0.331 | 15 | 3285.763 |
| 8 | 0.981 | 16 | 9364.353 |

### E. *Extension of Theorem 2 to more than two sensors*

Here, we seek to find $\mathcal{U}$ that minimizes the following cost:

$$\mathcal{J}_B(\mathcal{U}) = \mathbf{P}(W \neq \hat{W}), \tag{106}$$

where $\hat{W} = (\hat{X}_1, \dots, \hat{X}_N)$ is a $N$-tuple MAP estimate for $(X_1, \dots, X_N)$ obtained as follows:

$$\hat{W} = \mathcal{E}(Y), \tag{107}$$

where

$$\mathcal{E}(y) = \arg\max_{\alpha \in \mathbb{W}} \mathbf{P}\big((X_1, \dots, X_N) = \alpha \mid Y = y\big) \tag{108}$$

and $\mathbb{W} \stackrel{\text{def}}{=} \mathbb{X}_1 \times \cdots \times \mathbb{X}_N$.

*Theorem 6:* The $N$-tuple $\check{\mathcal{U}}$, formed by $\check{\mathcal{U}}_i = \mathcal{V}^1_{X_i}$ for all $i$ in $\{1, \dots, N\}$, minimizes $\mathcal{J}_B$.

*Proof:* See Appendix D. ■

*1) Example:* Consider the case of a system where $N$ sensors observe independent random variables $\{X_1, \dots, X_N\}$ that are identically distributed according to $p_X$. The total probability of error evaluated at the globally optimal solution $\mathcal{U}^\star = (\mathcal{V}^1_{X_1}, \dots, \mathcal{V}^1_{X_N})$ gives:

$$\begin{aligned}\mathcal{J}_B(\mathcal{U}^\star) = \ & 1 - N\big(q_X(1)\big)^{N-1}\big(1 - q_X(1) - q_X(2)\big) \\ & - \big(q_X(1) + q_X(2)\big)^N. \end{aligned} \tag{109}$$

When $p_X$ is Bernoulli, it follows immediately that the optimal cost is zero for any number of sensors. However, for $|\mathbb{X}| \geq 3$, the performance of the system degrades when the number of sensors $N$ increases and the optimal cost converges to one as $N$ tends to infinity.

We illustrate how the performance degrades with $N$ for the case in which $p_X$ is the Geometric pmf with parameter $\pi \in [0, 1]$. The optimal cost calculated for $N = 2, 4, 8, 16$ and $64$ is depicted in Fig. 5.

## VII. Conclusion and future work

We proposed a framework in which estimates of two or more independent discrete random variables are calculated based on information transmitted via a collision channel. In this framework, each random variable is observed by a sensor that decides whether it must stay silent or attempt to transmit its measurement to the remote estimator. The collision channel declares a collision when two or more sensors decide to transmit; and operates as a perfect link, otherwise.

A team decision problem is formulated that seeks to determine the transmission policies that minimize two types of cost. The first is a weighted sum of the probabilities of error of the estimates of each random variable. We also considered the case in which we seek to minimize the total probability of error. In both cases, we used a person-by-person optimality approach to obtain structural characterizations of globally optimal solutions. For the latter case, we explicitly found a globally optimal solution, while in the former we showed that there is a finite set of candidate solutions with a specific structure that contains at least a globally optimal solution. The number of candidate solutions grows exponentially ($3^N$) with the number of sensors, however, it does not depend on the distribution or cardinality of the support of the random variables, which can be countably infinite. The determination of which solution is optimal among the set of candidates, as well as the determination of the minimal cost, is carried out via simple calculations that depend only on the probabilities of the two most probable outcomes of each random variable. When there are two sensors, we show that the number of candidate solutions can be further reduced from nine to five. This suggests that $3^N$ may be, in general, a very conservative upper bound on the number of candidate

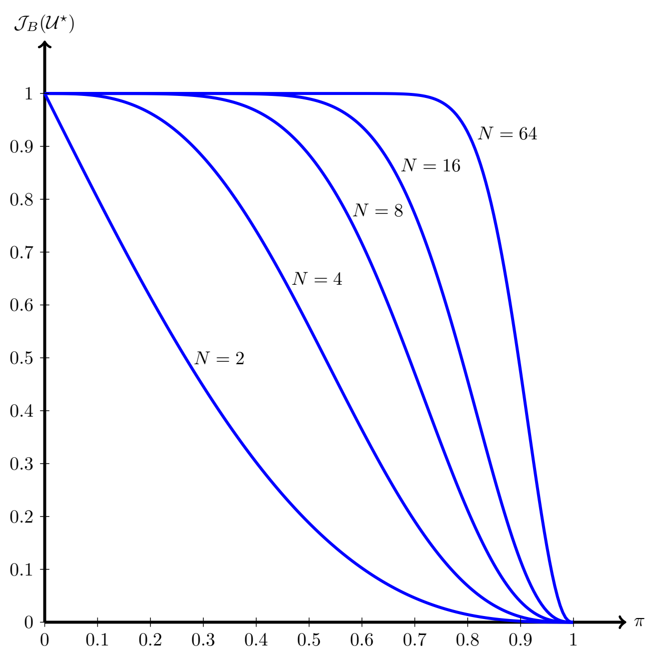


Fig. 5. Optimal performance of a team with $N$ sensors observing i.i.d. Geometric random variables with parameter $\pi$ and minimizing the total probability of error criterion.

solutions that need to be tested. Indeed, we showed that when the observations at the sensors are i.i.d., the complexity is quadratic in the number of sensors; and if, additionally, the weights are uniform, we obtained a globally optimal solution in closed-form for any number of sensors.

*Open problems:* There are several possible extensions of the work reported here. First, is to consider the case when the random variables are dependent or, as addressed in [27] for continuously distributed random variables, the measurements contain private and common components. The case in which $u$ is not available in Eq. (92) when there is a collision is a realistic and important problem that also remains unsolved. In this case, there is ambiguity on which sensors attempted to transmit when a collision occurs. In the case of arbitrarily distributed observations, it is also important to investigate systematic methods to reduce the number of candidate solutions that need to be tested when the cost is the aggregate probability of estimation error. In this regard, for problems with $N > 16$ sensors, either more sophisticated algorithms or suboptimal approaches will need to be developed. Finally, the sequential estimation of discrete Markov sources over the collision channel with feedback is yet another important unsolved problem.

## APPENDIX A
## COMMUNICATION COST AND OFF-SET TERMS

*Proof of Proposition 1:* First, we need to show that, for $i, j \in \{1, 2\}$ and $j \neq i$, the following holds:

$$\mathcal{E}_j\big((i, x_i)\big) = \mathcal{E}_j(\varnothing), \ \ x_i \in \mathbb{X}_i, \tag{110}$$

which implies that for the purpose of estimating $X_j$, observing $Y = (i, x_i)$ at the remote estimator is equivalent to receiving $Y = \varnothing$.

From the definition of the MAP estimator in Eq. (7), we have:

$$\begin{aligned}
\mathcal{E}_j(\varnothing) &= \arg\max_{\alpha \in \mathbb{X}_j} \mathbf{P}(X_j = \alpha, Y = \varnothing) && (111)\\
&= \arg\max_{\alpha \in \mathbb{X}_j} \mathbf{P}(X_j = \alpha, U_i = 0, U_j = 0) && (112)\\
&\overset{(a)}{=} \arg\max_{\alpha \in \mathbb{X}_j} \mathbf{P}(X_j = \alpha, U_j = 0). && (113)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{E}_j\big((i, x_i)\big) &= \arg\max_{\alpha \in \mathbb{X}_j} \mathbf{P}\big(X_j = \alpha, Y = (i, x_i)\big) && (114)\\
&= \arg\max_{\alpha \in \mathbb{X}_j} \mathbf{P}(X_j = \alpha, U_j = 0, U_i = 1, X_i = x_i) && (115)\\
&\overset{(b)}{=} \arg\max_{\alpha \in \mathbb{X}_j} \mathbf{P}(X_j = \alpha, U_j = 0) && (116)\\
&= \mathcal{E}_j(\varnothing). && (117)
\end{aligned}$$

The equalities $(a)$ and $(b)$ follow from Assumption 1 and the fact that $X_1$ and $X_2$ are independent. Consequently, the MAP estimator $\hat{X}_j = \mathcal{E}_j(Y)$ leads to:

$$\mathbf{P}(X_j \neq \hat{X}_j \mid U_i = 1, U_j = 0) = \mathbf{P}(X_j \neq \hat{X}_j \mid U_i = 0, U_j = 0). \tag{118}$$

Finally, given that $(U_i = 0, U_j = 1)$ and $Y = (j, X_j)$ define the same event, we also have:

$$\mathbf{P}(X_j \neq \hat{X}_j \mid U_i = 0, U_j = 1) = 0. \tag{119}$$

Using the law of total probability, Eqs. (118) and (119), we rewrite $\rho_{\tilde{\mathcal{U}}_j}$ as follows:

$$\begin{aligned} \rho_{\tilde{\mathcal{U}}_j} &= \mathbf{P}(X_j \neq \hat{X}_j, U_j = 1 \mid U_i = 1) && (120)\\ &= \mathbf{P}(X_j \neq \mathcal{E}_j(\mathfrak{C}), U_j = 1). && (121) \end{aligned}$$

Using the definition of MAP estimator and expressing the result in terms of the policy $\tilde{\mathcal{U}}_j$, we have

$$\rho_{\tilde{\mathcal{U}}_j} = \mathbf{P}(U_j = 1) - \max_{\alpha_j \in \mathbb{X}_j} \tilde{\mathcal{U}}_j(\alpha_j) p_{X_j}(\alpha_j). \tag{122}$$

Following similar steps, we can show that the off-set term $\theta_{\tilde{\mathcal{U}}_j}$ is given by:

$$\theta_{\tilde{\mathcal{U}}_j} = \mathbf{P}(X_j \neq \mathcal{E}_j(\varnothing), U_j = 0), \tag{123}$$

which expressed in terms of $\tilde{\mathcal{U}}_j$ is

$$\theta_{\tilde{\mathcal{U}}_j} = \mathbf{P}(U_j = 0) - \max_{x \in \mathbb{X}_j} (1 - \tilde{\mathcal{U}}_j(x)) p_{X_j}(x). \tag{124}$$

The proof is concluded by noticing that Eqs. (122) and (124) imply that $0 \leq \rho_{\tilde{\mathcal{U}}_j} \leq 1$ and $0 \leq \theta_{\tilde{\mathcal{U}}_j} \leq 1$. ■

## Appendix B
## Auxiliary results for the proof of Theorem 2

*Proof of Proposition 2:* The conditional probability of a correct estimate conditioned on the event of a collision can be computed as:

$$\begin{aligned} \mathbf{P}(W = \hat{W} \mid Y = \mathfrak{C}) &\overset{(a)}{=} \max_{\omega \in \mathbb{W}} \mathbf{P}(W = \omega \mid Y = \mathfrak{C}) \\ &\overset{(b)}{=} \max_{\omega \in \mathbb{W}} \mathbf{P}(W = \omega \mid U_1 = 1, U_2 = 1) && (125)\\ &\overset{(c)}{=} \max_{\alpha_1 \in \mathbb{X}_1} \mathbf{P}(X_1 = \alpha_1 \mid U_1 = 1) \\ &\quad \times \max_{\alpha_2 \in \mathbb{X}_2} \mathbf{P}(X_2 = \alpha_2 \mid U_2 = 1). && (126) \end{aligned}$$

where $(a)$ follows from our definition of $\hat{W}$; the equality in $(b)$ results from the fact that the events $(Y = \mathfrak{C})$ and $(U_1 = 1, U_2 = 1)$ are identical; finally, $(c)$ follows from the fact that $(U_1, X_1)$ and $(U_2, X_2)$ are independent.

The proof of the second equality can be derived from the fact that the events $(Y = \varnothing)$ and $(U_1 = 0, U_2 = 0)$ are identical, followed by the same steps as before. ■

*Proof of Proposition 3:* It suffices to show that for every $\alpha_i \in \mathbb{X}_i$, the following equalities hold:

$$\begin{aligned} \mathbf{P}\big(W = \hat{W} \mid Y = (i, \alpha_i)\big) &\overset{(a)}{=} \max_{\omega \in \mathbb{W}} \mathbf{P}\big(W = \omega \mid Y = (i, \alpha_i)\big) \\ &\overset{(b)}{=} \max_{\omega \in \mathbb{W}} \mathbf{P}(W = \omega \mid U_i = 1, X_i = \alpha_i, U_j = 0) && (127)\\ &\overset{(c)}{=} \max_{\alpha_j \in \mathbb{X}_j} \mathbf{P}(X_j = \alpha_j \mid U_j = 0). && (128) \end{aligned}$$

Where $(a)$ follows from our definition of $\hat{W}$; the equality in $(b)$ follows from the fact that the events $(Y = (i, \alpha_i))$ and $(U_i = 1, X_i = \alpha_i, U_j = 0)$ are equivalent; and $(c)$ follows from the fact that $(U_1, X_1)$ and $(U_2, X_2)$ are independent. ■

## APPENDIX C
## PROOF OF THEOREM 5

*Lemma 5:* Consider Problem 1 with an arbitrary number of sensors $N \geq 2$. Let $\eta_k = 1/N$ and $X_k \sim p_X$, $k \in \{1, \cdots, N\}$. A globally optimal solution can be found by solving the following optimization problem:

$$\begin{aligned} \underset{n_0, n_1 \in \mathbb{Z}}{\text{minimize}} \quad & f\big(n_0, n_1\big) \\ \text{subject to} \quad & n_0 + n_1 \leq N \\ & n_0, n_1 \geq 0, \end{aligned} \tag{129}$$

where

$$f(n_0, n_1) = \Big[n_0 \cdot \big(1 - q_X(1)\big) + (N - n_0 - n_1) \cdot t_X + \\ n_1 \cdot t_X \cdot \Big(1 - \big(q_X(1)\big)^{n_1 - 1}\big(1 - q_X(2)\big)^{N - n_0 - n_1}\Big)\Big]/N. \tag{130}$$

*Proof:* Theorem 3 implies that the search for globally optimal solutions can be constrained to $N$-tuples of policies $\mathcal{U} = (\mathcal{U}_1, \cdots, \mathcal{U}_N)$ where $\mathcal{U}_k \in \{\mathcal{V}_X^0, \mathcal{V}_X^1, \mathcal{V}_X^2\}$, $k \in \{1, \cdots, N\}$. Let $n_0$ be the number of sensors using policy $\mathcal{V}_X^0$; $n_1$ be the number of sensors using policy $\mathcal{V}_X^1$; and $n_2$ be the number of sensors using policy $\mathcal{V}_X^2$. These quantities are such that

$$n_0 + n_1 + n_2 = N. \tag{131}$$

Assuming that $\eta_k = \frac{1}{N}$, Proposition 4 implies that the cost can be written as the following function:

$$\tilde{\mathcal{J}}_A(n_0, n_1, n_2) \overset{\text{def}}{=} \Big[n_0 \cdot \big(1 - q_X(1)\big) + n_1 \cdot t_X \Big(1 - \big(q_X(1)\big)^{n_1 - 1}\big(1 - q_X(2)\big)^{n_2}\Big) + n_2 \cdot t_X\Big]/N. \tag{132}$$

Therefore, Problem 1 reduces to the following equivalent nonlinear integer program:

$$\begin{aligned} \underset{n_0, n_1 \in \mathbb{Z}}{\text{minimize}} \quad & \tilde{\mathcal{J}}_A\big(n_0, n_1, N - n_0 - n_1\big) \\ \text{subject to} \quad & n_0 + n_1 \leq N \\ & n_0, n_1 \geq 0. \end{aligned} \tag{133}$$

Define the function $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$ as

$$f(n_0, n_1) \overset{\text{def}}{=} \tilde{\mathcal{J}}_A\big(n_0, n_1, N - n_0 - n_1\big). \tag{134}$$

■

*Lemma 6:* The function $f(n_0, n_1)$ is monotone decreasing in $n_0$.

*Proof:* To establish this fact, let $n_0 > 0$ and $n_1 \geq 0$ such that $n_0 + n_1 \leq N$. Consider the following:

$$N \cdot \big(f(n_0, n_1) - f(n_0 - 1, n_1)\big) = -q_X(2)\Big(1 - n_1 t_X q_X(1)^{n_1 - 1}(1 - q_X(2))^{N - n_0 - n_1}\Big). \tag{135}$$

Then notice that the following strict inequality holds

$$n_1 q_X(1)^{n_1-1}(1-q_X(2))^{N-n_0-n_1} < n_1 q_X(1)^{n_1-1}. \tag{136}$$

Furthermore,

$$\begin{aligned} n_1 q_X(1)^{n_1-1} &< 1 + q_X(1) + \cdots + q_X(1)^{n_1-1} &(137)\\ &< \sum_{j=0}^{\infty} q_X(1)^j = \frac{1}{1-q_X(1)} &(138)\\ &< \frac{1}{1-q_X(1)-q_X(2)} = \frac{1}{t_X}. &(139)\end{aligned}$$

Therefore,

$$f(n_0, n_1) > f(n_0 - 1, n_1). \tag{140}$$

■

*Lemma 7:* Let $\alpha \in (0,1)$. Consider the sequence $p(n)$ defined as:

$$p(n) \overset{\text{def}}{=} \begin{cases} n\alpha^{n-1}(1-\alpha)^2 & n = 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases} \tag{141}$$

The following holds:

$$\arg\max p(n) = \left\lfloor \frac{\alpha}{1-\alpha} \right\rfloor + 1. \tag{142}$$

*Proof:* The sequence $p(n+1)$ is the probability mass function of a Negative Binomial random variable with parameters 2 and $\alpha$. This pmf has mode equal to $\lfloor \alpha/(1-\alpha) \rfloor$. ■

*Proof of Theorem 5:* Lemma 5 implies that a globally optimal solution to Problem 1 can be found by solving the equivalent optimization problem in Eq. (129). From Lemma 6, the monotonicity of $f(n_0, n_1)$ in $n_0$ implies that, without loss of optimality, $n_0^\star = 0$.

In order to find the optimal value of $n_1$ that minimizes $f(n_0, n_1)$, it suffices to optimize the following function:

$$f(0, n_1) = t_X\Big(1 - n_1 q_X(1)^{n_1-1}\big(1 - q_X(2)\big)^{N-n_1}/N\Big), \tag{143}$$

which is equivalent to solving

$$\begin{aligned} \underset{n_1 \in \mathbb{Z}}{\text{maximize}} \quad & n_1 q_X(1)^{n_1-1}\big(1-q_X(2)\big)^{N-n_1} \\ \text{subject to} \quad & 0 \le n_1 \le N. \end{aligned} \tag{144}$$

Notice that the objective function in Eq. (144) is directly proportional to

$$n_1 \left(\frac{q_X(1)}{1-q_X(2)}\right)^{n_1-1} \left(1 - \frac{q_X(1)}{1-q_X(2)}\right)^2. \tag{145}$$

The result follows from Lemma 7. ■

## APPENDIX D
## PROOF OF THEOREM 6

*Proof of Theorem 6:* The fact that $\hat{W}$ is the MAP estimate leads to the following equalities:

$$\mathbf{P}(W=\hat{W} \mid Y=\varnothing)=\prod_{k=1}^{N}\max_{\alpha_k\in\mathbb{X}_k}\mathbf{P}(X_k=\alpha_k \mid U_k=0) \tag{146}$$

and

$$\mathbf{P}(W=\hat{W} \mid Y=(j,\tilde{x}))=\prod_{k\neq j}\max_{\alpha_k\in\mathbb{X}_k}\mathbf{P}(X_k=\alpha_k \mid U_k=0). \tag{147}$$

The fact that the estimator receives $U$ from $\chi$ when a collision occurs, leads to:

$$\mathbf{P}(W=\hat{W} \mid Y=(\mathfrak{C},\mu))=\prod_{k=1}^{N}\max_{\alpha_k\in\mathbb{X}_k}\mathbf{P}(X_k=\alpha_k \mid U_k=\mu_k). \tag{148}$$

for any $\mu$ in $\{0,1\}^N$. For any given policy $\mathcal{U}$, we can use the total probability law to express the cost as:

$$\begin{aligned}\mathcal{J}_B(\mathcal{U})=1-\prod_{k=1}^{N}&\max_{\alpha_k\in\mathbb{X}_k}\mathbf{P}(X_k=\alpha_k,U_k=0)\\&-\sum_{j=1}^{N}\Big(\prod_{k\neq j}\max_{\alpha_k\in\mathbb{X}_k}\mathbf{P}(X_k=\alpha_k,U_k=0)\Big)\mathbf{P}(U_j=1)\\&-\sum_{\mu\in\mathbb{L}_{N,2}}\Big(\prod_{k=1}^{N}\max_{\alpha_k\in\mathbb{X}_k}\mathbf{P}(X_k=\alpha_k,U_k=\mu_k)\Big).\end{aligned} \tag{149}$$

where $\mathbb{L}_{N,k}\stackrel{\text{def}}{=}\{\mu\in\{0,1\}^N \mid \sum_{i=1}^{N}\mu_i\geq k\}$.

The cost can be re-expressed as:

$$\begin{aligned}\mathcal{J}_B(\mathcal{U})=1-\prod_{k=1}^{N}\max_{\alpha_k\in\mathbb{X}_k}(1-\mathcal{U}_k(\alpha_k))p_{X_k}(\alpha_k)-\sum_{j=1}^{N}\Big(\prod_{k\neq j}\max_{\alpha_k\in\mathbb{X}_k}(1-\mathcal{U}_k(\alpha_k))p_{X_k}(\alpha_k)\Big)\\\times\sum_{\alpha_j\in\mathbb{X}_j}\mathcal{U}_j(\alpha_j)p_{X_j}(\alpha_j)-\sum_{\mu\in\mathbb{L}_{N,2}}\Big(\prod_{k:\mu_k=0}\max_{\alpha_k\in\mathbb{X}_k}(1-\mathcal{U}_k(\alpha_k))p_{X_k}(\alpha_k)\Big)\\\times\Big(\prod_{k:\mu_k=1}\max_{\alpha_k\in\mathbb{X}_k}\mathcal{U}_k(\alpha_k)p_{X_k}(\alpha_k)\Big).\end{aligned} \tag{150}$$

For any arbitrary choice of $i$ in $\{1,\dots,N\}$, and for any given fixed $\tilde{\mathcal{U}}_{-i}$, we have:

$$\begin{aligned}\mathcal{J}_B(\mathcal{U}_i,\tilde{\mathcal{U}}_{-i})=1-\tilde{\tau}_{-i}&\max_{\alpha_i\in\mathbb{X}_i}\mathcal{U}_i(\alpha_i)p_{X_i}(\alpha_i)\\&-\tilde{\varrho}_{-i}\sum_{\alpha_i\in\mathbb{X}_i}\mathcal{U}_i(\alpha_i)p_{X_i}(\alpha_i)\\&-(\tilde{\varrho}_{-i}+\tilde{\beta}_{-i})\max_{\alpha_i\in\mathbb{X}_i}(1-\mathcal{U}_i(\alpha_i))p_{X_i}(\alpha_i),\end{aligned} \tag{151}$$

where the coefficients $\tilde{\varrho}_{-i},\tilde{\tau}_{-i}$ and $\tilde{\beta}_{-i}$ are given by:

$$\tilde{\varrho}_{-i}\stackrel{\text{def}}{=}\prod_{k\neq i}\max_{\alpha_k\in\mathbb{X}_k}(1-\tilde{\mathcal{U}}_k(\alpha_k))p_{X_k}(\alpha_k); \tag{152}$$

$$\tilde{\tau}_{-i} \stackrel{\text{def}}{=} \sum_{\mu_{-i} \in \mathbb{L}_{N-1,1}} \left( \prod_{k:\mu_k=0} \max_{\alpha_k \in \mathbb{X}_k} (1 - \tilde{\mathcal{U}}_k(\alpha_k)) p_{X_k}(\alpha_k) \right) \times \left( \prod_{k:\mu_k=1} \max_{\alpha_k \in \mathbb{X}_k} \tilde{\mathcal{U}}_k(\alpha_k) p_{X_k}(\alpha_k) \right); \tag{153}$$

and

$$\tilde{\beta}_{-i} \stackrel{\text{def}}{=} \sum_{\mu_{-i} \in \mathbb{L}_{N-1,2}} \left( \prod_{\substack{k \neq i: \\ u_k=0}} \max_{\alpha_k \in \mathbb{X}_k} (1 - \tilde{\mathcal{U}}_k(\alpha_k)) p_{X_k}(\alpha_k) \right) \times \left( \prod_{\substack{k \neq i: \\ \mu_k=1}} \max_{\alpha_k \in \mathbb{X}_k} \tilde{\mathcal{U}}_k(\alpha_k) p_{X_k}(\alpha_k) \right)$$
$$+ \sum_{j \neq i} \left( \prod_{k \notin \{i,j\}} \max_{\alpha_k \in \mathbb{X}_k} (1 - \tilde{\mathcal{U}}_k(\alpha_k)) p_{X_k}(\alpha_k) \right) \times \sum_{\alpha_j \in \mathbb{X}_j} \tilde{\mathcal{U}}_j(\alpha_j) p_{X_j}(\alpha_j). \tag{154}$$

After a few calculations, we arrive at the following:

$$\tilde{\tau}_{-i} - \tilde{\beta}_{-i} = \sum_{j \neq i} \left( \prod_{k \notin \{i,j\}} \max_{\alpha_k \in \mathbb{X}_k} (1 - \tilde{\mathcal{U}}_k(\alpha_k)) p_{X_k}(\alpha_k) \right)$$
$$\times \left( \max_{\alpha_j \in \mathbb{X}_j} \tilde{\mathcal{U}}_j(\alpha_j) p_{X_j}(\alpha_j) - \sum_{\alpha_j \in \mathbb{X}_j} \tilde{\mathcal{U}}_j(\alpha_j) p_{X_j}(\alpha_j) \right) \leq 0. \tag{155}$$

From the inequality above, which implies that $\tilde{\tau}_{-i} \leq \tilde{\beta}_{-i}$, and from the fact that if $\tilde{\mathcal{U}}_{-i}$ is fixed then Eq. (151) has the same structure as $\mathscr{J}_B$ in Eq. (81), we can conclude from Lemma 4 that $\mathcal{J}_B(\mathcal{V}^1_{X_i}, \tilde{\mathcal{U}}_{-i}) \leq \mathcal{J}_B(\mathcal{U}_i, \tilde{\mathcal{U}}_{-i})$ holds for any $\mathcal{U}_i$ in $\mathbb{U}_i$. Since, the choice of $i$ and $\tilde{\mathcal{U}}_{-i}$ was arbitrary, we conclude that $\check{\mathcal{U}}$, formed by $\check{\mathcal{U}}_i = \mathcal{V}^1_{X_i}$ for all $i$ in $\{1, \ldots, N\}$, is globally optimal. ∎

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