STA 351: Probability Models and Inference

Module 1: Foundations of Multiple Random Variables
Joint Distributions

Exam-Focused Study Notes

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Contents

Introduction **2**

1 Joint Distributions **2**
- 1.1 Joint PMF (Discrete Case) . . . 2
- 1.2 Joint PDF (Continuous Case) . . . 3
- 1.3 Joint CDF and Properties . . . 4
- 1.4 Independence of Random Variables . . . 5

2 Marginal Distributions **6**
- 2.1 Finding Marginal PMF/PDF . . . 6
- 2.2 Relationship Between Joint and Marginal . . . 8

3 Conditional Distributions **8**
- 3.1 Conditional PMF and PDF . . . 8
- 3.2 Conditional Expectation . . . 9
- 3.3 Law of Iterated Expectation (Tower Property) . . . 10

4 Covariance and Correlation **11**
- 4.1 Definition and Computation of Covariance . . . 11
- 4.2 Correlation Coefficient . . . 12
- 4.3 Uncorrelated vs Independent (KEY DISTINCTION!) . . . 12
- 4.4 Variance of Sums . . . 13

5 Summary Table: Discrete vs Continuous **14**

6 Exam Tips and Common Mistakes **15**

7 Practice Problems **16**
- 7.1 Problem 1: Joint PMF (Table-Based) . . . 16
- 7.2 Problem 2: Finding Marginals and Checking Independence . . . 17
- 7.3 Problem 3: Conditional Expectation . . . 17
- 7.4 Problem 4: Covariance and Independence . . . 17

8 Python Code Snippets (Optional) **17**
- 8.1 Computing Joint PMF from Data . . . 17
- 8.2 Verifying Covariance Formula . . . 18
- 8.3 Simulating to Check Independence . . . 18

Appendix: Quick Reference 19

Introduction

This document provides a comprehensive, exam-focused review of Module 1 for STA 351: Probability Models and Inference. The material covers joint distributions of multiple random variables, including both discrete and continuous cases.

Study Strategy

- **Focus on understanding** the intuition before memorizing formulas
- **Practice problems** are your best preparation
- **Pay attention to warnings** about common mistakes
- **Know the difference** between discrete and continuous cases
- **Memorize key formulas** highlighted in boxes

1 Joint Distributions

1.1 Joint PMF (Discrete Case)

Joint Probability Mass Function (PMF)

For discrete random variables X and Y, the **joint PMF** is defined as:

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

Intuition: Think of the joint PMF as a 2D probability table where each cell (x, y) tells you the probability that X takes value x AND Y takes value y simultaneously.

Properties:

1. **Non-negativity:** $p_{X,Y}(x, y) \geq 0$ for all x, y
2. **Normalization:** $\sum_x \sum_y p_{X,Y}(x, y) = 1$ (sum over all possible values)

Visual Representation:

XY	y_1	y_2	y_3	$p_X(x)$
x_1	$p_{X,Y}(x_1, y_1)$	$p_{X,Y}(x_1, y_2)$	$p_{X,Y}(x_1, y_3)$	$\sum_y p_{X,Y}(x_1, y)$
x_2	$p_{X,Y}(x_2, y_1)$	$p_{X,Y}(x_2, y_2)$	$p_{X,Y}(x_2, y_3)$	$\sum_y p_{X,Y}(x_2, y)$
$p_Y(y)$	$\sum_x p_{X,Y}(x, y_1)$	$\sum_x p_{X,Y}(x, y_2)$	$\sum_x p_{X,Y}(x, y_3)$	1

Example 1.1: Joint PMF

Problem: Two fair dice are rolled. Let X be the value on the first die and Y be the value on the second die. Find $p_{X,Y}(3,4)$.
Solution:

$$\begin{aligned} p_{X,Y}(3,4) &= P(X=3, Y=4) \\ &= P(\text{first die shows 3 AND second die shows 4}) \\ &= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \end{aligned}$$

Since the dice are independent, the joint probability is the product of individual probabilities.

1.2 Joint PDF (Continuous Case)

Joint Probability Density Function (PDF)

For continuous random variables X and Y, the **joint PDF** $f_{X,Y}(x,y)$ satisfies:

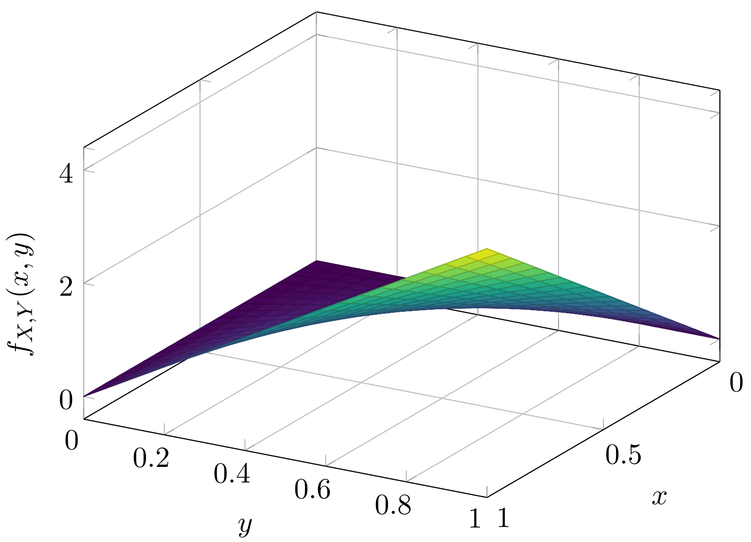
$$P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

Geometric Interpretation: The probability is the **volume under the surface** $z = f_{X,Y}(x,y)$ over the region A in the xy-plane.

Properties:

1. **Non-negativity:** $f_{X,Y}(x,y) \geq 0$ for all x, y
2. **Normalization:** $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

Visualization:



Example 1.2: Uniform Distribution over a Region

Problem: Let (X, Y) be uniformly distributed over the triangular region $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le x\}$. Find the joint PDF.

Solution:

Step 1: Find the area of the region D:

$$\text{Area}(D) = \int_0^1 \int_0^x dy\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

Step 2: For a uniform distribution, the PDF is constant over D:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\text{Area}(D)} = 2 & \text{if } (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

Step 3: Verify normalization:

$$\int_0^1 \int_0^x 2\, dy\, dx = 2 \cdot \frac{1}{2} = 1 \quad \checkmark$$

1.3 Joint CDF and Properties

Joint Cumulative Distribution Function (CDF)

The **joint CDF** is defined as:

$$F_{X,Y}(x, y) = P(X \le x, Y \le y)$$

For discrete: $F_{X,Y}(x, y) = \sum_{s \le x} \sum_{t \le y} p_{X,Y}(s, t)$

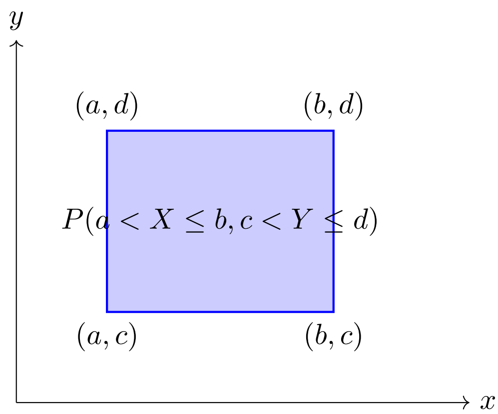
For continuous: $F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s, t)\, dt\, ds$

Key Properties:

1. **Limits at boundaries:**
   - $F_{X,Y}(-\infty, y) = 0$
   - $F_{X,Y}(x, -\infty) = 0$
   - $F_{X,Y}(\infty, \infty) = 1$
2. **Non-decreasing:** $F_{X,Y}$ is non-decreasing in each argument
3. **Right-continuity:** $F_{X,Y}$ is right-continuous in each argument

Rectangle Formula:

$$P(a < X \le b, c < Y \le d) = F_{X,Y}(b, d) - F_{X,Y}(a, d) - F_{X,Y}(b, c) + F_{X,Y}(a, c)$$



1.4 Independence of Random Variables

Independence

Random variables X and Y are **independent** if and only if:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) \quad \text{for all } x, y$$

Or equivalently: $F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$ for all x, y.
Intuition: Knowing the value of X tells you **nothing** about the value of Y.

Test for Independence:

1. Find the marginal distributions $f_X(x)$ and $f_Y(y)$
2. Check if $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ for **all** (x, y)
3. If the equation holds for all (x, y), then X and Y are independent
4. If it fails for even **one** pair (x, y), they are **not** independent

Common Mistake: Independence Test

WARNING: Checking only that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ is **NOT sufficient** to prove independence!
This condition only shows that X and Y are **uncorrelated**, which is a weaker condition than independence.
Remember:

$$\text{Independent} \Rightarrow \text{Uncorrelated}$$
$$\text{Uncorrelated} \not\Rightarrow \text{Independent}$$

You must verify the factorization of the joint distribution!

Example 1.3: Testing Independence

Problem: Consider the following joint PMF for X and Y:

$X\backslash Y$	0	1	$p_X(x)$
0	0.2	0.3	0.5
1	0.3	0.2	0.5
$p_Y(y)$	0.5	0.5	1

Are X and Y independent?
Solution:
Check if $p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)$ for all (x,y):
For $(0,0)$: $p_{X,Y}(0,0) = 0.2$ but $p_X(0) \cdot p_Y(0) = 0.5 \times 0.5 = 0.25$
Since $0.2 \neq 0.25$, the factorization fails.
Conclusion: X and Y are **NOT** independent.

2 Marginal Distributions

2.1 Finding Marginal PMF/PDF

Marginal Distributions

The **marginal distribution** of X is obtained by "summing out" (discrete) or "integrating out" (continuous) the other variable Y.
Discrete Case:

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

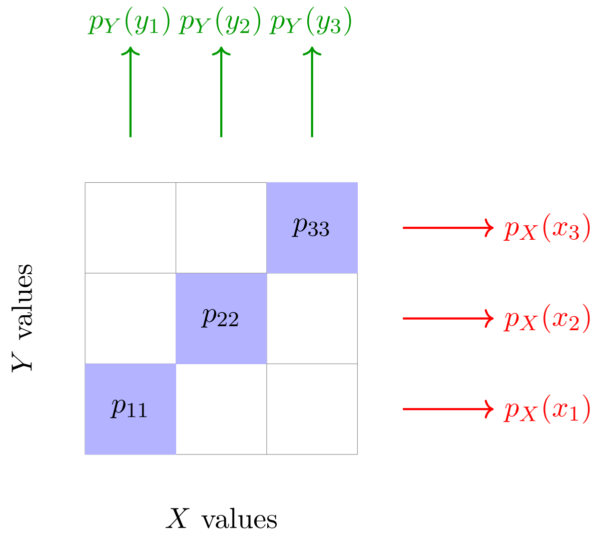
Continuous Case:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Intuition: Marginals are the "shadows" or "projections" of the joint distribution onto each axis.

Visual Interpretation:



Example 1.4: Finding Marginals (Discrete)

Problem: Given the joint PMF:

$X \backslash Y$	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Find the marginal PMFs of X and Y.
Solution:
For X:

$$p_X(1) = 0.1 + 0.1 + 0.2 = 0.4$$
$$p_X(2) = 0.2 + 0.3 + 0.1 = 0.6$$

For Y:

$$p_Y(1) = 0.1 + 0.2 = 0.3$$
$$p_Y(2) = 0.1 + 0.3 = 0.4$$
$$p_Y(3) = 0.2 + 0.1 = 0.3$$

Example 1.5: Finding Marginals (Continuous)

Problem: Let $f_{X,Y}(x, y) = 2$ for $0 \leq y \leq x \leq 1$, and 0 otherwise. Find $f_X(x)$ and $f_Y(y)$.
Solution:
For $f_X(x)$ where $0 \leq x \leq 1$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \int_0^x 2\, dy = 2x$$

For $f_Y(y)$ where $0 \leq y \leq 1$:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx = \int_y^1 2\, dx = 2(1 - y)$$

2.2 Relationship Between Joint and Marginal

Key Insight About Joint and Marginal Distributions

Important Exam Concept:

1. You can **ALWAYS** find marginal distributions from the joint distribution (by summing/integrating)
2. You **CANNOT** recover the joint distribution from the marginals alone (unless the variables are independent)
3. **Exception:** If X and Y are independent, then:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$$

   So marginals completely determine the joint in this special case.

This is a common exam question!

3 Conditional Distributions

3.1 Conditional PMF and PDF

Conditional Distributions

The **conditional distribution** of Y given $X = x$ is:
Discrete Case:

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x, y)}{p_X(x)} \quad \text{for } p_X(x) > 0$$

Continuous Case:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} \quad \text{for } f_X(x) > 0$$

Intuition: The conditional distribution is a "slice" of the joint distribution at a fixed value of X, normalized to be a proper probability distribution.

Connection to Bayes' Theorem:

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$$

Example 1.6: Conditional Distribution

Problem: Using the joint PMF from Example 1.4, find the conditional distribution of Y given $X = 1$.
Solution:
From Example 1.4, we know $p_X(1) = 0.4$.
For each value of Y:

$$p_{Y|X}(1|1) = \frac{p_{X,Y}(1,1)}{p_X(1)} = \frac{0.1}{0.4} = 0.25$$
$$p_{Y|X}(2|1) = \frac{p_{X,Y}(1,2)}{p_X(1)} = \frac{0.1}{0.4} = 0.25$$
$$p_{Y|X}(3|1) = \frac{p_{X,Y}(1,3)}{p_X(1)} = \frac{0.2}{0.4} = 0.50$$

Verify: $0.25 + 0.25 + 0.50 = 1$

3.2 Conditional Expectation

Conditional Expectation

The **conditional expectation** of Y given $X = x$ is:
Discrete:

$$\mathbb{E}[Y|X = x] = \sum_y y \cdot p_{Y|X}(y|x)$$

Continuous:

$$\mathbb{E}[Y|X = x] = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\, dy$$

Note: $\mathbb{E}[Y|X = x]$ is a **function of** x.
We can also write $\mathbb{E}[Y|X]$ as a **random variable** (a function of the random variable X).

Example 1.7: Conditional Expectation

Problem: Find $\mathbb{E}[Y|X = 1]$ using the conditional distribution from Example 1.6.
Solution:

$$\begin{aligned}\mathbb{E}[Y|X = 1] &= \sum_y y \cdot p_{Y|X}(y|1) \\ &= 1(0.25) + 2(0.25) + 3(0.50) \\ &= 0.25 + 0.50 + 1.50 \\ &= 2.25\end{aligned}$$

3.3 Law of Iterated Expectation (Tower Property)

Law of Iterated Expectation

THE KEY FORMULA:

$$\boxed{\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]}$$

In words: The **expectation of** Y equals the **expectation of the conditional expectation of** Y **given** X.
Intuition: You can find the average of Y by:

1. First computing the average of Y for each value of X
2. Then averaging these conditional averages over all values of X

This is also called the **Tower Property**.

Why This is Powerful:
Sometimes computing $\mathbb{E}[Y]$ directly is difficult, but:

- Computing $\mathbb{E}[Y|X = x]$ for each x is easier
- Then we average over X to get $\mathbb{E}[Y]$

Exam Alert: Law of Iterated Expectation

This appears frequently in exam problems!
Common applications:

- Finding expectations of complicated random variables
- Proving properties of expectations
- Working with conditional distributions

Formula to memorize: $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$
Also useful: $\mathbb{E}[g(X,Y)|X] = g(X, \mathbb{E}[Y|X])$ when g is linear in Y.

Example 1.8: Law of Iterated Expectation

Problem: Let $N \sim \text{Poisson}(\lambda)$, and given $N = n$, let $Y \sim \text{Binomial}(n, p)$. Find $\mathbb{E}[Y]$.
Solution:
Using the law of iterated expectation:
Step 1: Find $\mathbb{E}[Y|N]$:

$$\mathbb{E}[Y|N = n] = np \quad \text{(binomial mean)}$$

So $\mathbb{E}[Y|N] = Np$ (as a random variable).
Step 2: Apply the law of iterated expectation:

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y|N]] \\ &= \mathbb{E}[Np] \\ &= p\mathbb{E}[N] \\ &= p\lambda\end{aligned}$$

Answer: $\mathbb{E}[Y] = p\lambda$

4 Covariance and Correlation

4.1 Definition and Computation of Covariance

Covariance

The **covariance** between random variables X and Y is:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$

where $\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$.
Computational Formula:

$$\boxed{\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}$$

This is usually easier to compute!

Properties of Covariance:

1. $\text{Cov}(X, X) = \text{Var}(X)$
2. $\text{Cov}(X, Y) = \text{Cov}(Y, X)$ (symmetric)
3. $\text{Cov}(aX + b, Y) = a\text{Cov}(X, Y)$ (bilinearity)
4. $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$ (additivity)
5. If X and Y are independent, then $\text{Cov}(X, Y) = 0$

Example 1.9: Computing Covariance

Problem: Let X and Y have the following joint PMF:

$X \backslash Y$	0	1
0	0.3	0.2
1	0.2	0.3

Find $\text{Cov}(X, Y)$.
Solution:
Step 1: Find marginals:

$$p_X(0) = 0.5, \quad p_X(1) = 0.5$$
$$p_Y(0) = 0.5, \quad p_Y(1) = 0.5$$

Step 2: Compute expectations:

$$\mathbb{E}[X] = 0(0.5) + 1(0.5) = 0.5$$
$$\mathbb{E}[Y] = 0(0.5) + 1(0.5) = 0.5$$
$$\mathbb{E}[XY] = 0 \cdot 0 \cdot 0.3 + 0 \cdot 1 \cdot 0.2 + 1 \cdot 0 \cdot 0.2 + 1 \cdot 1 \cdot 0.3 = 0.3$$

Step 3: Apply formula:

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.3 - (0.5)(0.5) = 0.3 - 0.25 = 0.05$$

4.2 Correlation Coefficient

Correlation Coefficient

The **correlation coefficient** (or Pearson correlation) between X and Y is:

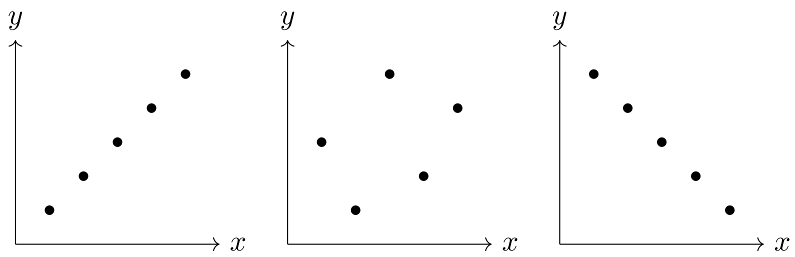
$$\rho(X,Y) = \text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

where $\sigma_X = \sqrt{\text{Var}(X)}$ and $\sigma_Y = \sqrt{\text{Var}(Y)}$.
Interpretation: ρ measures the strength of the **linear relationship** between X and Y.

Properties:

1. $-1 \leq \rho(X,Y) \leq 1$ (always!)
2. $\rho = 1$: Perfect positive linear relationship ($Y = aX + b$ with $a > 0$)
3. $\rho = -1$: Perfect negative linear relationship ($Y = aX + b$ with $a < 0$)
4. $\rho = 0$: No linear relationship (uncorrelated)
5. ρ is **dimensionless** (scale-invariant)

Visual Interpretation:



$\rho = +1$ $\rho = 0$ $\rho = -1$

4.3 Uncorrelated vs Independent (KEY DISTINCTION!)

EXAM FAVORITE: Uncorrelated $\not\Rightarrow$ Independent

CRITICAL DISTINCTION:

	Independent	**Uncorrelated**
Definition	$f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for all x, y	$\text{Cov}(X,Y) = 0$ (or $\rho = 0$)
Implication	Independent $\Rightarrow$ Uncorrelated **Uncorrelated $\not\Rightarrow$ Independent**	

Always true: If X and Y are independent, then they are uncorrelated.
NOT always true: If X and Y are uncorrelated, they may still be dependent!

Example 1.10: Classic Counterexample

Problem: Let X be uniformly distributed on $\{-1, 0, 1\}$, each with probability $1/3$. Define $Y = |X|$. Show that X and Y are uncorrelated but dependent.
Solution:
Step 1: Show they are **dependent**:
The joint distribution is:

$X \backslash Y$	0	1
-1	0	$1/3$
0	$1/3$	0
1	0	$1/3$

Marginals: $p_Y(0) = 1/3$, $p_Y(1) = 2/3$
Check independence: $p_{X,Y}(0, 1) = 0$ but $p_X(0) \cdot p_Y(1) = (1/3)(2/3) = 2/9 \neq 0$
Therefore, X and Y are **dependent**.
Step 2: Show they are **uncorrelated**:
Compute expectations:

$$\begin{aligned}
\mathbb{E}[X] &= (-1)(1/3) + 0(1/3) + 1(1/3) = 0 \\
\mathbb{E}[Y] &= 0(1/3) + 1(2/3) = 2/3 \\
\mathbb{E}[XY] &= (-1)(1)(1/3) + 0(0)(1/3) + 1(1)(1/3) = -1/3 + 1/3 = 0
\end{aligned}$$

Therefore:

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0 - 0 \cdot (2/3) = 0$$

Conclusion: X and Y are **uncorrelated but dependent**!
This is a classic example showing that zero correlation does not imply independence.

4.4 Variance of Sums

Variance of Sums

General Formula:

$$\boxed{\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)}$$

Special Cases:

1. If X and Y are **uncorrelated** (i.e., $\text{Cov}(X, Y) = 0$):
$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$
2. If X and Y are **independent** (which implies uncorrelated):
$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$
3. For n **pairwise uncorrelated** random variables $X_1, \ldots, X_n$:
$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i)$$

Connection to Practice Problems:

This formula is crucial for Problem 4 in the practice set, where you need to find the variance of a sum of exponential random variables.

Example 1.11: Variance of Sum of Exponentials

Problem: Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with rate λ. Find $\mathrm{Var}(X_1 + X_2 + \cdots + X_n)$.
Solution:
Since the X_i are independent, they are uncorrelated, so:

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^n X_i\right) &= \sum_{i=1}^n \mathrm{Var}(X_i) \\ &= \sum_{i=1}^n \frac{1}{\lambda^2} \quad \text{(variance of exponential)} \\ &= \frac{n}{\lambda^2}\end{aligned}$$

Answer: $\mathrm{Var}(X_1 + \cdots + X_n) = n/\lambda^2$

5 Summary Table: Discrete vs Continuous

Concept	**Discrete**	**Continuous**
Joint Distribution	PMF: $p_{X,Y}(x,y)$	PDF: $f_{X,Y}(x,y)$
Normalization	$\sum_x \sum_y p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$
Marginal of X	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$
Marginal of Y	$p_Y(y) = \sum_x p_{X,Y}(x,y)$	$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$
Conditional Dist.	$p_{Y\|X}(y\|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$	$f_{Y\|X}(y\|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$
Independence	$p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for all x, y	$f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x, y
Cond. Expectation	$\mathbb{E}[Y\|X=x] = \sum_y y \cdot p_{Y\|X}(y\|x)$	$\mathbb{E}[Y\|X=x] = \int_{-\infty}^{\infty} y \cdot f_{Y\|X}(y\|x)\,dy$

6 Exam Tips and Common Mistakes

What to Memorize

Key Formulas:

1. Covariance: $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
2. Correlation: $\rho(X, Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$
3. Variance of sum: $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$
4. Law of iterated expectation: $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$
5. Independence test: $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for ALL (x, y)

Common Mistakes to Avoid

1. **Confusing uncorrelated with independent**
   - Independent $\Rightarrow$ Uncorrelated (TRUE)
   - Uncorrelated $\not\Rightarrow$ Independent (FALSE!)
2. **Forgetting to check all (x, y) pairs for independence**
   - Must verify $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for **every** pair
   - If it fails for even one pair, not independent!
3. **Mixing up integration limits for marginals**
   - Pay attention to the support of the joint distribution
   - Don't integrate from $-\infty$ to ∞ if support is bounded!
4. **Not normalizing conditional distributions**
   - Remember: $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$
   - The denominator is crucial for normalization!
5. **Forgetting the $2\text{Cov}(X, Y)$ term in $\text{Var}(X + Y)$**
   - Only drops out when X and Y are uncorrelated
   - Don't assume uncorrelated unless stated or proven!

Problem-Solving Strategies

1. **For joint PMF/PDF problems:**
   - First, identify the support (where the distribution is non-zero)
   - Draw a picture if continuous (sketch the region)
   - Always verify normalization as a check
2. **For finding marginals:**
   - Sum/integrate over the other variable
   - Be careful with limits of integration
   - Verify that marginal integrates/sums to 1
3. **For independence:**
   - Find marginals first
   - Check if joint = product of marginals
   - Only need one counterexample to show dependence
4. **For covariance/correlation:**
   - Use computational formula: $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
   - Compute all expectations separately first
   - Remember: ρ is dimensionless, between -1 and 1
5. **For conditional expectation:**
   - Find conditional distribution first
   - Then compute expectation using conditional PMF/PDF
   - Consider using law of iterated expectation if useful

7 Practice Problems

7.1 Problem 1: Joint PMF (Table-Based)

Problem: Consider the following joint PMF for random variables X and Y:

$X \backslash Y$	1	2	3	4
1	0.05	0.10	0.05	0.10
2	0.10	0.15	0.10	0.05
3	0.05	0.05	0.10	0.10

(a) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.

(b) Find $P(X \leq 2, Y > 2)$.

(c) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Hint: For (a), sum across rows for p_X and down columns for p_Y.

7.2 Problem 2: Finding Marginals and Checking Independence

Problem: Let (X, Y) have joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} 6xy & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that this is a valid PDF (check normalization).

(b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(c) Are X and Y independent? Justify your answer.

Hint: For independence, check if $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all (x, y) in the support.

7.3 Problem 3: Conditional Expectation

Problem: Let X and Y have joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} e^{-x} & 0 < y < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $f_X(x)$ and $f_{Y|X}(y|x)$.

(b) Compute $\mathbb{E}[Y|X = x]$.

(c) Use the law of iterated expectation to find $\mathbb{E}[Y]$.

Hint: For (a), integrate over y from 0 to x to find $f_X(x)$.

7.4 Problem 4: Covariance and Independence

Problem: Let X be uniformly distributed on $\{-1, 0, 1\}$, and define $Y = X^2$.

(a) Find the joint PMF $p_{X,Y}(x, y)$.

(b) Compute $\text{Cov}(X, Y)$.

(c) Are X and Y independent? Are they uncorrelated?

(d) Explain why your answer demonstrates the relationship between independence and correlation.

Hint: This is similar to Example 1.10. Note that Y can only take values 0 and 1.

8 Python Code Snippets (Optional)

8.1 Computing Joint PMF from Data

```
import numpy as np

# Sample data
X = np.array([1, 1, 2, 2, 2, 3, 3, 3, 3])
Y = np.array([1, 2, 1, 2, 3, 2, 3, 3, 3])

# Compute joint PMF
```

```
unique_x = np.unique(X)
unique_y = np.unique(Y)

joint_pmf = np.zeros((len(unique_x), len(unique_y)))

for i, x in enumerate(unique_x):
    for j, y in enumerate(unique_y):
        joint_pmf[i, j] = np.sum((X == x) & (Y == y)) / len(X)

print("Joint PMF:")
print(joint_pmf)
```

8.2 Verifying Covariance Formula

```
import numpy as np

# Generate random data
np.random.seed(42)
X = np.random.normal(0, 1, 1000)
Y = 2*X + np.random.normal(0, 0.5, 1000)

# Method 1: Definition
mean_X = np.mean(X)
mean_Y = np.mean(Y)
cov_def = np.mean((X - mean_X) * (Y - mean_Y))

# Method 2: Computational formula
cov_comp = np.mean(X * Y) - np.mean(X) * np.mean(Y)

# NumPy's covariance
cov_numpy = np.cov(X, Y, bias=True)[0, 1]

print(f"Covariance (definition): {cov_def:.4f}")
print(f"Covariance (computational): {cov_comp:.4f}")
print(f"Covariance (NumPy): {cov_numpy:.4f}")
```

8.3 Simulating to Check Independence

```
import numpy as np
import matplotlib.pyplot as plt

# Independent random variables
np.random.seed(42)
X_indep = np.random.normal(0, 1, 1000)
Y_indep = np.random.normal(0, 1, 1000)

# Dependent random variables
X_dep = np.random.normal(0, 1, 1000)
Y_dep = 2*X_dep + np.random.normal(0, 0.3, 1000)

# Plot
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 5))

ax1.scatter(X_indep, Y_indep, alpha=0.5)
ax1.set_title('Independent: Cov = 0, rho = 0')
```

```
ax1.set_xlabel('X')
ax1.set_ylabel('Y')
ax1.grid(True)

ax2.scatter(X_dep, Y_dep, alpha=0.5)
ax2.set_title('Dependent: Cov != 0, rho != 0')
ax2.set_xlabel('X')
ax2.set_ylabel('Y')
ax2.grid(True)

plt.tight_layout()
plt.savefig('independence_check.png')
print("Plot saved as 'independence_check.png'")
```

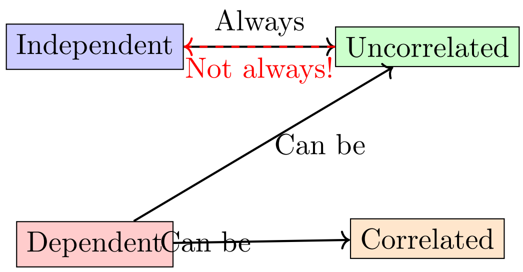
Appendix: Quick Reference

Must-Know Formulas

Essential Formulas for Module 1

1. **Marginal from Joint:**
   - Discrete: $p_X(x) = \sum_y p_{X,Y}(x, y)$
   - Continuous: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
2. **Conditional Distribution:**
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$
3. **Independence:**
$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) \text{ for all } x, y$$
4. **Covariance:**
$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$
5. **Correlation:**
$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$$
6. **Variance of Sum:**
$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$
7. **Law of Iterated Expectation:**
$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$$

Key Relationships



Good luck with your exam!

Remember: Practice problems are the key to success.

Understanding ¿ Memorization