Problem Set 7

Due: 4:00 pm Wednesday, March 3

1/4. Solve the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{1}$$

on the periodic domain $0 \leq x < 2\pi$ with initial condition $u(x, t = 0) = 5\sin(x)$ and $\nu = 0.4$. Use the Galerkin method with a Fourier basis as was done in class. Plot the solution, u, as a function of x at four times $t = [0, 0.2, 0.4, 0.6]$.

Hint: Use a fourth-order Runge-Kutta scheme for time advancement. You can solve it with whatever parameters you like but I would recommend using $N = 64$ and a time step of $\Delta t = 0.004$, this should give you a stable solution and you will be able to plot the solution at the 0th, 50th, 100th, and 150th time steps respectively to match the desired times.

Note: Though not required it is quite interesting to play around with the various parameters to see how the solution behaves and how the numerical method performs. e.g.: Adjusting ν so that we form a shock; How how sensitive is the largest stable time step for numerical stability to the choice of N?

2/4. Consider the PDE

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

for $u(x, t)$ on the domain $-1 \leq x \leq 1$ subject to the boundary condition $u(-1, t) = 0$ and the initial condition

$$u(x, 0) = \begin{cases} \sin(2\pi x) + \frac{1}{3}\sin(6\pi x) & \text{if } -1 \leq x \leq -0.5 \\ 0 & \text{if } -0.5 < x \leq 1 \end{cases}$$

In solving this equation using the Tau method, we substitute u by its truncated Chebyshev expansion $u(x, t) = \sum_{n=0}^{N} a_n(t) T_n(x)$, which will give us the system of ODEs

$$\frac{da_n}{dt} = -\frac{2}{c_n} \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{N} p a_p, \qquad n = 0, 1, \cdots, N,$$

where $c_0 = 2$ and $c_n = 1$ for $n > 0$. Solve this equation two ways

(a) Using the Tau method (*i.e.*, drop the last equation and replace it by another equation to account for the boundary condition)

(b) Without considering the boundary condition.

Plot $u(x,t)$ at $t = 0, 0.25, 0.5, 1$. Compare this with the exact solution

$$u(x,t) = \begin{cases} \sin(2\pi(x-t)) + \frac{1}{3}\sin(6\pi(x-t)) & \text{if } -1 \leq x - t \leq -0.5 \\ 0 & \text{if otherwise} \end{cases}$$

and comment on what you observe.

Hint: Use a fourth-order Runge-Kutta scheme for time advancement.

3/4. For the functions $y_1 = \sin(4x) + 0.15\sin(14x)$ and $y_2 = \sin(8x) + 0.2\cos(14x)$, compute the Fourier transform of the product $y_1 y_2$ using $N = 32$ grid points with $0 \leq x < 2\pi$ and discuss the aliasing error.

NOTE: It is not required but if you have time you could also first take the Fourier transform of y_1 and y_2 and then perform the convolution in wave space to get the Fourier coefficients of the product and see that aliasing does not occur.

4/4. Investigate the aliasing properties of the following analytic identity:

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^2}{2}\right)$$

Compute i) $y\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{y^2}{2}\right)$ for the function $y = \sin(4x) + 0.15\sin(14x)$ using $N = 32$ grid points with $0 \leq x < 2\pi$ and show the difference due to aliasing. Perform a de-aliased calculation and check if i) and ii) are the same.