MTL106 Probability and Stochastic Processes≤

Sem-II 2016-17

TUTORIAL-4 Function of Random Variables

Solutions

1. X ~ U[0,10]
   Y = max{2, min{4,X}}

   Case I: 0 ≤ X ≤ 2
   Y = max{2, X} = 2

   Case II: 2 < X ≤ 4
   Y = max{2, X} = X

   Case III: X > 4 and X ≤ 10
   Y = max{2, 4} = 4

   $$Y = \begin{cases} 2 & 0 \le X < 2 \\ X & 2 \le X < 4 \\ 4 & 4 \le X \le 10 \end{cases}$$

   CDF of Y will be

   $$F_Y(y) = \begin{cases} 0 & y < 2 \\ (y-2)/_{10} + 1/_5 & 2 \le y < 4\le \\ 1 & y \ge 4 \end{cases}$$

2. f$(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}$, x > 0 where θ > 0 and α > 0
   Y = ln $(\frac{x}{\theta})$
   P(Y ≤ y) = P(ln $(\frac{x}{\theta}) \le$ y) = P$(\frac{x}{\theta} \le e^y)$ = P$(x \le \theta e^y)$
   $F_Y(y) = F_X(\theta e^y)$
   $f_Y(y) = f_X(\theta e^y)\,\theta e^y = \frac{\alpha\theta^\alpha}{\theta^{\alpha+1}(1+e^y)^{\alpha+1}}\theta e^y = \frac{\alpha e^y}{(1+e^y)^{\alpha+1}}, \quad y\,\varepsilon\,(-\infty,\infty)$

3. $f_X(x) = e^{-x}$, x > 0

   $$Y = \begin{cases} X, & X < 1 \\ \frac{1}{X}, & X \ge 1 \end{cases}, \quad 0 < Y \le 1$$

   P$(Y \le y)$ = P(X ≤ y) + P$(1/_X \le y)$ = P(X ≤ y) + P$(X \ge 1/_y)$

$F_Y(y) = F_X(y) + 1 - F_X(1/y)$

$f_Y(y) = f_X(y) - f_X(1/y)(-1/y^2) = e^{-y} + 1/y^2\, e^{-1/y}, \quad 0 < y < 1$

4. X be a random variable denoting marks obtained

   $X \sim N(76, 225)$

   Let minimum marks to obtain A grade be m_1

   $P(X \geq m_1) = 0.15$

   $P(X \leq m_1) = 0.85$

   $$\int_{-\infty}^{m_1} \frac{1}{\sqrt{2\pi * 225}} e^{-\frac{(x-76)^2}{2*225}} dx = 0.85$$

   Substituting $\frac{x-76}{15} = y$, we get Y as standard normal distributed random variable and

   $$\int_{-\infty}^{\frac{m_1-76}{15}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dx = 0.85$$

   From table of standard normal distribution value of upper limit of integral for which above integral evaluates to 0.85 is 1.04

   $\frac{m_1-76}{15} = 1.04$, $m_1 = 91.60$

   Let m_2 be minimum marks for passing the course

   Similarly we have, $P(X \leq m_2) = 0.10$

   $$\int_{-\infty}^{m_2} \frac{1}{\sqrt{2\pi * 225}} e^{-\frac{(x-76)^2}{2*225}} dx = 0.10$$

   Following similar approach as above, $\frac{m_2-76}{15} = -1.29$ which gives $m_2 = 56.65$

5. $Y = \begin{cases} X^{\frac{1}{2}}, & X > 0 \\ -|X|^{\frac{1}{2}}, & X < 0 \end{cases}$

   $X \sim N(0,1)$

   When $X < 0$

   $Y = -|X|^{\frac{1}{2}}$ and $Y < 0$

   $Y^2 = |X|,$

   $Y^2 = X\ (Not\ possible)$

   $or\ Y^2 = -X, \quad X = -Y^2, monotonically\ decreasing\ function$

   $f_Y(y) = \left|-\frac{\partial Y^2}{\partial Y}\right| \times \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\Big|_{x=y^2} = |2y| \times \frac{e^{\frac{-y^4}{2}}}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}|y|e^{\frac{-y^4}{2}}, y < 0$

   $x > 0$, $Y = X^{1/2}, X = Y^2$

   Following same approach as above

$f_Y(y) = \sqrt{\frac{2}{\pi}}|y|e^{\frac{-y^4}{2}}\ when\ \ y > 0$

So $f_Y(y) = \sqrt{\frac{2}{\pi}}|y|e^{\frac{-y^4}{2}}\ ,\ -\infty < y < \infty$

6. X ~ U(a,b),
   $Y = \frac{X - \mu}{\sigma}$
   $\mu = E[X] = {}^{1}\!/_{b - a}$
   $\sigma^2 = Var[X] = {}^{(b-a)^2}\!/_{12}$

   $X = Y\sigma + \mu$
   $\frac{\partial X}{\partial Y} = \sigma$

   $f_Y(y) = |\sigma| \times f_X(x = Y\sigma + \mu) = {}^{(b-a)}\!/_{2\sqrt{3}} \times {}^{1}\!/_{(b-a)} = {}^{1}\!/_{2\sqrt{3}}$

   to find range of Y
   when x = a, Y = $-\sqrt{3}$
   when x = b, Y = $\sqrt{3}$

   $$f_Y(y) = \begin{cases} {}^{1}\!/_{2\sqrt{3}}\ , & -\sqrt{3} < y < \sqrt{3} \\ 0\ , & otherwise \end{cases}$$

7. X ~ Bin(n, p)

   LHS – $E({}^{1}\!/_{X+1}) = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}p^k(1-p)^{n-k}$

   $= \sum_{k=0}^{n} \frac{1}{k+1}\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$

   $= \frac{1}{p(n+1)}\sum_{k=0}^{n} \frac{n!}{(k+1)!(n-k)!}p^k(1-p)^{n-k}p(n+1)$

   $= \frac{1}{p(n+1)}\sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!}p^{k+1}(1-p)^{n-k}$

   $= \frac{1}{p(n+1)}\left(\binom{n+1}{1}p(1-p)^n + \cdots + \binom{n+1}{n+1}p^{n+1}(1-p)^0 \right)$

$= \frac{1}{p(n+1)} \left(\binom{n+1}{1} p(1-p)^n + \cdots + \binom{n+1}{n+1} p^{n+1}(1-p)^0 + \binom{n+1}{0} p^0 (1-p)^{n+1} - \binom{n+1}{0} p^0 (1-p)^{n+1} \right)$

$= \frac{1}{p(n+1)} \left((p + 1 - p)^{n+1} - (1-p)^{n+1} \right)$

$= \frac{1 - (1-p)^{n+1}}{p(n+1)}$

8. $Y = F_X(x)$

   Since F_X is monotonically increasing function, so is F_X^{-1} and we can apply theorem directly

   $f_Y(y) = \left| \frac{\partial F_X^{-1}(y)}{\partial y} \right| \times f_X(F_X^{-1}(y))$

   $= \frac{\partial F_X^{-1}(y)}{\partial y} \times \frac{\partial F_X(x)}{\partial X} \Big|_{x = F_X^{-1}(y)}$

   $= \frac{\partial F_X^{-1}(F_X(x))}{\partial y} \times \frac{\partial Y}{\partial X} \quad Since\ Y = F_X(x)$

   $= \frac{\partial x}{\partial y} \times \frac{\partial y}{\partial x}$

   $= 1$

a. Since value of distribution function always lie between 0 and 1, $Y \sim U(0,1)$

b. $Var[Y] = {(b-a)^2}/{12} = {1}/{12}$

9. $X \sim exp(\lambda)$

   $P(\,|X-1| > 1 \mid X > 1) = P(\,(X-1 > 1 \text{ or } X - 1 < -1) \mid X > 1)$

   $= P(\,(X > 2 \text{ or } X < 0) \mid X > 1)$

   $= P(\,X > 2 \mid X > 1)$

   $= \frac{P(X>2)}{P(X>1)}$

   $= \frac{1 - F_X(2)}{1 - F_X(1)} = \frac{1 - (1 - e^{-2\lambda})}{1 - (1 - e^{-\lambda})} = e^{-\lambda}$