A LITTLE NUMBER THEORY

D. SEABOLD

Definition. A natural number $p > 1$ is *prime* if the only positive divisors of p are 1 and p.

Example. The numbers $2, 3, 5, 7, 11$, and 13 are prime. The numbers $9(= 3 \cdot 3)$ and $10(= 2 \cdot 5)$ are not.

Lemma 1. *Every natural number greater than 1 has a prime divisor.*

Proof. Let $n > 1$ be a natural number. If n is prime, we are done. If not, then n can be factored into a product of two positive divisors other than 1 and n, say $n = n_1 a_1$. If n_1 is prime, then, since it divides n, we are done. If not, then we can factor n_1 into the product $n_2 a_2$. If n_2 is prime, then since n_2 divides n, we done. If not, we can factor n_2 into the product $n_3 a_3$. Continuing in this fashion, we obtain a sequence of smaller and smaller positive numbers n_1, n_2, $n_3, \ldots$ that each divide n. Since any strictly decreasing sequence of positive numbers must be finite, we must finally obtain a number n_i which is prime and which divides n. □

Example. The prime divisors of 65 are 5 and 13. The prime divisors of $180(= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5)$ are 2, 3, and 5.

Theorem 2. *There are infinitely many prime numbers.*

Proof. Suppose to the contrary that there are only finitely prime numbers, $p_1, p_2, p_3, \ldots, p_n$. We will show that this leads to a contradiction. Consider the number $m = p_1 \cdot p_2 \cdot p_3 \cdots p_{n+1}$. Note that p_1 does not divide m since m is one more than a multiple of p_1 (hence the remainder on division by p_1 is nonzero). In fact, for each $i = 1, 2, 3, \ldots, n$, we find that m is one more than a multiple of p_i, so p_i is not a divisor of m. Thus m has no prime divisor. This contradicts Lemma **??**, which states that every integer $m > 1$ has a prime divisor. □

Definition. Two prime numbers are *twin primes* if they differ by 2.

Example. The integers 3 and 5 are twin primes. The integers 11 and 13 are twin primes.

Conjecture 3. *There are infinitely many pairs of twin primes.*