Tom Mulvey — Homework 5

3/27/18

5.12) Prove that there is no largest negative rational number.

Proof :

Lean towards the contrary that there is a largest negative rational number. Let's call that largest negative number r, and let $r \in \mathbb{Q}$. Since $r \in \mathbb{Q}$, it follows that r can be represented as $\frac{-a}{b}$ where a , $b \in \mathbb{Z}$ (by definition).
Now let's divide $\frac{-a}{b}$ by 2, resulting in $\frac{-a}{2b}$.
Since $-a$,$2b \in \mathbb{Z}$ and $\frac{-a}{b} < \frac{-a}{2b}$, we have reached a contradiction, thus proving there is no largest negative

rational number.

5.20) Let a be an irrational number and r be a nonzero rational number. Prove that if s is a real number, then either ar+s or ar-s is irrational.

Proof :

Lean towards the contrary that either ar+s or ar-.s is rational. There are 3 cases: Both ar+s and ar-.s are rational (Case 1); only ar+s is rational (Case 2); only ar-s is rational (Case 3).

CASE 1:

By definiton, if ar+s and ar-.s are rational, then ar+s and ar-.s can be represent as $\frac{p}{q}$ and $\frac{v}{w}$ respectfully, such that $p, q, v, w \in \mathbb{Z}$.

It MUST follow that:

$$ar + s = \frac{p}{q} \text{ and } ar - s = \frac{v}{w}.$$

$$\iff ar = \frac{p}{q} - s \text{ and } ar = \frac{v}{w} + s.$$

$\iff ar = \frac{p}{q} - \frac{sq}{q}$ and $ar = \frac{v}{w} + \frac{sw}{w}$.

$ar = \frac{p-sq}{q}$ and $ar = \frac{v+sw}{w}$.

$a = \frac{p-sq}{rq}$ and $a = \frac{v+sw}{rw}$.

We have reached a contradiction. Recall in the defintion of a, we claimed that a was irrational, but since $\frac{p-sq}{rq}, \frac{v+sw}{rw} \in \mathbb{Z}$, a is not irrational.

This concludes case 1.

Cases 2 and 3 can be ommitted because the proofs of cases 2 and 3 can be found in Case 1. By following $ar + s$ and $ar - s$ seperately, these prove case 2 and 3 respectfully.

5.22) Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Proof : I will prove this by the contradiction method.

Lemma 1. *$\sqrt{6}$ is irrational.*

For the lemma, lean towards the contrary. Assume $\sqrt{6} \in \mathbb{Q}$, which means $\sqrt{6} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. The square of a rational number is still rational, thus $\frac{a}{b} * \frac{a}{b} = \sqrt{6} * \sqrt{6}$.

$\iff \frac{a^2}{b^2} = 6.$

$\iff a^2 = 6b^2.$

$6b^2 = 2(3(b^2))$, and since $3(b^2) \in \mathbb{Z}$, it follows that a must be even (If the square of an even number is even). Thus $\exists$ c s.t $a = 2c$ where $c \in \mathbb{Z}$. Replacing a with c gives $(2c)^2 = 6b^2$. $\iff 4c^2 = 6b^2$.

$\iff 2c^2 = 3b^2$. Now we see $3b^2$ must be even since $2c^2$ is even.

If both a and b are even, they share atleast a common divisor of 2. We have reached a contradiction (by the definition of rational numbers, a and b must be coprime, which they are not if they share 2 as a factor). This ends the lemma, demonstrating that $\sqrt{6}$ is irrational.

Assume that $\sqrt{2} + \sqrt{3}$ is rational. Since $\sqrt{2} + \sqrt{3} \in \mathbb{Q}$, then the square of $\sqrt{2} + \sqrt{3}$ is $\in \mathbb{Q}$ as well (this is obvious).
Expressing the square of $\sqrt{2} + \sqrt{3}$ yields $(\sqrt{2} + \sqrt{3})^2$.
$\iff$ $2 + \sqrt{2} * \sqrt{3} + \sqrt{2} * \sqrt{3} + 3$.
$\iff$ $5 + 2\sqrt{6}$.
We have reached a contradiction. If $\sqrt{2} + \sqrt{3}$ was rational, the result of the square would be as well. But since $\sqrt{6}$ is irrational, by the lemma, then it is impossible $\sqrt{2} + \sqrt{3}$ is rational. (2 times an irrational number is still irrational).

5.46) Prove there exists a unique real number solution to the equation $x^3 + x^2 - 1 = 0$, let's call this f(x), between $x = \frac{2}{3}$ and $x = 1$.

Proof :

I will prove this using the Intermediate Value Theorem.
Plugging in $\frac{2}{3}$ into the equation, we get :
$\frac{2^3}{3^3} + \frac{2^2}{3^2}$ - 1.
$\iff$ $\frac{8}{27} + \frac{4}{9}$ - 1.
$\iff$ $\frac{8}{27} + \frac{12}{27}$ - 1.
$\iff$ $\frac{20}{27}$ - $\frac{27}{27}$.
$\iff$ $\frac{-7}{27}$.

Now plugging in 1 in the equation, we get :

1^3 + 1^2 -1.

$\iff$ 1 + 1 - 1.

$\iff$ 1.

Because $x^3 + x^2 - 1$ is continous on the interval [$\frac{2}{3}$, 1]
($\frac{d}{dx}[x^3 + x^2 - 1]$=$x^2 + 2x$, which is strictly increasing), and zero is between the outputs of f($\frac{2}{3}$) and f(1), by the I.V.T there must be a real number c s.t. f(c) = 0. This proves a real number solution exists.

Let $r \in$ [$\frac{2}{3}$, 1] and f(r)=0.

It's the case such that : $r^3 + r^2 - 1 = 0 = c^3 + c^2 - 1$.

$\iff r^3 - c^3 + r^2 - c^2 = 0$.

$\iff (r - c)(r^2 + rc + c^2) + (r - c)(r + c) = 0$.

$\iff (r - c)[(r^2 + rc + c^2) + (r + c)] = 0$.

$\iff (r - c) = 0$.

$\iff r = c$.

This proves the real solution is unique.