THE MATHEMATICS OF PHYSICAL QUANTITIES

PART II: QUANTITY STRUCTURES AND DIMENSIONAL ANALYSIS

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INTRODUCTION

1. Mathematical models. In setting up a scientific theory, one chooses a framework in which the phenomena to be discussed may be expressed. This framework is given some kind of structure, which is then compared with various experiments. If this structure is given in a precise manner, so that one may draw rigorous conclusions about it, it is called a "mathematical model." The investigation then takes two separate directions:

(a) One studies the model itself, and draws various conclusions that are consequences of the chosen structure of the model.

(b) One tests the relation between the model and experiment.

The construction and employment of mathematical models is now one of the most important functions of applied mathematics, ever growing in scope. In fact, the use of models has been at the basis of scientific theories throughout history, though they have generally not been thought of quite in these terms. A very early model was the system of natural numbers, used in counting, and studied mathematically by Dedekind and Peano in the last century. The use of the "aether" as a model in which electromagnetic waves travel comes to mind; but this gave way to a more purely mathematical model to describe such phenomena.

Newton's laws of motion are expressed in terms of mass, length and time; these terms may be used to designate actual experimental findings, or concepts related to such findings. In writing equations, the latter meaning is best used; then we are building a mathematical model in which the laws take on precise meaning. Let us call a model of this nature a "quantity structure"; it is the kind of model studied in this paper. Though it is obviously basic to the expression of classical physics, it is curious that this sort of model seems never to have been set up in a complete and strict fashion. The essential features will be found in Drobot, [2].

The present Part II may be read independently of Part I (see [5]).

2. The kind of model studied. Classically, the results of measurements have been expressed in terms of numbers. The various sets of numbers that are important for us are:

N, the natural numbers; I, the integers (positive, zero or negative);

Q, the rationals; R^+, the positive reals; R, the real numbers.

Here is a typical statement in physics: The velocity of light is

$$c = 3 \times 10^{10} \frac{\text{cm}}{\text{sec}} \cdot \tag{2.1}$$

Such a statement is often interpreted as follows: The numbers express the "measure"; the "cm" and "sec" shows the units which are used. If such an interpretation is necessary, what does the equal sign mean? How does one work with equations, anyway? Obviously we cannot be on secure ground unless we give things precise meanings; we wish these meanings to fit the desired purpose, of course. It is said that you cannot "really" divide distance by time. Perhaps you cannot with actual experimental distance and time. But if we let "cm" and "sec" be elements of a mathematical model, we may introduce division and anything else we please; then the experimentalist may fit our model to the phenomena as he sees best.

Our first job is to choose the elements that will make up the model. Let us call these "quantities"; they correspond to physical quantities.

Next, we wish to add and subtract quantities; but only if they are of the same "type," corresponding to some physical dimension. (Then one of the quantities is a real multiple of the other.)

Certain physical quantities, like mass, are inherently positive; others, like time intervals, may be positive or negative, but may have a natural "positive direction." We suppose that each element of our model may be called positive, zero, or negative.

The positive quantities of a single type form a "ray"; R^+ is a typical example of a ray. All the quantities of a single type form a "biray"; this is simply an oriented one-dimensional vector space, with R as typical example. One may define the concepts of rays and birays from simple assumptions about addition. One may then define $3x$ to be $x+x+x$, and $(1/3)x$ to be y, where $3y=x$; continuing in this manner, one introduces R as a domain of operators on a biray; see [5]. In contrast with this, we assume a knowledge of R here, and addition is not mentioned in the definition of a quantity structure; instead, we *define* $ax+bx$ to be $(a+b)x$ (see Section 4).

We wish to be able to multiply any two quantities in our model; the result is a quantity, generally of a new type. Any nonzero quantity may be raised to any integral power; we may then for instance define cm ÷ sec to be cm × sec^{-1}.

If we measure the height h upwards, a falling stone travels a distance

$$h = -(16 \text{ ft/sec}^2)t^2$$

in the time t. Solving for t gives

$$t = \sqrt{-h/16 \text{ ft}} \text{ sec}.$$

Thus we must be able to take square roots of positive quantities. (Note that h must be negative in the above; this corresponds to the fact that the stone does not fall upwards.) More generally, we may allow any rational exponent of a positive quantity in our model, thus using Q as power operators; or we might allow R itself here.

We get in this manner certain mathematical quantities that might not appear experimentally; for example, $\mathrm{cm}^{5/7}$, or $\sec^{\pi}$. In fact, $2\ \mathrm{cm}/\mathrm{cm}=2$ is in our model, though it is not commonly thought of as a physical quantity.

The operations now introduced into our model must be subject to certain laws. These are in fact similar to those about numbers, and have always been used by scientists without worry. Some samples are:

$$3\ \mathrm{cm} + 4\ \mathrm{cm} = (3+4)\ \mathrm{cm}; \qquad 6\ \mathrm{cm} \div 2\ \mathrm{sec} = 3\ \mathrm{cm/sec};$$
$$(9\ \mathrm{cm/sec}^2)^{1/2} = 9^{1/2}\ \mathrm{cm}^{1/2}\ (\mathrm{sec}^{-2})^{1/2} = 3\ \mathrm{cm}^{1/2}/\mathrm{sec}.$$

3. Outline of the paper. In Chapter I, we define quantity structures, and study their general properties; in Chapter II, we illustrate their use through various examples, in particular, through "dimensional analysis."

Rays and birays are defined (Sections 4, 5) through the use of R^+ and R as operator domains. Quantity structures are defined postulationally (Sect. 6); that is, any given structure is by definition a quantity structure provided it satisfies certain given conditions. The elementary facts about these structures appear naturally.

Each element $x \neq 0$ of a given quantity structure S is on a biray $[x] = \{ax: a \in \mathrm{R}\}$ (Sect. 7), consisting of all elements of the same type as x; the positive part $[x]^+$ of $[x]$ is a ray. Thus $[\mathrm{in}] = [\mathrm{ft}] = [-100\ \mathrm{ft}]$, called the "dimension" of length. Considering each $[x]$ as an element of a new system gives us in a natural manner a vector space $[S]$ (Sect. 8) written multiplicatively; this is the space of possible "physical dimensions." If $[x_1], \cdots, [x_n]$ forms a base for $[S]$, then each element of S may be written in the form

$$x = a x_1^{\alpha_1} \cdots x_n^{\alpha_n}; \tag{3.1}$$

the representation is unique if $x \neq 0$ (see Sect. 9).

An isomorphism ϕ of S onto itself is a "similarity" if it preserves type: $[\phi(x)] = [x]$ (and preserves positiveness). With a base as above, the general similarity is defined by $\phi(x_i) = \lambda_i x_i$ for some positive numbers $\lambda_1, \cdots, \lambda_n$ (Sect. 11).

A quantity structure may be constructed from given base elements $x_1, \cdots, x_n$, or from the corresponding rays (Sections 12, 13). For example, choosing rays M, L, T (which represent mass, length and time), we build the model used in most problems in mechanics.

We end Chapter I by showing how algebra and calculus are carried out directly in a quantity structure; all terms in equations are elements of S, rather than just real numbers.

In Chapter II we give first some elementary applications of quantity structures, showing how one may "keep the units" in equations. Some remarks on "dropping units" are made; the end of Section 21 shows the danger in this. We then take up a powerful tool in many investigations, "dimensional analysis." This illustrates the fact that using a mathematical model (a quantity structure) enables one to be precise in statement, thus clearing up much of the confusion one normally sees in the subject. The key theorem is the "Pi Theorem" (Sect. 23): Let $f(x_1, \cdots, x_s)$ be a function for $x_i \in \Gamma_i$, where the Γ_i are rays in S, and let the values of f be in a ray Γ. Then if f is assumed "homogeneous," i.e. invariant under similarities of S, considerable knowledge about the form of f follows.

The use of dimensional analysis is quite standard; we illustrate it through a discussion of pendulums, of satellites, of the flow of water, and of torque of a wire.

CHAPTER I. QUANTITY STRUCTURES

4. Rays. We first consider operations on one set by another.

DEFINITION 4A. Let A and X be sets. We say A *operates* on X if for each $a \in A$ and $x \in X$ there is a fixed corresponding element y of X. (Thus we have a function from $A \times X$ to X.) We say A is *multiplicative* if y is written as ax, and is a *power* operation if y is written as x^a. We say $x \in X$ *generates* X if each $y \in X$ is ax for some a; x generates X *simply* if this a is unique, for each y.

DEFINITION 4B. A *ray* is a set X with an operation by R^+, such that:
(P_1) $a(bx) = (ab)x$ for all $a, b \in R^+$ and $x \in X$.
(P_2) There is an element $w \in X$ which generates X simply.
Note that R^+ is a ray; it operates on itself by multiplication.
We give some elementary properties of a ray X. First:

$$1x = x \qquad (x \in X). \tag{4.1}$$

For choosing w by (P_2), we may write $x = cw (c \in R^+)$. Now

$$1x = 1(cw) = (1c)w = cw = x.$$

Next:

$$\text{Each } x \in X \text{ generates } X \text{ simply.} \tag{4.2}$$

For we may write $x = cw$. Set $c' = 1/c$; then $c'x = c'(cw) = (c'c)w = 1w = w$. Now for any $y = dw$, $y = d(c'x) = (dc')x$. We must show still that

$$\text{If } a \neq b \text{ then } ax \neq bx \qquad (a, b \in R^+, x \in X). \tag{4.3}$$

With $x = cw$, we have $ax = (ac)w$, $bx = (bc)w$; since w generates X simply, $ac = bc$, and therefore $a = b$.

DEFINITION OF ADDITION. Choose w as before; we define

$$aw + bw = (a + b)w. \tag{4.4}$$

We now show that

$$(a + b)x = ax + bx \qquad (a, b \in R^+; x \in X). \tag{4.4'}$$

For, writing $x = cw$, we have

$$(a + b)x = (a + b)(cw) = [(a + b)c]w = (ac + bc)w$$
$$= (ac)w + (bc)w = a(cw) + b(cw) = ax + bx.$$

Thus the definition of addition is independent of the w used.

The *commutative* and *associative* laws for addition follow at once. For instance, $aw+bw=(a+b)w=(b+a)w=bw+aw$. We give a second distributive law:

$$a(x + y) = ax + ay \qquad (a \in \mathrm{R}^+;\, x, y \in X). \tag{4.5}$$

For writing $x=cw$, $y=dw$, we have

$$a(x + y) = a[(c + d)w] = [a(c + d)]w = acw + adw = ax + ay.$$

There is a natural *order* in any ray, defined as follows:

$$x < y \text{ means } x + z = y \text{ for some } z. \tag{4.6}$$

Using x as generator in X gives: $ax<bx$ if $ax+cx=bx$ for some c; thus (writing "iffif" for "if and only if")

$$ax < bx \text{ iffif } a < b \qquad (a, b \in \mathrm{R}^+;\, x \in X). \tag{4.7}$$

We clearly have *trichotomy*: For each x, y, exactly one of $x<y$, $x=y$, $y<x$ is true. Also order is transitive: $x<y$ and $y<z$ imply $x<z$. Thus X is *simply ordered.*

We now show that *any two rays are isomorphic.* By definition, a *homomorphism* ϕ of a system X into a system X' of the same sort is a mapping which preserves all the properties of the systems. It is an *isomorphism* if it is one-one, and is an isomorphism *onto* X' if the image of X is all of X'.

THEOREM 4C. *Let X and X' be rays, and suppose $w\in X$, $w'\in X'$. Then there is a unique mapping ϕ of X into X' such that*

$$\phi(ax) = a\phi(x) \qquad (a \in \mathrm{R}^+,\, x \in X), \tag{4.8}$$

$$\phi(w) = w. \tag{4.9}$$

This ϕ is an isomorphism of X onto X', and hence preserves addition and order:

$$\phi(x + y) = \phi(x) + \phi(y), \tag{4.10}$$

$$\phi(x) < \phi(y) \text{ iffif } x < y. \tag{4.11}$$

First, since ϕ must satisfy $\phi(aw)=a\phi(w)=aw'$, ϕ is unique. Use this to define ϕ. Then for any $x=cw$ and any $a\in\mathrm{R}^+$,

$$\phi(x) = \phi(cw) = c\phi(w) = cw', \quad \phi(ax) = \phi(acw) = ac\phi(w) = acw' = a\phi(x);$$

thus ϕ is a homomorphism.

If $\phi(aw)=\phi(bw)$, then $aw'=bw'$, and hence $a=b$, by (4.3) in X'; thus ϕ is one-one. Since w' generates X', ϕ is onto X'. Since addition and order are defined in terms of the operation by R^+, these are preserved; a direct verification is also immediate.

COROLLARY 4D. *For each $x\in X$ there is a unique isomorphism ϕ of* R^+ *onto X such that $\phi(1)=x$; then $\phi(a)=ax$.*

COROLLARY 4E. *The isomorphisms of X into itself are in one-one correspondence with the numbers $\lambda \in \mathrm{R}^+$; they are given by $\phi_\lambda(x) = \lambda x$.*

For, taking a fixed $w \in X$, any $w' \in X$ may be written uniquely as λw for some λ; now apply Theorem 4C.

5. Birays. We give first a simple characterization of one-dimensional vector spaces.

THEOREM 5A. *Let V be a set on which* R *operates. Suppose* (P_1) *and* (P_2) *of Section* 4 *hold, with* R^+ *replaced by* R. *Then V, with addition defined as in* (4.4), *is a one-dimensional vector space over* R.

Choose $w \in V$ by (P_2), and set $O = 0w$. Now given $x \in V$, we may write $x = cw$, and

$$0x = 0(cw) = (0c)w = 0w = O.$$

Later we shall write O for the zero in any ray.

Also, for $a \in \mathrm{R}$, $aO = a(0w) = (a0)w = 0w = O$. Using also the proof of (4.1), we now have

$$1x = x, \quad 0x = aO = O \qquad (a \in \mathrm{R}, x \in V). \tag{5.1}$$

If $x \neq O$, then $x = cw$ with $c \neq 0$; the proof of (4.2) shows that:

$$\text{If } x \neq O \text{ then } x \text{ generates } V \text{ simply.} \tag{5.2}$$

In particular, for $x \neq 0$, if $a \neq 0$ then $ax \neq O$; hence, recalling (5.1),

$$ax = O \text{ iffif } a = 0 \text{ or } x = O \qquad (a \in \mathrm{R}, x \in V). \tag{5.3}$$

As before, the definition of addition is independent of the element $w \neq O$ used, and the associative, commutative and two distributive laws hold. Using (5.1) also shows that V is a vector space over R. Because of (P_2), $\dim(V) = 1$.

Recall that in a vector space, we may define *negatives* of elements and *subtraction* by setting

$$-x = (-1)x, \quad x - y = x + (-y). \tag{5.4}$$

The usual laws for minus signs follow. For instance,

$$\begin{aligned} -(x - y) &= (-1)[x + (-1)y] = (-1)x + (-1)[(-1)y] \\ &= [(-1)(-1)]y + (-x) = 1y - x = y - x. \end{aligned}$$

DEFINITION 5B. A one-dimensional vector space V is *oriented* if a subset V^+ of V is designated as the set of *positive* elements of V; we assume that for some positive element w, aw is positive iffif $a > 0$. This is then true of any positive element.

The negatives of the positive elements form the set V^- of *negative* elements of V. Clearly each element of V is exactly one of the following: positive, negative, or zero (i.e. $=O$).

DEFINITION 5C. Let V be oriented. Define

$$x < y \text{ iffif } x + z = y \quad \text{for some } z \in V^+. \tag{5.5}$$

Now V is simply ordered; the proof is very similar to that of Section 4.

THEOREM 5D. *If V is oriented, then V^+, operated on by* R$^+$, *is a ray.*

If x is positive, then for $a>0$, ax is positive; thus R$^+$ operates on V. The rest of the proof is trivial.

DEFINITION 5E. A *biray* is an oriented one-dimensional vector space.

We now show that any two birays are isomorphic.

THEOREM 5F. *Let B and B' be birays, and suppose $w \in B^+$, $w' \in B'$. Then there is a unique mapping ϕ of B into B' such that $\phi(ax)=a\phi(x)$ ($x \in B$, $a \in$* R*), and $\phi(w)=w'$;* (4.10) *holds. If $w' \neq O'$, then ϕ is one-one onto B'. If $w' \in B'^+$, then ϕ is an isomorphism onto B', which (therefore) preserves order.*

If $w'=O'$, the requirement on ϕ gives: $\phi(x)=O'$, all x. Otherwise, the proof of Theorem 4C applies, giving the present theorem.

COROLLARY 5G. *For each $w \in B^+$ there is a unique isomorphism of* R *onto B such that $\phi(1)=w$; then $\phi(a)=aw$.*

COROLLARY 5H. *The isomorphisms of B onto itself are in one-one correspondence with the numbers $\lambda \in$* R$^+$*; they are given by $\phi_\lambda(x)=\lambda x$.*

See the proof of Corollary 4E.

6. Quantity structures. We now give our basic definition, telling us if a given model is a quantity structure S. The existence of such structures will be proved through a construction process, in Section 12.

As operators on S by powers, one might choose either Q or R; to leave open both (and possibly other) cases, we use the symbol R*.

DEFINITION 6A. A *quantity structure* consists of a set S, which contains the real numbers R, a subset S^+ of S, a binary operation of multiplication in S, and a power operation on S^+ by a set R* (either Q or R), subject to the following conditions:

Properties of multiplication:

(Q_1) Multiplication has its usual meaning in the subset R of S.

(Q_2) $1x=x$, $0x=0$. We define: $-x=(-1)x$.

(Q_3) Multiplication is associative and commutative.

Properties of subsets:

(Q_4) $R \cap S^+=$ R$^+$.

(Q_5) For each $x \neq 0$, just one of x, $-x$ is in S^+.
(Q_6) If $x, y \in S^+$ then $xy \in S^+$.

Properties of powers ($x^\alpha \in S^+$ *if* $x \in S^+$, $\alpha \in \mathrm{R}^*$):
(Q_7) a^α has its usual meaning for $a \in \mathrm{R}^+$.
(Q_8) $x^1 = x$, $x^0 = 1$.
(Q_9) $(xy)^\alpha = x^\alpha y^\alpha$, $x^{\alpha+\beta} = x^\alpha x^\beta$, $(x^\alpha)^\beta = x^{\alpha\beta}$.

DEFINITION 6B. The elements of S^+ are called *positive*; their negatives form the set S^- of *negative* elements of S.

Example 6C. R, operating on itself by multiplication, and operating on its positive part by powers, is a quantity structure (which is contained in any other one with $\mathrm{R}^* = \mathrm{R}$).

We now prove some elementary properties of a quantity structure. We shall use letters $x, y, \cdots$ for elements of S (including R), $a, b, \cdots$ for numbers, and $\alpha, \beta, \cdots$ for elements of R^*.

Since $(-1)[(-1)x] = [(-1)(-1)]x = 1x = x$ and $[(-1)x]y = (-1)(xy)$,

$$(6.1) \qquad -(-x) = x, \qquad (-x)y = -(xy), \qquad (-x)(-y) = xy.$$

Trichotomy: Each element of S is either positive, negative or zero, and is only one of these. For if $x \neq 0$, then by (Q_5), x is either positive or negative, but not both. By (Q_4), $0 \notin S^+$; and since $-0 = 0$, $0 \notin S^-$.

The *rule of signs* is: If x and y are both positive or both negative, then xy is positive; if one of x, y is positive and the other negative, then xy is negative. This follows from (6.1), (Q_5) and (Q_6).

Since $\mathrm{R} \subset S$ and (Q_4) holds, a consequence of the rule of signs is:

$$(6.2) \qquad \text{If } x \in S^+ \text{ and } a \in \mathrm{R}, \text{ then } ax \in S^+ \text{ iffif } a \in \mathrm{R}^+.$$

If $x, y \neq 0$, then the rule of signs gives $xy \neq 0$. Hence, using (Q_2) and (Q_3),

$$(6.3) \qquad xy = 0 \text{ iffif } x = 0 \text{ or } y = 0.$$

Any element $x \neq 0$ has a *multiplicative inverse*, namely, x^{-1}, if x is positive, and $-(-x)^{-1}$ if x is negative. For if $x \in S^+$, then by (Q_8) and (Q_9), $xx^{-1} = x^1x^{-1} = x^{1-1} = x^0 = 1$, and if $x \in S^-$, then $-x \in S^+$, and $x(-(-x)^{-1}) = (-x)(-x)^{-1} = 1$. We may set $x^{-1} = -(-x)^{-1}$ if $x \in S^-$.

We prove the cancellation law:

$$(6.4) \qquad \text{If } xz = yz \text{ and } z \neq 0 \text{ then } x = y.$$

For, taking u so that $zu = 1$, $xz = yz$ gives $xzu = yzu$, $x1 = y1$, $x = y$. (Note that we do not yet have addition in S, so we do not form $(x-y)z$.)

We may form *integral powers of any* $x \neq 0$. First, $x^2 = xx$, $x^3 = xxx$, etc. Next, we defined $x^{-1} = u$ above if $x \neq 0$; it is unique, by (6.4). Now set $x^{-n} = u^n$ for $n \in N$. It is easily verified that the properties in (Q_9) (with integral powers) continue to hold. For instance,

$$x^2x^{-4} = xxuuuu = xuuu = uu = x^{-2} = x^{2-4},$$
$$(x^{-3})^{-2} = (u^3)^{-2} = [(u^3)^{-1}]^2 = (x^3)^2 = x^{3\cdot 2} = x^{(-3)(-2)}.$$

7. Rays and birays in S. Given any $x \neq 0$ in the quantity structure S, set

$$[x] = \{ax : a \in \mathrm{R}\}, \qquad [x]^+ = [x] \cap S^+. \tag{7.1}$$

By (Q_3), (P_1) of Section 4 holds with R in place of R^+. Now x generates $[x]$, and by (6.4), the generation is simple. Then, with addition in $[x]$ defined by (4.4), Theorem 5A shows that $[x]$ is a one-dimensional vector space. By (Q_5), we may set $y=x$ or $y=-x$, and have $y \in S^+$; by (6.2), $ay \in S^+$ iffif $a \in \mathrm{R}^+$. Thus $[x]$ *is a biray*, with $[x]^+$ as positive part; and by Theorem 5D, $[x]^+$ *is a ray*.

In particular, there is a *natural order* in $[x]$, and in particular, in $[x]^+$.

We now compare two birays in S. First note that any biray $[x]$ contains $0x=0$ as its zero element. We now prove:

$$\text{For any } x, y \neq 0, \text{ either } [x] = [y] \text{ or } [x] \cap [y] = \{0\}. \tag{7.2}$$

For suppose $z \in [x] \cap [y]$, $z \neq 0$. Then we have $z=ax=by$ for some $a, b \neq 0$; hence $x=cz$, $y=dz$, and $y=dax$, $x=cby$, showing that $[y]=[x]$. We now see at once that (letting $\emptyset$ denote the null set) if $x \neq 0$, $y \neq 0$,

$$\text{either } [x]^+ = [y]^+ \text{ or } [x]^+ \cap [y]^+ = \emptyset; \tag{7.3}$$

$$[x]^+ = [y]^+ \text{ iffif } [x] = [y]; \tag{7.4}$$

$$[x] = [y] \text{ iffif } y \in [x]. \tag{7.5}$$

We have two distributive laws, (4.4′) and (4.5), in each biray or ray, with $a, b \in \mathrm{R}$ or $a, b \in \mathrm{R}^+$. We give now a more general distributive law:

$$z(x + y) = zx + zy \text{ if } x \text{ and } y \text{ are both in some } [w]. \tag{7.6}$$

This is true if $z=0$; suppose $z \neq 0$. Say $x=aw$, $y=bw$. Now by (Q_3) and the definition of addition in $[w]$,

$$zx = z(aw) = a(zw), \qquad zy = b(zw),$$
$$z(x + y) = z[(a + b)w] = (a + b)(zw).$$

By (6.3), $zw \neq 0$, and thus the three elements shown are all in $[zw]$. The definition of addition in $[zw]$ now gives (7.6).

REMARK 7A. In any sum of terms, we will be careful to keep all the terms in one biray. Due to 0 being in all birays, other sums, for instance $x+(y-y)$, may have meaning; but note that $(x+y)-y$ in general is not defined. It would be possible to introduce any finite formal sum as a new sort of element, and add such elements componentwise; but this would not be useful for our purposes. (But see the end of Sect. 6 of [5].)

8. The type space of a quantity structure. In mechanics, the physical dimension of length divided by the physical dimension of time gives the physical dimension of velocity. We shall give corresponding definitions in S.

DEFINITION 8A. In the quantity structure S, let each biray $[x]$ be considered as an element of a new structure $[S]$. We define multiplication and the power operation by R^* in $[S]$ as follows:

$$(8.1) \qquad [x][y] = [xy], \qquad [x]^\alpha = [x^\alpha] \qquad (\text{taking } x \in S^+).$$

REMARKS. Henceforth we shall nearly always use elements x of S^+ to specify any biray $[x]$. Because of (7.4), we could use $[x]^+$ in place of $[x]$ in the definition.

We must show that the definitions (8.1) are independent of the elements x, y chosen. Suppose $[x]=[x']$, $[y]=[y']$ (all elements in S^+). Then we can write $x'=ax$, $y'=by$, and we have

$$x'y' = abxy \in [xy], \qquad x'^\alpha = a^\alpha x^\alpha \in [x^\alpha],$$

and hence, by (7.5), $[x'y']=[xy]$, $[x'^\alpha]=[x^\alpha]$.

THEOREM 8B. *$[S]$ is a multiplicative vector space over R^*, with identity element $[1]$ and inverses $[x]^{-1}=[x^{-1}]$.*

First, since $[1][x]=[1x]=[x]$ and $[x^{-1}][x]=[x^{-1}x]=[1]$, the last statements are true. Next, (Q_3) shows that multiplication in $[S]$ is associative and commutative. We must show that the operation by R^* on $[S]$ has the properties

$$(\Gamma\Delta)^\alpha = \Gamma^\alpha\Delta^\alpha, \qquad \Gamma^{\alpha+\beta} = \Gamma^\alpha\Gamma^\beta, \qquad \Gamma^{\alpha\beta} = (\Gamma^\alpha)^\beta, \qquad \Gamma^1 = \Gamma.$$

(With the additive notation, one recognizes the usual requirements for a vector space.) These properties follow at once from (Q_9) and (Q_8). For instance,

$$([x][y])^\alpha = [xy]^\alpha = [(xy)^\alpha] = [x^\alpha y^\alpha] = [x^\alpha][y^\alpha] = [x]^\alpha[y]^\alpha.$$

REMARK 8C. Going back to the actual birays, we have

$$[xy] = \{uv : u \in [x], v \in [y]\};$$

but $[x]^2 \neq \{u^2 : u \in [x]\}$, since u^2 is never negative.

9. Finitely generated structures. First we recall some definitions from the theory of vector spaces, written multiplicatively.

A set of elements $\Gamma_1, \cdots, \Gamma_n$ *generates*, or *spans*, $[S]$ iffif each element of $[S]$ can be written in the form $\Gamma_1^{\alpha_1} \cdots \Gamma_n^{\alpha_n}$ (with the α_i in R^*). The elements $\Gamma_1, \cdots, \Gamma_n$ are *independent* iffif

$$\Gamma_1^{\alpha_1} \cdots \Gamma_n^{\alpha_n} = [1] \quad \text{implies} \quad \alpha_i = 0 \quad (\text{all } i),$$

or equivalently,

$$\Gamma_1^{\alpha_1} \cdots \Gamma_n^{\alpha_n} = \Gamma_1^{\beta_1} \cdots \Gamma_n^{\beta_n} \quad \text{implies} \quad \alpha_i = \beta_i \quad (\text{all } i).$$

A *base* for $[S]$ is an independent generating set.

DEFINITION 9A. Given the quantity structure S, the elements $x_1, \cdots, x_n$ of S^+ *generate* S iffif $[x_1], \cdots, [x_n]$ generate $[S]$. Similarly for independent sets and bases.

We justify these definitions through the following theorems.

THEOREM 9B. *The elements $x_1, \cdots, x_n$ generate S iffif every $x \in S$ can be written in the form*

$$x = ax_1^{\alpha_1} \cdots x_n^{\alpha_n} \qquad (a \in \mathrm{R}; \alpha_1, \cdots, \alpha_n \in \mathrm{R}^*). \tag{9.1}$$

If $[x_1], \cdots, [x_n]$ generate $[S]$, then for any $x \neq 0$, $[x] \in [S]$, and we can write $[x] = [x_1]^{\alpha_1} \cdots [x_n]^{\alpha_n}$; thus $x \in [x_1^{\alpha_1} \cdots x_n^{\alpha_n}]$, and (9.1) is possible. The converse is clear.

THEOREM 9C. *The elements $x_1, \cdots, x_n$ of S^+ are independent iffif*

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathrm{R} \quad \textit{implies} \quad \alpha_i = 0 \quad (\textit{all } i), \tag{9.2}$$

or equivalently, any expression of the form (9.1) *is unique, if* $x \neq 0$.

The first part of (9.2) gives $[x_1]^{\alpha_1} \cdots [x_n]^{\alpha_n} = [1]$; hence if the x_i are independent, all α_i are 0. The converse is clear. The last statement follows at once.

THEOREM 9D. *Given $x_1, \cdots, x_n \in S^+$, any $x \neq 0$ may be written uniquely in the form* (9.1) *iffif the x_i form a base for S.*

This follows from the last two theorems.

DEFINITION 9E. S is *finitely generated* if it has a (finite) generating set. Then it is a standard fact about vector spaces that S has a base, and all bases have the same number of elements. We call this number the *type dimension* dim (S) of S. We also recall from vector space theory:

THEOREM 9F. *Any independent set in a finitely generated S is part of a base.*

We now *compare two bases* $\{x_1, \cdots, x_n\}$, $\{y_1, \cdots, y_n\}$ in S. By Theorem 9D, we may write uniquely

$$y_i = a_i x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}, \qquad x_i = b_i y_1^{\beta_{i1}} \cdots y_n^{\beta_{in}} \qquad (i = 1, \cdots, n). \tag{9.3}$$

Substituting the first expression in the second gives

$$\begin{aligned} x_i &= b_i(a_1 x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}})^{\beta_{i1}} \cdots (a_n x_1^{\alpha_{n1}} \cdots x_n^{\alpha_{nn}})^{\beta_{in}} \\ &= b_i a_1^{\beta_{i1}} \cdots a_n^{\beta_{in}} x_1^{\beta_{i1}\alpha_{11} + \cdots + \beta_{in}\alpha_{n1}} \cdots x_n^{\beta_{i1}\alpha_{1n} + \cdots + \beta_{in}\alpha_{nn}}; \end{aligned}$$

because of uniqueness, this gives (using the similar relation with the x_i and y_i interchanged)

$$\|\beta_{ij}\| = \|\alpha_{ij}\|^{-1}, \tag{9.4}$$

$$(9.5)\qquad a_i = b_1^{-\alpha_{i1}} \cdots b_n^{-\alpha_{in}}, \qquad b_i = a_1^{-\beta_{i1}} \cdots a_n^{-\beta_{in}} \qquad (i = 1, \cdots, n).$$

If an element is expressed in terms of the x_i, (9.3) gives at once its expression in terms of the y_i:

$$(9.6)\qquad cx_1^{\gamma_1} \cdots x_n^{\gamma_n} = cb_1^{\gamma_1} \cdots b_n^{\gamma_n} y_1^{\gamma_1\beta_{11}+\cdots+\gamma_n\beta_{n1}} \cdots y_n^{\gamma_1\beta_{1n}+\cdots+\gamma_n\beta_{nn}}.$$

Now let us consider a *type preserving* change of base: We require $[y_i]=[x_i]$ for each i. Then (9.3) reduces to

$$(9.7)\qquad y_i = \lambda_i x_i, \qquad x_i = \lambda_i^{-1} y_i \qquad (i = 1, \cdots, n);$$

the λ_i are positive, since the base elements are in S^+. Now (9.6) becomes

$$(9.8)\qquad cx_1^{\gamma_1} \cdots x_n^{\gamma_n} = c\lambda_1^{-\gamma_1} \cdots \lambda_m^{-\gamma_n} y_1^{\gamma_1} \cdots y_n^{\gamma_n}.$$

In the expression (9.1), we may call a the *coefficient* of x in terms of the given base. With (9.8), one finds the rule for finding the coefficient in terms of the new base from that in terms of the old. See, in this connection, Section 21 and the example of John's race in Section 18.

REMARK 9G. In the case of the trivial quantity structure, $S=R$, $[S]$ consists of a single element $[1]$, and $\dim(S)=0$; a base is the null set.

10. Homomorphisms of quantity structures. We shall show, among other things, that two structures of the same type dimension are isomorphic.

DEFINITION 10A. A mapping ϕ of a quantity structure S into another, S', is a *homomorphism* if it preserves the structural properties, keeping R fixed. That is:

(H_1) If x is positive then $\phi(x)$ is positive.

(H_2) $\phi(xy)=\phi(x)\phi(y)$.

(H_3) $\phi(x^\alpha)=(\phi(x))^\alpha (x\in S^+)$.

(H_4) $\phi(a)=a$ $(a\in \mathrm{R})$.

Note that (H_2) and (H_4) give, for $x\in S$ and $a\in \mathrm{R}$,

$$(10.1)\qquad \phi(ax) = a\phi(x), \qquad \phi(0) = 0.$$

We shall consider now only finitely generated structures.

THEOREM 10B. *Let S and S' be structures with the same power operators* R^*, *and let $\{x_1, \cdots, x_n\}$ be a base in S. Then for each set of elements $x_1', \cdots, x_n'$ of S'^+ there is one and only one homomorphism ϕ of S into S' such that $\phi(x_i)=x_i'$ for each i; it is given by*

$$(10.2)\qquad \phi(ax_1^{\alpha_1} \cdots x_n^{\alpha_n}) = ax_1'^{\alpha_1} \cdots x_n'^{\alpha_n}.$$

Recalling Theorem 9D, (10.2) defines ϕ uniquely. The verification that ϕ is a homomorphism is immediate. For instance,

$$\phi((ax_1^{\alpha_1} \cdots x_n^{\alpha_n})^\gamma) = \phi(a^\gamma x_1^{\alpha_1\gamma} \cdots x_n^{\alpha_n\gamma}) = a^\gamma x_1'^{\alpha_1\gamma} \cdots x_n'^{\alpha_n\gamma}$$

$$= (ax_1'^{\alpha_1} \cdots x_n'^{\alpha_n})^\gamma = (\phi(ax_1^{\alpha_1} \cdots x_n^{\alpha_n}))^\gamma.$$

THEOREM 10C. *In the last theorem, ϕ is an isomorphism iffif $K' = \{x_1', \cdots, x_n'\}$ is independent; ϕ is onto S' iffif K' generates S'; ϕ is an isomorphism onto S' iffif K' is a base for S'.*

If K' is independent and $\phi(x) = \phi(y)$, then comparing the corresponding right hand sides of (10.2) and applying Theorem 9C shows that $x = y$, provided that $\phi(x) \neq 0$; in the contrary case, $x = y = 0$. The rest of the theorem is clear.

Recalling that a base consists of positive elements, the last theorem gives:

THEOREM 10D. *There is an isomorphism of S onto S' iffif* $\dim(S) = \dim(S')$; *any such isomorphism carries each base in S onto a base in S'.*

REMARK 10E. In Theorem 10B, take $S' = \mathrm{R}$ and $x_i' = 1$ (all i); then ϕ gives the coefficient of an element in terms of the given base (see Sect. 9).

11. Similarities of S. Invariance under similarities is at the heart of dimensional analysis; see Section 22.

DEFINITION 11A. A *similarity* ϕ of the quantity structure S is an isomorphism of S onto itself which preserves type:

$$[\phi(x)] = [x]. \tag{11.1}$$

THEOREM 11B. *A homomorphism of S into itself satisfying* (11.1) *is a similarity. Also if $x \in S^-$ then $\phi(x) \in S^-$.*

If $x \in S^-$, then $-x \in S^+$, hence $\phi(-x) \in S^+$, by (H_1), and taking $a = -1$ in (10.1) shows that $\phi(x) = -\phi(-x) \in S^-$. Now suppose $\phi(x) = \phi(y) = u$. If $u \neq 0$, then $x, y \neq 0$, by (10.1), and $[x] = [\phi(x)] = [\phi(y)] = [y]$, so that $x = ay$ for some a; now

$$\phi(x) = \phi(ay) = a\phi(y) = a\phi(x), \qquad a = 1, \qquad x = y.$$

If $u = 0$, then $x = y = 0$ by what is already proved. Thus ϕ is an isomorphism. Finally, given x, $\phi(x) = ax$ for some $a > 0$; now $\phi((1/a)x) = x$, and ϕ is onto S.

THEOREM 11C. *Let $\{x_1, \cdots, x_n\}$ be a base for S. Then there is a one-one correspondence between ordered sets $\lambda = (\lambda_1, \cdots, \lambda_n)$ of positive numbers and similarities ϕ_λ of S, such that*

$$\phi_\lambda(x_i) = \lambda_i x_i \qquad (i = 1, \cdots, n); \tag{11.2}$$

we have

$$\phi_\lambda(x) = \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} x \quad \textit{if } x \in [x_1^{\alpha_1} \cdots x_n^{\alpha_n}]. \tag{11.3}$$

First, given λ, $\{\lambda_1 x_1, \cdots, \lambda_n x_n\}$ is a base, and by Theorem 10C, the isomorphism ϕ_λ is determined. If $x = ax_1^{\alpha_1} \cdots x_n^{\alpha_n}$, then (10.2) gives

$$\phi_\lambda(x) = a(\lambda_1 x_1)^{\alpha_1} \cdots (\lambda_n x_n)^{\alpha_n},$$

and (11.3) follows. This shows that (11.1) holds; thus ϕ is a similarity. Clearly changing the λ_i changes the similarity. Finally, given any similarity ϕ, $\phi(x_i) \in [x_i] \cap S^+$, hence $\phi(x_i) = \lambda_i x_i$ for some $\lambda_i > 0$ (all i), and $\phi = \phi_\lambda$.

We now consider similarities which keep the elements of a substructure S' fixed.

DEFINITION 11D. A *substructure* S' of S is a subset of S containing R, which is closed under the operations of S. That is, if $x, y \in S'$, $a \in$ R, and $\alpha \in$ R*, then xy, ax and x^α are in S'. Thus S' is a quantity structure. By definition, $S'^+ = S' \cap S^+$.

If S' is a substructure of S, then $[S']$ is a subspace of $[S]$, and we may form the factor space $[S]$ mod $[S']$. Each $x \neq 0$ in S determines a biray $[x]$ in S, i.e. an element of $[S]$, and this has a coset

$$[x][S'] = \{[x]\Gamma : \Gamma \in [S']\} = \{[xy] : y \in S'^+\} \subset [S],$$

which is an element of $[S]$ mod $[S']$. (An attempt to define S mod S' brings in complications, since we must include R in a structure.)

Note that the minimal substructure of S is R; also $[S]$ mod $[\mathrm{R}]$ is just $[S]$ itself.

DEFINITION 11E. For any set $K = \{x_1, \cdots, x_n\}$ in S^+, set $K_{S'} = \{[x_1][S'], \cdots, [x_n][S']\}$ in $[S]$ mod $[S']$. We say K is *independent* mod S' if $K_{S'}$ is independent; K *generates* S mod S' if $K_{S'}$ generates $[S]$ mod $[S']$; and K is a *base* for S mod S' if $K_{S'}$ is a base for $[S]$ mod $[S']$.

THEOREM 11F. *Let S' be a substructure of S, and let $\{x_1, \cdots, x_m\}$ be a base for S mod S'. Then there is a one-one correspondence between ordered sets $\lambda = (\lambda_1, \cdots, \lambda_m)$ of positive numbers and similarities ϕ_λ of S which keep all elements of S' fixed, such that* (11.2) *holds. We have*

$$(11.4) \qquad \phi_\lambda(x) = \lambda_1^{\alpha_1} \cdots \lambda_m^{\alpha_m} x \quad \text{if } x \in [x_1^{\alpha_1} \cdots x_m^{\alpha_m} x'] \quad \text{for some } x' \in S'.$$

Choose a base $\{y_1, \cdots, y_k\}$ for S'; then clearly the x_i and y_j form a base for S. Applying Theorem 11C to this base gives the present theorem.

12. Construction of S from a set of base elements. Suppose objects $x_1, \cdots, x_n$ are given; we wish to construct a quantity structure S from them, so that they will form a base for S. The elements of S will be the expressions

$$(12.1) \qquad a x_1^{\alpha_1} \cdots x_n^{\alpha_n} \qquad (a \in \mathrm{R}; \alpha_1, \cdots, \alpha_n \in \mathrm{R}^*),$$

with the identification

$$(12.2) \qquad 0 x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0 x_1^0 \cdots x_n^0 \qquad (\text{all } \alpha_1, \cdots, \alpha_n \in \mathrm{R}^*).$$

Certain of these elements we consider to be new names for elements of R:

$$ax_1^0 \cdots x_n^0 = a \qquad (a \in \mathrm{R}). \tag{12.3}$$

The elements of S^+ are those in (12.1) with $a>0$. We define multiplication and powers by

$$(ax_1^{\alpha_1} \cdots x_n^{\alpha_n})(bx_1^{\beta_1} \cdots x_n^{\beta_n}) = (ab)x_1^{\alpha_1+\beta_1} \cdots x_n^{\alpha_n+\beta_n}, \tag{12.4}$$

$$(ax_1^{\alpha_1} \cdots x_n^{\alpha_n})^{\gamma} = a^{\gamma} x_1^{\alpha_1\gamma} \cdots x_n^{\alpha_n\gamma} \qquad (a > 0). \tag{12.5}$$

Note that if either element on the left of (12.4) has several names, then $a=0$ or $b=0$, hence $ab=0$, and the right hand side is well defined.

We now show that S is a quantity structure. Note first that

$$(bx_1^0 \cdots x_n^0)(ax_1^{\alpha_1} \cdots x_n^{\alpha_n}) = (ba)x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \tag{12.6}$$

Taking the α_i to be 0, (Q_1) follows. Using $b=1$ and $b=0$ in (12.6) gives (Q_2). Associativity and commutativity of multiplication in S follows directly from the same properties of multiplication in R and of addition in R*. (Q_4) follows from the definition of S^+ and (12.3), and (Q_5) is proved with the help of (12.6), using $b=-1$. (Q_6) is clear.

Taking the α_i to be 0 in (12.5) gives (Q_7). Since (for $a>0$) $a^1=a$, $\alpha 1=\alpha$, and $a^0=1$, $\alpha 0=0$, (Q_8) holds. The proof of (Q_9) is straightforward. For instance, representing xy by (12.4), we have

$$(xy)^{\gamma} = (abx_1^{\alpha_1+\beta_1} \cdots x_n^{\alpha_n+\beta_n})^{\gamma} = (ab)^{\gamma} x_1^{(\alpha_1+\beta_1)\gamma} \cdots x_n^{(\alpha_n+\beta_n)\gamma},$$
$$x^{\gamma} y^{\gamma} = (a^{\gamma} x_1^{\alpha_1\gamma} \cdots x_n^{\alpha_n\gamma})(b^{\gamma} x_1^{\beta_1\gamma} \cdots x_n^{\beta_n\gamma}) = (a^{\gamma} b^{\gamma})x_1^{\alpha_1\gamma+\beta_1\gamma} \cdots x_n^{\alpha_n\gamma+\beta_n\gamma},$$

and these are equal.

We now introduce further shorthand notations: In (12.1), any x_i with the exponent 0 may be omitted. This agrees with (12.3). Also, if $a=1$ we may omit this symbol (unless all x_i have been omitted). For instance,

$$x_i \text{ means } 1x_1^0 \cdots x_i^1 \cdots x_n^0.$$

Now x_i to the power α is just x_i^{α}, for

$$(1x_1^0 \cdots x_i^1 \cdots x_n^0)^{\alpha} = 1x_1^0 \cdots x_i^{\alpha} \cdots x_n^0.$$

We may now interpret the expressions (12.1) as products of powers. For, taking $n=3$ and x, y, z as base elements for short, we have

$$a(x)^{\alpha}(y)^{\beta}(z)^{\gamma} = (ax^0y^0z^0)(1x^{\alpha}y^0z^0)(1x^0y^{\beta}z^0)(1x^0y^0z^{\gamma})$$
$$= ax^{0+\alpha+0+0} \cdots = ax^{\alpha}y^{\beta}z^{\gamma}.$$

In particular, the x_i are now elements of S. Moreover, comparing (12.1) with (9.1) and Theorem 9D shows that the x_i form a base for S.

13. Construction of S from base rays. Let the rays $\Gamma_1, \cdots, \Gamma_n$ be given (according to Definition 4B); we shall construct the quantity structure S from these. (One could equally well use birays in place of rays.) We could choose $x_i \in \Gamma_i$ and apply the definition of the last section; but we prefer to give an intrinsic definition. It is: The elements of S are the expressions

$$(13.1) \qquad a x_1^{\alpha_1} \cdots x_n^{\alpha_n} \ (a \in \mathrm{R};\ x_i \in \Gamma_i,\ \alpha_i \subset \mathrm{R}^*, \text{ for all } i);$$

we make the following identifications. For any positive numbers $\lambda_1, \cdots, \lambda_n$,

$$(13.2) \qquad a(\lambda_1 x_1)^{\alpha_1} \cdots (\lambda_n x_n)^{\alpha_n} = (a\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}) x_1^{\alpha_1} \cdots x_n^{\alpha_n};$$

we also make the identification (12.2), and introduce the abbreviation (12.3). We let the expression (13.1) be in S^+ iffif $a > 0$, and introduce multiplication and powers by (12.4) and (12.5).

To show that these definitions make sense and result in a quantity structure, we proceed as follows: Choose fixed elements $z_i \in \Gamma_i$, and define S_0 in terms of these, by Section 12. We now interpret the expressions shown above as products of powers in S_0. Say $x_i = c_i z_i$ $(i = 1, \cdots, n)$; then (13.2) becomes

$$a(\lambda_1 c_1 z_1)^{\alpha_1} \cdots (\lambda_n c_n z_n)^{\alpha_n} = (a\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n})(c_1 z_1)^{\alpha_1} \cdots (c_n z_n)^{\alpha_n},$$

which we recognize as a true equation in S_0. Thus each element of S is put into correspondence with a unique element of S_0. Since no identification other than (13.2) and (12.2) have been made, the elements of S may be identified with those of S_0. Now the definitions of positive elements, multiplication and powers agree in S and in S_0; thus S is precisely S_0. Therefore S is a quantity structure; moreover, if the definition of Section 12 is used to construct S, with chosen $x_i \in \Gamma_i$, it is independent of the choice of the x_i.

Since the x_i (in the Γ_i) form a base for S, the Γ_i form a set of base rays for S; equivalently, they form a base for $[S]$.

Here are three particular cases:

(a) No base rays at all: S consists of all $a (a \in \mathrm{R})$; here, $S = \mathrm{R}$.

(b) There is one base ray, say L. The elements of S are the expressions $al^\alpha (a \in \mathrm{R}, l \in L, \alpha \in \mathrm{R}^*)$. For instance, if L represents lengths, then L^2 represents areas, L^3 represents volumes, etc.

(c) There are three base rays: M, L and T. Interpreting these as representing mass, length and time, we have the basic quantity structure in Newtonian mechanics.

14. Structure of S^+. With a fixed base $x_1, \cdots, x_n$ for S, we may write the elements of S^+ in the form

$$(14.1) \qquad e^\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} = (\alpha, \alpha_1, \cdots, \alpha_n) \qquad (\alpha \in \mathrm{R};\ \alpha_1, \cdots, \alpha_n \in \mathrm{R}^*).$$

The operations in S^+ are given by

$$(14.2)\quad (\alpha, \alpha_1, \cdots, \alpha_n)(\beta, \beta_1, \cdots, \beta_n) = (\alpha + \beta, \alpha_1 + \beta_1, \cdots, \alpha_n + \beta_n),$$

$$(14.3)\quad (\alpha, \alpha_1, \cdots, \alpha_n)^\gamma = (\gamma\alpha, \gamma\alpha_1, \cdots, \gamma\alpha_n).$$

Thus, taking R^* to be R, S^+ is isomorphic with the vector space R^{n+1} over R. Note that the isomorphism depends on the choice of a base; the different isomorphisms differ by similarities in S. Also replacing e by some other $e' > 1$ would change the isomorphism.

The rays in S^+ are composed of the elements $(\alpha, \bar{\alpha}_1, \cdots, \bar{\alpha}_n)$ with fixed $\bar{\alpha}_i$. In this form, the definition of addition does not appear naturally; it is introduced through the definition

$$(14.4)\quad (\alpha, \bar{\alpha}_1, \cdots) + (\beta, \bar{\alpha}_1, \cdots) = (\log(e^\alpha + e^\beta), \bar{\alpha}_1, \cdots).$$

15. Algebra in a quantity structure. We shall show how parts of elementary algebra fit into quantity structures; we consider simultaneous linear equations, quadratic equations, and roots of polynomials. *We now allow any symbol to denote elements of S.*

First we introduce division: Recalling that y^{-1} is defined if $y \neq 0$ (see the end of Section 6), we set

$$(15.1)\quad x \div y = x/y = xy^{-1} \qquad (x, y \in S, y \neq 0).$$

Then

$$(15.2)\quad x/y \in [x][y]^{-1}.$$

The usual rules follow (assuming certain elements to be nonzero):

$$(15.3)\quad \frac{x}{y} \cdot \frac{z}{w} = (xy^{-1})(zw^{-1}) = (xz)(yw)^{-1} = \frac{xz}{yw},$$

$$(15.4)\quad \frac{x}{y} \div \frac{z}{w} = (xy^{-1})(zw^{-1})^{-1} = xy^{-1}z^{-1}w = \frac{xw}{yz}.$$

Also, assuming x and y positive if α is not integral, since $(y^\beta)^\gamma = y^{\beta\gamma} = (y^\gamma)^\beta$,

$$(15.5)\quad (x/y)^\alpha = (xy^{-1})^\alpha = x^\alpha(y^{-1})^\alpha = x^\alpha(y^\alpha)^{-1} = x^\alpha/y^\alpha.$$

Let us consider a triple of equations in three unknowns. Suppose we have birays $\Gamma_1, \Gamma_2, \Gamma_3, \Delta_1, \Delta_2, \Delta_3, \Delta$ in S, such that

$$(15.6)\quad \Delta_1\Gamma_1 = \Delta_2\Gamma_2 = \Delta_3\Gamma_3 = \Delta.$$

Then the linear equations

$$(15.7)\quad a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 = c_i \qquad (i = 1, 2, 3)$$

have meaning, if we require

$$(15.8)\quad a_{ij} \in \Delta_j, \qquad x_j \in \Gamma_j, \qquad c_i \in \Delta.$$

If the equations are true, then multiplying them by the elements

$$a_{22}a_{33} - a_{23}a_{32}, \qquad -a_{12}a_{33} + a_{13}a_{32}, \qquad a_{12}a_{23} - a_{13}a_{22}$$

of $\Delta_2\Delta_3$ respectively and adding gives

$$Dx_1 = D_1, \tag{15.9}$$

where

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \in \Delta_1\Delta_2\Delta_3, \qquad D_1 = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix} \in \Delta\Delta_2\Delta_3; \tag{15.10}$$

similarly for x_2 and x_3. Thus if $D \neq 0$, we must have

$$x_1 = D_1/D \in (\Delta\Delta_2\Delta_3)(\Delta_1\Delta_2\Delta_3)^{-1} = \Delta\Delta_1^{-1} = \Gamma_1, \tag{15.11}$$

etc. Conversely, these x_i satisfy (15.7).

Next we consider quadratic equations.

LEMMA 15A. *If $x \neq 0$, then $x^2 \in S^+$, $1/x^2 = x^{-2} \in S^+$.*

This follows from the rule of signs, Section 6.

DEFINITION 15B. For $x \in S^+$, $\sqrt{x} = x^{1/2}$. (Recall that $x^\alpha \in S^+$, by definition, if α is not an integer.)

THEOREM 15C. *For $a \in S^+$, $x^2 = a$ has just two solutions, $\sqrt{a}$ and $-\sqrt{a}$.*

For if $x = \sqrt{a}$ and $y^2 = a$ also, then $y \in S^+$ implies $(y^2)^{1/2} = (x^2)^{1/2}$, $y^1 = x^1$, $y = x$, while if $y \in S^-$, then $-y \in S^+$ and $(-y)^2 = x^2$, and hence $-y = x$. Alternatively,

$$[y] = [y^2]^{1/2} = [x^2]^{1/2} = [x],$$

so that we may write $y^2 - x^2 = (y-x)(y+x)$, and (6.3) gives $y - x = 0$ or $y + x = 0$.

We now solve the general quadratic equation:

$$P(x) = ax^2 + bx + c = 0, \tag{15.12}$$

where

$$a \neq 0, \qquad a \in \Delta, \qquad b \in \Delta\Gamma, \qquad c \in \Delta\Gamma^2. \tag{15.13}$$

THEOREM 15D. *If $D = b^2 - 4ac \in S^-$, then* (15.12) *has no solutions. If $D = 0$, then the only solution is $x = -b/2a \in \Gamma$. If $D \in S^+$, then there are just two solutions, namely,*

$$\frac{1}{2a}(-b + \sqrt{D}), \qquad \frac{1}{2a}(-b - \sqrt{D}) \in \Gamma. \tag{15.14}$$

For

$$\frac{1}{a} P(x) = \left(x + \frac{b}{2a}\right)^2 - \frac{1}{4a^2}(b^2 - 4ac),$$

and the facts follow from Lemma 15A and Theorem 15C.

We end with a factorization theorem.

THEOREM 15E. *Suppose we define*

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n, \tag{15.15}$$

where

$$x \in \Gamma, \qquad a_i \in \Gamma^i \qquad (i = 1, \cdots, n). \tag{15.16}$$

Suppose $x_1 \in \Gamma$ *and* $P(x_1) = 0$. *Then there is a polynomial*

$$Q(x) = x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-1} \qquad (x \in \Gamma, b_i \in \Gamma^i \textit{ for all } i),$$

such that $P(x) = (x - x_1)Q(x)$ $(x \in \Gamma)$.

This follows at once from the equations

$$P(x) - P(x_1) = (x^n - x_1^n) + a_1(x^{n-1} - x_1^{n-1}) + \cdots + a_{n-1}(x - x_1),$$
$$x^i - x_1^i = (x - x_1)(x^{i-1} + x_1 x^{i-2} + \cdots + x_1^{i-2} x + x_1^{i-1}).$$

16. Calculus in a quantity structure. Even more than with algebra, it is commonly felt that calculus in actual fact can only deal with real numbers. This is a mistake; all the ordinary processes can be used in a quantity structure S. We indicate the main features here.

Functions. Let Γ and Δ be birays in S; we consider functions from Γ to Δ. In actual practice, the domain of the function will often be restricted to some subset of Γ. For our purpose, it is not necessary to go into this matter here.

Recall that algebraic operations carry us into new birays; this is of course true of physical quantities in general. There are certain important "elementary functions," the exponential, logarithmic and trigonometric functions for instance; these are from R to R. (See Sect. 19 for an example of the use of the exponential function.)

Absolute values. For any $x \in S$, define $|x|$ to be whichever of x, $-x$ is not in S^-; in particular, $|0| = 0$. By Theorem 15C, $|x| = \sqrt{x^2}$. The rule of signs shows that $|xy| = |x||y|$. Recalling that there is a natural order in any biray, the usual proof gives

$$|x + y| \leqq |x| + |y| \qquad (x \text{ and } y \text{ in the same biray}).$$

It is important to keep in mind that $|x|$ *is an element of* $[x]$, not a real number, in general; in fact, $|x| \in [x]^+$ if $x \neq 0$.

Limits. If $x_1, x_2, \cdots$ is a sequence of points on the biray Γ, we define $\lim x_i$ to be that element x of Γ (if it exists) such that for each $\gamma \in \Gamma^+$ there is an integer

N such that

$$|x - x_i| < \gamma \quad \text{if } i > N.$$

If the limit exists, it is unique. The usual rules are true; for instance: If $x_i \in \Gamma$, $y_i \in \Delta$, and $\lim x_i = x$, $\lim y_i = y$, then $\lim x_i y_i = xy \in \Gamma\Delta$. The usual proofs may be interpreted directly in S.

Given $f: \Gamma \to \Delta$, $\lim f(x)$ as $x \to x_0$ is defined similarly: It is $y \in \Delta$ iffif for each $\delta \in \Delta^+$ there is a $\gamma \in \Gamma^+$ such that

$$|y - f(x)| < \delta \quad \text{if } |x - x_0| < \gamma, \qquad x \neq x_0.$$

Continuous functions are defined as usual. A continuous function in a bounded closed interval of a biray takes on its minimum and maximum values, and all values between.

Differentiation. Let L and T be birays in S, and set $V = LT^{-1} = L/T$. We use this notation to suggest length, time and velocity. Let $f: T \to L$ be given. Then f is *differentiable* at $t_0 \in T$ iffif

$$\lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \in V$$

exists; if so, this limit is the *derivative* of f at t_0. The elementary theorems of differentiation, the differentiation of polynomials (compare Sect. 15), and so on, are found as usual.

Rolle's Theorem and the Theorem of the Mean are proved in the normal manner. As a corollary, if a function has the derivative 0 in an interval, it is constant in the interval. *Expansion in an infinite series* may be used.

Integration. Let $f: T \to V$ be continuous; suppose $VT = L$. We define

$$I(t', t'') = \int_{t'}^{t''} f(t)dt \in L$$

as follows. Given any subdivision $\mathfrak{S}$ of the interval (t', t'') in T, by the points $t_0 = t'$, $t_1, \cdots, t_n = t''$, set

$$F(\mathfrak{S}) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1});$$

then $F(\mathfrak{S}) \in L$. One shows as usual that there is an element $l \in L$ such that for any $\lambda \in L^+$, $|F(\mathfrak{S}) - l| < \lambda$ if the largest interval of $\mathfrak{S}$ is small enough. This l is defined to be the integral.

The "Fundamental Theorem of the calculus" holds, etc.

17. Calculus in 3-space. Here, we must put quantity structures S into relation with vectors in a vector space (the space of translations of 3-space). This is done by letting the *components* of vectors be elements of S. Letting u, v be vectors and p, q be elements of S, we may use the following relations:

$$p(u+v) = pu + pv, \qquad p(qv) = (pq)v, \qquad 1v = v.$$

If the coefficients of two vectors are in the same biray of S, we may add them; in particular,

$$pv + qv = (p+q)v \qquad (p, q \text{ in the biray } \Gamma).$$

We have here a structure that is a generalization of a vector space over a field, and of a module over a ring. (Since we shall not use this structure, we shall not attempt a full definition here.)

We may form tensors, and in particular, differential forms. We give one example.

Consider a *flow of fluid*. At the point P and time t, the velocity $v(P, t)$ and density $\rho(P, t)$ are defined. Suppose σ is a surface, with a given orientation across it; we wish to find a formula for the rate of flow of fluid through σ (in mass per unit time) at the time t. This is

$$I(\sigma, t) = \iint_\sigma \rho(P, t)v(P, t)\cdot d\sigma.$$

Note that, with the birays M, L and T of mass, length and time, we have, for coefficients,

$$\rho \in ML^{-3}, \qquad v \in LT^{-1}, \qquad d\sigma \in L^2,$$
$$I(\sigma, t) \in (ML^{-3})(LT^{-1})L^2 = MT^{-1}.$$

We remark that in relativity theory, going over to 4-dimensional space-time, one loses the distinction between the birays L and T. Still, an observer (under no acceleration) may observe the velocity of light and obtain the relation (2.1); he distinguishes L and T here.

If one wishes to put Maxwell's equations into intrinsic form, using quantity structures, there are other problems. For instance, one commonly differentiates between E and H, one being "axial" and the other "polar." With proper interpretations, this is not necessary. See, in this connection, Flanders, [**3**].

CHAPTER II. APPLICATIONS

In the first sections of this chapter we give some familiar examples of equations involving geometric or physical quantities. They may be read mostly without reference to Chapter I. When interpreted in terms of quantity structures, their meaning is clear and precise.

18. Some elementary problems. Here we use only simple algebra in a quantity structure.

Areas of rectangles. If we know that the area of a rectangle is the product of the lengths of the sides, then if these lengths are 2 ft and 5 ft respectively, the area is

$$A = 2\text{ ft} \times 5\text{ ft} = 2 \times 5 \times \text{ft} \times \text{ft} = 10\text{ ft}^2.$$

If we know the area is 10 ft^2 and one side is 2 ft, then the other side is

$$l = 10\text{ ft}^2 \div 2\text{ ft} = 10 \times 2^{-1} \times \text{ft}^{2-1} = 5\text{ ft}.$$

Velocities. John ran a quarter of a mile in two and a quarter minutes. What was his average speed, in ft per sec? Knowing that this is, by definition, the distance divided by the time, we have

$$v = (\tfrac{1}{4}\text{ mile})(2\tfrac{1}{4}\text{ min})^{-1} = (\tfrac{1}{4} \times 5280\text{ ft})(2\tfrac{1}{4} \times 60\text{ sec})^{-1}$$
$$= \tfrac{88}{9}\text{ ft sec}^{-1} = 9.78\text{ ft sec}^{-1}\text{ approximately.}$$

Here, we are working in a quantity structure constructed from the rays L for length and T for time; the speed lies in the ray LT^{-1}.

Flight of the ball. A ball is thrown up with the velocity 40 ft per sec. When is it 20 ft. high? We neglect friction of the air, and assume it is acted on by gravity only. Then, from elementary physics, we must solve

$$h = v_0 t - \tfrac{1}{2}gt^2 = 20\text{ ft},$$

with $v_0 = 40$ ft sec^{-1}, $g = 32$ ft sec^{-2}. Inserting gives

$$(40\text{ ft sec}^{-1})t - (16\text{ ft sec}^{-2})t^2 = 20\text{ ft},$$

or multiplying by $\text{sec}^2/4$ ft,

$$4t^2 - 10\text{ sec } t + 5\text{ sec}^2 = 0.$$

The discriminant is (see Theorem 15D) $(10\text{ sec})^2 - 4\times4\times5\text{ sec}^2 = 20\text{ sec}^2$, which is in $T^2 \cap S^+$; hence, by Theorem 15D, there are two solutions (given by the usual formula):

$$t = \frac{5-\sqrt{5}}{4}\text{ sec} \quad\text{or}\quad t = \frac{5+\sqrt{5}}{4}\text{ sec}.$$

19. Some problems using calculus. For a *volume* problem, see Section 20.

A *pendulum* consists of a point mass m at the end of a string of length l. Using the arc length s and the angle $\theta = s/l$, the component of gravity along s is $-g\sin\theta$, or, keeping the amplitude small, $-g\theta$. By Newton's law of motion, $md^2s/dt^2 = -mg\theta$, or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

Note that

$$t \in T; \qquad l, s \in L; \qquad \theta \in \mathrm{R}; \qquad g \in LT^{-2};$$

and (see Sect. 16) the equation to be solved has its terms on the biray T^{-2}. The solution is

$$\theta = \alpha \sin(\sqrt{g/l}\, t - \beta),$$

with unknown constants α, $\beta \in \mathrm{R}$. We have $\sqrt{g/l}\, t \in \mathrm{R}$.

Consider the *growth of bacteria*, according to the law

$$\frac{du}{dt} = ku.$$

Here we use a quantity structure constructed from T and a ray Γ of quantities of bacteria. Since $du/dt \in \Gamma T^{-1}$, we must have $k \in T^{-1}$. The solution is

$$u = u_0 e^{kt}; \qquad u, u_0 \in \Gamma, \qquad kt \in \mathrm{R}.$$

20. Proportionality. The volume V of a cylinder is proportional to the height h and to the square of the radius r of the base; it follows that $V = chr^2$ for some constant c. We give here a general theorem of this nature, first without using a quantity structure. We consider three independent variables for simplicity.

THEOREM 20A. *Let* Γ, Γ_1, Γ_2, Γ_3 *be rays, and let*

$$f(x_1, x_2, x_3) \qquad (x_i \in \Gamma_i;\ \textit{values of } f \textit{ in } \Gamma)$$

be a given function. Suppose there are numbers α_1, α_2, α_3 *such that, for any* $\lambda > 0$,

$$f(\lambda x_1, x_2, x_3) = \lambda^{\alpha_1} f(x_1, x_2, x_3), \tag{20.1}$$

and similarly for the second and third variables. Then

$$f(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} f(x_1, x_2, x_3). \tag{20.2}$$

For

$$\begin{aligned} f(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) &= \lambda_1^{\alpha_1} f(x_1, \lambda_2 x_2, \lambda_3 x_3) = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} f(x_1, x_2, \lambda_3 x_3) \\ &= \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3} f(x_1, x_2, x_3). \end{aligned}$$

REMARK 20B. We can weaken the hypothesis in (20.1). We show, in fact, that if (for all $\lambda \in \mathrm{R}^+$ and all x in a given ray)

$$f(\lambda x) = \phi(\lambda) f(x), \qquad \phi \text{ continuous},$$

then $\phi(\lambda) = \lambda^\alpha$, α being defined by $\phi(e) = e^\alpha$. For, first of all,

$$\phi(\lambda\mu) f(x) = f((\lambda\mu)x) = f(\lambda(\mu x)) = \phi(\lambda) f(\mu x) = \phi(\lambda)\phi(\mu) f(x),$$

and hence $\phi(\lambda\mu) = \phi(\lambda)\phi(\mu)$, by (4.3). Next, with n terms in the products,

$$e^\alpha = \phi(e) = \phi(e^{1/n} \cdots e^{1/n}) = \phi(e^{1/n}) \cdots \phi(e^{1/n}),$$

and hence $\phi(e^{1/n}) = (e^\alpha)^{1/n}$. It follows that $\phi(e^{k/n}) = (e^\alpha)^{k/n}$, and since ϕ is continuous, $\phi(e^u) = (e^\alpha)^u = (e^u)^\alpha$, if $u > 0$. Considering $\phi(1 \cdot 1)$ and $\phi(\lambda\lambda^{-1})$ gives the remaining cases.

The result (20.2) is expressed in terms of real variables λ_1, λ_2, λ_3. We now suppose the rays lie in a quantity structure, and express the result directly in this structure. We allow also subsidiary variables, in the form of a parameter u.

THEOREM 20C. *Let* $\Gamma, \Gamma_1, \cdots, \Gamma_r$ *be rays in a quantity structure* S, *let* U *be a set, and let* $f: \Gamma_1 \times \cdots \times \Gamma_r \times U \to \Gamma$ *be given. Suppose there are numbers* $\alpha_1, \cdots, \alpha_r$ *such that for each* $\lambda > 0$,

$$(20.3) \qquad f(x_1, \cdots, \lambda x_i, \cdots, x_r, u) = \lambda^{\alpha_i} f(x_1, \cdots, x_i, \cdots, x_r, u).$$

Then there is a uniquely defined function $F: U \to \Delta = \Gamma\Gamma_1^{-\alpha_1} \cdots \Gamma_r^{-\alpha_r}$ *such that*

$$(20.4) \qquad f(x_1, \cdots, x_r, u) = x_1^{\alpha_1} \cdots x_r^{\alpha_r} F(u).$$

If U *is not used,* $F(u)$ *is replaced by a constant* $c \in \Delta$.

REMARK. In the applications, we often have $\Delta = \mathrm{R}^+$.

To show this, choose $z_i \in \Gamma_i$ for each i, and set

$$(20.5) \qquad F(u) = z_1^{-\alpha_1} \cdots z_r^{-\alpha_r} f(z_1, \cdots, z_r, u).$$

Now given $x_1, \cdots, x_r$ and u, there are positive numbers $\lambda_1, \cdots, \lambda_r$ such that $x_i = \lambda_i z_i$ for each i. Taking out the λ_i from $f(\lambda_1 x_1, \cdots)$ as before gives the result. Uniqueness is clear.

As noted in the case of the cylinder, the theorem does not give the actual constant. Let us find the constant, in the case of the *volume* V *of a pyramid* of base area A and height h; this will illustrate the use of calculus. Let the cross section at the distance h' from the vertex have the area $A(h')$. In the quantity structure constructed from the ray L representing length, we have: $h, h' \in L$; $A, A(h') \in L^2$; $V \in L^3$. From simple geometrical hypotheses, we then have $A(h') = kh'^2$ for some $k \in \mathrm{R}^+$. In particular, $A = kh^2$; hence $k = A/h^2$, and

$$V = \int_0^h A(h')dh' = Ah^{-2}\int_0^h h'^2 dh' = \tfrac{1}{3}Ah^{-2}h'^3 \Big|_0^h = \tfrac{1}{3}Ah.$$

21. Dropping and changing units. In working problems in mechanics, one may choose to express all quantities in terms of grams, centimeters and seconds. In the corresponding quantity structure S, this means the following: With the base gr, cm, sec (see Sect. 9), all quantities may be expressed in the form $a\,\mathrm{gr}^\alpha \mathrm{cm}^\beta \mathrm{sec}^\gamma$; the expression is unique if $a \neq 0$. Now if there are considerable computations to be made, one commonly drops the units, retaining only the coefficient a. At the end, one puts the units back in.

In terms of S, this means the following: By Remark 10E, there is a unique homomorphism ϕ of S into R such that $\phi(\mathrm{gr}) = \phi(\mathrm{cm}) = \phi(\mathrm{sec}) = 1$; the image of any element is its coefficient. Since ϕ is a homomorphism, it preserves the operations of addition, multiplication, and powers of positive elements. When a numerical answer is obtained, i.e. we have the coefficient a of the desired quantity, we simply put the proper units back in (since we are supposed to know the physical dimension of the final quantity).

If we wish to change units, this means that we must find the coefficients in terms of a new base. If the new base elements are of the same physical dimension as the old, i.e. (9.7) holds, the change in the coefficients is given by (9.8); in the general case, we must use (9.6).

In elementary work, it is very doubtful if units should be omitted, especially if units are to be changed. For instance, let us give the answer in the problem of John's race, in Section 18, as follows: "John ran the distance $\frac{1}{4}$ in the time $2\frac{1}{4}$, in terms of miles and minutes; his average speed was $\frac{88}{9}$, in terms of feet and seconds." The student will find it conceptually difficult to understand this, and more so to derive it. How does he remember in which direction the formulas for changing units work? Keeping in the units, all such difficulties disappear.

22. Dimensional analysis. In Newton's laws of motion, there is no particular constant of mass, length or time, or any combination of these appearing. When expressed in terms of a quantity structure S, the laws are invariant under transformations of S which leave all properties of S unchanged; that is, the laws are invariant under similarities of S. A common way of expressing this is: The laws do not depend on the choice of units in which they are expressed. (In S, one needs no units to express the laws.)

Suppose, in a given situation, a quantity q depends on other quantities $p_1, \cdots, p_s$: $q=f(p_1, \cdots, p_s)$, as a direct consequence of the laws of mechanics. Then a similarity ϕ of S, changing each p_i into $\phi(p_i)$, changes q into $\phi(q)$. Knowing this, one derives from the "Pi Theorem" considerable information about the function f. This is the method of "dimensional analysis," of great use in parts of mechanics, especially in hydrodynamics. A classical book on this subject is that of Bridgman, [1]. A more modern treatment, with a very detailed study of applications, will be found in Sedov, [4].

A great deal has been written on dimensional analysis, much of it of a rather vague and even philosophical nature: what a "dimension" really is, etc. Here, the advantages of using a mathematical model are clear; one can see precisely what physical assumptions are used, which require the function considered to be invariant under similarities; then the form of the function deduced is a mathematical consequence.

An important point to note is that there is not one model fixed for all considerations; one may choose a particular quantity structure for a particular application. We illustrate this in later sections.

23. The Pi Theorem. This appellation comes from the use of certain products, namely those in (23.3). We suppose we have a quantity structure S; actually, only the positive part, S^+, will be used.

DEFINITION 23A. Let $\Gamma, \Gamma_1, \cdots, \Gamma_s$ be rays in S^+. The function

$$f: \Gamma_1 \times \cdots \times \Gamma_s \to \Gamma$$

is *homogeneous* if it is invariant under similarities ϕ of S; that is,

$$f(\phi(x_1), \cdots, \phi(x_s)) = \phi(f(x_1, \cdots, x_s)). \tag{23.1}$$

If a substructure S' of S is given, we say f is *homogeneous* mod S' if it is invariant under similarities which keep all elements of S' fixed (see Sect. 11).

Note that "homogeneous mod R" is the same as "homogeneous."

LEMMA 23B. *Let the above f be homogeneous* (*or homogeneous* mod S'). *Then* Γ *is in the substructure generated by* $\Gamma_1, \cdots, \Gamma_s$ (*or by these* Γ_i *and* S').

Suppose not. Then we may choose a base $\{z_1, \cdots, z_n\}$ for S such that $z_1, \cdots, z_{n-1}$ generate a substructure containing the Γ_i and S', and $\Gamma = [z_n]$. By Theorem 11C, there is a similarity ϕ of S such that $\phi(z_i) = z_i$ for $i < n$, and $\phi(z_n) = 2z_n$. Now $\phi(x_i) = x_i$, $x_i \in \Gamma_i$ (all i), and ϕ leaves S' fixed; hence

$$f(x_1, \cdots, x_n) = f(\phi x_1, \cdots, \phi x_n) = \phi f(x_1, \cdots, x_n) = 2f(x_1, \cdots, x_n),$$

a contradiction (since all elements are in S^+).

We now give the Pi Theorem. To allow for subsidiary variables, we suppose a set U given; this may be omitted.

THEOREM 23C. *Let* $\Gamma, \Gamma_1, \cdots, \Gamma_s$ *be rays of* S^+, *let* S' *be a substructure, and suppose the function* $f: \Gamma_1 \times \cdots \times \Gamma_s \times U \to \Gamma$ *is homogeneous* (mod S') *for each fixed* $u \in U$. *Take a maximal subset* $\Gamma_1, \cdots, \Gamma_r$ *of the* Γ_i (*after renumbering*) *which are independent* (mod S'); *then* (*see the lemma*) *for some* Γ' *and* Γ_j' $(j > r)$ *in* S', *we may write*

$$\Gamma = \Gamma_1^{\beta_1} \cdots \Gamma_r^{\beta_r}\Gamma', \qquad \Gamma_j = \Gamma_1^{\alpha_{j1}} \cdots \Gamma_r^{\alpha_{jr}}\Gamma_j' \qquad (j > r). \tag{23.2}$$

Then there is a function $F: \Gamma_{r+1}' \times \cdots \times \Gamma_s' \times U \to \Gamma'$ *with the following property. Given* $x_i \in \Gamma_i$ (*all* i) *and* $u \in U$, *if we set*

$$\rho_j = x_j x_1^{-\alpha_{j1}} \cdots x_r^{-\alpha_{jr}} \in \Gamma_j' \qquad (j > r), \tag{23.3}$$

then

$$f(x_1, \cdots, x_s, u) = x_1^{\beta_1} \cdots x_r^{\beta_r} F(\rho_{r+1}, \cdots, \rho_s, u). \tag{23.4}$$

REMARK 23D. If S' is not used, then the rays Γ', Γ_j' are replaced by R^+ (and hence may be omitted from (23.2)); then the ρ_j and F have real values. If $r = s$, i.e. the Γ_i are independent (mod S'), then $F = F(u)$, and the theorem (without S') reduces to Theorem 20C, as is easily seen; if, moreover, there is no U, then F is replaced by a constant (in Γ' or in R^+).

REMARK 23E. In many applications, the function f will not be defined throughout the rays, but only on certain convex subsets; then the similarities used must be reasonably near the identity. We shall not study this type of question here.

To prove the theorem, choose elements $z_i \in \Gamma_i$ $(i = 1, \cdots, r)$, and set

$$z_j = z_1^{\alpha_{j1}} \cdots z_r^{\alpha_{jr}} \in \Gamma_1^{\alpha_{j1}} \cdots \Gamma_r^{\alpha_{jr}} = \Gamma_j \Gamma_j'^{-1} \qquad (j > r). \tag{23.5}$$

Since $\rho_j z_j \in \Gamma_j$ if $\rho_j \in \Gamma_j'$, we may define F by

$$(23.6)\qquad F(\rho_{r+1}, \cdots, \rho_s, u) = z_1^{-\beta_1} \cdots z_r^{-\beta_r} f(z_1, \cdots, z_r, \rho_{r+1}z_{r+1}, \cdots, \rho_s z_s, u).$$

Now given the x_i and u and hence the ρ_j we may write

$$(23.7)\qquad x_i = \lambda_i z_i, \qquad \lambda_i \in \mathrm{R}^+ \qquad (i \leqq r).$$

We may find a base for S consisting of $z_1, \cdots, z_r$, some elements forming a base for S' (if S' is used), and some further elements perhaps. Applying Theorem 11C shows that there is a similarity ϕ of S such that

$$\phi(x) = x \text{ (all } x \in S'), \qquad \phi(z_i) = \lambda_i z_i = x_i \qquad (i \leqq r).$$

Applying (11.3) to $\rho_j z_j \in \Gamma_j$ gives, using (23.2) and (23.3),

$$\phi(\rho_j z_j) = \lambda_1^{\alpha_{j1}} \cdots \lambda_r^{\alpha_{jr}} \rho_j z_j = \rho_j \lambda_1^{\alpha_{j1}} \cdots \lambda_r^{\alpha_{jr}} z_1^{\alpha_{j1}} \cdots z_r^{\alpha_{jr}}$$
$$= \rho_j x_1^{\alpha_{j1}} \cdots x_r^{\alpha_{jr}} = x_j \qquad (j > r).$$

Therefore, using (11.3) for ϕf in Γ and (23.6),

$$\begin{aligned} f(x_1, \cdots, x_s, u) &= f(\phi z_1, \cdots, \phi(\rho_{r+1}z_{r+1}), \cdots, u) \\ &= \phi f(z_1, \cdots, \rho_{r+1}z_{r+1}, \cdots, u) \\ &= \lambda_1^{\beta_1} \cdots \lambda_r^{\beta_r} z_1^{\beta_1} \cdots z_r^{\beta_r} F(\rho_{r+1}, \cdots, \rho_s, u), \end{aligned}$$

and using (23.7) gives (23.4).

24. An elementary area problem. One may construct different quantity structures to study a single problem; that one in which we have invariance under the largest class of similarities gives the most information. We illustrate this with the problem of finding the area A of a rectangle whose sides are of lengths x_1 and x_2: $A = f(x_1, x_2)$.

First, suppose we have no knowledge about area. Then we use $S = \mathrm{R}$ (thus assigning real numbers to all quantities). In the Pi Theorem, $\Gamma = \Gamma_1 = \Gamma_2 = \mathrm{R}^+$; $r = 0$ (each ray is dependent). The only similarity of S is the identity. Given $x_i \in \Gamma_i$ $(i = 1, 2)$, (23.3) reads $\rho_j = x_j$ $(j = 1, 2)$, and (23.4) gives $f(x_1, x_2) = F(x_1, x_2)$, an unknown function.

Next, try calling the "dimension" of each side L, and the "dimension" of A, L^2. Then $\Gamma_1 = \Gamma_2 = L$, $\Gamma = L^2$, and $r = 1$, $s = 2$. Now (23.3) reads $\rho_2 = x_2 x_1^{-1}$, and (23.4) gives

$$f(x_1, x_2) = x_1^2 F(x_2/x_1).$$

The assumption about similarities, in this case, is that multiplying all lengths by the same constant multiplies the area by that constant squared.

Finally, construct S from the rays Γ_1 and Γ_2, and take A in $\Gamma_1\Gamma_2$; then $r = s = 2$. The assumption about the homogeneity of f may be expressed as follows:

$$f(\lambda x_1, x_2) = \lambda f(x_1, x_2) = f(x_1, \lambda x_2),$$

i.e. the area is proportional to each side separately. Both the Pi Theorem and Theorem 20C give $f(x_1, x_2)=cx_1x_2$.

REMARK 24 A. With the last S above, if we have no knowledge about area, we may set $S'=S$, which allows only the trivial similarity; for the second case, in which Γ_1, and Γ_2 are dependent, we may let S' be generated by $\Gamma_2\,\Gamma_1^{-1}$; for the last case, use $S'=\mathrm{R}$. The same results are obtained.

25. A falling body problem. If a ball is thrown up with the velocity v, when does it land on the ground (neglecting air resistance)? As in Section 18, we may solve $vt-16$ ft $\sec^{-2}t^2=0$ with $t>0$, giving $t=cv$, with $c=\sec^2/16$ ft. (Hence if $v=40$ ft/sec, then $t=2.5$ sec.)

Now let us try dimensional analysis on the problem. We first assume known that the acceleration g of gravity enters in, but that it is a constant:

$$g = 32 \text{ ft sec}^{-2} \in LT^{-2}.$$

Assuming mass does not enter in, we construct S from L and T; we look for $t=f(v)$. Let g generate the substructure S'; we use only similarities keeping S' fixed. With the notations of Section 23, we have $r=s=1$, and

$$\Gamma_1 = [v] = LT^{-1}, \qquad \Gamma = [t] = T = \Gamma_1^{\beta}\Gamma' = (LT^{-1})^{\beta}(LT^{-2})^{\gamma}$$

for some β and γ. This gives $\beta+\gamma=0$, $-\beta-2\gamma=1$, and hence $\beta=1$, $\gamma=-1$. The physical hypothesis is that f is homogeneous mod S'; assuming this, (23.4) gives $t=f(v)=cv$.

Now, travelling to other planets, we find that g is not constant; hence we look for a function $t=f(v, g)$. In this case, $r=s=2$, and there is no S'; we have

$$\Gamma_1 = LT^{-1}, \qquad \Gamma_2 = LT^{-2}, \qquad \Gamma = \Gamma_1^{1}\Gamma_2^{-1};$$

thus the expression for Γ is the same as before, with $\Gamma^{-\frac{1}{2}}$ replacing Γ'. Again assuming homogeneity, (23.4) now gives $t=f(v, g)=cv/g$.

26. Pendulum problems. What is the period P of a pendulum of mass m and length l? We assume that the motion is periodic (an experimental fact), and include the amplitude α as a subsidiary variable. Let us first treat g as constant; we use the quantity structure generated by M, L and T, and let S' be generated by g. We wish $P=f(m, l, \alpha)$; we have $r=s=2$. Now

$$\Gamma_1 = M, \qquad \Gamma_2 = L, \qquad \Gamma = \Gamma_1^{\beta}\Gamma_2^{\beta'}\Gamma'^{\gamma} = M^{\beta}L^{\beta'}(LT^{-2})^{\gamma};$$

hence $\beta=0$, $\beta'=1/2$, $\gamma=-1/2$, and hence, assuming f homogeneous mod S',

$$P = \sqrt{l}\,F(\alpha). \tag{26.1}$$

Now consider g as variable and look for $f(m, l, g, \alpha)$. Here, $\gamma=-1/2$ enters in, and we have $P=\sqrt{l/g}\,F(\alpha)$.

REMARK. The same analysis, and hence the same formula (with a different $F(\alpha)$) applies if we assume the pendulum is a rod of length l.

Now consider a more complex pendulum problem. We assume the pendulum consists of a rod, suspended at an interior point, with the length l below and the length l' above this point; we look for $P=f(l, l', g, \alpha)$ (omitting m). Here, $r=2$, $s=3$, and we take

$$\Gamma_1 = [l] = L, \qquad \Gamma_2 = [g] = LT^{-2}, \qquad \Gamma_3 = [l'] = L,$$

and $\Gamma=\Gamma_1^{1/2}\Gamma_2^{-1/2}$ as before. Now (23.3) becomes $\rho_3=l'/l$, and hence

$$P = \sqrt{l/g}\,F(l'/l, \alpha). \tag{26.2}$$

In place of l and l', we could have used for instance $l+l'$ and l'; the same analysis and formula would have resulted.

Note that if l' is small, and it seems a reasonable assumption that it may therefore be neglected, then it drops out of the above, and the preceding formula results. This illustrates the fact that in the applications, one must choose the model and its use to fit the individual case.

27. Satellites. First consider the motion of a pair of masses m_1 and m_2 about their common center of mass; we assume their distance apart r is constant. We recall the law of gravitation:

$$\text{force} = -\gamma m_1 m_2/r^2.$$

Hence

$$[\gamma] = [\text{force}][r^2][m_1m_2]^{-1} = (MLT^{-2})L^2M^{-2} = M^{-1}L^3T^{-2}. \tag{27.1}$$

We look for the period $P=f(m_1, m_2, r, \gamma)$; we could omit γ, and keep S', generated by $[\gamma]$, constant. Taking m_1, r, γ as independent, we use $\rho=m_2/m_1$, and

$$[P] = T = M^{\beta}L^{\beta'}(M^{-1}L^3T^{-2})^{\beta''},$$

giving $\beta=-1/2$, $\beta'=3/2$, $\beta''=-1/2$. The conclusion is

$$P = \sqrt{\frac{r^3}{m_1\gamma}}\,F\left(\frac{m_2}{m_1}\right).$$

Now assume that m_2 is small compared with m_1; then we may suppose that f is independent of m_2. In the case of a satellite about the earth, not far away from the earth, we should take account of a new variable, the radius of the earth. If we suppose that the force of gravity is the same as if all the mass of the earth were concentrated at the center, and we let r be the distance from the satellite to this center, then we may drop m_2 from the preceding analysis, giving (if γ is omitted) $P=c\sqrt{r^3/m_1}$.

As an application of this, note that if we know the mass and radius of the earth and the period of a satellite about the earth, and measure this period about the moon at a certain distance from the moon, we can deduce the mass of the moon from the above formula.

28. Water flowing from a reservoir. We assume there is a V-shaped opening of angle α through which the water runs out; we wish to find the rate r of flow of water in volume per unit time, as a function of the height h above the bottom of the V. We look for $r=f(h, g, \alpha)$ (we of course could take g as constant). Now

$$[r] = L^3T^{-1}, \qquad [h] = L, \qquad [g] = LT^{-2}, \qquad [r] = [h]^{5/2}[g]^{1/2}.$$

Now assume that water is a "perfect fluid." This means, for us, that f is homogeneous. Then the Pi Theorem gives

$$r = \sqrt{h^5 g}\, F(\alpha).$$

Actually, the formula is fairly accurate; thus the physical assumption is borne out.

29. Torque on a wire. We illustrate here a combination of methods, and the separation of two different kinds of length into different types (physical dimensions). First, consider a very short rod, of base A and length l. With the base fixed, its top is pulled a distance d (sideways) by the force F. It is an obvious physical assumption that F is proportional to A and to d, and inversely proportional to l; hence (Sect. 20)

$$F = \eta A d/l,$$

η being the rigidity of the substance. Take $d \in L$, $A \in L^2$, and $l \in L'$; then

$$[\eta] = [F][Ad]^{-1}[l] = [F]L^{-3}L'.$$

Now consider a rod (of the same substance) of radius r and length l, twisted through the (small) angle θ; we wish to find the torque τ needed. Clearly τ is proportional to θ (we could introduce a new dimension $[\theta]$ and obtain this; but this would render the physical assumption less clear); thus we now write

$$\tau = \theta f(r, l, \eta).$$

With $[\tau]=[F]L$, $[\theta]=[1]$, $[r]=L$, $[l]=L'$, and the same η, we have

$$[\tau] = [F]L = L^\alpha L'^\beta([F]L^{-3}L')^\gamma;$$

hence $\gamma=1$, $\beta=-1$, $\alpha=4$, and we obtain

$$\tau = c\eta\, r^4\theta/l.$$

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