Tutorial 8 solutions

1. E(X) = E(A0) + t E(A1) + t2 E(A2) = 0

(using linearity of expectation, and that t is constant while taking the expectation).

Let t<s

*Cov*(*X*(*s*)*, X*(*t*)) = *E*[*X*(*s*)*\* X*(*t*)] *− E*[*X*(*s*)]*E*[*X*(*t*)]

= *E*[*X*(*s*)*, X*(*t*)] (as expectation is 0)

*Cov*(*X*(*s*)*, X*(*t*))= E[ (A0 + A1t + A2t2) (A0 + A1s + A2s2)]

E(A0 A1) = E(A0)\* E(A1) = 0 (because of uncorrelation)

Var(A0) = E(A02) – E2(A0) = 1

1 = E(A02) – 0 = E(A02)

From this, *Cov*(*X*(*s*)*, X*(*t*)) = 1 + *st* + *s*2*t*2

1. *Y* (*t*) = *X*(*t*)cos(2*πwt* + Θ)

E ( Y ) = E (X) \* E(cos(2*πwt* + Θ)) (because of independence)

E ( Y ) = 0

Let t<s

Cov(Yt, Ys) = E(YtYs)– E(Yt)E(Ys)

= E(YtYs)

E(YtYs) = E(XtXs) E (cos(2*πwt* + Θ)\* cos(2*πws* + Θ))

Using the identity

Cos(A)cos(B) = ½(cos(A+B) + cos(A-B))

and integrating we get

Using the formula for cos(2θ), and integrating, we get

as

So Y(t) is wide sense stationary.

1. ))

E(Xt) = ½

Since X(0) takes value 0 and 1 with probability ½ , so

Cov(XtXs) = E(XtXs) – ¼

E(XtX­s) = ¼[(1-E((-1)X(0)+Y(t) ) (1-(-1)X(0)+Y(s) )) ] – ¼

E(XtX­s) = ¼[1 + E(-1X(0) + X(0)Y(t) + X(0)Y(s) + Y(s)Y(t) )]

Simplifying the above, we get based on the fact that X(0) takes both 0 and 1 values with ½ probability

E(XtX­s) = ¼ - ¼ = 0

E(Xt2) = ¼\*E((-1)X(0)+Y(t) )\* (-1)X(0)+Y(t) ))

E(Xt2) = ½ < ∞

So Xt is wide sense stationary.

If we can express P(Xn+1 = x | Xn, Xn-1, …. X0) = f(Xn) + Y, where Y is a random variable independent of Xn, then Xn is a markov chain.

If xi­ is a RV which denotes number of people in ith step, then

Xi = Max(Xn-1 – 1, 0 ) + xi

Since xi is clearly satisfying the conditions mentioned above, Xn is a markov chain.

1. A)

Let P(Xn+1  = 0 | jk ) mean P(Xn+1  = 0 | Xn-1  = j, Xn  = k )

P(Xn+1  = 0 | 00 ) = q00  (1)

P(Xn+1  = 0 | 01 ) = q01  (2)

P(Xn+1  = 0 | 10 ) = q10

P(Xn+1  = 0 | 11 ) = q11

P(Xn+1  = 0 | Xn = 0 ) = q00  + q10

This doesn’t equal either of (1) or (2), so this isn’t a markov chain.

B) We say that machine is in state (*j,k*) on day *n* if the machine is in state *j* on day (*n*−1) and in state *k* on day *n*. We have to show that with this changed state space the system is a discrete time Markov chain.

Define new variable Yn = (Xn,Xn-1)

For this new variable Yn, we need to see if it is a Markov chain or not.

To prove : P(Yn+1 = x | Yn, Yn-1, …. Y1) = P(Yn+1 = x | Yn)

Here, given Yn, we know Xn and Xn-1 , and so from *P* (*X*(*n*+1) = 0*/X*(*n*−1) = *j , X*(*n*) = *k*) = *qjk ,*

We have that Yn+1 , ie (Xn+1, Xn) only depends on Xn+1, Xn, Xn-1 , hence given Yn, we know (Xn,Xn-1) and so {Yn} is a Markov chain.

C)

1. A) Π2 = Π0 \* P2

From here, after solving, read the corresponding probability from the two step transition matrix.

Ans = 0.212

Another way to solve is to condition on the previous steps, and using the stationary distribution.

B) P ( X3 = 2, X2 = 3, X1 = 3, X0 = 2) = P ( X3 = 2 | X2 = 3)\* P( X2 = 3| X1 = 3)\* P( X1 = 3| X0 = 2)\*P(X0 = 2)

P ( X3 = 2, X2 = 3, X1 = 3, X0 = 2) = 0.4\*0.1\*0.2\*0.2

P ( X3 = 2, X2 = 3, X1 = 3, X0 = 2) = 0.0016

1. Classify states using definitions
2. Xn = Xn-1 + Y, where Y is a random variable

Y = 1, with prob =1/2 and Y=-1, with prob=1/2

Here, Xn = f(Xn-1) + Y, so it is a Markov chain.

1. If n is the period for some state, then n+1 is also a period because Pi,I > 0, so d(x)=gcd{n∈N+:Pn(x,x)>0} = 1 for every state.
2. For absorption at nth step given initial state j, the last step must necessarily be from 1 to 0.

For the rest of the n-1 steps, we can choose which steps are not self loops, so we get n-1 choose j-1. Hence

fjo(n) = (n-1,j-1) pn-j qj if j<=n

= 0 if j>n

1. Because P00 = 1, so chain is aperiodic as gcd of the time periods is 1.

It can be easily proved that all states are positively recurrent since

For steady state distribution

πP=π, and solve for π

π*i* = *pi*(1 *− p*) for all i=0,1,2,3,4,……