Bi-fidelity approximation applied to Burgers' equation

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Abstract

This document describes the application of the bi-fidelity approximation [1] to solve Burgers' equation with uncertainty in the initial parameters. Domain-specific information is used to augment the approximation, in order to eliminate oscillations seen in the reconstructed solution, involving two applications of the matrix interpolative decomposition.

1 Introduction

Consider the inviscid Burgers' equation in 1D:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{1}$$

defined on the domain $x \in [0, 2]$, with periodic boundaries and a Gaussian initial condition

$$u(x, t = 0) = \exp\left(-s(x - 0.5)^2\right) \tag{2}$$

Here, $s \sim U[1, 25]$ is a random variable. The PDE is discretized using a first-order upwind finite difference, and integrated in time using forward Euler.

From Doostan et al.[1]: "...the cost of propagating uncertainty may be significantly reduced when two models with different levels of fidelity, one low (cheap to simulate) and one high (expensive to evaluate), are used. The low-fidelity model is employed to learn a reduced basis and an interpolation rule that can be used, along with a small number of high-fidelity model evaluations, to approximate the high-fidelity solution at arbitrary samples."

In this problem, the low-fidelity (LF) model is a 32-node discretization, and the high-fidelity (HF) model is a 256-node discretization.

2 The bi-fidelity approximation

The bi-fidelity approximation involves the following steps:

1. Compute the LF solution at N realizations of the random variable.
2. Construct a matrix U_L with columns that contain the state vector (or quantity of interest) for all N realizations. If the state vector for the LF solution — here, the velocity at all grid points at time $t = 1$ — has length M, then $U_L \in \mathbb{R}_{M\times N}$.
3. Perform the matrix interpolative decomposition (ID) of U_L, using an algorithm such as the one by Liberty et al.[2], to get
$$U_L = U_L^C \Lambda_L \tag{3}$$
where $U_L^C \in \mathbb{R}_{M\times K}$, $K < M, N$.

4. Compute the HF solution at the K samples of the random variable in U_L^C to get U_H^C.
5. The ID for the HF solution is given by $U_H = U_H^C \Lambda_H$. Assuming $\Lambda_H = \Lambda_L$, we reconstruct the HF solution at N samples using

$$U_H \approx U_H^C \Lambda_L = U_B \tag{4}$$

where $U_H \in \mathbb{R}_{P \times N}$, $P > M$.

The columns of U_L that make up U_L^C provide a low-rank basis for the LF model, and Λ_L provides an interpolation rule for the remaining $(N-K)$ samples of the HF model. In our physical problem, we expect the reduced basis to capture the variance in the solution, but not the physical features in the solution.

The bi-fidelity approximation only requires K evaluations of the HF model, rather than N, and if a suitably low-rank structure can be found in the LF model, the cost is greatly reduced (assuming the time for LF evaluations is negligibly small compared to the HF evaluations). The tolerance to which the ID is computed is a free parameter that must be selected based on the required accuracy of the reconstruction and the computing budget. This is called the *fixed precision problem*, as opposed to solving for a given numerical rank - the *fixed rank problem*.

3 Results and discussion

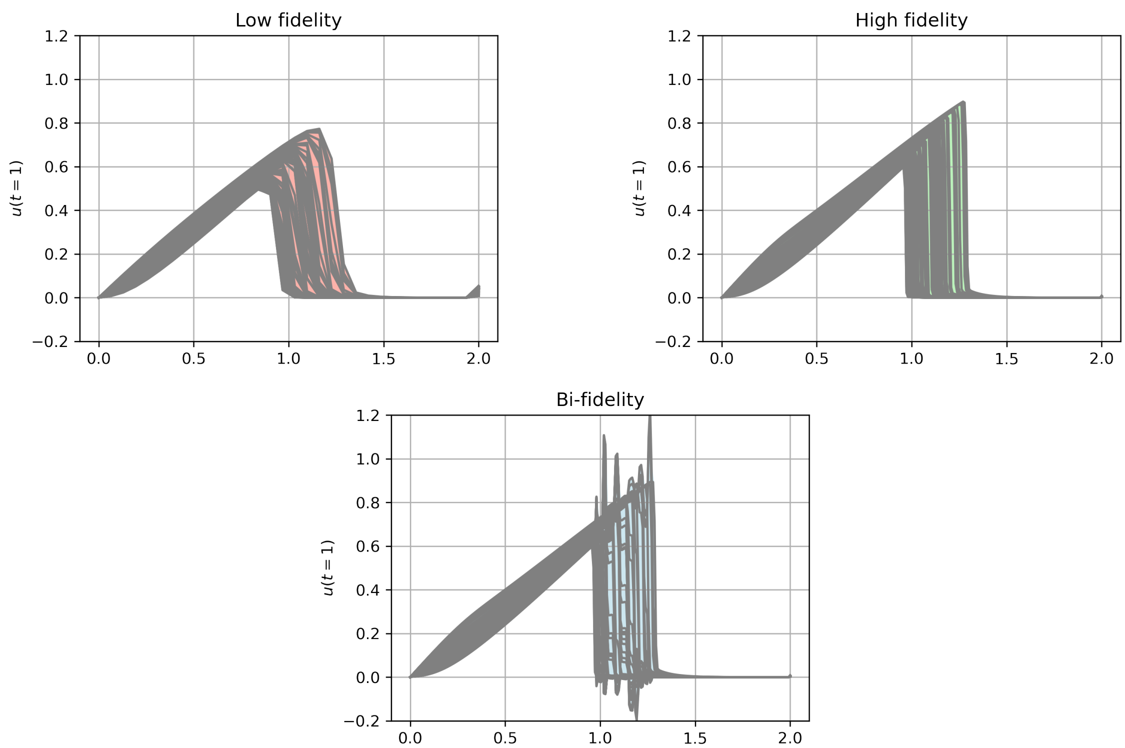

Figure 1: Comparison of LF, HF and BF solutions with $\epsilon = 10^{-6}$

Figure 1 shows the results of the LF, HF and reconstructed bi-fidelity (BF) approximation. All 100 evaluations of the HF solution are computed and plotted for reference; however, note that only those evaluations (columns) that form the low rank basis for the LF solution matrix are used in the BF matrix construction.

While the overall uncertainty envelope is captured quite accurately, one observes that several of the reconstructed solutions exhibit spurious oscillations, which will adversely impact any sensitivity studies, for example, that may be performed with the data. In addition, the primary feature of interest in the Burgers' equation - the shock - is not accurately captured, with the approximation producing a "staircasing" rather than a single, sharp discontinuity.

3.1 A change of variables

In order to improve the reconstruction, we apply a change of variable that transforms the problem into a domain wherein the interpolation rule for the shocks is straightforward: namely, one where the shocks are aligned. This is achieved by a linear transformation that operates on the columns of U_L, i.e. $U_L^T = U_L T$, where $T \in \mathbb{R}_{N \times N}$ aligns all the shocks (identified by the location of the largest gradient) at a spatial coordinate value of 1.

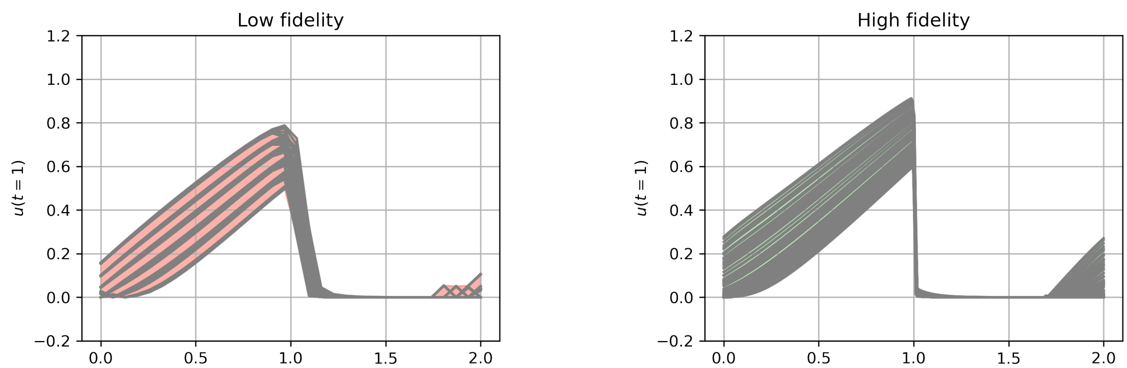

Figure 2: LF and HF solutions in transformed domain

Figure 2 shows the LF and HF solutions after the change of variables. The BF approximation is then obtained as normal from the LF solution, and the transformation is then inverted (it is trivial to note that T here is invertible). In practice, the transform is simply achieved by "rolling" each column of the LF matrix by the required number of steps until the shock is at $x = 1$, and the inverse is an "unrolling" operation.

The result of this transformation on the reconstruction is shown in Figure 3. One notes a considerable mitigation of the oscillations seen in the standard bi-fidelity approximation. Yet, the solution suffers from two new errors hitherto unseen: the entire profile is spatially shifted along the $-x$ direction, and the individual solution evaluations now appear in close clusters. Both errors can be traced to the same source: the transformation has led to an irrecoverable loss of information, since the HF evaluations that make up the low rank basis no longer have the information about the variance in the solution (i.e. the shock locations). This information now comes from the less accurate LF matrix, and this coarse graining has resulted in the additional errors.

3.2 A two-stage approach

One may combine elements of both the standard and transformed BF approximations sequentially to remove both errors. The steps involved are as follows:

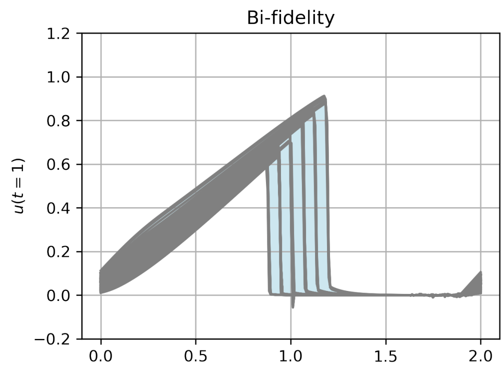

Figure 3: Bi-fidelity approximation with change of variables

1. Compute the standard BF matrix and store the shock locations.
2. Transform the LF matrix to the shock-aligned domain, and compute the BF matrix.
3. Using the shock locations from the standard BF matrix, invert the transformed BF matrix from step 2.

The results from the two-stage approach are shown in Figure 4. The results show satisfactory improvement on both counts - the spurious oscillations, and the poor shock location reconstructions - for the cost of two ID computations rather than one. The singular value spectrum for the LF matrix in the standard and transformed domains are shown in Figure 5. A steeper drop-off of the singular values is initially observed in the transformed coordinates, which indicates that the problem permits a lower rank representation in the new coordinates.

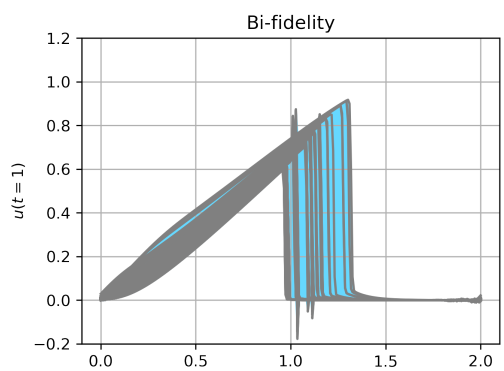

Figure 4: The BF solution using the two-stage approach

It is important to note that the transformation as applied to the Burgers' equation is by no means generalizable; however, the problem is instructive as a proof-of-concept for the exploration of lower rank structure within the QOI matrices, with potential future research into how such structures may be uncovered systematically.

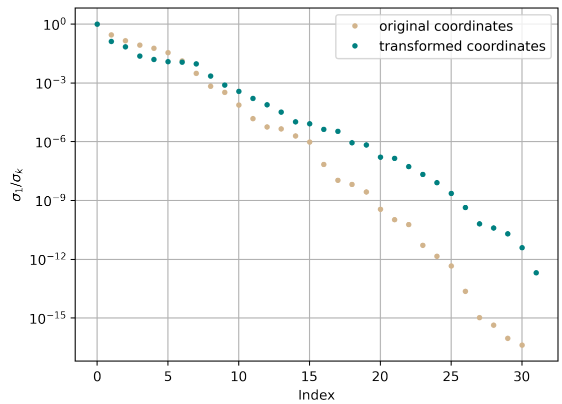

Figure 5: Singular value decay comparison between original and transformed LF solutions

Appendix

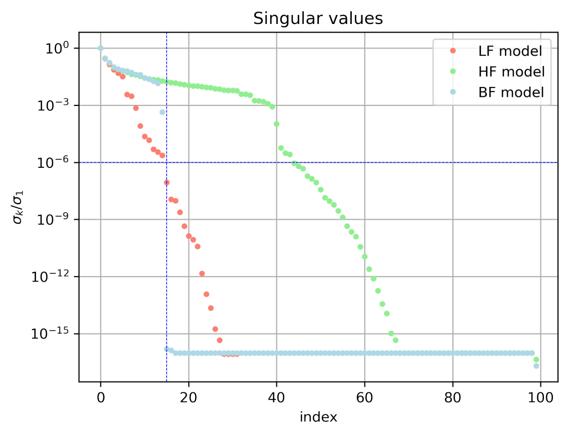

Figure 6: Comparison of singular value decays for LF, HF, and BF solutions

References

[1] Alireza Doostan, Gianluca Geraci, and Gianluca Iaccarino. "A Bi-Fidelity Approach for Uncertainty Quantification of Heat Transfer in a Rectangular Ribbed Channel". In: Turbo Expo: Power for Land, Sea, and Air 2C: Turbomachinery (June 2016). URL: `https://doi.org/10.1115/GT2016-58092`.

[2] Edo Liberty et al. "Randomized algorithms for the low-rank approximation of matrices". In: *Proceedings of the National Academy of Sciences* 104.51 (2007), pp. 20167–20172. ISSN: 0027-8424. URL: `https://www.pnas.org/content/104/51/20167`.