Please Duplicate 1-12

TransactionNumber: 392242

System:	**OCLC**	**Call #:**
OCLC Number:	**9738167**	**Location:**
ILL Number:	**142351767**	
MaxCost:	**30.00IFM**	
LenderString:	***DLC,DLC**	

LIBRARY USE ONLY

This book has been lent to SAN DIEGO STATE UNIVERSITY, MALCOLM A LOVE LIBRARY - ILL for reading room use only.
Off-site use is not authorized.

Interlibrary Loan Request Form

Article Information

Journal Title: Reports and memoranda /

Volume: 766 **Issue:**
Month/Year: 1922 **Pages:** 358-

Article Author: Southwell and Gough

Article Title: Southwell and Gough 'On the free transverse vibrations of airscrew blades'

Loan Information

Loan Title: Reports and memoranda / Southwell and Gough 'On the free transverse vibrations of airscrew blades'

Loan Author: Southwell and Gough

Publisher: London : H.M.S.O., [1939?-]
Place:
Date: 1922
Imprint:

Borrower Information

Borrowing Institution: CDS - SAN DIEGO STATE UNIVERSITY
MALCOLM A LOVE LIBRARY - ILL
5500 CAMPANILE DRIVE
SAN DIEGO CA 92182-8050

Phone: (619) 594-6730
Fax: (619) 265-0414
Email: ill@rohan.sdsu.edu
Odyssey: 130.191.17.111

JD

A.3.d. Airscrews, 61
(T. 1641.)

AERONAUTICAL RESEARCH COMMITTEE.

15

REPORTS AND MEMORANDA, No. 766.
(Ae. 26.)

ON THE FREE TRANSVERSE VIBRATIONS OF AIR-SCREW BLADES. By R. V. SOUTHWELL AND BARBARA S. GOUGH.

OCTOBER, 1921.

LONDON:
PUBLISHED BY HIS MAJESTY'S STATIONERY OFFICE.

To be purchased through any Bookseller or directly from
H.M. STATIONERY OFFICE at the following addresses:
IMPERIAL HOUSE, KINGSWAY, LONDON, W.C. 2;
28, ABINGDON STREET, LONDON, S.W. 1;
37, PETER STREET, MANCHESTER;
1, ST. ANDREW'S CRESCENT, CARDIFF;
23, FORTH STREET, EDINBURGH;
or from EASON & SON, LTD., 40 & 41, LOWER SACKVILLE STREET, DUBLIN.

1922.

Price 3d. Net.

ON THE FREE TRANSVERSE VIBRATIONS OF AIRSCREW BLADES.

By R. V. SOUTHWELL and BARBARA S. GOUGH.

Reports and Memoranda 766 (*Ae.* 26). *October*, 1921.

SUMMARY.—(1) The gravest frequency of free transverse vibration, for an airscrew blade of any shape whatever, may be estimated with very fair accuracy from the formula

$$p^2 = p_1^2 + \omega^2, \qquad \text{(A)}$$

where ω is the angular velocity, and p_1 the gravest frequency in the absence of rotation; it can also be proved that the estimate given by (A) will be of the nature of a low limit to the correct result. (2) By an application of the methods of Lord Rayleigh, a value of the constant α can be determined, for use in the formula

$$p^2 = p_1^2 + \alpha\omega^2, \qquad \text{(B)}$$

which will ensure that the estimate given by this formula is of the nature of an upper limit. (3) When the blade shape is such that p_1 cannot be determined by exact calculation, use may be made of a graphical construction published previously.

Examples are treated in detail which indicate the order of accuracy to be expected in applying the foregoing methods.

§ 1. The vibrations of an airscrew blade are of practical interest, first, because of the possibility that they may become a cause of fracture in the air, and secondly, because they contribute to the noise which, for several reasons, is an objectionable feature of aeroplanes. It would appear that they are excited in some way by the aerodynamic forces which act upon the blade when in motion, and recent work on the allied problem of "singing" in wires exposed to the wind* suggests that the process is one of considerable complexity; but it is practically certain that one of the problems which will arise in the course of their complete investigation will be the determination of the modes and frequencies of free vibration natural to a blade of given material and shape, and this may be treated by the standard methods of the Theory of Sound.

In an airscrew blade of ordinary design, the principal axes of the cross-section vary in orientation along the length of the blade. The normal modes of vibration will thus involve, in general, a

* *Cf.* T.1564 and 1564a (R. G. Harris, on "Rafwire Vibrations"), and E. F. Relf, "On the Sound produced by Circular Wires in an Air Current," *Phil. Mag.*, July, 1921.

(B6421) Wt. 63—720. 500. 6/22. Gp. 32.

combination of torsion and flexure, and if we assume them to be purely flexural we are in effect, by neglecting the torsion, overestimating the "stiffness" of the elastic system. This assumption will necessarily lead to excessive values for the natural frequencies*: but it greatly reduces the difficulties of analysis, and so may be regarded as justified in a preliminary examination of blade vibration, although in a detailed investigation more accurate methods may prove to be necessary. For the present, therefore, we may identify our problem with that of finding the natural frequencies of transverse vibration for a bar of given shape, when this rotates *in vacuo* about an axis through one end, and one principal axis of every cross-section is parallel to the axis of rotation.

§ 2. The following is a summary of recent papers dealing with the problem thus stated :—R. & M. 486† pointed out the necessity of taking the effects of rotation into consideration; R. & M. 488‡ investigated these effects for a uniform rod, and in two cases which corresponded fairly to airscrew blades of typical design found that rotation increased the frequency by 34 per cent. and 61 per cent. respectively; R. & M. 626§ gave corresponding analysis for a blade of wedge form (with flexural rigidity proportional to the cube of the distance from the tip) rotating about its base, and proposed for general use the formula||

$$p^2 = p_1^2 + 1\cdot3\omega^2, \qquad (1)$$

where p_1 is the gravest frequency of vibration in the absence of rotation, and p the gravest frequency when the blade is rotating with angular velocity ω. Finally, Miss L. M. Swain¶ has suggested a line of attack, based upon Rayleigh's method of an assumed type,** for use when it is desired to represent the blade more accurately by means of a truncated instead of by a knife-edged wedge.

* Rayleigh, "Theory of Sound," vol. i, §§ 74 and 88. A. A. Griffith and B. Hague ("On the Shape of Propeller Blades," R. & M. 452) have shown that the best type of blade, from a stress standpoint, is one in which transverse vibrations involve practically no torsion; *cf.* R. & M. 451 (A. A. Griffith), § 1.

† R. V. Southwell.—"Note on Mr. Griffith's Formula for Calculating the Vibration Speeds of Propellers, with a Solution for the Frequencies of Vibration in a Rotating Heavy String."

‡ A. Berry.—"On the Vibrations of a Uniform Rod Rotating Uniformly about one End, which is Encastré."

§ H. A. Webb and Lorna M. Swain, on the "Vibration Speeds of Airscrew Blades."

|| The notation has been changed to agree with that of the present paper.

¶ "On the Period of Vibration of the Gravest Mode of a Thin Rod, in the Form of a Truncated Wedge, when in Rotation about its Base."—*Phil. Mag.*, vol. xii (1921), pp. 259–266.

** *See* § 4.

§ 3. The purpose of the present paper is to illustrate the application to this problem of certain simplifications in method which have been shown to be legitimate in the analogous problem of free transverse vibrations in a rotating circular disc.* The special difficulty of both problems arises from the fact that flexural rigidity and centrifugal force are both operative, and that the *normal modes, as well as the frequencies, depend upon their relative importance.* But it was shown, in the investigation referred to, that approximations sufficient for practical purposes could be obtained by separating the two restoring systems, and calculating the frequencies, p_1 and p_2, which occur when each system acts separately; the theorem was proved, that if the restoring forces which control the vibrations of an elastic system can be separated into two or more groups which affect the potential energy independently, and if the gravest frequency of vibration be found on the assumption that each group in turn acts separately (the inertia being unchanged), then the sum of the squares of the frequencies thus found is approximately equal to, *but necessarily less than*, the square of the gravest frequency which can occur when all the groups act simultaneously—*i.e.*, the square of the gravest frequency natural to the system.†

The frequency p_1 is, in the present problem, the quantity thus denoted in equation (1). Further, it was shown in R. & M. 486 that the gravest frequency of vibration when the centrifugal forces act alone (so that the blade may be treated as a heavy string) is given by

$$p_2 = \omega, \qquad (2)$$

and that the corresponding mode may be regarded as a rotation of the blade about the same centre, but in a new plane inclined to the first.‡ These results were obtained on the assumption of uniform cross-section; but since the mode involves no curvature of the blade, it will be equally characteristic of a blade having any variation of cross-section along its length, and equation (2) will therefore hold in all cases.

Thus the foregoing theorem, as applied to our present problem, states that

$$p^2 \mathrel{\dot{>}} p_1^2 + \omega^2, \qquad (3)$$

the dots attached to the sign of inequality indicating *that the inequality will be slight*; p and p_1 are the quantities thus denoted in equation (1), the empirically established accuracy of which supports the conclusion (3). It will be evident that the latter

* H. Lamb and R. V. Southwell, on "The Vibrations of a Spinning Disc."—*Proc. Roy. Soc.* (*A*), vol. 99 (1921), pp. 272–280.

† *Loc. cit.*, p. 278.

‡ R. & M. 486, p. 5, and footnote to p. 6.

formula is an important extension of the results obtained in R. & M. 486, which showed merely that

$$\text{and} \quad \left.\begin{array}{l} p > p_1, \\ p > \omega. \end{array}\right\} \qquad (4)$$

§ 4. We shall find that the formula (3) (which is quite general, and has, as we have shown, a logical foundation) is of sufficient accuracy for most practical purposes, and may be employed to estimate the value of p, after p_1 has been determined either by calculation or by experiment. But to establish this result we shall require some idea of the possible error, and here we may make use of the fact that (3) necessarily gives a lower limit; for Rayleigh's well-known method of an assumed type* enables us to obtain an alternative estimate which is necessarily of the nature of an upper limit, and hence *we can determine the value of p within known limits of error.*

The principle of Rayleigh's method may be briefly reviewed here. If in a vibrating system, which is free from dissipative forces, we assume the type of a free vibration, and calculate the instantaneous values, V and T, of the potential and kinetic energies corresponding to a vibration of this type, on the assumption that the displacement is simply harmonic in time, then a close estimate of the gravest frequency of vibration may be obtained from the energy equation

$$V + T = \text{const.} \qquad (5)$$

For the form of the general expression which represents the frequency as thus obtained shows that it has a stationary value when the assumed type of vibration is identical with one of the "normal modes" of free vibration appropriate to the system considered; hence, if the assumed type differs but slightly from a normal mode, the calculated frequency will differ by a small quantity of the second order from the natural frequency corresponding to that normal mode. The general expression also shows that the calculated frequency cannot be less than the gravest frequency natural to the system, so that if the assumed type is in reasonably good agreement with the gravest normal mode *we shall obtain an estimate of the gravest frequency of free vibration which will be (a) very close, and (b) necessarily an upper limit.*

In applying this method to the present problem, we take the transverse displacement to be represented by an expression of the form

$$u = \mathrm{U} f\left(\frac{x}{l}\right) \sin pt, \qquad (6)$$

where U is an undetermined constant, and $f\left(\frac{x}{l}\right)$ a function of the single variable (x/l), the form of which we may assume

* "Theory of Sound," vol. i, §§ 88, 89.

arbitrarily, provided only that it satisfies the requisite conditions at $x = 0$; x denotes the distance of the section considered from the axis of rotation, and l the length of the blade.

The potential energy V may be divided into two parts—V_1, due to the flexural rigidity, and V_2, due to the centrifugal forces. Clearly

$$T = \frac{1}{2g}\int_o^l \rho\sigma\left(\frac{\partial u}{\partial t}\right)^2 dx,$$

where ρ and σ denote the density of the material and the area of the cross-section. Hence, substituting from (6), we have

$$T = \frac{1}{2g}\rho U^2 p^2 \cos^2 pt \int_o^l \sigma\left\{f\left(\frac{x}{l}\right)\right\}^2 dx. \quad . \quad . \quad (7)$$

The potential energy of flexure is

$$V_1 = \frac{1}{2}\int_o^l EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx,$$

$$= \frac{1}{2}\frac{E}{l^4}U^2 \sin^2 pt \int_o^l I\left\{f''\left(\frac{x}{l}\right)^2\right\} dx, \quad . \quad . \quad (8)$$

if we substitute from (6).

Finally, the potential energy of the centrifugal forces is

$$V_2 = \int_o^l \rho\sigma\frac{\omega^2}{g} x\,\Delta\, dx,$$

where Δ denotes the displacement of the section x, towards the centre, which results from the deflection u. It is clear that

$$\Delta = \int_o^x \left\{1 - \cos\left(\frac{\partial u}{\partial x}\right)\right\} dx,$$

$$= \frac{1}{2}\int_o^x \left(\frac{\partial u}{\partial x}\right)^2 dx, \text{ approximately},$$

$$= \frac{1}{2}\frac{U^2}{l^2}\sin^2 pt \int_o^x \left\{f'\left(\frac{x}{l}\right)\right\}^2 dx,$$

if we substitute from (6). Hence we have

$$V_2 = \frac{1}{2}\frac{\rho\omega^2}{gl^2}U^2 \sin^2 pt \int_o^l \sigma x \int_o^x \left\{f'\left(\frac{x}{l}\right)\right\}^2 dx dx. \quad . \quad . \quad (9)$$

On substituting from (7)–(9) in (5), we obtain from this relation the frequency equation

$$p^2 = \frac{gE}{\rho l^4}\frac{\int_o^l I\left\{f''\left(\frac{x}{l}\right)\right\}^2 dx}{\int_o^l \sigma\left\{f\left(\frac{x}{l}\right)\right\}^2 dx} + \frac{\omega^2}{l^2}\frac{\int_o^l \sigma x\int_o^x \left\{f'\left(\frac{x}{l}\right)\right\}^2 dx dx}{\int_o^l \sigma\left\{f\left(\frac{x}{l}\right)\right\}^2 dx}, \quad (10)$$

in which the integrals can be found, by exact or approximate calculations, when the form of $f\left(\frac{x}{l}\right)$ is assumed.

We may remark in passing, as a conclusion drawn from the form of (10), that Rayleigh's principle gives strong support to a formula of the type (1), for use as an approximate expression of the exact relation between p and ω. Since the sign of ω is immaterial, and since p^2 tends to the limit $p_1{}^2$ as $\omega \rightarrow 0$, we may regard the exact relation as expressible in the form

$$p^2 = p_1{}^2 + \alpha\omega^2 + \beta\omega^4 + \gamma\omega^6 + \ldots \text{ \&c.}, \quad . \quad (11)$$

α, β, . . . &c., being constant coefficients of which the values depend upon the form of the blade. If rotation had no effect upon the *mode* of vibration, an exact expression for the frequency could be obtained from (10) by giving to $f\left(\frac{x}{l}\right)$ the form which this function assumes in the gravest mode of vibration natural to the non-rotating blade; the first term on the right of (10) would then reduce to $p_1{}^2$, the second term would fix the value of α in (11), and the coefficients β, γ, . . . &c., would vanish severally. Obviously, then, the terms in ω^4, ω^6, ... &c., on the right of (11), arise merely because the rotation modifies the *mode* of the vibration; but such modifications are, by Rayleigh's principle, comparatively unimportant in their effect on the frequency, and hence the first two terms on the right of (11) are of dominant importance.

A more important conclusion can, however, be drawn. If we determine α in the manner just described, *we are assuming the type of the vibration*; the assumed form tends to a correct limit as $\omega \rightarrow 0$, but is in error when ω is finite. Rayleigh's theorem asserts that in the latter case we shall, as a consequence, slightly over-estimate the gravest frequency, and hence, finally, we may write

$$p^2 \lessdot p_1{}^2 + \alpha\omega^2,$$

where α is found from the formula

$$\alpha = \frac{\int_o^l \sigma x \int_o^x \left\{f'\left(\frac{x}{l}\right)\right\}^2 dx dx}{l^2\int_o^l \sigma\left\{f\left(\frac{x}{l}\right)\right\}^2 dx}, \quad . \quad . \quad (12)$$

by giving to $f\left(\frac{x}{l}\right)$ the form which this function assumes in the gravest mode of vibration for the non-rotating blade. The formula (12) is exact when ω is infinitesimal; in all cases, when used in conjunction with (3), it enables us to determine p^2 between known limits of error.

§ 5. The conclusions of the last section may be tested by applying them to special cases of the problem which have been previously investigated in the papers cited above.

(a) *Uniform Blade*.—The determination of p_1 is simple.* The blade is to be regarded as a cantilever, clamped at $x = 0$, and free at $x = l$. In notation almost identical with that of R. & M. 488 (Berry), we have the equation

$$\left.\begin{aligned} &1 + \cosh \mu_1 \cos \mu_1 = 0, \\ \text{where} \qquad &\mu_1{}^4 = p_1{}^2 \cdot \frac{\rho\sigma l^4}{g\mathrm{EI}}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

and the gravest value of p_1 corresponds to the root

$$\left.\begin{aligned} &\mu_1 = 1{\cdot}875 \ldots \\ \text{or} \qquad &\mu_1{}^4 = 12{\cdot}36 \ldots \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

The corresponding expression for the transverse displacement is given by (6), where

$$f\left(\frac{x}{l}\right) = \left(\cosh \mu_1 + \cos \mu_1\right)\left(\sinh \mu_1 \frac{x}{l} - \sin \mu_1 \frac{x}{l}\right)$$
$$- \left(\sinh \mu_1 + \sin \mu_1\right)\left(\cosh \mu_1 \frac{x}{l} - \cos \mu_1 \frac{x}{l}\right) \quad . \quad (15)$$

Writing, with Berry,

$$\left.\begin{aligned} \lambda &= \tfrac{1}{2}\rho\sigma\omega^2 l^4/g\mathrm{EI}, \\ \mu^4 &= \rho\sigma p^2 l^4/g\mathrm{EI}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

we obtain from (3) the formula

$$\mu^4 \gtrsim 12{\cdot}36 + 2\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In calculating the coefficient α of the alternative formula (12), we may take the constant quantity σ outside the integrals, before substituting for $f(x/l)$ from (15). By straightforward though lengthy integration, we find that

$$\alpha = 1{\cdot}193 \ldots$$

Then from (12) we have, corresponding to (17), the formula

$$\mu^4 \lesssim 12{\cdot}36 + 2{\cdot}386\lambda, \text{ approximately}, \quad . \quad . \quad (18)$$

in which the coefficient of λ agrees well with the value 2·38 found by Berry, as the result of a somewhat laborious integration in series of the differential equation of motion.

* *Cf.* Rayleigh, "Theory of Sound," vol. i, § 173. Rayleigh's symbol m corresponds to our μ_1: Berry's x is our x/l.

If the form of $f(x/l)$ had not permitted direct integration, we should, of course, have had to use graphical methods in conjunction with (12). As an illustration of the order of accuracy attainable by their means, it may be mentioned that in this example they gave a value 1·195 for α. It should be possible in any ordinary example to keep the error within similarly fine limits, provided that $f'(x/l)$ and $f(x/l)$ are accurately known; to achieve this end it is desirable that $f'(x/l)$ should be determined, whenever possible, by analytical rather than by graphical differentiation.

Comparing (17) with (18), we see that the error involved in this example by the use of the formula (3)—which involves no additional calculation, when $\mu_1{}^4$ is known—vanishes with λ, and cannot under any circumstances be greater than $0{\cdot}386\lambda$. The fractional error involved in μ^4 cannot therefore exceed the value

$$\frac{0{\cdot}386\lambda}{12{\cdot}36 + 2{\cdot}386\lambda}, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which, when $\lambda = 10$ (an extreme figure, according to R. & M. 626*), has the value 0·1066. This gives the maximum error in p^2; thus the frequency, as calculated from (3), will in no ordinary case be more than 5 per cent. low, and in general the error will be considerably less.

By using (12) to determine an upper limit, the absolute magnitude of the possible error can be reduced. For if the mean of (17) and (18) be taken as a working formula, the maximum fractional error in μ^4 becomes

$$\frac{0{\cdot}193\lambda}{12{\cdot}36 + 2\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which, when $\lambda = 10$, has the value 5·964. The corresponding maximum error in the frequency is less than 3 per cent.

(b) *Knife-edged Wedge*.—The exact solution of Kirchhoff, for a blade of this form which does not rotate, has been quoted in R. & M. 626, § 4. In the notation of the present paper we have

$$\mu_1{}^4 = 28{\cdot}28, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

if in the expressions (13) or (16) we take σ and I to denote the values of these quantities at the axis of rotation. Then the formula (3) gives us

$$\mu^4 \gtrsim 28{\cdot}28 + 2\lambda. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The paper quoted does not investigate the best value to take for the coefficient of λ in an approximate formula of this form, but by elimination of undetermined coefficients from a solution in series of the exact equation of vibration it obtains values for μ^4 corresponding to a series of values of λ. The following table gives

* Berry, in two typical blades, obtains the values 4 and 7 for λ.

a comparison of the exact results, obtained in this way, with the approximate estimates obtainable from (22) :—

TABLE 1.

λ =	0	2	4	6	8	10
μ^4: Exact values (R. & M. 626) ...	28·28	34·10	39·86	45·49	51·02	56·51
μ^4: approximate values from (22)	28·28	32·28	36·28	40·28	44·28	48·28
Error in μ^4	0	1·82	3·58	5·21	6·74	8·23
Percentage error in μ^4	0	5·34	8·98	11·45	13·21	14·56
Percentage error in frequency	0	2·64	4·39	5·57	6·39	7·03

In view of the tedious nature of the calculations described in R. & M. 626, it is interesting to notice how closely the frequencies could have been estimated from (3), without any calculations whatever in addition to those which give the value of $\mu_1{}^4$.

Had the analysis of R. & M. 626 not been available, limits of error could have been determined, as in the last example, by calculating the value of α from the formula (12). The curve of deflection corresponding to the solution (21) for a non-rotating blade is, in the notation of the present paper,*

$$\left.\begin{aligned} u = \frac{\mathrm{U}}{\sqrt{1-\frac{x}{l}}}\Big[\mathrm{I}_1(2\mu_1)\mathrm{J}_1\Big(2\mu_1\sqrt{1-\frac{x}{l}}\Big) \\ -\mathrm{J}_1(2\mu_1)\mathrm{I}_1\Big(2\mu_1\sqrt{1-\frac{x}{l}}\Big)\Big], \\ \text{where} \quad \mathrm{I}_1(2\mu_1)\mathrm{J}_1{}'(2\mu_1) = \mathrm{J}_1(2\mu_1)\mathrm{I}_1{}'(2\mu_1), \end{aligned}\right\} \quad (23)$$

and U is an undetermined constant. J_1 and I_1 in this expression denote the Bessel functions for which these symbols are customarily employed. We have to substitute for $f(x/l)$, in (12), the above expression for u, and write

$$\left.\begin{aligned} \mathrm{I} &= \mathrm{I}_0\Big(1-\frac{x}{l}\Big)^3, \\ \text{and } \sigma &= \sigma_0\Big(1-\frac{x}{l}\Big). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (24)$$

The resulting integrals do not appear to be tractable mathematically, but are easily evaluated by graphical methods; for accurate results, the form of $f'(x/l)$ should be found, as before,

* *Cf.* R. & M. 626, §§ 3 and 4. In that paper, x is used for the quantity here denoted by $(1-x/l)$, and μ_0 for our μ_1.

by analytical in preference to graphical differentiation. Using tables to determine the values of the functions for ten equidistant points along the blade, a value of 1·455 was obtained for α, whence we have the formula

$$\mu^4 \lessdot 28{\cdot}28 + 2{\cdot}91\lambda. \quad . \quad . \quad . \quad . \quad (25)$$

Thus, without having recourse to the calculations of R. & M. 626, we can say that the error in μ^4 involved by the use of our simple formula (3) cannot exceed a fractional value of

$$\frac{0{\cdot}91\lambda}{28{\cdot}28 + 2{\cdot}91\lambda}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

or 15·86 per cent., when λ has the extreme value 10; this corresponds to a maximum error in the frequency of about 7·6 per cent. Alternatively, by taking the mean of (22) and (25), we can reduce the extreme fractional error in μ^4 to

$$\frac{0{\cdot}455\lambda}{28{\cdot}28 + 2\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or 9·42 per cent., when $\lambda = 10$; the maximum error in frequency is then about 4·6 per cent.

Table 2 has been constructed for the formula (25) in the same way as Table 1 for (22). It indicates that the accuracy obtainable by our method of applying Lord Rayleigh's principles is substantially better than that of the simple formula (3), so that in special cases the additional labour may be justified. Another conclusion is that better accuracy is likely to be obtained directly from the "high limit" formula than by taking a mean of the high and low limits: but these deductions, although they accord with what we should expect from physical considerations, would not appear to be capable of rigorous proof.

TABLE 2.

λ =	0	2	4	6	8	10
μ^4: Exact values (R. & M. 626) ...	28·28	34·10	39·86	45·49	51·02	56·51
μ^4: Approximate values from (25)	28·28	34·10	39·92	45·74	51·56	57·38
Error in μ^4	0	0	0·06	0·25	0·54	0·87
Percentage error in μ^4	0	0	0·15	0·55	1·06	1·54
Percentage error in frequency	0	0	0·07	0·27	0·53	0·76

§ 6. In both of the examples treated above, an exact solution was available for the extreme case of no rotation. In general, approximate methods of calculation will be required for the estimation of p_1^2 in (3) or (12), and in order that a general formula (applicable to the rotating blade) may be deduced by the procedure of this paper, the calculations must also give the curve of deflection.

A graphical method which satisfies these requirements has been published by one of the present authors.* In the example treated, it was found to give results well within 1 per cent. of the correct figure, and it proceeds by the actual construction of curves of deflection. We have found that an extension can be devised whereby it will take account of any specified rotational speed; but the greater practical utility of a general formula, applicable to all speeds, is thought to outweigh the disadvantage of a slight decrease in accuracy. As a method of attack suited to our general problem, we therefore propose that p_1^2 be determined by the graphical method in its existing form; that the curve of deflection which this method affords be used to determine the coefficient α in equation (12); and that the formulæ (3) and (12) be used to fix high and low limits to the value of the frequency. We add the suggestion that the true value is likely to lie closer to (12) than to (3).

The essentials of the graphical method are fully described in the paper referred to. Two points require to be noticed in the present connection: first, the procedure required to ensure that the curve of deflection shall have zero slope at a specified section was not described in detail, but may be gathered from the discussion of a similar construction given in R. & M. 741†; secondly, to ensure the greatest possible accuracy in the estimation of α in (12), it is desirable that the form of $f'(x/l)$ should be determined by graphical integration of the curve of bending moments, rather than by graphical differentiation of the curve of deflections.

As an indication of the accuracy attainable, we have used the graphical method to calculate μ_1^4 for the knife-edged wedge, and thereby to derive "high and low limit formulæ" for comparison with (22) and (25). The test is a very severe one, owing to the evanescence of I and σ, as shown by (24), at the tip, and the consequent rapid increase of deflection near this point: but even so the graphical method gave a value of 25·9 for μ_1^4, which is only 8·4 per cent. in error (corresponding to an error of 4·3 per cent. in the frequency); the corresponding diagram of deflections was used to determine α from equation (12), and gave the value 1·524, which agrees well with the value 1·455 found previously.

* R. V. Southwell, "On a Graphical Method for Determining the Frequencies of Lateral Vibration, or Whirling Speeds, for a Rod of Non-Uniform Cross-Section."—*Phil. Mag.*, vol. xli (1921), pp. 419–431.

† Barbara S. Gough, "A graphical method for the determination of bending moments and deflections in an aeroplane spar."

Printed under the authority of His Majesty's Stationery Office,
By Sir Joseph Causton & Sons, Limited, 9, Eastcheap, London, E.C. 3.

A. 3. a. Aerofoils—general, 100.
(T. 1593.)

AERONAUTICAL RESEARCH COMMITTEE.

REPORTS AND MEMORANDA, No. 767.
(Ae. 27.)

THE CALCULATION OF THE CHARACTERISTICS OF TAPERED WINGS.—By H. GLAUERT, of The Royal Aircraft Establishment. Presented by THE DIRECTOR OF RESEARCH.

MAY, 1921.

LONDON:
PUBLISHED BY HIS MAJESTY'S STATIONERY OFFICE.
To be purchased through any Bookseller or directly from
H.M. STATIONERY OFFICE at the following addresses:
Imperial House, Kingsway, London, W.C. 2;
28, Abingdon Street, London, S.W. 1;
37, Peter Street, Manchester;
1, St. Andrew's Crescent, Cardiff;
23, Forth Street, Edinburgh;
or from EASON & SON, Ltd., 40 & 41, Lower Sackville Street, Dublin.
1922.
Price 3d. Net.