1.

$$P(|X-(\lambda+1)| > (\lambda+1)) < \frac{E[X-(\lambda+1)]^2}{(\lambda+1)^2}$$

$$P[|X-(\lambda+1)| < (\lambda+1)] \geqslant 1 - \frac{E[X-(\lambda+1)]^2}{(\lambda+1)^2}$$

$$P[-(\lambda+1) < X-(\lambda+1) < (\lambda+1)] \geqslant 1 - \frac{E[X-(\lambda+1)]^2}{(\lambda+1)^2}$$

$$P\{0 < X < 2(\lambda+1)\} \geqslant 1 - \frac{E[X-(\lambda+1)]^2}{(\lambda+1)^2} \quad - (1)$$

$$\begin{aligned} E[X-(\lambda+1)]^2 &= E\{X^2+(\lambda+1)^2-2X(\lambda+1)\} \\ &= EX^2+(\lambda+1)^2-2(\lambda+1)E(X) \\ &= EX^2+(\lambda+1)^2-2(\lambda+1)\lambda \\ &= \lambda^2+\lambda+(\lambda+1)^2-2\lambda(\lambda+1) \\ &= \cancel{\lambda^2}+\lambda+\cancel{\lambda^2}+1+2\cancel{\lambda}-2\lambda^2-2\cancel{\lambda} \\ &= 1+\lambda \end{aligned}$$

$$P\{0<X<2(\lambda+1)\} \geqslant 1-\frac{\cancel{1+\lambda}}{(1+\lambda)^{\cancel{2}}} = 1-\frac{1}{1+\lambda} = \frac{\lambda}{\lambda+1}$$

Thus, $$P\{0<X<2(\lambda+1)\} \geqslant \frac{\lambda}{\lambda+1}$$

2.

$$P\{\mu-2\sigma \le X \le \mu+2\sigma\} = 0.6$$

$$P\{|X-\mu| \le 2\sigma\} = 0.6$$

$$P\{|X-\mu| \geqslant 2\sigma\} = 0.4 \quad - (1)$$

$$P\{|X-\mu| > 2\sigma\} < E\frac{(X-\mu)^2}{(2\sigma)^2} = \frac{E(X-\mu)^2}{4\sigma^2} = \frac{\sigma^2}{4\sigma^2} = 0.25 \quad - (2)$$

Using (1) and (2) gives a contradiction

3: $N \sim$ geometric $(1/4) \to p_1$

$k \sim$ geometric $(1/10^5) \to p_2$

\# bits in a page

Let N be the number of pages in the transmission

Thus the total number of bits K can be written as

$$K\big|_N = K_1 + K_2 + \cdots + K_N$$

where $K_i \sim$ geom (p_2)

$$MGF(K) = E[e^{tK}] = E\{E\{e^{tK} | N\}\}$$

$$= E\{E\{e^{t(K_1+K_2+\cdots+K_n)}\}\}$$

$$= E\{E\{e^{tK_1} \cdot e^{tK_2} \cdots e^{tK_n}\}\}$$

but the # bits in a page is independent of other

thus, by using independence, we have

$$= E\{[E(e^{tK_1})]^N\} = E\{(m(t))^N\}$$

where $m(t) = MGF(K_1) = \dfrac{p_2 e^t}{1-(1-p_2)e^t}$

$$= \frac{p_1 m(t)}{1-(1-p_1)m(t)} = \frac{p_1\left\{\frac{p_2 e^t}{1-(1-p_2)e^t}\right\}}{1-\frac{(1-p_1)p_2 e^t}{1-(1-p_2)e^t}}$$

$$= \frac{p_1 p_2 e^t}{1-(1-p_2)e^t - (1-p_1)p_2 e^t}$$

$$= \frac{p_1 p_2 e^t}{1-e^t + \cancel{p_2 e^t} - \cancel{p_2 e^t} + p_1 p_2 e^t}$$

$$= \frac{p_1 p_2 e^t}{1-(1-p_1 p_2)e^t}$$

thus K is geometric with parameter $p_1 p_2$

4. Let $N(t)$ denote the # bacteria at time t

Let $X = \begin{cases} 0 & , 0.25 \\ 1 & , 0.25 \\ 2 & , 0.5 \end{cases}$

be a random variable

then $N(t+1) \mid N(t) = X_1 + X_2 + \cdots + X_{N(t)}$

$$PGF(N(t+1))_{(s)} = E\left(s^{N(t+1)}\right) = E\left\{ E\left\{ s^{N(t+1)} \mid N(t) \right\} \right\}$$

$$= E\left\{ E\left(s^{X_1 + X_2 + \cdots + X_{N(t)}} \right) \right\}$$ but each X_i is independent of other since bacterias act independently

$$= E\left\{ \left[E\left(s^{X_1}\right) \right]^{N(t)} \right\}$$

$$= E\left\{ \left(m(s) \right)^{N(t)} \right\}$$ where $m(s) = PGF(X_1) = \frac{1}{4} + \frac{1}{4}s + \frac{s^2}{2}$

$$= PGF(N(t))[m(s)]$$

it is the same as PGF of $N(t)$ evaluated at $m(s)$

$$= PGF(N(t-1))[m(m(s))]$$

which is the same as PGF of $N(t-1)$ evaluated at $m(m(s))$

Thus, $$PGF[N(t+1)](s) = PGF(N(0))[\underbrace{m(m \cdots (m(s))}_{t+1 \text{ times}})]$$

Assuming that at $t=0$, # bacteria at

$$E[N(t+1) \mid N(t)] = E\left(X_1 + X_2 + \cdots + X_{N(t)}\right)$$
$$= E(X_1) + E(X_2) + \cdots + E(X_{N(t)})$$
$$= N(t)\, E(X_1)$$
$$= 1000\left\{ \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{2}(2) \right\}$$
$$= 1000\left(\frac{5}{4}\right) = \underline{1250}$$

5.

$$MGF(S_N) = E(s^{S_N})$$

$$= E\left[E(s^{S_N} \mid N)\right] = E\{E(s^{X_1+X_2+\cdots+X_N})\}$$

$$= E\{E(s^{X_1} \cdot s^{X_2} \cdots s^{X_N})\}$$

now $X_1, X_2, \ldots, X_N$ are independent of each other given N

$$= E\{[E(s^{X_1})]^n\}$$

$$= E\{(m(s))^n\}$$

where $m(s)$ is the PGF of X_1

and now $= PGF(N)[m(s)]$

which is same as PGF of N evaluated at $m(s)$

a) $$E(S_N) = E[E(S_N \mid N)] = E\{E(X_1+X_2+\cdots+X_N \mid N)\}$$

$$= E[E(X_1) + E(X_2) + \cdots + E(X_N) \mid N]$$

$$= E(N)\,E(X)$$

b) $$Var(S_N) = E(S_N^2) - [E(S_N)]^2$$

$$= E[E(S_N^2 \mid N)] - E(N)^2 E(X)^2$$

$$= E\,E\{(X_1+X_2+\cdots+X_N)^2 \mid N\} - E(N)^2 E(X)^2$$

$$= E\{E(X_1^2 + X_2^2 + \cdots + X_N^2 + 2X_1X_2 + \cdots + 2X_NX_{N-1}) \mid N\} - E(N)^2 E(X)^2$$

$$= E\{E(X_1^2) + E(X_2^2) + \cdots + E(X_N^2) + 2E(X_1X_2) + \cdots + 2E(X_NX_{N-1})\} - E(N)^2 E(X)^2$$

$$= E\{N\,E(X_1)^2 + 2E(X_1)E(X_2) + \cdots + 2E(X_N)E(X_{N-1})\} - E(N)^2 E(X)^2$$

$$= E\{N\,E(X_1)^2 + N(N-1)\,E(X_1)^2\} - E(N)^2 E(X)^2$$

$$= E(N)\,E(X_1^2) + E(N^2)\,E(X_1)^2 - E(N)\,E(X_1)^2 - E(N)^2 E(X)^2$$

$$= E(N)E(X_1^2) + E(X_1)^2\{Var(N) + \cancel{E(N)^2}\} - E(N)E(X_1)^2 - \cancel{E(N)^2 E(X)^2}$$

$$= E(N)E(X_1^2) + Var(N)E(X_1)^2 - E(N)E(X_1)^2$$

$$= E(N)\,Var(X) + Var(N)\,E(X)^2$$

6. $E(Y/X) = 1$

$$Var(XY) = E(X^2Y^2) - [E(XY)]^2$$

$$= E[E(X^2Y^2|X)] - \{E[E(XY|X)]\}^2$$

$$= E[X^2 E(Y^2|X)] - E(X)^2$$

$$Var(Y/X) = E(Y^2/X) - [E(Y/X)]^2 \geq 0$$

$$E(Y^2/X) \geq [E(Y/X)]^2 = 1$$

$$E(Y^2/X) \geq 1$$

thus $E[X^2 E(Y^2/X)] - E(X)^2 \geq E(X^2) - E(X)^2$

thus $Var(XY) \geq Var(X)$

7. # games played forms a geometric distribution

Let the # games played $= N \sim geom(p)$ where $p = 1/3$

Let $X_i = \begin{cases} 2 ; & 1/3 \\ 1 ; & 1/3 \\ 0 ; & 1/3 \end{cases}$ denote the points outcome of the i^{th} game

Then, total number of points earned Y can be written as

$$Y|N = X_1 + X_2 + \cdots + X_N$$

$$MGF(Y) = E(e^{tY}) = E\{E(e^{tY}|N)\}$$

$$= E\{E(e^{t(X_1+X_2+\cdots+X_N)}|N)\}$$

$$= E\{E(e^{tX_1}e^{tX_2}\cdots e^{tX_N}|N)\}$$

Now X_i's are all independent given N

$$= E[(E\,e^{tX_1})^N]$$

$$= E[(m(t))^N]$$

now $m(t) = MGF(X) = \frac{e^{2t} + e^t + 1}{3}$

$$= \frac{p e^t}{1-(1-p)e^t}, \quad p = 1/3$$

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8: $X \sim Bin(n, p)$

$$X = X_1 + X_2 + \cdots + X_n$$

where each X_i is Bernoulli distributed with parameter p and all are independent of each other

$$E(X) = np$$
$$Var(X) = np(1-p)$$

thus $Z = \dfrac{X - np}{\sqrt{np(1-p)}}$ can be considered standard normal using CLT

thus
$$P\left(X \geq n/2\right) = P\left(\frac{X - np}{\sqrt{n p(1-p)}} \geq \frac{n(1/2 - p)}{\sqrt{np(1-p)}}\right)$$
$$= P\left(Z \geq \frac{n(1/2 - p)}{\sqrt{np(1-p)}}\right)$$

thus the value for which $P(X \geq n/2) \leq 1-\alpha$ is same as that of for which $P\left(Z \geq \dfrac{n(1/2 - p)}{\sqrt{np(1-p)}}\right) \leq 1-\alpha$

9. $T =$ total time to failure

$$T = T_1 + T_2 + \cdots + T_{30}$$

where T_i denotes the failure time of i^{th} device

$$T_i \sim exp(0.1)$$

$$E(T) = NE(T_1) = 30/(0.1) = 300$$
$$Var(T) = N\,Var(T) = 30/(0.1)^2 = 3000$$

$$P(T \geq 350) = P\left(\frac{T - 300}{\sqrt{3000}} \geq \frac{350 - 300}{\sqrt{3000}}\right)$$
$$= P\left(Z \geq \frac{50}{\sqrt{3000}}\right) \text{ using CLT}$$

11. $S = X_1 + X_2 + \cdots + X_{30}$

$X_i \sim \text{Poiss}(\lambda = 0.01)$

$E(S) = 30\,E(X_i) = 30(0.01) = 0.3$

$Var(S) = 30\,Var(X_i) = 30(0.01) = 0.3$

a) $P(S \geq 3) = P\left(\frac{S - 0.3}{\sqrt{0.3}} \geq \frac{3 - 0.3}{\sqrt{0.3}}\right)$

$= P\left(Z \geq \frac{2.7}{\sqrt{0.3}}\right) \approx 0$

b) Now sum of Poissons is also Poisson

$S \sim \text{Poiss}(30\lambda) = \text{Poiss}(0.3)$

$P(S \geq 3) = 1 - P(S=0) - P(S=1) - P(S=2)$

$= 1 - \frac{(0.3)^0 e^{-0.3}}{0!} - \frac{e^{-0.3}(0.3)^1}{1!} - \frac{e^{-0.3}(0.3)^2}{2!}$

12. $P(|S_n - p| \leq 0.01) \geq 1 - \frac{E(S_n - p)^2}{(0.01)^2}$

So we have

$1 - \frac{E(S_n - p)^2}{(0.01)^2} = 0.95$

$\frac{E(S_n - p)^2}{(0.01)^2} = 0.05$

$S_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$

$S_n - p = \frac{X_1 + X_2 + \cdots + X_n}{n} - p = \frac{(X_1 - p) + (X_2 - p) + \cdots + (X_n - p)}{n}$

$E(S_n - p)^2 = \frac{1}{n^2} E\left((X_1 - p) + (X_2 - p) + \cdots + (X_n - p)\right)^2$

$= \frac{1}{n^2}\left\{n\,E(X_1 - p)^2 + n(n-1)\cancel{E(X_1 - p)}^{0}\,\cancel{E(X_2 - p)}^{0}\right\}$

$= \frac{n}{n^2} E(X_1 - p)^2 = \frac{1}{n} p(1-p)$

$\frac{p(1-p)}{n(0.01)^2} = 0.05$

solve to get the # of samples that we need to take.

13. $Y = \sum_{k=1}^{100} X_k$

a) $P(Y \geq 900) < \frac{E(Y)}{900} = \frac{100}{900} = 1/9$

b) $E(Y) = 100\, E(X_k) = 100$

$Var(Y) = 100\, Var(X_k) = 16 \times 10^4$

$P(Y \geq 900) = P\left(\frac{Y - 100}{\sqrt{16 \times 10^4}} \geq \frac{900 - 100}{\sqrt{16 \times 10^4}} \right) =$

$= P\left(Z \geq \frac{800}{400}\right)$

$= P(Z \geq 2) = 1 - \phi(2)$