Tom Mulvey — Homework 6

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6.8) Find a formula for $1 + 4 + 7 + ... + (3n - 2)$ for positive integers n and then verify your formula by mathematical induction.

Proof :

Notice that $1 + 4 + 7 + ... + (3n - 2) = \frac{n(3n-1)}{2}$.

We employ induction. Since $1 = \frac{1(3(1)-1)}{2} = \frac{(3-1)}{2} = \frac{2}{2}$, the initial statement holds when n=1. Assume that

$$1 + 4 + 7 + ... + (3k - 2) = \frac{k(3k-1)}{2} \text{ where } k \in \mathbb{Z}+$$

We show that

$$1 + 4 + 7 + ... + (3k - 2) + (3(k + 1) - 2) = \frac{(k+1)(3(k+1)-2)}{2}.$$

Thus $1+4+7+...+(3k-2)+(3(k+1)-2) = \frac{k(3k-1)}{2}+(3(k+1)-2)$.

$\iff \frac{k(3k-1)}{2} + (3(k + 1) - 2) = \frac{k(3k-1)}{2} + (3k + 1)$.

$\iff \frac{k(3k-1)}{2} + (3k + 1) = \frac{k(3k-1)}{2} + \frac{2(3k+1)}{2}$.

$\iff \frac{k(3k-1)}{2} + \frac{2(3k+1)}{2} = \frac{k(3k-1)+2(3k+1)}{2}$.

$\iff \frac{k(3k-1)+2(3k+1)}{2} = \frac{3k^2-k+6k+2}{2}$.

$\iff \frac{3k^2-1k+6k+2}{2} = \frac{3k^2+5k+2}{2}$.

$\iff \frac{3k^2-1k+6k+2}{2} = \frac{(k+1)(3k+2)}{2}$.

$\iff \frac{3k^2-1k+6k+2}{2} = \frac{(k+1)(3(k+1)-1)}{2}$.

By the principle of mathematical induction, $1 + 4 + 7 + ... + (3n - 2) = \frac{n(3n-1)}{2} \forall n \in \mathbb{Z}+$

6.22) Prove that $3^n > n^2$ $\forall n$ s.t. $n \in \mathbb{Z}+$

Proof :

Observe that $3^1 > (1)^2 \Rightarrow 3 > 1$, thus the statement holds when $n = 1$.

We will prove this statement using mathematical induction.

Assume that $3^k > k^2$ where $k \in \mathbb{Z}+$.

We show that $3^{(k+1)} > (k+1)^2$.

$\iff \ 3 * 3^k > (k+1)^2 = k^2 + 2k + 1$.

$\iff \ 3 * 3^k > 2k^2 > k^2 + 2k + 1$ (when $k \geq 3$).

Since the inequality will only hold when $k \geq 3$, I will show two more base cases besides n = 1.

n = 2 $\Rightarrow 3^2 > 2^2 \Rightarrow 9 > 4$.

n = 3 $\Rightarrow 3^3 > 2^3 \Rightarrow 27 > 9$.

Since n = 1, 2, and all n $\geq$ 3 all hold true, then by mathematical induction $3^n > n^2 \ \forall n$ s.t. $n \in \mathbb{Z}+$.

6.26) Prove that $81 | (10^{n+1} - 9n - 10) \ \forall \ n \in \mathbb{N}$.

Proof :

We will employ weak induction. See that when

$n = 0$, $81 | (10^{0+1} - 9(0) - 10)$. $\Rightarrow n = 0$, $81 | 10 - 10) \Rightarrow 81 | 0$.

This is obvious because 81 * 0 = 0.

$81 | (10^{n+1} - 9n - 10) \Rightarrow \exists \ x \ s.t \ 81x = (10^{n+1} - 9n - 10)$.

Solving for 10^{n+1} yields $81x + 9n + 10 = 10^{n+1}$.

We must show that $81 | (10^{k+1+1} - 9(k+1) - 10)$, where $k \in \mathbb{N}$.

$(10^{k+1+1} - 9(k+1) - 10) \iff (10 * 10^{k+1} - 9k - 9 - 10)$.

$(10 * 10^{k+1} - 9n - 9 - 10) \iff (10 * (81x + 9k + 10) - 9n - 19)$.

(recall solving for 10^{k+1})

$(10 * (81x + 9k + 10) - 9k - 19) \iff 810x + 90k + 100 - 9k - 19$.

$810x + 90k + 100 - 9n - 19 \iff 810x + 81k + 81$.

$810x + 81k + 81 \iff 81(10x + k + 1)$,

and since $(10x + k + 1) \in \mathbb{Z}$, 81 divides $(10x + k + 1)$.

By the mathematical induction $81|(10^{n+1} - 9n - 10)$ $\forall n$ s.t. $n \in \mathbb{N}$.

6.34) A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$ and $a_2 = 2$ and $a_n = a_{n-1} + 2 * a_{n-}$ $\forall n \geq 3$. Conjecture a formula for a_n and validate it.

$a_n = 2^{k-1}$.

Proof :

We will utilize strong induction! I First observe that

$a_1 = 2^0 = 1$, and $a_2 = 2^1 = 2$.

This shows that a_n holds when n=1 and n=2.

Assume the formula holds from 1 to k where $k \geq 1$.

$a_i = 2^{i-1}$ $\forall$ $i \in \mathbb{Z}$ where 1≤i≤k.

We will prove that $a_{k+1} = 2^{(k+1)-1} = 2^k$.

Notice each recursion call traces back two steps, so

we will solve when $k \geq 3$. This is fine

since k=1 and k=2 are base cases.

Now $a_{k+1} = 2^{k-1} + 2 * 2^{k-2}$.

$\iff$ $a_{k+1} = 2^{k-1} + 2^{k-2+1} = 2 * 2^{k-1} = 2^k$.

By the mathematical induction .