Department of Mathematics

Tutorial Sheet 6

MTL 106

Q.1 Let Time taken by Aditya (X) ~ exp(1/5)

Time taken by Ayush (Y) ~ exp(1/3)

(a)

$$P(X < Y) = \int_0^\infty \int_0^y f(x,y)\,dx\,dy = \frac{3}{8}$$

(b)

$$P(X < Y / X > 1) = \frac{P(1 < X < Y)}{P(X > 1)}$$

$$= \frac{\int_1^\infty \int_1^y f(x,y)\,dx\,dy}{\int_0^\infty \int_1^\infty f(x,y)\,dx\,dy}$$

(c)

$$P(|X - Y| \geq 1) = P(1 + Y \geq X \geq Y - 1)$$

$$= \int_0^1 \int_0^{x+1} f(x,y)\,dy\,dx + \int_1^\infty \int_{x-1}^{x+1} f(x,y)\,dy\,dx$$

Q.2 (a)

$$W = X_1 + X_2$$

$$Z = X_1 - X_2$$

$$J = \begin{vmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{vmatrix} = 0.5$$

$$f(w,z) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\left(\frac{w+z}{2}\right)^2} \quad \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{w-z}{2}\right)^2}$$

$$= \left(\frac{1}{\sqrt{4\pi}} e^{-w^2/2}\right)\left(\frac{1}{\sqrt{4\pi}} e^{-z^2/2}\right)$$

$$= f(w) \cdot f(z)$$

Hence Independent.

(b)

$$E\left(\frac{(W^2+Z^2}{2} / W=t\right)=E\left(\frac{Z^2+t^2}{2}\right)=\frac{t^2}{2}+1$$

(c) We know, $\frac{W}{\sqrt{2}} \sim N(0,1)$

$$\begin{aligned} E\left(W^4 / Z\right) &= E\left(W^4\right) \qquad \because W \text{ is independent to } Z \\ &= 4.E\left(\left[W/\sqrt{2}\right]^4\right) \\ &= 12 \end{aligned}$$

Q.3

$$\begin{aligned} \operatorname{var}\left(\frac{X+Y}{2}\right) &= \frac{1}{4}\operatorname{var}(X)+\frac{1}{4}\operatorname{var}(Y)+\frac{1}{2}\operatorname{cov}(X,Y) \\ &= 0.5\operatorname{var}(X)+0.5\operatorname{var}(X)\rho \\ &= 0.5\operatorname{var}(X)(1+\rho) \\ &\le \operatorname{var}(X) \qquad \because (1+\rho<2) \end{aligned}$$

Q.4 $Var(X-3Y) = Var(X) + 9\,Var(Y) - 6\,cov(X,Y) = 31$

Q.5(a) P(1/4<min<3/4) = P(all>1/4)-P(all>3/4) = $\left(\frac{3}{4}\right)^5 - \left(\frac{1}{4}\right)^5$

(b) $\frac{3!}{5!}$ $(Favourable\ Permutations/Total\ Permutations)$

Q.6

$$f(x,y)=\begin{cases} 2 & 0<x<1,\ 0<y<x \\ 0 & otherwise \end{cases}$$

$$E((X-Y)^2 / X)=\int_Y (x-y)^2 \frac{2}{f_X(2)}=\frac{x^2}{3}$$

Q.7

$$E(y/x)=\int \frac{y\,f(x,y)}{f(x)}dy$$

$$=\int_0^{\infty}\frac{\frac{y^2}{(1+x)^4}e^{\frac{-y}{1+x}}}{\frac{1}{(1+x)^2}}dy$$

$$=2(1+x)$$

Q.8 Given

$$f(x,y)=f(y/x)\cdot f(x)$$

$$f(y)=\int f(y/x)\cdot f(x)\,dx$$

$$E(X/y)=\int_{-\infty}^{\infty}x\frac{f(x,y)}{f(y)}dx$$

$$=\int_{-\infty}^{\infty}x\frac{f(x,y)}{f(y)}dx$$

$$=\frac{a+y}{n+a+b}$$

Q.9 $\mathrm{E(X1/X2>0)=np}\left(\frac{1-(1-q)^{n-1}}{1-(1-q)^{n}}\right);\ \ p=q=1/3$

$$\mathrm{E(X1)}=P(X2>0)\mathrm{E(X1/X2>0)}+P(X2=0)\mathrm{E(X1/X2=0)}$$

Hint : $\mathrm{E(X1/X2>0)}=\dfrac{\mathrm{E(X1)}-P(X2=0)\mathrm{E(X1/X2=0)}}{1-P(X2=0)}$

$$=\frac{(n/3)-(2/3)^{n}(n/2)}{1-(2/3)^{n}}=(n/3)\frac{1-(2/3)^{n-1}}{1-(2/3)^{n}}$$

Q.10 (a)

$$\begin{aligned}\operatorname{cov}(X,E(Y/X))&=E(X\,E(Y/X))-E(X)E(E(Y/X))\\&=E(E(XY/X))-E(X)E(Y)\\&=E(XY)-E(X)E(Y)\\&=\operatorname{cov}(X,Y)\end{aligned}$$

(b) Put E(Y/X)=a+bX in the equation $\operatorname{cov}(X, E(Y/X)) = \operatorname{cov}(X, Y)$ => b= $\dfrac{\operatorname{cov}(X,Y)}{\operatorname{var}(X)}$

Q.11

$$E(Y^k / X) = \int_0^x y^k \frac{1/x}{1} dy = \frac{x^k}{k+1},\ 0 \le x \le 1$$

$$E(Y^k) = E(E(Y^k / X)) = \frac{1}{(k+1)^2}$$

Q.12

$$\begin{aligned} P(X \ge a) &= \int_a^\infty f(x)\, dx \\ &= \frac{1}{b}\int_a^\infty b\, f(x)\, dx \\ &\le \frac{1}{b}\int_a^\infty g(x)\, f(x)\, dx \\ &\le \frac{1}{b} E(g(x)) \end{aligned}$$