Stability in State Space

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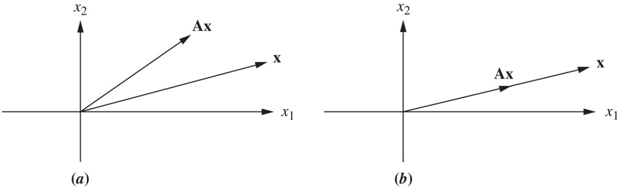
Background: Eigenvectors and Eigenvalues

Before addressing the question of stability, define

Eigenvector — A vector $\bar{x}_i \neq \bar{0}$ is an eigenvector of A if

$$A\bar{x}_i = \lambda_i \bar{x}_i \tag{1}$$

That is, an eigenvector under the transformation A becomes a scalar multiple of itself



Vector in (a) is not an eigenvector, whereas (b) shows an eigenvector

Eigenvalue — A scalar $\lambda_i \neq 0$ is an eigenvalue of A if it satisfies (1)

Background — Eigenvectors and Eigenvalues (Cont'd)

Use $A\bar{x} = \lambda\bar{x}$ to find the eigenvalues of A

$$
\begin{aligned}
\lambda\bar{x} - A\bar{x} &= \bar{0} \\
(\lambda I - A)\bar{x} &= \bar{0} \\
\bar{x} &= (\lambda I - A)^{-1}\bar{0} \\
\bar{x} &= \frac{\text{adj}(\lambda I - A)}{\det(\lambda I - A)}\bar{0}
\end{aligned}
$$

The only way $\bar{x} \neq \bar{0}$ is for $\det(\lambda I - A) = 0$

Solutions of $\det(\lambda I - A) = 0$ are the eigenvalues

Stability in State Space Criterion

Converting state-space representation to transfer function yields

$$\det(sI - A)$$

in the denominator.

- Look elsewhere for derivation

The eigenvalues of the system matrix, A, and the poles of the transfer function are the same

$\therefore$ A system is stable if its system matrix has eigenvalues with negative real components

Example 1

Determine the stability of a system with the state-space representation:

$$\dot{\bar{x}} = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 4 & 4 \\ 3 & 2 & 12 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \bar{x}$$

Calculate eigenvalues of A as solutions to $\det(\lambda I - A) = 0$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 & -4 \\ 0 & \lambda - 4 & -4 \\ -3 & -2 & \lambda - 12 \end{vmatrix}$$

$$= \lambda^3 - 19\lambda^2 + 76\lambda - 72 = 0$$

$$\lambda_{1,2,3} = 1.4, 3.7, 14$$

$\therefore$ unstable

Example 2

Determine the stability of a system with the state-space representation:

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}$$

Calculate eigenvalues of A as solutions to $\det(\lambda I - A) = 0$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 10 & \lambda + 5 \end{vmatrix} = \lambda^2 + 5\lambda + 10 = 0$$
$$\lambda_{1,2} = -2.5 \pm j1.9$$

$\therefore$ stable

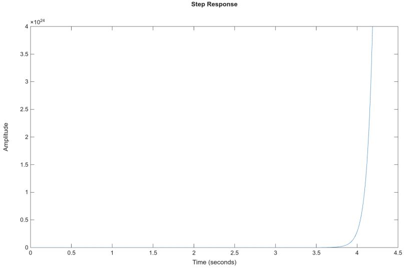
Examples Revisited Using MATLAB

MATLAB supports state-space models

`eig()` calculates eigenvectors and eigenvalues

$$\dot{\bar{x}} = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 4 & 4 \\ 3 & 2 & 12 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \bar{x}$$

```
>> A = [3 0 4; 0 4 4; 3 2 12];
>> eig(A)
ans = 1.4; 3.7; 14
>> B = [0; 0; 9];
>> C = [1 0 0];
>> sys = ss(A, B, C, []);
>> step(sys)
```



$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}$$

```
>> A = [0 1; -10 -5];
>> eig(A)
ans = -2.5 + 1.9i; -2.5 - 1.9i
>> B = [0; 4];
>> C = [1 0];
>> sys = ss(A, B, C, []);
>> step(sys)
```