Introduction to State-Space Modeling

KENNESAW STATE UNIVERSITY

State-Space, A.K.A. Modern, A.K.A. Time-Domain Approach

Frequency-domain, A.K.A. classical approach

- Pros: Simplified because differential equations replaced w/algebraic; rapid I.D. of stability, transient response, and effects of changing parameters
- Cons: Limited to LTI systems

State-space approach

- Pros: Can model nonlinear, non-zero initial conditions, time-varying, multiple-input and multiple-output (MIMO)
- Cons: Not as transparent/intuitive, designs more sensitive to parameter variations

Objectives for state-space in this course

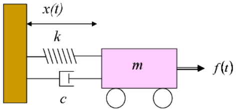
- Model dynamic systems
- Analyze stability
- Design controllers

State-Space Modeling

1. Select particular subset of variables $\longrightarrow$ *state variables*
2. For n^{th}-order system: write n first-order diff. eqs
   - n state variables
   - *State equations*
3. Given initial conditions and input, solve for state variables
   - Modeling does not necessarily require solving diff. eqs
4. *Algebraically* combine state variables with input to find other system variables
   - *Output equations*
5. Together the state and output equations constitute the *state-space representation*

Example 1 — Spring-Damper System

Write state-space representation of system shown given that the mass is zero.



Start with equation of motion from Newton's 2nd law

$$f - kx - c\dot{x} = 0$$

First-order system ∴ need

- One 1st-order differential equation and
- One state variable (we'll choose position, x)

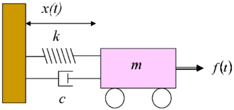
$$\dot{x} = -\frac{k}{c}x + \frac{1}{c}f$$

This is the state equation. Now write the output equation.

Example 1 — Spring-Damper System (Cont'd)

State equation: $\dot{x} = -\frac{k}{c}x + \frac{1}{c}f$

State-space representation also requires output equation



In this example, we have choice of output parameter.

- Must be written in terms of state variable and input

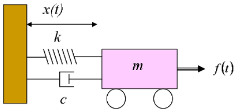
For example,

- Spring force: $f_s = kx$
- Damper force: $f_c = -kx + f$
- Velocity: $\dot{x} = -\frac{k}{c}x + \frac{1}{c}f$

We now have a state-space representation.

Example 2 — Spring-Mass-Damper System

Repeat previous example but use non-zero mass and both spring and damper forces as output



Start with equation of motion from Newton's 2nd law

$$f - kx - c\dot{x} = m\ddot{x}$$

Second-order system $\therefore$ need

- Two 1st-order differential equations and
- Two state variables (we'll choose position and velocity)

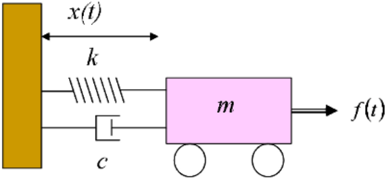
$$\dot{x} = v$$
$$\dot{v} = -\frac{k}{m}x - \frac{c}{m}v + \frac{1}{m}f$$

Next write output equations

Example 2 — Spring-Mass-Damper System (Cont'd)

State equations:

$$\dot{x} = v$$
$$\dot{v} = -\frac{k}{m}x - \frac{c}{m}v + \frac{1}{m}f$$



Output equations:

$$f_s = kx$$
$$f_c = cv$$

Place state equations and output equations in matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f$$
$$\begin{bmatrix} f_s \\ f_c \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

Choosing the State Vector

Define a state vector, $\bar{x}$

- One approach is to start with the differential equations governing the system
  - Define a state variable for each of the variables that have their time-derivative in the diff. eqs

The goal is to transform (at least one) n^{th}-order differential equation into n first-order diff. eqs

State variables must be linearly independent

- For example, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is not a valid state vector if $x_3 = 5x_1 + 4x_2$.

The General State-Space Representation

A system is represented in state space by

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$
$$\bar{y} = C\bar{x} + D\bar{u}$$

where

$\bar{x}$ = state vector

$\bar{y}$ = output vector

$\bar{u}$ = input vector

A = system matrix

B = input matrix

C = output matrix

D = feedforward matrix

Subsequent examples: SISO and $D = 0$

Example 3

Create state-space representation in general form for system shown below. Use θ_3 as output.

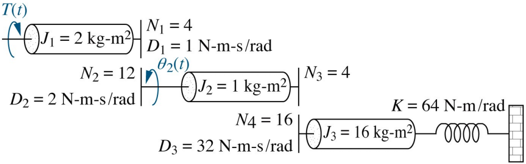
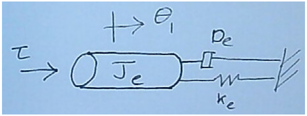


Figure P2.19

First find the equivalent system seen at input.



An equivalent impedance at input shaft is given (via *reflection*) by

$$Z_e = Z_1 + \left(\frac{N_1}{N_2}\right)^2 (Z_2) + \left(\frac{N_1}{N_2}\right)^2 \left(\frac{N_3}{N_4}\right)^2 (Z_3)$$

$$\therefore\ J_e = 2 + \left(\frac{4}{12}\right)^2 (1) + \left(\frac{4}{12}\right)^2 \left(\frac{4}{16}\right)^2 (16) = 2.22$$

$$D_e = 1 + \left(\frac{4}{12}\right)^2 (2) + \left(\frac{4}{12}\right)^2 \left(\frac{4}{16}\right)^2 (32) = 1.44$$

$$k_e = 0 + \left(\frac{4}{12}\right)^2 (0) + \left(\frac{4}{12}\right)^2 \left(\frac{4}{16}\right)^2 (64) = 0.444$$

Example 3 (Cont'd)

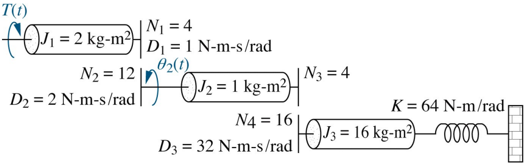
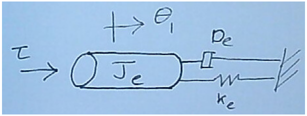


Figure P2.19



Governing equation from Newton's 2nd law

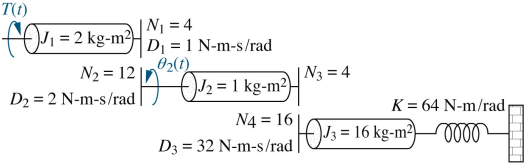
$$\tau - D_e\dot{\theta}_1 - k_e\theta_1 = J_e\ddot{\theta}_1$$

Need two state variables and two first-order differential equations

$$\dot{\theta}_1 = \omega_1$$
$$\dot{\omega}_1 = -\frac{k_e}{J_e}\theta_1 - \frac{D_e}{J_e}\omega_1 + \frac{1}{J_e}\tau$$

The output is given by the gear ratio as $\theta_3 = \frac{N_1}{N_2}\cdot\frac{N_3}{N_4}\cdot\theta_1$

Example 3 (Cont'd)



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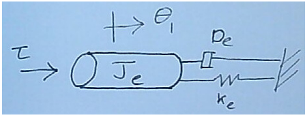

Put equations into the general form for SISO with no feedforward

$$\dot{\bar{x}} = A\bar{x} + Bu = \begin{bmatrix} 0 & 1 \\ -\frac{k_e}{J_e} & -\frac{D_e}{J_e} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \omega_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J_e} \end{bmatrix} \tau$$

$$= \begin{bmatrix} 0 & 1 \\ -0.200 & -0.650 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \omega_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.450 \end{bmatrix} \tau$$

$$y = C\bar{x} = \begin{bmatrix} \frac{N_1}{N_2} \cdot \frac{N_3}{N_4} & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \omega_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0833 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \omega_1 \end{bmatrix}$$