

A GENERAL SUBSTRUCTURE SYNTHESIS METHOD FOR THE DYNAMIC SIMULATION OF COMPLEX STRUCTURES[†]

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In this paper a general substructure synthesis method is developed for the dynamic analysis of complex flexible structures. The motion of each substructure is represented by a given number of substructure admissible functions. Substructure admissible functions are often low-order polynomials, and hence computationally easy to work with. The otherwise disjoint substructures are connected together to form a whole structure by imposing approximate geometric compatibility conditions by means of the method of weighted residuals. The behavior of the estimated eigenvalues obtained by the substructure synthesis method can be ascertained by means of a bracketing theorem. The estimated eigenvalues do converge to the actual eigenvalues of the original structure, although it is necessary to consider two limiting processes, one in which the number of substructure admissible functions is increased and the other in which the number of internal boundary weighting functions is increased.

1. INTRODUCTION

This paper is concerned with the dynamic simulation of complex flexible structures. The interest is primarily in obtaining accurate estimates of the lower natural frequencies and associated natural modes of vibration of the structure. One way to model complex structures is by the finite element method [1]. Whereas the finite element method lends itself to modeling very complex structures, it has the major disadvantage that it often requires a very large number of degrees of freedom to obtain accurate estimates of the lower natural frequencies and associated natural modes, where the number can reach into the tens of thousands. Because it is impractical to work with such a large number of degrees of freedom, methods to reduce the number of degrees of freedom are desirable. Quite often, the complex structure can be conveniently regarded as an assemblage of a few simpler substructures. In this case, it is possible to construct a mathematical model containing a substantially reduced number of degrees of freedom as compared to the finite element method. To this end, one can distinguish among four somewhat similar methods: namely, (1) component mode synthesis [2, 3], (2) branch mode analysis [4], (3) component mode substitution [5], and (4) coupled free-free component modes [6, 7]. In all of these methods one represents each substructure by a given number of substructure modes. The number of modes used for each substructure affects the accuracy of the estimated lower natural frequencies and natural modes. However, experience has shown that accurate estimates may be obtained by using a small number of component modes. The methods differ in the type of substructure modes used. Boundary conditions at an internal boundary between substructures are specified to be fixed, free, or to consist of inertial and/or stiffness loading. In each of these methods it is necessary to produce first a suitable set of substructure modes

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by solving an eigenvalue problem for each substructure or, in the case of the branch mode analysis, for each branch. Quite often, however, this is no easy matter and, in most cases, substructure eigenvalue problems cannot be defined uniquely. Moreover, even if one can obtain substructure eigensolutions they are often very difficult to work with computationally.

In the papers on substructure modes mentioned above it is hinted that suitable modes might be found without solving substructure eigenvalue problems, but the idea is never developed. It is pointed out in references [8] and [9] that for distributed parameter substructures, in the spirit of Rayleigh-Ritz, one need use only admissible functions and that substructure modes are not really necessary, although substructure modes are certainly suitable admissible functions. Of course, the advantages of admissible functions are that they are appreciably more plentiful and that they are often low-order polynomials, and hence easy to work with.

In addition to the problem of representing each substructure, there is the problem of coupling together adjacent substructures to form a whole structure. The coupling process requires the satisfaction of certain geometric compatibility conditions at every point of an internal boundary between two adjacent substructures. In references [2-9] an internal boundary is considered to consist of a finite set of points. This may not always be the case. An internal boundary between two adjacent substructures might consist of a line or a surface: i.e., of an infinite number of points. In this case, it is reasonable to select a finite number of points along the internal boundary and require that the geometric compatibility conditions be satisfied at each one of these points. Because the satisfaction of the geometric compatibility conditions at a finite set of points does not ensure their satisfaction at every point of the infinite number of points along the internal boundary, such a procedure is only an approximate one. Of course, the nature of the approximation affects the estimated dynamic characteristics of the structure. This problem has not been addressed in the literature.

This paper is concerned with the dynamic simulation of complex structures consisting of an assemblage of substructures. For the sake of developing a general method, the substructures are represented by distributed parameter mathematical models. A general substructure synthesis method in which the motion of each substructure is represented by a given number of substructure admissible functions is developed. Substructure modes, if they can be found, may be used as substructure admissible functions but they are not really necessary. The otherwise disjoint substructures are connected together to form a whole structure by imposing approximate geometric compatibility conditions by means of the method of weighted residuals. A structure that is no longer disjoint but whose internal boundary conditions are only approximations to the actual ones is referred to as an "intermediate structure". The intermediate structure represents a mathematical concept defined by the type of weighting functions used and their number. In the limit, as the number of weighting functions is increased, the intermediate structure will approach the original structure. Note that if the weighting functions are chosen in the form of spatial Dirac delta functions then the net effect is to satisfy the compatibility conditions at a finite set of points.

The approximate eigensolution obtained by using the substructure synthesis method is discussed. A bracketing theorem is presented and used to show the nature of convergence of the approximate eigenvalues of an intermediate structure to the eigenvalues of the original structure. Two limits are considered, one in which the number of substructure admissible functions is increased and the other in which the number of internal boundary weighting functions is increased. A numerical example is given which illustrates the two convergence processes. The substructure synthesis method of this paper should be of both theoretical and practical significance in structural dynamics.

2. SUBSTRUCTURE KINETIC AND POTENTIAL ENERGY

Assume that a given structure is divided into a number of substructures and consider a certain substructure s . For the moment, one may assume that substructure s satisfies any geometric boundary conditions imposed on the whole structure but that it acts independently of all other substructures. Coupling of the substructures to act as a whole will be discussed in the next section. The substructure is regarded as having distributed mass and stiffness. Then, denoting by $\mathbf{u}_s(P, t)$ the displacement vector of any point on the substructure relative to an inertial space, where P is the nominal position of the point in question, one can write the kinetic energy of substructure s in the symbolic form

$$T_s = \frac{1}{2} \int_{D_s} \rho \dot{\mathbf{u}}_s^T \dot{\mathbf{u}}_s dD_s = \frac{1}{2} (\sqrt{\rho \dot{\mathbf{u}}_s}, \sqrt{\rho \dot{\mathbf{u}}_s}), \quad (1)$$

where $(\sqrt{\rho \dot{\mathbf{u}}_s}, \sqrt{\rho \dot{\mathbf{u}}_s})$ is recognized as the inner product of $\sqrt{\rho \dot{\mathbf{u}}_s}$ with itself, in which ρ represents the mass distribution. Similarly, the potential energy may be written in the symbolic form

$$V_s = \frac{1}{2} [\mathbf{u}_s, \mathbf{u}_s], \quad (2)$$

where $[\mathbf{u}_s, \mathbf{u}_s]$ can be identified as the energy inner product [10] of the vector \mathbf{u}_s with itself. The energy inner product involves derivatives of \mathbf{u}_s with respect to the spatial variables through order p , as well as integration over the domain D_s . One obtains the energy inner product for substructure s by considering the substructure to be separate from all other substructures. Note that, whereas one may write the potential energy in the form of an energy inner product for all types of structural members, the form of the energy inner product can vary from one type of structural member to another. For simplicity, the discussion may be restricted to structural elements for which $p \leq 2$.

In the spirit of the Rayleigh-Ritz method, one may assume that the displacement $\mathbf{u}_s(P, t)$ can be approximated by a linear combination of space-dependent admissible functions multiplied by time-dependent generalized co-ordinates,

$$\mathbf{u}_s(P, t) = \sum_{i=1}^{N_s} \phi_{s_i}(P) \zeta_{s_i}(t), \quad (3)$$

where ϕ_{s_i} ($i = 1, 2, \dots, N_s$) are vector admissible functions from a complete set. Equation (3) can be conveniently written in the matrix form

$$\mathbf{u}_s(p, t) = \Phi_s(P) \zeta_s(t), \quad (4)$$

where Φ_s is a $3 \times N_s$ matrix of admissible functions and ζ_s is an N_s -vector. Note that admissible functions must satisfy only the geometric boundary conditions of the disjoint substructure s . In theory, the dimension of ζ_s approaches infinity, but for practical reasons it must be taken as finite.

Introducing equation (4) into equation (1), one obtains

$$T_s = \frac{1}{2} \zeta_s^T M_s \zeta_s, \quad (5)$$

where

$$M_s = (\sqrt{\rho \Phi_s}, \sqrt{\rho \Phi_s}) \quad (6)$$

is an $N_s \times N_s$ matrix, which can be easily identified as the substructure mass matrix. Similarly, introducing equation (4) into equation (2), one obtains

$$V_s = \frac{1}{2} \zeta_s^T K_s \zeta_s, \quad (7)$$

where

$$K_s = [\Phi_s, \Phi_s] \quad (8)$$

is an $N_s \times N_s$ matrix, recognized as the *substructure stiffness matrix*. The process of transforming the kinetic and potential energies in their distributed forms, equations (1) and (2), to their discrete forms, equations (5) and (7), is known as *discretization*.

3. THE EIGENVALUE PROBLEM FOR THE ASSEMBLED STRUCTURE

If the structure consists of m substructures, $s = 1, 2, \dots, m$, and if the substructures are assumed for the time being to act independently of one another, then the combined kinetic energy is

$$T = \sum_{s=1}^m T_s = \frac{1}{2} \zeta_d^T M_d \zeta_d, \quad (9)$$

where

$$\zeta_d = [\zeta_1^T | \zeta_2^T | \dots | \zeta_m^T]^T \quad (10)$$

is the N -dimensional *disjoint configuration vector*, $N = \sum_{s=1}^m N_s$, and M_d is the $N \times N$ block-diagonal matrix

$$M_d = \text{block-diag } M_s, \quad s = 1, 2, \dots, m. \quad (11)$$

Similarly, the combined potential energy can be written as

$$V = \sum_{s=1}^m V_s = \frac{1}{2} \zeta_d^T K_d \zeta_d, \quad (12)$$

where K_d is the $N \times N$ block-diagonal matrix

$$K_d = \text{block-diag } K_s, \quad s = 1, 2, \dots, m. \quad (13)$$

Letting

$$\zeta_d(t) = \zeta_d f(t) \quad (14)$$

where ζ_d is a constant vector and $f(t)$ a harmonic function of time, one can write the disjoint Rayleigh quotient as

$$R_d = \zeta_d^T K_d \zeta_d / \zeta_d^T M_d \zeta_d. \quad (15)$$

In reality, however, the substructures do not act independently and they are subject to forces exerted by one substructure upon another at boundaries shared by two or more substructures. In this context it is convenient to distinguish between external and internal boundaries. External boundaries are the physical substructure boundaries that coincide with the boundary of the whole structure, whereas internal boundaries are fictitious boundaries between any two adjacent substructures. They are fictitious in the sense that they are the result of imagining the structure as being subdivided into substructures, which quite often is an arbitrary process. Of course, there can be substructures with only internal boundaries.

At every point of a boundary there are two types of boundary conditions, geometric and natural. Geometric boundary conditions express geometric compatibility conditions, such as conditions on displacements and displacement gradients. On the other hand, natural boundary conditions express force and moment balance. Consistent with the Rayleigh-Ritz method, at external boundaries the admissible functions need satisfy only the geometric

boundary conditions, as natural boundary conditions are accounted for automatically in the energy expressions [10]. At internal boundaries, the forces and moments on each side of a boundary balance out to zero or combine to yield terms contributing to the overall potential energy. The latter is true when two adjacent substructures are connected through elements capable of storing potential energy, such as elastic restraints. On the other hand, geometric compatibility implies that the displacements and derivatives of the displacements normal to the boundary through order $p - 1$ on each side of an internal boundary must be such that the structural integrity is preserved. For example, considering two adjacent substructures r and s with $p = 2$, one must have

$$\mathbf{u}_s = \mathbf{u}_r, \quad \partial\mathbf{u}_s/\partial\mathbf{n}_s = \partial\mathbf{u}_r/\partial\mathbf{n}_r, \quad r, s = 1, 2, \dots, m, \quad s \neq r, \quad (16a, b)$$

at every point of the internal boundary between substructures r and s . In equation (16b), $\partial\mathbf{u}_s/\partial\mathbf{n}_s$ denotes the partial derivative of the displacement vector with respect to the outward normal of the internal boundary on substructure s . Of course, if $p = 1$ one needs to satisfy only equation (16a) at every point of the internal boundary.

Clearly, for a general boundary and general sets of substructure admissible functions, equations (16) can be satisfied only approximately. This can be done conveniently by the method of weighted residuals [10], according to which certain weighted averages are set equal to zero. Note that if the weighting functions are chosen in the form of spatial Dirac delta functions, then the net effect is to satisfy equations (16) at discrete points of the internal boundary. The subject of internal boundary conditions is discussed in the next section.

Equations (16) can be used to connect the otherwise disjoint substructures. Recalling equation (3) and assuming that the compatibility conditions (16) are satisfied approximately by the application of the weighted residuals method, one obtains certain relations between the components of the disjoint vector ζ_d . Suppose, for example, that there are M_{rs} such constraint equations connecting adjacent substructures r and s . Then, with M_c denoting the total number of constraint equations, it follows that the system has only $n = N - M_c$ degrees of freedom. Letting ζ be the n -dimensional vector of independent generalized coordinates, one can write the relation between the disjoint vector ζ_d and the unconstrained vector ζ in the form

$$\zeta_d = C\zeta, \quad (17)$$

where C is an $N \times n$ rectangular matrix. Introducing equation (17) into equation (15) one obtains the Rayleigh quotient

$$R = \zeta^T K \zeta / \zeta^T M \zeta, \quad (18)$$

where

$$K = C^T K_d C, \quad M = C^T M_d C \quad (19a, b)$$

are the $n \times n$ stiffness and mass matrices for the assembled structure.

The process of rendering R stationary yields the eigenvalue problem

$$KU = MU\Lambda, \quad (20)$$

where U is the modal matrix and Λ is the diagonal matrix of the eigenvalues for the assembled structure.

4. THE INTERNAL BOUNDARY CONDITIONS

Suppose now that it is possible to satisfy the compatibility conditions (16) exactly, i.e., at every point of the internal boundary. One can then use conditions (16) to assemble a set of functions consisting of functions which are admissible for the entire structure. Such functions may be called global admissible functions. If one takes $p = 2$, the global admissible functions are twice differentiable over the entire structure and satisfy the external geometric boundary conditions. Within a substructure s , global admissible functions consist of linear combinations of the admissible functions $\phi_{s_i}(P)$ used for substructure s . The particular linear combination within substructure s is determined by the way in which the compatibility conditions are used to connect functions in one substructure with functions in an adjacent substructure and is expressed implicitly by equation (17). Of course, independently of the way in which the compatibility conditions are used, one generates a set of global admissible functions spanning the same function space. Note that this function space is defined by the admissible functions ϕ_s for each substructure s and the compatibility conditions (16). Note also that global admissible functions are never generated explicitly, nor is there any need for such explicit expressions. They are, however, generated implicitly by the process used in section 3 to connect the substructures together.

To demonstrate that global admissible functions are generated implicitly, consider the transverse vibration of a uniform slender beam of length l which is clamped at both ends. Let the domain $0 \leq x < l/2$ be substructure 1 and the domain $l/2 \leq x \leq l$ be substructure 2. The internal boundary between the substructures consists of the single point $x = l/2$. The geometric compatibility conditions which must be satisfied at the point of connection $x = l/2$ are

$$u_1(l/2) = u_2(l/2), \quad \partial u_1 / \partial x|_{x=l/2} = \partial u_2 / \partial x|_{x=l/2}, \quad (21a, b)$$

where $u_i(x)$ ($i = 1, 2$) is the transverse displacement of the beam. For simplicity, let the single admissible function

$$\phi_{11} = x^2, \quad 0 \leq x \leq l/2, \quad (22)$$

represent substructure 1 and the two admissible functions

$$\phi_{21} = (l - x)^2, \quad \phi_{22} = (l - x)^3, \quad l/2 \leq x \leq l, \quad (23a, b)$$

represent substructure 2. Note that $N_1 = 1$ and $N_2 = 2$ in equation (3). In view of equations (22) and (23) the disjoint mass and stiffness matrices are found to be

$$M_d = \frac{\rho l^5}{32} \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & l/12 \\ 0 & l/12 & l^2/28 \end{bmatrix}, \quad K_d = lEI \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3l/2 \\ 0 & 3l/2 & 3l^2/2 \end{bmatrix}, \quad (24a, b)$$

where ρ is the mass per unit length and EI is the bending stiffness of the beam. By substituting equation (3) into equations (21) and using equations (22) and (23), the constraint matrix C in equation (17) can be obtained in the form

$$C = [1 \quad 5 \quad -8/l]^T. \quad (25)$$

Finally, substituting equations (24) and (25) into equations (19) gives the 1×1 mass and stiffness matrices for the assembled structure as

$$M = (63/1680)\rho l^5, \quad K = 28 lEI. \quad (26a, b)$$

Hence, for this example, the substructure synthesis method produces the eigenvalue problem (20) with M and K as in equations (26). The same eigenvalue problem (20), which is obtained by the substructure synthesis method, can also be obtained by introducing the single global admissible function

$$\phi_1(x) = \begin{cases} x^2, & 0 \leq x \leq l/2 \\ 5(l-x)^2 - (8/l)(l-x)^3, & l/2 < x \leq l \end{cases} \quad (27)$$

in the eigenvalue problem [10]

$$\sum_{j=1}^n \{ [\phi_i, \phi_j] - \Lambda^n (\sqrt{\rho} \phi_i, \sqrt{\rho} \phi_j) \} a_i = 0, \quad i = 1, 2, \dots, n, \quad (28)$$

where Λ^n denotes the estimated eigenvalue of the whole structure, the functions ϕ_i ($i = 1, 2, \dots, n$) are global admissible functions, ρ is the mass density at any point in the structure, and the inner products involve integrations over the domain of the whole structure. Note that $\phi_1(x)$ is two times differentiable everywhere in the interval $0 \leq x \leq l$ and satisfies the geometric boundary conditions for the beam.

The implication of the above developments is that the substructure synthesis method is equivalent to a Rayleigh-Ritz method provided that the compatibility conditions between substructures are satisfied exactly. It is obvious that when the substructures are one-dimensional it is particularly easy to satisfy the compatibility conditions (16). Moreover, when the internal boundary consists of a finite number of points, the process of satisfying the compatibility conditions is also well defined. However, when the internal boundary consists of a line or a surface, the situation is more complicated. It may still be possible to satisfy the compatibility conditions at every point of the internal boundary. One way is to use very special sets of substructure admissible functions which can be easily matched on the internal boundary. Such admissible functions are known as conformable in the finite element theory. But the commonly used finite elements generally have much simpler geometric shapes than arbitrary substructures. Hence, whereas conformable admissible functions may exist in theory, in practice they are difficult to obtain for arbitrary substructures.

Consider again compatibility conditions (16), which must be satisfied at an infinite number of points along the internal boundary. To eliminate the spatial dependence, one can multiply equations (16) by a function g_{rsi} and integrate the results over the internal boundary S_{rs} shared by substructures r and s to obtain

$$\int_{S_r} g_{rsi} (\mathbf{u}_s - \mathbf{u}_r) dS_{rs} = \mathbf{0}, \quad r, s = 1, 2, \dots, m, \quad s \neq r, \quad (29a)$$

$$\int_{S_{rs}} g_{rsi} \left(\frac{\partial \mathbf{u}_s}{\partial \mathbf{n}_s} - \frac{\partial \mathbf{u}_r}{\partial \mathbf{n}_r} \right) dS_{rs} = \mathbf{0}, \quad r, s = 1, 2, \dots, m. \quad s \neq r. \quad (29b)$$

Note that equations (16) as well as equations (29) are vector equations. In general, one requires that equations (29) hold for all functions g_{rsi} . However, one can approximate the compatibility conditions (16) by requiring that equations (29) hold for only a finite number M_{rs} of functions g_{rsi} ($i = 1, 2, \dots, M_{rs}$). This procedure is essentially the weighted residuals method, where the functions g_{rsi} are weighting functions. As mentioned earlier, if the weighting functions are chosen in the form of spatial Dirac delta functions, then the net effect is to satisfy equations (16) at a discrete number of points. Moreover, as the number of weighting functions M_{rs} is increased to infinity, one expects that equations (16) will be satisfied at every point provided the weighting functions satisfy certain criteria to be discussed later in this paper.

Now consider the consequences of satisfying the geometric compatibility conditions only approximately. It is clear that if conditions (16) are not required to be satisfied at every point of the internal boundary, then it may be possible for discontinuities in the displacement and slope to occur on the internal boundary. In other words, the global functions which are generated implicitly via the approximate compatibility conditions may not be global admissible functions for the entire structure. Hence, one proposition of the Rayleigh-Ritz method for the original structure is violated along the internal boundaries: namely, that the admissible functions be p times differentiable in the domain of the entire structure. This violation is not as serious as it may seem at first.

For fixed numbers M_{rs} and particular sets of weighting functions g_{rsi} , a fictitious structure is defined which is subject only to approximate geometric compatibility conditions. This new structure is called an *intermediate structure* because it lies between the disjoint structure and the original structure. If the weighting functions g_{rsi} are spatial Dirac delta functions, then the intermediate structure has a simple physical form. It consists of the m substructures joined together at finite sets of points. The Rayleigh-Ritz method is valid for the intermediate structure. Hence, *the eigenvalues and eigenvectors obtained from application of the substructure synthesis method converge to the eigensolution of the intermediate structure.*

The intermediate structure is intimately related to the original structure. For practical reasons, one chooses M_{rs} to be finite. However, if M_{rs} approaches infinity, then under certain circumstances the eigensolution of the intermediate structure approaches the eigensolution of the original structure. In choosing a finite number of weighting functions, two limiting processes must now be considered: the convergence of the Rayleigh-Ritz method for a particular intermediate structure and the convergence of the intermediate structure to the original structure. However, estimates of the eigenvalues and eigenvectors of the original problem are still obtained by the substructure synthesis method.

5. THE FIRST MONOTONICITY PRINCIPLE AND THE BRACKETING THEOREM

One can now examine the qualitative behavior of the eigenvalues obtained from the substructure synthesis method. To this end, the following theorems [11] are relevant.

Theorem 1 (First monotonicity principle). Let V_2 be a Hilbert space with the norm $|u| = [u, u]^{1/2}$ and let (u, u) be a bounded quadratic functional on V_2 . Let V_1 be a subspace of V_2 . If $\Lambda_1^{(2)} \leq \Lambda_2^{(2)} \leq \dots$ are the stationary values of the Rayleigh quotient $[u, u]/(u, u)$ on V_2 and $\Lambda_1^{(1)} \leq \Lambda_2^{(1)} \leq \dots$ are the stationary values of the same Rayleigh quotient with u restricted to lie in V_1 , then

$$\Lambda_n^{(2)} \leq \Lambda_n^{(1)}, \quad n = 1, 2, \dots \quad (30)$$

Theorem 2 (Bracketing theorem). If there are linear functionals l_1, l_2, \dots, l_r on V_2 so that $V_1 = \{u \in V_2 | l_1(u) = \dots = l_r(u) = 0\}$ then

$$\Lambda_n^{(2)} \leq \Lambda_n^{(1)} \leq \Lambda_{n+r}^{(2)}, \quad n = 1, 2, \dots \quad (31)$$

Theorems 1 and 2 place the monotonicity and bracketing properties of eigenvalues in a considerably more general setting than that of the Rayleigh-Ritz method. The dimension of the space V_1 is not required to be finite and it may be infinite. One may take V_2 to be the space of functions defined over the domain of the entire structure which are admissible within each substructure and which may be discontinuous on the internal boundaries between substructures. Correspondingly, one may take the energy norm on this infinite

dimensional space of admissible functions to be given by

$$|\mathbf{u}|^2 = \sum_{i=1}^m [\mathbf{u}_i, \mathbf{u}_i], \quad (32)$$

where $[\mathbf{u}_s, \mathbf{u}_s]$ is the energy inner product for substructure s . In addition, the functional (\mathbf{u}, \mathbf{u}) is considered to be given by

$$(\mathbf{u}, \mathbf{u}) = \sum_{i=1}^m (\sqrt{\rho_i} u_i, \sqrt{\rho_i} u_i). \quad (33)$$

Then, the quotient $|\mathbf{u}|^2 / (\mathbf{u}, \mathbf{u})$ is seen to be the Rayleigh quotient for the disjoint structure. Note that the integrations implied by equations (32) and (33) exclude the internal boundaries. The disjoint structure is assumed to be positive definite. The space V_2 is spanned by functions which are both infinite in number and discontinuous along internal boundaries. An intermediate structure is defined by imposing M_c compatibility conditions along the internal boundaries. One takes as a subspace V_1 of V_2 the space of functions which are admissible within each substructure and which satisfy the M_c compatibility conditions. Note that this subspace V_1 is also an infinite-dimensional space. Theorem 1 then assures that the eigenvalues of the disjoint structure are less than or equal to the respective eigenvalues of the intermediate structure.

Other subspaces are also of interest in the substructure synthesis method. They are obtained by restricting the dimension of the function space to be finite as well as by imposing compatibility conditions along the internal boundaries. One can regard an n -degree-of-freedom system as the result of imposing two types of constraints on the infinite-dimensional disjoint structure. First, an N -degree-of-freedom disjoint system is obtained by imposing the constraints

$$\zeta_{s, N_s+1} = \zeta_{s, N_s+2} = \dots = 0, \quad s = 1, 2, \dots, m. \quad (34)$$

Then, the $n = N - M_c$ degree of freedom system is obtained by imposing a total of M_c constraints in the form of compatibility conditions (29) on the disjoint system. As in the Rayleigh-Ritz method, the imposition of constraints has a tendency to raise the estimated eigenvalues. But here the estimated eigenvalues provide upper bounds for the true eigenvalues of the *disjoint structure* and not of the original structure. The estimated eigenvalues do, however, approximate the eigenvalues of the original structure although they do not provide upper bounds.

It is consequently of interest to consider how the estimated eigenvalues behave in the limit. By adding a term to the series (3) for a substructure s , the number of constraints imposed on the system is reduced by one or the dimension of the system is increased by one. Let V_2 be the $n + 1 = N + 1 - M_c$ dimensional space obtained by reducing the number of constraints (34) and V_1 be the $n = N - M_c$ dimensional space. Then, by Theorem 2,

$$\Lambda_1^{(2)} \leq \Lambda_1^{(1)} \leq \Lambda_2^{(2)} \leq \dots \leq \Lambda_n^{(2)} \leq \Lambda_n^{(1)} \leq \Lambda_{n+1}^{(2)}, \quad (35)$$

which is equivalent to the classical inclusion principle for vibrating self-adjoint systems [10]. Equation (35) permits one to conclude that the estimated eigenvalues tend to decrease with each additional degree of freedom obtained by adding a term to the series (3). Next, consider the effect of increasing the number of compatibility conditions from M_c to $M_c + 1$. Now let V_2 be the $n = N - M_c$ dimensional space and V_1 be the $n - 1 = N - 1 - M_c$ dimensional space obtained by increasing the number of compatibility conditions. One again finds that equation (35) holds, from which one concludes that the estimated eigenvalues tend to increase as one decreases the number of degrees of freedom by adding com-

patibility conditions. It is observed that adding terms to the series (3) tends to decrease the estimated eigenvalues while adding compatibility conditions tends to increase the estimated eigenvalues. In the limit, for a fixed number M_c of compatibility conditions, one decreases the estimated eigenvalues as far as possible by taking an infinite series for each substructure s : i.e., taking $N_s = \infty$ ($s = 1, 2, \dots, m$). (This is equivalent to applying the Rayleigh-Ritz method to the intermediate structure defined as the disjoint structure with the M_c compatibility conditions.) Then one increases the number of compatibility conditions M_c to infinity causing the estimated eigenvalues to approach the true eigenvalues of the original structure from below. Therefore, whereas in the Rayleigh-Ritz method one wishes to find the minima that the stationary values of Rayleigh's quotient can take, in the substructure synthesis method one wishes to find the maxima of these minima.

Note that the bracketing theorem will yield inequalities similar to inequalities (35) when V_1 is obtained by imposing more than one constraint. This is particularly useful for monitoring convergence of the estimated eigenvalues obtained by using the substructure synthesis method. It is convenient to add one term, or several terms, to the series (3) for each substructure thereby increasing the number of degrees of freedom from n to at least $n + m$. Also, it is convenient to use the same weighting functions g_{rsi} ($i = 1, 2, \dots, M_{rs}$) at all similar internal boundaries and to increase their number M_{rs} by one. This has the effect of reducing the number of degrees of freedom by p times the number of similar internal boundaries. The bracketing of the estimated eigenvalues by using Theorem 2 with more than one constraint will be demonstrated in the numerical example.

Implicit in this discussion is the need to approximate a distributed parameter structure by a discrete one. The distributed parameter structure is one that, either by convenience or necessity, is divided into a number of distributed parameter substructures. Of course, the object is to obtain as accurate an approximation to the lowest eigenvalues and eigenfunctions as possible, while using only a small number N_s of functions in equation (3) and a small number M_{rs} of weighting functions in equations (29). The question remains as to how to select the admissible functions $\phi_{si}(P)$ ($i = 1, 2, \dots, N_s$) and weighting functions g_{rsi} ($i = 1, 2, \dots, M_{rs}$).

6. ON THE SELECTION OF ADMISSIBLE FUNCTIONS AND WEIGHTING FUNCTIONS

The first thing that comes to mind is to select the substructure admissible functions as the eigenfunctions of the various substructures. This approach has three drawbacks: (1) the solution of the substructure eigenvalue problem may not be so easy to obtain, (2) the substructure eigenfunctions (if they are obtainable) may not be easy to work with, and (3) an eigenvalue problem for the substructure cannot in general be defined uniquely. For instance, one might consider the substructure eigenvalue problem to be either free or fixed at the internal boundaries. Fortunately, whereas substructure eigenfunctions (if they can be found) would make a suitable set of admissible functions, there are other sets of functions that can yield equally good results, provided the functions satisfy certain criteria as discussed below.

As discussed in section 4, the representation of substructure elastic motion in terms of admissible functions and the connection of otherwise disjoint substructures via compatibility conditions is equivalent to a Rayleigh-Ritz procedure for the entire structure, provided the compatibility conditions are satisfied exactly. If the compatibility conditions are satisfied only approximately, then a Rayleigh-Ritz procedure for an intermediate structure is obtained. A set of global admissible functions is generated implicitly. For convergence, in the case of a particular intermediate structure, the global admissible functions must be such that they

satisfy certain criteria for the intermediate structure. The following criteria are proved in reference [12] to be sufficient for convergence for a particular closed domain \bar{D} :

- (1) any finite number of admissible functions must be linearly independent;
- (2) the set of admissible functions must be complete in the energy space of the domain \bar{D} .

For these criteria to be satisfied by global admissible functions for a particular intermediate structure, it is sufficient that the same criteria be satisfied by the set of admissible functions for the closed domain \bar{D}_s of each substructure s ($s = 1, 2, \dots, m$). A proof of this statement can be obtained by considering a Cauchy sequence of global admissible functions for the intermediate structure. Note that functions in the energy space of a substructure are not required to satisfy any boundary conditions on the internal boundaries of the substructure.

The second of these criteria applied to a substructure requires that one be able to identify a set of admissible functions that is complete in the energy space for the substructure. To this end, the following statement is useful: if A and B are positive definite operators, and H_A and H_B are the corresponding energy spaces, and if the space H_A and H_B contain the same elements, then any set that is complete in H_B is complete in H_A . Therefore, by letting B have a simple form for which a complete set is easily found, one can use that set as admissible functions for the operator A . This justifies the earlier statement that it is not necessary to use substructure eigenfunctions. Indeed, for an involved operator A it should be possible to find a very simple operator B such that H_A and H_B contain the same elements and a complete set of admissible functions for H_A is found readily without any of the complexities inherent in working with the operator A .

The question remains as to how to choose the weighting functions g_{rsi} ($i = 1, 2, \dots, M_{rs}$) in equations (29). It is obvious that they are required to be defined and integrable on the internal boundary between substructures r and s . To satisfy the geometric compatibility conditions (16) exactly, one requires that conditions (29) hold for an infinity of weighting functions g_{rsi} . For practical reasons one must take the number M_{rs} of functions to be finite. However, for conditions (16) to be satisfied exactly in the limit, it is sufficient to take the weighting functions g_{rsi} to be from a complete set of integrable functions defined on the internal boundary. If the weighting functions are not from a complete set of functions, then the intermediate structures do not necessarily converge to the original structure although approximations to the original structure may still be obtained. The use of spatial Dirac delta weighting functions can be justified by the fact that it yields an intermediate structure with a simple physical meaning. In addition, one requires that for a particular r and s any finite number of compatibility conditions (29) must be linearly independent. Moreover, for computational purposes, one must be able to solve the M_{rs} linearly independent compatibility conditions uniquely for M_{rs} generalized co-ordinates of the structure in terms of the remaining generalized co-ordinates.

In the above discussion nothing has been said about the rate of convergence of approximate eigenvalues and eigenfunctions for particular sets of substructure admissible functions and weighting functions satisfying the above criteria. In fact, the approximate eigensolution will not necessarily converge very rapidly to the true solution. However, in practice, one is concerned with finding only the lower eigenvalues and eigenfunctions for the entire structure. Because eigenfunctions corresponding to lower eigenvalues will generally be smoother than eigenfunctions corresponding to higher eigenvalues, relatively simple substructure admissible functions should yield rapid convergence to the lower eigenvalues and associated eigenfunctions for many problems. Moreover, because the displacement associated with the lower eigenfunctions along internal boundaries will also be smooth it is expected that a small set of relatively simple weighting functions will be sufficient. Therefore, the lack of analytical knowledge about the rate of convergence of the approximate eigensolution is not a deterrent to the use of the method. Indeed, by comparing the approximate lower eigen-

values and eigenfunctions obtained by using several different choices of substructure admissible functions and/or weighting functions, one should be able to find substructure admissible functions and weighting functions yielding good convergence.

7. NUMERICAL EXAMPLE

To provide a numerical example, consider a thin rectangular membrane extending over a closed domain \bar{D} defined by $0 \leq x \leq 9$ and $0 \leq y \leq 5$. The external boundary of the domain consists of the straight lines $x = 0.9$ and $y = 0.5$ and the membrane is assumed to be clamped at these boundaries. In its equilibrium position the membrane lies entirely in the xy plane under a uniform tension T . The displacement u of any point on the membrane is normal to the xy plane and is assumed to be sufficiently small that the tension remains constant. The mass distribution ρ of the membrane is taken to be

$$\rho = \rho(x, y) = \frac{3}{4}(1 + x/9)(1 + 3y/50). \quad (36)$$

The eigenvalue problem for the membrane is described by

$$T(\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2) = \lambda \rho(x, y) u = \lambda \frac{3}{4}(1 + x/9)(1 + 3y/50)u, \quad (37)$$

subject to the boundary conditions $u(x, y) = 0$ along $x = 0.9$ and $y = 0.5$. Note that $p = 1$ and that the boundary conditions are purely geometric. The energy norm for the system is defined by

$$\|u\|^2 = \int_0^5 \int_0^9 T \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy. \quad (38)$$

Because the mass density ρ is non-uniform, a closed-form eigensolution is not readily obtainable. Instead, an approximate eigensolution for the membrane is found by using the Rayleigh-Ritz method. The algebraic eigenvalue problem (28) of order 36 can be obtained by considering as admissible functions all possible products of the functions $\sin(j\pi x/9)$ and $\sin(k\pi y/5)$ ($j, k = 1, 2, \dots, 6$). The resulting estimates of the first 15 eigenvalues are displayed in the last column of Table 1. It is expected that these estimated

TABLE 1
Eigenvalue estimates for the non-uniform membrane

A_n/T n	Substructure synthesis method			Rayleigh-Ritz method, $n = 36$
	$n = 46$	$n = 44$	$n = 42$	
1	0.404564	0.444874	0.450832	0.450831
2	0.629964	0.752104	0.778708	0.778675
3	0.736775	1.103316	1.328738	1.325334
4	1.416013	1.477428	1.482865	1.482156
5	1.477547	1.716410	1.808183	1.810001
6	1.794718	1.882658	2.105430	2.090936
7	2.017901	2.103477	2.336327	2.356659
8	2.307191	2.569380	3.117528	3.078728
9	2.876558	2.959163	3.210240	3.122261
10	3.300413	3.535434	3.658482	3.201052
11	3.529254	3.650579	3.985428	3.528897
12	3.644286	3.945602	4.185389	4.075555
13	3.923404	4.000128	4.530043	4.110054
14	4.034297	4.467813	4.659121	4.482323
15	4.478857	5.006243	5.302714	4.841157

eigenvalues are very accurate and that one may use them to determine the accuracy of estimated eigenvalues obtained by using the substructure synthesis method.

Next, assume that the same membrane is divided into three equally-sized substructures one, two and three with the domains \bar{D}_1 , \bar{D}_2 and \bar{D}_3 , respectively, where \bar{D}_1 is the rectangle $0 \leq x \leq 3, 0 \leq y \leq 5$, \bar{D}_2 is the rectangle $3 \leq x \leq 6, 0 \leq y \leq 5$, and \bar{D}_3 is the rectangle $6 \leq x \leq 9, 0 \leq y \leq 5$. The internal boundary S_{12} between substructures one and two is the line $x = 3, 0 \leq y \leq 5$, while the internal boundary S_{23} between substructures two and three is the line $x = 6, 0 \leq y \leq 5$. For consistency with the original membrane, the external boundaries for all substructures are assumed to be clamped. The geometric compatibility conditions, to be satisfied on $x = 3$ and $x = 6$, are

$$u_1(3, y) = u_2(3, y), \quad u_2(6, y) = u_3(6, y), \quad 0 \leq y \leq 5, \quad (39a, b)$$

where $u_s(x, y)$ ($s = 1, 2, 3$) is the displacement of a point (x, y) in substructure s . For substructure admissible functions, one may take functions which are separable in the x and y co-ordinates. To this end, each substructure admissible function ϕ_{s_k} ($k = 1, 2, \dots, N_s, s = 1, 2, 3$) can be written as

$$\phi_{s, N_{s_y}(i-1)+j} = \chi_{s_i}(x) \psi_{s_j}(y), \quad i = 1, 2, \dots, N_{s_x}, \quad j = 1, 2, \dots, N_{s_y}, \quad s = 1, 2, 3, \quad (40)$$

where N_{s_x} and N_{s_y} ($s = 1, 2, 3$) are the total numbers of functions of the x and y co-ordinates, respectively. To represent each substructure s , one takes the functions $\chi_{s_i}(x)$ and $\psi_{s_j}(y)$ to have the explicit expressions

$$\chi_{1i}(x) = (x/3)^i, \quad i = 1, 2, \dots, N_{1x}, \quad 0 \leq x \leq 3, \quad (41a)$$

$$\psi_{1j}(y) = (y/5)^j (1 - y/5), \quad j = 1, 2, \dots, N_{1y}, \quad 0 \leq y \leq 5, \quad (41a)$$

$$\chi_{2i}(x) = (2 - x/3), \quad \chi_{2i}(x) = (1 - x/3)^{i-1}, \quad i = 2, 3, \dots, N_{2x}, \quad 3 \leq x \leq 6, \quad (41b)$$

$$\psi_{2j}(y) = (y/5)(1 - y/5)^j, \quad j = 1, 2, \dots, N_{2y}, \quad 0 \leq y \leq 5, \quad (41b)$$

$$\chi_{3i}(x) = (3 - x/3)^i, \quad i = 1, 2, \dots, N_{3x}, \quad 6 \leq x \leq 9, \quad (41c)$$

$$\psi_{3j}(y) = (y/5)^j (1 - y/5), \quad j = 1, 2, \dots, N_{3y}, \quad 0 \leq y \leq 5. \quad (41c)$$

For convenience let $N_{1x} = N_{2x} = N_{3x} = N_x$ and $N_{1y} = N_{2y} = N_{3y} = N_y$ so that

$$N = \sum_{i=1}^3 N_s = 3 N_x N_y.$$

By using equations (40) and (41), the disjoint mass and stiffness matrices M_d and K_d can be calculated for any choice of N_x and N_y . These matrices are not presented here for brevity.

The substructures are coupled together via the compatibility conditions (39). For convenience, one may use the same weighting functions on each internal boundary, so that $g_{12i} = g_{23i} = g_i(y)$ ($i = 1, 2, \dots, M_{12} = M_{23}$). Therefore, multiplying conditions (39) by the function $g_i(y)$ and integrating along the internal boundaries, one obtains

$$\int_0^5 g_i(y) [u_1(3, y) - u_2(3, y)] dy = 0, \quad i = 1, 2, \dots, M_{12}, \quad (42a)$$

$$\int_0^5 g_i(y) [u_2(6, y) - u_3(6, y)] dy = 0, \quad i = 1, 2, \dots, M_{23}. \quad (42b)$$

By substituting equations (4) into equations (42) and using equations (40) and (41), a relationship between the disjoint vector ζ_d and the unconstrained vector ζ , equation (17), can be obtained for a particular choice of the weighting functions $g_i(y)$, their number

$M_{12} = M_{23}$, the number N_x of functions $\chi_{s_i}(x)$ ($s = 1, 2, 3$), and the number N_y of functions $\psi_{s_i}(y)$ ($s = 1, 2, 3$). For any particular choice of these quantities, the eigenvalue problem (20) can be solved numerically for the estimated eigenvalues and eigenfunctions.

One now may demonstrate, first, the raising of the estimated eigenvalues by increasing the number M_c of compatibility conditions while ignoring all other effects. As weighting functions $g_i(y)$, consider the $M_{12} = M_{23} = 3$ spatial Dirac delta functions

$$g_1(y) = \delta(y - 5/4), \quad g_2(y) = \delta(y - 5/2), \quad g_3(y) = \delta(y - 15/4), \quad (43)$$

where the Dirac delta function $\delta(y - P)$ is defined by $\int \delta(y - P) f(y) dy = f(P)$. Let $N_x = N_y = 4$ so that the dimension N of the disjoint structure is 48. Three eigenvalue problems are then solved in which one uses one, two, and three weighting functions, respectively, along each internal boundary. Because there are two internal boundaries, M_c takes the values 2, 4, and 6, respectively. Moreover, the dimensions $n = N - M_c$ of the respective eigenvalue problems are 46, 44, and 42. The lowest 15 eigenvalues obtained by taking $n = 46, 44$, and 42 are displayed in columns 1, 2, and 3 of Table 1, respectively. Note that, by adding a weighting function at each internal boundary, one decreases the dimension of the system by two: i.e., one imposes two constraints on the system. Theorem 2 of section 5 for two constraints tells one that

$$\begin{aligned} A_n^{(46)} &\leq A_n^{(44)} \leq A_{n+2}^{(46)}, & n = 1, 2, \dots, 44, \\ A_n^{(44)} &\leq A_n^{(42)} \leq A_{n+2}^{(44)}, & n = 1, 2, \dots, 42. \end{aligned} \quad (44)$$

These bracketing relationships may be verified from examination of Table 1. In addition, comparing the third and fourth columns of Table 1 one sees that good results for the lower eigenvalues are obtained by the substructure synthesis method when the substructures are assumed to be connected at only three points. This can be attributed to the smoothness of the lower eigenfunctions in the y direction. The results in the fourth column appear better than those in the third column, in spite of the fact that fewer admissible functions have been used. This can be attributed to the fact that the admissible functions used for the Rayleigh-Ritz method are smoother overall than the (implicitly generated) global admissible functions used for the substructure synthesis. Plots of the first five eigenfunctions as estimated by using three weighting functions at each internal boundary are displayed in Figure 1. Note that the geometric compatibility is excellent for the lower modes, although it is not so good for the higher modes. To improve the geometric compatibility for the higher modes it is necessary to increase the number of weighting functions.

Next, consider the simultaneous lowering of the estimated eigenvalues by increasing the number of substructure admissible functions and raising of the estimated eigenvalues by adding weighting functions. To this end, consider the following cases: (1) $N_x = 2, N_y = 2, N = 12$; (2) $N_x = 3, N_y = 2, N = 18$; (3) $N_x = 3, N_y = 3, N = 27$; (4) $N_x = 4, N_y = 3, N = 36$; (5) $N_x = 4, N_y = 4, N = 48$. The requirement in section 6 that compatibility conditions at an internal boundary be linearly independent allows one to use at most N_y of the three weighting functions (43) for any case. The first three respective eigenvalue estimates for each case are tabulated in Table 2, as obtained with 1, 2, and 3 weighting functions, respectively, at each internal boundary. Note that the estimated eigenvalues decrease from left to right as more substructure admissible functions are used and increase from top to bottom as more weighting functions are used. Table 3 contains the first three respective eigenvalue estimates for each case when the weighting functions

$$g_i(y) = (y/5)^{i-1} \quad (45)$$

are used. Comparison of Tables 2 and 3 suggests that using the weighting functions (45) results in slightly better compatibility when one or two compatibility conditions are im-

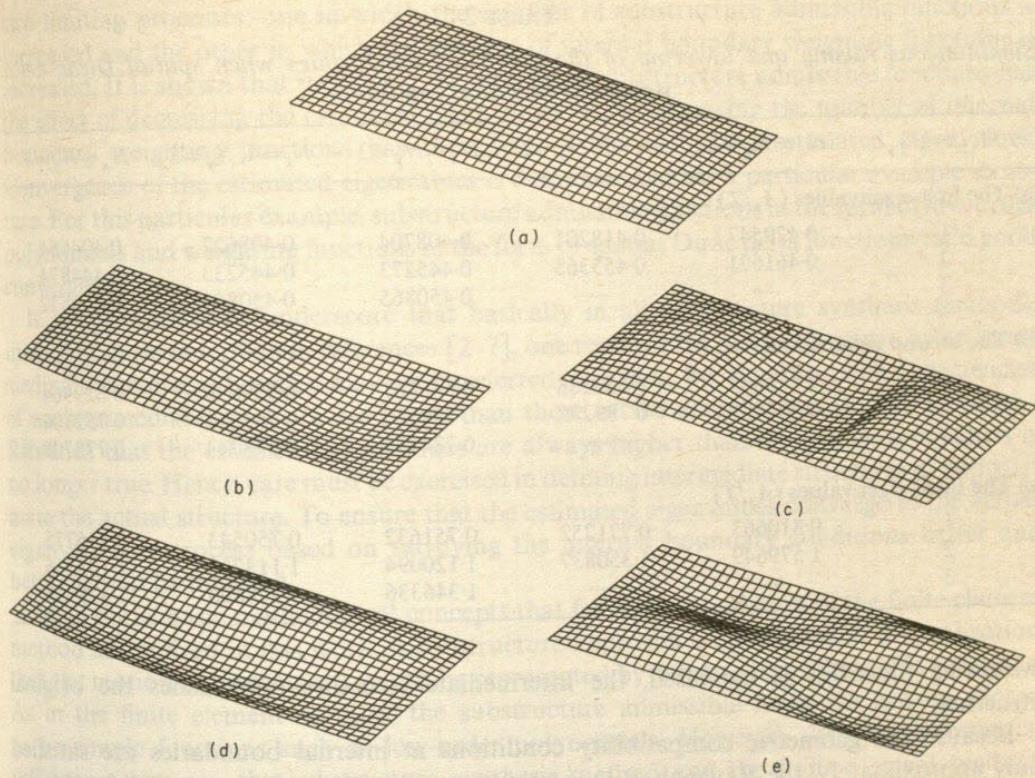


Figure 1. (a) The first estimated eigenfunction, $\Lambda_1 = 0.450832 T \text{ (rad s}^{-1}\text{)}; (b)$ the second estimated eigenfunction, $\Lambda_2 = 0.778708 T \text{ (rad s}^{-1}\text{)}; (c)$ the third estimated eigenfunction, $\Lambda_3 = 1.328738 T \text{ (rad s}^{-1}\text{)}; (d)$ the fourth estimated eigenfunction, $\Lambda_4 = 1.482865 T \text{ (rad s}^{-1}\text{)}; (e)$ the fifth estimated eigenfunction, $\Lambda_5 = 1.808183 T \text{ (rad s}^{-1}\text{).}$

posed at each internal boundary. However, the use of three conditions at each internal boundary appears to be sufficient to insure very close compatibility regardless of the type of weighting functions used, at least in this particular example.

8. SUMMARY AND CONCLUSIONS

In this paper a general substructure synthesis method for the dynamic analysis of complex structures has been described. The motion of each substructure is represented by a given number of substructure admissible functions. Substructure modes, if they are readily available, can be used, as they are a special class of substructure admissible functions. The advantages of ordinary substructure admissible functions are that they are more plentiful than substructure modes and that they are often low-order polynomials, and hence computationally easy to work with. The otherwise disjoint substructures are connected together to form a whole structure by approximating the geometric compatibility conditions by means of the method of weighted residuals. A structure which is no longer disjoint but whose internal boundary conditions are only approximations to the actual ones is referred to as an "intermediate structure". An intermediate structure represents a mathematical concept defined by the type of weighting function used and their number. If the weighting functions are chosen in the form of spatial Dirac delta functions, then the intermediate structure has a simple physical representation. It consists of the otherwise disjoint substructures joined together at a finite set of points. In the limit, as the number of

TABLE 2

Simultaneous raising and lowering of the estimated eigenvalues when spatial Dirac delta weighting functions are used

$M_{12} = M_{23}$	$N_x = N_y = 2$	$N_y = 3, N_y = 2$	$N_x = N_y = 3$	$N_x = 4, N_y = 3$	$N_x = N_y = 4$
(a) The first eigenvalues (Λ_1/T)					
1	0.429547	0.418201	0.408704	0.408627	0.404564
2	0.461671	0.455365	0.445273	0.445233	0.444874
3	—	—	0.450865	0.450855	0.450832
(b) The second eigenvalues (Λ_2/T)					
1	0.670663	0.656188	0.640861	0.639099	0.629964
2	0.796409	0.789272	0.759002	0.753554	0.752104
3	—	—	0.784771	0.778744	0.778708
(c) The third eigenvalues (Λ_3/T)					
1	0.810663	0.771252	0.751632	0.750543	0.736775
2	1.579649	1.350837	1.120094	1.113730	1.103316
3	—	—	1.346336	1.328890	1.328738

weighting functions is increased, the intermediate structure approaches the original structure.

Because the geometric compatibility conditions at internal boundaries are satisfied only approximately, the Rayleigh-Ritz requirements on the displacement and its derivatives for the original structure are generally violated at the internal boundaries. Hence, the substructure synthesis method does not provide estimated eigenvalues which are upper bounds for the actual eigenvalues of the original structure, but this problem is not serious. The behavior of the estimated eigenvalues obtained by the substructure synthesis method can be ascertained by means of a bracketing theorem. The estimated eigenvalues do converge to the actual eigenvalues of the original structure, although it is necessary to consider

TABLE 3

Simultaneous raising and lowering of the estimated eigenvalues when the weighting functions $1, y/5$ and $(y/5)^2$ are used

$M_{12} = M_{23}$	$N_x = N_y = 2$	$N_x = 3, N_y = 2$	$N_x N_y = 3$	$N_x = 4, N_y = 3$	$N_x = N_y = 4$
(a) The first eigenvalues (Λ_1/T)					
1	0.461593	0.455267	0.448874	0.448845	0.448817
2	0.461671	0.455365	0.448980	0.448952	0.448927
3	—	—	0.450865	0.450855	0.450826
(b) The second eigenvalues (Λ_2/T)					
1	0.795914	0.788754	0.775652	0.769534	0.769469
2	0.796409	0.789272	0.776147	0.770019	0.769972
3	—	—	0.784771	0.778744	0.778679
(c) The third eigenvalues (Λ_3/T)					
1	1.458914	1.337955	1.265000	1.252535	1.249957
2	1.579649	1.350837	1.270994	1.258073	1.257883
3	—	—	1.346336	1.328890	1.328487

two limiting processes, one in which the number of substructure admissible functions is increased and the other in which the number of internal boundary weighting functions is increased. It is shown that increasing the number of substructure admissible functions has the effect of decreasing the estimated eigenvalues while increasing the number of internal boundary weighting functions has the effect of increasing the estimated eigenvalues. Convergence of the estimated eigenvalues is demonstrated for a particular example structure. For this particular example, substructure admissible functions in the form of low-order polynomials and weighting functions in the form of spatial Dirac delta functions yield good convergence.

It is appropriate to underscore that basically in all substructure synthesis methods, including those described in references [2-7], one replaces the actual structure by an intermediate structure, although it may not be referred to as such. Because the natural frequencies of an intermediate structure are lower than those of the actual structure, the Rayleigh-Ritz idea that the estimated eigenvalues are always higher than the actual eigenvalues is no longer true. Hence, care must be exercised in defining intermediate structures to approximate the actual structure. To ensure that the estimated eigenvalues converge to the actual eigenvalues, a process based on satisfying the internal boundary conditions better and better should be initiated.

Some of the basic mathematical concepts that form the foundation of the finite element method are utilized in this work. A substructure approach is adopted with the realization that the motion of a substructure can be represented by substructure admissible functions. As in the finite element method, the substructure admissible functions can be chosen to be simple functions such as low-order polynomials. However, there are substantial differences between the substructure synthesis method and the finite element method. In the substructure synthesis method, substructure geometry is not arbitrary and is determined *a priori*. The substructures generally do not consist of simple geometric shapes. Hence, satisfaction of the compatibility conditions between adjacent substructures is more of a problem than in the finite element method. Moreover, in the substructure synthesis method, the number of substructures is held fixed. Convergence of the estimated eigenvalues is obtained by increasing the number of substructure admissible functions and the number of internal boundary weighting functions rather than the number of substructures.

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