ME 408/CME 322 Take-home Final Exam

Due 5:00pm, Friday March 19th

Name: ______________________________

The Stanford University Honor Code

1. The Honor Code is an undertaking of the students, individually and collectively:

   (a) that they will not give or receive aid in examinations; that they will not give or receive unpermitted aid in class work, in the preparation of reports, or in any other work that is to be used by the instructor as the basis of grading;

   (b) that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.

2. The faculty on its part manifests its confidence in the honor of its students by refraining from proctoring examinations and from taking unusual and unreasonable precautions to prevent the forms of dishonesty mentioned above. The faculty will also avoid, as far as practicable, academic procedures that create temptations to violate the Honor Code.

3. While the faculty alone has the right and obligation to set academic requirements, the students and faculty will work together to establish optimal conditions for honorable academic work.

I acknowledge and accept the honor code.

(Signed) ______________________________

This exam is to be completed individually; you may not collaborate with others. Be sure to discuss the numerical results of the programming exercises, and please include your code when submitting your exam.

1. **Navier-Stokes** (50 points)
   Solve the 2D Navier-Stokes equations with periodic boundary conditions in a domain $[-\pi, \pi] \times [-\pi, \pi]$. Use $\nu = 1$ and the initial conditions

$$u = 0.5 [\sin(y + x) + \sin(y - x)],$$
$$v = -0.5 [\sin(x + y) + \sin(x - y)].$$

   Compute the solution $u(x, y, t)$ and $v(x, y, t)$ spectrally using a Fourier expansion in both directions, and take $N = 16, 32$ and 64. Use the 2nd-order Adams-Bashforth explicit time advancement scheme, and discuss the timestep Δt required for a stable and time-accurate solution. In addition, discuss the computational cost of your treatment of the nonlinear term. At $t = 0.5, 1.0, 1.5,$ and 2.0, plot contours of the velocities and pressure, as well as the vorticity $\omega = \partial v/\partial x - \partial u/\partial y$. In addition, compute and plot the total energy in the domain as a function of time, and calculate its rate of change.

2. **Eigenvalue problems** (50 points)

   Spectral methods are very good for solving eigenvalue problems and can provide accurate eigenvalues with relatively few grid points when compared to finite difference methods for example. A useful model problem to consider is the eigenvalue problem corresponding to Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{n^2}{x^2}y = -\lambda y \tag{1}$$

   on the domain $0 \leq x \leq 1$ with boundary conditions

$$y(1) = 0 \tag{2}$$
$$y(0) = \text{bounded}. \tag{3}$$

   The eigenvalues, λ, are the squares of the zeros of the Bessel functions of order n, $J_n(x)$.

   The Bessel equation is a good model problem to test out numerical methods for three reasons: i) its eigenvalues are accurately known, ii) it is similar in structure to more complex problems of interest (in fluid mechanics for example), iii) it has a singularity at the origin which appears in applications that use cylindrical coordinate systems.

   In many settings the singularity at the origin can cause numerical difficulties, this is the so-called pole problem. Here we will briefly explore the eigenvalue problem associated with Bessel's equation and some ways the pole problem can be treated in order to yield more accurate eigenvalues.

We will approximate the extension of the function $y(x)$ about $x = 0$ onto the domain $-1 \leq x \leq 1$ by a finite series of Chebyshev polynomials

$$y(x) \approx \sum_{m=0}^{N} a_m T_m(x). \tag{4}$$

(a). Solve the eigenvalue problem defined by Eqn. (1) for $n = 7$ with the boundary condition $y(1) = 0$. Use N =10, 18, 26 and plot the relative error in the smallest eigenvalue, $\lambda_{7,1}$, vs N on a log-log plot. Use the matrix form of the Chebyshev collocation derivative.

- There are multiple ways to enforce the boundary condition. Either you carry the value on the boundary as part of your system or you remove it since it is no-longer an unknown. If you carry the value on the boundary in order to enforce the boundary condition you will need to enforce the corresponding value on the right hand side to be zero. To do this within the framework of an eigenvalue problem you can pose it as a generalized eigenvalue problem of the form $A\vec{x} = \lambda B\vec{x}$ where you can replace one of the rows of matrix B with zeros. The eigenvalue solver in programs such as Matlab can solve this type of eigenvalue problem and take both A and B as inputs.
- $\lambda_{7,1}$ refers to the smallest non-zero eigenvalue of the system and is the square of the first zero of $J_7(x)$. The exact value is $\lambda_{7,1} = 122.907600204$ to 12 significant figures. Note, however, that due to the presence of spurious eigenvalues $\lambda_{7,1}$ may not be the smallest eigenvalue that you find. In that case you may need to just select the eigenvalue closest to the exact $\lambda_{7,1}$ given above.
- Because this method remains in physical space and you will have a grid point at $x = 0$ (for even N), this may create some infinite values in your matrix which might cause the eigenvalue solver to complain. In order to proceed you can add a small number on the order of machine precision to x at the origin which will no-longer produce infinity but will just produce a very large number when you compute $1/x$.

(i). Do you observe convergence?

(ii). Repeat the exercise but with an odd N, e.g. N =11, 19, 27. If you observe convergence what is the rate? Is it exponential?

(b). One way of minimizing the impact the pole has on the eigenvalue problem is to use a different form of the equation. Bessel's equation can be multiplied through by x^2 to produce an alternate form without any $1/x$ or $1/x^2$ terms

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2 y = -\lambda x^2 y \tag{5}$$

which is solved as a generalized eigenvalue problem $A\vec{y} = \lambda B\vec{y}$ with B being the coefficient $-x^2$ in operator form.

(i). Solve the eigenvalue problem defined by Eqn. (5) for $n = 7$ with the boundary condition $y(1) = 0$. Use $N = 10$, 18, 26. Plot the relative error in the smallest eigenvalue vs N on a log-log plot, the exact value is

$$\lambda_{7,1} = 122.907600204 \tag{6}$$

(ii). Comment on what you observe. Does this method converge? If so, is it exponential?

(iii). How does the performance of the two forms of the equations (Eqn. (1) vs Eqn. (5)) compare if you were to repeat this exercise using an odd N?

(c). We now want to see how well finite difference performs compared to the spectral method we have been using. Solve the eigenvalue problem defined by Eqn. (5) for $n = 7$ with the boundary condition $y(1) = 0$ using second order finite difference. How many grid points do you need in order to obtain similar level of accuracy in the estimation of $\lambda_{7,1}$ as was obtained in the previous problem using $N = 10$? (Note that this ratio increases as N increases, try to match the error for $N = 14$ for example). Comment on the implications of this for solving eigenvalue problems.

- You can once again solve it on the domain $-1 \leq x \leq 1$ but with finite difference this time.