Department of Mathematics

Tutorial Sheet 6

MTL 106

Q.1 (a) Since X and Y are independent

$$p(4,2) = p(x = 4) \times p(y = 2)$$

$$p(x = 4) = \frac{0.18}{0.6} = 0.3$$

$$Also, \qquad p(x = 1) + p(x = 2) + p(x = 4) = 1$$

$$So, \qquad p(x = 1) = 0.2$$

Similarly get all other values.

Q.1 (b)

$$P(|X - Y| \geq 2) = p(4,2) + p(4,2) + p(3,1) = 0.5$$

Q.2 (a)

$$P(X = j / Y = k) = \frac{P(X = j, Y = k)}{P(Y = k)}$$

$$P(Y = k / X = j) = \frac{P(X = j, Y = k)}{P(X = j)}$$

$$\begin{aligned}
P(X) &= \sum_{Y} P(X, Y) \\
&= \sum_{k} q^{k-j} p^{j} \\
&= \left(\frac{p}{q}\right)^{j} \sum_{k=j+1}^{\infty} q^{k} \\
&= \left(\frac{p}{q}\right)^{j} \left(\frac{q^{j+1}}{1-q}\right) \\
&= qp^{j-1}
\end{aligned}$$

$$Similarly,$$
$$P(Y) = q^k \left(\frac{t - t^k}{1 - t} \right)$$
$$where,\ t = \frac{p}{q}$$
$$P(X = j / Y = k) = \left(\frac{p}{q} \right)^j \frac{1 - t}{t - t^k}$$
$$P(Y = k / X = j) = pq^{k-j-1}$$

(b)

$$Similarly,$$
$$P(X = j) = \frac{15!}{j!(15 - j)!} \left(\frac{1}{2} \right)^j \left(\frac{1}{2} \right)^{15-j} = Bin(15,0.5)$$
$$P(Y = k) = \frac{15!}{k!(15 - k)!} \left(\frac{1}{3} \right)^k \left(\frac{2}{3} \right)^{15-k} = Bin(15, 1/3)$$
$$P(X = j / Y = k) = Bin\left(15 - k, 3/4\right)$$
$$P(Y = k / X = j) = Bin\left(15 - j, 2/3\right)$$

Q.3 Consider the rectangle ABCD with points as A(0.5,0.5) B(1.5,0.5) C(1.5,1.5) D(0.5,1.5). Now the probability P(R) that the point lies in this rectangle is F(C)+F(A)-F(B)-F(D) where F is the cdf. (Because F(C) is the probability that x<1.5 and y<1.5 and P(R)=P(0.5<x<1.5,0.5<y<0.15)) For the given cdf, it comes out to be
P(R)=1+0-1-1=-1<0.
Hence not a cdf.

Q.4 $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Integrating and solving, we get k=12.

Now $P(0 \le x \le 1, 0 \le y \le 2) = \int_0^2\int_0^1 f(x,y)\,dx\,dy = (1 - e^{-8})(1 - e^{-3})$

Q.5 Let X be the event that 1 occurred, and N be the total number of transsmissions
By total probability theorem,

$$P(X=k)=\sum_n P(X=k/N=n)\times P(N=n)$$

Here N is a Poisson process, and P(X/N) is binomial with probability of success (1-p) because we consider 1 to be success.

Putting in the values we get,

$$P(X=k)=\sum_n {}^nC_k\,(1-p)^k\,p^{n-k}\times\frac{e^{-\lambda}\lambda^n}{n!}$$

Evaluating we get,

$$P(X=k)=Poisson(\lambda(1-p))$$

Q.6 Let,

$$W=X+Y$$

$$Z=X$$

$$J=1$$

$$f(w,z)=\begin{cases}1 & 0<z,w-z<1\\ 0 & otherwise\end{cases}$$

$$f(w,z)=\begin{cases}1 & 0<z<1, z<w<1+z\\ 0 & otherwise\end{cases}$$

We need to find f(w)

$$f(w)=\begin{cases}\int_0^w f(w,z)dz & 0<w<1\\ \int_{w-1}^1 f(w,z)dz & 1<w<2\end{cases}$$

$$f(w)=\begin{cases}w & 0<w<1\\ 2-w & 1<w<2\end{cases}$$

(The limits of this integral could also be understood by plotting the constraints on w and z)

Q.7 Let,

$$W=X-Y$$

$$Z=X$$

$$J=1$$

$$f(w,z)=\begin{cases}\lambda^2e^{-\lambda(2z-w)} & z>0, z-w>0\\ 0 & otherwise\end{cases}$$

We need to find f(w)

$$f(w) = \begin{cases} \int_0^{\infty} f(w,z)dz & w < 0 \\ \int_w^{\infty} f(w,z)dz & w > 0 \end{cases}$$

$$f(w) = \begin{cases} \frac{\lambda}{2} e^{\lambda w} & w < 0 \\ \frac{\lambda}{2} e^{-\lambda w} & w > 0 \end{cases}$$

(The limits of this integral could be understood by plotting the constraints on w and z)

Q.8 Given X,Y,Z ~ U(0,1)

$$V = XY$$

$$f_v(v) = ?$$

$$let\ \ V = XY$$

$$T = X$$

$$So,\ \ Jacobian = \frac{1}{T}.$$

$$f(v,t) = \frac{1}{t}\ \ ,\ \ 0 < t < 1,\ \ 0 < v < w$$

$$f(v) = \int_v^1 \frac{1}{t} dt = -\log(v),\ 0 < v < 1$$

$$Also,\ \ W = Z^2,$$

$$f(w) = \frac{1}{2\sqrt{w}},\ 0 < w < 1$$

Now since X,Y,Z are independent, V and W are also independent.

So $f(v,w) = \frac{-\log(v)}{2\sqrt{w}},\ \ 0 < v, w < 1$

(b)

$$R^2 = X^2 + Y^2$$

$$\tan(\theta) = Y / X$$

$$X = R\cos(\theta)$$

$$Y = R\sin(\theta)$$

$$f_{R,\theta}(r,\theta) = \begin{cases} r, & 0 < \theta < \pi/4, 0 < r < \sec\theta \text{ or } \pi/4 < \theta < \pi/2, 0 < r < cosec\theta \\ 0, & \text{otherwise} \end{cases}$$

$$f_\theta(\theta) = \begin{cases} \frac{1}{2}\sec^2\theta, & 0 < \theta < \pi/4 \\ \\ \frac{1}{2}cosec^2\theta, & \pi/4 < \theta < \pi/2 \end{cases}$$

$$f_R(r) = \begin{cases} \frac{\pi}{2}r, & 0 < r < 1 \\ \\ r(cosec^{-1}(r) - \sec^{-1}(r)), & 1 < r < \sqrt{2} \end{cases}$$

Q.9 $P(|Y| > |X| + 1) = P([Y| - |X| > 1)$

Plot the graph to get the required probability. P=1/8

Q.10 First find k, by integrating over domain and equating to 1. k=1/8.

$$P(X < 1/Y = 0.5) = \frac{P(X < 1, Y = 0.5)}{P(Y = 0.5)}$$

The terms in numerator and denominator could be calculated using simple integration.

$$P(X < 1/Y = 0.5) = \frac{5/4}{81/4} = \frac{5}{81}$$

Similarly,

$$P(Y < 3/2, X = 1) = 1$$

Q.11 A,B,C ~ U(0,1)

We need to find probability that B^2-4AC >= 0.

$$X = B^2$$

$$Y = 4AC$$

We need distribution of W=X-Y, then find probability of P(W>=0)

$$f(x) = \frac{1}{2\sqrt{x}},\ 0 < x < 1$$

$$f(y) = -\log(y)\ ,\ 0 < y < 1$$

$$f(x,y) = \frac{-\log(y)}{2\sqrt{x}},\ 0 < x < 1, 0 < y < 1$$

$$W = X - Y$$

$$Z = X$$

$$J = 1$$

$$f(w,z) = \begin{cases} \dfrac{-\log(z-w)}{2\sqrt{z}} & 1 > z > 0, 1 > z - w > 0 \\ 0 & otherwise \end{cases}$$

Now we need to find f(w):

$$f(w) = \begin{cases} \int_0^w f(w,z)dz & w < 0 \\ \int_w^1 f(w,z)dz & w > 0 \end{cases}$$

$$P(W > 0) = \int_0^\infty f(w)dw$$

$$= \int_0^\infty \int_w^1 f(w,z)dz\, dw$$

$$= \frac{5}{16} + \frac{1}{12}\ln 4$$