Problem Set 8

Due: 4:00 pm Wednesday, March 10

1/7. Consider the method for solving a Poisson equation using Chebyshev polynomials as discussed in section 7.3 of the class notes (equations 68-70). Let $N = 16$, $\alpha = 0.5$ and $\hat{f} = -\hat{g} = 0.75$. Solve this equation for this right-hand side vector:

$$q = \begin{cases} 2/n^2 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Use both the standard Gaussian elimination (`sge.m`) and the modified Gaussian elimination (`mge.m`). As explained in the notes, the modified Gaussian elimination changes the order of the variables (moving a_0 to the end of the vector of unknowns) to deal with the round-off problem. These routines are available on Canvas under the folder "problem set 8". Use single precision variables. Compare your results with MATLAB's linear solver (e.g. $x = A\backslash b$) (Or the equivalent in whatever language you are using). Provide your solution for the Chebyshev coefficients using all methods and comment on the accuracy of the results and the computational expense, i.e. the number of arithmetic operations required to obtain them.

2/7. Consider the heat equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

for $x \in [-1, 1]$ with the initial condition $u(x, 0) = u_0(x) = 1 - x^8$, the Dirichlet boundary conditions $u(\pm 1, t) = 0$, and $\nu = 0.5$. Compute the solution $u(x, t)$ using the algorithms in sections 7.3-7.4 of the class notes (modified for Dirichlet boundary conditions). Use an N = 16 (N +1 = 17 term) Chebyshev expansion. Plot $u(x, t)$ at several t, e.g. 0, 1/2, 1, 3/2 and 2, on the same graph. Use the trapezoidal rule for time advancement and choose a timestep Δt that provides a time-accurate solution. Justify your choice of Δt. To control round-off error and re-use your code from the previous problem, you may use the modified Gaussian elimination code (`sge.m`) provided on the class web page.

3/7. Solve the differential equation

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0, \quad c(x) = \frac{1}{10} + \frac{1}{2}\sin^2(x - 2)$$

for $x \in [0, 2\pi]$, $t > 0$, with periodic boundary conditions. As an initial condition we take $u(x, 0) = u_0(x) = e^{-100(x-2)^2}$. Use both Fourier collocation and second order centered finite differences. For time advancement, use fourth-order Runge-Kutta and comment on the stability limit of this scheme for the spectral and finite-difference solutions. Notice that although $u_0(x)$ is not periodic, the function and its derivatives are so small at the boundaries that it can be treated as periodic for this problem. In your write-up, include a 3D plot of $u(x, t)$ for $x \in [0, 2\pi]$ and $t \in [0, 16]$. Notice that the solution is the function depicted in the cover of Trefethen's Spectral Methods in MATLAB.

NOTE: Make sure to comment on whether or not you see dispersive errors and if this goes away with grid or timestep refinement.

4/7. (3.10 Trefethen) The solution of Problem 3 is periodic in time: for a certain $T \sim 26$, $u(x, T) = u(x, 0)$ for all x. Determine T analytically by evaluating an appropriate integral. Then modify your code for Problem 3 to compute $u(x, T)$ instead of $u(x, 16)$. (Make sure t stops exactly at T, not at some nearby number determined by round.)

For $N = 32, 64, \cdots, 256$, determine

$$\max_j |u(x_j, T) - u(x_j, 0)|,$$

and plot this error on a log-log scale as a function of N. What is the rate of convergence? How could it be improved?