COL202: Discrete Mathematical Structures. I semester, 2017-18.
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Tutorial Sheet 3
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Important: The boxed question is to be submitted at the beginning of class on a plain sheet of paper with your name, entry number and the tutorial sheet number clearly written at the top of the sheet. **Note:** Try to use generating functions to solve all problems, even the ones which do not explicitly mention them. It could also be useful to attempt some of the counting or summation problems both with and without generating function.

Q1. In how many ways can $3r$ balls be chosen from $2r$ red balls, $2r$ blue ballls and $2r$ green balls?

Q2. Evaluate the following sums:

Q2.1.

$$\binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + i \cdot \binom{n}{i} + \cdots + n \cdot \binom{n}{n}$$

Q2.2. Given that $k \leq m$ and $k \leq n$

$$\binom{n}{0} \cdot \binom{m}{k} + \binom{n}{1} \cdot \binom{m}{k-1} + \binom{n}{2} \cdot \binom{m}{k-2} + \cdots + \binom{n}{k} \cdot \binom{m}{0},$$

Q2.3.

$$\binom{2n}{n} + \binom{2n-1}{n-1} + \cdots + \binom{2n-i}{n-i} + \cdots + \binom{n}{0}$$

Q3. Prove that

$$\binom{r}{1}^2 + \binom{r}{2}^2 + \cdots + \binom{r}{i}^2 + \cdots + \binom{r}{r}^2 = \binom{2r}{r}$$

And also that the generating funtion for $a_r = \binom{2r}{r}$ is

$$\hat{a}(z) = (1-4z)^{-1/2}$$

Q4. (Derangements revisited) Recall that a permutation π of the numbers 1 to n is called a *derangement* if $\pi(i) \neq i$ for all $1 \leq i \leq n$. Let the number of derangements of n numbers be denoted D_n.

Q4.1. Using simple counting arguments argue that

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

Q4.2. Using the identity established in Q4.1. and a generating function argument argue that

$$D_n = n! \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i!}.$$

Q5. **[Wilf94]** In each part below the sequence $\{a_n\}_{n\geq 0}$ satisfies the given recurrence. Find the power series generating function in each case and solve to find a_n where possible.

Q5.1.

$$a_{n+1} = 3a_n + 2, (n \geq 0, a_0 = 0).$$

Q5.2.

$$a_{n+1} = \alpha a_n + \beta, (n \geq 0, a_0 = 0).$$

Q5.1.

$$a_{n+2} = 2a_{n+1} - a_n, (n \geq 0, a_0 = 0, a_1 = 1).$$

Q5.1.

$$a_{n+1} = a_n/3 + 1, (n \geq 0, a_0 = 0).$$

Q6. **[Wilf94]** Let $f(n)$ be the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive integers. Find a recurrence for $f(n)$ and try to solve it to the extent possible using generating functions.