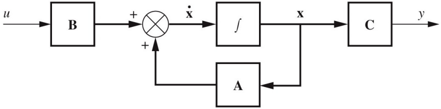
Controller Design

Contents

State-Variable Feedback

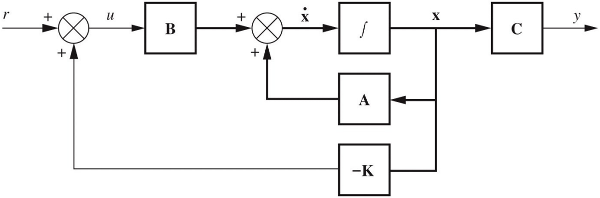
Given a plant with the following state-space representation

$$\dot{\bar{x}} = A\bar{x} + Bu$$
$$y = C\bar{x}$$

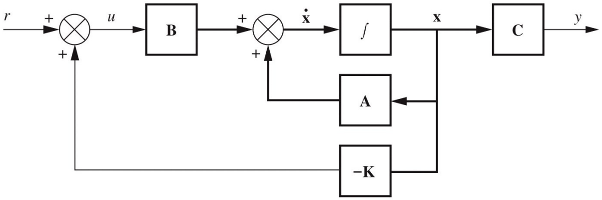


Feedback based on *state variables*:

- Product of gain matrix, K, and state vector



Gain Matrix



The resulting state-space representation is

$$\dot{\bar{x}} = A\bar{x} + Bu = A\bar{x} + B(-K\bar{x} + r) = \boxed{(A - BK)\bar{x} + Br}$$

$$y = \boxed{C\bar{x}}$$

System matrix $(A - BK)$ can have different eigenvalues than A. 👍

Since we're dealing with single-input systems, K, is a row vector:

$$K = \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix}$$

- Provides n gains to adjust *all* pole locations

Characteristic Equation

A system's *characteristic equation* determines its natural response.

- Roots of transfer-function denominator
- Can also be expressed using the system matrix

$$\det(sI - A) = 0$$

(For continuity with classical approach, use s instead of λ)

Review: check the stability of a system described as follows

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \bar{x}$$

$$\det(sI - A) = \begin{vmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{vmatrix}$$

$$= s^3 - 6s^2 - 7s - 52 = 0$$

The roots/poles/eigenvalues are $-0.882 \pm j2.43$ and 7.76

$\therefore$ unstable

System Matrix Example for Phase-Variable Feedback

With feedback, our system matrix is $A - BK$.

- For example, third-order system using phase-variable form

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_3) \end{bmatrix}$$

Has characteristic equation, $\det(sI - (A - BK)) = 0$,

$$\begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a_0 + k_1 & a_1 + k_2 & s + a_2 + k_3 \end{vmatrix} = s^3 + (a_2 + k_3)s^2 + (a_1 + k_2)s + (a_0 + k_1) = 0$$

Can see that $K \neq 0$ changes the characteristic equation and thus pole locations

Pole Placement for Plants in Phase-Variable Form

Choosing feedback gains:

1. Represent plant in phase-variable form
   - Other forms work too, this one just makes calculations simpler
2. Feed back product of gain matrix, K, and phase variables
3. Find the characteristic equation with unknown gains

$$s^n + (a_{n-1} + k_n)s^{n-1} + \ldots + (a_1 + k_2)s + (a_0 + k_1) = 0$$

4. Identify desired closed-loop characteristic equation

$$s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0 = 0$$

5. Equate like coefficients from 3 and 4 and solve for gains

$$d_i = a_i + k_{i+1}$$
$$k_{i+1} = d_i - a_i$$

Second-Order Systems

High-order systems ($n > 2$) have complicated transient response

- Desired poles often based on second-order approximation
  - Two *dominant* poles from desired transient response
  - Remaining poles at least $5\times$ farther from imaginary axis
    - (Or cancelling any zeros)

Before controller-design example, review second-order behavior

$$s^2 + as + b = 0$$

$$\omega_n = \sqrt{b}$$

$$\zeta = \frac{a/2}{\omega_n}$$

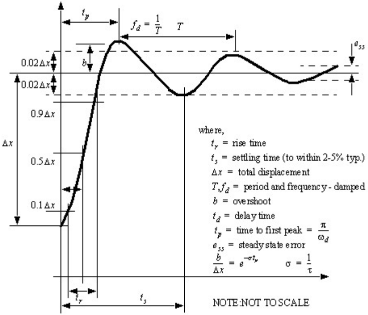
$$\therefore s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

$$\therefore \zeta = -\frac{\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$$T_s = 4/(\zeta\omega_n)$$



Gains for 8% OS, $T_s = 0.84$ s, and plant $G(s) = \frac{25(s+3)}{s(s+2)(s+4)}$

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & -6 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 75 & 25 & 0 \end{bmatrix} \bar{x}$$

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(8+k_2) & -(6+k_3) \end{bmatrix}$$

$$\det(sI - (A - BK)) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(8+k_2) & -(6+k_3) \end{vmatrix}$$

$$= s^3 + (6+k_3)s^2 + (8+k_2)s + k_1 = 0$$

zero at -3, $\zeta = 0.627$, $\omega_n = 7.6 \rightarrow (s+3)(s^2 + 9.53s + 57.8) = 0$

$$s^3 + (6+k_3)s^2 + (8+k_2)s + k_1 = s^3 + 12.5s^2 + 86.3s + 173$$

$$\therefore K = \begin{bmatrix} 173 & 86.3 - 8 & 12.5 - 6 \end{bmatrix} = \boxed{\begin{bmatrix} 173 & 78.3 & 6.50 \end{bmatrix}}$$