Tom Mulvey — Homework 4

3/8/18

4.8) Let $x \in \mathbb{Z}$. Prove that if $2|(x^2-5)$ then $4|(x^2-5)$.
Assume $2|(x^2-5)$ is true. Since 2 divides (x^2-5), (x^2-5) is an even integer that can represented as $2k$ for some $k \in \mathbb{Z}$.

Thus $(x^2-5) = 2k$
$x^2 = 2k+5$
$x^2 = 2(k+2)+1$, and since x^2 is odd, x is too by theorem 3.12

Since x is odd, $x = 2p+1$ for some $p \in \mathbb{Z}$
The expression can be rewritten as ...

$(2p+1)^2 - 5 = 4p^2 + 4p + 1 - 4 = 4(p^2+p-1)$.
Since $p^2+p-1 \in \mathbb{Z}$, it shows that x^2-5 is divisble by 4. ■

4.18) Let $m, n \in \mathbb{N}$ and $m|n$. Prove if $a, b \in \mathbb{Z}$ s.t. if $a \equiv b \bmod n$ then $a \equiv b \bmod m$.

Assume $a \equiv b \bmod n$. By the definition of congruence, it can be rewritten as $n|a-b$. By the same means, $a \equiv b \bmod m$ can be written as $m|a-b$

Now let another integer, t, equal $a-b$. $n|a-b$ can now be represented as $n|t$, and $m|a-b$ now equals $m|t$
Recall that $m|n$. We also just stated $n|t$. Using the theorem in 4.1, (if $a|b$ and $b|c$, then $a|c$), the same logic can be applied here. So Since $m|n$ and $n|t$,it follows that $m|t$.

Unsubbing t, we get $m|t = m|a-b$. Now using the definition of congruence, we get $a \equiv b \bmod m$, which is the result wanted. ■

4.40) Let A and B be sets, Prove that $A \cup B = (A-B) \cup (B-A) \cup (A \cap B)$

For the backward direction,

let $x \in (A - B) \cup (B - A) \cup (A \cap B)$ This is logically equivalent to $x \in (A - B) \vee x \in (B - A) \vee x \in (A \cap B)$
These terms can be expressed as..
$(x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B) \vee (x \in A \wedge x \in B)$
The first and third can be simplified. No reason to have two $x \in A$.
$(x \in A \wedge (x \in B \vee x \notin B)\ ) \vee (x \notin A \wedge x \in B)$.
Now it's always true x is either in B or it is not, so that can be omitted.
Thus we get $x \in A \vee (x \notin A \wedge x \in B)$.
Now we have either x is in A, or x is in B and not A. The $x \notin A$ is logically redundant, because if x is not in A, it is in B. The result is $x \in A \vee x \in B$, which can be rewritten as $x \in A \cup B$. This implies that $(A - B) \cup (B - A) \cup (A \cap B) \subseteq A \cup B$.

For the forward direction, we will prove directly. Suppose $x \in A \cup B$.
There are three possibilities.
1) $x \in A$ but $x \notin B$ or
2) $x \notin A$ but $x \in B$ or
3) $x \in A$ and $x \in B$.
Writting in logic, the product is $(x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)$ $\vee\ (x \in A \wedge x \in B)$.
This is equivalent to $x \in (A - B) \vee x \in (B - A) \vee x \in (A \cap B)$
Which is the exact same as $x \in (A - B) \cup (B - A) \cup (A \cap B)$.
$A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$ ■

4.46) Let A and B be sets. Prove $A \cup B = A \cap B \iff A = B$

For the reverse direction, assume $A = B$.
Thus $A \cup B = A \cap B$ is logically equivalent to $A \cup A = A \cap A$. Replacing B with A, from our assumption, gives $A \cup A = A$ and that $A \cap A = A$. The result is A=A, and obivously $A \subseteq A$ and vice versa.

For the forward direction, assume $A \cup B = A \cap B$.
It is the case such that $A \subseteq A \cup B \subseteq A \cap B \subseteq B$.
And similarly with B, $B \subseteq A \cup B \subseteq A \cap B \subseteq A$.
$A \subseteq B$ and $B \subseteq A$. ■