Controllability

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Definition

If an input to a system can be found that takes every state variable from a desired initial state to a desired final state the system is said to be *controllable*.

- If can't control behavior of a state variable then can't place all closed-loop poles
- Pole-placement is a valid technique only for controllable systems
- Input can influence a state variable either directly or through coupling of variables.

Determining Controllability By Inspection

To consider controllability, examine the following systems:

Example 1

$$\dot{\bar{x}} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

or

$$\dot{x}_1 = -a_1 x_1 + u$$

$$\dot{x}_2 = -a_2 x_2 + u$$

$$\dot{x}_3 = -a_3 x_3 + u$$

Controllable because input drives all state variables

Example 2

$$\dot{\bar{x}} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

or

$$\dot{x}_1 = -a_1 x_1$$

$$\dot{x}_2 = -a_2 x_2 + u$$

$$\dot{x}_3 = -a_3 x_3 + u$$

Uncontrollable because x_1 not influenced by the input

Determining Controllability in General

In general, state matrix is not diagonal

- Relationships between x_i and u not so clear

Use *controllability matrix* for testing general systems:

The n^{th}-order with state equation

$$\dot{\bar{x}} = A\bar{x} + Bu$$

is controllable if the matrix

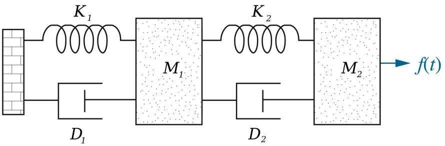
$$C_M = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

is of rank n.

Controllability Example

Determine whether or not the system shown below is controllable.

Use $K_1 = 2$, $D_1 = 3$, $M_1 = 1$, $K_2 = 9$, $D_2 = 4$, and $M_2 = 1$.



Start by writing equations of motion for the two masses

$$-(K_1 + K_2)x_1 - (D_1 + D_2)\dot{x}_1 + K_2 x_2 + D_2 \dot{x}_2 = M_1 \ddot{x}_1$$

$$\ddot{x}_1 = -\frac{K_1 + K_2}{M_1}x_1 - \frac{D_1 + D_2}{M_1}\dot{x}_1 + \frac{K_2}{M_1}x_2 + \frac{D_2}{M_1}\dot{x}_2$$

$$= -11x_1 - 7\dot{x}_1 + 9x_2 + 4\dot{x}_2$$

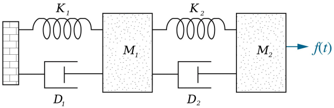
$$K_2 x_1 + D_2 \dot{x}_1 - K_2 x_2 - D_2 \dot{x}_2 + f = M_2 \ddot{x}_2$$

$$\ddot{x}_2 = 9x_1 - 4\dot{x}_1 - 9x_2 - 4\dot{x}_2 + f$$

Choose position and velocity of both masses for state variables

$$\dot{\underline{x}} = \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}$$

Controllability Example (Cont'd)



$$\ddot{x}_1 = -11x_1 - 7\dot{x}_1 + 9x_2 + 4\dot{x}_2$$
$$\ddot{x}_2 = 9x_1 - 4\dot{x}_1 - 9x_2 - 4\dot{x}_2 + f$$

State equation:

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -11 & -7 & 9 & 4 \\ 0 & 0 & 0 & 1 \\ 9 & 4 & -9 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} f$$

Determine controllability:

$$C_M = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 & -35 \\ 0 & 4 & -35 & 257 \\ 0 & 1 & -4 & 23 \\ 1 & -4 & 23 & -160 \end{bmatrix}$$

The rank of C_M is 4 $\therefore$ the system is controllable

There exists some $f(t)$ that can get $\bar{x} = \begin{bmatrix} x_1 & v_1 & x_2 & v_2 \end{bmatrix}^T$ to any desired final value.