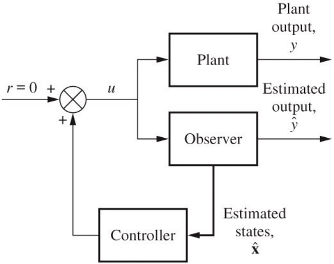
Observer Design

Contents

Observer Introduction

Measuring all state variables sometimes impractical

- Estimate missing states



Example implementation of observer (or *estimator*)

This implementation has significant disadvantage

Plant

$$\dot{\bar{x}} = A\bar{x} + Bu$$
$$y = C\bar{x}$$

Observer

$$\dot{\hat{x}} = A\hat{x} + Bu$$
$$\hat{y} = C\hat{x}$$

Taking the difference yields

$$\dot{\bar{x}} - \dot{\hat{x}} = A(\bar{x} - \hat{x})$$
$$y - \hat{y} = C(\bar{x} - \hat{x})$$

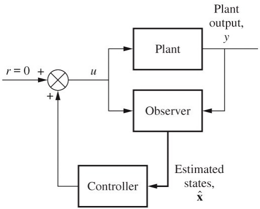
These are unforced, differences will approach zero

Observer and plant have same characteristic equation

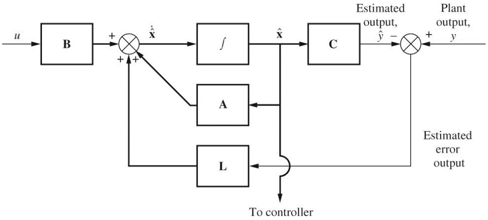
- Convergence is too slow

Improved Observer Dynamics

Use feedback to increase estimator convergence speed



Concept



Detail

Dynamics of estimate error vector:

Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

Plant

$$\dot{\bar{x}} = A\bar{x} + Bu$$
$$y = C\bar{x}$$

$\bar{e}_x = \bar{x} - \hat{x}$

$$\begin{aligned}\dot{\bar{x}} - \dot{\hat{x}} &= A(\bar{x} - \hat{x}) - L(y - \hat{y}) \\ &= (A - LC)(\bar{x} - \hat{x}) \\ \dot{\bar{e}}_x &= (A - LC)\bar{e}_x\end{aligned}$$

This equation is unforced

$\therefore$ e_x will decay to 0

Observer Design

Want error in state estimate to decay quickly relative to plant dynamics

Place roots of observer C.E. for faster response than closed-loop system

$$\det(sI - (A - LC)) = 0$$

Simplest to use *observer canonical form*

- Left companion state matrix

Converting TF to Observer Canonical Form, Example

Represent example system in observer canonical form

$$\frac{C(s)}{R(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24} = \frac{\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}}{1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}}$$

$$\left(\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}\right)R(s) = \left(1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}\right)C(s)$$

$$C(s) = \frac{1}{s}\left(R(s) - 9C(s) + \frac{1}{s}\left(7R(s) - 26C(s) + \frac{1}{s}(2R(s) - 24C(s))\right)\right)$$

Using outputs of integrators as state variables ($x_1 = c(t)$) yields:

$$\begin{aligned} \dot{x}_1 &= -9x_1 + x_2 + r \\ \dot{x}_2 &= -26x_1 + x_3 + 7r \\ \dot{x}_3 &= -24x_1 + 2r \end{aligned}$$

Thus we have the following state-space representation.

$$\dot{\bar{x}} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix}\bar{x} + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\bar{x}$$

Feedback Gain Matrix, L

Place roots of observer C.E. for faster response than closed-loop system

$$\det(sI - (A - LC)) = 0$$

L is a column vector of length n

$$A - LC = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ -a_1 & 0 & 0 & & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-1} \\ l_n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\det(sI - (A - LC)) = \left| \begin{bmatrix} s & 0 & \cdots & 0 & 0 \\ 0 & s & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & s & 0 \\ 0 & 0 & \cdots & 0 & s \end{bmatrix} - \begin{bmatrix} -(a_{n-1} + l_1) & 1 & 0 & \cdots & 0 \\ -(a_{n-2} + l_2) & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ -(a_1 + l_{n-1}) & 0 & 0 & & 1 \\ -(a_0 + l_n) & 0 & 0 & \cdots & 0 \end{bmatrix} \right|$$

$$= s^n + (a_{n-1} + l_1)s^{n-1} + (a_{n-2} + l_2)s^{n-2} + \cdots + (a_1 + l_{n-1})s + a_0 + l_n = 0$$

Form desired characteristic equation and solve for l_i by comparing terms

Example Observer Design

A controller was designed for the following plant so that the closed-loop poles are at -4 and $-1 \pm j2$

Design an observer to estimate the state variables, assuming they are in observer canonical form

Start by converting to state space

$$G(s) = \frac{s+4}{(s+1)(s+2)(s+5)} = \frac{s+4}{s^3+8s^2+17s+10}$$

$$\dot{\hat{x}} = A\hat{x} + Bu = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} u$$

$$\hat{y} = C\hat{x} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \hat{x}$$

Using an observer gain matrix, $L = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^{\mathsf{T}}$, write the C.E.

$$\det(sI - (A - LC)) = s^3 + (8 + l_1)s^2 + (17 + l_2)s + 10 + l_3 = 0$$

Example Observer Design (Cont'd)

Using the characteristic equation for the observer error

$$\det(sI - (A - LC)) = s^3 + (8 + l_1)s^2 + (17 + l_2)s + 10 + l_3 = 0$$

Place poles at $-5 \pm j2$ and -10, then calculate l_i

$$(s + 10)(s^2 + 10s + 29) = s^3 + 20s^2 + 129s + 290 = 0$$
$$\therefore L = \begin{bmatrix} 12 & 112 & 280 \end{bmatrix}^\mathsf{T}$$