Problem Set 9

Due: 4:00 pm Wednesday, March 17

1/2. Let's consider the 2 dimensional heat equation with a forcing term

$$\frac{\partial u}{\partial t} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + S(x, y)$$

for $x, y \in [-1, 1]$ with periodic boundary conditions in both directions. Use the initial condition

$$u(x, y, 0) = u_0(x, y) = \left(1 - x^2\right)^4 \left(1 - y^2\right)^4,$$

the source term

$$S(x, y) = 100 \exp\left[-(x^2 - 1)^6 - (y^2 - 1)^6\right] - 60,$$

and $\nu = 100$. Compute the solution $u(x, y, t)$ using Fourier expansions in both directions. Take $N = 16$ in both directions. Use the 4th-order Runge-Kutta scheme, with and without the integrating factor trick discussed in class, for explicit time advancement. For both cases, discuss the timestep Δt required for a stable and time-accurate solution. Plot slices of the solution along $x = 0$, $y = 0$ and $x = y$ at $t = 2.5 \times 10^{-4}, 5.0 \times 10^{-4}, 7.5 \times 10^{-4}, 1.0 \times 10^{-3}$, and 1.0×10^{-2}. Does the integrating factor approach help in gaining higher accuracy with larger time steps?

2/2. Consider the following partial differential equation

$$\frac{\partial u}{\partial t} + 3u\frac{\partial u}{\partial x} + \frac{1}{4}\frac{\partial^3 u}{\partial x^3} = 0$$

for $x \in [-3, 3]$ with periodic boundary conditions. Use the initial condition

$$u = \frac{64 \exp(-12 - 8x)}{\left[1 + \exp(-12 - 8x)\right]^2} + \frac{4\pi^2 \exp(\pi - 2\pi x)}{\left[1 + \exp(\pi - 2\pi x)\right]^2}.$$

The initial condition describes a superposition of two solitons—wavepackets that maintain their shape as they move across the domain at a constant speed. Compute the solution $u(x, t)$ spectrally using a Fourier expansion in x, and take $N = 64, 128$ and 256. Use the 4th-order Runge-Kutta scheme, with and without the integrating factor trick discussed in class, for explicit time advancement. Perform the integrating factor trick on only the linear term. For both cases and for each N, discuss the timestep Δt required for a stable and time-accurate solution. Plot the solution at $t = 0.1, 0.2, 0.3$, and 0.4.