# Converting Transfer Functions to State Space

KENNESAW STATE UNIVERSITY

# Contents

- Phase variables
- Converting transfer function with no zeros
- Converting transfer function with zeros
    - Lower order than denominator

# Phase Variables

The $n^{\text{th}}$-order transfer function has a single input and output.

- Choose state variables, $x_i$, to be the output and its $(n-1)$ derivatives — known as *phase variables*
- Example: phase variables for a system with output, $y$

$$
\begin{aligned}
x_1 &= y \\
x_2 &= \frac{dy}{dt} \\
x_3 &= \frac{d^2y}{dt^2} \\
&\vdots \\
x_n &= \frac{d^{n-1}y}{dt^{n-1}}
\end{aligned}
$$

# Phase Variables in State Equation

If the relationship between the output, $y$, and input, $u$, is described by the following differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

then taking the first derivative of the phase variables yields

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + b_0 u
\end{aligned}$$

# State-Space Representation in Phase Variable Form

Putting this in matrix form yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

which is the phase-variable form of the state equation.

Based on our definition of the state variables, the output equation is

$$y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

# Example 1

Start with transfer function

$$G(s) = \frac{C(s)}{R(s)} = \frac{43}{s^2 + 9s + 24}$$

Cross multiply to get

$$(s^2 + 9s + 24)C(s) = 43R(s)$$

Take inverse Laplace transform (assuming zero initial conditions)

$$\ddot{c} + 9\dot{c} + 24c = 43r$$

Use phase variables as state variables

$$x_1 = c$$
$$x_2 = \dot{c}$$

Write state equations (first-order derivatives of state variables)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -24x_1 - 9x_2 + 43r$$

# Example 1 (Continued)

In summary, the transfer function

$$G(s) = \frac{C(s)}{R(s)} = \frac{43}{s^2 + 9s + 24}$$

has state-space representation in phase-variable form of

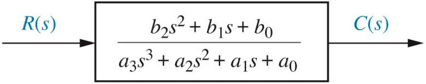
$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -9 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 43 \end{bmatrix} r$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}$$

where the state variables are

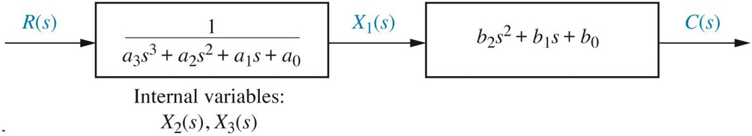
$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c \\ \dot{c} \end{bmatrix}$$

# Transfer Functions With Non-Zeroth-Order Numerator

Consider the transfer function shown in (a) as two blocks cascaded as shown in (b)




**(*a*)**




**(*b*)**

Choose $x_1$ and its derivatives (phase variables) as state variables

$$\bar{x} = \begin{bmatrix} x_1 & \dot{x}_1 & \ddot{x}_1 \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$$

The second block gives the output $(c)$ in terms of phase variables

$$y = b_0 x_1 + b_1 \dot{x}_1 + b_2 \ddot{x}_1 = b_0 x_1 + b_1 x_2 + b_2 x_3 = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \bar{x}$$

# Example 2

Start with transfer function

$$G(s) = \frac{C(s)}{R(s)} = \frac{43s + 2}{s^2 + 9s + 24}$$

Treat this as two cascaded transfer functions, $G_1(s)$ and $G_2(s)$

$$G_1(s) = \frac{1}{s^2 + 9s + 24}, \quad G_2(s) = 43s + 2$$

Cross multiply $\frac{X_1(s)}{R(s)} = \frac{1}{s^2+9s+24}$ and take inverse Laplace:

$$\ddot{x}_1 + 9\dot{x}_1 + 24x_1 = r$$

Write state equation using phase variables of $X_1(s)$

$$\dot{\bar{x}} = A\bar{x} + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ -24 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r}$$

Use $Y(s) = C(s) = X_1(s)G_2(s)$ to form output equation

$$y = 2x_1 + 43\dot{x}_1 = C\bar{x} = 2x_1 + 43x_2 = \boxed{\begin{bmatrix} 2 & 43 \end{bmatrix} \bar{x}}$$

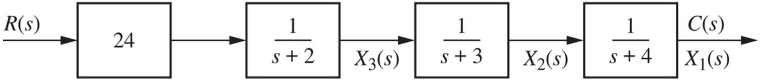
# Example 3 — Alternative Representations

The so-called *phase-variable form* is far from the only state-space representation possible.

- Cascade form, parallel form, controller canonical form, observer canonical form, . . .

For example, represent $G(s) = \frac{24}{(s+2)(s+3)(s+4)}$ in cascade form

Imagine $G(s)$ as the cascade form of four transfer functions




Taking the output of each block from denominator as a state variable yields

$$
\begin{aligned}
\dot{x}_1 &= -4x_1 + x_2 \\
\dot{x}_2 &= -3x_2 + x_3 \\
\dot{x}_3 &= -2x_3 + 24r \\
y &= c(t) = x_1
\end{aligned}
$$

$$
\dot{\bar{x}} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \bar{x}
$$