

Direct Path Method for Flexible Multibody Spacecraft Dynamics

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The dynamic equations of motion of a multibody flexible spacecraft with topological tree configuration are derived by the direct path method. The terminal bodies are treated as flexible and the interconnected bodies as rigid. Bodies are connected by gimbal hinges supplied with torsional springs and dampers. The three gimbal rotational axes may be either completely free or locked sequentially. Three approaches are suggested in this analysis, namely; the Lagrangian-Newtonian approach, the complete Newtonian approach, and the complete Lagrangian approach. The direct paths, the direct position vectors, and the incidence matrix are the key features of this analysis. They efficiently govern the contribution from the single body to the overall dynamic equations of the system. From the physical point of view, the complete Newtonian approach is the most straightforward. Torque equations are obtained from directly taking the moment about the hinge points rather than going through the lengthier differentiation of the angular momentum. This approach also gives better physical understanding to the interaction among motions of the individual bodies. From the mathematical point of view, the complete Lagrangian approach is the fastest. The lengthy elimination of the interacting forces and torques does not appear in this approach.

Nomenclature

D^j	= rate of change in damping energy dissipation of body j
e_ξ^e	= unit vectors along x_ξ^e ($\xi = 1, 2, 3$)
e_β^e	= unit vectors along x_β^e ($\beta = 1, 2, 3$)
e_γ^e	= unit vectors along x_γ^e ($\gamma = 1, 2, 3$)
$e_\eta^{r'}$	= base vectors along $x_\eta^{r'}$ ($\eta = 1, 2, 3$)
$e_\zeta^{r'}$	= reciprocal base vectors associated with $e_\eta^{r'}$ ($\zeta = 1, 2, 3$)
e_λ^{vg}	= base vectors along x_λ^{vg} ($\lambda = 1, 2, 3$)
e_μ^{vg}	= reciprocal base vectors associated with e_λ^{vg} ($\mu = 1, 2, 3$)
f^{je}	= $f_\beta^j e_\beta^e$ = force vector acting on the unit mass in body j due to the environmental disturbances
F^j	= $F_\beta^j e_\beta^e$ = force vector acting on body j at O_j from its limb body
$g^{v(j)}$	= $g_\beta^{v(j)} e_\beta^e$ = direct position vector $\overline{O_v Q_j}$
G^{jr}	= $e_\beta^j G_{\beta\gamma}^{jr} e_\gamma^{r'}$ = Euler transformation tensor between e_β^j and $e_\gamma^{r'}$
G^v	= $e_\beta^v G_{\beta\alpha}^v e_\alpha^{vg}$ = gimbal transformation tensor between e_β^v and e_α^{vg}
$G^{u(T)}$	= $e_\mu^{ug} G_{\mu\beta}^{u(T)} e_\beta^u$ = transpose gimbal transformation tensor between e_μ^{ug} and e_β^u
h^{jm}	= $h_\beta^{jm} e_\beta^j$ = position vector $\overline{O_j O_m}$
K^j	= kinetic energy of body j
K_{kl}^j	= generalized stiffness matrix of body j ($k, l = 1, 2, \dots, n_j$)
K_σ^j	= torsional spring constants about the gimbal rotational axes between body j and its limb body ($\sigma = 1, 2, 3$)
N	= total number of bodies
n_j	= number of the natural vibration modes for body j
O_e	= origin of the Earth coordinates
O_j	= origin of the body j coordinates
O_r	= origin of the spacecraft reference coordinates

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P_j, P'_j	= positions of the mass element dm^j before and after elastic deformation
Q_j'	= instantaneous mass center of body j
$q_j^i(t)$	= time function associated with $\phi_j^i(r^j)$
r^j	= $r_\beta^j e_\beta^j$ = position vector $\overline{O_j P_j}$
R^j	= $R_\gamma^j e_\gamma^r$ = position vector $\overline{O_r O_j}$
R'	= $R_\xi^r e_\xi^e$ = position vector $\overline{O_e O_r}$
S^j	= strain energy of body j
T^j	= $T_\beta^j e_\beta^j$ = torque vector acting on body j at O_j from its limb body
u^j	= $u_\beta^j e_\beta^j$ = displacement vector $\overline{P_j P'_j}$
V_{kl}^j	= generalized damping matrix of body j ($k, l = 1, 2, \dots, n_j$)
V_σ^j	= torsional damping constants about the gimbal rotational axes between body j and its limb body ($\sigma = 1, 2, 3$)
x_ξ^e	= Earth coordinates
x_β^e	= body j coordinates
x_γ^e	= spacecraft reference coordinates
$x_\eta^{r'}$	= Euler rotational axes between x_γ^e and x_β^e
x_λ^{vg}	= gimbal rotational axes between x_β^v and x_α^{vg}
x_α^{vg}	= nominal (mounting) coordinates of body v on its limb body
ϵ^{vj}	= incidence matrix ($v, j = 1, 2, \dots, N$)
ρ^j	= $\rho_\xi^j e_\xi^e$ = position vector $\overline{O_e P'_j}$
θ_η^{jr}	= Euler angles about x_η^{jr} ($\eta = 1, 2, 3$)
θ_λ^v	= gimbal rotational angles about x_λ^{vg} ($\lambda = 1, 2, 3$)
ω^j	= $\omega_\beta^j e_\beta^j$ = angular velocity vector of body j
ω'	= $\omega_\gamma^r e_\gamma^r$ = angular velocity vector of the reference coordinates
$\phi_j^i(r^j)$	= $\phi_{j\beta}^i e_\beta^j$ = natural vibration modal displacement vector functions of body j based on: "Point O_j is constrained from any linear or rotational displacements; and the rest of the boundary is free of force and torque" ($l = 1, 2, \dots, n_j$)

I. Introduction

THIS paper formulates the dynamic equations of motion of a flexible multibody spacecraft with arbitrary topological tree configuration. The terminal bodies are

treated as flexible and the interconnected bodies as rigid. Bodies are connected by gimballed hinges supplied with torsional springs and dampers. The three gimbal rotational axes may be either completely free or locked sequentially. A general-purpose simulation computer program can be developed from this analysis.

Dynamic equations of motion of the multibody system have been investigated by many people in recent years.¹⁻¹⁷ Fletcher, Rongved, and Yu¹ first treated the simple case of two rigid bodies. Hooker and Margulies² derived the dynamic equations of the n -rigid-body system. Later, Hooker³ presented the modification technique to eliminate the interacting constraint torques. In their papers, the concept of "barycenter" was introduced, which can be interpreted as the new mass center obtained by loading each joint of the body with the residual mass of the subsystem of bodies connected to that point. Roberson and Wittenburg⁴ treated the n -rigid-body independently and derived the dynamic equations in matrix form. In this paper, the interacting torques were not eliminated.

Ness and Farrenkopf⁵ and Ho and Bluck⁶ extended the model into the flexible n -body system. Ness and Farrenkopf, using the flexible-terminal/rigid-interconnected model, chose the unified approach to deal with the nonlinear dynamic equations of the total motion of the system. Ho and Gluck, using the all-flexible model, chose the perturbation approach to deal with the linearized dynamic equations of the perturbed motion. They considered the total motion to be the superposition of a small perturbed motion about the nominal motion with all the flexible bodies temporarily assumed to be rigid. In Refs. 5 and 6, the dynamic equations of a single body are first derived analytically. Then, they are combined with the equations of the adjacent body through a computerized inductive method following the combining algorithm. The interacting forces and torques are eliminated at each combining step. Modifications for the case of the locked gimbal axes are performed simultaneously. The drawback of the computerized inductive method is that the loading and reloading of the coefficient matrices in the dynamic equations of the subsystems of bodies will, undoubtedly, take up much computer time. And also it does not provide the analytical overall dynamic equations of the complete system.

The direct path method was originally developed in 1973.⁴ At the same time Hooker¹⁵ and Likins¹⁶ independently developed their analysis using the barycenter approach. The differences between the direct path approach and the barycenter approach will be discussed in Sec. X. After numerous technical discussions with the author, Hooker abandoned the barycenter approach in favor of the direct path approach and developed another analysis.¹⁷ We agreed that a great deal of unnecessary calculations can be eliminated.

In the direct path approach, we try to predetermine analytically the contribution from the motion of each body to the certain slots of the coefficient matrices of the overall dynamic equations in the most efficient way. Three approaches are suggested in this analysis, namely; the Lagrangian-Newtonian approach, the complete Newtonian approach, and the complete Lagrangian approach. In the first two approaches, the dynamic equations of each body are first derived from the Lagrangian approach or the Newtonian approach. Since bodies are connected to each other by gimbal hinges, the position vector, the angular velocity vector, and their time derivatives for each body are related to those of the other bodies through kinematic relations.

It is necessary to transform the variables in the dynamic equations of all the bodies to a common set of variables of a specific body in the topological tree system. Evidently, the main body, which carries the payload, is the most logical choice. The direct path is a vectorial path oriented from the main body directly toward each body in the system. The direct position vectors, which lead from the hinge points to the in-

stantaneous mass centers of the other bodies within the same direct path, are widely used throughout the analysis. They serve as the grouping devices of the position vectors embedded in the bodies along the direct path, and transform the influence of the motion of one body to the others. The angular velocity and acceleration vectors of each body are also transformed into those of the main body and the reference coordinates of the system. The relative angular velocity and acceleration vectors between adjacent bodies along the direct path are also induced into the equations. The incidence matrix used in the analysis can not only uniquely define the specific topological tree configuration, but also systematically govern the transformation of variables of each body. The dynamic equations of each body are rearranged in such a way that the elimination of the interacting forces and torques at all the hinge points will be completed on one single summation step. This part of the derivation is considered to be Newtonian in nature.

In the complete Lagrangian approach, we first derive the kinetic energy of each body in its own body coordinates, and then transform the variables before obtaining the total kinetic energy of the system. The strain energy and the rate of change in damping energy dissipation in the flexible terminal bodies and at all the gimbal hinges are obtained in a rather simple manner. The overall dynamic equations are directly obtained from LaGrange's equations.

II. System Definition for the Topological Tree

Figure 1 shows a flexible multibody spacecraft with topological tree configuration. Bodies are branching out from the main body into different levels. The main body is named as body 1; the next N_1 level-1 bodies are named as body 2, body 3,...body $(N_1 + 1)$; and then the next N_2 level-2 bodies are named as body $(N_1 + 2)$, body $(N_1 + 3)$,...,body $(N_1 + N_2 + 1)$ and so forth. Each body has its own body coordinates, x_β^j ($j = 1, 2, \dots, N$) ($\beta = 1, 2, 3$). There are also the arbitrarily chosen spacecraft reference coordinates x_γ^r ($\gamma = 1, 2, 3$), and the Earth coordinates, x_ξ^e ($\xi = 1, 2, 3$).

III. Dynamic Equations of a Flexible Terminal Body

The dynamic equations of a flexible terminal body may be derived from either the Lagrangian approach or the Newtonian approach.

A. Lagrangian Approach

The position vector of a mass element dm^j in body j at the deformed state with respect to the origin of the Earth coordinates, point O_e , as shown in Fig. 2, is

$$\rho^j = R^r + R^j + r^j + u^j \quad (1)$$

where

$$u^j = \sum_{l=1}^{n_j} q^j_l \phi_l^j \quad (2)$$

The velocity vector of the mass element dm^j is

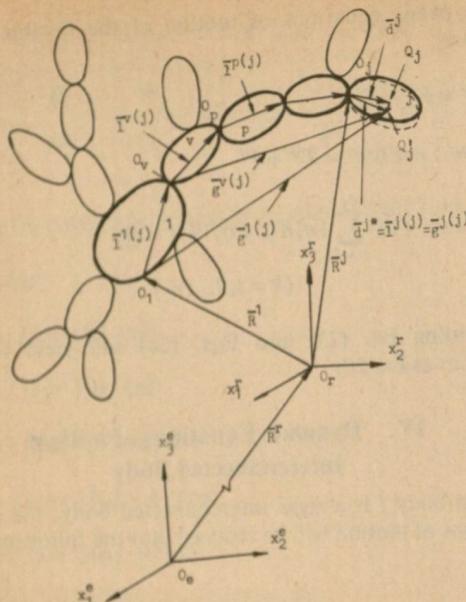
$$v^j = \dot{R}^r + \dot{R}^j + \omega^r \times R^j + \omega^j \times r^j + \sum_{l=1}^{n_j} (\dot{q}^j_l \phi_l^j + q^j_l \omega^j \times \phi_l^j) \quad (3)$$

The kinetic energy of body j takes the following form

$$K^j = \frac{1}{2} \int v^j \cdot v^j dm^j \quad (4)$$

Substituting Eq. (3) into Eq. (4), we have

$$K^j = \frac{1}{2} m^j [\dot{R}^r + \dot{R}^j - R^j \times \omega^r + \omega^j \times \dot{q}_k^j] \quad (5)$$

Fig. 3 Direct paths of body j and direct position vectors.

introduce the "incidence matrix," ϵ^{vj} ($v, j = 1, 2, \dots, N$). The values of ϵ^{vj} are either 1 or 0. They govern the appearance of body v in the direct path $1-j$ for the specific topological tree. The incidence matrix for the topological tree in Fig. 1 is shown in Table 1.

The direct position vector is written as

$$\mathbf{g}^{v(j)} = \epsilon^{vj} \sum_{p=1}^j \epsilon^{pj} \mathbf{l}^{p(j)} \quad (26)$$

where $\mathbf{l}^{p(j)} = l_p^{p(j)} \mathbf{e}_p^p$ = position vector $\overline{O_p O_s}$, where body s is the adjacent branch body of body p ; and both body p and body s are in the direct path $v-j$ ($p \neq j$) and also $\mathbf{l}^{p(j)} = \mathbf{d}^j = \mathbf{g}^{j(j)}$.

$$\mathbf{R}^j = \mathbf{R}^I + \sum_{p=1}^j \epsilon^{pj} \mathbf{l}^{p(j)} - \mathbf{l}^{(j)} \quad (27a)$$

$$\dot{\mathbf{R}}^j + \omega^r \times \mathbf{R}^j = \dot{\mathbf{R}}^I + \omega^r \times \mathbf{R}^I + \sum_{p=1}^j \epsilon^{pj} [\omega^p \times \mathbf{l}^{p(j)}] - \omega^j \times \mathbf{l}^{(j)} \quad (27b)$$

$$\ddot{\mathbf{R}}^j = \ddot{\mathbf{R}}^I + \sum_{p=1}^j \epsilon^{pj} [\dot{\omega}^p \times \mathbf{l}^{p(j)} + \omega^p (\omega^p \times \mathbf{l}^{p(j)})] - \dot{\omega}^j \times \mathbf{l}^{(j)} - \omega^j \times (\omega^j \times \mathbf{l}^{(j)}) \quad (27c)$$

Since

$$\omega^p = \omega^r + \sum_{v=1}^p \epsilon^{vp} [\mathbf{G}^v \cdot \dot{\theta}^v] \quad (28a)$$

$$\dot{\omega}^p = \dot{\omega}^r + \sum_{v=1}^p \epsilon^{vp} [\dot{\mathbf{G}}^v \cdot \dot{\theta}^v + (\dot{\mathbf{G}}^v + \omega^v \times \mathbf{G}^v) \cdot \dot{\theta}^v] \quad (28b)$$

$$= \dot{\omega}^r + \sum_{v=1}^p \epsilon^{vp} [\mathbf{G}^v \cdot \dot{\theta}^v]^o$$

where

$$\dot{\theta}^v = \dot{\theta}_\lambda^v \mathbf{e}_\lambda^{vg} \quad \mathbf{G}^v = \mathbf{e}_\beta^v \mathbf{G}_\beta^v \mathbf{e}_\mu^{vg*} \quad (29)$$

\mathbf{e}_λ^{vg} ($\lambda = 1, 2, 3$ are the base vectors along the gimbal rotational axes x_λ^{vg} between body v and its limb body; and \mathbf{e}_μ^{vg*} ($\mu = 1, 2, 3$) are the reciprocal base vectors. Their relations are similar to those in Eqs. (12) and (13).

Table 1 Incidence matrix for the topological tree in Fig. 1^a

v	j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2		1						1	1							
3			1					1	1							
4				1					1							
5					1					1						
6						1					1					
7							1					1				
8								1					1			
9									1							
10										1						
11											1					
12												1				
13													1			
14														1		
15															1	

^aThe empty slots are zeros.

Combining Eqs. (28), (29), and (26) yields the following equation

$$\begin{aligned} \ddot{\mathbf{R}}^j &= \ddot{\mathbf{R}}^I - (\mathbf{g}^{I(j)} - \mathbf{g}^{j(j)}) \times \dot{\omega}^r \\ &+ \sum_{v=1}^j \epsilon^{vj} [\mathbf{g}^{v(j)} - \mathbf{g}^{j(j)}] \times [\mathbf{G}^v \cdot \dot{\theta}^v]^o \\ &+ \sum_{v=1}^j \epsilon^{vj} [\omega^v \times (\omega^v \times \mathbf{l}^{v(j)})] - \omega^j (\omega^j \times \mathbf{l}^{(j)}) \end{aligned} \quad (30)$$

VI. Contribution from Single-Body Motion to Overall Dynamic Equations

Before determining the contribution from the single-body motion to the overall dynamic equations of the system, let us reexamine the physical meaning of Eqs. (20a,b,c): Equation (20a)—force equation of body j ; Equation (20b)—torque equation of body j about point O_j ; and Equation (20c)—independent vibration equations of body j . The overall dynamic equations of the system will take the following form:

- 1) the overall force equation of the system;
- 2) the torque equations about all the hinge points, O_u ($u = 1, 2, \dots, N$); and
- 3) the independent vibration equation of all the flexible terminal bodies.

The motion of a single body will definitely influence the motion of the whole system. In this direct path method, we have found that we do not really need all the interrelations among bodies. The motion of body j will only contribute to the overall force equation and the torque equations about the hinge points which belong to the *direct path* $1-j$. We may rearrange Eqs. (20a,b,c) in the following manner

$$\left\{ \begin{array}{l} \text{Eq. (20a)} \\ \text{--- } \epsilon^{uj} [\text{Eq. (20b)} + (\mathbf{g}^{u(j)} - \mathbf{g}^{j(j)}) \times \text{Eq. (20a)}] \\ \text{--- } \epsilon^{sj} [\text{Eq. (20c)}] \end{array} \right\} \quad (u = 1, 2, \dots, N; s = \text{flexible terminal bodies})$$

Using the previous rearrangement and the transformation equations (28) and (30), the contribution from the motion of body j to the overall dynamic equations of the system will be

$$m^j \begin{bmatrix} \delta \bullet & -\mathbf{g}^{v(j)} \epsilon^{vj} \times & \Phi^j / \epsilon^{ij} \\ \epsilon^{uj} \mathbf{g}^{u(j)} \times & \epsilon^{uj} \mathbf{g}^{uv(j)} \epsilon^{vj} & \epsilon^{uj} \omega_j^u \epsilon^{ij} \\ \epsilon^{sj} \Phi_k^j \bullet & \epsilon^{sj} \omega_k^v \epsilon^{vj} & \epsilon^{sj} M_{kl}^j \epsilon^{ij} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{R}}^I + \dots \\ \mathbf{G}^v \times \dots \\ \mathbf{M}_{kl}^j \times \dots \end{bmatrix}$$

$$\begin{aligned}
& +m^j \begin{Bmatrix} -\mathbf{g}^{u(j)} \times \\ \epsilon^{uj} \mathbf{g}^{u(j)} \cdot \\ \epsilon^{sj} \omega_k^{l(j)} \cdot \end{Bmatrix} \dot{\omega}^r + m^j \begin{Bmatrix} 0 \\ \epsilon^{uj} \omega^j \times \mathbf{B}^{j\Delta} \cdot \omega^j \\ -\epsilon^{sj} \omega^j \cdot \mathbf{C}_k^{j\Delta} \cdot \omega^j \end{Bmatrix} \\
& +m^j \begin{Bmatrix} \delta \cdot \\ \epsilon^{uj} \mathbf{g}^{u(j)} \times \\ \epsilon^{sj} \Phi_k^j \cdot \end{Bmatrix} \epsilon^{uj} [\omega^v \times (\omega^v \times \mathbf{l}^{v(j)})] \\
& +2m^j \begin{Bmatrix} \omega^j \times \Phi_k^j \\ \epsilon^{uj} \mathbf{J}_l^{u(j)} \cdot \omega^j \\ -\epsilon^{sj} \mathbf{Z}_{kl}^j \cdot \omega^j \end{Bmatrix} \epsilon^{uj} \dot{q}_l^j + m^j \begin{Bmatrix} 0 \\ 0 \\ \epsilon^{sj} (K_{kl}^j q_l^j + V_{kl}^j \dot{q}_l^j) \end{Bmatrix} \\
& = \begin{Bmatrix} \mathbf{F}^{je} \\ \epsilon^{uj} \mathbf{T}^{u(je)} \\ \epsilon^{sj} Q_k^{je} \end{Bmatrix} + \begin{Bmatrix} \mathbf{F}^j \\ \epsilon^{uj} [\mathbf{T}^j + (\mathbf{g}^{u(j)} - \mathbf{g}^{j(j)}) \times \mathbf{F}^j] \\ 0 \end{Bmatrix} \quad (31a) \\
& \quad (31b) \quad (31c)
\end{aligned}$$

$u=1,2,\dots,N$; s = flexible terminal bodies; $k=1,2,\dots,n_j$; sum on $v=1,2,\dots,N$; t = flexible terminal bodies; and $l=1,2,\dots,n_j$.

where

$$\mathbf{g}^{u(j)} = \mathbf{B}^{j\Delta} - \mathbf{g}^{u(j)} \times (\mathbf{g}^{v(j)} \times \quad (32a)$$

$$\mathbf{W}_l^{u(j)} = \mathbf{W}_l^{j\Delta} + \mathbf{g}^{u(j)} \times \Phi_l^j \quad (32b)$$

$$\mathbf{W}_k^{v(j)} = \mathbf{W}_k^{j\Delta} - \Phi_k^j \times \mathbf{g}^{v(j)} \quad (32c)$$

$$\mathbf{J}_l^{u(j)} = \mathbf{C}_l^j - (\mathbf{g}^{u(j)} - \mathbf{g}^{j(j)}) \times (\Phi_l^j \times \quad (32d)$$

VII. Overall Dynamic Equations of System

The overall dynamic equations of the multibody system can be obtained by the following simple summation.

$$\sum_{j=1}^N \left\{ \mathbf{F}^j - \sum_m^{(j)} \mathbf{F}^m \right\} = \mathbf{F}' = 0 \quad (34)$$

$$\begin{aligned}
& \sum_{j=u}^N \epsilon^{uj} \left\{ \mathbf{T}^j + (\mathbf{g}^{u(j)} - \mathbf{g}^{j(j)}) \times \mathbf{F}^j \right. \\
& \quad \left. - \sum_m^{(j)} [\mathbf{T}^m + (\mathbf{g}^{u(j)} - \mathbf{g}^{j(j)} + \mathbf{h}^{jm}) \times \mathbf{F}^m] \right\} = \mathbf{T}^u \quad (35)
\end{aligned}$$

\mathbf{T}^u is the interacting torque vector at point O_u and its components T_β^u ($\beta=1,2,3$) are associated with the orthogonal unit vectors \mathbf{e}_β^u in the body u coordinates. It can be transformed into the gimbal torque \mathbf{T}^{ug} by premultiplying Eq. (35) with the transpose gimbal transformation tensor $\mathbf{G}^{u(T)} = \mathbf{e}_\mu^{ug} \mathbf{G}_{\mu\beta}^{u(T)} \mathbf{e}_\beta^u$. Since the components of the gimbal torque \mathbf{T}^{ug} , i.e., T_λ^{ug} ($\lambda=1,2,3$), are associated with the nonorthogonal base vectors \mathbf{e}_λ^{ug} ($\lambda=1,2,3$) along the gimbal rotational axes x_λ^{ug} ($\lambda=1,2,3$), we have the following relation

$$\begin{aligned}
\mathbf{T}^{ug} &= \mathbf{e}_\lambda^{ug} T_\lambda^{ug} = \mathbf{G}^{u(T)} \cdot \mathbf{T}^u = \mathbf{e}_\mu^{ug} \mathbf{G}_{\mu\beta}^{u(T)} T_\beta^u \\
&= \mathbf{e}_\lambda^{ug} G_{\lambda\zeta}^{u(-l)} G_{\zeta\mu}^{u(T)(-l)} G_{\mu\lambda}^{u(T)} T_\beta^u \\
&= \mathbf{e}_\lambda^{ug} G_{\lambda\beta}^{u(-l)} T_\beta^u = \mathbf{e}_\lambda^{ug} (-K_\lambda^{ug} \theta_\lambda^u - V_\lambda^{ug} \dot{\theta}_\lambda^u) \quad (36)
\end{aligned}$$

The interacting forces and torques are now completely eliminated, because θ_λ^u and $\dot{\theta}_\lambda^u$ ($\lambda=1,2,3$) are part of the state variables in the overall dynamic equations which will take the following form

$$\begin{aligned}
& m \begin{bmatrix} \delta \cdot & -\mathbf{D}^v \times \mathbf{G}^v \cdot & \Psi_l^t \\ \mathbf{G}^{u(T)} \times \mathbf{D}^u \cdot & \mathbf{G}^{u(T)} \cdot \mathbf{J}^{uv} \cdot \mathbf{G}^v \cdot & \mathbf{G}^{u(T)} \cdot \mathbf{H}_l^{ut} \\ \Psi_k^s & \mathbf{H}_k^{sv} \cdot \mathbf{G}^v \cdot & S_{kl}^{st} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{R}}^t \\ \ddot{\theta}^v \\ \ddot{q}_l^t \end{Bmatrix} = -m \begin{bmatrix} \delta \cdot & -\mathbf{D}^v \times \\ \mathbf{G}^{u(T)} \times \mathbf{D}^u \cdot & \mathbf{G}^{u(T)} \cdot \mathbf{J}^{uv} \cdot \\ \Psi_k^s \cdot & \mathbf{H}_k^{sv} \cdot \end{bmatrix} \\
& \left\{ \begin{array}{l} \ddot{\mathbf{R}}^r + 2\omega^r \times \dot{\mathbf{R}}^r \\ + \dot{\omega}^r \times \mathbf{R}^r + \omega^r \times (\omega^r \times \mathbf{R}^r) \\ - (\dot{\mathbf{G}}^v + \omega^v \times \mathbf{G}^v) \cdot \theta^v \end{array} \right\} - m \begin{Bmatrix} -\mathbf{D}' \times \\ \mathbf{G}^{u(T)} \cdot \mathbf{J}^{ul} \cdot \\ \mathbf{H}_k^{lr} \cdot \end{Bmatrix} \dot{\omega}^r - m \begin{Bmatrix} \mathbf{L} \\ \mathbf{G}^{u(T)} \cdot \mathbf{P}^u \\ X_k^s \end{Bmatrix} + \begin{Bmatrix} 0 \\ \mathbf{T}^{ug} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \mathbf{F}^{(e)} \\ \mathbf{G}^{u(T)} \cdot \mathbf{T}^{u(e)} \\ Q_k^{s(e)} \end{Bmatrix} \quad (37a) \\
& \quad (37b) \quad (37c)
\end{aligned}$$

where

$$m = \sum_{j=1}^N m^j \quad (38a)$$

$$\mu^j = m^j/m \quad (38b)$$

$$\mathbf{D}^v = \sum_{j=v}^N \mu^j \epsilon^{vj} \mathbf{g}^{v(j)} \quad (38c)$$

$$\Psi_l^t = \mu^t \Phi_l^t \quad (38d)$$

$$\mathbf{J}^{uv} = \sum_{j=\max(u,v)}^N \mu^j \epsilon^{uj} \mathbf{g}^{u(j)} \epsilon^{vj} \quad (38e)$$

$$\mathbf{H}_l^{uv} = \epsilon^{ui} \mu^{is} \mathbf{W}_l^{u(i)} \quad (38f)$$

$$\mathbf{H}_k^{sv} = \epsilon^{vs} \mu^{st} \mathbf{W}_k^{v(s)} \quad (38g)$$

$$S_{kl}^{st} = \epsilon^{st} \mu^t M_{kl}^t \quad (38h)$$

$u,v=1,2,\dots,N$; s,t = flexible terminal bodies; $k=1,2,\dots,n_s$; and $l=1,2,\dots,n_t$.

$$\begin{aligned}
& \mathbf{B}^{j\Delta} = \mathbf{B}^j + \mathbf{d}^{j*} \times (\mathbf{d}^{j*} \times \quad (32e) \\
& \mathbf{W}_k^{j\Delta} = \mathbf{W}_k^j + \Phi_k^j \times \mathbf{d}^{j*} \quad (32f) \\
& \mathbf{C}_k^{j\Delta} = \mathbf{C}_k^j + \Phi_k^j \times (\mathbf{d}^{j*} \times \quad (32g)
\end{aligned}$$

($u,v=1,2,\dots,N$; $k=1,2,\dots,n_j$)

$$\mathbf{T}^{u(je)} = \mathbf{T}^{je} + (\mathbf{g}^{u(j)} - \mathbf{g}^{j(j)}) \times \mathbf{F}^{je} \quad (33)$$

($u=1,2,\dots,N$)

When body j is a rigid interconnected body, there will be no vibration equations, and all the terms associated with q_l^j and \dot{q}_l^j will vanish. The interacting forces and torques will be

$$\left\{ \begin{array}{l} \mathbf{F}^j - \sum_m^{(j)} \mathbf{F}^m \\ \epsilon^{uj} \{ \mathbf{T}^j + (\mathbf{g}^{u(j)} - \mathbf{g}^{j(j)}) \times \mathbf{F}^j \\ - \sum_m^{(j)} [\mathbf{T}^m + (\mathbf{g}^{u(j)} - \mathbf{g}^{j(j)} + \mathbf{h}^{jm}) \times \mathbf{F}^m] \} \end{array} \right\}$$

$$L = \sum_{j=1}^N \mu^j \sum_{v=1}^N \epsilon^{vj} \omega^v \times (\omega^v \times I^{v(j)}) + 2 \sum_{l=1}^{(F)} \mu^l \sum_{l=1}^{n_l} \dot{q}_l^l \omega^l \times \Phi_l^l \quad (39a)$$

$$\begin{aligned} P^u &= \sum_{j=u}^N \mu^j \epsilon^{uj} \omega^j \times B^{\Delta} \cdot \omega^j + \sum_{j=u}^N \mu^j \epsilon^{uj} \sum_{v=1}^N \epsilon^{vj} g^{uv} \\ &\quad \times [\omega^v \times (\omega^v \times I^{v(j)})] + 2 \sum_{l=1}^{(F)} \mu^l \epsilon^{ul} \sum_{l=1}^{n_l} \dot{q}_l^l J_l^{u(l)} \cdot \omega^l \end{aligned} \quad (39b)$$

$$\begin{aligned} X_k^s &= \mu^s \left\{ -\omega^s \cdot C_k^s \cdot \omega^s + \Phi_k^s \cdot \sum_{v=1}^N \epsilon^{vs} \omega^v \times (\omega^v \times I^{v(s)}) \right. \\ &\quad \left. - 2 \sum_{l=1}^{n_s} \dot{q}_l^s Z_{kl}^s \cdot \omega^s + \sum_{l=1}^{n_s} \left[q_l^s K_{kl}^s + \dot{q}_l^s V_{kl}^s \right] \right\} \end{aligned} \quad (39c)$$

$u=1,2,\dots,N$; s = flexible terminal bodies; and $k=1,2,\dots,n_s$.

$$F^{(e)} = \sum_{j=1}^N F^{je} \quad T^{u(e)} = \sum_{j=u}^N \epsilon^{uj} T^{u(je)} \quad (40)$$

VIII. Complete Lagrangian Approach

We now propose an alternative approach, called the complete Lagrangian approach. The first step is to obtain the total kinetic energy of the system.

$$\mathcal{K} = \sum_{j=1}^N \mathcal{K}^j \quad (41)$$

Substituting the transformation equations, (27) and (28) into Eqs. (5) and (41), we have

$$\mathcal{K} = \frac{1}{2} m [\dot{R}^r + \dot{R}^l \cdot \omega^r + \dot{\theta}^u \cdot G^{u(T)} + \dot{q}_k^s]$$

$$\begin{bmatrix} \delta \cdot & -D^l \times & -D^u \times & \Psi_l^s \\ D^l \times & J^{ll} \cdot & J^{lu} \cdot & H_l^u \\ D^u \times & J^{ul} \cdot & J^{uu} \cdot & H_l^u \\ \Psi_k^s \cdot & H_k^{ls} \cdot & H_k^{us} \cdot & S_{kl}^s \end{bmatrix} \begin{bmatrix} \dot{R}^r + \dot{R}^l \\ \omega^r \\ G^u \cdot \dot{\theta}^u \\ \dot{q}_l^s \end{bmatrix}$$

(42)

$$\begin{aligned} m \begin{bmatrix} G^{u(T)} \cdot [J^{uv} + D^u \times (D^v \times)] \cdot G^v & G^{u(T)} \cdot [H_l^u - D^u \times \Psi_l^s] \\ [H_k^{ls} + \Psi_k^s \times D^v] \cdot G^v & S_{kl}^s - \Psi_k^s \cdot \Psi_l^s \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}^v \\ \dot{q}_l^s \end{bmatrix} &= -m \begin{bmatrix} G^{u(T)} \cdot [J^{uv} + D^u \times (D^v \times)] \\ H_k^{ls} + \Psi_k^s \times D^v \end{bmatrix} \cdot [\dot{G}^v + \omega^v \times G^v] \cdot \dot{\theta}^v \\ -m \begin{bmatrix} G^{u(T)} \cdot [J^{ul} + D^u \times (D^l \times)] \\ H_k^{ls} + \Psi_k^s \times D^l \end{bmatrix} \cdot \dot{\omega}^r - m \begin{bmatrix} G^{u(T)} \cdot [P^u - D^u \times L] \\ X_k^s - \Psi_k^s \cdot L \end{bmatrix} &+ \begin{bmatrix} T^{ug} \\ 0 \end{bmatrix} + \begin{bmatrix} G^{u(T)} \cdot [T^{u(e)} - D^u \times F^{(e)}] \\ Q_k^{s(e)} - \Psi_k^s \cdot F^{(e)} \end{bmatrix} \end{aligned} \quad (47)$$

$u,v=1,2,\dots,N$; s,t = flexible terminal bodies; $k=1,2,\dots,n_s$; $l=1,2,\dots,n_l$, where

$$\dot{R}^l = \dot{R}^l + \omega^r \times R^l \quad (43)$$

The total strain energy of the system is

$$S = \frac{1}{2} m \sum_j^{(F)} \sum_{k=1}^{n_j} \sum_{l=1}^{n_l} q_k^l q_l^l K_{kl}^s + \frac{1}{2} \sum_{u=2}^N \sum_{\sigma=1}^3 \theta_\sigma^u K_\sigma^{ug} \theta_\sigma^u \quad (44)$$

$$\mathcal{D} = \frac{1}{2} m \sum_j^{(F)} \sum_{k=1}^{n_j} \sum_{l=1}^{n_l} \dot{q}_k^l \dot{q}_l^l V_{kl}^s + \frac{1}{2} \sum_{u=2}^N \sum_{\sigma=1}^3 \dot{\theta}_\sigma^u V_\sigma^{ug} \dot{\theta}_\sigma^u \quad (45)$$

The Lagrange's equations of the complete system are

$$\frac{d}{dt} \left(\frac{\partial \mathcal{K}}{\partial \dot{R}^l} \right) - \frac{\partial \mathcal{K}}{\partial R^l} + \frac{\partial S}{\partial R^l} + \frac{\partial \mathcal{D}}{\partial R^l} = Q^R_l \quad (46a)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{K}}{\partial \dot{\theta}_\sigma^u} \right) - \frac{\partial \mathcal{K}}{\partial \theta_\sigma^u} + \frac{\partial S}{\partial \theta_\sigma^u} + \frac{\partial \mathcal{D}}{\partial \theta_\sigma^u} = Q_\sigma^{\theta_u} \quad (46b)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{K}}{\partial \dot{q}_k^s} \right) - \frac{\partial \mathcal{K}}{\partial q_k^s} + \frac{\partial S}{\partial q_k^s} + \frac{\partial \mathcal{D}}{\partial q_k^s} = Q_k^q \quad (46c)$$

$\gamma=1,2,3$; $u=1,2,\dots,N$; $\sigma=1,2,3$; s = flexible terminal bodies; $k=1,2,\dots,n_s$. Since the interacting forces and torques at the hinge points always come in equal and opposite pairs, they will not contribute to any virtual work associated with the virtual displacements. Consequently, they will not appear in the generalized forces of Lagrange's equations. Substituting Eqs. (42), (44), and (45) into Eqs. (46), we shall obtain exactly the same dynamic equations of the multibody system as Eqs. (37).

IX. Limited Degrees of Freedom

When a gimbal rotational axis at a certain hinge point is locked, we simply strike out the torque component equation and the associated angle of rotation about that axis. Projection from the body torque equations to the gimbal torque equations makes the modification for the limited degree of freedom rather simple. [see Eq. (37b)].

X. Discussion and Conclusion

The vector D^u is a position vector from the hinge point O_u to the instantaneous mass center O_u^* of the subsystem of bodies of body u , including itself. The torque equations in Eq. (37b) are taken about the hinge point O_u ($u=1,2,\dots,N$). We may perform the following shifting process to eliminate the force equation (37a)

$$\left. \begin{array}{l} \text{Eq.(37b)} - G^{u(T)} \times D^u \cdot \text{Eq.(37a)} \\ \text{Eq.(37c)} - \Psi_k^s \cdot \text{Eq.(37a)} \end{array} \right\}$$

The overall dynamic equations of the system will be

$u=1,2,\dots,N$; s,t = flexible terminal bodies; $k=1,2,\dots,n_s$; $v=1,2,\dots,N$; t = flexible terminal bodies; $l=1,2,\dots,n_l$.

When the direct path method was first developed by the author, Hooker asked two very important questions. The first

Table 2 Incidence matrix of an 8-body system^a

v	j	1	2	3	4	5	6	7	8
1		1	1	1	1	1	1	1	1
3			1						
3				1					
4					1				
5						1			
6							1		
7								1	
8									1

^a The empty slots are zeros

Table 3 Contribution from the body motion to the coefficient matrices in the overall dynamic equations of an 8-body system

1,2,3,4 5,6,7,8	1,2,3,4 5,6,7,8	2	3,5,6	4,7,8	5	6	7,8	8	2	5	6	8
1,2,3,4 5,6,7,8	1,2,3,4 5,6,7,8	2	3,5,6	4,7,8	5	6	7,8	8	2	5	6	8
2	2	2	0	0	0	0	0	0	2	0	0	0
3,5,6	3,5,6	0	3,5,6	0	5	6	0	0	0	5	6	0
4,7,8	4,7,8	0	0	4,7,8	0	0	7,8	8	0	0	0	0
5	5	0	5	0	5	0	0	0	0	5	0	0
6	6	0	6	0	0	6	0	0	0	0	6	0
7,8	7,8	0	0	7,8	0	0	7,8	8	0	0	0	8
8	8	0	0	8	0	0	8	8	0	0	0	8
2	2	2	0	0	0	0	0	0	2	0	0	0
5	5	0	5	0	5	0	0	0	0	5	0	0
6	6	0	6	0	0	6	0	0	0	0	6	0
8	8	0	0	8	0	0	8	8	0	0	0	8

one was, why couldn't we eliminate the overall force equation completely? The second one was due to the fact that the motion of each body should definitely influence the torque equations of the other bodies throughout the topological tree system, "Why don't these interrelations appear in Eqs. (37)?"

Eliminating the overall force equation does have the advantage because we then have three less force component equations to solve. The disadvantage is that the translational motion of the system is lost in this shifting process. It is really a very important part of the dynamic behavior of the multibody system. Unless the system is completely symmetric with respect to the main body, it would be better to maintain the force equation, (37a).

As far as the torque equations are concerned, it depends entirely on where the reference points are from which we take the moment. In the direct path approach, each torque equation represents physically the summation of moments about each hinge point. There are actually two ways of obtaining the torque equation about the same hinge point. The first one is to sum up the contributions from the motion of all branch bodies of this point. The other one is to sum up the contributions from the motion of all the rest of the bodies in the system. Naturally, we would like to choose the first one, which is simpler. The physical meaning of the torque equations in Eq. (47) is that we are taking moment about the instantaneous mass centers O_u^* of the subsystems of bodies, including body u . The motion of all the bodies in the topological tree system are completely "coupled." This complicates the computer calculations. Furthermore, this shifting process also complicates the coefficient matrices in the vibration equations, where they are completely uncoupled in Eq. (37c).

To illustrate the contribution from the motion of the bodies in the topological tree system to the coefficient matrices of the overall dynamic equations, we choose the 8-body system given in Table 2.

From Table 3 we find that only a few slots of the coefficient matrices require summation of the contribution from the motion of several bodies. Most of the slots require only single-body entry or zero entry. It also shows the uncoupling characteristics of the equations. When Eq. (47) is used, all the zero slots will be filled with nonzero numbers. We conclude that this shifting process certainly adds an unnecessary complication to the problem.

Comparing the three approaches presented in this analysis, the complete Newtonian approach is the most straight-

forward. It provides better physical understanding of the interaction among motions of the individual bodies. After obtaining the dynamic equations of a single body, there are two major steps of derivation we must go through. The first one is to transform the variables. The second one is to transform the effect of the single-body motion to the torque equations about other hinge points. If we reverse the order of steps or anxiously go into the final dynamic equations, the systematic definitions of the time-independent and time-dependent mass properties, [Eqs. (6) and (7)] and the transformed mass properties [Eqs. (32)] will not be shown clearly. From the computer programming point of view, calculating all properties of each body separately before calculating the overall properties of the system [Eqs. (38-40)] is very efficient.

In the complete Lagrangian approach, transforming the velocity is certainly easier than transforming the acceleration. We do not have to worry about the elimination of the interacting forces and torques. Carrying out the partial differentiations in Lagrange's equations is very straightforward from the mathematical point of view. However, knowing the definitions of the transformed mass properties [Eqs. (32)] beforehand makes the derivation easier.

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