

CHAPTER 4

Vector Differential Calculus

The vector differential calculus extends the basic concepts of (ordinary) differential calculus to vector functions, by introducing derivative of a vector function and the new concepts of gradient, divergence and curl.

4.1 Vector Calculus

Definition. A vector function $\vec{r}(t)$ is a function whose domain is a set of real numbers and whose range is a set of vectors. Any vector $\vec{r}(t)$ can be expressed in the component form

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

Where f , g and h are three scalar functions of t .

4.1.1 Limit

The way we define limits of vector-valued functions is similar to the way we define limits of real-valued functions.

Definition. Let $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ be a vector function and \mathbf{L} is a vector. We say \vec{r} has limit \mathbf{L} as t approach and write

$$\lim_{t \rightarrow t_o} \vec{r}(t) = \mathbf{L}$$

The equation

$$\lim_{t \rightarrow t_0} \vec{r}(t) = \lim_{t \rightarrow t_0} f(t)\hat{i} + \lim_{t \rightarrow t_0} g(t)\hat{j} + \lim_{t \rightarrow t_0} h(t)\hat{k}$$

provides a practical way to calculate limits of a vector functions.

Example 4.1. Find the limit of the following vector-valued functions at the indicated value of t

1.

$$\lim_{t \rightarrow \frac{\pi}{4}} (\cos(t)\hat{i} + \sin(t)\hat{j} + t\hat{k})$$

Solution.

$$\begin{aligned} \lim_{t \rightarrow \frac{\pi}{4}} \vec{r}(t) &= \lim_{t \rightarrow \frac{\pi}{4}} (\cos(t)\hat{i} + \sin(t)\hat{j} + t\hat{k}) \\ &= \lim_{t \rightarrow \frac{\pi}{4}} \cos(t)\hat{i} + \lim_{t \rightarrow \frac{\pi}{4}} \sin(t)\hat{j} + \lim_{t \rightarrow \frac{\pi}{4}} t\hat{k} \\ &= \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j} + \frac{\pi}{4}\hat{k} \end{aligned}$$

2.

$$\lim_{t \rightarrow 3} \left(\frac{2t-4}{t+1}\hat{i} + \frac{t}{t^2+1}\hat{j} + (4t-3)\hat{k} \right)$$

Solution.

$$\begin{aligned} \lim_{t \rightarrow 3} \vec{r}(t) &= \lim_{t \rightarrow 3} \left(\frac{2t-4}{t+1}\hat{i} + \frac{t}{t^2+1}\hat{j} + (4t-3)\hat{k} \right) \\ &= \lim_{t \rightarrow 3} \frac{2t-4}{t+1}\hat{i} + \lim_{t \rightarrow 3} \frac{t}{t^2+1}\hat{j} + \lim_{t \rightarrow 3} (4t-3)\hat{k} \\ &= \frac{1}{2}\hat{i} + \frac{3}{10}\hat{j} + 9\hat{k} \end{aligned}$$

Exercise. Calculate $\lim_{t \rightarrow -2} r(t)$ for the function

$$r(t) = \sqrt{t^2 - 3t - 1}\hat{i} + (4t + 3)\hat{j} + \frac{\sin(t+1)\pi}{2}\hat{k}$$

4.1.2 Vector Differentiation

Differentiating a vector function is a simple extension of differentiating scalar quantities.

Definition. *The derivative of a vector-valued function $\hat{r}(t)$ is*

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

provided the limit exists. If $r'(t)$ exists, then r is differentiable at t . If $r'(t)$ exists for all t in an open interval (a, b) , then r is differentiable over the interval (a, b) .

Some Results on Differentiation

If $\vec{r}(t)$ and $\vec{s}(t)$ are vector valued functions, then

$$1. \frac{d}{dt}\{c\vec{r}\} = c\frac{d\vec{r}}{dt} \text{ where } c \text{ is constant.}$$

$$2. \frac{d}{dt}\{\vec{r} + \vec{s}\} = \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt}$$

$$3. \frac{d}{dt}\{\vec{r}\vec{s}\} = \vec{s}\frac{d\vec{r}}{dt} + \vec{r}\frac{d\vec{s}}{dt}$$

$$4. \frac{d}{dt}\{\vec{r} \times \vec{s}\} = \vec{s}\frac{d\vec{r}}{dt} \times \vec{r}\frac{d\vec{s}}{dt}$$

Example 4.2. *Use the definition to calculate the derivative of the function*

$$\mathbf{r}(t) = (3t + 4)\hat{i} + (t^2 - 4t + 3)\hat{j}$$

Solution.

$$\begin{aligned} r'(t) &= \lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\left[(3(t + \Delta t) + 4)\hat{i} + ((t + \Delta t)^2 - 4(t + \Delta t) + 3)\hat{j} \right] - \left[(3t + 4)\hat{i} + (t^2 - 4t + 3)\hat{j} \right]}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{(3t + 3\Delta t + 4)\hat{i} - (3t + 4)\hat{i} + (t^2 + 2t\Delta t + (\Delta t)^2 - 4t - 4\Delta t + 3)\hat{j} - (t^2 - 4t + 3)\hat{j}}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{(3\Delta t)\hat{i} + (2t\Delta t + (\Delta t)^2 - 4\Delta t)\hat{j}}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} (3\hat{i} + (2t + \Delta t - 4)\hat{j}) \\ &= 3\hat{i} + (2t - 4)\hat{j}. \end{aligned}$$

Velocity:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

Acceleration:

$$a = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Example 4.3. If $\vec{r}(t) = \sin(t)\hat{i} + e^{-t}\hat{j} + 3\hat{k}$, find $\frac{d\vec{r}}{dt}$

Solution. Given $\vec{r}(t) = \sin(t)\hat{i} + e^{-t}\hat{j} + 3\hat{k}$. Hence

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d}{dt} (\sin(t)\hat{i} + e^{-t}\hat{j} + 3\hat{k}) \\ &= \frac{d}{dt}(\sin(t))\hat{i} + \frac{d}{dt}(e^{-t})\hat{j} + \frac{d}{dt}(3)\hat{k} \\ &= \cos(t)\hat{i} - e^{-t}\hat{j}\end{aligned}$$

Example 4.4. A particle moves along the curve

$$f(t) = 4\cos(t), \quad g(t) = 4\sin(t), \quad h(t) = 6t$$

Find the velocity and acceleration at time $t = 0$.

Solution.

First we can write the position as a vector \vec{r}

$$\vec{r}(t) = 4\cos(t)\hat{i} + 4\sin(t)\hat{j} + 6t\hat{k}$$

Then,

$$\vec{v} = \frac{d\vec{r}}{dt} = -4\sin(t)\hat{i} + 4\cos(t)\hat{j} + 6\hat{k}$$

at $t=0$, $\vec{v} = 4\hat{j} + 6\hat{k}$

$$a = \frac{d^2\vec{r}}{dt^2} = -4\cos(t)\hat{i} - 4\sin(t)\hat{j}$$

at $t=0$, $a = -4\hat{i}$

4.1.3 Integrals of Vector-Valued Functions

Definition. Let f , g , and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The indefinite integral of a vector-valued function $r(t) = f(t)\hat{i} + g(t)\hat{j}$ is

$$\int [f(t)\hat{i} + g(t)\hat{j}] dt = \left[\int f(t) dt \right] \hat{i} + \left[\int g(t) dt \right] \hat{j}.$$

The definite integral of a vector-valued function is

$$\int_a^b [f(t)\hat{i} + g(t)\hat{j}] dt = \left[\int_a^b f(t) dt \right] \hat{i} + \left[\int_a^b g(t) dt \right] \hat{j}.$$

2. The indefinite integral of a vector-valued function

$$r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

is

$$\int [f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}] dt = \left[\int f(t) dt \right] \hat{i} + \left[\int g(t) dt \right] \hat{j} + \left[\int h(t) dt \right] \hat{k}.$$

The definite integral of a vector-valued function is

$$\int_a^b [f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}] dt = \left[\int_a^b f(t) dt \right] \hat{i} + \left[\int_a^b g(t) dt \right] \hat{j} + \left[\int_a^b h(t) dt \right] \hat{k}.$$

Example 4.5. Calculate each of the following integrals:

- 1.

$$\int [(3t^2 + 2t)\hat{i} + (3t - 6)\hat{j} + (6t^3 + 5t^2 - 4)\hat{k}] dt$$

Solution.

$$\begin{aligned} & \int [(3t^2 + 2t)\hat{i} + (3t - 6)\hat{j} + (6t^3 + 5t^2 - 4)\hat{k}] dt \\ &= \int [(3t^2 + 2t)dt] \hat{i} + \int [(3t - 6)dt] \hat{j} + \int [(6t^3 + 5t^2 - 4)dt] \hat{k} \\ &= (t^3 + t^2)\hat{i} + \left(\frac{3}{2}t^2 - 6t\right)\hat{j} + \left(\frac{3}{2}t^4 + \frac{5}{3}t^3 - 4t\right)\hat{k} + C \end{aligned}$$

2.

$$\int_0^{\frac{\pi}{3}} \left[\sin 2t \hat{i} + \tan t \hat{j} + e^{-2t} \hat{k} \right] dt$$

Solution.

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} \left[\sin 2t \hat{i} + \tan t \hat{j} + e^{-2t} \hat{k} \right] dt \\ &= \int_0^{\frac{\pi}{3}} [\sin 2t dt] \hat{i} + \int_0^{\frac{\pi}{3}} [\tan t dt] \hat{j} + \int_0^{\frac{\pi}{3}} [e^{-2t} dt] \hat{k} \\ &= \left[-\frac{1}{2} \cos t \right]_0^{\frac{\pi}{3}} \hat{i} + [\ln(\cos t)]_0^{\frac{\pi}{3}} \hat{j} + \left[-\frac{1}{2} e^{-2t} \right]_0^{\frac{\pi}{3}} \hat{k} \\ &= \frac{3}{4} \hat{i} + \ln 2 \hat{j} + \left(\frac{1}{2} - \frac{1}{2} e^{-\frac{2\pi}{3}} \right) \hat{k} \end{aligned}$$

4.2 Curves and Their Length

Definition. A space **curve** is a range of a continuous vector - valued function on an interval of real numbers.

When a particle moves through space during a time interval I, then the particles' coordinates can be considered as function defined on I. $x = f(t)$, $y = g(t)$, $z = h(t)$

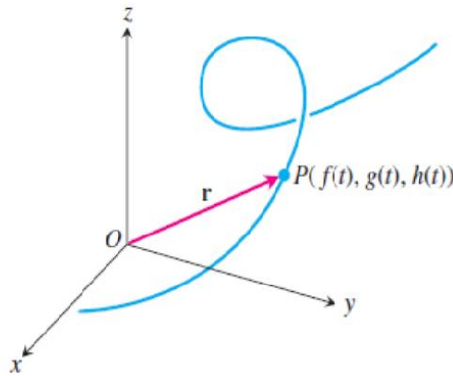


Figure 4.1: A parametric representation of a curve

We denoted a curve by C , and the vector-valued function whose range is the Curve by \vec{r} . Here we say that C is **parameterized** by \vec{r} , or that r is the **parametrization** of C .

Examples of Parameterizations of some Curves:

1. A straight line L through a point A with position vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ in the direction of a constant vector $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ can be represented by in the form

$$\vec{r}(t) = \vec{a} + t\vec{b} = (a_1 + tb_1)\hat{i} + (a_2 + tb_2)\hat{j} + (a_3 + tb_3)\hat{k}$$

2. The vector fuction $\vec{r}(t) = a \cos(t)\hat{i} + a \sin(t)\hat{j} + ct\hat{k}$ is defined for all real values of t . The curve traced by \vec{r} is a **circular helix** (from an old Greek word for "spiral") that winds around the circular cylinder $x^2 + y^2 = 1$ (Fig 2). The curve lies on the cylinder because the \hat{i} and \hat{j} components of \vec{r} , being the x and y coordinates of the tip of \vec{r} , satisfy the cylinder's equation: $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$

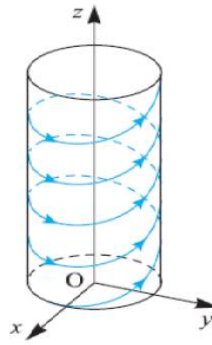


Figure 4.2: circular helix

3. The vector function $\vec{r}(t) = (x_0 + a \cos t)\hat{i} + (y_0 + b \sin t)\hat{j}$ represents an **ellipse** in the xy -plane with center (x_0, y_0) and the principal axis in the direction of the x and y axis. If $a = b$, then $\vec{r}(t) = (x_0 + a \cos t)\hat{i} + (y_0 + a \sin t)\hat{j}$ is an equation of **circle** centered at (x_0, y_0) with radius a .

Definition. A **plane curve** is a curve that lies in a plane. A curve that is not a plane curve is called a **twisted curve**.

Example 4.6. A graph of continuous function of one variable $y = f(x)$ is a plane curve; whereas a circular helix is a twisted curve.

Definition. A curve C is **closed** if and only if it has a parametrization \vec{r} on $[a, b]$ such that $\vec{r}(a) = \vec{r}(b)$. In other words a curve is **closed** on $[a, b]$ iff its initial and terminal points coincide.

Example 4.7. The unit circle $C : \vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$, for $0 \leq t \leq 2\pi$ is closed since $\vec{r}(0) = \vec{r}(2\pi)$ whereas line-segments, cycloids, circular helix, etc are not closed.

Definition. The curve traced by \vec{r} is **smooth** if $\frac{d\vec{r}}{dt}$ is continuous and never 0, i.e if f , g , and h have continuous first derivatives that are not simultaneously 0.

Definition. The curve, C , traced by a continuous vector-valued function \vec{r} defined on an interval I is said to be **piece-wise smooth** iff I is composed of finite number of sub intervals on each of which C is smooth and \vec{r} has one-sided derivative at each interior point of I .

Example 4.8. The circular helix $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + t\hat{k}$ is smooth for $-\infty < t < \infty$ since $\frac{d\vec{r}}{dt} = -\sin(t)\hat{i} + \cos(t)\hat{j} + \hat{k} \neq 0$ for all $t \in \mathbb{R} = (-\infty, \infty)$.

4.2.1 Arc Length of a Curve

One of the special features of smooth space curves is that they have a measurable length.

Definition. Suppose C is a piece-wise smooth curve with parametrization of $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ on $[a, b]$. The length L of C is defined by

$$L = \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| dt$$

Example 4.9.

1. Calculate the arc length for each of the following vector-valued functions:

(a) $\vec{r}(t) = t\hat{i} + \frac{\sqrt{6}t^2}{2}\hat{j} + t^3\hat{k}, \quad \text{for } -1 \leq t \leq 1$

Solution.

The vector function is $\vec{r}(t) = t\hat{i} + \frac{\sqrt{6}t^2}{2}\hat{j} + t^3\hat{k}$.

So

$$\frac{d\vec{r}}{dt} = \hat{i} + \sqrt{6}t\hat{j} + 3t^2\hat{k}$$

Thus the arc length is

$$\begin{aligned} L &= \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| dt \\ &= \int_{-1}^1 \sqrt{1 + 6t^2 + 9t^4} dt \\ &= \int_{-1}^1 (3t^2 + 1) dt \\ &= t^3 + t \Big|_{-1}^1 = 4 \end{aligned}$$

(b) $r(t) = (3t - 2)\hat{i} + (4t + 5)\hat{j}, \quad 1 \leq t \leq 5$

Solution. Using the definition,

$$r'(t) = 3\hat{i} + 4\hat{j}$$

So,

$$\begin{aligned} L &= \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| dt \\ &= \int_1^5 \sqrt{3^2 + 4^2} dt \\ &= \int_1^5 5 dt \\ &= 5t \Big|_1^5 = 20 \end{aligned}$$

2. Find the length of one turn of the helix

$$\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + t\hat{k}$$

Solution.

The helix makes one full turn as L runs from 0 to 2π .

$$\frac{d\vec{r}}{dt} = -\sin(t)\hat{i} + \cos(t)\hat{j} + \hat{k}$$

Using the length formula, the length of this portion of this curve is:

$$\begin{aligned} L &= \int_0^{2\pi} \left\| \frac{d\vec{r}}{dt} \right\| dt \\ &= \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1} dt \\ &= \int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t) + 1} dt \\ &= \int_0^{2\pi} \sqrt{2} dt \\ &= 2\sqrt{2}\pi \text{ units} \end{aligned}$$

Exercise. Find the arc length for each of the following vector-valued functions:

1. $r(t) = \langle t \cos t, t \sin t, 2t \rangle$, $0 \leq t \leq 2\pi$
2. $r(t) = \langle 2t^2 + 1, 2t^2 - 1, t^3 \rangle$, $0 \leq t \leq 3$

Arc-Length Parameterizations

Definition. Let C be a smooth curve parameterized by

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

on an interval I , and let $a \in I$ be fixed number. The **arc length function**, s , is defined by

$$s(t) = \int_a^t \left\| \frac{d\vec{r}}{du} \right\| du = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2} du$$

Note: The arc length parametrization possess some advantages. By the fundamental theorem of calculus, we then have

$$\frac{ds}{dt} = \left\| \frac{d\vec{r}}{dt} \right\|$$

Example 4.10. Find the arc length function for the following vector-valued functions

$$1. \vec{r}(t) = (3 - 3t)\hat{i} + 4t\hat{j}, 0 \leq t \leq 1$$

Solution. First

$$\begin{aligned} \frac{d\vec{r}}{dt} &= -3\hat{i} + 4\hat{j} \\ \left\| \frac{d\vec{r}}{dt} \right\| &= \sqrt{(-3)^2 + (4)^2} = 5 \end{aligned}$$

Then,

$$\begin{aligned} s &= \int_0^t \left\| \frac{d\vec{r}}{du} \right\| du \\ &= \int_0^t 5 du \\ &= 5t \end{aligned}$$

$$2. \vec{r}(t) = 4 \cos t \hat{i} + 4 \sin t \hat{j}, t \geq 0$$

Solution. First

$$\begin{aligned} \frac{d\vec{r}}{dt} &= -4 \sin t \hat{i} + 4 \cos t \hat{j} \\ \left\| \frac{d\vec{r}}{dt} \right\| &= \sqrt{(-4 \sin t)^2 + (4 \cos t)^2} = 4 \end{aligned}$$

Then,

$$\begin{aligned} s(t) &= \int_0^t \left\| \frac{d\vec{r}}{du} \right\| du \\ &= \int_0^t 4 du \\ &= 4t \end{aligned}$$

Example 4.11. *Reparametrize the curves*

1.

$$\vec{r}(t) = \frac{t^2}{2}\hat{i} + \frac{t^3}{3}\hat{k}, \quad 0 \leq t \leq 2$$

Solution.

$$\frac{d\vec{r}}{dt} = t\hat{i} + t^2\hat{k}$$

$$\begin{aligned} s(t) &= \int_0^t \left\| \frac{d\vec{r}}{du} \right\| du \\ &= \int_0^t \sqrt{u^2 + u^4} \\ &= \frac{(t^2 + 1)^{\frac{3}{2}} - 1}{3} \end{aligned}$$

Inverting this produces

$$t = [(3s + 1)^{\frac{2}{3}} - 1]^{\frac{1}{2}}$$

and the new parametrization is

$$\vec{r}(s) = \frac{(3s + 1)^{\frac{2}{3}} - 1}{2}\hat{i} + \frac{[(3s + 1)^{\frac{2}{3}} - 1]^{\frac{3}{2}}}{3}\hat{k}$$

2.

$$\vec{r}(t) = (2 \cos t)\hat{i} + (2 \sin t)\hat{j} \quad 0 \leq t \leq 2\pi$$

Solution. *Exercise!!!*

Note: If we take s the parameter in place of t then the magnitude of the tangent vector, i.e., $\left\| \frac{d\vec{r}}{ds} \right\| = 1$.

4.3 Tangent, Curvature and Torsion

Tangent to a Curve

Definition. Let C be a curve traces by \vec{r} defined on an interval I . If the vector $\frac{d\vec{r}}{dt} \neq 0$, we call $\frac{d\vec{r}}{dt}$ is a **tangent vector** to the space curve C . The corresponding unit vector is **unit tangent vector** denoted by, \vec{T} , is given by

$$\vec{T}(t) = \frac{d\vec{r}/dt}{\|d\vec{r}/dt\|}$$

Example 4.12.

1. Find the unit tangent vector of the helix

$$\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$$

Solution.

The vector equation of the curve is

$$\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k}$$

$$d\vec{r}/dt = -\sin t \hat{i} + \cos t \hat{j} + \hat{k}$$

$$\Rightarrow \|d\vec{r}/dt\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

Then,

$$\begin{aligned} \vec{T}(t) &= \frac{d\vec{r}/dt}{\|d\vec{r}/dt\|} \\ &= \frac{-\sin t}{\sqrt{2}} \hat{i} + \frac{\cos t}{\sqrt{2}} \hat{j} + \frac{1}{\sqrt{2}} \hat{k} \end{aligned}$$

2. Determine the unit tangent vector at the point (2, 4, 7) for the curve with parametric equations

$$x = 2t, \quad y = t^2 + 3, \quad z = 2t^2 + 5$$

Solution.

First we see that the point (2, 4, 7) corresponds to $t=1$.

The vector equation of the curve is

$$\vec{r}(t) = 2t \hat{i} + (t^2 + 3) \hat{j} + (2t^2 + 5) \hat{k}$$

Therefore,

$$d\vec{r}/dt = 2\hat{i} + 2t\hat{j} + 4t\hat{k}$$

and at $t=1$,

$$d\vec{r}/dt = 2\hat{i} + 2\hat{j} + 4\hat{k}$$

Hence

$$\|d\vec{r}/dt\| = \{4 + 4 + 16\}^{\frac{1}{2}} = 2\sqrt{6}$$

Then

$$\begin{aligned}\vec{T}(t) &= \frac{d\vec{r}/dt}{\|d\vec{r}/dt\|} \\ &= \frac{2\hat{i} + 2\hat{j} + 4\hat{k}}{2\sqrt{6}} \\ &= \frac{1}{\sqrt{6}}\{\hat{i} + \hat{j} + 2\hat{k}\}\end{aligned}$$

Exercise.

1. Find the unit tangent vector of the curve

$$\vec{r}(t) = (\cos t + t \sin t)\hat{i} + (\sin t - t \cos t)\hat{j}, \quad t > 0$$

.

2. Find the unit tangent vector at the point $(2, 0, \pi)$ for the curve with parametric equations

$$x = 2 \sin \theta, \quad y = 3 \cos \theta, \quad z = 2\theta$$

Note:

1. **Principal Normal.** If $d\vec{T}/dt$ is not the zero vector, we define a unit vector \mathbf{N} to be $d\vec{T}/dt$ divided by its own magnitude,

$$\mathbf{N} = \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|}$$

This vector is called the *principal normal*.

2. **Binormal.** A third unit vector \mathbf{B} defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, is called the *binormal*. Since \mathbf{T} and \mathbf{N} are unit vectors, \mathbf{B} is also a unit vector perpendicular to both \mathbf{T} and \mathbf{N} .

Example 4.13. Find \mathbf{T} and \mathbf{N} for the circular motion

$$\vec{r}(t) = \cos 2t\hat{i} + \sin 2t\hat{j} \quad (0 \leq t \leq 2\pi)$$

Solution.

To find $\mathbf{T}(t)$ and $\mathbf{N}(t)$, first we must compute $d\vec{r}/dt$ and $\|d\vec{r}/dt\|$. Doing so, we obtain

$$d\vec{r}/dt = -2 \sin 2t\hat{i} + 2 \cos 2t\hat{j}$$

and

$$\|d\vec{r}/dt\| = \sqrt{(-2 \sin 2t)^2 + (2 \cos 2t)^2} = 2$$

so,

$$\begin{aligned} \mathbf{T} &= \frac{d\vec{r}/dt}{\|d\vec{r}/dt\|} \\ &= \frac{-2 \sin 2t\hat{i} + 2 \cos 2t\hat{j}}{2} \\ &= -\sin 2t\hat{i} + \cos 2t\hat{j} \end{aligned}$$

From this we find

$$d\mathbf{T}/dt = -2 \cos 2t\hat{i} - 2 \sin 2t\hat{j}$$

and

$$\|d\mathbf{T}/dt\| = \sqrt{(-2 \cos 2t)^2 + (-2 \sin 2t)^2} = 2$$

So,

$$\begin{aligned} \mathbf{N} &= \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|} \\ &= \frac{-2 \cos 2t\hat{i} - 2 \sin 2t\hat{j}}{2} \\ &= -\cos 2t\hat{i} - \sin 2t\hat{j} \end{aligned}$$

Definition. Let C be a curve that is parameterized by

$$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}.$$

Then,

(1) The **curvature**, κ , is given by

$$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r''(t)\|^3} = \frac{\|T'(t)\|}{\|r'(t)\|}$$

(2) The **torsion**, τ is given by,

$$\tau = \frac{\begin{vmatrix} f' & g' & h' \\ f'' & g'' & h'' \\ f''' & g''' & h''' \end{vmatrix}}{\|r'(t) \times r''(t)\|^2}$$

Example 4.14. Find the curvature and torsion of the curve

$$x = a \cos t, \quad y = a \sin t \quad z = ct$$

.

Solution. The vector equation of the curve is $\vec{r}(t) = a \cos t \hat{i} + a \sin t \hat{j} + ct \hat{k}$.
Therefore,

$$dr/dt = -a \sin t \hat{i} + a \cos t \hat{j} + c \hat{k}$$

and

$$r''(t) = -a \cos t \hat{i} - a \sin t \hat{j}$$

So,

$$r'(t) \times r''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin t & a \cos t & c \\ -a \cos t & -a \sin t & 0 \end{vmatrix} = ac \sin t \hat{i} + ac \cos t \hat{j} + a^2 \hat{k}$$

Then,

$$\|r'(t) \times r''(t)\| = \sqrt{a^2 c^2 + a^4} = a \sqrt{a^2 + c^2}$$

and

$$\|r'(t)\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + c^2} = \sqrt{a^2 + c^2}$$

Therefore,

$$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r''(t)\|^3} = \frac{a \sqrt{a^2 + c^2}}{(\sqrt{a^2 + c^2})^3} = \frac{a}{\sqrt{a^2 + c^2}}$$

Also

$$\begin{aligned}\tau &= \frac{\begin{vmatrix} f' & g' & h' \\ f'' & g'' & h'' \\ f''' & g''' & h''' \end{vmatrix}}{\|r'(t) \times r''(t)\|^2} \\ &= \frac{\begin{vmatrix} -\sin t & a \cos t & c \\ -a \cos t & -a \sin t & 0 \\ a \sin t & -a \cos t & 0 \end{vmatrix}}{a^2(a^2 + c^2)} = \frac{c}{a^2 + c^2}\end{aligned}$$

Exercise

1. Find \mathbf{T} , \mathbf{N} , \mathbf{B} , κ and τ for $\vec{r}(t) = 6 \sin 2t \hat{i} + 6 \cos 2t \hat{j} + 5t \hat{k}$
2. Find the torsion of $C: \vec{r}(t) = \langle t, t^2, t^3 \rangle$
3. Show that the helix $\vec{r}(t) = \langle a \cos t, a \sin t, ct \rangle$ can be represented by $\langle a \cos(s/k), a \sin(s/k), cs/k \rangle$, where $k = \sqrt{a^2 + c^2}$ and s is the arc length. Show that $\tau = c/k^2$.

4.4 Scalar and Vector Fields

1. If to each point P of a region \mathbf{R} in space there corresponds a definite scalar denoted by $f(P)$, then $f(P)$ is called a **scalar point function** in \mathbf{R} . The region \mathbf{R} so defined is called a **scalar field**.

The temperature at any instant, density of a body and potential due to gravitational matter are all examples of scalar point functions.

2. If to each point P of a region \mathbf{R} in space there corresponds a definite vector denoted by $\mathbf{F}(P)$, then it is called the **vector point function** in \mathbf{R} . The region \mathbf{R} so defined is called a **vector field**.

The velocity of a moving fluid at any instant, the gravitational intensity of force are examples of vector point functions.

Definition. A vector field \mathbf{F} in \mathbb{R}^2 is an assignment of a two-dimensional vector $\mathbf{F}(x, y)$ to each point (x, y) of a subset D of \mathbb{R}^2 . The subset D is the domain of the vector field.

A vector field \mathbf{F} in \mathbb{R}^3 is an assignment of a three-dimensional vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) of a subset D of \mathbb{R}^3 . The subset D is the domain of the vector field.

- **Vector Fields in \mathbb{R}^2**

A vector field in \mathbb{R}^2 can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$$

The second way is to use the standard unit vectors:

$$\mathbf{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j}.$$

A vector field is said to be continuous if its component functions are continuous.

- **Vector Fields in \mathbb{R}^3**

We can represent vector fields in \mathbb{R}^3 with component functions. We simply need an extra component function for the extra dimension. We write either

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

or

$$\mathbf{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$$

4.5 Gradients of Scalar Fields

Definition. The vector function ∇f is defined as the gradient of the scalar point function $f(x, y, z)$ and is written as $\text{grad } f$.

$$\text{Thus} \quad \text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

It should be noted that ∇f is a vector whose three components are $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$. Thus, if f is a scalar point function, then ∇f is a vector point function.

Definition. Let f be a differentiable function of x and y and $\vec{u} = u_1\hat{i} + u_2\hat{j}$ be a unit vector. Then the directional derivative of f at (x_0, y_0) in the direction of \vec{u} , denoted by $D_{\vec{u}}f(x, y)$, is defined by

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

provided that this limit exists.

Theorem. Let f be differentiable at (x_0, y_0) . Then f has a directional derivative at (x_0, y_0) in every direction. More over, if $\vec{u} = a_1\hat{i} + a_2\hat{j}$ is a unit vector, then

$$D_{\vec{u}}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)a_1 + \frac{\partial f}{\partial y}(x_0, y_0)a_2$$

Hence the directional derivative in the direction of any unit vector \vec{u} is

$$D_{\vec{u}}f(x, y) = \nabla f \cdot \vec{u}$$

Note: The gradient is not merely a notational device to simplify the formula for the directional derivative; we will see that the length and direction of the gradient ∇f provide important information about the function f and the surface $z = f(x, y)$

Theorem. Let f be a function of two variables that is differentiable at (x, y) . Suppose that $\nabla f \neq 0$ and $D_{\vec{u}}f(x, y) = \nabla f(x, y) \cdot \vec{u} = \|\nabla f(x, y)\| \|\vec{u}\| \cos \theta = \|\nabla f(x, y)\| \cos \theta$, where θ is the angle between $\nabla f(x, y)$ and \vec{u} .

1. The maximum value of $D_{\vec{u}}f(x, y)$ is $\|\nabla f(x, y)\|$, and this maximum occurs when $\theta = 0$, that is, when \vec{u} is in the direction of $\nabla f(x, y)$.
2. The minimum value of $D_{\vec{u}}f(x, y)$ is $-\|\nabla f(x, y)\|$, and this minimum occurs when $\theta = \pi$, that is, when \vec{u} is in the oppositely directed to $\nabla f(x, y)$.

Theorem. If f is differentiable at (x_0, y_0) and if $\nabla f(x_0, y_0) \neq 0$, then $\nabla f(x_0, y_0)$ is normal to the level surface of f through (x_0, y_0) .

Example 4.15. If $f = 3x^2y - y^3z^2$, find $\text{grad } f$ at the point $(1, -2, -1)$.

Solution.

$$\begin{aligned} \text{grad } f = \nabla f &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (3x^2y - y^3z^2) \\ &= \hat{i} \frac{\partial}{\partial x} (3x^2y - y^3z^2) + \hat{j} \frac{\partial}{\partial y} (3x^2y - y^3z^2) + \hat{k} \frac{\partial}{\partial z} (3x^2y - y^3z^2) \\ &= 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\hat{k} \end{aligned}$$

$$\begin{aligned} \text{grad } f(1, -2, -1) &= 6(1)(-2)\hat{i} + (3(1) - 3(4)(1))\hat{j} - 2(-8)(-1)\hat{k} \\ &= -12\hat{i} - 9\hat{j} - 16\hat{k} \end{aligned}$$

Example 4.16. Find the directional derivative of $f = x^y + y^2z + z^2x$ at the point $(1, -1, 2)$ in the direction of the vector $4\hat{i} + 2\hat{j} - 5\hat{k}$.

Solution. Because $f = x^2y + y^2z + z^2x$

$$\nabla f = (2xy + z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + 2xz)\hat{k}$$

At $(1, -1, 2)$,

$$\nabla f = 2\hat{i} - 3\hat{j} + 5\hat{k}$$

$$\begin{aligned}\vec{a} = 4\hat{i} + 2\hat{j} - 5\hat{k} &\implies \|\vec{a}\| = \sqrt{16 + 4 + 25} \\ &\implies 3\sqrt{5}\end{aligned}$$

Let \vec{u} be the unit vector in the given direction.

Then

$$\vec{u} = \frac{4\hat{i} + 2\hat{j} - 5\hat{k}}{3\sqrt{5}}$$

\therefore Directional derivative

$$\begin{aligned}D_{\vec{u}}f(1, -1, 2) &= \nabla f(1, -1, 2) \cdot \vec{u} \\ &= (2\hat{i} - 3\hat{j} + 5\hat{k}) \cdot \frac{1}{3\sqrt{5}}(4\hat{i} + 2\hat{j} - 5\hat{k}) \\ &= \frac{1}{3\sqrt{5}}(8 - 6 - 25) \\ &= \frac{-23}{3\sqrt{5}}\end{aligned}$$

Example 4.17. Let $f(x, y) = x^2e^y$. Find the maximum value of a directional derivative at $(-2, 0)$, and find the unit vector in the direction in which the maximum value occurs.

Solution. Since

$$\nabla f(x, y) = f_x(x, y)\hat{i} + f_y(x, y)\hat{j} = 2xe^y\hat{i} + x^2e^y\hat{j}$$

the gradient at $(-2, 0)$ is

$$\nabla f(-2, 0) = -4\hat{i} + 4\hat{j}$$

By theorem above, the maximum value of the directional derivative is

$$\|\nabla f(-2, 0)\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2, 0)$. The unit vector in this direction is

$$\begin{aligned}\vec{u} &= \frac{1}{4\sqrt{2}}(-4\hat{i} + 4\hat{j}) \\ &= -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\end{aligned}$$

Example 4.18. Find the unit normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

Solution. Since $f(x, y, z) = x^2y + 2xz$, and

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\hat{i} + f_y(x, y, z)\hat{j} + f_z(x, y, z)\hat{k} \\ &= (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k}\end{aligned}$$

the gradient at $(2, -2, 3)$ is

$$\nabla f(2, -2, 3) = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

This vector is normal to the level surface $x^2y + 2xz = 4$. So, a unit vector normal to the surface

$$\begin{aligned}&= \frac{\nabla f(2, -2, 3)}{\|\nabla f(2, -2, 3)\|} \\ &= \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4 + 16 + 16}} \\ &= \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{6} = \frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})\end{aligned}$$

Exercise

1. Find ∇f when $f = (x^2 + y^2 + z^2)e^{-\sqrt{x^2+y^2+z^2}}$
2. Find the directional derivative of $f = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$. In what direction the directional derivative will be maximum and what is its magnitude? Also find a unit normal to the surface $x^2yz + 4xz^2 = 6$ at the point $(1, -2, -1)$.

4.6 Divergence and Curl of Vector Fields

We will now define two important operations on vector fields in 3-space the *divergence* and the *curl* of the field. These names originate in the study of fluid flow, in which case the divergence relates to the way in which fluid flows towards to or away from a point and the curl relates to the rotational properties of the fluid at a point.

Definition. Let $\mathbf{F}(x, y, z) = f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k}$ be a continuously differentiable vector point function. Then

1. The divergence of \mathbf{F} is denoted by $\text{div } \mathbf{F}$ is a scalar function and is defined by the equation

$$\begin{aligned}\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(f(x, y, z)\hat{i} + g(x, y, z)\hat{j} + h(x, y, z)\hat{k} \right) \\ &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\end{aligned}$$

2. The curl of \mathbf{F} is denoted by $\text{curl } \mathbf{F}$ is a vector function and is defined by the equation

$$\begin{aligned}\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \hat{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \hat{k}\end{aligned}$$

Example 4.19. If $\mathbf{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$ then find $\text{div } \mathbf{F}$ and $\text{curl } \mathbf{F}$ at the point $(1, -1, 1)$.

Solution. We have $\mathbf{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial}{\partial x} (xy^2) + \frac{\partial}{\partial y} (2x^2yz) + \frac{\partial}{\partial z} (-3yz^2) \\ &= y^2 + 2x^2z - 6yz\end{aligned}$$

So, $\text{div } \mathbf{F}(1, -1, 1) = (-1)^2 + 2(1)^2(1) - 6(-1)(1) = 9$

Again,

$$\begin{aligned}\text{curl } \mathbf{F} &= \text{curl}[xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}] \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y} (-3yz^2) - \frac{\partial}{\partial z} (2x^2yz) \right) \hat{i} + \left(\frac{\partial}{\partial z} (xy^2) - \frac{\partial}{\partial x} (-3yz^2) \right) \hat{j} + \left(\frac{\partial}{\partial x} (2x^2yz) - \frac{\partial}{\partial y} (xy^2) \right) \hat{k} \\ &= (-3z^2 - 2x^2y)\hat{i} + (0 - 0)\hat{j} + (4xyz - 2xy)\hat{k} \\ &= (-3z^2 - 2x^2y)\hat{i} + (4xyz - 2xy)\hat{k}\end{aligned}$$

So, $\text{curl } \mathbf{F}(1, -1, 1) = -\hat{i} - 2\hat{k}$

Exercise

1. Find the divergence and curl of the vector $(x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$.
2. Find $\text{div } \mathbf{F}$ and $\text{curl } \mathbf{F}$, where $\mathbf{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$.

Definition. Given a vector field \mathbf{F} .

1. \mathbf{F} is said to be **divergence free** or **Solenoidal**, if $\nabla \cdot \mathbf{F} = 0$.
2. \mathbf{F} is said to be **irrotational**, if $\nabla \times \mathbf{F} = 0$

Example 4.20.

1. If vector $\mathbf{F} = 3x\hat{i} + (x + y)\hat{j} - az\hat{k}$ is solenoidal. Find a. (**Ans.** a=4)
2. Find the constants a, b, c so that $\mathbf{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is irrotational. (**Ans.** a=4, b=2, c=-1)

4.7 Physical Interpretation

4.7.1 Physical Interpretation of Divergence

A revealing interpretation of the divergence of a vector field arises from the study of fluid flowing through a region. Suppose \vec{V} represents the velocity field of fluid, for instance air flowing through a surface, such as a screen. Then in physical terms, $\nabla \cdot \vec{V}(x, y, z)$ represents the rate (with respect to time) of mass flow per unit volume of the fluid from the point (x,y,z).

Note.

1. If $\nabla \cdot \vec{V}(x, y, z) > 0$, then the point (x,y,z) is called a **source** of the fluid.
2. If $\nabla \cdot \vec{V}(x, y, z) < 0$, then the point (x,y,z) is called a **sink** of the fluid.
3. If $\nabla \cdot \vec{V}(x, y, z) = 0$, then there are neither sources nor sinks in the region. A fluid whose velocity field is divergence free is called **incompressible**

Example 4.21. Let $\vec{V}(x, y, z) = x^3yz^2\hat{i} + x^2y^2z^2\hat{j} + x^2yz^3\hat{k}$ be the velocity field of a fluid at point (x,y,z). Determine which points in space are sources, sinks? Where the fluid is incompressible?

Ans.

- Source: $\forall_y > 0$ and $\forall_{x,y} \neq 0$;
- Sink: $\forall_y < 0$ and $\forall_{x,y} \neq 0$;
- Incompressible: $\forall_{x,y,z} = 0$

4.7.2 Physical Interpretation of curl

Suppose the $\vec{V}(x, y, z)$ represents the velocity of a fluid flowing through a solid region. Then it turns out that curl of $\vec{V}(x, y, z)$, i.e., $\nabla \times \vec{V}(x, y, z)$ measures the tendency of the fluid to curl or rotate about an axis. More specifically particles in the fluid tend to rotate about an axis in the direction of $\nabla \times \vec{V}(x, y, z)$ and the length of $\nabla \times \vec{V}(x, y, z)$ measures the swiftness of the motion of the particles around the axis.