MODELING ROTATING CANTILEVER BEAMS:
MATHEMATICAL DEVELOPMENT AND
NUMERICAL ANALYSIS OF TECHNIQUES

A Thesis

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In Partial Fulfillment

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Master of Science in Applied Mathematics

with a Concentration in

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by

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Modeling Rotating Cantilever Beams:

Mathematical Development and

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DEDICATION

Dedicated to my friends and family. Without you, this would be impossible.

To my parents - you have provided me with love and support, and taught me to work hard and never give up.

To my sister - we share in our afflictions, and I fight to give you hope while you pick me up when I am down.

To my Aunt Shannon and Uncle Jim - you have always been supportive and loving.

To my friends - you helped me through many rough spots in which I was lucky to survive.

In mathematics, you don't understand things. You just get used to them.

– Johann von Neumann

ABSTRACT OF THE THESIS

Modeling Rotating Cantilever Beams:
Mathematical Development and
Numerical Analysis of Techniques
by
James Anthony Mullinix
Master of Science in Applied Mathematics with a Concentration in Dynamical Systems
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This document is a summary of some relevant commands needed to create a Master's thesis for the Department of Mathematics and Statistics using LaTeX. Included are examples of equations, figures, tables, and theorems. The formats listed in this document have been approved by the Department of Mathematical Sciences and the Graduate Division and Research. If you have any difficulties with any of the driver or style files, please see your graduate adviser.

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I would like to thank Dr. Gauss for allowing me to work on this thesis even though he has been dead for so many years. I would also like to thank Dr. Bolzano for not having any comments on this thesis, and I would like to thank Dr. Knuth for being the only living person on my thesis committee, and for writing the wonderful TeX.

This thesis is partially protected against the evil forces of the Montezuma Publishing thesis reviewers by the magic of the Department of Mathematics and Statistics Master's Thesis LaTeX Template.

CHAPTER 1
EQUATIONS OF MOTION

The equations of motion of the rotating cantilever beam system are developed through the principles of variational approximation as covered in Appendix A. The discussion begins by describing the energy equations of the system, then applies variational approximation to formulate the approximating form and applies the discretizing interpolation for a single element two-node system. Finally, the system is expanded to N-dimensional form, and the system matrices are formulated.

1.1 ENERGY EQUATIONS

The energy equations consist of kinetic energy, denoted by T, and potential energy, denoted by V. These are precisely the quantities discussed in Appendix A, p.46. In this case, the beam is discretized as a coupled mass-spring system. The coupling is determined by the equations of motion resultant from integration, simplification, and interpolation. The system that shall be considered is the system discussed by Ref. [3]. In this system, it is assumed that the beam is long compared to the height and thickness, so that shear deformations may be neglected.

The beam is affixed to a central hub which is rotated at a frequency Ω. The direction which is in-plane with the rotations and along the length of the beam is referred herein as the axial direction, labeled u. The direction which is in-plane with rotations and orthogonal to the u direction is referred to as the chord-wise direction, and is labeled v. The out-of-plane direction which is orthogonal to both u, v is referred to as the flap-wise direction and is labeled w.

**Figure 1.1. Cantilever Beam with Rotational Oscillation [3]**
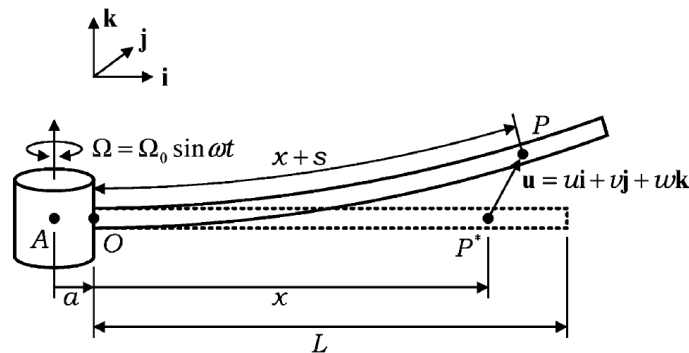
1.1.1   Stretch Coordinate Frame

Any displacement will cause elongation or shortening of the beam, which shall be named stretch and assigned the variable s. The stretch is affected by all displacement coordinates, u, v, w. Consequently, accounting for stretch from displacements in the Cartesian reference frame becomes unwieldy to maintain. In order to remedy this, it is necessary to create a coordinate relation between the displacements and the stretch. This section follows closely to the work done in Ref. [8].

Consider the following diagram. In this diagram, the u direction has been replaced by a dummy variable η in order to prevent confusion during the derivation of the stretch term, s.

Figure 1.2. Stretch Coordinate Relation Diagram [8]
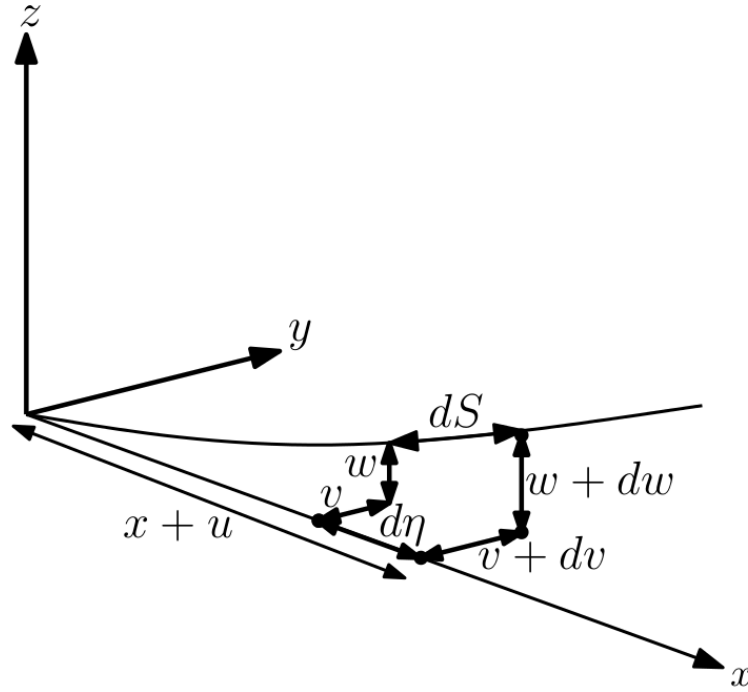
From Pythagoras, we have the length of the hypotenuse of the infinitesimal displacements $\mathrm{d}\eta, \mathrm{d}v, \mathrm{d}w$ given as

$$\mathrm{d}S = \sqrt{(\mathrm{d}\eta)^2 + (\mathrm{d}v)^2 + (\mathrm{d}w)^2}, \tag{1.1}$$

and integrating both sides of Eqn. 1.1 yields

$$S = \int_0^{x+u} \sqrt{(\mathrm{d}\eta)^2 + (\mathrm{d}v)^2 + (\mathrm{d}w)^2}, \tag{1.2}$$

which is known as the arclength integral in $\mathbb{R}^3$. Notice that the integration is done with respect to η, and varies over 0 to $x + u$ as in the diagram. *Note:* the arclength S is not equivalent to

stretch s. This integral is complicated to compute in this form, so a change of variable is introduced. Let

$$\phi = \eta - u \tag{1.3}$$

so that

$$\mathrm{d}\phi = \mathrm{d}\eta - \mathrm{d}u, \tag{1.4}$$

and

$$\mathrm{d}\eta = \mathrm{d}\phi + \mathrm{d}u. \tag{1.5}$$

The integration limits are converted by considering that at the center of rotation, the displacement will be zero in all directions, so that stretch is zero. For the upper limit, consider that $\eta = u + x$ so that $\phi = x$.

Substituting Eqn. 1.5 into Eqn. 1.1 yields

$$\mathrm{d}S = \sqrt{(\mathrm{d}\phi + \mathrm{d}u)^2 + (\mathrm{d}v)^2 + (\mathrm{d}w)^2}. \tag{1.6}$$

By the chain rule, this can be rewritten as

$$\mathrm{d}S = \left[\left(\mathrm{d}\phi + \frac{\partial u}{\partial \phi} \mathrm{d}\phi \right)^2 + \left(\frac{\partial v}{\partial \phi} \mathrm{d}\phi \right)^2 + \left(\frac{\partial w}{\partial \phi} \mathrm{d}\phi \right)^2 \right]^{1/2}, \tag{1.7}$$

factoring out $\mathrm{d}\phi$, replacing the integration limits, and substitution into Eqn. 1.2 yields

$$S = \int_0^x \left[\left(1 + \frac{\partial u}{\partial \phi} \right)^2 + \left(\frac{\partial v}{\partial \phi} \right)^2 + \left(\frac{\partial w}{\partial \phi} \right)^2 \right]^{1/2} \mathrm{d}\phi. \tag{1.8}$$

The first term in Eqn. 1.8 is expanded as

$$\left(1 + \frac{\partial u}{\partial \phi} \right)^2 = 1 + 2\frac{\partial u}{\partial \phi} + \left(\frac{\partial u}{\partial \phi} \right)^2, \tag{1.9}$$

and considering the infinitesimals,

$$1 + 2\frac{\partial u}{\partial \phi} \gg \left(\frac{\partial u}{\partial \phi} \right)^2, \tag{1.10}$$

we approximate Eqn. 1.9 as

$$\left(1 + \frac{\partial u}{\partial \phi} \right)^2 \approx 1 + 2\frac{\partial u}{\partial \phi}. \tag{1.11}$$

Let

$$f(y) = (1 + y)^{1/2}. \tag{1.12}$$

The McLaurin expansion of f is given as

$$
\begin{aligned}
f(y) &= f(0) + f'(0)y + \frac{1}{2}f''(0)y^2 + \cdots \\
&= (1+0)^{1/2} + \frac{1}{2}(1+0)^{-1/2}y - \frac{1}{4}(1+0)^{-3/2}y^2 + \cdots \\
&= 1 + \frac{y}{2} - \frac{y^2}{4} + \cdots,
\end{aligned}
\tag{1.13}
$$

and assuming small variations so that $y^2 \ll y$, we approximate f near 0 by

$$
f(y) \approx 1 + \frac{y}{2}.
\tag{1.14}
$$

Plugging Eqn. 1.11 in to Eqn. 1.8 yields

$$
S = \int_0^x \left[1 + 2\frac{\partial u}{\partial \phi} + \left(\frac{\partial v}{\partial \phi}\right)^2 + \left(\frac{\partial w}{\partial \phi}\right)^2 \right]^{1/2} \mathrm{d}\phi.
\tag{1.15}
$$

Observe that this can be rewritten as

$$
S = \int_0^x \left[1 + \left\{ 2\frac{\partial u}{\partial \phi} + \left(\frac{\partial v}{\partial \phi}\right)^2 + \left(\frac{\partial w}{\partial \phi}\right)^2 \right\} \right]^{1/2} \mathrm{d}\phi,
\tag{1.16}
$$

and for

$$
y = 2\frac{\partial u}{\partial \phi} + \left(\frac{\partial v}{\partial \phi}\right)^2 + \left(\frac{\partial w}{\partial \phi}\right)^2,
\tag{1.17}
$$

the approximation in Eqn. 1.14 may be applied to Eqn. 1.16 so that arclength may be written as

$$
S = \int_0^x 1 + \frac{1}{2}\left[2\frac{\partial u}{\partial \phi} + \left(\frac{\partial v}{\partial \phi}\right)^2 + \left(\frac{\partial w}{\partial \phi}\right)^2 \right] \mathrm{d}\phi.
\tag{1.18}
$$

Distribution of the constant $1/2$ and integration over the first two terms yields

$$
S = x + u + \frac{1}{2}\int_0^x \left(\frac{\partial v}{\partial \phi}\right)^2 \mathrm{d}\phi + \frac{1}{2}\int_0^x \left(\frac{\partial w}{\partial \phi}\right)^2 \mathrm{d}\phi.
\tag{1.19}
$$

We now define the stretch term s in relation to the arclength S by

$$
S = x + s,
\tag{1.20}
$$

so that stretch can be written as

$$
s = u + \frac{1}{2}\int_0^x \left(\frac{\partial v}{\partial \phi}\right)^2 \mathrm{d}\phi + \frac{1}{2}\int_0^x \left(\frac{\partial w}{\partial \phi}\right)^2 \mathrm{d}\phi.
\tag{1.21}
$$

For ease of notation, let

$$h_v = \frac{1}{2} \int_0^x \left(\frac{\partial v}{\partial \phi} \right)^2 \mathrm{d}\phi \qquad (1.22)$$

$$h_w = \frac{1}{2} \int_0^x \left(\frac{\partial w}{\partial \phi} \right)^2 \mathrm{d}\phi \qquad (1.23)$$

so that

$$s = u + h_v + h_w. \qquad (1.24)$$

The time rate of change of stretch can thus be expressed as

$$\dot{s} = \dot{u} + \dot{h}_v + \dot{h}_w, \qquad (1.25)$$

where the chain rule yields

$$\dot{h}_v = \int_0^x \frac{\partial v}{\partial \phi} \frac{\partial \dot{v}}{\partial \eta} \mathrm{d}\phi \qquad (1.26)$$

$$\dot{h}_w = \int_0^x \frac{\partial w}{\partial \phi} \frac{\partial \dot{w}}{\partial \eta} \mathrm{d}\phi. \qquad (1.27)$$

With the book-keeping surrounding stretch completed, it is now possible to ascertain the equations of motion. Note, however, that there were many assumptions that will require the numerics to include a sufficiently fine grid in both time and space such that the assumptions are valid. A discussion of what "sufficiently fine" pertains to is given in Section 1.2.3.

1.1.2 Kinetic Energy

The system's kinetic energy is given by the classical scalar kinetic energy formulation

$$T = \frac{1}{2} m \vec{\mathbf{v}}^2. \qquad (1.28)$$

The generalized integral representation of this equation is given by

$$T = \frac{1}{2} \int_0^L \rho A \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \, \mathrm{d}x, \qquad (1.29)$$

where L is the length of the cantilever beam, ρ and A are the point density and cross section area, respectively. The units for ρ are $\frac{kg}{m^3}$, and A is m^2, and since Equation 1.29 integrates upon a singular axis, we recover mass along a single axis by the product ρA at each point along the beam.

If density and cross section are uniform along the entire beam, Equation 1.29 can be written as

$$T = \frac{1}{2}\rho A \int_0^L \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \, dx, \tag{1.30}$$

where for each point on the beam, Equation 1.30 is precisely the same as Equation 1.28.

From the rotation, and deformation corresponding to bending, the velocity at the point P is

$$\vec{\mathbf{v}}_p = (\dot{u} - \Omega v)\mathbf{i} + [\dot{v} + \Omega(a + x + u)]\mathbf{j} + \dot{w}\mathbf{k}. \tag{1.31}$$

Rotation induces angular momentum, and allowing for stretch along the axial direction yields that the momentum from the rotational speed Ω is calculated by

$$L = \Omega \cdot (a + x + u)A\rho, \tag{1.32}$$

where $(a + x)$ is the point position along the axial direction when the system is at rest, u is the displacement along the axial direction when the system is rotating at frequency Ω.

The corresponding direction of rotation is the same direction as v, so that the chord-wise component of velocity is increased by this momentum. Assuming conserved momentum, a change in the chord-wise direction must result in a change in the axial direction. Then an increase in momentum in the chord-wise direction must result in a decrease in momentum in the axial direction, so that the $-\Omega v$ term in Equation 1.31 is required. The flap-wise direction is unaffected, and it is not present in Equation 1.32. The assumption of conserved momentum is valid since the system achieves stability; otherwise, the momentum would grow rapidly and the beam would break.

Plugging Equation 1.31 into Equation 1.30 yields

$$T = \frac{1}{2}\rho A \int_0^L \langle (\dot{u} - \Omega v)\mathbf{i} + [\dot{v} + \Omega(a + x + u)]\mathbf{j} + \dot{w}\mathbf{k}\rangle^2 \, dx, \tag{1.33}$$

$$= \frac{1}{2}\rho A \int_0^L (\dot{u} - \Omega v)^2 + [\dot{v} + \Omega(a + x + u)]^2 + \dot{w}^2 \, dx. \tag{1.34}$$

1.1.3 Potential Energy

The potential energy of the system is taken to be strain energy. This is because when deformed, the beam will try to restore itself to its original shape; the restoring force can be thought of as stored energy, and typically this is called strain energy. This assumption is taken because of the properties of the material that shall be considered, typically metals such as steel and aluminum.

1.1.3.1 BUILDING THE STRAIN EQUATIONS

By considering an applied force to the beam, and applying Equation B.23 to Equation B.26, we rewrite the total strain energy as

$$U = \int_0^L \int_A \frac{1}{2E} \left(\frac{F}{A} \right)^2 \, \mathrm{d}A \, \mathrm{d}x, \tag{1.35}$$

and integrating over the area A yields

$$U = \int_0^L \frac{F^2}{2EA} \, \mathrm{d}x. \tag{1.36}$$

The axial stress resultant from stretch can be expressed by considering the stress-strain relationship in Equation B.25, and applying the strain definition from Equation B.10, we write

$$\sigma = E \frac{\partial s}{\partial x} \tag{1.37}$$

thus

$$U_{\text{stretch}} = \int_0^L \frac{EA}{2} \left(\frac{\partial s}{\partial x} \right)^2 \, \mathrm{d}x. \tag{1.38}$$

The strain induced by bending is given by

$$\varepsilon = -\frac{y}{\rho}, \tag{1.39}$$

where y is the distance from the neutral plane. For axisymmetric materials, maximum strain occurs at the boundary (or for a cylinder, the radius). ρ is given as the radius of curvature; that is, the radius of the osculating circle that best fits the curvature of the bend. The radius of curvature in 2D orthogonal (or cartesian) coordinates is expressed as the arclength of the path traveling along the rod. The arclength integral is computed to be

$$\rho = \left| \frac{(1 + y'^2)^{3/2}}{y''} \right|. \tag{1.40}$$

Here, y' is the derivative of y with respect to x. We now assume the bending is small so that $y' \ll 1$ and approximate the radius of curvature to be

$$\rho = \frac{1}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}}. \tag{1.41}$$

This assumption is valid for the metallic alloys we are considering as a result of material properties. Additionally, we are in a coordinate frame in which we will consider the directions u, v, w with respect to x.

Substituting Eqn. 1.41 into Eqn. 1.39, and adjusting into the flap-wise coordinate frame yields

$$\varepsilon = -y \left(\frac{\partial^2 v}{\partial x^2} \right), \tag{1.42}$$

and using the stress-strain relationship given in Eqn. B.25, we find

$$\sigma = -Ey \left(\frac{\partial^2 v}{\partial x^2} \right). \tag{1.43}$$

Bending moment, M, is defined to be

$$M = -\int_A y\sigma \, \mathrm{d}A = \int_A Ey^2 \left(\frac{\partial^2 v}{\partial x^2} \right) \mathrm{d}A \tag{1.44}$$

For each subsection along x, both E and $\frac{\partial^2 v}{\partial x^2}$ are considered constant with respect to the area, and we write the integral as

$$M = E \left(\frac{\partial^2 v}{\partial x^2} \right) \int_A y^2 \, \mathrm{d}A. \tag{1.45}$$

The quantity resultant of the area integral is known as the *moment of inertia*, and in this case it is the second moment about z, written as

$$I_{zz} = \int_A y^2 \, \mathrm{d}A. \tag{1.46}$$

With this simplification, we write the bending moment as

$$M_z = EI_{zz} \left(\frac{\partial^2 v}{\partial x^2} \right). \tag{1.47}$$

Upon substitution for $\frac{\partial^2 v}{\partial x^2}$, we may rewrite Eqn. 1.43 as

$$\sigma = \frac{M_z y}{I_{zz}}. \tag{1.48}$$

Plugging this stress into the strain integral from Eqn. B.26 we have

$$U_{\text{bending}} = \int_0^L \int_A \frac{1}{2} \frac{\sigma^2}{E} \, \mathrm{d}A \, \mathrm{d}x = \int_0^L \int_A \frac{1}{2E} \left(\frac{M_z y}{I_{zz}} \right)^2 \mathrm{d}A \, \mathrm{d}x \tag{1.49}$$

Distributing the power, we use the principle from Eqn. 1.45 to pull out the terms not involving y, and integrating over the cross-section area cancels one moment of inertia term I_{zz}, which yields

$$U_{\text{bending}} = \int_0^L \frac{M_z^2}{2EI_{zz}} \, \mathrm{d}x, \tag{1.50}$$

and substituting for M_z yields

$$U_{\text{bending}} = \int_0^L \frac{EI_{zz}}{2} \left(\frac{\partial^2 v}{\partial x^2} \right)^2 \mathrm{d}x. \tag{1.51}$$

This term holds for bending in two directions, flap-wise (v) and chord-wise (w). In the case of chord-wise bending, the bending moment is written as M_y, the moment of inertia is I_{yy}, and the inverse radius of curvature is given as $\frac{\partial^2 w}{\partial x^2}$. Otherwise, the formulation is precisely the same.

The full strain energy equation is a combination of these terms, given as

$$U = \int_0^L \frac{E}{2} \left[I_{zz} \left(\frac{\partial^2 v}{\partial x^2} \right)^2 + I_{yy} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + A \left(\frac{\partial s}{\partial x} \right)^2 \right] \, \mathrm{d}x. \tag{1.52}$$

1.2 APPLYING VARIATIONAL APPROXIMATION

Now that the kinetic and potential energy equations have been determined, the principles of variational approximation may be used to approximate the equations of motion. We begin by writing down the Lagrangian of the system, then rearranging the system by way of integration by parts to develop the Euler-Lagrange equations for the system. The non-vanishing terms that are not included in the Euler-Lagrange formulation are used to formulate approximation functions which both satisfy the boundary conditions and vanish as required to permit the use of variational approximation. Additionally, the properties of the physical system are considered when choosing the class of test functions which might be appropriate.

1.2.1 Developing the Euler-Lagrange Equations

The Lagrangian is given to be

$$L = T - V = T - U. \tag{1.53}$$

Applying variations as in Appendix A, we have the functional

$$J = \int_{t_0}^{t_1} L \, \mathrm{d}t \tag{1.54}$$

to be minimized as

$$\delta J = \delta \int_{t_0}^{t_1} L \, \mathrm{d}t. \tag{1.55}$$

By the linearity of δ we may move the variation inside the integral as

$$\delta J = \int_{t_0}^{t_1} \delta L \, \mathrm{d}t = \int_{t_0}^{t_1} \delta(T - U) \, \mathrm{d}t = \int_{t_0}^{t_1} \delta T - \delta U \, \mathrm{d}t. \tag{1.56}$$

For T, U, each has a spatial integral. We define the subfunctions $\hat{T}, \hat{U}$ as

$$T = \int_0^L \hat{T} \, \mathrm{d}x, \tag{1.57}$$

$$U = \int_0^L \hat{U} \, \mathrm{d}x, \tag{1.58}$$

and write

$$\delta J = \int_{t_0}^{t_1} \int_0^L \delta \hat{T} - \delta \hat{U} \, \mathrm{d}x \, \mathrm{d}t. \tag{1.59}$$

The chain rule for the variational operator requires that each independent variable of the function under variation is considered. For the kinetic energy, we have that

$$\hat{T} = \hat{T}(u, \dot{u}, v, \dot{v}, w, \dot{w}), \tag{1.60}$$

which yields

$$\delta \hat{T} = \sum_{p_i} \frac{\partial \hat{T}}{\partial p_i} \delta p_i + \frac{\partial \hat{T}}{\partial \dot{p}_i} \delta \dot{p}_i, \qquad p_i \in \{u, v, w\}. \tag{1.61}$$

Note that there isn't explicitly a function w in $\hat{T}$, but including this term for ease of notation is harmless since $\frac{\partial \hat{T}}{\partial w} = 0$. For the potential energy function, we have that

$$\hat{U} = \hat{U}(s_x, v_{xx}, w_{xx}), \tag{1.62}$$

which yields

$$\delta \hat{U} = \sum_{q_j} \frac{\partial \hat{U}}{\partial q_j} \delta q_j, \qquad q_j \in \{s_x, v_{xx}, w_{xx}\}. \tag{1.63}$$

In order to properly formulate the Lagrangian of the system, we must transform these coordinates so that terms are given in $\delta s, \delta v, \delta w$. Integration by parts shall be employed in various ways in order to derive the Lagrangian in this form.

We first introduce a tool to transform products by way of integration by parts. The assumptions are strict and specific, but are well suited for problems in this domain, since they always apply to cantilever beams.

Lemma 1.1. *Let f be an integrable function in the domain $[0, L]$ and define $F = \int f(x) \, \mathrm{d}x$ on $[0, L]$. If $F(0) = f(L) = 0$, then given the integrable function g,*

$$I = \int_0^L \left(\int_0^x g(\xi) \, \mathrm{d}\xi \right) f(x) \, \mathrm{d}x = \int_0^L \left(\int_L^x f(\xi) \, \mathrm{d}\xi \right) g(x) \, \mathrm{d}x. \tag{1.64}$$

Proof. We perform integration by parts as follows. Let

$$\alpha = \int_0^x g(\xi) \, \mathrm{d}\xi.$$

Then

$$\mathrm{d}\alpha = g(x) \, \mathrm{d}x$$

by the Fundamental Theorem of Calculus. Let

$$\mathrm{d}\beta = f(x) \, \mathrm{d}x.$$

Then

$$\beta = F(x).$$

Applying the integration by parts formula, we have

$$
\begin{aligned}
I &= \int_0^x g(\xi) \, \mathrm{d}\xi F(x)\big|_0^L - \int_0^L F(x)g(x) \, \mathrm{d}x \\
&= \int_0^L g(x) \, \mathrm{d}x F(L) - \int_0^L F(x)g(x) \, \mathrm{d}x \quad \text{(since } F(0) = 0) \\
&= \int_0^L F(L)g(x) \, \mathrm{d}x - \int_0^L F(x)g(x) \, \mathrm{d}x \quad \text{(since } F(L) \text{ is constant)} \\
&= \int_0^L [F(L) - F(x)]g(x) \, \mathrm{d}x \quad \text{(by linearity)} \\
&= \int_0^L \left(\int_L^x f(\xi) \, \mathrm{d}\xi \right) g(x) \, \mathrm{d}x \quad \text{(by definition of } F)
\end{aligned}
$$

$\square$

Since time, t and space, x are independent of each other, the iterated integrals may be switched, and we may write the variation on $\hat{T}$ as follows.

$$\int_{t_0}^{t_1} \int_0^L \delta\hat{T} \, \mathrm{d}x \, \mathrm{d}t = \int_0^L \int_{t_0}^{t_1} \delta\hat{T} \, \mathrm{d}t \, \mathrm{d}x. \tag{1.65}$$

The term for $q_i = u$ has special properties and we shall consider those properties later; we first analyze $q_i \in \{v, w\}$. The $\dot{q}_i$ term is integrated by using integration by parts. This is accomplished by utilizing the time integral; the spatial integral shall be excluded for brevity. First, write out the full variation on the kinetic energy:

$$\int_{t_0}^{t_1} \delta\hat{T}(\dot{q}_i, q_i) \, \mathrm{d}t = \int_{t_0}^{t_1} \frac{\partial\hat{T}}{\partial\dot{q}_i} \delta\dot{q}_i \, \mathrm{d}t + \int_{t_0}^{t_1} \frac{\partial\hat{T}}{\partial q_i} \delta q_i \, \mathrm{d}t. \tag{1.66}$$

Then, consider the first integral on the right hand side, and let

$$\alpha = \frac{\partial\hat{T}}{\partial\dot{q}_i}, \tag{1.67}$$

then

$$\mathrm{d}\alpha = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial\hat{T}}{\partial\dot{q}_i} \right) \mathrm{d}t. \tag{1.68}$$

Next, let

$$\mathrm{d}\beta = \delta\dot{q}_i \, \mathrm{d}t, \tag{1.69}$$

then

$$\beta = \int \frac{\mathrm{d}}{\mathrm{d}t} (\delta q_i) \, \mathrm{d}t = \delta q_i. \tag{1.70}$$

Finally, we may write the first integral as

$$\int_{t_0}^{t_1} \frac{\partial\hat{T}}{\partial\dot{q}_i} \delta\dot{q}_i \, \mathrm{d}t = \frac{\partial\hat{T}}{\partial\dot{q}_i} \delta q_i \bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial\hat{T}}{\partial\dot{q}_i} \right) \delta q_i \, \mathrm{d}t. \tag{1.71}$$

Variational approximation requires that

$$\frac{\partial \hat{T}}{\partial \dot{q}_i} \delta q_i \bigg|_{t_0}^{t_1} = 0, \tag{1.72}$$

thus the first integral is given by

$$\int_{t_0}^{t_1} \frac{\partial \hat{T}}{\partial \dot{q}_i} \delta \dot{q}_i \, \mathrm{d}t = -\int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat{T}}{\partial \dot{q}_i}\right) \delta q_i \, \mathrm{d}t. \tag{1.73}$$

Substitution of Eqn. 1.73 into Eqn. 1.66 yields

$$\int_{t_0}^{t_1} \delta \hat{T}(\dot{q}_i, q_i) \, \mathrm{d}t = \int_{t_0}^{t_1} \left\{ -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \hat{T}}{\partial \dot{q}_i}\right] + \frac{\partial \hat{T}}{\partial q_i} \right\} \delta q_i \, \mathrm{d}t. \tag{1.74}$$

We now reintroduce the spatial integral and switch the order of integration so that $\hat{T}, \hat{U}$ are integrating in the same order, which yields the following expression.

$$\int_{t_0}^{t_1} \int_0^L \delta \hat{T}(\dot{q}_i, q_i) \, \mathrm{d}x \, \mathrm{d}t = \int_{t_0}^{t_1} \int_0^L \left\{ -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \hat{T}}{\partial \dot{q}_i}\right] + \frac{\partial \hat{T}}{\partial q_i} \right\} \delta q_i \, \mathrm{d}x \, \mathrm{d}t. \tag{1.75}$$

We now continue the process for strain energy.

The strain energy U yields differing variation fields which must be converted to be in the q_i form.

The dynamic expressions in U are given in Eqn. 1.52 as v_{xx}, w_{xx}, and s_x. The generalized coordinates are q_{xx_i}, and q_{x_i}, where the subscript x represents the derivative with respect to x, and the subscript i represents the i-th direction.

The chain rule applied to the strain term U is given by

$$\delta U(q_{xx_i}, q_{x_i}) = \frac{\partial U}{\partial q_{xx_i}} \delta q_{xx_i} + \frac{\partial U}{\partial q_{x_i}} \delta q_{x_i}. \tag{1.76}$$

Once again, we aim to modify this expression to that the terms on the right are given as variations of q_i. The full variation of the strain energy is given as

$$\int_{t_0}^{t_1} \delta U(q_{xx_i}, q_{x_i}) \, \mathrm{d}t = \int_{t_0}^{t_1} \frac{\partial U}{\partial q_{xx_i}} \delta q_{xx_i} \, \mathrm{d}t + \int_{t_0}^{t_1} \frac{\partial U}{\partial q_{x_i}} \delta q_{x_i} \, \mathrm{d}t. \tag{1.77}$$

We now rewrite U as follows.

$$U = \int_0^L \hat{U} \, \mathrm{d}x, \tag{1.78}$$

so that

$$\delta U = \delta \int_0^L \hat{U} \, \mathrm{d}x = \int_0^L \delta \hat{U} \, \mathrm{d}x. \tag{1.79}$$

The dynamic components of U are the same as that of $\hat{U}$, so the chain rule applies in the same way and the full variation applied to this new expression is given as

$$\int_{t_0}^{t_1} \int_0^L \delta \hat{U}(q_{xx_i}, q_{x_i}) \, \mathrm{d}x \, \mathrm{d}t = \int_{t_0}^{t_1} \int_0^L \frac{\partial \hat{U}}{\partial q_{xx_i}} \delta q_{xx_i} \, \mathrm{d}x \, \mathrm{d}t + \int_{t_0}^{t_1} \int_0^L \frac{\partial \hat{U}}{\partial q_{x_i}} \delta q_{x_i} \, \mathrm{d}x \, \mathrm{d}t. \quad (1.80)$$

Following the same form, we integrate by parts. First, consider the first integral on the right side. For brevity, we shall not write the time integral, and focus on the spatial integral. As before, let

$$\alpha = \frac{\partial \hat{U}}{\partial q_{xx_i}}. \quad (1.81)$$

Then

$$\mathrm{d}\alpha = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial \hat{U}}{\partial q_{xx_i}} \right) \mathrm{d}x. \quad (1.82)$$

Let

$$\mathrm{d}\beta = \delta q_{xx_i} \, \mathrm{d}x. \quad (1.83)$$

Then

$$\beta = \int \frac{\mathrm{d}}{\mathrm{d}x} \delta q_{x_i} \, \mathrm{d}x = \delta q_{x_i}. \quad (1.84)$$

Applying the integration by parts formula, we have

$$\int_0^L \frac{\partial \hat{U}}{\partial q_{xx_i}} \delta q_{xx_i} \, \mathrm{d}x = \left. \frac{\partial \hat{U}}{\partial q_{xx_i}} \delta q_{x_i} \right|_0^L - \int_0^L \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial \hat{U}}{\partial q_{xx_i}} \right) \delta q_{x_i} \, \mathrm{d}x. \quad (1.85)$$

In this case, the first expression will not necessarily vanish. This provides the means for ensuring the approximating functions will satisfy the physical boundary conditions. Furthermore, the second term is still not in proper form, as the variation is acting on q_{x_i}, not q_i. Thus, we must integrate the second term by parts. Let

$$\alpha = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial \hat{U}}{\partial q_{xx_i}} \right). \quad (1.86)$$

Then

$$\mathrm{d}\alpha = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(\frac{\partial \hat{U}}{\partial q_{xx_i}} \right) \mathrm{d}x. \quad (1.87)$$

Let

$$\mathrm{d}\beta = \delta q_{x_i} \, \mathrm{d}x. \quad (1.88)$$

Then

$$\beta = \int \frac{\mathrm{d}}{\mathrm{d}x} \delta q_i \, \mathrm{d}x = \delta q_i. \quad (1.89)$$

Applying the integration by parts formula yields

$$\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{U}}{\partial q_{xx_i}}\right)\delta q_{x_i}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{U}}{\partial q_{xx_i}}\right)\delta q_i\bigg|_0^L - \int_0^L \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \hat{U}}{\partial q_{xx_i}}\right)\delta q_i\,\mathrm{d}x. \tag{1.90}$$

Once again, the first term is a boundary condition term. Plugging this into Eqn. 1.85 yields

$$\int_0^L \frac{\partial \hat{U}}{\partial q_{xx_i}}\delta q_{xx_i}\,\mathrm{d}x = \frac{\partial \hat{U}}{\partial q_{xx_i}}\delta q_{x_i}\bigg|_0^L - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{U}}{\partial q_{xx_i}}\right)\delta q_i\bigg|_0^L + \int_0^L \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \hat{U}}{\partial q_{xx_i}}\right)\delta q_i\,\mathrm{d}x. \tag{1.91}$$

Similarly, we integrate the second term of Eqn. 1.80 by parts. Let

$$\alpha = \frac{\partial \hat{U}}{\partial q_{x_i}}. \tag{1.92}$$

Then

$$\mathrm{d}\alpha = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{U}}{\partial q_{x_i}}\right)\,\mathrm{d}x. \tag{1.93}$$

Let

$$\mathrm{d}\beta = \delta q_{x_i}\,\mathrm{d}x. \tag{1.94}$$

Then

$$\beta = \int \frac{\mathrm{d}}{\mathrm{d}x}\delta q_i\,\mathrm{d}x = \delta q_i. \tag{1.95}$$

Applying the integration by parts formula, we have

$$\int_0^L \frac{\partial \hat{U}}{\partial q_{x_i}}\delta q_{x_i}\,\mathrm{d}x = \frac{\partial \hat{U}}{\partial q_{x_i}}\delta q_i\bigg|_0^L - \int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{U}}{\partial q_{x_i}}\right)\delta q_i\,\mathrm{d}x. \tag{1.96}$$

Again, the lead term is for the boundary conditions. Replacing Eqn. 1.91 and Eqn. 1.96 into Eqn. 1.80 yields

$$\int_{t_0}^{t_1}\int_0^L \delta\hat{U}(q_{xx_i}, q_{x_i})\,\mathrm{d}x\,\mathrm{d}t = \int_{t_0}^{t_1}\int_0^L \left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial \hat{U}}{\partial q_{xx_i}}\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial \hat{U}}{\partial q_{x_i}}\right]\right\}\delta q_i\,\mathrm{d}x\,\mathrm{d}t + \text{B.C.'s}, \tag{1.97}$$

where the term "B.C.'s" is given by

$$\text{B.C.'s} = \frac{\partial \hat{U}}{\partial q_{xx_i}}\delta q_{x_i}\bigg|_0^L - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \hat{U}}{\partial q_{xx_i}}\right)\delta q_i\bigg|_0^L + \frac{\partial \hat{U}}{\partial q_{x_i}}\delta q_i\bigg|_0^L. \tag{1.98}$$

Rewriting $\hat{U}$ in terms of U yields

$$\int_{t_0}^{t_1} \delta U(q_{xx_i}, q_{x_i})\,\mathrm{d}t = \int_{t_0}^{t_1} \left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial U}{\partial q_{xx_i}}\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial U}{\partial q_{x_i}}\right]\right\}\delta q_i\,\mathrm{d}t + \text{B.C.'s}, \tag{1.99}$$

with "B.C.'s" given by

$$\text{B.C.'s} = \frac{\partial U}{\partial q_{xx_i}} \delta q_{x_i} \bigg|_0^L - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial U}{\partial q_{xx_i}} \right) \delta q_i \bigg|_0^L + \frac{\partial U}{\partial q_{x_i}} \delta q_i \bigg|_0^L. \tag{1.100}$$

Combining the kinetic and strain energy variations, we have

$$\int_{t_0}^{t_1} \delta L \, \mathrm{d}t = \int_{t_0}^{t_1} \left\{ -\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial T}{\partial \dot{q}_i} \right] + \frac{\partial T}{\partial q_i} - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left[\frac{\partial U}{\partial q_{xx_i}} \right] + \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\partial U}{\partial q_{x_i}} \right] \right\} \delta q_i \, \mathrm{d}t + \text{B.C.'s.} \tag{1.101}$$

This formulation, provided the B.C.'s are properly satisfied, is to be minimized. Thus, we set

$$\int_{t_0}^{t_1} \delta L \, \mathrm{d}t = 0, \tag{1.102}$$

so that, given the arbitrary nature of the test function q_i, it must be true that

$$-\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial T}{\partial \dot{q}_i} \right] + \frac{\partial T}{\partial q_i} - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left[\frac{\partial U}{\partial q_{xx_i}} \right] + \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\partial U}{\partial q_{x_i}} \right] = 0. \tag{1.103}$$

These are known as the *Euler-Lagrange Equations*. This expression describes the motion of the system.

1.2.2 System Analysis

Considering that the displacement u is a function of s, h_v, h_w, we must analyze the special case of δu. The Chain Rule for Variations yields that given that

$$u = u(s, h_v, h_w) = s - (h_v + h_w), \tag{1.104}$$

we write the variation of u as

$$\delta u = \frac{\partial u}{\partial s} \delta s + \frac{\partial u}{\partial h_v} \delta h_v + \frac{\partial u}{\partial h_w} \delta h_w = \delta s - \delta h_v - \delta h_w. \tag{1.105}$$

Similarly, given that

$$\dot{u} = \dot{u}(\dot{s}, \dot{h}_v, \dot{h}_w) = \dot{s} - (\dot{h}_v + \dot{h}_w), \tag{1.106}$$

we write the variation of $\dot{u}$ as

$$\delta \dot{u} = \frac{\partial \dot{u}}{\partial \dot{s}} \delta \dot{s} + \frac{\partial \dot{u}}{\partial \dot{h}_v} \delta \dot{h}_v + \frac{\partial \dot{u}}{\partial \dot{h}_w} \delta \dot{h}_w = \delta \dot{s} - \delta \dot{h}_v - \delta \dot{h}_w. \tag{1.107}$$

The variations $\delta h_v, \delta h_w, \delta \dot{h}_v$ and $\delta \dot{h}_w$ are not in the proper frame; that is, we need variations on s, v, w only. Since h_v, h_w are functions of v, w respectively, we can calculate the variations as follows.

$$\delta h_v = \delta \frac{1}{2} \int_0^x \left(\frac{\partial v}{\partial \eta} \right)^2 \mathrm{d}\eta. \tag{1.108}$$

Let

$$\tau = \left(\frac{\partial v}{\partial \eta}\right)^2. \tag{1.109}$$

Then

$$\delta h_v = \delta \frac{1}{2} \int_0^x \tau \, d\eta, \tag{1.110}$$

and the Chain Rule for Variations yields

$$\delta h_v = \frac{1}{2} \int_0^x \frac{\partial \tau}{\partial v_\eta} \delta v_\eta \, d\eta = \int_0^x v_\eta \delta v_\eta \, d\eta, \tag{1.111}$$

where v_η is the partial derivative of v with respect to η. Note that the variation is not in terms of v, but in terms of v_η; we shall leave the variation in this form for simplicity. The reason shall be clear upon application to the system.

The process is identical for δh_w, thus plugging into Eqn. 1.105 we have

$$\delta u = \delta s - \int_0^x \frac{\partial v}{\partial \eta} \delta v_\eta \, d\eta - \int_0^x \frac{\partial w}{\partial \eta} \delta w_\eta \, d\eta. \tag{1.112}$$

For the terms $\delta \dot{h}_v$ and $\delta \dot{h}_w$, we consider the full terms as follows. First, we shall consider $\delta \dot{h}_v$.

$$-\delta \dot{h}_v \ = \ -\delta \frac{\partial h_v}{\partial t} \tag{1.113}$$

$$= \ -\frac{\partial}{\partial t} \delta h_v \tag{1.114}$$

$$= \ -\frac{\partial}{\partial t} \left[-\int_0^x v_{\eta\eta} \delta v \, d\eta \right] \tag{1.115}$$

$$= \ \int_0^x \frac{\partial}{\partial t} \left[\frac{\partial^2 v}{\partial \eta^2} \delta v \right] \, d\eta \tag{1.116}$$

$$= \ \int_0^x \frac{\partial^3 v}{\partial t \partial \eta^2} \delta v + \frac{\partial^2 v}{\partial \eta^2} \delta \dot{v} \, d\eta \tag{1.117}$$

However, this is multiplied by the partial of T on $\dot{u}$, which yields

$$\int_{t_0}^{t_1} \int_0^L \left(\int_0^x \frac{\partial}{\partial t} \left[\frac{\partial^2 v}{\partial \eta^2} \delta v \right] \, d\eta \right) \frac{\partial \hat{T}}{\partial \dot{u}} \, dx \, dt. \tag{1.118}$$

We apply Lemma 1.1 and find

$$\int_{t_0}^{t_1} \int_0^L \left(\int_L^x \frac{\partial \hat{T}}{\partial \dot{u}} \, d\eta \right) \frac{\partial}{\partial t} \left[\frac{\partial^2 v}{\partial x^2} \delta v \right] \, dx \, dt \tag{1.119}$$

$$= \int_{t_0}^{t_1} \int_0^L \left(\int_L^x \frac{\partial \hat{T}}{\partial \dot{u}} \, d\eta \right) \left[\frac{\partial^3 v}{\partial t \partial x^2} \delta v + \frac{\partial^2 v}{\partial x^2} \delta \dot{v} \right] \, dx \, dt.$$

The term involving $\delta\dot{v}$ must be reconfigured so that the variation is in terms of v instead of $\dot{v}$. To accomplish this, we perform integration by parts, as below. Consider the integral

$$\int_{t_0}^{t_1} \frac{\partial^2 v}{\partial x^2} \delta\dot{v} \, dt. \tag{1.120}$$

Let

$$\alpha = \frac{\partial^2 v}{\partial x^2}. \tag{1.121}$$

Then

$$d\alpha = \frac{\partial}{\partial t}\frac{\partial^2 v}{\partial x^2} dt = \frac{\partial^3 v}{\partial t \partial x^2} dt. \tag{1.122}$$

Let

$$d\beta = \delta\dot{v} \, dt. \tag{1.123}$$

Then

$$\beta = \int \delta \frac{\partial v}{\partial t} dt = \int \frac{\partial}{\partial t}\delta v \, dt = \delta v. \tag{1.124}$$

Applying the integration by parts formula, we have

$$\int_{t_0}^{t_1} \frac{\partial^2 v}{\partial x^2} \delta\dot{v} \, dt = \left.\frac{\partial^2 v}{\partial x^2}\delta v\right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{\partial^3 v}{\partial t \partial x^2}\delta v \, dt, \tag{1.125}$$

and the first term on the right side vanishes by definition of δ, so we have that

$$\int_{t_0}^{t_1} \frac{\partial^2 v}{\partial x^2} \delta\dot{v} \, dt = -\int_{t_0}^{t_1} \frac{\partial^3 v}{\partial t \partial x^2}\delta v \, dt, \tag{1.126}$$

and clearly

$$\frac{\partial^2 v}{\partial x^2}\delta\dot{v} = -\frac{\partial^3 v}{\partial t \partial x^2}\delta v. \tag{1.127}$$

Substituting this result in Eqn. 1.119 yields

$$\int_{t_0}^{t_1}\int_0^L \left(\int_L^x \frac{\partial \hat{T}}{\partial \dot{u}} \, d\eta\right)\left[\frac{\partial^3 v}{\partial t \partial x^2}\delta v - \frac{\partial^3 v}{\partial t \partial x^2}\delta v\right] dx \, dt = 0. \tag{1.128}$$

The result is precisely the same for $\delta\dot{h}_w$, so that

$$\delta\dot{u} = \delta\dot{s}. \tag{1.129}$$

In our analysis we may now substitute these results for $\delta u, \delta\dot{u}$. We shall now evaluate the terms $\frac{\partial \hat{T}}{\partial u}$ and $\frac{\partial \hat{T}}{\partial \dot{u}}$.

$$\frac{\partial \hat{T}}{\partial u} = \frac{\partial \hat{T}}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial \hat{T}}{\partial h_v}\frac{\partial h_v}{\partial u} + \frac{\partial \hat{T}}{\partial h_w}\frac{\partial h_w}{\partial u}. \tag{1.130}$$

Since

$$u = s - (h_v + h_w), \tag{1.131}$$

we have

$$s = u + (h_v + h_w), \tag{1.132}$$

$$h_v = s - (u + h_w), \tag{1.133}$$

$$h_w = s - (u + h_v), \tag{1.134}$$

so that

$$\frac{\partial s}{\partial u} = 1, \tag{1.135}$$

$$\frac{\partial h_v}{\partial u} = -1, \tag{1.136}$$

$$\frac{\partial h_w}{\partial u} = -1, \tag{1.137}$$

which yields

$$\frac{\partial \hat{T}}{\partial u} = \frac{\partial \hat{T}}{\partial s} - \left(\frac{\partial \hat{T}}{\partial h_v} + \frac{\partial \hat{T}}{\partial h_w} \right). \tag{1.138}$$

Similarly,

$$\frac{\partial \hat{T}}{\partial \dot{u}} = \frac{\partial \hat{T}}{\partial \dot{s}} - \left(\frac{\partial \hat{T}}{\partial \dot{h}_v} + \frac{\partial \hat{T}}{\partial \dot{h}_w} \right). \tag{1.139}$$

Given the linearity of both $\frac{\partial \hat{T}}{\partial u}$ and $\frac{\partial \hat{T}}{\partial \dot{u}}$, we may calculate these in terms of $u, \dot{u}$ and substitute for $u = s - (h_v + h_w)$ upon conclusion of the calculation.

We begin the analysis with the first term of Eqn. 1.103.

$$-\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial T}{\partial \dot{q}_i} \right]\bigg|_{q_i = u} = -\left(\frac{\rho A}{2} \right) \frac{\mathrm{d}}{\mathrm{d}t} \int_0^L [2\dot{u} - 2\Omega v] \, \mathrm{d}x, \tag{1.140}$$

$$= -\rho A \int_0^L [\ddot{u} - \dot{\Omega} v - \Omega \dot{v}] \, \mathrm{d}x, \tag{1.141}$$

Continuing, we have

$$-\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial T}{\partial \dot{q}_i} \right]\bigg|_{q_i = v} = -\left(\frac{\rho A}{2} \right) \frac{\mathrm{d}}{\mathrm{d}t} \int_0^L [2\dot{v} + 2\Omega(a + x + u)] \, \mathrm{d}x, \tag{1.142}$$

$$= -\int_0^L \rho A [\ddot{v} + \dot{\Omega}(a + x + u) + \Omega \dot{u}] \, \mathrm{d}x. \tag{1.143}$$

In w,

$$-\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial T}{\partial \dot{q}_i} \right]\bigg|_{q_i = w} = -\left(\frac{\rho A}{2} \right) \frac{\mathrm{d}}{\mathrm{d}t} \int_0^L [2\dot{w}] \, \mathrm{d}x, \tag{1.144}$$

$$= -\int_0^L \rho A [\ddot{w}] \, \mathrm{d}x. \tag{1.145}$$

Next we shall analyze the second term. In u,

$$\left.\frac{\partial T}{\partial q_i}\right|_{q_i=u} = \frac{\rho A}{2}\int_0^L 2\dot{v}\Omega + 2\Omega^2(u+a+x)\,\mathrm{d}x, \tag{1.146}$$

$$= \rho A\int_0^L [\dot{v}\Omega + \Omega^2(u+a+x)]\,\mathrm{d}x. \tag{1.147}$$

In v,

$$\left.\frac{\partial T}{\partial q_i}\right|_{q_i=v} = \frac{\rho A}{2}\int_0^L [-2\Omega\dot{u} + 2\Omega^2 v]\,\mathrm{d}x, \tag{1.148}$$

$$= \rho A\int_0^L [\Omega^2 v - \Omega\dot{u}]\,\mathrm{d}x. \tag{1.149}$$

Finally, in w,

$$\left.\frac{\partial T}{\partial q_i}\right|_{q_i=w} = 0 \tag{1.150}$$

The third and fourth terms are from strain energy. The third term is analyzed as follows. In s,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial U}{\partial q_{xx_i}}\right]\Bigg|_{q_i=s} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}[0], \tag{1.151}$$

$$= 0. \tag{1.152}$$

In v,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial U}{\partial q_{xx_i}}\right]\Bigg|_{q_i=v} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^L \frac{EI_{zz}}{2}[2v_{xx}]\,\mathrm{d}x, \tag{1.153}$$

$$= -E\int_0^L [I_{zz}]_{xx}[v_{xx}] + 2[I_{zz}]_x[v_{xxx}] + \tag{1.154}$$
$$[I_{zz}][v_{xxxx}]\,\mathrm{d}x.$$

We have already assumed constant cross-section area A, and we shall also consider that moment of inertia is constant, so that $I_{qq_x} = 0$, $qq \in \{yy, zz\}$. Thus we write

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial U}{\partial q_{xx_i}}\right]\Bigg|_{q_i=v} = -EI_{zz}\int_0^L v_{xxxx}\,\mathrm{d}x. \tag{1.155}$$

In w,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial U}{\partial q_{xx_i}}\right]\Bigg|_{q_i=w} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^L \frac{EI_{yy}}{2}[2w_{xx}]\,\mathrm{d}x \tag{1.156}$$

$$= -EI_{yy}\int_0^L w_{xxxx}\,\mathrm{d}x. \tag{1.157}$$

Finally, the fourth term is given as follows. In s,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial U}{\partial q_{x_i}}\right]\bigg|_{q_i=s} = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^L \frac{EA}{2}\left[2s_x\right]\,\mathrm{d}x, \tag{1.158}$$

$$= EA\int_0^L s_{xx}\,\mathrm{d}x. \tag{1.159}$$

In v,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial U}{\partial q_{x_i}}\right]\bigg|_{q_i=v} = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^L \frac{EA}{2}\left[0\right]\,\mathrm{d}x \tag{1.160}$$

$$= 0. \tag{1.161}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial U}{\partial q_{x_i}}\right]\bigg|_{q_i=w} = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^L \frac{EA}{2}\left[0\right]\,\mathrm{d}x \tag{1.162}$$

$$= 0. \tag{1.163}$$

It is important to note that the equations are broken up into their respective coordinates purposefully. One reason is to illuminate the zero evaluations. The other is that the Euler-Lagrange equations must be evaluated for each coordinate, s, v, w. We will summarize these results by grouping the non-zero terms into their respective coordinate references.

We first consider the Lagrangian for the various coordinates. We use the linearity to write the derivatives of the potential function T in terms of u. Furthermore, we show the Lagrangian in terms of the three coordinates s, v, w as separate equations. The expanded Lagrangian under variation is written as

$$\delta\int_{t_0}^{t_1} L\,\mathrm{d}t\bigg|_s = \int_{t_0}^{t_1}\left\{-\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial \dot{u}}\right] + \frac{\partial T}{\partial u} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial U}{\partial s_{xx}}\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial U}{\partial s_x}\right]\right\}\delta s\,\mathrm{d}t, \tag{1.164}$$

$$\delta\int_{t_0}^{t_1} L\,\mathrm{d}t\bigg|_v = \int_{t_0}^{t_1}\left\{-\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial \dot{v}}\right] + \frac{\partial T}{\partial v} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial U}{\partial v_{xx}}\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial U}{\partial v_x}\right]\right\}\delta v\,\mathrm{d}t \tag{1.165}$$

$$-\int_{t_0}^{t_1}\frac{\partial T}{\partial u}\left[\int_0^x v_\eta\delta v_\eta\,\mathrm{d}\eta\right]\,\mathrm{d}t,$$

$$\delta\int_{t_0}^{t_1} L\,\mathrm{d}t\bigg|_w = \int_{t_0}^{t_1}\left\{-\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T}{\partial \dot{w}}\right] + \frac{\partial T}{\partial w} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\partial U}{\partial w_{xx}}\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial U}{\partial w_x}\right]\right\}\delta w\,\mathrm{d}t \tag{1.166}$$

$$-\int_{t_0}^{t_1}\frac{\partial T}{\partial u}\left[\int_0^x w_\eta\delta w_\eta\,\mathrm{d}\eta\right]\,\mathrm{d}t,$$

where we have appended the terms for $\delta h_v, \delta h_w$ into the v and w variation directions.

Now we write down the s coordinate equations, after plugging in for $u = s - (h_v + h_w)$.

$$\int_{t_0}^{t_1} \left[-\rho A \int_0^L \left[\ddot{s} - \frac{\partial}{\partial t}(\dot{h}_v + \dot{h}_w) - \dot{\Omega}v - \Omega\dot{v} \right] dx \right.$$
$$+ \rho A \int_0^L [\{s - (h_v + h_w) + a + x\}\Omega^2 + \dot{v}\Omega] \, dx \qquad (1.167)$$
$$\left. + EA \int_0^L s_{xx} \, dx \right] \delta s \, dt$$

Applying the variation, canceling integrals and simplifying yields the following.

$$\rho A \left[\ddot{s} - \frac{\partial}{\partial t}(\dot{h}_v + \dot{h}_w) - \dot{\Omega}v + 2\Omega\dot{v} - \{s - (h_v + h_w) + a + x\}\Omega^2 \right] = EAs_{xx}. \qquad (1.168)$$

Next, we write the v coordinate equations after substituting for u.

$$\int_{t_0}^{t_1} \left[-\rho A \int_0^L [\ddot{v} + \dot{\Omega}(a + x + s - (h_v + h_w)) + \Omega(\dot{s} - (\dot{h}_v + \dot{h}_w))] \, dx \right.$$
$$+ \rho A \int_0^L [-\Omega(\dot{s} - (\dot{h}_v + \dot{h}_w)) + \Omega^2 v] \, dx + EI_{zz} \int_0^L v_{xxxx} \, dx \right] \delta v \, dt \quad (1.169)$$
$$- \int_{t_0}^{t_1} \rho A \int_0^L [\{s - (h_v + h_w) + a + x\}\Omega^2 + \dot{v}\Omega] \left[\int_0^x v_\eta \delta v_\eta \, d\eta \right] dx \, dt$$

We cannot apply the variation unless the variation is in the proper domain. We transform the variation to be in terms of v using integration by parts. To solve this, we first apply Lemma 1.1, which yields

$$\int_{t_0}^{t_1} \left[-\rho A \int_0^L [\ddot{v} + \dot{\Omega}(a + x + s - (h_v + h_w)) + \Omega(\dot{s} - (\dot{h}_v + \dot{h}_w))] \, dx \right.$$
$$+ \rho A \int_0^L [-\Omega(\dot{s} - (\dot{h}_v + \dot{h}_w)) + \Omega^2 v] \, dx + EI_{zz} \int_0^L v_{xxxx} \, dx \qquad (1.170)$$
$$\left. - \rho A \int_0^L \left[\int_L^x \{s - (h_v + h_w) + a + \eta\}\Omega^2 + \dot{v}\Omega \, d\eta \right] v_x \delta v_x \, dx \right] dt.$$

We now apply integration by parts on the last term. Let

$$\alpha = \left[\int_L^x \{s - (h_v + h_w) + a + \eta\}\Omega^2 + \dot{v}\Omega \, d\eta \right] v_x, \qquad (1.171)$$

then

$$d\alpha = \frac{\partial}{\partial x} \left\{ \left[\int_L^x (s - (h_v + h_w) + a + \eta)\Omega^2 + \dot{v}\Omega \, d\eta \right] v_x \right\}. \qquad (1.172)$$

Let

$$d\beta = \delta v_x \, dx, \qquad (1.173)$$

then

$$\beta = \delta v. \tag{1.174}$$

Application of the integration by parts formula yields

$$\int_0^L \left[\int_L^x (s - (h_v + h_w) + a + \eta)\Omega^2 + \dot{v}\Omega \, \mathrm{d}\eta \right] v_x \delta v_x \, \mathrm{d}x \tag{1.175}$$

$$= \alpha\delta v \Big|_0^L - \int_0^L \frac{\partial}{\partial x} \left\{ \left[\int_L^x (s - (h_v + h_w) + a + \eta)\Omega^2 + \dot{v}\Omega \, \mathrm{d}\eta \right] v_x \right\} \delta v \, \mathrm{d}x,$$

where the term $\alpha\delta v$ vanishes on the boundary by definition of δ. Applying the variation, canceling integrals and simplifying yields the following.

$$-\rho A[\ddot{v} + \dot{\Omega}(a + x + s - (h_v + h_w)) + 2\Omega(\dot{s} - (\dot{h}_v + \dot{h}_w)) - \Omega^2 v] \tag{1.176}$$

$$+\rho A \frac{\partial}{\partial x} \left\{ \left[\int_L^x (s - (h_v + h_w) + a + \eta)\Omega^2 + \dot{v}\Omega \, \mathrm{d}\eta \right] v_x \right\} = -EI_{zz}v_{xxxx}$$

Lastly, we write the w coordinate equations.

$$\int_{t_0}^{t_1} \left[-\rho A \int_0^L \ddot{w} \, \mathrm{d}x + EI_{yy} \int_0^L w_{xxxx} \, \mathrm{d}x \right] \delta w \, \mathrm{d}t \tag{1.177}$$

$$+ \int_{t_0}^{t_1} \rho A \int_0^L [(s - (h_v + h_w) + a + x)\Omega^2 + \dot{v}\Omega] \left[\int_0^x w_\eta \delta w_\eta \, \mathrm{d}\eta \right] \mathrm{d}x \, \mathrm{d}t$$

Again, we must apply Lemma 1.1 to move the variation.

$$\int_{t_0}^{t_1} \left[-\rho A \int_0^L \ddot{w} \, \mathrm{d}x + EI_{yy} \int_0^L w_{xxxx} \, \mathrm{d}x \right] \delta w \, \mathrm{d}t \tag{1.178}$$

$$+ \int_{t_0}^{t_1} \rho A \int_0^L \int_L^x [(s - (h_v + h_w) + a + \eta)\Omega^2 + \dot{v}\Omega] \, \mathrm{d}\eta \, w_x \delta w_x \, \mathrm{d}x \, \mathrm{d}t$$

We perform integration by parts in the same way as in the case of v, which yields

$$\int_0^L \left[\int_L^x (s - (h_v + h_w) + a + \eta)\Omega^2 + \dot{v}\Omega \, \mathrm{d}\eta \right] w_x \delta w_x \, \mathrm{d}x \tag{1.179}$$

$$= \alpha\delta w \Big|_0^L - \int_0^L \frac{\partial}{\partial x} \left\{ \left[\int_L^x (s - (h_v + h_w) + a + \eta)\Omega^2 + \dot{v}\Omega \, \mathrm{d}\eta \right] w_x \right\} \delta w \, \mathrm{d}x,$$

where the term $\alpha\delta w$ vanishes on the boundary by definition of δ. Applying the variation, canceling integrals and simplifying yields the following.

$$-\rho A\ddot{w} + \rho A \frac{\partial}{\partial x} \left\{ \left[\int_L^x (s - (h_v + h_w) + a + \eta)\Omega^2 + \dot{v}\Omega \, \mathrm{d}\eta \right] w_x \right\} = -EI_{yy}w_{xxxx} \tag{1.180}$$

Note that the product of $s, h_v, h_w,$ and $\dot{v}$ with either v_x or w_x yields nonlinear terms. Furthermore, recall that h_v, h_w and their derivatives are nonlinear. We shall exclude these

terms. Additionally, we shall *ad hoc* add the forcing terms p_v, p_w. This simplification yields the following system of linear differential equations. This is an approximation to the equations of motion of a rotating cantilever beam.

$$\rho A \left[\ddot{s} - \dot{\Omega} v + 2\Omega \dot{v} - (s + a + x)\Omega^2 \right] = EAs_{xx}. \tag{1.181}$$

$$-\rho A[\ddot{v} + \dot{\Omega}(s+a+x) + 2\Omega\dot{s} - \Omega^2 v] + \rho A \frac{\partial}{\partial x} \left\{ \int_L^x \Omega^2(a+\eta) \, \mathrm{d}\eta v_x \right\} = p_v - EI_{zz}v_{xxxx} \tag{1.182}$$

$$-\rho A\ddot{w} + \rho A \frac{\partial}{\partial x} \left\{ \int_L^x \Omega^2(a+\eta) \, \mathrm{d}\eta w_x \right\} = p_w - EI_{yy}w_{xxxx} \tag{1.183}$$

1.2.3 Discretizing the System

We now have the final equations of motion, with the caveat that 'suitable' test functions q_i must be selected. 'Suitable' means that the functions are in H^2 in the interval $[0, L]$, and that all boundary and initial conditions are satisfied by the functions. There are standard test functions that are frequently used; choosing which class of test functions to use is dependent upon the physics of the problem. Furthermore, the test functions should match the behavior of the spatial displacements of the system. Polynomial shapes are reasonable approximations to how we expect the beam to displace from bending and stretching, so choosing polynomials for test functions is logical.

In order that we might find suitable interpolating polynomials for the discretized system, we begin by considering a rod consisting of one element and two nodes.

Figure 1.3. Single Element, Two Node Beam
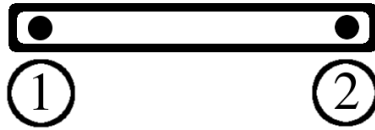
The displacement directions are s, v, w, corresponding to stretch, chordwise and flapwise bending respectively. Stretch is uni-axial and thus only has one degree of freedom, which is from displacement. Bending, however, has both displacement and bending angle, so the flapwise and chordwise displacements each have two degrees of freedom: one from displacement, the other from bending angle.

The stretch degrees of freedom applied to the two node single element beam yields one degree of freedom at two distinct nodes for a total of two degrees of freedom. The smallest polynomial which may represent this system has two coefficients and is therefore linear.

Figure 1.4. Single Element, Two Node Stretch Displacement Diagram
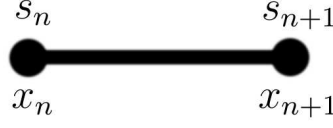
Considering the n-th two node element along the rod, we represent the stretch degrees of freedom (DOF) as s_n, s_{n+1} as shown in Fig. 1.4. The interpolating stretch polynomial can be expressed as

$$s(x) = c_0 + c_1 x, \tag{1.184}$$

or, in vector form,

$$s(x) = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}. \tag{1.185}$$

In this form, we express the polynomial vector $\vec{p}$ and coefficient vector $\vec{c}$ as follows.

$$\vec{p} = \begin{pmatrix} 1 \\ x \end{pmatrix}, \tag{1.186}$$

$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}, \tag{1.187}$$

so that the polynomial can be written as

$$s(x) = \vec{p}^{\top} \vec{c}. \tag{1.188}$$

Plugging in constraints on the rod so that $x_n = 0$ and $x_{n+1} = L$, we can write the polynomial from Eqn. 1.184 as

$$s(0) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = c_0 = s_0, \tag{1.189}$$

$$s(L) = \begin{pmatrix} 1 & L \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = c_0 + c_1 L = s_1. \tag{1.190}$$

We then write the element vector $\vec{s}$ as

$$\vec{s} = \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}, \tag{1.191}$$

and the matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & L \end{pmatrix}, \tag{1.192}$$

so that we can write the system as

$$\vec{s} = \mathbf{A}\vec{c}. \tag{1.193}$$

Solving for the coefficients $\vec{c}$ yields

$$\vec{c} = \mathbf{A}^{-1}\vec{s} = \begin{pmatrix} 1 & 0 \\ -1/L & 1/L \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}. \tag{1.194}$$

Plugging the $\vec{c}$ coefficient vector from Eqn. 1.194 into Eqn. 1.188 yields

$$s(x) = \vec{p}^{\top}\mathbf{A}^{-1}\vec{s}. \tag{1.195}$$

The product $\vec{p}^{\top}\mathbf{A}^{-1}$ yields the following vector

$$\vec{H}^{\top} = \vec{p}^{\top}\mathbf{A}^{-1} = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/L & 1/L \end{pmatrix} = \begin{pmatrix} 1 - x/L & x/L \end{pmatrix}, \tag{1.196}$$

so that

$$\vec{H}(x) = \begin{pmatrix} 1 - x/L \\ x/L \end{pmatrix}. \tag{1.197}$$

The first entry in $\vec{H}$, say $H_1(x)$, is a polynomial in x. This polynomial is a scaled first order Hermitian polynomial. The scaling is from L, the arbitrary length. If $L = 1$, this is precisely the Hermitian polynomial of order one. Together with the second term, they form the Hermitian interpolating basis polynomials for the stretch component of our physical system.

The product $\vec{H}^{\top}\vec{s}$ yields the system stretch polynomial in terms of the interpolating polynomials and the stretch displacements at the nodal positions. When at the first node, $H_2 = 0$ so that the stretch displacement s is precisely equal to the measured stretch at the first node; similarly, when at the second node, $H_1 = 0$ so that the stretch displacement s is precisely equal to the measured stretch at the second node. Elsewhere, the stretch is a linear combination of the measured stretch terms.

In this case, we chose the terminal position to be L, the total length of the rod. However, this formulation works at the n-th element along the rod, where L is replaced by the distance l between the n-th and $(n + 1)$-th nodes. For the general element n, the starting point is still 0, since we may index each element and treat it as a two node single element rod. Thus all nodes are accounted for and the entire rod can be interpolated by the elemental polynomial functions. This provides a discretization for the entire rod and we can represent the stretch displacement with a matrix of size $(n + 1) \times (n + 1)$ for n elements.

In a similar fashion, we now describe the bending displacements v, w. For each of these displacements, there is also a bending angle. Thus, for each node there are two degrees of freedom, and for a single element two node beam there will be four degrees of freedom. The smallest interpolatory polynomial that can account for four DOF has degree three.

Figure 1.5. Single Element, Two Node Bending Displacement Diagram
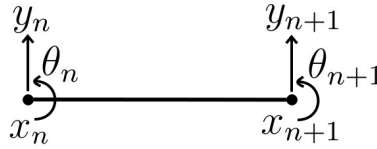
Considering the n-th two node element along the rod, we represent the bending DOF as $y_n, \theta_n, y_{n+1}, \theta_{n+1}$ as shown in Fig. 1.5. Here, bending is represented by y as a generic coordinate frame to represent either v or w bending. We shall plug v, w in for y at the conclusion of the calculation.

The interpolating bending polynomial can be expressed as

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3. \tag{1.198}$$

The coefficient vector $\vec{c}$ and polynomial vector $\vec{p}$ are expressed as

$$\vec{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{1.199}$$

$$\vec{p} = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}. \tag{1.200}$$

The bending angle θ_i at each node can be locally approximated if the bending angle is small. Large bending angles would certainly permanently deform or break the beam, so this assumption is valid relative to the physical system. Thus, the bending approximation is given to be equal to the slope at the node, or

$$\theta(x) \approx \frac{\mathrm{d}y}{\mathrm{d}x} = c_1 + 2c_2 x + 3c_3 x^2 \tag{1.201}$$

Following a similar path to that taken in the stretch interpolation, we plug in $x = 0$ and $x = L$, solve the system for the coefficient vector $\vec{c}$, and back substitute to find the Hermitian polynomial vector $\vec{H}(x)$.

The vector form of the bending function is expressed as

$$y(x) = \vec{p}^{\top}\vec{c}. \tag{1.202}$$

Plugging in the limits $0, L$ into Eqn. 1.198 and Eqn. 1.201, we have

$$y(0) = c_0 = y_1 \tag{1.203}$$
$$\theta(0) = c_1 = \theta_1 \tag{1.204}$$
$$y(L) = c_0 + c_1 L + c_2 L^2 + c_3 L^3 = y_2 \tag{1.205}$$
$$\theta(L) = c_1 + 2c_2 L + 3c_2 L^2 = \theta_2 \tag{1.206}$$

We label the vector $\vec{y}$ as

$$\vec{y} = \begin{pmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{pmatrix} \tag{1.207}$$

and the matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{pmatrix}. \tag{1.208}$$

We now have the system

$$\vec{y} = \mathbf{A}\vec{c}, \tag{1.209}$$

and we can find the coefficients $\vec{c}$ as

$$\vec{c} = \mathbf{A}^{-1}\vec{y}, \tag{1.210}$$

where

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{pmatrix} \tag{1.211}$$

Plugging this back into Eqn. 1.202, we have

$$y(x) = \vec{p}^{\top}\mathbf{A}^{-1}\vec{y}. \tag{1.212}$$

The product $\vec{p}^\top \mathbf{A}^{-1}$ yields the Hermite interpolating polynomials, given as

$$\vec{H} = \begin{pmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\ \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ -\frac{x^2}{L} + \frac{x^3}{L^2} \end{pmatrix}. \tag{1.213}$$

Once again, if we consider the discretized rod of n elements, each element can be treated as a single element two node rod, and this interpolation will satisfactorily approximate the beam deflection between the nodes.

We now have elemental representations of stretch and bending corresponding to the displacements s, v, w. What remains is to describe how the elements fit into the system of equations we found.

To be clear, we shall state the requirements of the approximation technique we shall employ. Firstly, consider the results from Appendix A. We have a system of equations derived through variations, and is thusly in variational form, so the Ritz approximation technique may be employed. However, we add in *ad hoc* non-conservative terms to the governing equations once the variational form is procured. Since the new equations did *not* come from applying variations, we are no longer guaranteed that Ritz approximations will properly emulate the system. Thus, we must employ Ritz-Galerkin (or Galerkin-Petrov) weighted residual minimization. This technique uses the approximating test functions as the weighting functions. In order to employ this technique, it is prudent to ensure that our test functions satisfy the requirements of the test functions as well as the weighting functions.

The test functions must satisfy the following criteria. For test functions of the form

$$s(x) \approx \sum_{i=1}^{n} c_i^s \phi_i^u(x) + \phi_0^u(x) \tag{1.214}$$

$$v(x) \approx \sum_{i=1}^{n} c_i^v \phi_i^v(x) + \phi_0^v(x) \tag{1.215}$$

$$w(x) \approx \sum_{i=1}^{n} c_i^w \phi_i^w(x) + \phi_0^w(x), \tag{1.216}$$

- ϕ_0^α $(\alpha = s, v, w)$ should satisfy the specified essential boundary conditions corresponding to their displacement direction

- ϕ_i^α $(\alpha = s, v, w; i = 1, \ldots, n)$ should satisfy

    - continuity in accordance with the variational method

    - the homogeneous form of the specified essential boundary conditions

    - linear independence and completeness.

Our test functions shall be Hermite polynomials of order 1 or greater. The Hermite polynomials are continuous on the boundary, complete, and linearly independent. The essential boundary conditions have yet to be described. We will address the essential boundary condition requirements after applying the approximating functions to the system.

The bending element displacement and acceleration vectors from Eqn. 1.207 can be expressed as

$$\vec{y} = \begin{pmatrix} y_i \\ \theta_i \\ y_{i+1} \\ \theta_{i+1} \end{pmatrix}, \tag{1.217}$$

$$\ddot{\vec{y}} = \begin{pmatrix} \ddot{y}_i \\ \ddot{\theta}_i \\ \ddot{y}_{i+1} \\ \ddot{\theta}_{i+1} \end{pmatrix}, \tag{1.218}$$

where y_i represents the displacement at node i, and θ_i represents the bending angle at node i. Similarly, we express the Hermitian vector from Eqn. 1.213 as

$$\vec{H} = \begin{pmatrix} H_1(x) \\ H_2(x) \\ H_3(x) \\ H_4(x) \end{pmatrix}. \tag{1.219}$$

We write the function $y(x)$ from Eqn. 1.198 in terms of the vectors $\vec{y}$ and $\vec{H}$:

$$y(x) = \sum_{j=1}^{4} H_j(x) y_j = \vec{H}^\top \vec{y}, \tag{1.220}$$

where y_j is the j-th element of $\vec{y}$. This is an expression for bending displacement interpolation functions for v, w, and by simply writing v, w in place of y, we may express v, w as follows.

$$v(x) = \sum_{j=1}^{4} H_j(x) v_j = \vec{H}^\top \vec{v}, \tag{1.221}$$

$$w(x) = \sum_{j=1}^{4} H_j(x) w_j = \vec{H}^\top \vec{w}. \tag{1.222}$$

We wish to apply these interpolating approximations to the Euler-Lagrange equations found as Eqns. 1.181, 1.182, 1.183. Unfortunately, we have a fourth order spatial derivative in both v and w, but v and w are approximated by third degree polynomials. To remedy this, we

are inclined to use integration by parts to reduce the order of the derivatives. Integration by parts done when products of functions are inside an integral. Following this logic, we shall multiply the equations by their respective displacement direction interpolation functions as follows.

$$[\rho A(\ddot{s} - 2\Omega\dot{v} - \Omega^2 s - \dot{\Omega}v) - EAs_{xx}] \cdot s = \rho A\Omega^2(a + x) \cdot s, \tag{1.223}$$

$$\left(\rho A(\ddot{v} + 2\Omega\dot{s} - \Omega^2 v + \dot{\Omega}s) + EI_{zz}v_{xxxx} - \rho A\Omega^2 \frac{\partial}{\partial x} \left\{ \left[a(L - x) \right. \right. \tag{1.224}$$

$$\left. \left. + \frac{1}{2}(L^2 - x^2) \right] \frac{\partial v}{\partial x} \right\} \right) \cdot v = [p_v - \rho A\dot{\Omega}(a + x)] \cdot v,$$

$$\left(\rho A\ddot{w} + EI_{yy}w_{xxxx} - \rho A\Omega^2 \frac{\partial}{\partial x} \left\{ \left[a(L - x) \right. \right. \tag{1.225}$$

$$\left. \left. + \frac{1}{2}(L^2 - x^2) \right] \frac{\partial w}{\partial x} \right\} \right) \cdot w = p_w \cdot w.$$

We now integrate each equation over the element. First consider the stretch coordinate s.

$$\int_0^l \rho A(\ddot{s} - 2\Omega\dot{v} - \Omega^2 s - \dot{\Omega}v)s - EAs_{xx}s \, \mathrm{d}x = \int_0^l \rho A\Omega^2(a + x)s \, \mathrm{d}x. \tag{1.226}$$

The interpolation function for s is a first degree polynomial, and we have a second order derivative. We will employ integration by parts to alleviate this situation. Consider

$$\int_0^l -EAs_{xx}s \, \mathrm{d}x. \tag{1.227}$$

Let

$$\alpha = -EAs, \tag{1.228}$$

then

$$\mathrm{d}\alpha = \frac{\mathrm{d}}{\mathrm{d}x}(-EAs) \, \mathrm{d}x. \tag{1.229}$$

Let

$$\mathrm{d}\beta = s_{xx} \, \mathrm{d}x, \tag{1.230}$$

then

$$\beta = \int \frac{\mathrm{d}}{\mathrm{d}x}s_x \, \mathrm{d}x = s_x. \tag{1.231}$$

Then, by the integration by parts formula,

$$\int_0^l -EAs_{xx}s \, \mathrm{d}x = \left. -EAss_x \right|_0^l + \int_0^l EAs_x^2 \, \mathrm{d}x. \tag{1.232}$$

The first term on the right vanishes on the boundaries, so we have

$$-\int_0^l EAs_{xx}s \, \mathrm{d}x = \int_0^l EAs_x^2 \, \mathrm{d}x. \tag{1.233}$$

Applying this result, we have

$$\int_0^l \rho A(\ddot{s} - 2\Omega\dot{v} - \Omega^2 s - \dot{\Omega}v)s + EAs_x^2 \,\mathrm{d}x = \int_0^l \rho A\Omega^2(a+x)s \,\mathrm{d}x. \tag{1.234}$$

Immediately an inconsistency emerges. We have mixed terms in s, v, but the vector forms of s, v do not allow for the other terms. We now expand our definitions of s, v into one combined form so that this equation has a sensible definition. At the same time, for ease of notation and clarity, we shall roll the w term into the combined term. Define the combined displacement vector $\vec{d}$ as follows.

$$\vec{d} = \begin{pmatrix} s_i \\ v_i \\ \theta_i \\ w_i \\ \phi_i \\ s_{i+1} \\ v_{i+1} \\ \theta_{i+1} \\ w_{i+1} \\ \phi_{i+1} \end{pmatrix}. \tag{1.235}$$

Next, we define a combined interpolation vector $\vec{N}$ as follows.

$$\vec{N} = \begin{pmatrix} H_{s1}(x) \\ H_{v1}(x) \\ H_{v2}(x) \\ H_{w1}(x) \\ H_{w2}(x) \\ H_{s2}(x) \\ H_{v3}(x) \\ H_{v4}(x) \\ H_{w3}(x) \\ H_{w4}(x) \end{pmatrix}, \tag{1.236}$$

where the subindices s, v, w indicate the displacement direction and $1, 2, 3, 4$ indicate the index of the interpolatory vector from each displacement direction formulation. The vector $\vec{N}$ can be expressed as the sum of the directional vectors $\vec{N}_s, \vec{N}_v, \vec{N}_w$ as follows.

$$\vec{N} = \vec{N}_s + \vec{N}_v + \vec{N}_w, \tag{1.237}$$

where the directional vectors are defined as follows.

$$\vec{N}_s = \begin{pmatrix} H_{s1}(x) & 0 & 0 & 0 & 0 & H_{s2}(x) & 0 & 0 & 0 & 0 \end{pmatrix}^\top \tag{1.238}$$

$$\vec{N}_v = \begin{pmatrix} 0 & H_{v1}(x) & H_{v2}(x) & 0 & 0 & 0 & H_{v3}(x) & H_{v4}(x) & 0 & 0 \end{pmatrix}^\top \tag{1.239}$$

$$\vec{N}_w = \begin{pmatrix} 0 & 0 & 0 & H_{w1}(x) & H_{w2}(x) & 0 & 0 & 0 & H_{w3}(x) & H_{w4}(x) \end{pmatrix}^\top. \tag{1.240}$$

We can now write the functions s, v terms of the combined displacement vector $\vec{d}$ and the respective interpolatory vectors $\vec{N}_s, \vec{N}_v$ as follows.

$$s = \vec{N}_s^\top \vec{d}, \tag{1.241}$$

$$v = \vec{N}_v^\top \vec{d}. \tag{1.242}$$

$$w = \vec{N}_w^\top \vec{d}. \tag{1.243}$$

We may now use Eqn. 1.234 sensibly. Next, we analyze the bending coordinate v. Before we can integrate over the element, we must recall the conditions of the integral

$$\int_x^L (a + \eta) \, \mathrm{d}\eta. \tag{1.244}$$

This integral treats x on the full coordinate system, not just in the element. To consider the element, recall that

$$\int_a^b f(x) \, \mathrm{d}x = \int_0^{b-a} f(x + a) \, \mathrm{d}x. \tag{1.245}$$

In our case,

$$f(x) = \int_x^L a + \eta \, \mathrm{d}\eta, \tag{1.246}$$

so that under the revised coordinates, we transform the elemental integration from the limits $x_e \to x_{e+1}$ to $0 \to l$ for $l = x_{e+1} - x_e$. For f, we have

$$f(x + x_e) = \int_{x+x_e}^L a + \eta \, \mathrm{d}\eta = a(L - (x + x_e)) + \frac{1}{2}(L^2 - (x + x_e)^2). \tag{1.247}$$

It is not apparent, but the displacement approximation polynomials were already described in the space $0 \to l$ instead of $x_e \to x_{e+1}$, so we don't need to adjust s, v, w. We only require a modification to the function f as described above. Thus, the v direction is expressed as

$$\int_0^l \rho A(\ddot{v} + 2\Omega \dot{s} - \Omega^2 v + \dot{\Omega}s)v + EI_{zz}v_{xxxx}v - \rho A\Omega^2 \left[a(L - (x + x_e)) \right. \tag{1.248}$$

$$\left. + \frac{1}{2}(L^2 - (x + x_e)^2) \right] \frac{\partial^2 v}{\partial x^2} v \, \mathrm{d}x = \int_0^l [p_v - \rho A\dot{\Omega}(a + x)]v \, \mathrm{d}x.$$

This equation has two terms to be simplified, the fourth order spatial derivative and the third term which must be simplified. Integration by parts twice on the fourth order term yields

$$\int_0^l EI_{zz}\frac{\partial^4 v}{\partial x^4}v\,\mathrm{d}x = EI_{zz}v\frac{\partial^3 v}{\partial x^3}\Big|_0^l - EI_{zz}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\Big|_0^l + \int_0^l EI_{zz}v_{xx}^2\,\mathrm{d}x, \tag{1.249}$$

and the first two terms on the right cancel due to boundary conditions, yielding

$$\int_0^l EI_{zz}\frac{\partial^4 v}{\partial x^4}v\,\mathrm{d}x = \int_0^l EI_{zz}v_{xx}^2\,\mathrm{d}x. \tag{1.250}$$

The third term is manipulated as follows. Consider

$$\int_0^l -\rho A\Omega^2\frac{\partial}{\partial x}\left\{\left[a(L-(x+x_e))+\frac{1}{2}(L^2-(x+x_e)^2)\right]\frac{\partial v}{\partial x}\right\}v\,\mathrm{d}x. \tag{1.251}$$

Let

$$\alpha = -\rho A\Omega^2 v, \tag{1.252}$$

then

$$\mathrm{d}\alpha = \frac{\mathrm{d}}{\mathrm{d}x}(-\rho A\Omega^2 v)\,\mathrm{d}x = -\rho A\Omega^2\frac{\partial v}{\partial x}. \tag{1.253}$$

Let

$$\mathrm{d}\beta = \frac{\partial}{\partial x}\left\{\left[a(L-(x+x_e))+\frac{1}{2}(L^2-(x+x_e)^2)\right]\frac{\partial v}{\partial x}\right\}\,\mathrm{d}x, \tag{1.254}$$

then

$$\beta = \left[a(L-(x+x_e))+\frac{1}{2}(L^2-(x+x_e)^2)\right]\frac{\partial v}{\partial x}. \tag{1.255}$$

The integration by parts formula yields

$$\int_0^l -\rho A\Omega^2\frac{\partial}{\partial x}\left\{\left[a(L-(x+x_e))+\frac{1}{2}(L^2-(x+x_e)^2)\right]\frac{\partial v}{\partial x}\right\}v\,\mathrm{d}x$$
$$= -v\left[a(L-(x+x_e))+\frac{1}{2}(L^2-(x+x_e)^2)\right]\frac{\partial v}{\partial x}\Big|_0^l \tag{1.256}$$
$$+ \int_0^l \rho A\Omega^2\left[a(L-(x+x_e))+\frac{1}{2}(L^2-(x+x_e)^2)\right]v_x^2\,\mathrm{d}x,$$

where the first term on the right side vanishes due to boundary conditions, yielding

$$-\int_0^l \rho A\Omega^2\frac{\partial}{\partial x}\left\{\left[a(L-(x+x_e))+\frac{1}{2}(L^2-(x+x_e)^2)\right]\frac{\partial v}{\partial x}\right\}v\,\mathrm{d}x \tag{1.257}$$
$$= \int_0^l \rho A\Omega^2\left[a(L-(x+x_e))+\frac{1}{2}(L^2-(x+x_e)^2)\right]v_x^2\,\mathrm{d}x.$$

The full v bending integral equation is written as

$$\int_0^l \rho A(\ddot{v} + 2\Omega\dot{s} - \Omega^2 v + \dot{\Omega}s)v + EI_{zz}v_{xx}^2 \tag{1.258}$$

$$+ \quad \rho A\Omega^2 \left[a(L - (x + x_e)) + \frac{1}{2}(L^2 - (x + x_e)^2) \right] v_x^2 \, \mathrm{d}x = \int_0^l [p_v - \rho A\dot{\Omega}(a + x)]v \, \mathrm{d}x.$$

The mixed terms are already handled by the combined displacement vector $\vec{d}$ as well as the interpolation vectors $\vec{N}_s, \vec{N}_v$

The integral equation for the w bending direction is handled similarly to the v bending direction. There is a fourth order term and the third term that can be simplified. The process is exactly the same with w written in place of v. The result is written as follows.

$$\int_0^l \rho A\ddot{w}w + EI_{zz}w_{xx}^2 + \rho A\Omega^2 \left[a(L - (x + x_e)) \right. \tag{1.259}$$

$$+ \quad \left. \frac{1}{2}(L^2 - (x + x_e)^2) \right] w_x^2 \, \mathrm{d}x = \int_0^l p_w w \, \mathrm{d}x.$$

From the application of the discretization to the equations of motion, we now have the essential boundary conditions. We shall show that the boundary conditions are met by the approximating functions $s(x), v(x), w(x)$.

From Eqn. 1.232, we have that

$$s\frac{\partial s}{\partial x}\bigg|_0^L = 0. \tag{1.260}$$

From the physical system, we know that at $x = 0$, the beam has zero displacement in each displacement direction. However, at the end of the beam ($x = L$), the beam may certainly be displaced in every direction. We conclude, then, that this condition is satisfied if $s_x(L) = 0$.

The formulation of $v(x), w(x)$ is exactly the same, so the essential boundary conditions are also the same. For either case, we define $\xi(x), \xi \in \{v, w\}$ for ease of discussion. We have from Eqn. 1.249

$$\xi\frac{\partial^3 \xi}{\partial x^3}\bigg|_0^L = 0 \tag{1.261}$$

and

$$\frac{\partial \xi}{\partial x}\frac{\partial^2 \xi}{\partial x^2}\bigg|_0^L = 0, \tag{1.262}$$

and from Eqn. 1.256

$$\xi\left[a(L - x) + \frac{1}{2}(L^2 - x^2)\right]\frac{\partial \xi}{\partial x}\bigg|_0^L = 0. \tag{1.263}$$

As in the case of s, the displacement at $x = 0$ must be zero; additionally, the bending must be zero. Thus we have $\xi(0) = 0, \xi_x(0) = 0$. For the first condition, we must have that $\xi_{xxx}(L) = 0$, which represents the vanishing shear force for a free beam; indeed this agrees with the physical system. For the second condition, we must have that $\xi_{xx}(L) = 0$, which represents the vanishing of the bending moment for a free beam; this also agrees with our physical system. The third condition is zero at $x = L$, so there is no more work to do.

The partial derivatives applied to the vector form of s, v, w can be expressed in the following manner.

$$z_x = [\vec{N}_z]_x^\top \vec{d}, \tag{1.264}$$

$$z_{xx} = [\vec{N}_z]_{xx}^\top \vec{d}, \tag{1.265}$$

$$\dot{z} = [\vec{N}_z]^\top \dot{\vec{d}}, \tag{1.266}$$

$$\ddot{z} = [\vec{N}_z]^\top \ddot{\vec{d}}, \tag{1.267}$$

where z is s, v or w and the subscripts x, xx indicate derivative with respect to x and the second derivative with respect to x, respectively. The over dot represents differentiation with respect to time. The differentiation operation on the vector applies the derivative to each element of the vector and results in a vector of the same size.

Under this new notation, the s direction integral equation (Eqn. 1.234) is written as

$$\int_0^l \rho A \left(\vec{N}_s^\top \ddot{\vec{d}} - 2\Omega \vec{N}_v^\top \dot{\vec{d}} - \Omega^2 \vec{N}_s^\top \vec{d} - \dot{\Omega} \vec{N}_v^\top \vec{d} \right) \vec{N}_s^\top \vec{d} + EA \left([\vec{N}_s]_x^\top \vec{d} \right)^2 \, dx \tag{1.268}$$

$$= \int_0^l \rho A \Omega^2 (a + x) \vec{N}_s^\top \vec{d} \, dx.$$

Observe that

$$\left([\vec{N}_s]_x^\top \vec{d} \right)^2 = \left([\vec{N}_s]_x^\top \vec{d} \right) \left([\vec{N}_s]_x^\top \vec{d} \right) = \left([\vec{N}_s]_x^\top \vec{d} \right) [\vec{N}_s]_x^\top \vec{d} = \left([\vec{N}_s]_x^\top \vec{d} [\vec{N}_s]_x^\top \right) \vec{d}. \tag{1.269}$$

With this result, we may pull $\vec{d}$ out to the right on all terms, which yields

$$\int_0^l \left\{ \rho A \left(\vec{N}_s^\top \ddot{\vec{d}} - 2\Omega \vec{N}_v^\top \dot{\vec{d}} - \Omega^2 \vec{N}_s^\top \vec{d} - \dot{\Omega} \vec{N}_v^\top \vec{d} \right) \vec{N}_s^\top \right. \tag{1.270}$$

$$\left. + EA \left([\vec{N}_s]_x^\top \vec{d} [\vec{N}_s]_x^\top \right) \right\} \vec{d} \, dx = \int_0^l \left\{ \rho A \Omega^2 (a + x) \vec{N}_s^\top \right\} \vec{d} \, dx.$$

Lemma 1.2. *Given nontrivial vectors $\{\alpha, \beta, \gamma\} \in \mathbb{R}^n$,*

$$\int \vec{\alpha}^\top \vec{\gamma} \, d\xi = \int \vec{\beta}^\top \vec{\gamma} \, d\xi \Leftrightarrow \int \vec{\alpha}^\top \, d\xi = \int \vec{\beta}^\top \, d\xi. \tag{1.271}$$

Proof.

$$
\begin{aligned}
\int \vec{\alpha}^\top \vec{\gamma}\, \mathrm{d}\xi &= \int \vec{\beta}^\top \vec{\gamma}\, \mathrm{d}\xi \\
\Leftrightarrow \quad \vec{\alpha}^\top \vec{\gamma} &= \vec{\beta}^\top \vec{\gamma} \\
\Leftrightarrow \quad \left(\vec{\alpha}^\top - \vec{\beta}^\top\right)\vec{\gamma} &= 0 \\
\Leftrightarrow \quad \left(\vec{\alpha}^\top - \vec{\beta}^\top\right) &= 0 \qquad (\vec{\gamma}\text{ is nontrivial}) \\
\Leftrightarrow \quad \vec{\alpha}^\top &= \vec{\beta}^\top \\
\Leftrightarrow \quad \int \vec{\alpha}^\top\, \mathrm{d}\xi &= \int \vec{\beta}^\top\, \mathrm{d}\xi
\end{aligned}
\tag{1.272}
$$

$\square$

The integral of the vector is not defined. Let this notation imply that we shall integrate each element of the vector. Since $\vec{d}$ is non-trivial, we have

$$
\int_0^l \rho A \left(\vec{N_s}^\top \ddot{\vec{d}} - 2\Omega \vec{N_v}^\top \dot{\vec{d}} - \Omega^2 \vec{N_s}^\top \vec{d} - \dot{\Omega}\vec{N_v}^\top \vec{d} \right) \vec{N_s}^\top + EA[\vec{N_s}]_x^\top \vec{d}[\vec{N_s}]_x^\top\, \mathrm{d}x \tag{1.273}
$$

$$
= \int_0^l \rho A \Omega^2 (a+x)\vec{N_s}^\top\, \mathrm{d}x.
$$

Consider the vectors $\left\{\vec{\alpha}, \vec{\beta}, \vec{\gamma}\right\} \in \mathbb{R}^n$ of the same length. Note that

$$
\left(\vec{\alpha}^\top \vec{\gamma}\right)\vec{\beta}^\top = \left[\vec{\beta}\left(\vec{\gamma}^\top \vec{\alpha}\right)\right]^\top = \left[\left(\vec{\beta}\vec{\alpha}^\top\right)\vec{\gamma}\right]^\top. \tag{1.274}
$$

We shall use this property to organize Eqn. 1.273. At the same time, we shall group the terms by time derivatives of the vector $\vec{d}$. We have

$$
\int_0^l \left\{ \rho A \vec{N_s}\vec{N_s}^\top \ddot{\vec{d}} - \rho A 2\Omega \vec{N_s}\vec{N_v}^\top \dot{\vec{d}} \right.
$$

$$
\left. + \left(\rho A \left[-\Omega^2 \vec{N_s}\vec{N_s}^\top - \dot{\Omega}\vec{N_s}\vec{N_v}^\top\right] + EA[\vec{N_s}]_x[\vec{N_s}]_x^\top\right)\vec{d}\right\}\, \mathrm{d}x \tag{1.275}
$$

$$
= \int_0^l \rho A \Omega^2 (a+x)\vec{N_s}\, \mathrm{d}x.
$$

The v bending integral equation (Eqn. 1.258) may be discretized in a similar fashion.

$$
\int_0^l \left\{ \rho A \left(\vec{N_v}^\top \ddot{\vec{d}} + 2\Omega \vec{N_s}^\top \dot{\vec{d}} - \Omega^2 \vec{N_v}^\top \vec{d} + \dot{\Omega}\vec{N_s}^\top \vec{d}\right)\vec{N_v}^\top \vec{d} \right.
$$

$$
\left. + EI_{zz}\left([\vec{N_v}]_{xx}^\top \vec{d}\right)^2 + \rho A \Omega^2 \left[a(L-x) + \frac{1}{2}(L^2 - x^2)\right]\left([\vec{N_v}]_x^\top \vec{d}\right)^2 \right\}\, \mathrm{d}x \tag{1.276}
$$

$$
= \int_0^l [p_v - \rho A \dot{\Omega}(a+x)]\vec{N_v}^\top \vec{d}\, \mathrm{d}x.
$$

Using the observation of Eqn. 1.269 and "cancelling" the non-trivial vector $\vec{d}$ yields

$$\int_0^l \left\{ \rho A \left(\vec{N_v}^\top \ddot{\vec{d}} + 2\Omega \vec{N_s}^\top \dot{\vec{d}} - \Omega^2 \vec{N_v}^\top \vec{d} + \dot{\Omega} \vec{N_s}^\top \vec{d} \right) \vec{N_v}^\top \right.$$

$$\left. + EI_{zz}[\vec{N_v}]_{xx}^\top \vec{d}[\vec{N_v}]_{xx}^\top + \rho A \Omega^2 \left[a(L-x) + \frac{1}{2}(L^2 - x^2) \right] [\vec{N_v}]_x^\top \vec{d}[\vec{N_v}]_x^\top \right\} \, \mathrm{d}x \quad (1.277)$$

$$= \int_0^l [p_v - \rho A \dot{\Omega}(a+x)]\vec{N_v}^\top \, \mathrm{d}x.$$

Applying the observation of Eqn. 1.274 to reorganize, we have

$$\int_0^l \left\{ \rho A \vec{N_v} \vec{N_v}^\top \ddot{\vec{d}} + \rho A 2\Omega \vec{N_v} \vec{N_s}^\top \dot{\vec{d}} + \left(\rho A [-\Omega^2 \vec{N_v} \vec{N_v}^\top + \dot{\Omega} \vec{N_v} \vec{N_s}^\top] \right. \right.$$

$$\left. \left. + EI_{zz}[\vec{N_v}]_{xx}[\vec{N_v}]_{xx}^\top + \rho A \Omega^2 \left[a(L-x) + \frac{1}{2}(L^2 - x^2) \right] [\vec{N_v}]_x [\vec{N_v}]_x^\top \right) \vec{d} \right\} \, \mathrm{d}x \quad (1.278)$$

$$= \int_0^l [p_v - \rho A \dot{\Omega}(a+x)]\vec{N_v} \, \mathrm{d}x.$$

Similarly, the w integral equation (Eqn. 1.259) may be discretized.

$$\int_0^l \left\{ \rho A \vec{N_w}^\top \ddot{\vec{d}} \vec{N_w}^\top \vec{d} + EI_{yy} \left([\vec{N_w}]_{xx}^\top \vec{d} \right)^2 \right. \quad (1.279)$$

$$\left. + \rho A \Omega^2 \left[a(L-x) + \frac{1}{2}(L^2 - x^2) \right] \left([\vec{N_w}]_x^\top \vec{d} \right)^2 \right\} \, \mathrm{d}x = \int_0^l p_w \vec{N_w}^\top \vec{d} \, \mathrm{d}x.$$

Using the observation of Eqn. 1.269 and "cancelling" the non-trivial vector $\vec{d}$ yields

$$\int_0^l \left\{ \rho A \vec{N_w}^\top \ddot{\vec{d}} \vec{N_w}^\top + EI_{yy}[\vec{N_w}]_{xx}^\top \vec{d}[\vec{N_w}]_{xx}^\top \right. \quad (1.280)$$

$$\left. + \rho A \Omega^2 \left[a(L-x) + \frac{1}{2}(L^2 - x^2) \right] [\vec{N_w}]_x^\top \vec{d}[\vec{N_w}]_x^\top \right\} \, \mathrm{d}x = \int_0^l p_w \vec{N_w}^\top \, \mathrm{d}x.$$

Applying the observation of Eqn. 1.274 to reorganize, we have

$$\int_0^l \left\{ \rho A \vec{N_w} \vec{N_w}^\top \ddot{\vec{d}} + \left(EI_{yy}[\vec{N_w}]_{xx}[\vec{N_w}]_{xx}^\top \right. \right. \quad (1.281)$$

$$\left. \left. + \rho A \Omega^2 \left[a(L-x) + \frac{1}{2}(L^2 - x^2) \right] [\vec{N_w}]_x [\vec{N_w}]_x^\top \right) \vec{d} \right\} \, \mathrm{d}x = \int_0^l p_w \vec{N_w} \, \mathrm{d}x.$$

We now combine Eqn. 1.275, Eqn. 1.278 and Eqn. 1.281 to form the full system, grouping the terms by time derivatives of $\vec{d}$.

$$\int_0^l \left\{ \rho A \left[\vec{N_s}\vec{N_s}^\top + \vec{N_v}\vec{N_v}^\top + \vec{N_w}\vec{N_w}^\top \right] \ddot{\vec{d}} + \rho A 2\Omega \left[\vec{N_v}\vec{N_s}^\top - \vec{N_s}\vec{N_v}^\top \right] \dot{\vec{d}} \right.$$

$$+ \left(-\rho A \Omega^2 \left[\vec{N_s}\vec{N_s}^\top + \vec{N_v}\vec{N_v}^\top \right] + \rho A \dot{\Omega} \left[\vec{N_v}\vec{N_s}^\top - \vec{N_s}\vec{N_v}^\top \right] + EA[\vec{N_s}]_x[\vec{N_s}]_x^\top \right.$$

$$+ \rho A \Omega^2 \left[a(L - x) + \frac{1}{2}(L^2 - x^2) \right] \left[[\vec{N_v}]_x[\vec{N_v}]_x^\top + [\vec{N_w}]_x[\vec{N_w}]_x^\top \right] \quad (1.282)$$

$$\left. \left. + E \left[I_{zz}[\vec{N_v}]_{xx}[\vec{N_v}]_{xx}^\top + I_{yy}[\vec{N_w}]_{xx}[\vec{N_w}]_{xx}^\top \right] \right) \vec{d} \right\} \, dx$$

$$= \int_0^l \rho A \Omega^2 (a + x)\vec{N_s} + [p_v - \rho A \dot{\Omega}(a + x)]\vec{N_v} + p_w \vec{N_w} \, dx.$$

We have the equations of motion for the discretized system represented by Eqn. 1.282, although it may not be very clear. To clarify this expression, first note that

$$\int_0^l \vec{d} \, dx = \int_0^l dx \vec{d}, \quad (1.283)$$

and the same applies to the time derivatives $\ddot{\vec{d}}, \dot{\vec{d}}$, since the elements of these vectors are constant with respect to x (they are nodal DOF). We now define element matrices to simplify the expression and illuminate the equations of motion. Let

$$[\mathbf{p}] = \rho A \int_0^l \vec{N_s}\vec{N_s}^\top + \vec{N_v}\vec{N_v}^\top \, dx \quad (1.284)$$

$$[\mathbf{m}] = [\mathbf{p}] + \rho A \int_0^l \vec{N_w}\vec{N_w}^\top \, dx, \quad (1.285)$$

$$[\mathbf{g}] = \rho A \int_0^l \vec{N_v}\vec{N_s}^\top - \vec{N_s}\vec{N_v}^\top \, dx, \quad (1.286)$$

$$[\mathbf{k}] = E \int_0^l A[\vec{N_s}]_x[\vec{N_s}]_x^\top + I_{zz}[\vec{N_v}]_{xx}[\vec{N_v}]_{xx}^\top + I_{yy}[\vec{N_w}]_{xx}[\vec{N_w}]_{xx}^\top \, dx, \quad (1.287)$$

$$[\sigma] = \rho A \int_0^l \Omega^2 \left[a(L - x) + \frac{1}{2}(L^2 - x^2) \right] \left[[\vec{N_v}]_x[\vec{N_v}]_x^\top + [\vec{N_w}]_x[\vec{N_w}]_x^\top \right] \, dx, (1.288)$$

$$\vec{f} = \int_0^l \rho A \Omega^2 (a + x)\vec{N_s} + [p_v - \rho A \dot{\Omega}(a + x)]\vec{N_v} + p_w \vec{N_w} \, dx. \quad (1.289)$$

Then we may express Eqn. 1.282 as

$$[\mathbf{m}]\ddot{\vec{d}} + 2\Omega[\mathbf{g}]\dot{\vec{d}} + ([\mathbf{k}] + \Omega^2([\sigma] - [\mathbf{p}]) + \dot{\Omega}[\mathbf{g}])\vec{d} = \vec{f}. \quad (1.290)$$

The matrices in Eqn. 1.290 are related to the physics and are given special names. The matrix $[\mathbf{m}]$ is called the **mass matrix**, $[\mathbf{k}]$ is called the **stiffness matrix**, and $[\mathbf{g}]$ is called the

gyroscopic matrix. Note that the total stiffness depends on $[\sigma]$, $[\mathbf{p}]$, and $[\mathbf{g}]$, where the term involving $[\sigma]$ and $[\mathbf{p}]$ describes Coriolis stiffening, while the term involving $[\mathbf{g}]$ describes inertial stiffening. If the cantilever beam is not rotating (*i.e.* $\Omega = 0$) we recover the standard cantilever beam system

$$[\mathbf{m}]\ddot{\vec{d}} + [\mathbf{k}]\vec{d} = \int_0^l p_v \vec{N_v} + p_w \vec{N_w} \, dx. \tag{1.291}$$

The computed matrices are given below.

$$[\mathbf{p}] = \frac{l\rho A}{420}
\begin{pmatrix}
140 & 0 & 0 & 0 & 0 & 70 & 0 & 0 & 0 & 0 \\
0 & 156 & 22l & 0 & 0 & 0 & 54 & -13l & 0 & 0 \\
0 & 22l & 4l^2 & 0 & 0 & 0 & 13l & -3l^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
70 & 0 & 0 & 0 & 0 & 140 & 0 & 0 & 0 & 0 \\
0 & 54 & 13l & 0 & 0 & 0 & 156 & -22l & 0 & 0 \\
0 & -13l & -3l^2 & 0 & 0 & 0 & -22l & 4l^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$[\mathbf{m}] = \frac{l\rho A}{420}
\begin{pmatrix}
140 & 0 & 0 & 0 & 0 & 70 & 0 & 0 & 0 & 0 \\
0 & 156 & 22l & 0 & 0 & 0 & 54 & -13l & 0 & 0 \\
0 & 22l & 4l^2 & 0 & 0 & 0 & 13l & -3l^2 & 0 & 0 \\
0 & 0 & 0 & 156 & 22l & 0 & 0 & 0 & 54 & -13l \\
0 & 0 & 0 & 22l & 4l^2 & 0 & 0 & 0 & 13l & -3l^2 \\
70 & 0 & 0 & 0 & 0 & 140 & 0 & 0 & 0 & 0 \\
0 & 54 & 13l & 0 & 0 & 0 & 156 & -22l & 0 & 0 \\
0 & -13l & -3l^2 & 0 & 0 & 0 & -22l & 4l^2 & 0 & 0 \\
0 & 0 & 0 & 54 & 13l & 0 & 0 & 0 & 156 & -22l \\
0 & 0 & 0 & -13l & -3l^2 & 0 & 0 & 0 & -22l & 4l^2
\end{pmatrix}$$

$$[\mathbf{g}] = \frac{l\rho A}{60}\begin{pmatrix} 0 & -21 & -3l & 0 & 0 & 0 & -9 & 2l & 0 & 0 \\ 21 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ 3l & 0 & 0 & 0 & 0 & 2l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -9 & -2l & 0 & 0 & 0 & -21 & 3l & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 & 21 & 0 & 0 & 0 & 0 \\ -2l & 0 & 0 & 0 & 0 & -3l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[\mathbf{k}] = \frac{E}{l^3}\begin{pmatrix} Al^2 & 0 & 0 & 0 & 0 & -Al^2 & 0 & 0 & 0 & 0 \\ 0 & 12I_{zz} & 6I_{zz}l & 0 & 0 & 0 & -12I_{zz} & 6I_{zz}l & 0 & 0 \\ 0 & 6I_{zz}l & 4I_{zz}l^2 & 0 & 0 & 0 & -6I_{zz}l & 2I_{zz}l^2 & 0 & 0 \\ 0 & 0 & 0 & 12I_{yy} & 6I_{yy}l & 0 & 0 & 0 & -12I_{yy} & 6I_{yy}l \\ 0 & 0 & 0 & 6I_{yy}l & 4I_{yy}l^2 & 0 & 0 & 0 & -6I_{yy}l & 2I_{yy}l^2 \\ -Al^2 & 0 & 0 & 0 & 0 & Al^2 & 0 & 0 & 0 & 0 \\ 0 & -12I_{zz} & -6I_{zz}l & 0 & 0 & 0 & 12I_{zz} & -6I_{zz}l & 0 & 0 \\ 0 & 6I_{zz}l & 2I_{zz}l^2 & 0 & 0 & 0 & -6I_{zz}l & 4I_{zz}l^2 & 0 & 0 \\ 0 & 0 & 0 & -12I_{yy} & -6I_{yy}l & 0 & 0 & 0 & 12I_{yy} & -6I_{yy}l \\ 0 & 0 & 0 & 6I_{yy}l & 2I_{yy}l^2 & 0 & 0 & 0 & -6I_{yy}l & 4I_{yy}l^2 \end{pmatrix}$$

Let

$$
\begin{aligned}
\alpha_1 &= L^2 - x_e^2, \\
\alpha_2 &= 2a(L - x_e) \\
\alpha_3 &= l(a + x_e), \\
\beta_1 &= -72l^2 + 252(\alpha_1 + \alpha_2 - \alpha_3), \\
\beta_2 &= -15l^3 + 21l(\alpha_1 + \alpha_2 - 2\alpha_3), \\
\beta_3 &= 6l^3 + 21l(\alpha_1 + \alpha_2), \\
\beta_4 &= -4l^4 + 14l^2(2\alpha_1 + 2\alpha_2 - \alpha_3), \\
\beta_5 &= 3l^4 + 7l^2(-\alpha_1 - \alpha_2 + \alpha_3), \\
\beta_6 &= -18l^4 + 14l^2(2\alpha_1 + 2\alpha_2 - 3\alpha_3),
\end{aligned}
$$

where L is (global) beam length, a is distance from the center of the axis of rotation to the beginning of the beam, x_e is the position of the e-th node, and l is the length of the element.

The matrix $[\sigma]$ can be expressed as

$$[\sigma] = \frac{\rho A}{420l} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & \beta_2 & 0 & 0 & 0 & -\beta_1 & \beta_3 & 0 & 0 \\ 0 & \beta_2 & \beta_4 & 0 & 0 & 0 & -\beta_2 & \beta_5 & 0 & 0 \\ 0 & 0 & 0 & \beta_1 & \beta_2 & 0 & 0 & 0 & -\beta_1 & \beta_3 \\ 0 & 0 & 0 & \beta_2 & \beta_4 & 0 & 0 & 0 & -\beta_2 & \beta_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_1 & -\beta_2 & 0 & 0 & 0 & \beta_1 & -\beta_3 & 0 & 0 \\ 0 & \beta_3 & \beta_5 & 0 & 0 & 0 & -\beta_3 & \beta_6 & 0 & 0 \\ 0 & 0 & 0 & -\beta_1 & -\beta_2 & 0 & 0 & 0 & \beta_1 & -\beta_3 \\ 0 & 0 & 0 & \beta_3 & \beta_5 & 0 & 0 & 0 & -\beta_3 & \beta_6 \end{pmatrix}.$$

Naturally, all of the matrices are symmetric, except the matrix $[\mathbf{g}]$, which is skew-symmetric. In other words,

$$[\mathbf{g}] = -[\mathbf{g}]^\top. \tag{1.292}$$

Lemma 1.3. *Skew symmetric matrices have purely imaginary eigenvalues (i.e. $Re(\lambda) = 0$).*

Proof. Let $\mathbf{A}$ be a real skew symmetric matrix, with eigenvalue λ and eigenvector $\vec{\xi}$. Then, by definition,

$$\mathbf{A}\vec{\xi} = \lambda\vec{\xi}, \tag{1.293}$$

and

$$\mathbf{A} = -\mathbf{A}^\top. \tag{1.294}$$

Let $()^*$ denote the conjugate operator. We have

$$
\begin{array}{rrcll}
 & (\mathbf{A}\vec{\xi})^* & = & (\lambda\vec{\xi})^* & \\
\Leftrightarrow & \mathbf{A}\vec{\xi}^* & = & \lambda^*\vec{\xi}^* & (\mathbf{A}\text{ is real}) \\
\Leftrightarrow & (\mathbf{A}\vec{\xi}^*)^\top & = & (\lambda^*\vec{\xi}^*)^\top & \\
\Leftrightarrow & (\mathbf{A}\vec{\xi}^*)^\top\vec{\xi} & = & (\lambda^*\vec{\xi}^*)^\top\vec{\xi} & \\
\Leftrightarrow & \vec{\xi}^{*\top}\mathbf{A}^\top\vec{\xi} & = & \lambda^*\vec{\xi}^{*\top}\vec{\xi} & \\
\Leftrightarrow & \vec{\xi}^{*\top}(-\mathbf{A})\vec{\xi} & = & \lambda^*\vec{\xi}^{*\top}\vec{\xi} & (\mathbf{A}\text{ is skew-symmetric}) \\
\Leftrightarrow & \vec{\xi}^{*\top}(-\lambda)\vec{\xi} & = & \lambda^*\vec{\xi}^{*\top}\vec{\xi} & (\text{Eqn. 1.293}) \\
\Leftrightarrow & -\lambda\vec{\xi}^{*\top}\vec{\xi} & = & \lambda^*\vec{\xi}^{*\top}\vec{\xi} & (\text{scalars commute}) \\
\Leftrightarrow & -\lambda\vec{\xi}^{*\top} & = & \lambda^*\vec{\xi}^{*\top} & (\vec{\xi}\text{ is nontrivial}) \\
\Leftrightarrow & -\lambda & = & \lambda^* & (\vec{\xi}\text{ is nontrivial}) \\
\Leftrightarrow & Re(\lambda) & = & 0 &
\end{array}
\tag{1.295}
$$

$\square$

As a result of the configuration of [g], the contribution of the $\dot{\vec{d}}$ term does not add nor remove energy from the system — and important distinction, as that would imply that the physical system would always decay to zero or grow unbounded, and we know the physical system does not do that.

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APPENDIX A

VARIATIONAL APPROXIMATION AND CALCULUS OF VARIATIONS

VARIATIONAL APPROXIMATION AND CALCULUS OF VARIATIONS

In this appendix a treatment on variational approximation is presented as the background and basis for the finite element approximation technique that is employed in solving the cantilever beam problem.

A.1 INTRODUCTION TO VARIATIONS

Let $a, b, \alpha, \beta, \gamma \in \mathbb{R}$. Suppose there is a function

$$\eta(x) : [a, b] \to [\alpha, \beta], \tag{A.1}$$

with $\eta \in \mathcal{C}^2[a, b]$, such that

$$\eta(a) = \eta(b) = 0. \tag{A.2}$$

Suppose there is also a function

$$y(x) : [a, b] \to [\alpha, \beta], \tag{A.3}$$

with $y \in \mathcal{C}^2[a, b]$, where

$$y(a) = \alpha, \quad y(b) = \beta. \tag{A.4}$$

For the parameter γ, define:

$$Y(x; \gamma) = y(x) + \gamma \eta(x). \tag{A.5}$$

Clearly, as $\gamma \to 0$, $Y(x; \gamma) \to y(x)$. Furthermore, for any η and γ,

$$Y(a; \gamma) = y(a), \quad Y(b; \gamma) = y(b). \tag{A.6}$$

By the chain rule,

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \frac{\partial Y}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \gamma \frac{\partial Y}{\partial y}\frac{\mathrm{d}\eta}{\mathrm{d}x} \tag{A.7}$$

$$= 1 \cdot y' + 1 \cdot \gamma \eta' \tag{A.8}$$

$$= y' + \gamma \eta'. \tag{A.9}$$

Since $y, \eta \in \mathcal{C}^2[a, b]$, this result has that $Y \in \mathcal{C}^2[a, b]$, and we write $Y' = \frac{\mathrm{d}Y}{\mathrm{d}x}$. We now define the variation, with operator δ.

Definition A.1. *Given Y as defined above, we define:*

$$\delta Y = Y(x; \gamma) - Y(x; 0) = \gamma \eta(x), \tag{A.10}$$

and call this the **variation of** Y*, and call δ the* **variational operator**.

From this definition, we see that for $0 < \gamma \ll 1$, $\delta Y \approx 0$, and, moreover,

$$\lim_{\gamma \to 0} \delta Y = 0. \tag{A.11}$$

In order to understand the linearity of δ, observe that:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\delta Y] = \frac{\mathrm{d}Y(x; \alpha)}{\mathrm{d}x} - \frac{\mathrm{d}Y(x; 0)}{\mathrm{d}x} \tag{A.12}$$

$$= y' + \gamma \eta' - y', \tag{A.13}$$

$$= \gamma \eta', \tag{A.14}$$

and

$$\delta \left[\frac{\mathrm{d}Y}{\mathrm{d}x} \right] = \delta \left[y' + \gamma \eta' \right] \tag{A.15}$$

$$= y' + \gamma \eta' - y', \tag{A.16}$$

$$= \gamma \eta', \tag{A.17}$$

so clearly

$$\frac{\mathrm{d}}{\mathrm{d}x}[\delta Y] = \delta \left[\frac{\mathrm{d}Y}{\mathrm{d}x} \right]. \tag{A.18}$$

Similarly,

$$\delta \int Y \mathrm{d}x = \int Y(x; \gamma) \mathrm{d}x - \int Y(x; 0) \mathrm{d}x \tag{A.19}$$

$$= \int Y(x; \gamma) - Y(x; 0) \mathrm{d}x \tag{A.20}$$

$$= \int \delta Y \, \mathrm{d}x. \tag{A.21}$$

A full treatment of this property has been presented in Refs. [7, 11].

Definition A.2. *Let y be defined as above. Consider the following integral:*

$$J = \int_a^b f(x, y, y') \, \mathrm{d}x, \tag{A.22}$$

for some function f in $\mathcal{C}^2[a, b]$. Suppose that there is some function y which minimizes J, where the function y is unknown. Suppose furthermore that there is some other function Y which is known and is "near" y. In order to find y, we construct Y as above, and write

$$\hat{J}(\gamma) = \int_a^b f(x, Y, Y') \, \mathrm{d}x. \tag{A.23}$$

*$\hat{J}$ is called a **functional**, and $\gamma = 0 \Rightarrow Y = y$ extremizes $\hat{J}$ in the function space.*

Definition A.3. *Consider:*

$$\hat{J}' = \frac{d\hat{J}}{d\gamma} = \int_a^b \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \gamma} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \gamma} \ dx = \int_a^b \frac{\partial f}{\partial Y}\eta + \frac{\partial f}{\partial Y'}\eta' \ dx. \tag{A.24}$$

Allowing $\gamma \to 0$, we have

$$\hat{J}'(0) = \int_a^b \frac{\partial f}{\partial y}\eta - \frac{\partial f}{\partial y'}\eta' \ dx, \tag{A.25}$$

and integrating by parts provides:

$$\hat{J}'(0) = \frac{\partial f}{\partial y'}\eta \bigg|_a^b + \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta \ dx = \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta \ dx. \tag{A.26}$$

Since we are minimizing $\hat{J}$, we set this integral to zero; that is:

$$\int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta \ dx = 0, \tag{A.27}$$

and since this requirement must hold for any η, it must be true that (see Ref. [11])

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{A.28}$$

These are known as the **Euler-Lagrange Equations***.*

The term *Euler-Lagrange Equations* is plural because, in spite of the fact that Eqn. A.28 appears as a singular equation, this extends to n-dimensional space for $y, y' \in \mathbb{R}^n$. Thus, Eqn. A.28 is actually a system of equations. A full treatment of this is given in Ref. [7].

A.2 VARIATIONAL MINIMIZATION

Alternatively, consider

$$\delta\hat{J} = \int_a^b \delta f \ dx, \tag{A.29}$$

and, following from Ref. [7],

$$\delta f = f(x, Y(x; 0), Y'(x; 0)) - f(x, Y(x; \gamma), Y'(x; \gamma)) = \gamma\left[\frac{\partial f}{\partial Y}\eta + \frac{\partial f}{\partial Y'}\eta'\right]. \tag{A.30}$$

Using this, we write

$$\delta\hat{J} = \int_a^b \delta f \ dx, \tag{A.31}$$

$$= \gamma \int_a^b \left[\frac{\partial f}{\partial Y}\eta + \frac{\partial f}{\partial Y'}\eta'\right] \ dx, \tag{A.32}$$

$$= \gamma \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta \ dx. \tag{A.33}$$

Thus, the variation of our functional can be expressed as a scalar multiple of the rate of change of our functional with respect to the parameter γ. We conclude that the variation of the functional can be used to minimize the functional. It can be shown that this formulation holds for n-dimensional x, Y, and Y' [7].

A.3 THE PRINCIPLE OF LEAST ACTION

The Euler-Lagrange equations are often used to describe the energy of a system. The more common notation associated with energy is

$$\mathcal{L} = L(t, x, \dot{x}), \tag{A.34}$$

and the integral

$$S = \int_{t_0}^{t_1} \mathcal{L} \, \mathrm{d}t = \int_{t_0}^{t_1} L(t, x, \dot{x}) \, \mathrm{d}t \tag{A.35}$$

is called the *action*. The Principle of Least Action states that as system will take the path of least action. In this case, we can determine the path of least action by taking the variation of the action functional equal to zero:

$$\delta S = \delta \int_{t_0}^{t_1} L(t, x, \dot{x}) \, \mathrm{d}t = 0, \tag{A.36}$$

or

$$\int_{t_0}^{t_1} \delta L(t, x, \dot{x}) \, \mathrm{d}t = 0 \Leftrightarrow \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0. \tag{A.37}$$

The *Lagrangian* of a system is given as

$$\mathcal{L} = T - V, \tag{A.38}$$

where T is the kinetic energy of the system and V is the potential energy in the system. For this discussion, it shall be assumed that all virtual work is stored as energy in V, and will derive the work in its potential form. This is valid for mass-conserving systems and the work is described as the conservative force minus the change in potential energy, which leads immediately to [9]:

$$\delta W = -\delta V, \tag{A.39}$$

where in this case W represents virtual work.

A.4 THE RITZ-RAYLEIGH METHOD

While the aforementioned variational approximation is useful for approximating the "mass action" of the system, it is not enough for computing true motion of the system. In order to do so, a discretization technique must be employed. Since we know that the energy equations are useful in variational approximation, we are motivated to utilize the energy

equations to elucidate an approximation for the equations of motion. We shall employ Ritz's method to approximate the system so that we may calculate the discrete equations of motion. In essence, Ritz's method provides us with an appropriate solution, and in doing so, yields a discretized system. By construction, the error (or residual) is minimized, and the boundary values are all accounted for. In finite difference techniques, a template is applied to a problem, and stability, well-posedness, numerical and system boundary conditions all need to be computed and verified. Finite element methods, including Ritz's method, account for all of these ingredients, so long as appropriate test functions are chosen [1].

The immediate question to be posed is: "How does one choose appropriate test functions?" The answer, however, is not nearly straightforward. In practice, it is easy to determine the appropriate function space and calculate the test functions. In terms of proving that the test functions are appropriate, the theory is quite complicated. It is beyond the scope of this paper, and shall not be thoroughly discussed. A full treatment can be found in Brenner's text [2]. The foundations, however, shall be stated.

Definition A.4. *Let $f \in C^2[a, b]$ be such that*

$$\int_a^b |f(x)|^n \ dx < \infty, \tag{A.40}$$

and define

$$||f||_n = \left(\int_a^b |f(x)|^n \ dx \right)^{1/n} \tag{A.41}$$

*as the n-th L-norm. Then all f meeting the requirement in Eqn. A.40, coupled with the corresponding L-norm, forms the n-**th Hilbert space**, $H^n([a, b])$. That is,*

$$H^n([a, b]) = \left\{ f \in C^2[a, b] : \int_a^b |f(x)|^n \ dx < \infty \right\}, \tag{A.42}$$

with corresponding L-norm.

Of particular interest, for the Finite Element application, is H^2.

Note: This is a *particular* definition of a Hilbert Space. There are more generic definitions, but this definition is quite suitable for our application. For more information on Hilbert Spaces, and other norms/normed spaces, refer to Ref. [2].

We now consider an extension of above ideas.

Definition A.5. *Let*

$$Y_n(x; \vec{\gamma}) = y(x) + \sum_{k=1}^n \gamma_k \eta_k(x), \tag{A.43}$$

where η_i, $i = 1, \ldots, n$ are as above — i.e. $\eta_i(a) = \eta_i(b) = 0$ for each i. Let

$$J = \int_a^b f(x, Y_n, Y_n') \ dx. \tag{A.44}$$

Then

$$\delta J = 0 \Leftrightarrow \frac{\partial f}{\partial Y_n} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y_n'}\right) = 0. \tag{A.45}$$

If this is satisfied, then J is minimized by Y^*, say, and there must be some Y_n^* comprised of the resulting solution parameters $\vec{\gamma}^*$ that satisfies

$$||Y^* - Y_n^*||_2 \le ||Y^* - V||_2, \text{ for any } V \in H^2([a,b]). \tag{A.46}$$

This is known as **Ritz-Rayleigh Minimization**.

This may not look very useful, but indeed we have the result that a *finite* dimensional function can be used to approximate an *infinite* dimensional problem. And, we have information about how to pick such finite dimension functions — they need to be in H^2 and they must satisfy the boundary criteria. Thus, we may use well known and characterized approximation functions to discretize our system, and at the same time, we will have found equations of motion of our system from the energy functions.

Perhaps more interestingly, what this statement says is that if we can find a function Y_n^* which satisfies the criteria — it meets the boundary conditions and is in H^2, then providing that the function is consistent with the problem (*i.e.* is periodic for periodic systems, etc.) — then we can be assured that the test function is as close to the minimizing function as any other function in the function space.

Note: Not all problems can be solved using Ritz's method. The problem must have a variational formulation; in other words, it must satisfy the Euler-Lagrange equations. If not, such as the case of non-conservative work, the variation cannot be used to minimize the functional, and the problem must be solved using Galerkin's method of the weighted residual. See Ref. [4] for a description of this method.

One final observation can be made. The Ritz method does not need to be applied when discretizing a system that is in variational form. Any of the Galerkin weighted residual methods will also work; however, when applying those methods to a problem in variational form, the result will always reduce to the same form as the Ritz approximation [10]. This is important to note because many authors publish that their approximation is done via weighted methods, but in the end, given that their system is in variational form, the final form will be the same as the resultant form of the Ritz approximation method. Furthermore, the notation behind weighted methods is sometimes simpler, and for that reason, one might be compelled to setup the problem in a weighted form; this result allows for such a substitution.

APPENDIX B

STRAIN EQUATIONS

STRAIN EQUATIONS

This appendix derives the strain equations from first principles of structural mechanics for use in building the stress and strain terms in the cantilever beam problem.

B.1 NORMAL STRAIN

The strain energy of the beam is proportional to the deformation and deforming force. Mathematically, this is expressed as

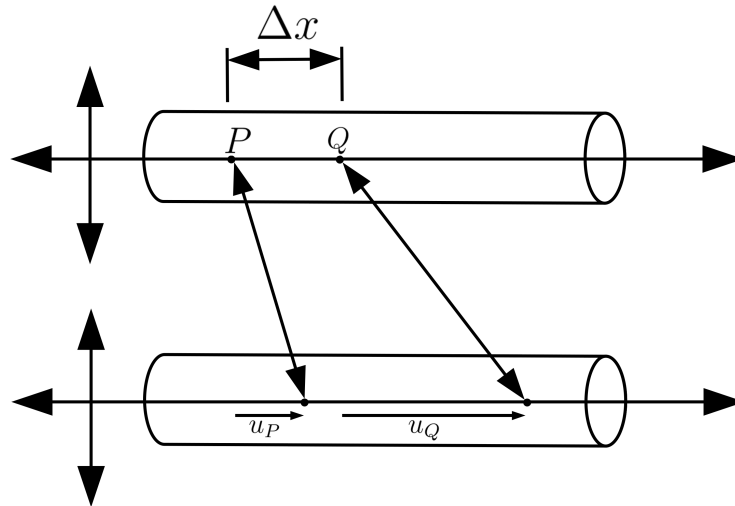$$U = \int_0^\delta F \, \mathrm{d}\delta, \tag{B.1}$$

where U is strain energy, F is the applied force, and δ is the displacement.

The **average** *Normal strain* is given by

$$\varepsilon_{\text{avg}} = \frac{\Delta L}{L}, \tag{B.2}$$

where L is the length and Δ is the usual difference operator, and the average normal strain is a dimensionless ratio. This strain is normal to the applied force.

Figure B.1. Normal Strain in a 1D Rod

The **point** normal strain in 1D is constructed as follows. Suppose two points are examined along a rod as in Figure B.1, say P and Q. The distance between the points is denoted

$$\Delta x = Q - P, \tag{B.3}$$

and the displacements of the two points P and Q are denoted as u_P and u_Q so that the displacement is given as

$$\Delta u = u_Q - u_P. \tag{B.4}$$

By setting

$$u = u_P \tag{B.5}$$

and adding zero

$$u_Q = u_Q + u_P - u_P = u_P + (u_Q - u_P) \tag{B.6}$$

we find

$$u_Q = u + \Delta u, \tag{B.7}$$

and we can write the difference quotient

$$\frac{(u + \Delta u) - u}{\Delta x}. \tag{B.8}$$

The point normal strain is given as the limit of this quotient

$$\varepsilon_p = \lim_{\Delta x \to 0} \frac{(u + \Delta u) - u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\mathrm{d}u}{\mathrm{d}x} \tag{B.9}$$

In the case of higher dimensions, the displacement functions are functions of higher dimension, and the point normal strain in each direction would be a partial derivative, *e.g.*

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{B.10}$$

for strain in the x-direction. The subscript, in this case, is given in *index* notation for the displacement vector u and the spatial displacement in x.

The mixed case is more general, and is a combination of multiple terms, given as

$$\varepsilon_{ij} = \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\} + \frac{1}{2} \left\{ \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right\} \tag{B.11}$$

in the orthogonal ijk coordinate frame.

It is important to note that this relationship only holds for small strains. The linearity of the displacement-strain relationship is a local effect. Large strains often result in nonlinear deformations. This requires consideration of the higher order terms, which yields *e.g.*

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right\} \tag{B.12}$$

for a 3D system.

B.2 SHEAR STRAIN

The *Shear strain* is an angular change. It is often thought of as the change in angle between two vectors. The study of shear strain involves the consideration of an infinitesimal volume that is cubic in shape. By considering only one of the faces of the cube, it is possible to discuss the angular relationship between the bottom and left sides of the cube. Before deformation, they are $\pi/2$ radians apart; once deformed, they will be $\pi/2 - \gamma$ radians apart, where $\gamma = \gamma_1 + \gamma_2$ is the final angular deformation. Here γ_1, γ_2 represent the angular displacements of the bottom and left edges with respect to their initial positions.

Figure B.2. Shear Strain over a Square [5]
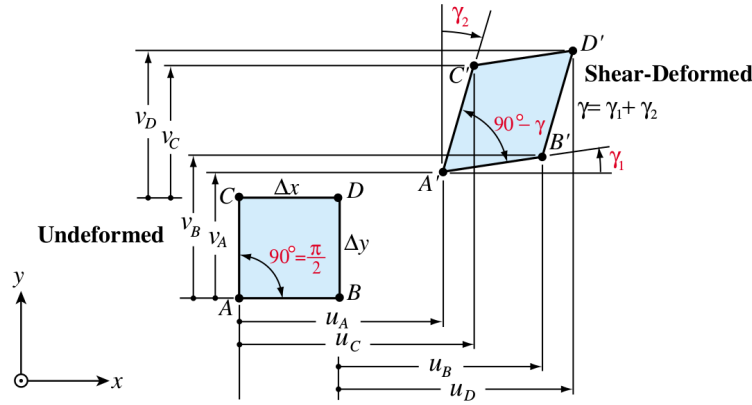
The *Average shear strain* is given as

$$\gamma_{\text{avg}} = \frac{\pi}{2} - \gamma, \tag{B.13}$$

where γ is the total angular displacement of the edges.

In general, it is also possible for the cube to be translated from its original position, and thus a general form will need to be implemented to find the displacements γ_1, γ_2.

From Figure B.2, we consider the right triangle formed with hypotenuse $A'B'$. The angle γ_1 is known to be the arctangent of the vertical over the horizontal legs. Using the corresponding notation, we have

$$\tan(\gamma_1) = \left(\frac{v_B - v_A}{u_B - u_A} \right) \tag{B.14}$$

and we write the numerator as a difference, while noting that the denominator is a change in the distance of segment AB. For small strains, this change will be trivial, so we approximate

$$u_B - u_A \approx \Delta x. \tag{B.15}$$

Plugging Equation B.15 into Equation B.14, and writing the numerator as a difference, we have

$$\tan\left(\gamma_1\right) = \left(\frac{\Delta v_{BA}}{\Delta x}\right). \tag{B.16}$$

Furthermore, by assuming small strains we have

$$\gamma_1 \ll 1 \tag{B.17}$$

so that

$$\tan\left(\gamma_1\right) \approx \gamma_1. \tag{B.18}$$

Following a similar process for γ_2, we construct

$$\gamma_1 = \frac{\Delta v}{\Delta x}, \qquad \gamma_2 = \frac{\Delta u}{\Delta y}, \tag{B.19}$$

and write

$$\gamma = \gamma_1 + \gamma_2 = \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y}. \tag{B.20}$$

We now write the definition of the *Point shear strain* as

$$\gamma_{xy} = \lim_{\Delta x, \Delta y \to 0} \gamma = \lim_{\Delta x, \Delta y \to 0} \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \tag{B.21}$$

Linearity of ∂ implies that $\gamma_{xy} = \gamma_{yx}$.

This is easily extended to 3D so that all faces of the cube can be analyzed for shear strain.

B.3 STRAIN ENERGY DENSITY

Average strain energy density is given as

$$\bar{U}_d = \int \sigma \, d\varepsilon, \tag{B.22}$$

where σ is **stress**, and ε is **strain**. *Normal stress* is defined as

$$\sigma = \frac{F}{A}, \tag{B.23}$$

where F is the applied force, and A is the surface area upon which the force is applied. *Shear stress* is defined as

$$\tau = \frac{T}{A}, \tag{B.24}$$

where T is the applied torque, and A is the surface area upon which the torque is applied. F is a force normal to the cross section, while T is tangential to the cross section.

Hooke's Law relates stress to strain by the linear relationship

$$\sigma = E\varepsilon, \tag{B.25}$$

where E represents Young's modulus, or the elastic modulus, and is a material property. Hooke's law holds for a small range on most materials, but some materials are nonlinear and do not have a linear relationship. For our purposes, the materials are linear for small strains; this is true for many common metallic alloys such as aluminum and steel [6].

Plugging Equation B.25 into Equation B.22 yields

$$\bar{U}_d = \int E\varepsilon \, \mathrm{d}\varepsilon = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\left(\frac{\sigma^2}{E}\right). \tag{B.26}$$

Total strain energy of a rod or beam is given by

$$U = \int_V \bar{U}_d \, \mathrm{d}V = \int_0^L \int_A \frac{1}{2}\left(\frac{\sigma^2}{E}\right) \, \mathrm{d}A \, \mathrm{d}x. \tag{B.27}$$

Similar to Hooke's law, there is a linear relationship between shear stress and shear strain given by

$$\tau = G\gamma, \tag{B.28}$$

where γ is shear strain, G is the material property called shear modulus, and τ is shear stress. The same procedure as above can be applied to calculate the strain energy from shear.

Poisson's Ratio is the absolute value of the lateral strain over the axial strain, is a dimensionless value, and is denoted by ν. E, G, and ν are related by

$$E = 2G(1 + \nu). \tag{B.29}$$