NUMERICAL METHODS

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Preface

Numerical Methods form an integral part of the mathematical background required for students of Mathematics, Science and Engineering.

This book, *Numerical Methods*, is an extension of our long experience of teaching this subject to various courses. The invaluable experience of using computer based numerical techniques for research and projects has helped value add to this book. Clarity and utility has been our constant goal. We have especially designed this book to achieve this objective. A vast number of solved examples followed by properly graded problems have been provided. Many examples and problems have been selected from recent question papers of various University and Engineering examinations. We have tried to make each chapter an independent unit so that the various topics could be studied without loss of continuity.

This book in its 12 chapters deals with Empirical Laws, Curve Fitting, Theory of Equations, Solution to Algebraic & Transcendental equations, Simultaneous Linear Algebraic Equations, Finite Differences, Interpolation with Equal Intervals, Central Difference Interpolation Formulae, Interpolation with Unequal Intervals, Numerical Differentiation and Integration, Difference Equations, Numerical Solution to Ordinary Differential Equations and Numerical Solution to Partial Differential Equation.

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CHAPTER 1

Empirical Laws and Curve Fitting

1.1 INTRODUCTION

Very often, the students of Engineering and Science are required to obtain data involving two variables, say, x and y (y dependent on x) from experimental observations. The data obtained is required to be expressed in the form of a law connecting the two variables x and y. In such cases, the corresponding values (x_i, y_i) of the given data are plotted on a graph paper and a smooth curve is drawn passing through the plotted points. Such a curve is called an *approximating curve*. Its equation, say, $y = f(x)$ is known as an *empirical equation*. Since it is possible to draw a number of such curves through or near the points, different empirical equations can be obtained to express the data. Now the problem is to find the equation of the curve which is best suited to predicting the unknown values. The process of finding such an equation of the 'best fit' is known as *curve fitting*.

1.2 THE LINEAR LAW

Let the empirical law between two variables, say, x and y, of certain data be linear, i.e. of the form

$$y = mx + c \tag{1.1}$$

where m and c are constants to be determined.

Since Eqn (1.1) is a straight line, m is the slope and c is the y intercept. If we plot the corresponding values of the data (x_i, y_i) with a suitable scale, they should lie on a straight line. But in practice the values obtained from experimental observations may not lie exactly on a straight line. In such cases, draw the straight line of 'best fit' such that the points are evenly distributed about the line. Choose two points (x_1, y_1) and (x_2, y_2) which are set apart on the line so that $m = (y_2 - y_1)/(x_2 - x_1)$. c can be found by substituting any point on the line or by knowing the y-intercept (i.e. value of y when $x = 0$). Substituting m and c in Eqn (1.1), we get the required linear law.

1.3 LAWS REDUCIBLE TO LINEAR LAW

Some laws which connect the two variables x and y, and which are non-linear can be reduced to linear form by suitable substitutions as explained below.

(i) *Law of the type $y = ax^n$, where a and n are constants.* Taking logarithms on both sides, we get

$$\log_{10} y = \log_{10} a + n \log_{10} x \quad \text{or} \quad Y = A + nX \qquad (1.2)$$

where $\log_{10} y = Y$, $\log_{10} a = A$ and $\log_{10} x = X$.

Here, Eqn (1.2) is linear in X and Y. A and n are constants to be determined.

(ii) *Law of the type $y = ae^{bx}$, where a and b are constants.* Taking logarithms on both sides, we get

$$\log_{10} y = \log_{10} a = bx \log_{10} e \quad \text{or} \quad Y = A + Bx$$

which is linear in X and Y. Here, $A = \log_{10} a$ and $B = b \log_{10} e$.

(iii) *Law of the type $y = ab^x$, where a and b are constants.* Taking logarithms on both sides, we get

$$\log_{10} y = \log_{10} a + x \log_{10} b \quad \text{or} \quad Y = A + Bx$$

which is linear in x and Y, where $A = \log_{10} a$ and $B = \log_{10} b$.

(iv) *Law of the type $xy = ax + by$.* Writing it as $y = a + b(y/x)$ and setting $X = y/x$, we get the linear form $y = a + bX$.

(v) *Law of the type $y = ax^n + b$.* Taking $x^n = X$, we get $y = aX + b$ which is linear in y and X.

By choosing $n = -1, \frac{1}{2}, 2$ we get the particular cases of above as

$$y = a/x + b,\ y = a\sqrt{x} + b \quad \text{and} \quad y = ax^2 + b.$$

Similarly, the laws of the type

$$xy = ax + b\,,\ y = ax^n + b \log x$$

etc. can be reduced to the linear form.

Example 1.1 The following values of y and x are possibly connected by a law of the type $y = ax^2 + b$. Test if this is so and find the law.

x	12	16	20	22	24	26	30
y	6.0	7.5	9.0	9.8	10.7	11.5	14

Solution The probable law is

$$y = ax^2 + b \tag{i}$$

Taking $x^2 = X$, Eqn (i) becomes

$$y = aX + b \tag{ii}$$

Table for X and y is

X	144	256	400	484	576	676	900
y	6.0 (P_1)	7.5 (P_2)	9.0 (P_3)	9.8 (P_4)	10.7 (P_5)	11.5 (P_6)	14 (P_7)

These points are plotted and the line of 'best fit' is drawn as shown in Fig. 1.1.

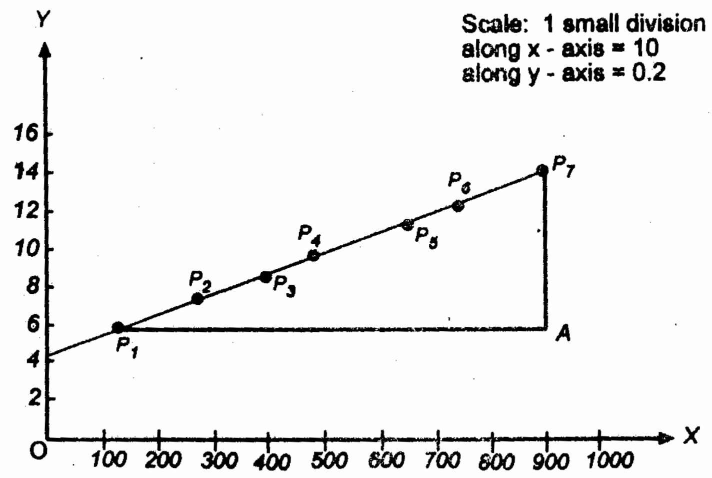

Fig 1.1

As these points are lying almost along a straight line, the given law is nearly accurate. Now the slope of this line is

$$a = \frac{AP_7}{P_1A} = \frac{14-6}{900-144} = \frac{8}{756} = 0.010582$$

$\because$ P_1 (144, 6.0) lies on (2).

$6 = (0.010582)(144) + b$ or $b = 4.48$ (approx)

which can also be observed from the graph.

Hence, the curve of 'best fit' is

$$y = 0.010582\, X + 4.48$$

i.e. $$y = 0.010582\, x^2 + 4.48$$

Example 1.2 The values of x and y obtained in an experiment are as follows:

x	2.3	3.1	4	4.92	5.91	7.20
y	33	39.1	50.3	67.2	85.6	125

Find a law of the type $y = ae^{bx}$.

Solution Given law type is

$$y = ae^{bx} \qquad \text{(i)}$$

Taking logarithms (to base 10) on both sides, we get

$$\log_{10} y = \log_{10} a + bx \log_{10} e$$

or $$Y = A + Bx \qquad \text{(ii)}$$

where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b\log_{10} e$

Table for x and Y is as under :

x	2.3	3.1	4	4.92	5.91	7.20
$Y = \log_{10} y$	1.52 (P_1)	1.59 (P_2)	1.7 (P_3)	1.83 (P_4)	1.93 (P_5)	2.1 (P_6)

Plotting these points on the graph paper we can see that almost all the points are lying on or near the straight line (Fig 1.2).

Now slope of this line is

$$B = \frac{P_6A}{P_2A} = \frac{2.1 - 1.59}{7.2 - 3.1} = 0.1244$$

Since P_2 (3.1, 1.59) lies on (ii),

$$1.59 = A + (0.1244)(3.1)$$

or $$A = 1.2044$$

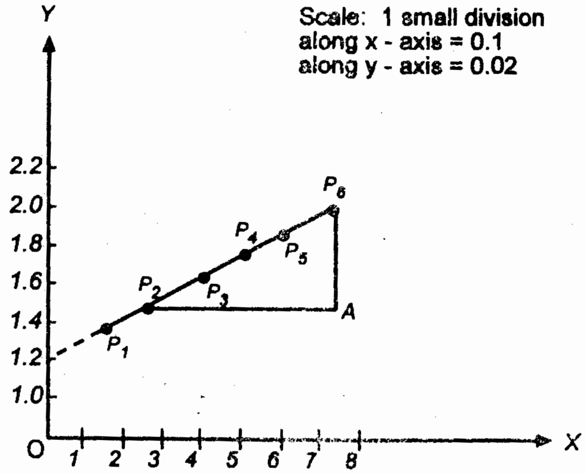

Fig 1.2

But $$b = \frac{B}{\log_{10} e} = \frac{0.1244}{0.4343} = 0.2864$$

and $\log_{10} a = A \therefore a = \text{anti} \log_{10} A = 16.01$

Hence the curve of 'best fit' is $y = 16.01\, e^{0.2864x}$

EXERCISE 1.1

1. The law of machine is $P = aW + b$, where P is the effort and W, the load in *lb*. Sketch a graph showing the relation between P and W, given

P	60	75	100	125	145
W	225	300	430	560	600

Find P when $W = 500$.

2. R is the resistance to motion of a train at speed V. Find a law of the type $R = aV^2 + b$ connecting R and V using the following data

R kg/ton	8	10	15	21	30
V (km/hr)	10	20	30	40	50

3. The resistance R of a carbon filament lamp was measured at various values of voltage V and the following observations were made.

V	62	70	78	84	92
R	73	70.7	69.2	67.8	66.3

(Ranchi B.Tech 1986)

Assuming a law of the form $R = a/V + b$, find by graphical method the best values of a and b.

4. μ, the co-efficient of friction between a belt and pulley and v, the velocity of the belt in ft/min, are connected as shown in the following table:

v	500	1000	2000	4000	6000
μ	0.29	0.33	0.38	0.45	0.51

The probable law is $\mu = a + b\sqrt{v}$. Test graphically the accuracy of this law and if it is true, find the values of a and b.

5. Fit a curve of the form $y = ae^{bx}$ to the following data:

x	1	2	3	4	5	6
y	14	27	40	55	68	300

6. The following observations are corresponding to pressure and specific volume of dry saturated steam. Fit a curve of the form $PV^n = k$ by graphical method.

V	38.4	20	8.51	4.44	3.03	2.31
P	10	20	50	100	150	200

ANSWERS

1. $P = 0.21W + 12$, $P = 117$
2. a = 0.0085, b = 7.35
3. a = 1120, b = 55.1
4. a = 0.2 b = 0.0044
5. $y = 7.943\, e^{0.5419x}$
6. $PV^{1.073} = 501$

1.4 METHOD OF GROUP OF AVERAGES

Let the straight line

$$y = a + bx \tag{1.3}$$

fit a set of n observations (x_i, y_i) $i = 1, 2, \ldots, n$ very closely.

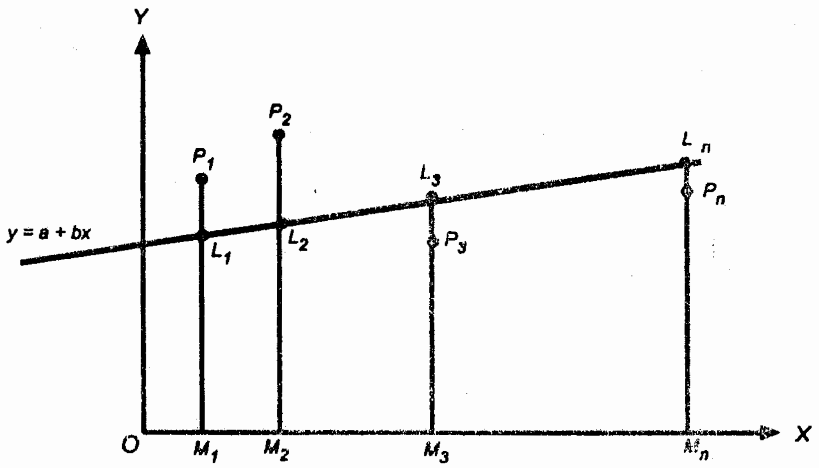

Fig 1.3

Now, from Fig. 1.3, when $x = x_1$, the observed value of $y = y_1 = P_1M_1$ and the expected value from Eqn (1.3) is $y = a + bx_1 = L_1M_1$.

Then,

$$\begin{aligned} d_1 &= \text{observed value at } L_1 - \text{expected value at } L_1 \\ &= y_1 - (a + bx_1) = P_1M_1 - L_1M_1 = P_1L_1 \end{aligned}$$

which is called the ***residual*** or ***error*** at x_1.

Similarly, the residuals for remaining observations are :

$$\begin{aligned} d_2 &= y_2 - (a + bx_2) = P_2L_2 \\ d_3 &= y_3 - (a + bx_3) = P_3L_3 \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \\ d_n &= y_n - (a + bx_n) = P_nL_n. \end{aligned}$$

From Fig. (1.3) it is clear that some of these residuals may be positive while others are negative. The ***method of group of averages*** is based on the assumption that the ***sum of the residuals is zero***, i.e. $\sum_{i=1}^{n} d_i = 0$.

To find the constants a and b in Eqn (1.3) we require two equations connecting them. Therefore, we divide the given data into two groups:

(i) The first group containing k observations $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ and

(ii) the second group containing the remaining $n - k$ observations $(x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2}), \ldots, (x_n, y_n)$.

Assuming that the sum of the residuals in each group is zero, we get

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} [y_i - (a + bx_i)] = 0$$

and
$$\sum_{i=k+1}^{n} d_i = \sum_{i=k+1}^{n} [y_i - (a + bx_i)] = 0$$

which gives on simplification,

$$\overline{y_1} = a + b\overline{x_1} \tag{1.4}$$

and
$$\overline{y_2} = a + b\overline{x_2} \tag{1.5}$$

where

$$\overline{x_1} = \frac{1}{k}\sum_{i=1}^{k} x_i, \quad \overline{y_1} = \frac{1}{k}\sum_{i=1}^{k} y_i$$

are the averages of x's, y's of the first group and

$$\overline{x_2} = \frac{1}{n-k}\sum_{i=k+1}^{n} x_i, \quad \overline{y_2} = \frac{1}{n-k}\sum_{i=k+1}^{n} y_i$$

are the averages of x's, y's of the second group.

This shows that the Eqns (1.4) and (1.5) are obtained from Eqn (1.3) by replacing the averages of x's, y's of the two groups in the places of x and y. Solving Eqns (1.4) and (1.5) we get a and b and hence, Eqn (1.3).

Note: Since grouping of observations can be done in different ways the values of a and b are not unique. This is the main defect of this method. To overcome this drawback, we generally divide the data into two groups so that both contain approximately equal number of observations.

Example 1.3 The following table gives corresponding values of x and y. Obtain an equation of the form $y = a + bx$ using the method of grouping.

x	0	5	10	15	20	25
y	12	15	17	22	24	30

Solution Let us divide the given data into two groups as follows :

Group I		Group II	
x	y	x	y
0	12	15	22
5	15	20	24
10	17	25	30
$\Sigma x = 15$	$\Sigma y = 44$	$\Sigma x = 60$	$\Sigma y = 76$

$\therefore$ The averages of Group I are $\bar{x}_1 = \dfrac{15}{3} = 5$; $\bar{y}_1 = \dfrac{44}{3} = 14.667$

and for Group II, $\bar{x}_2 = \dfrac{60}{3} = 20$; $\bar{y}_2 = \dfrac{76}{3} = 25.333$

Substituting the averages of x's , y's of the two groups in the required line $y = a + bx$, we get

$$\bar{y}_1 = a + b\bar{x}_1, \text{ i.e. } 14.667 = a + 5b$$

and $$\bar{y}_2 = a + b\bar{x}_2 \text{ i.e. } 25.333 = a + 20b$$

Solving these two equations, we get $a = 11.1117$ and $b = 0.7111$, approximately.

Hence, the required line is

$$y = 11.1117 + 0.7111x$$

Example 1.4 The head H (ft) and the quantity Q (ft^3) of water flowing per second are related by the law $Q = CH^n$. Find the best values of C and n by the method of group averages for the following data.

H	1.2	1.4	1.6	1.8	2.0	2.4	2.6
Q	4.2	6.1	8.5	11.5	14.9	23.5	27.1

Solution The given law can be written in linear form as

$$y = a + nx \qquad \text{(i)}$$

where $y = \log_{10}Q$, $a = \log_{10}C$ and $x = \log_{10}H$.

Now considering the first four observations as one group and remaining three observations as second group, we have the following table:

Group I		Group II	
$x = \log_{10} H$	$y = \log_{10} Q$	$x = \log_{10} H$	$y = \log_{10} Q$
0.07918	0.62325	0.30103	1.17319
0.14163	0.78533	0.38021	1.37107
0.20412	0.92942	0.41497	1.43297
0.25527	1.06070		
$\Sigma x = 0.6802$	$\Sigma y = 3.3987$	$\Sigma x = 1.09621$	$\Sigma y = 3.97723$

$\therefore$ The averages of Group I are

$$\bar{x}_1 = \frac{0.6802}{4} = 0.17005 \; ; \; \bar{y}_1 = \frac{3.3987}{4} = 0.84968$$

and for Group II,

$$\bar{x}_2 = \frac{1.09621}{3} = 0.3654 \; ; \; \bar{y}_2 = \frac{3.97723}{3} = 1.432574$$

Substituting the averages of x's, y's of the two groups in (i), we get

$$0.84968 = a + 0.17005\, n$$

and $$1.32574 = a + 0.3654\, n$$

Solving these equations, we get

$a = 2.4369593$; $n = -9.3342$ (approximately)

But $C = \text{antilog}_{10} a = 273.50124$

which are the best values as per the method.

Note : By taking three observations in first group and four observations in second group, we get different values of C and n.

1.5 EQUATIONS INVOLVING THREE CONSTANTS

Sometimes, it is required to fit the observed data to the curves of the form (i) $y = a + bx + cx^2$, (ii) $y = ax^b + c$ and (iii) $y = ae^{bx} + c$ which involve three constants a, b and c. In such cases, we reduce them to a linear form by reducing the number of constants from three to two. We will see below how these curves can be fitted.

(i) The curve of the form $y = a + bx + cx^2$. Let (x_1, y_1) be a particular point on the curve satisfying the data.

Then, $$y_1 = a + bx_1 + cx_1^2$$

$\therefore$ $$y - y_1 = b(x - x_1) + c(x^2 - x_1^2)$$

or $$\frac{y-y_1}{x-x_1} = b + c(x+x_1) \text{ or } Y = b + cX \tag{1.6}$$

where $Y = \dfrac{y-y_1}{x-x_1}$, and $X = x + x_1$

Now Eqn (1.6) is linear in Y and X, and can be easily fitted.

Example 1.5 Using the method of averages, fit a parabola $y = a + bx + cx^2$ to the following data :

x	20	40	60	80	100	120
y	5.5	9.1	14.9	22.8	33.3	46.0

(M.U, B.E., 1992)

Solution Let point (20,5.5) lies on the parabola

$$y = a + bx + cx^2 \quad \text{(i)}$$

$$\therefore \quad 5.5 = a + 20b + 400c \quad \text{(ii)}$$

(i) – (ii) gives,

$$\frac{y-5.5}{x-20} = b + c(x+20) \quad \text{(iii)}$$

or
$$Y = b + cX \quad \text{(iv)}$$

where $Y = \dfrac{y-5.5}{x-20}$, $X = x + 20$

Choosing first three observations as one group and last three observatons as another group, we make the following table for the values of X and Y.

For Group I

x	y	$x-20$	$y-5.5$	$X=x+20$	$Y=\dfrac{y-5.5}{x-20}$
20	5.5	0	0	–	–
40	9.1	20	3.6	60	0.18
60	14.9	40	9.4	80	0.235
				$\Sigma X = 140$	$\Sigma Y = 0.415$

$$\therefore \quad \bar{X}_1 = \frac{140}{2} = 70; \quad \bar{Y}_1 = \frac{0.415}{2} = 0.2075$$

For Group II

x	y	$x-20$	$y-5.5$	X	Y
80	22.9	60	17.3	100	0.288
100	33.3	80	27.8	120	0.348
120	46	100	40.5	140	0.405
				$\Sigma X = 360$	$\Sigma Y = 1.041$

$\therefore\ \bar{X}_2 = \dfrac{360}{3} = 120\ ;\ \bar{Y}_2 = \dfrac{1.041}{3} = 0.347$

Substituting the averages in Eqn (iv) for the two groups, we get

$$\bar{y}_1 = b + c\bar{X}_1 \quad \text{i.e.}\ \ 0.2075 = b + c(70)$$

and $$\bar{y}_2 = b + c\bar{X}_2 \quad \text{i.e.}\ \ 0.347 = b + c(120)$$

Solving these two equations,

$$b = 0.0122\ ;\ c = 0.00279$$

Substituting in Eqn (iii), we get

$$\frac{y-5.5}{x-20} = 0.0122 + 0.00279\,(x+20)$$

or $$y = 0.00279x^2 + 0.0122x + 4.14$$

which is the required curve.

(ii) The curve of the form $y = ax^b + c$. It can be written as

$$y - c = ax^b \tag{1.7}$$

or $$Y = A + bX \tag{1.8}$$

where $Y = \log_{10}(y-c)$, $A = \log_{10}a$ and $X = \log_{10}x$.

Now Eqn (1.8) is linear in X and Y.

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three particular points on the curve (1.7) such that x_1, x_2, x_3 are in geometric progression, i.e. $x_2^2 = x_1x_3$.

$\therefore\ y_1 - c = ax_1^b\ ;\ y_2 - c = ax_2^b$ and $y_3 - c = ax_3^b$

$\therefore\ (y_1 - c)(y_3 - c) = a^2(x_1x_3)^b = (ax_2^b)^2 = (y_2 - c)^2$

and on simplication,

$$c = \frac{y_1y_3 - y_2^2}{y_1 + y_3 - 2y_2}$$

Using this equation, values for Y can be determined and hence, the required curve.

Example 1.6 In a magnetic arc at constant arc length, voltage v consumed by the arc is observed for values of the current i.

i	0.5	1	2	4	8	12
v	160	120	94	75	62	56

If v and i are connected by a relation of the form $v = ai^b + c$, find a, b and c and hence, the curve by the method of group averages.

Solution Take $i_1 = 1$, $i_2 = 2$ and $i_3 = 4$ which are in GP to find c. Then $v_1 = 120$, $v_2 = 94$ and $v_3 = 75$

$$\therefore \qquad c = \frac{v_1 v_3 - v_2^2}{v_1 + v_3 - 2v_2} = \frac{(120)(75) - (94)^2}{120 + 75 - 2(94)} = 23.4$$

$$\therefore\ v = ai^b + 23.4 \ \text{ or } \ v - 23.4 = ai^b \tag{i}$$

or

$$Y = A + bX \tag{ii}$$

where $Y = \log_{10}(v - 23.4)$, $A = \log_{10} a$ and $X = \log_{10} i$.

Choosing first three observations as one group and other three as second group, we have the following table:

For Group I

i	v	$X = \log_{10} i$	$Y = \log_{10}(v - 23.4)$
0.5	160	$\bar{1}.6990$	2.1354
1	120	0.0000	1.9850
2	24	0.3010	1.8488
		$\Sigma x = 0.0000$	$\Sigma y = 5.9692$

$$\overline{X}_1 = \frac{\Sigma X}{3} = 0\ ; \quad \overline{Y}_1 = \frac{\Sigma Y}{3} = 1.9897$$

For Group II

i	v	$X = \log_{10} i$	$Y = \log_{10}(v - 23.4)$
4	75	0.6021	1.7126
8	62	0.9031	1.5866
12	56	1.0792	1.5132
		$\Sigma X = 2.5844$	$\Sigma Y = 4.8124$

$$\bar{X}_2 = \frac{2.5844}{3} = 0.8615 \;;\quad \bar{Y}_2 = \frac{4.8124}{3} = 1.6041$$

Substituting the averages of X's, Y's of two groups in Eqn (ii), we get
$1.9897 = A + b(0)$ $\therefore A = 1.9897$
and $1.6041 = A + 0.8615\,b$ $\therefore b = -0.4476$

But $a = \text{antilog}_{10} A = 97.6562$

$\therefore a = 97.66,\ b = -0.45,\ c = 23.4$, approximately,

and the curve is $v = 97.66\, t^{-0.45} + 23.4$

(iii) The curve of the form $y = ae^{bx} + c$. This equation can be written as

$$y - c = a\,e^{bx} \tag{1.9}$$

and also in linear form as

$$Y = A + Bx \tag{1.10}$$

where $Y = \log_{10}(y - c)$, $A = \log_{10} a$ and $B = b \log_{10} e$.

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three particular points on the curve such that x_1, x_2 and x_3 are in arithmetic progression, i.e. $x_1 + x_3 = 2x_2$.

Substituting these points in Eqn (1.9), we have

$$y_1 - c = a\,e^{bx_1},\quad y_2 - c = a\,e^{bx_2}, \text{ and } \quad y_3 - c = a\,e^{bx_3}$$

$$\therefore \quad (y_1 - c)(y_3 - c) = a^2 e^{b(x_1 + x_3)} = a^2 e^{b(2x_2)}$$

$$= a^2 (e^{bx_2})^2 = (y_2 - c)^2$$

which gives on simplication,

$$c = \frac{y_1 y_3 - y_2^2}{y_1 + y_3 - 2y_2}$$

Using this, values of Y can be determined and hence, the required curve.

Note : The same procedure can be adopted to the curve of the form $y = ab_x + c$.

Example 1.7 The variables s and t are connected by the relation $s = a + \mathrm{b}\, e^{nt}$ and their corresponding values are given in the following table:

t	1	2	6	8	11
s	12.71	12.46	11.65	11.34	10.99

Fit the curve by the method of group averages.

Solution The given curve can be written as

$$s - a = b\,e^{nt} \tag{i}$$

and in linear form as

$$Y = B + Nt \tag{ii}$$

where $Y = \log_{10}(s - a)$, $B = \log_{10} b$ and $N = n \log_{10} e$.

Take $t_1 = 1$, $t_2 = 6$ and $t_3 = 11$ such that $x_1 + x_3 = 2x_2$.

The corresponding values of s are $s_1 = 12.71$, $s_2 = 11.65$ and $s_3 = 10.99$.

$$\therefore \quad a = \frac{s_1 s_3 - s_2^2}{s_1 + s_3 - 2s_2} = \frac{(12.71)(10.99) - (11.65)^2}{12.71 + 10.99 - 2(11.65)}$$

$$= \frac{3.9604}{0.4} = 9.901$$

Now choosing the first three observations as one group and other two observations as second group, we have the following table :

Group I			Group II		
t	s	$Y = \log_{10}(s - 9.901)$	t	s	$Y = \log_{10}(s - 9.901)$
1	12.71	0.4486	8	11.34	0.1581
2	12.46	0.4081	11	10.99	0.0370
6	11.65	0.2428			
$\Sigma t = 9$		$\Sigma Y = 1.0995$	$\Sigma t = 19$		$\Sigma Y = 0.1951$

$\therefore$ Averages of the first group are

$$\bar{t}_1 = \frac{\Sigma t}{3} = \frac{9}{3} = 3 \; ; \; \bar{Y}_1 = \frac{\Sigma Y}{3} = \frac{1.0995}{3} = 0.3665$$

and the averages of the second group are

$$\bar{t}_2 = \frac{19}{2} = 9.5 \; ; \; \bar{Y}_2 = \frac{0.1951}{2} = 0.0976$$

Substituting the averages of t's and Y's of the two groups in (ii), we get $0.3665 = B + 3N$ and $0.0976 = B + 9.5N$

On solving, $B = 0.4907$; $N = -0.0414$

Now $b = \text{antilog}_{10} B = 3.097$ and $n = \dfrac{N}{\log_{10} e} = -0.0953$

Hence, the required curve is $s = 9.901 + 3.097\, e^{-0.0953t}$

EXERCISE 1.2

1. The weights of a calf taken at weekly intervals are given below. Fit a straight line using the method of group averages.

Age in weeks	1	2	3	4	5	6	7	8	9	10
weight	52.5	58.7	65	70.2	75.4	81.1	87.2	95.5	102.2	108.4

2. Fit a curve of the form $y = ax^n$ to the following set of observations by the method of group averages:

x	10	20	30	40	50	60	70	80
y	1.06	1.33	1.52	1.68	1.81	1.91	2,01	2.11

3. Fit a curve of the form $y = ab^x$ using the method of group averages for the following data:

x	2	4	6	8	10	12
y	7.32	8.24	9.20	10.19	11.01	12.05

4. Convert the equation $y = b/x(x - a)$ to a linear form and hence, determine a and b which will fit the following data using the method of group averages:

x	8	10	15	20	30	40
y	13	14	15.4	16.3	17.2	17.8

5. The following data represent test values obtained while testing a centrifugal pump. Assuming the relation to be $H = a + bQ + cQ^2$, where Q is the discharge in liter per second and H, head in meter of water, find the relation by the method of group averages.

Q	2	2.5	3	3.5	4	4.5	5	5.5	6
H	18	17.8	17.5	17	15.8	14.8	13.3	11.7	9

(M.U, B.E., 1971)

6. The temperature θ of a vessel of cooling water and the time t in minutes since the beginning of observation are connected by the law of the form $\theta = ae^{bt} + c$. The corresponding values of t and θ are given by:

t	0	1	2	3	5	7	10	15	20
θ	52.2	48.8	46.0	43.5	39.7	36.5	33.0	28.7	26.0

Find the best values of a, b and c using the method of group averages.

7. Fit a curve of the form $y = a + bx^c$ to the following data using the method of group averages.

x	1	2	4	6	10	16
y	15	45	165	364	1004	2564

8. Fit a curve of the form $y = a + bc^x$ to the following data using the method of group averages.

x	0	1	2	3	4	5	6	7	8
y	2.4	3.2	3.7	5.1	7.8	13.2	23.6	44.8	87

ANSWERS

1. $y = 46.048 + 6.104\,x$
2. $y = 0.4851\,x^{0.3354}$
3. $y = (6.7468)(1.0505)^x$
4. $a = 0.2039,\ b = 0.051$
5. $H = 1.58 + 2.1\,Q - 0.5\,Q^2$
6. $a = 29.5393,\ b = -0.09968,\ c = 21.98$
7. $y = 5 + 10x^2$
8. $y = 2.26 + (0.3)(2.07)^x$

1.6 PRINCIPLES OF LEAST SQUARES

So far we have studied the curve fitting by (i) Graphical method, and (ii) Method of group averages. The values of the constants in the first

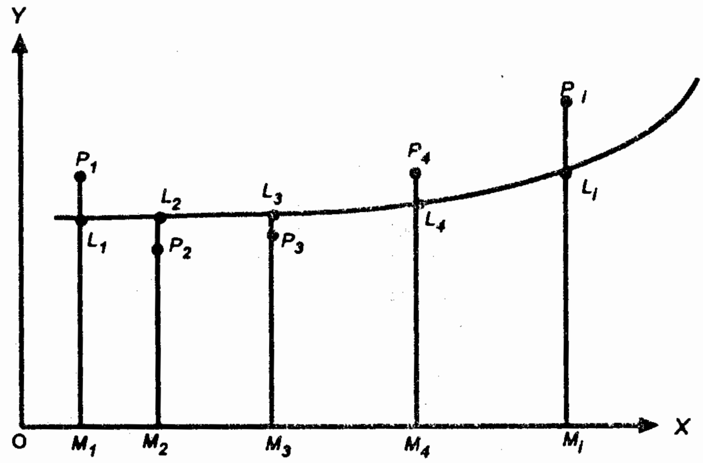

Fig 1.4

method depends upon the choice of points on the line, and in the second method on grouping the observations. Hence, both the methods fail to give a unique curve of fit. So we require some other method to get the best values of the constants. *Principle of least squares provides a unique set of values to the constants and hence, suggests a curve of best fit to the given data.*

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the n sets of observations and let $y = f(x)$ be the relation suggested between x and y.

When $x = x_i$, the observed value of y_i is $y_i = P_i M_i$.

Expected value $= L_i M_i = f(x_i)$

$\therefore$ Residual (or error) at $x = x_i$ is $e_i = y_i - f(x_i)$

It is obvious that some of e_i's may be +ve or –ve. Thus by giving equal weightage to each residuals, consider

$$E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} [y_i - f(x_i)]^2 \tag{1.11}$$

Now, if $E = 0$ then $y_i = f(x_i)$ $\forall$ i, i.e. all the points lie on the curve. Otherwise, the minimum of E results the best fitting curve to the data. Thus the curve of best fit is that one for which the sum of squares of the residuals is minimum. Using this principle, we shall fit the following curves:

(i) A straight line, $y = ax + b$

(ii) A parabola, $y = ax^2 + bx + c$

(iii) The exponential curve, $y = ae^{bx}$ and

(iv) The curve, $y = ax^b$

1.7 FITTING A STRAIGHT LINE

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be n sets of observations of related data and

$$y = ax + b \tag{1.12}$$

be the straight line to be fitted.

As explained in Section (1.6), the residual at $x = x_i$ is

$$e_i = y_i - f(x_i) = y_i - (ax_i + b)\,,\ i = 1, 2, \ldots, n$$

$$\therefore \qquad E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$$

By the principle of least squares, E is minimum.

$$\therefore \qquad \frac{\partial E}{\partial a} = 0 \text{ and } \frac{\partial E}{\partial b} = 0$$

$$\therefore \qquad \sum_{i=1}^{n} 2(y_i - ax_i - b)(-x_i) = 0 \text{ or } \sum_{i=1}^{n} (y_i x_i - ax_i^2 - bx_i) = 0$$

and $$\sum_{i=1}^{n} 2(y_i - ax_i - b)(-1) = 0 \text{ or } \sum_{i=1}^{n}(y_i - ax_i - b) = 0$$

$\therefore$ $$\sum_{i=1}^{n} y_i x_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i \tag{1.13}$$

and $$\sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} x_i + nb \tag{1.14}$$

Since x_i, y_i are known, Eqns (1.13) and (1.14) result in two equations in a and b. Solving these the best values for a and b can be known and hence, Eqn (1.12).

Note :

1. Dropping the suffix i, we can write Eqns (1.13) and (1.14) as

$$\Sigma xy = a\,\Sigma x^2 + b\,\Sigma x \quad \text{and} \quad \Sigma y = a\,\Sigma x + nb.$$

These are known as *normal equations* to the curve $y = ax + b$.

2. If the numbers in the given data are large, we can change the origin and scale with the substitutions $X = \dfrac{x-A}{h}$, $Y = \dfrac{y-B}{k}$ for the sake of convenience and ease in calculation work.

Example 1.8 By the method of least squares, find the straight line that best fits the following data:

x	1	2	3	4	5
y	14	27	40	55	68

(*M.U, B.E., 1993*)

Solution Let the straight line of best fit be

$$y = ax + b \tag{i}$$

The normal equations are

$$\Sigma xy = a\Sigma x^2 + b\Sigma x$$

(ii)

and $$\Sigma y = a\Sigma x + 5b \tag{iii}$$

The values of $\sum x$, $\sum y$, $\sum x^2$ and $\sum xy$ are calculated as shown below:

x	y	x^2	xy
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
$\sum x = 15$	$\sum y = 204$	$\sum x^2 = 55$	$\sum xy = 748$

$\therefore$ Eqns (ii) and (iii) become $748 = 55a + 15b$ and $204 = 15a + 5b$
Solving these, we have $a = 13.6$ and $b = 0$.
Putting these values in Eqn (i), we get the line of best fit as $y = 13.6x$.

Aliter: Let $X = x - 3$, $Y = y - 40$ and let the straight line in new variables be

$$Y = Ax + B \tag{i}$$

Now the normal equations are

$$\sum XY = A\sum X^2 + B\sum X \tag{i}$$

$$\sum Y = A\sum X + B \tag{ii}$$

x	y	$X = x - 3$	$Y = y - 40$	X^2	XY
1	14	–2	–26	4	52
2	27	–1	–13	1	13
3	40	0	0	0	0
4	55	1	15	1	15
5	68	2	28	4	56
		$\sum X = 0$	$\sum Y = 4$	$\sum X^2 = 10$	$\sum XY = 136$

Substituting $\sum X = 0$, $\sum Y = 4$, $\sum XY = 136$ and $\sum X^2 = 10$ in Eqns (ii) and (iii), we get

$$136 = 10A \text{ and } 4 = 5B \text{ or } A = 13.6 \text{ and } B = 0.8$$

$\therefore$ From Eqn (i), $Y = 13.6X + 0.8$
(i.e) $y - 40 = 13.6(x - 3) + 0.8$
or $y = 13.6\,x$ which is the same as seen before.

1.8 FITTING A PARABOLA

Let

$$y = ax^2 + bx + c \tag{1.15}$$

be a parabola to be fitted for the data (x_i, y_i), $i = 1, 2, \cdots, n$. As explained in Section 1.6, the residual at $x = x_i$ is

$$e_i = y_i - f(x_i) = y_i - (ax_i^2 + bx_i + c)$$

$$E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} [y_i - (ax_i^2 + bx_i + c)]^2$$

Now E should be minimum for the best values of a, b and c (Principle of least squares).

$$\therefore \qquad \frac{\partial E}{\partial a} = 0, \quad \frac{\partial E}{\partial a} = 0 \text{ and } \frac{\partial E}{\partial c} = 0$$

i.e.,
$$\sum_{i=1}^{n} 2[y_i - (ax_i^2 + bx_i + c)](-x_i^2) = 0$$

$$\sum_{i=1}^{n} 2[y_i - (ax_i^2 + bx_i + c)](-x_i) = 0$$

and
$$\sum_{i=1}^{n} 2[y_i - (ax_i^2 + bx_i + c)](-1) = 0$$

which results, on simplification after dropping the suffix, as

$$\sum x^2y = a\sum x^4 + b\sum x^3 + c\sum x^2$$

$$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x$$

and
$$\sum y = a\sum x^2 + b\sum x + nc$$

where $\sum_{i=1}^{n}$. These are called the *normal equations* of the parabola (1.15). Solving these we get the best values of the constants a, b and c and hence, best fit for Eqn (1.15).

Example 1.9 Fit a parabola $y = ax^2 + bx + c$ in least square sense to the data:

x	10	12	15	23	20
y	14	17	23	25	21

Solution The normal equations to the curve are

$$\sum y = a\sum x^2 + b\sum x + 5c$$

$$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x$$

and
$$\sum x^2y = a\sum x^4 + b\sum x^3 + c\sum x^2$$

The values of $\sum x$, $\sum x^2$ etc. are calculated by means of the following table:

x	y	x^2	x^3	x^4	xy	x_2y
10	14	100	1000	10000	140	1400
12	17	144	1728	20736	204	2448
15	23	225	3375	50625	345	5175
23	25	529	12167	279841	575	13225
20	21	400	8000	160000	420	8400
Σx =80	Σy = 100	Σx^2 = 1398	Σx^3 = 26270	Σx^4 = 521202	Σxy = 1684	Σx^2y = 30648

Substituting the obtained values from the table in normal equations we have

$$100 = 1398a + 80b + 5c$$
$$1684 = 26270a + 1398b + 80c$$
$$30648 = 521202a + 26270b + 1398c$$

On solving, $a = -0.07$, $b = 3.03$, $c = -8.89$.

$\therefore$ the required equation is

$$y = -0.07x^2 + 3.03x - 8.89$$

Aliter: Let $x = x - 16$ and $y = y - 20$ and the parabola of fit be $Y = AX^2 + BX + C$ in the new variables, whose normal equations are

$$\begin{aligned} \Sigma Y &= A\Sigma X^2 + B\Sigma X + 5c \\ \Sigma XY &= A\Sigma X^3 + B\Sigma X^2 + C\Sigma X \\ \Sigma X^2Y &= A\Sigma X^4 + B\Sigma X^3 + C\Sigma X^2 \end{aligned}$$

The values of ΣX etc., are calculated as below:

x	y	X	Y	X^2	X^3	X^4	XY	X^2Y
10	14	–6	–6	36	–216	1296	36	–216
12	17	–4	–3	16	–64	256	12	–48
15	23	–1	3	1	1	1	–3	3
23	25	7	5	49	343	2401	35	245
20	21	4	1	16	64	256	4	16
		ΣX = 0	ΣY = 0	ΣX^2 = 118	ΣX^3 = 128	ΣX^4 = 4120	ΣXY = 84	ΣX^2Y = 0

$\therefore$ The normal equations become

$$\begin{aligned} 0 &= 118A + 5C \\ 84 &= 128A + 118B \\ 0 &= 4210A + 128B + 118C \end{aligned}$$

On solving, $A = -0.07$, $B = 0.79$ and $C = 1.67$

$$\therefore \quad Y = -0.07X^2 + 0.79X + 1.67$$

or $\qquad (y-20) = -0.07(x-16)^2 + 0.79(x-16) + 1.67$

which simplifies to

$y = -0.07x^2 + 3.03x - 8.89$ as observed earlier.

Example 1.10 Fit a second degree parabola to the following data using method of least squares.

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

(*M.U.,B.E., 1996*)

Solution Let $\quad y = ax^2 + bx + c \qquad$ (i)

be the parabola. The normal equations are

$$\Sigma y = a\Sigma x^2 + b\Sigma x + 5c$$
$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$

and $\qquad \Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$

The values Σx, Σx^2, $\cdots$ etc., are calculated by means of the following table:

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
Σx =10	Σy = 12.9	Σx^2 = 30	Σx^3 = 100	Σx^4 = 354	Σxy =37.1	Σx^2y =130.3

Substituting the obtained values in the normal equations, we get

$12.9 \quad = 30a + 10b + 5c$

$37.1 \quad = 100a + 30b + 10c$

$130.3 \quad = 354a + 100b + 30c$

On solving, we get

$$a = 0.55,\ b = -1.07 \text{ and } c = 1.42$$

$\therefore$ The required parable is

$$y = 0.55x^2 - 1.07x + 1.42$$

1.9 FITTING AN EXPONENTIAL CURVE

Consider the equation $y = ae^{bx}$.

Taking logrithms on both sides, we get

$$\log_{10} y = \log_{10} a + bx\log_{10} e$$

i.e.,
$$Y = A + Bx \tag{1.16}$$

where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b \log_{10} e$

The normal equations for (1.16) are

$$\sum Y = nA + B\sum x \quad \text{and} \quad \sum xY = A\sum x + B\sum x^2$$

Solving these, we get A and B.

Then $a = \text{antilog } A$ and $b = B / \log_{10} e$.

1.10 FITTING THE CURVE $y = ax^b$

Taking logarithms on both sides, we get

$$\log_{10} y = \log_{10} a + b\log_{10} x$$

i.e.
$$Y = A + bX \tag{1.17}$$

where $Y = \log_{10} y$, $A = \log_{10} a$ and $X = \log_{10} x$.

The normal equations to (1.17) are

$$\sum Y = nA + b\sum X \text{ and } \sum XY = A\sum X + b\sum X^2$$

which gives A and b on solving and $a = \text{antilog } A$.

Example 1.11 Find the curve of best fit of the type $y = ae^{bx}$ to the following data by the method of least squares.

x	1	5	7	9	12
y	10	15	12	15	21

Solution The curve to be fitted is $y = ae^{bx}$ or $Y = A + Bx$, where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b\log_{10} e$.

$\therefore$ The normal equations are

$$\sum Y = 5A + B\sum x \quad \text{and} \quad \sum xY = A\sum x + B\sum x^2$$

x	y	$Y = \log_{10} y$	x^2	xY
1	10	1.0000	1	1
5	15	1.1761	25	5.8805
7	12	1.0792	49	7.5544
9	15	1.1761	81	10.5849
12	21	1.3222	144	15.8664
$\sum x = 34$		$\sum Y = 5.7536$	$\sum x^2 = 300$	$\sum xY = 40.8862$

Substituting the values of $\sum x$, etc., calculated by means of above table in the normal equations, we get

$5.7536 = 5A + 34B$ and $40.8862 = 34A + 300B$

On solving, $A = 0.9766$; $B = 0.02561$

$\therefore$ $a = \text{antilog}_{10} A = 9.4754$; $b = B / \log_{10} e = 0.059$

Hence, the required curve is $y = 9.4754\, e^{0.059x}$

Example 1.12 The methc d of least squares to determine the constants a and b such that $y = ae^{bx}$ fits the following data

x	0.0	0.5	1.0	1.5	2.0	2.5
y	0.10	0.45	2.15	9.15	40.35	180.75

Solution The curve to be fitted is $y = ae^{bx}$ or $Y = A + Bx$, where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b\log_{10} e$.

$\therefore$ The normal equations are

$$\sum Y = 6A + B\sum x \text{ and } \sum xY = A\sum x + B\sum x^2$$

x	y	$Y = \log_{10} y$	x^2	xY
0	0.10	−1	0	0
0.5	0.45	−0.3468	0.25	−0.1734
1.0	2.15	0.3324	1.0	0.3324
1.5	9.15	0.9614	2.25	1.4421
2.0	40.35	1.6058	4.0	3.2116
2.5	180.75	2.2571	6.25	5.6428
$\sum x = 7.5$		$\sum Y = 3.8099$	$\sum x^2 = 13.75$	$\sum xY = 10.4555$

Substituting the values of $\sum x$, $\sum x^2$, $\cdots$, etc., in the normal equations, we get

$3.8099 = 6A + 7.5B$ and $10.4555 = 7.5A + 13.75B$

On solving, $A = -0.9916$; $B = 1.3013$

$\therefore a = \text{antilog}_{10} A = 0.1019$; $b = B / \log_{10} e = 2.9963$

and the curve is $y = 0.1019 \exp(2.9963x)$

Example 1.13 Obtain a relation of the form $y = ab^x$ for the following data by the method of least squares.

x	2	3	4	5	6
y	8.3	15.4	33.1	65.2	127.4

Solution The curve to be fitted is $y = ab^x$ or $Y = A + Bx$, where $A = \log_{10} a$, $B = \log_{10} b$ and $Y = \log_{10} y$.

$\therefore$ The normal equations are

$$\sum Y = 5A + B\sum x \quad \text{and} \quad \sum xY = A\sum x + B\sum x^2$$

x	y	$Y = \log_{10} y$	x^2	xY
2	8.3	0.9191	4	1.8382
3	15.4	1.1872	9	3.5616
4	33.1	1.5198	16	6.0792
5	65.2	1.8142	25	9.0710
6	127.4	2.1052	36	12.6312
$\sum x = 20$		$\sum Y = 7.5455$	$\sum x^2 = 90$	$\sum xY = 33.1812$

Substituting the values of $\sum x$, etc. from the above table in normal equations, we get

$7.5455 = 5A + 20B$ and $33.1812 = 20A + 90B$.

On solving, $A = 0.31$ and $B = 0.3$

$\therefore$ a = antilog $A = 2.04$ and b = antilog $B = 1.995$

Hence, the required curve is $y = 2.04\,(1.995)^x$

Example 1.14 Fit $y = ab^x$ by the method of least squares to the data given below.

x	0	1	2	3	4	5	6	7
y	10	21	35	59	92	200	400	610

Solution The curve to be fitted is $y = ab^x$ or $Y = A + Bx$, where $A = \log_{10} a$, $B = \log_{10} b$ and $Y = \log_{10} y$.

$\therefore$ The normal equations are

$$\sum Y = 8A + B\sum x \quad \text{and} \quad \sum xY = A\sum x + B\sum x^2$$

x	y	$Y = \log_{10} y$	x^2	xY
0	10	1.0000	0	0
1	21	1.3222	1	1.3222
2	35	1.5441	4	3.0882
3	59	1.7708	9	5.3124
4	92	1.9638	16	7.8552
5	200	2.3010	25	11.5050
6	400	2.6021	36	15.6126
7	610	2.7853	49	19.4971
$\sum x = 28$		$\sum Y = 15.2893$	$\sum x^2 = 140$	$\sum xY = 64.1927$

Substituting the values of Σx, etc., from the table in normal equations, we get $15.2893 = 8A + 28B$ and $64.1927 = 28A + 140B$.

On solving, $A = 1.02115$, $B = 0.2542892$

$\therefore$ $a = \text{antilog } A = 10.499$ and $B = \text{antilog } B = 1.7959$

Hence, the required curve is $y = 10.499\,(1.7959)^x$.

1.11 SUM OF SQUARES OF RESIDUALS

(i) *Straight line* In fitting a straight line (see Section 1.7), we have seen that the sum of the squares of the residuals E is given by

$$\begin{aligned} E &= \Sigma\,[y - (ax + b)]^2 \text{ (dropping the suffixes)} \\ &= \Sigma\,[y - (ax + b)][y - (ax + b)] \\ &= \Sigma\, y\,[y - (ax + b)] - a\Sigma x\,[y - (ax + b)] - b\Sigma[y - (ax + b)] \\ &= \Sigma\, y^2 = a\,\Sigma xy\ - b\Sigma y \end{aligned}$$

since the last two sums vanish due to normal equations (refer to Section 1.7).

$$\therefore \qquad E = \Sigma\, y^2 - a\,\Sigma\, xy\ - b\Sigma\, y$$

(ii) *Parabola* We have seen that in case of parabola (refer to Section 1.8) the sum of the squares of the residuals is

$$\begin{aligned} E &= \Sigma\,[\,y - (ax^2 + bx + c)]^2 && \text{(dropping the suffixes)} \\ &= \Sigma\, y^2 - a\Sigma x^2 y\ - b\Sigma xy - c\Sigma y && \text{(by using normal equations after expansion)} \end{aligned}$$

Example 1.15 For the data given in Ex.1.9, fit a straight line and the parabola and find out which one is most appropriate.

Solution (i) Fitting a straight line $y = ax + b$,
Normal equations are

$$\Sigma y = a\Sigma x + 5b \text{ and } \Sigma xy = a\Sigma x^2 + b\Sigma x$$

$\Sigma x = 80$, $\Sigma y = 100$, $\Sigma x^2 = 1398$, $\Sigma xy = 1684$ [from Ex.1.9]

$\therefore$ $100 = 80a + 5b$ and $1684 = 1398a + 80b$

On solving, $a = 0.7119$ and $b = 1.7559$

$\therefore$ The line of best fit is $y = 0.7119x + 1.7559$.

Let E_1 be the sum of the squares of the residuals

$$\begin{aligned} \therefore\ E_1 &= \Sigma y^2 - a\Sigma xy - b\Sigma y \\ &= 2080 - (0.7119)(1684) - (1.7559)(100) \\ &= 705.5704 \end{aligned}$$

(ii) We know from Example 1.9 that the parabola is

$$y = -0.07x^2 + 3.03x - 5.59$$

Let E_2 be the sum of the squares of the residuals

$\therefore E_2 = \Sigma y^2 - a\Sigma x^2 y - b\Sigma xy - c\Sigma y$

$= 2080 + 0.07(30648) - 3.03(1684) + 8.89(100)$

$= 11.84$

As $E_2 < E_1$, clearly the parabola is a better fit than the straight line.

EXERCISE 1.3

1. A simply supported beam carries a concentrated load P (lb) at its midpoint. Corresponding to various values of P, the maximum deflection Y (in) is measured. The data are given below. Find a law of the type $Y = a + bP$ by the method of least squares.

P	100	125	140	160	180	200
Y	0.45	0.55	0.60	0.70	0.80	0.85

(*Shivaji B.E., 1984*)

2. In the following table, y is the weight of potassium bromide which will dissolve in 100 gm of water at temperature x°C. Find a linear law between x and y using least square method.

x(°C)	0	10	20	30	40	50	60	70
y(gm)	53.5	59.5	65.2	70.6	75.5	80.2	85.5	90

3. By the method of least squares, find the curve $y = ax + bx^2$ that best fits the following data :

x	1	2	3	4	5
y	1.8	5,1	8.9	14.1	19.8

4. Find the parabola of the form $y = a + bx + cx^2$ which fits most closely with the following observations by the method of least squares.

x	–3	–2	–1	0	1	2	3
y	4.63	2.11	0.67	0.09	0.63	2.15	4.58

(*Kerala B.E., 1985*)

5. By the method of least squares, fit a second degree curve $y = a + bx + cx^2$ to the following data :

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

6. By the method of least squares, fit a parabola $y = a + bx + cx^2$ to the following data.

x	2	4	6	8	10
y	3.07	12.85	31.47	57.38	91.29

(*Mangalore B.E.. 1985*)

7. Fit an equation of the form $y = ae^{bx}$ to the following data by the method of least squares.

x	1	2	3	4
y	1.65	2.7	4.5	7.35

8. The voltage v across a capacitor at time t seconds is given by the following table. Use the principle of least squares to fit a curve of the form $v = a\,e^{kt}$ to the data:

t	0	2	4	6	8
v	150	63	28	12	5.6

9. Fit a curve of the form $y = ae^{bx}$ to the following data in least square sense:

x	0	2	4
y	5.012	10	31.62

10. Fit a curve of the form $y = ax^b$ to the data given below in square sense:

x	1	2	3	4	5
y	7.1	27.8	62.1	110	161

11. Fit a curve of the form $y = ax^b$ in least square sense to the following observations:

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

(*Calicut B.E., 1988*)

12. Fit a curve of the form $y = ab^x$ in least square sense to the data given below:

x	2	3	4	5	6
y	144	172.8	207.4	248.8	298.5

(*Karnataka B.E., 1993*)

13. Fit a curve of the form $y = ab^x$ in least square sense to the data given below:

x	1	2	3	4
y	4	11	35	100

14. Fit a straight line $y = ax + b$ and also a parabola $y = ax^2 + bx + c$ to the following set of observations:

x	0	1	2	3	4
y	1	5	10	22	38

Calculate the sum of squares of the residuals in each case and test which curve is more suitable to the data.

ANSWERS

1. $Y = 0.004P + 0.048$
2. $y = 54.35 + 0.5184x$
3. $y = 1.37x + 0.53x^2$
4. $y = 1.243 - 0.004x + 0.22x^2$
5. $y = -1 + 3.55x - 0.27x^2$
6. $y = 0.34 - 0.78x + 0.99x^2$
7. $y = e^{0.5x}$
8. $v = 146.3\, e^{-0.4118t}$
9. $y = 4.642\, e^{0.46x}$
10. $y = 7.173\, x^{1.952}$
11. $y = 0.5012\, x^{1.9977}$
12. $y = 99.86\,(1.2)^x$
13. $y = 1.33\,(2.95)^x$
14. $y = 9.1\,x - 3$; $y = 2.2x^2 + 0.3x + 1.4$

    $E_1 = 70.7$, $E_2 = 2.5$, $E_2 < E_1$, parabola is the best curve of fit.

1.12 METHOD OF MOMENTS

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be n sets of observations of related data, so that the x's are equally spaced.

i.e. $\quad x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = h$ (say).

We define the moments of the observed values of y as follows

The first moment $= \mu_1 = \Sigma\, yh = h\, \Sigma\, y$
The second moment $= \mu_2 = h\, \Sigma\, xy$
The first moment $= \mu_3 = h\, \Sigma\, x^2 y$ and so on.

Let $y = f(x)$ be a curve fitting the data. Then the moments of the expected values of y are

The first moment $= \gamma_1 = \int y\, dx = \int f(x)\, dx$
The second moment $= \gamma_2 = \int xy\, dx = \int x f(x)\, dx$
The third moment $= \gamma_3 = \int x^2 y dx = \int x^2 f(x)\, dx$ and so on.

This method is based on the assumption that the moments of the observed values of y are, respectively, equal to the moments of the expected values of y.

i.e. $$\mu_1 = \gamma_1 ;\ \mu_2 = \gamma_2 ;\ \mu_3 = \gamma_3 \tag{1.18}$$
and so on.

μ_i's are calculated from the tabulated values of x and y while γ_i's are computed as follows.

In Fig 1.5, the ordinate P_1 $(x = x_1)$ can be taken as the value of y at the midpoint of the interval $x_1 - h/2$, $x_1 + h/2$.

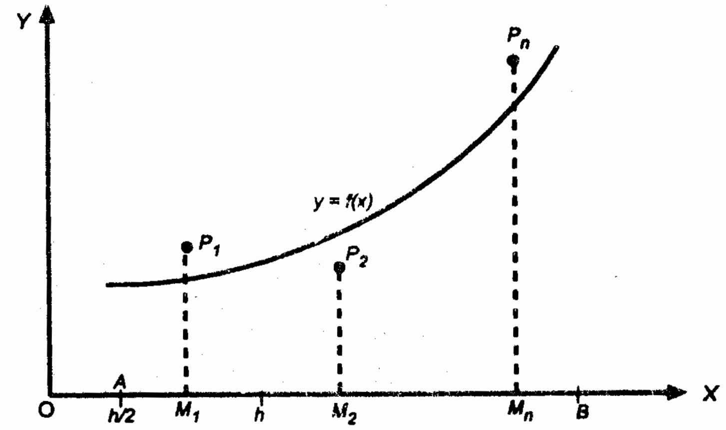

Fig 1.5

Similarly, $P_n(x = x_n)$ can be taken as the value of y at the midpoint of the interval $(x_n - h/2, x_n + h/2)$. If A and B be the points on ox such that $OA = x_1 - h/2$ and $OB = x_n + h/2$,

$$\gamma_1 = \int y\,dx = \int f(x)\,dx = \int_{x_1-h/2}^{x_n+h/2} f(x)\,dx$$

$$\gamma_2 = \int x\,f(x)\,dx = \int_{x_1-h/2}^{x_n+h/2} x\,f(x)\,dx$$

$$\gamma_3 = \int x^2 f(x)\,dx = \int_{x_1-h/2}^{x_n+h/2} x^2 f(x)\,dx \text{ and so on.}$$

Now using Eqn (1.18) [called observation equations], we can solve for the unknowns involved in $f(x)$.

Note: If $y = a + bx$ then it involves only two unknowns. Hence, the observation equations are $\mu_1 = \gamma_1$ and $\mu_2 = \gamma_2$.

For parabola $y = a + bx + cx^2$, the observation equations are $\mu_1 = \gamma_1$, $\mu_2 = \gamma_2$ and $\mu_3 = \gamma_3$ since it involves three unknowns. In general, *number of unknowns = number of observation equations.*

Example 1.16 Fit a straight line and a parabola to the following data

x	1	2	3	4	5	6
y	4	8	10	12	16	20

by the method of moments

Solution

(i) *Fitting a straight line.*

Let the equation be

$$y = a + bx \qquad \text{(i)}$$

This involves only two unknowns, a and b.

$\therefore$ Observation equations are

$$\mu_1 = \gamma_1 \qquad \text{(ii)}$$

and

$$\mu_2 = \gamma_2 \qquad \text{(iii)}$$

Here, $h = 1$,

$$\mu_1 = h\sum y = 1\,(4 + 8 + 10 + 12 + 16 + 20) = 70$$
$$\mu_2 = h\sum xy = 1\,(4 + 16 + 30 + 48 + 80 + 120) = 298$$

The limits of integration are $[1 - 1/2, 6 + 1/2]$, i.e $[0.5, 6.5]$.

$$\therefore \quad \gamma_1 = \int_{0.5}^{6.5} y\,dx = \int_{0.5}^{6.5}(a+bx) = \left[ax+\frac{bx^2}{2}\right]_{0.5}^{6.5} = 6a+21b.$$

$$\gamma_2 = \int_{0.5}^{6.5} xy\,dx = \int_{0.5}^{6.5}(ax+bx^2)\,dx$$

$$= \left[\frac{ax^2}{2}+\frac{bx^3}{3}\right]_{0.5}^{6.5} = 21a+91.5b$$

$\therefore$ From the observation equations (ii) and (iii), we have

$70 = 6a + 21b$ and $298 = 21a + 91.5b$

Solving these equations, we get

$a = 1.3611$ and $b = 2.9444$

Hence, the required equation is

$y = 1.3611 + 2.9444x$

(ii) Fitting a parabola. Let the equation be $y = a + bx + cx^2 \cdots$ (i)
involving three unknowns a, b and c.

$\therefore \mu_1 = \gamma_1$, $\mu_2 = \gamma_2$, and $\mu_3 = \gamma_3$ are the observation equations.

Here, $h = 1$. $\mu_1 = h\sum y = 70$, $\mu_2 = h\sum xy = 298$
and $\mu_3 = h\sum x^2y = 1.[4 + 32 + 90 + 192 + 400 + 720] = 1438$

Limits of integration is [0.5, 6.5].

$$\gamma_1 = \int_{0.5}^{6.5}(a+bx+cx^2)\,dx = \left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}\right]_{0.5}^{6.5}$$

$$= 6a+21b+91.5c$$

$$\gamma_2 = \int_{0.5}^{6.5} x(a+bx+cx^2)\,dx = \left[a\frac{x^2}{2}+b\frac{x^3}{3}+c\frac{x^4}{4}\right]_{0.5}^{6.5}$$

$$= 21a+91.5b+446.25c.$$

$$\gamma_3 = \int_{0.5}^{6.5} x^2(a+bx+cx^2)\,dx = 91.5a+446.25b+2320.575c.$$

Thus, the observation equations are

$$70 = 6a+21b+91.5c$$
$$298 = 21a+91.5b+446.25c$$
$$1438 = 91.5a+446.25b+2320.575c$$

Solving these equations, we get
$a = 1.2540$, $b = 3.0255$ and $c = -0.0116$

Hence, the parabola is

$$y = 1.254 + 3.0255x - 0.0116x^2$$

Example 1.17 By using the method of moments fit a parabola to the following data:

x	1	2	3	4
y	0.30	0.64	1.32	5.40

(M.U. B.E., 1997)

Solution Let the equation be

$$y = a + bx + cx^2 \qquad \text{(i)}$$

involving three constants a, b and c.

$\therefore$ The observation equations are

$\mu_1 = \gamma_1$, $\mu_2 = \gamma_2$ and $\mu_3 = \gamma_3$. Here $h = 1$.

$$\mu_1 = h\Sigma y = 1(0.30 + 0.64 + 1.32 + 5.40) = 7.66$$

$$\mu_2 = h\Sigma xy = 1(0.30 + 1.28 + 3.96 + 21.6) = 27.14$$

$$\mu_3 = h\Sigma x^2 y = 1(0.30 + 2.56 + 11.88 + 86.4) = 101.14$$

Limits of integration is [0.5, 4.5]

$$\text{Now } \gamma_1 = \int_{0.5}^{4.5} y\, dx = \int_{0.5}^{4.5} (a + bx + cx^2)\, dx$$

$$= \left[ax + bx^2/2 + cx^3/3\right]_{0.5}^{4.5}$$

$$\therefore \qquad \gamma_1 = 4a + 10b + 30.3333c$$

$$\gamma_2 = \int_{0.5}^{4.5} xy\, dx = \int_{0.5}^{4.5} x(a + bx + cx^2)\, dx$$

$$= \left[ax^2/2 + bx^3/3 + cx^4/4\right]_{0.5}^{4.5}$$

$$\therefore \qquad \gamma_2 = 10a + 30.3333b + 102.5c$$

$$\gamma_3 = \int_{0.5}^{4.5} x^2 y\, dx = \int_{0.5}^{4.5} x^2(a + bx + cx^2)\, dx$$

$$= \left[ax^3/3 + bx^4/4 + cx^5/5\right]_{0.5}^{4.5}$$

$$\therefore \qquad \gamma_3 = 30.3333a + 102.5b + 369.05c$$

Thus the observation equations are

$$7.66 = 4a + 10b + 30.3333c$$

$$27.14 = 10a + 30.3333b + 102.5c$$
$$101.14 = 30.3333a + 102.5b + 369.05c$$

Solving, we get

$$a = 1.399,\ b = -1.7856 \text{ and } c = 0.6567$$

$\therefore$ From (i) the required parabola is

$$y = 1.399 - 1.7856x - 0.6567x^2$$

EXERECISE 1.4

1. Use the method of moments to fit a straight line to the data given below :

x	1	3	5	7	9
y	1.5	2.8	4.0	4.7	6.0

(M.K.U.,1976)

2. Fit a parabola of the form $y = ax^2 + bx + c$ to the data

x	1	2	3	4
y	1.7	1.8	2.3	3.2

by the method of moments. *(Coimbatore, B.E.,1988)*

ANSWERS

1 $y = 1.1845 + 0.5231x$

2. $y = 0.74x^2 + 0.063x + 1.53$

CHAPTER 2

Theory of Equations

2.1 INTRODUCTION

An expression of the form
$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ where n is a positive integer and $a_0, a_1, a_2, \ldots, a_n$ are real constants, is called a *polynomial* in x of nth degree if $a_0 \neq 0$.

An equation of the form

$$f(x) = 0 \tag{2.1}$$

is called an *algebraic equation* in x or a *polynomial equation* in x of nth degree. If α is a root of the equation $f(x) = 0$, i.e. $f(\alpha) = 0$ then $(x - \alpha)$ is a factor of the polynomial $f(x)$.

Let us first recall some of the fundamental theorems (without proof) before we proceed further.

1. Every equation of the form $f(x) = 0$ has at least one root, either real or complex.
2. Every polynomial equation of nth degree has n and only n roots.
3. If $f(x) = 0$ is a polynomial equation and if $f(a)$ and $f(b)$ are of different signs, then $f(x) = 0$ must have atleast one real root in between a and b.
4. If $f(x) = 0$ is an equation of odd degree, then it has atleast one real root whose sign is opposite to that of the last term.
5. If $f(x) = 0$ is an equation of even degree whose last term is (–)ve, then it has atleast one (+)ve root and atleast one (–)ve root.
6. If $f(x) = 0$ has no real root between a and b $(a < b)$, then $f(a)$ and $f(b)$ are of the same sign.

2.2 RELATIONSHIPS BETWEEN ROOTS AND COEFFICIENTS

In this section, we will see the relationships between the roots and coefficients of the equation $f(x) = 0$. Consider the following equation:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \tag{2.2}$$

Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ be the roots of $f(x) = 0$. Then $(x - \alpha_1)$, $(x - \alpha_2)$, $(x - \alpha_3), \cdots, (x - \alpha_n)$ are the factors of Eqn (2.2).

$$\begin{aligned}\therefore f(x) &= a_0(x - \alpha_1), (x - \alpha_2), (x - \alpha_3), \cdots, (x - \alpha_n)\\ &= a_0[x^n - (\Sigma\alpha_1)\, x^{n-1} + (\Sigma\alpha_1\alpha_2)\, x^{n-2} - (\Sigma\alpha_1\alpha_2\alpha_3)\, x^{n-3}\\ &\quad + \cdots + (-1)^n (\alpha_1\alpha_2\alpha_3 \cdots \alpha_n)]\end{aligned} \tag{2.3}$$

where $\Sigma\alpha_1\alpha_2\cdots\alpha_k$ denotes the sum of the products of $\alpha_1, \alpha_2, \cdots, \alpha_k$ taken k at a time.

Equating the coefficients of like powers on both sides of Eqn (2.3) using Eqn (2.2), we get the required relations as

$$\begin{aligned}&\Sigma\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = -a_1/a_0\\ &\Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-1}\alpha_n = -a_2/a_0\\ &\Sigma\alpha_1\alpha_2\alpha_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3\alpha_4 + \cdots + \alpha_{n-2}\alpha_{n-1}\alpha_n = -a_3/a_0\\ &\quad\cdots \quad \cdots \quad \cdots \quad \cdots\\ &\quad\cdots \quad \cdots \quad \cdots \quad \cdots\\ &\alpha_1\alpha_2\alpha_3 \ldots\ldots \alpha_{n-1}\alpha_n = (-1)^n a_n/a_0.\end{aligned}$$

Note: (i) If α, β are the roots of $ax^2 + bx + c = 0$, where $a \neq 0$, then $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

(ii) If α, β, γ are the roots of $ax^3 + bx + c = 0$, where $a \neq 0$, then $\alpha + \beta + \gamma = -b/a$; $\alpha\beta + \beta\gamma + \gamma\alpha = c/a$ and $\alpha\beta\gamma = -d/a$.

(iii) If $\alpha, \beta, \gamma, \delta$ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a \neq 0$, then $\alpha + \beta + \gamma + \delta = -b/a$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c/a$

$\alpha\beta\gamma + \alpha\gamma\delta + \beta\gamma\delta + \alpha\beta\delta = -d/a$

and $\alpha\beta\gamma\delta = e/a$.

Example 2.1 Solve the equation $x^3 - 9x^2 + 26x - 24 = 0$ given that the roots are in arithmetic progression.

Solution Let the roots be

$$\alpha - \beta, \alpha, \alpha + \beta \tag{i}$$

Sum of the roots = $\alpha - \beta + \alpha + \alpha + \beta = 3\alpha = -(-9)/1$

$$\therefore \alpha = 3$$

Product of the roots is $(\alpha - \beta)\,\alpha\,(\alpha + \beta) = -(-24)/1$

or $$\alpha(\alpha^2 + \beta^2) = 24$$

or $\quad 3(9-\beta^2)=24 \quad (\because \alpha=3)$

$\therefore -\beta^2 = 24/3 - 9$ or $\beta = \pm 1$

Substituting $\alpha=3$, $\beta=+1$ in Eqn (i), we get the roots as 2, 3, 4.

Taking $\alpha=3$, $\beta=-1$ in Eqn (i), we get the roots as 4, 3, 2. (The same roots in either case.)

Example 2.2 Solve the equation $x^3+x^2-16x+20=0$ given that the difference between two roots is 7.

Solution Let the roots of the given equation be α, $\alpha+7$, β.

Sum of the roots is $\alpha+(\alpha+7)+\beta=-1$

or $\quad 2\alpha+\beta=-8 \qquad$ (i)

Sum of products of roots taken two at a time is

$$\alpha(\alpha+7)+(\alpha+7)\beta+\alpha\beta=-16$$

i.e. $$\alpha(\alpha+7)+\beta(2\alpha+7)=-16 \qquad \text{(ii)}$$

Eliminating β from Eqns (i) and (ii), we get

$$\alpha(\alpha+7)+(-8-2\alpha)(2\alpha+7)=-16$$

or $\quad 3\alpha^2+23\alpha+40=0$

or $\quad (\alpha+5)(3\alpha+8)=0 \quad \therefore \alpha=-5$ or $\alpha=-8/3$

But the product of the roots is $\alpha(\alpha+7)\beta=-20$, which is satisfied only by $\alpha=-5$

$\therefore \beta = -8-2(-5)=2.$

Hence the roots are $-5, 2, 2$.

Example 2.3 Find the condition in which the equation $x^4+px^3+qx^2+rx+s=0$ be such that the sum of two roots is equal to the sum of the other two roots.

Solution Let $\alpha, \beta, \gamma, \delta$ be the roots of the given equation. It is given that

$\alpha+\beta=\gamma+\delta=k$ (say)

$\therefore x^4+px^2+qx^2+rx+s=(x^2-kx+a)(x^2-kx+b)$

Equating like coefficients on both sides, we get

$$\begin{aligned} -2k &= p & \therefore\ & k=-p/2 \\ k^2+a+b &= q & \therefore\ & a+b=q-p^2/4 \\ -ka-kb &= r & \therefore\ & a+b=r/-k=2r/p \end{aligned}$$

By equating $a+b$, we get

$q-p^2/4=2r/p$, i.e. $4pq-p^3=8r$

which is the required condition.

Example 2.4 Solve $2x^3 - x^2 - 22x - 24 = 0$ such that the two roots are in the ratio 3 : 4.

Solution Let the roots of the given equation be 3α, 4α, β.

Sum of the roots is $3\alpha + 4\alpha + \beta = 1/2$, $\beta = 1/2 - 7\alpha$ (i)

Sum of product of the roots taken two at a time is

$3\alpha . 4\alpha + 4\alpha . \beta + \beta . 3\alpha = -11$

i.e. $12\alpha^2 + 7\alpha\beta = -11$

or $12\alpha^2 + 7\alpha(1/2 - 7\alpha) = -11$ [using Eqn (i)]

or $74\alpha^2 - 7\alpha - 22 = 0$

or $(2\alpha + 1)(37\alpha - 22) = 0$ $\therefore \alpha = 1/2$ or $22/37$

$\therefore$ From Eqn (i), $\beta = 4$ or $-271/74$

But the product of the roots is

$(3\alpha)(4\alpha)\beta = 12$ or $\alpha^2\beta = 1$

which is satisfied only by $\alpha = -1/2$, $\beta = 4$.

Therefore, the other set of values is dropped. Hence, the roots of the given equation are $-3/2, -2, 4$.

Example 2.5 Solve the equation $x^5 - 5x^4 - 5x^3 + 25x^2 + 4x - 20 = 0$ given that its roots are of the form $\pm a$, $\pm b$, c.

Solution Sum of the roots is

$$a - a + b - b + c = 5 \quad \therefore c = 5$$

But c is a root of the given equation. Hence, $x - 5$ is a factor of the given equation.

$$\begin{array}{r|rrrrrr} x = 5 & 1 & -5 & -5 & 25 & 4 & -20 \\ & 0 & 5 & 0 & -25 & 0 & 20 \\ \hline & 1 & 0 & -5 & 0 & 4 & \boxed{0} \end{array}$$

$\therefore$ The depressed equation (by synthetic division) is

$$x^4 - 5x^2 + 4 = 0 \text{ or } (x^2 - 4)(x^2 - 1) = 0$$

$$\therefore \qquad x = \pm 2, \pm 1$$

Hence, the roots are ± 2, ± 1 and 5.

Example 2.6 Find the conditions in which the roots of

$$ax^3 + bx^2 + cx + d = 0,$$

where $a \neq 0$, are in (i) arithmetical progression, (ii) geometrical progression, and (iii) harmonic progression.

Solution

(i) Let the roots of the given equation be in AP. Therefore, they can be $\alpha - \beta$, α, $\alpha + \beta$.

Sum of the roots, $\alpha - \beta + \alpha + \alpha + \beta = -b/a$

$\therefore \qquad 3\alpha = -b/a$ or $\alpha = -b/3a$

But α is a root of the given equation.

$$\therefore \qquad a\alpha^3 + b\alpha^2 + c\alpha + d = 0$$

$$\text{or} \qquad a\left(\frac{-b}{3a}\right)^3 + b\left(\frac{-b}{3a}\right)^2 + c\left(\frac{-b}{3a}\right) + d = 0$$

or $2b^3 - 9abc + 27da^2 = 0$, which is the required condition.

(ii) Let the roots of the given equation be in GP. Taking them as α/β, α, $\alpha\beta$, the product of the roots will be $(\alpha/\beta)(\alpha)(\alpha\beta) = -d/a$.

$$\therefore \qquad \alpha^3 = -d/a \qquad \text{(i)}$$

But α is a root of the given equation.

$$\therefore \qquad a\alpha^3 + b\alpha^2 + c\alpha + d = 0$$

$$a(-d/a) + \alpha(b\alpha + c) + d = 0$$

$$\therefore \qquad \alpha(b\alpha + c) = 0$$

$$\Rightarrow \alpha = -c/d \,(\alpha \neq 0) \qquad \text{(ii)}$$

Putting Eqn (ii) in Eqn (i), we get

$$\left(-\frac{c}{b}\right)^3 = -d/a \Rightarrow c^3a = b^3d$$

which is the required condition.

(iii) Let the roots of the given equation be in HP. Putting $x = 1/y$ in the given equation, it reduces to

$$dy^3 + cy^2 + by + a = 0 \qquad \text{(iii)}$$

Now, the roots of Eqn (iii) will be in AP. Let $\alpha - \beta$, α, $\alpha + \beta$ be the roots of Eqn (iii).

Sum of the roots $= 3\alpha = -c/d$ or $\alpha = -c/3d$

But α is a root of Eqn (iii).

$$\therefore \qquad d\alpha^3 + c\alpha^2 + b\alpha + a = 0$$

$$\text{or} \qquad d\left(-\frac{c}{3d}\right)^3 + c\left(-\frac{c}{3d}\right)^2 + b\left(-\frac{c}{3d}\right) + a = 0$$

or $\quad 2c^3 - 9bcd + 27d^2a = 0$, which is the required condition.

2.3 EQUATIONS WITH REAL COEFFICIENTS AND IMAGINARY ROOTS

In a polynomial equation $f(x) = 0$ with real coefficients, imaginary roots occur in conjugate pairs.

Let $\alpha + i\beta$ be a root of $f(x) = 0$, where α and β are real, and $\beta \neq 0$.

Then $[x - (\alpha + i\beta)][x - (\alpha - i\beta)] = (x - \alpha)^2 + \beta^2$

Now, when $f(x)$ is divided by $(x - \alpha)^2 + \beta^2$, the remainder must be an expression of first degree in x. Let $g(x)$ be the quotient and $Ax + B$ be the remainder.

Then $f(x) = [(x - \alpha)^2 + \beta^2]g(x) + Ax + B$

$$= [x - (\alpha + i\beta)][x - (\alpha - i\beta)]g(x) + Ax + B \tag{2.4}$$

But $f(\alpha + i\beta) = 0 \because \alpha + i\beta$ is a root of $f(x) = 0$

$\therefore f(\alpha + i\beta) = A(\alpha + i\beta) + B = 0$

$\Rightarrow A\alpha + B = 0$ and $\beta A = 0$

$\because \beta \neq 0 \Rightarrow A = 0$ and hence, $B = 0$.

$\therefore$ From Eqn (2.4), $f(x) = [x - (\alpha + i\beta)][x - (\alpha - i\beta)]g(x)$

$\Rightarrow x - (\alpha - i\beta)$ is a factor of $f(x)$.

$\Leftrightarrow x = \alpha - i\beta$ is a root of $f(x) = 0$.

Example 2.7 Solve the equation $x^3 + 6x + 20 = 0$, of which $1 - 3i$ is a root.

Solution Since $1 - 3i$ is a root, its conjugate $1 + 3i$ is also a root of the given equation.

$$\therefore \quad [x - (1 - 3i)][x - (1 + 3i)] = x^2 - 2x + 10$$

When the given polynomial is divided by $x^2 - 2x + 10$, the remainder is zero and the quotient must be an expression of first degree in x.

$$\therefore f(x) = x^3 + 6x + 20 = (x^2 - 2x + 10)(ax + b)$$
$$= ax^3 + (b - 2a)x^2 + (10a - 2b)x + 10b$$

Comparing the like coefficients on both sides, we get

$a = 1, b - 2a = 0, 10a - 2b = 6, 10b = 20$

$\Rightarrow a = 1$ and $b = 2$

$\therefore x^3 + 6x + 20 = (x^2 - 2x + 10)(x + 2)$

$\therefore$ The third root of $x^3 + 6x + 20 = 0$ is -2.

2.4 EQUATIONS WITH RATIONAL COEFFICIENTS AND IRRATIONAL ROOTS

In an equation with rational coefficients, irrational roots in the form of quadratic surds occur in conjugate pairs.

Let $f(x) = 0$ be the equation and $\alpha + \sqrt{\beta}$ be a root of it, where $\beta \neq 0$ and α, β are rational.

Then, $[x - (\alpha + \sqrt{\beta})]\,[x - (\alpha - \sqrt{\beta})] = (x - \alpha)^2 - \beta$

Therefore, when $f(x)$ is divided by $(x - \alpha)^2 - \beta$, let $h(x)$ be the quotient and the remainder be $ax + b$.

i.e. $f(x) = [(x-\alpha)^2 - \beta]h(x) + ax + b$

or $f(x) = [x - (\alpha + \sqrt{\beta})][x - (\alpha - \sqrt{\beta})]h(x) + ax + b$ (2.5)

Now $f(\alpha + \sqrt{\beta}) = 0 \because x = \alpha + \sqrt{\beta}$ is a root of $f(x) = 0$

i.e. $f(\alpha + \sqrt{\beta}) = a(\alpha + \sqrt{\beta}) + b = 0$

$\Rightarrow a\alpha + b = 0$ and $a\sqrt{\beta} = 0$

$\Leftrightarrow a = 0$ and $b = 0$ $(\because \beta \neq 0)$

$\therefore$ From Eqn (2.5), $f(x) = [x - (\alpha + \sqrt{\beta})][x - (\alpha - \sqrt{\beta})]h(x)$

$\Rightarrow x - (\alpha - \sqrt{\beta})$ is a factor of $f(x)$

In other words, $\alpha - \sqrt{\beta}$ is a root of $f(x) = 0$.

Example 2.8 Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$, given that $3 + 2\sqrt{2}$ is a root.

Solution Since $3 + 2\sqrt{2}$ is a root, its conjugate $3 - 2\sqrt{2}$ will also be a root of the given equation.

$\therefore [x - (3 + 2\sqrt{2})][x - (3 - 2\sqrt{2})] = x^2 - 6x + 1$

when $f(x) = x^4 - 10x^3 + 26x^2 - 10x + 1$ is divided by $x^2 - 6x + 1$, the quotient is a second degree expression and the remainder is zero.

Let $f(x) = x^4 - 10x^3 + 26x^2 - 10x + 1 = (x^2 - 6x + 1)(x^2 + ax + 1)$

Equating the coefficient of x^3 on both sides,

$$-10 = a - 6 \text{ or } a = -4.$$

$$\therefore f(x) = (x^2 - 6x + 1)(x^2 - 4x + 1).$$

Solving $x^2 - 4x + 1 = 0$, we get $x = 2 \pm \sqrt{3}$

Hence, the roots of $f(x)$ are $3 \pm 2\sqrt{2}$ and $2 \pm \sqrt{3}$.

EXERCISE 2.1

1. Solve $x^3 + 6x + 20 = 0$, one root being -2.
2. Solve $x^3 - 12x^2 + 39x - 28 = 0$, whose roots are in arithmetic progression. *(M.U., B.E. 1995)*
3. Solve $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, whose roots are in arithmetic progression.
4. Solve $27x^3 + 42x^2 - 28x - 8 = 0$, the roots of which are in geometric progression. *(M.U., B.E. 1994)*
5. Solve $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$, whose roots are in geometric progression.
6. Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$ whose roots are in harmonic progression.

7. Solve $15x^4 - 8x^3 - 14x^2 + 8x - 1 = 0$, whose roots are in harmonic progression.
8. Solve $x^3 - 8x^2 + 9x + 18 = 0$ given that two of its roots are in the ratio 1 : 2.
9. The equation $x^4 - 4x^3 + px^2 + 4x + q = 0$ has two pairs of equal roots. Find the values of p and q.
10. Solve the equation $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$, given that the sum of two of the roots is equal to the sum of the other two.
11. Solve $x^4 - 8x^3 + 23x^2 - 28x + 12 = 0$, given that the difference of two roots is equal to the difference of the other two.
12. Solve the equation $x^4 - 8x^3 + 7x^2 + 36x - 36 = 0$, given that product of two roots is negative of the product of the remaining two.
13. Solve $x^3 - 4x^2 - 20x + 48 = 0$, given that the relationship between two roots, α and β, is $\alpha + 2\beta = 0$.
14. Find the conditions in which the cubic $x^3 + px^2 + qx + r = 0$ should have its roots in

    (i) arithmetical progression (*M.U B.E., 1993, 1994*) (ii) geometrical progression and (iii) harmonic progression.
15. Solve the equation $3x^3 - 4x^2 + x + 88 = 0$, given that $2 - i\sqrt{7}$ is a root.
16. Solve $3x^5 - 4x^4 - 42x^3 + 56x^2 + 27x - 36 = 0$, given that $\sqrt{2} + \sqrt{5}$, $-\sqrt{2} - \sqrt{5}$ are two roots.

ANSWERS

1. $1 \pm 3i, 2$
2. $1, 4, 7$
3. $-4, -1, 2, 5$
4. $-2/9, -2/3, -2$
5. $-2, -4, -1, -8$
6. $-1/2, 2, 1/3$
7. $-1, 1, 1/3, 1/5$
8. $3, 6, -1$
9. $p = 2, q = 1$
10. $-1, 1, 3, 5$
11. $1, 2, 2, 3$
12. $3, -2, 1, 6$
13. $-7, 2, 6$
14. (i) $2p^3 - 9pq + 27r = 0$ (ii) $p^3r = q^3$ (iii) $2q^3 - 9pqr + 27r^2 = 0$
15. $2 \pm i\sqrt{7}, -8/3$
16. $\sqrt{2} \pm \sqrt{5}, -\sqrt{2} \pm \sqrt{5}, 4/3$

2.5 SYMMETRIC FUNCTIONS OF ROOTS

A symmetric function of the roots of an equation is a function in which all the roots are involved alike, so that the expression remains unaltered even if two of the roots are interchanged.

For example, if α, β and γ are the roots of a cubic equation then,

$$\Sigma\alpha = \Sigma\beta = \Sigma\gamma = \alpha + \beta + \gamma$$
$$\Sigma\alpha\beta = \Sigma\beta\gamma = \alpha\beta + \beta\gamma + \gamma\alpha$$

Example 2.9 If α, β and γ are the roots of $x^3 + px^2 + qx + r = 0$, find the value of

(i) $\Sigma\alpha^2$ (ii) $\Sigma\alpha^2\beta^2$
(iii) $\Sigma\alpha^2\beta\gamma$ (iv) $\Sigma(\alpha - \beta)^2$
(v) $\Sigma(\beta/\gamma + \gamma/\beta)$ (vi) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$

Solution If α, β and γ are the roots of $x^3 + px^2 + qx + r = 0$,

then
$$\Sigma\alpha = \alpha + \beta + \gamma = -p$$
$$\Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = -q$$
and
$$\alpha\beta\gamma = -r$$

(i) $\Sigma\alpha^2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$
$= (-p)^2 - 2q = p^2 - 2q$

(ii) $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$
$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$
$= q^2 - 2(-p)(-r) = q^2 - 2pr$

(iii) $\Sigma\alpha^2\beta\gamma = \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$
$= \alpha\beta\gamma(\alpha + \beta + \gamma)^2 = (-r)(-p) = rp$

(iv) $\Sigma(\alpha - \beta)^2 = (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$
$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= 2(p^2 - 2q) - 2q = 2p^2 - 6q$

(v) $\Sigma\left[\frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right] = \Sigma\left[\frac{\beta^2 + \gamma^2}{\beta\gamma}\right]$

$$= \frac{\beta^2 + \gamma^2}{\beta\gamma} + \frac{\gamma^2 + \alpha^2}{\gamma\alpha} + \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{1}{\alpha\beta\gamma}[\alpha(\beta^2 + \gamma^2) + \beta(\gamma^2 + \alpha^2) + \gamma(\alpha^2 + \beta^2)]$$

$$= \frac{1}{\alpha\beta\gamma}[(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma]$$

$$= (-pq + 3r)/r$$

(vi) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

$$= (\Sigma\alpha-\alpha)(\Sigma\alpha-\beta)(\Sigma\alpha-\gamma)$$
$$= (-p-\alpha)(-p-\beta)(-p-\gamma)$$
$$= -[p^3+p^2(\Sigma\alpha)+p(\Sigma\alpha\beta)+\alpha\beta\gamma]$$
$$= -[p^3+p^2(-p)+p(q)-r]$$
$$= r-pq$$

2.6 FORMATION OF EQUATIONS WHOSE ROOTS ARE GIVEN

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of $f(x) = 0$.

Now let us formulate the equation whose roots are $\beta_1, \beta_2, \cdots, \beta_n$, where $\beta_1, \beta_2, \cdots, \beta_n$ are the symmetric functions of $\alpha_1, \alpha_2, \cdots, \alpha_n$. Here, we first express β_1 as a function of α_1 alone, i.e. $\beta_1 = \phi(\alpha_1)$. Then clearly, $\beta_2 = \phi(\alpha_2), \cdots, \beta_n = \phi(\alpha_n)$. If y is a general root of the required equation and x, that of the given equation, then $y = \phi(x)$. Eliminating x between $f(x) = 0$ and $y = \phi(x)$, the resulting equation in y is the required one.

Example 2.10 If α, β and γ are the roots of $x^3 + ax^2 + bx + c = 0$, form the equation whose roots are

(i) $\alpha+\beta,\ \beta+\gamma,\ \gamma+\alpha$ and (ii) $\dfrac{\alpha}{\beta+\gamma}, \dfrac{\beta}{\gamma+\alpha}, \dfrac{\gamma}{\alpha+\beta}$

Solution Given that α, β, and γ are the roots of

$$x^3 + ax^2 + bx + c = 0 \tag{i}$$

$\therefore\ \alpha+\beta+\gamma = -a;\ \alpha\beta+\beta\gamma+\gamma\alpha = b;\ \alpha\beta\gamma = -c$

(i) Let the new root be denoted by y,

$y = \alpha+\beta = -a-r = -a-x$ [$\because \gamma$ is a root of Eqn (i)]

$\therefore\ x = -(a+y)$ satisfies Eqn (i).

i.e. $$-(a+y)^3 + a(a+y)^2 - b(a+y) + c = 0$$

or $$y^3 - 2ay^2 + (a^2+b)y + ab - c = 0$$

which is the required equation.

(ii) Let $y = \dfrac{\alpha}{\beta+\gamma} = \dfrac{\alpha}{-a-\alpha} = \dfrac{x}{-a-x}$ $(\because x = \alpha)$

$\therefore$ $$x = \frac{-ay}{1+y}.$$

Substituting in Eqn (i), we get

$$-\frac{a^3y^3}{(1+y)^3}+a\frac{a^2y^2}{(1+y)^2}-\frac{aby}{1+y}+c=0$$

or $\quad c(1+y)^3-aby(1+y)^2+a^3y^2(1+y)-a^3y^2=0$

or $\quad (c-ab)y^3+(3c-2ab+a^3)y^2+(3c-ab)y+c=0$

which is the required equation.

Example 2.11 If a, b and c are the roots of $x^3+px+r=0$, form the equation whose roots are

(i) $(b-c)^2, (c-a)^2, (a-b)^2$ and (ii) $bc+\frac{1}{a}, ca+\frac{1}{b}, ab+\frac{1}{c}$.

Solution Of the given equation,

$$x^3+px+r=0 \tag{i}$$

a, b, c are the roots. $\therefore a+b+c=0$

$$ab+bc+ca = p,\ abc=-r$$

(i) Let $y=(b-c)^2=(b+c)^2-4bc=(-a)^2-4(abc)/a$

$=a^2+(4r/a)=x^2+(4r/x)$ [$\because a=x$, root of Eqn (i)]

$$\therefore y=x^2-(4r/x) \text{ or } x^3-xy+4r=0 \tag{ii}$$

Subtracting Eqn (ii) from Eqn (i), we get

$$(p+y)x-3r=0 \quad \therefore x=3r/(p+y)$$

Substituting this value of x in Eqn (i), we get

$$\frac{27r^3}{(p+y)^3}+p\frac{3r}{(p+y)}+r=0$$

or $27r^2+3p(p+y)^2+r(p+y)^3=0$

or $y^3+6py^2+9p^2y+4p^3+27r^2=0$

which is the required equation.

(ii) Let $y=bc+\frac{1}{a}=\frac{abc}{a}+\frac{1}{a}$

$$=\frac{-r+1}{a}=\frac{-r+1}{x} \qquad \therefore x=\frac{-r+1}{y}$$

Substituting the value of x in Eqn (i), we get

$$\left[\frac{-r+1}{y}\right]^3+p\left[\frac{-r+1}{y}\right]+r=0$$

or $\quad ry^3+p(1-r)y^2+(1-r)^3=0$

which is the required equation.

2.7 TRANSFORMATION OF EQUATIONS

1. In order to find an equation whose roots are m times the roots of any given equation, consider Eqn (2.1) ($f(x) = 0$) to be the given equation.

Here, x is a root. So the new root will be $y = mx$ or $x = y/m$. Substituting this in Eqn (2.1), we get $f(y/m) = 0$ or $\phi(y) = 0$, which is the required equation.

Example 2.12 If α is a root of $x^3 - 3x^2 + 4x - 8 = 0$ then find an equation for which 3α is a root.

Solution Given that α is a root of $f(x) = x^3 - 3x^2 + 4x - 8 = 0$,

$$\therefore \quad f(\alpha) = \alpha^3 - 3\alpha^2 + 4\alpha - 8 = 0 \tag{i}$$

Let $y = 3\alpha$ be the new root $\therefore\ \alpha = y/3$

Putting in Eqn (i), we get

$$f(y/3) = (y/3)^3 - 3(y/3)^2 + 4(y/3) - 8 = 0$$

or $\quad y^3 - gy^2 + 36y - 216 = 0$

which is the required equation.

[**Note:** This is the same as multiplying the second term by 3, third by 3^2 and fourth by 3^3 (valid in general).]

2. To obtain a transformed equation whose roots are with opposite signs to those of any given equation,

put $x = -y$ in Eqn (2.1) [$f(x) = 0$].

For example, let the given equation be

$$4x^4 + x^3 + x^2 + 2x + 4 = 0$$

By putting $x = -y$ it transforms to $4y^4 - y^3 + y^2 - 2y + 4 = 0$ whose roots are with opposite signs to that of the given equation above.

3. In a similar fashion, putting $x = 1/y$ in any given equation[say, $f(x) = 0$], we obtain a transformed equation whose roots are reciprocal of the roots of the given equation.

For example, if $6x^3 - 11x^2 + 6x - 1 = 0$ is the equation then by putting $x = 1/y$ in it, we get

$$y^3 - 6y^2 + 11y - 6 = 0$$

whose roots are reciprocal to the original.

4. To diminish the roots of an equation $f(x) = 0$ by h, suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the the roots of $f(x) = 0$. Now we require the equation whose roots are $\alpha_1 - h, \alpha_2 - h, \ldots, \alpha_n - h$. Let $y = \alpha_i - h = x - h$.

Therefore, $x = y + h$. Substituting in the given equation, we obtain the transformed equation $f(y + h) = 0$.

The coefficients in the transformed equation can easily be found by synthetic division.

[**Note** : To increase the roots of $f(x) = 0$ by h, put $x = y - h$.]

Example 2.13 Diminish the roots of $x^4 - 8x^3 + 19x^2 - 12x + 2 = 0$ by 2 and hence solve it. *(M.U., B.E., 1996)*

Solution The given equation is

$$x^4 - 8x^3 + 19x^2 - 12x + 2 = 0 \quad \text{(i)}$$

Let $y = x - 2$ be the new root. Putting $x = y + 2$ in Eqn (i), we get

$$(y+2)^4 - 8(y+2)^3 + 19(y+2)^2 - 12(y+2) + 2 = 0.$$

or $y^4 - 5y^2 + 6 = 0$ as the transformed equation.

Now, $(y^2 - 3)(y^2 - 2) = 0$, i.e. $y = \pm\sqrt{3}$ or $y = \pm\sqrt{2}$

Hence the original roots are $x = y + 2$

i.e. $\pm\sqrt{3} + 2$; $\pm\sqrt{2} + 2$.

Alternative method: We can also diminish the roots of Eqn (i) by 2 using synthetic division in the following way, take $h = 2$.

$$\begin{array}{r|rrrrr}
2 & 1 & -8 & 19 & -12 & 2 \\
 & 0 & 2 & -12 & 14 & 4 \\
\hline
2 & 1 & -6 & 7 & 2 & \boxed{6} \\
 & 0 & 2 & -8 & -2 & \\
\hline
2 & 1 & -4 & -1 & \boxed{0} & \\
 & 0 & 2 & -4 & & \\
\hline
2 & 1 & -2 & \boxed{-5} & & \\
 & 0 & 2 & & & \\
\hline
 & 1 & \boxed{0} & & &
\end{array}$$

Now the transformed equation is $y^4 - 5y^2 + 6 = 0$, which is same as obtained above.

Example 2.14 Increase by 2 the roots of $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ and hence solve it.

Solution The given equation is

$$x^4 - x^3 - 10x^2 + 4x + 24 = 0 \quad \text{(i)}$$

Increasing Eqn (i) is equivalent to decreasing the same by -2. Using synthetic division, we have

−2	1	−1	−10	4	24
	0	−2	6	8	−24
−2	1	−3	−4	12	$\boxed{0}$
	0	−2	10	−12	
−2	1	−5	6	$\boxed{0}$	
	0	−2	14		
−2	1	−7	$\boxed{20}$		
	0	−2			
	1	$\boxed{-9}$			

Therefore, the transformed equation is

$$y^4 - 9y^3 + 20y^2 = 0 \quad (\text{where } y = x + 2)$$

or $\quad y^2 (y^2 - 9y + 20) = 0$ or $y^2 (y - 4)(y - 5) = 0$

$\therefore \qquad y = 0, 0, 4, 5$

Now, the roots of Eqn (i) are given by $x = y - 2$

$\therefore \qquad x = -2, -2, 2, 3$

5. Let us now see how to remove the second term in the transformed equation of $f(x) = 0$.

Let $\qquad f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$

If its roots are to be diminished by h, then

$$f(y + h) = a_0(y + h)^n + a_1(y + h)^{n-1} + \cdots + a_n = 0$$

or $\quad a_0 y^n + (a_0nh + a_1)y^{n-1} + \cdots = 0$

If the term y^{n-1} is to be absent, then

$$a_0nh + a_1 = 0, \quad \text{i.e. } h = -a_1/na_0.$$

The second term in the transformed equation vanishes if we diminish the roots of the original equation by h, where h = sum of the roots/degree.

Example 2.15 Remove the second term in the transformed equation of $x^4 - 8x^3 - x^2 + 68x + 60 = 0$ and hence solve it.

Solution Here, sum of the roots is $-(-8/1) = 8$

Degree = 4 $\therefore h = 8/4 = 2$.

Now, diminishing the given equation by 2 (using synthetic division), we have

2	1	−8	−1	68	60
	0	2	−12	−26	84
2	1	−6	−13	42	$\boxed{144}$
	0	2	−8	−42	
2	1	−4	−21	$\boxed{0}$	
	0	2	−4		
2	1	−2	$\boxed{-25}$		
	0	2			
	1	$\boxed{0}$			

Therefore, the transformed equation is

$$y^4 - 25y^2 + 144 = 0 \quad (\text{where } y = x - 2)$$

or $\quad (y^2 - 9)(y^2 - 16) = 0, \quad$ i.e. $y = \pm 3, \ \pm 4$

Hence, the roots of the given equation are

$$x = -1, -2, 5, 6.$$

2.8 MULTIPLE ROOTS

If $f(x) = 0$ has a multiple root α of multiplicity k, then $f'(x) = 0$ has a multiple root α of multiplicity $(k - 1)$.

Let $f(x) = 0$ be of degree n.

Then $f(x) = (x - \alpha)^k g(x)$, where $g(\alpha) \neq 0$.

$$\begin{aligned} \therefore f'(x) &= (x - \alpha)^k g'(x) + k(x - \alpha)^{k-1} g(x) \\ &= (x - \alpha)^{k-1}[(x - \alpha)g'(x) + kg(x)] \\ &= (x - \alpha)^{k-1} \phi(x) \end{aligned}$$

Evidently, $\phi(\alpha) = kg(\alpha) \neq 0$

Therefore, $f'(x) = 0$ has a root a of multiplicity $(k - 1)$.

Example 2.16 Solve $4x^3 + 8x^2 + 5x + 1 = 0$, given that it has a double root.

Solution Let $f(x) = 4x^3 + 8x^2 + 5x + 1 = 0$ possess a double root α.

Then $f'(x) = 12x^2 + 16x + 5 = 0$ has a root α.

$$12x^2 + 16x + 5 = (2x + 1)(6x + 5) = 0$$

$$\therefore \qquad x = -1/2 \text{ or } -5/6$$

But $\quad f(1/2) = -4/8 + 8/4 - 5/2 + 1 = 0$

$\therefore x = -1/2$ is the double root.

Hence, $\quad f(x) = (2x + 1)^2(x + 1) = 0$

$\therefore\ x = -1/2, -1/2, -1$ are the roots

EXERCISE 2.2

1. If α, β, and γ are the roots of the $x^3 + px + q = 0$, then find
   (i) $\Sigma\alpha^3$ *(M.U., B.E., 1994)* (ii) $\Sigma\alpha^2\beta$ and
   (iii) $\Sigma\alpha^4$
2. If α, β and γ are the roots of $x^3 + px^2 + qx + r = 0$, find the values of
   (i) $\Sigma\, 1/\alpha$ (ii) $\Sigma\, \alpha^3$
   (iii) $\Sigma\, \alpha^3\beta$ (iv) $\Sigma\, (\beta^2 + \beta\gamma + \gamma^2)$
   (v) $\Sigma\, (\beta + \gamma - \alpha)^3$, and (vi) $\Sigma\, (\alpha^2 + \beta\gamma)/(\beta + \gamma)$
3. If α, β and γ are the roots of $x^3 + px^2 + qx + \mathrm{r} = 0$, form the equation whose roots are
   (i) $\alpha^2, \beta^2, \gamma^2$ (ii) $\alpha\beta, \beta\gamma, \alpha\gamma$
   (iii) $\alpha(\beta + \gamma), \beta(\gamma + \alpha), \gamma(\alpha + \beta)$ and
   (iv) $\alpha + 1/\beta\gamma, \beta + 1/\alpha\gamma, \gamma + 1/\alpha\beta$.
4. If α, β and γ are the roots of the equation $x^2 + px + q = 0$, obtain the equation whose roots are
   (i) $\alpha + \beta - \gamma, \beta + \gamma - \alpha, \gamma + \alpha - \beta$ *(M.U., B.E., 1993)*
   (ii) $(\alpha + \beta)(\gamma + \alpha), (\beta + \gamma)(\alpha + \beta), (\gamma + \alpha)(\beta + \gamma)$
5. If α, β, and γ are the roots of $x^3 - 7x + 6 = 0$, form an equation whose roots are $(\beta - \gamma)^2, (\gamma - \alpha)^2, (\alpha - \beta)^2$. *(Raipur B.E., 1987)*
6. If α, β, and γ are the roots of $2x^3 + 3x^2 - x - 1 = 0$, obtain an equation whose roots are $(1 - \alpha)^{-1}, (1 - \beta)^{-1}, (1 - \gamma)^{-1}$. *(Kerala B. Tech, 1988)*
7. If α, β, and γ are the roots of $x^3 - 3x + 1 = 0$, form the equation whose roots are $\dfrac{(\alpha - 2)}{(\alpha + 2)}, \dfrac{(\beta - 2)}{(\beta + 2)}, \dfrac{(\gamma - 2)}{(\gamma + 2)}$.
8. If θ is a root of $x^3 + x^2 - 2x - 1 = 0$, then prove that $\theta^2 - 2$ is also a root. *(M.U, B.E., 1993)*
9. If α, β, and γ are the roots of $x^3 + 2x^2 + 3x + 3 = 0$, prove that
$$\frac{\alpha^2}{(\alpha+1)^2} + \frac{\beta^2}{(\beta+1)^2} + \frac{\gamma^2}{(\gamma+1)^2} = 13.$$
10. Find the equation whose roots are -3 times those of $x^4 - 3x^3 + x^2 - 6x + 4 = 0$.
11. Find the equation whose roots are with opposite signs to those of $x^5 - 4x^4 + 3x^3 - 5x^2 + x - 11 = 0$.
12. Find the equation whose roots are reciprocal of the roots of $x^5 - 11x^4 + 7x^3 - 8x^2 + 6x - 13 = 0$.

13. Diminish by 3 the roots of $x^4 + 3x^3 - 2x^2 - 4x - 3 = 0$.

14. Diminish the equation $x^4 - 8x^3 + 19x^2 - 12x + 2 = 0$ by 2 and hence solve it. *(M.U., B.E., 1996)*

15. Increase the roots of $3x^4 + 2x^3 - 10x^2 + 15x - 9 = 0$ by 4.

16. Find the equation whose roots are the roots of the eqution $x^3 - 4x^2 - 3x - 2 = 0$ increased by 2. *(M.U., B.E., 1995)*

17. Remove the second term in $x^4 - 8x^3 - x^2 + 68x + 60 = 0$ and solve it.

18. Diminish the roots of the equation $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$ by 1 and solve it.

19. Increase the roots of the equation $x^4 - 2x^3 - 10x^2 + 6x + 21 = 0$ by 2 and solve it.

20. Solve $x^3 - 4x^2 + 5x - 2 = 0$, given that it has a double root.

ANSWERS

1. (i) $-3q$ (ii) $3q$
   (iii) $2p^2$
2. (i) $-(q/r)$ (ii) $pq - 3r - p^3$
   (iii) $p^2q - 2q^2 - pr$ (iv) $2p^2 - 3q$
   (v) $24r - p^3$
3. (i) $y^3 + (2q - p^2)y^2 + (q^2 - 2pr)y - r^2 = 0$
   (ii) $y^3 - qy^2 + pry - r^2 = 0$
   (iii) $y^3 - 2qy^2 + (pr + q^2)y + (r^2 - prq) = 0$
   (iv) $y^3 + 2py^2 + (p^2 + q)y + pq - r = 0$
4. (i) $y^2 - 2py + 4q = 0$ (ii) $y^3 - py^2 - q^2 = 0$
5. $y^3 - 42y^2 + 441y - 400 = 0$
6. $3y^3 - 11y^2 + 9y - 2 = 0$
7. $y^3 + 33y^2 + 27y + 3 = 0$
10. $y^4 + 9y^3 + 9y^2 + 162y + 324 = 0$
11. $y^5 + 4y^4 + 3y^3 + 5y^2 + y + 11 = 0$
12. $13y^5 - 6y^4 + 8y^3 - 7y^2 + 11y - 1 = 0$
13. $y^4 + 15y^3 + 79y^2 + 173y + 129 = 0$
14. $y^4 + 5y^2 + 6 = 0, 2 \pm \sqrt{3}, 2 \pm \sqrt{2}$
15. $3y^4 - 46y^3 + 254y^2 - 577y + 411 = 0$
16. $y^3 - 10y^2 + 31y - 32 = 0$
17. $-1, -2, 5, 6$
18. $-2, -3, 2, 3$
19. $1 \pm 2\sqrt{2}, \pm\sqrt{3}$
20. $1, 1, 2$

2.9 RECIPROCAL EQUATIONS

An equation $f(x) = 0$ is called a *reciprocal equation* if it remains unaltered when x is replaced by $1/x$. That is, if $f(x) = 0$ has a root α then $1/\alpha$ is also a root of $f(x) = 0$, if it is reciprocal.

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ be the reciprocal equation.

Then

$$f(1/x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 = 0$$

But $f(x) = f(1/x)$, which implies

$$\frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \frac{a_2}{a_{n-2}} = \ldots = \frac{a_{n-1}}{a_1} = \frac{a_n}{a_0}$$

$$\therefore \qquad a_0^2 = a_n^2\,;\, a_1^2 = a_{n-1}^2\,,\, a_2^2 = a_{n-2}^2\,,\ldots$$

or $$a_0 = \pm a_n,\quad a_1 = \pm a_{n-1},\quad a_2 = \pm a_{n-2}, \ldots$$

There arises two cases.

Case 1 $a_0 = a_n,\ a_1 = a_{n-1},\ a_2 = a_{n-2}, \ldots$

Here, it is a reciprocal equation with like signs for its coefficients. The coefficients of the corresponding terms taken from the begining and end are equal.

Case 2 $a_0 = -a_n,\ a_1 = -a_{n-1},\ a_2 = -a_{n-2}, \ldots$

In this case, it is a reciprocal equation with unlike signs for its coefficients. The coefficients of the corresponding terms taken from the begining and end are equal in magnitude but opposite in sign.

Standard Types

There are four standard types of reciprocal equations.

Type I Reciprocal equation of odd degree with like signs for coefficients.

Example 2.17 Solve $x^5 + 8x^4 + 21x^3 + 21x^2 + 8x + 1 = 0$.

(M.U., B.E., 1993, 1994)

Solution Since the sums of alternative coefficients are equal, $x = -1$ is a root of the given equation. Dividing the equation by $(x + 1)$,

$x = -1$	1	8	21	21	8	1
	0	−1	−7	−14	−7	−1
	1	7	14	7	1	0

Now the equation can be written as

$$(x+1)(x^4+7x^3+14x^2+7x^2+1)=0 \qquad \text{(i)}$$

Consider $x^4+7x^3+14x^2+7x^2+1=0$
Dividing by x^2 on both sides, we get

$$x^2+7x+14+7/x+1/x^2=0$$

or
$$\left(x^2+\frac{1}{x^2}\right)+7\left(x+\frac{1}{x}\right)+14=0 \qquad \text{(ii)}$$

Let $x+\dfrac{1}{x}=t \quad \therefore x^2+\dfrac{1}{x^2}=t^2-2$

Putting in Eqn (ii), we get

$$(t^2-2)+7t+14=0 \quad \text{or} \quad t^2+7t+12=0$$

or
$$(t+3)(t+4)=0 \;\therefore\; t=-3 \text{ or } -4$$

Now
$$t=-3 \;\Rightarrow\; x+\frac{1}{x}=-3 \text{ or } x^2+3x+1=0$$

$$\therefore \quad x=\frac{-3\pm\sqrt{5}}{2}$$

Again, $t=-4 \Rightarrow x+\dfrac{1}{x}=-3$ or $x^2+3x+1=0$

$$\therefore \quad x=-2\pm\sqrt{3}$$

Hence, roots of the given equation are -1, $\dfrac{-3\pm\sqrt{5}}{2}$, $-2\pm\sqrt{3}$.

Type II Reciprocal equation of even degree with like signs for its coefficients.

Example 2.18 Solve $6x^4-25x^3+37x^2-25x+6=0$ (*M.U. B.E., 1989*)

Solution Dividing the given equation by x^2 on both sides and adjusting the terms, we get

$$6\left(x^2+\frac{1}{x^2}\right)-25\left(x+\frac{1}{x}\right)+37=0 \qquad \text{(i)}$$

Let $x+\dfrac{1}{x}=t$ so that $x^2+\dfrac{1}{x^2}=t^2-2$

Substituting in Eqn (i), we get

$$6(t^2-2)-25t+37=0$$

or $$6t^2 - 25t + 25 = 0$$

or $$(2t-5)(3t-5) = 0 \quad \therefore\ t = 5/2 \text{ or } 5/3$$

Now $$t = \frac{5}{2} \Rightarrow x + \frac{1}{x} = \frac{5}{2} \qquad \text{or } 2x^2 - 5x + 2 = 0$$

or $$(2x-1)(x-2) = 0 \ \therefore x = 1/2, 2$$

Again, $$t = \frac{5}{3} \Rightarrow x + \frac{1}{x} = \frac{5}{3} \qquad \text{or } 3x^2 - 5x + 3 = 0$$

$$\therefore \qquad x = \frac{5 \pm \sqrt{11}i}{6}$$

Hence, the roots of the given equation are

$$x = 1/2, 2, \frac{5 \pm \sqrt{11}i}{6}.$$

Type III Reciprocal equation of odd degree with unlike coefficients.

Example 2.19 Solve $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$

Solution Here, sum of all the coefficients is 0.

$\therefore x = 1$ is a root of the given equation.

$x = 1$						
	1	−5	9	−9	5	−1
	0	1	−4	5	−4	1
	1	−4	5	−4	1	0

Therefore, the equation can be written as

$$(x-1)(x^4 - 4x^3 + 5x^2 - 4x + 1) = 0 \qquad \text{(i)}$$

Consider $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$

Dividing by x^2 on both sides and adjusting the terms, we get

$$\left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0$$

put $x + \dfrac{1}{x} = t$ so that $x^2 + \dfrac{1}{x^2} = t^2 - 2$

$$\therefore \qquad t^2 - 2 - 4(t) + 5 = 0 \qquad \text{or } t^2 - 4t + 3 = 0$$

$$\therefore \qquad t = 1 \text{ or } 3$$

Now $t = 1 \Rightarrow x + 1/x = 1 \quad$ or $x^2 - x + 1 = 0$

$$\therefore \qquad x = \frac{1 \pm \sqrt{3}i}{2}$$

Again, $t = 3 \Rightarrow x + 1/x = 3$ or $x^2 - 3x + 1 = 0$

$$\therefore \qquad x = \frac{3 \pm \sqrt{5}}{2}.$$

Hence, the roots of the given equation are

$$x = 1, \frac{1 \pm \sqrt{3}i}{2}, \frac{3 \pm \sqrt{5}}{2}.$$

Type IV Reciprocal equation of even degree with unlike signs for the coefficients.

Example 2.20 Solve $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$.

(M.U., B.E., 1987, 1990, 1992, 1994)

Solution Here the sum of all the coefficients is zero and also, sums of alternative coefficients are equal. Therefore, $x = 1$ and $x = -1$ are two roots.

Dividing the given equation by $x = 1, -1$ synthetically, we have

$x = 1$	6	-25	31	0	-31	25	-6
	0	6	-19	12	12	-19	6
$x = -1$	6	-19	12	12	-19	6	0
	0	-6	25	-37	25	-6	
	6	-25	37	-25	6	0	

Therefore, the equation can be written as

$$(x - 1)(x + 1)(6x^4 - 25x^3 + 37x^2 - 25x + 6) = 0 \tag{i}$$

Now consider

$$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0 \tag{ii}$$

Dividing by x^2 on both sides and adjusting the terms, we get

$$6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0$$

Let $\quad x + \dfrac{1}{x} = t$ so that $x^2 + \dfrac{1}{x^2} = t^2 - 2$ (iii)

Putting in Eqn (iii), we get

$$6(t^2 - 2) - 25t + 37 = 0 \text{ or } 6t^2 - 25t + 25 = 0$$

or $\quad (2t - 5)(3t - 5) = 0 \Rightarrow t = 5/2$ or $t = 5/3$

$$\therefore \qquad x = 1/2, 2, \frac{5 \pm i\sqrt{11}}{6}$$ [refer to Example 2.18].

Hence, the roots of the given equation are

$$x = \pm 1, \frac{1}{2}, 2, \frac{5 \pm i\sqrt{11}}{6}$$

EXERCISE 2.3

1. Solve $x^5 + 4x^4 + x^3 + x^2 + 4x + 1 = 0$
2. Solve $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$
3. Solve $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$
   (M.U, B.E., 1987,1990, 1996)
4. Solve $2x^6 - 9x^5 + 10x^4 - 3x^3 + 10x^2 - 9x + 2 = 0$
5. Solve $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$ *(M.U, B.E., 1986)*
6. Solve $2x^4 + x^3 - 6x^2 + x + 2 = 0$ *(M.U, B.E., 1986)*
7. Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$
8. Solve $x^4 + 6x^3 - 5x^2 + 6x + 1 = 0$ *(M.U, B.E., 1988)*
9. Solve $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$ *(M.U, B.E., 1991)*
10. Solve $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$
11. Solve $x^6 + 2x^5 + 2x^4 - 2x^2 - 2x - 1 = 0$ *(M.U, B.E., 1994)*
12. Solve $6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0$ *(M.U, B.E., 1986)*
13. Solve $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$
14. Show that the equation $x^4 - 3x^3 + 4x^2 - 2x + 1 = 0$ transforms into a reciprocal equation by diminishing the root by 1. Hence solve it.
    (M.U, B.E., 1990)
15. Show that $x^4 - 10x^3 + 23x^2 - 6x - 15 = 0$ can be transformed into a reciprocal equation by diminishing the roots by 2. Hence solve it.
    (M.U, B.E., 1993, Coimbatore B.E., 1988)

ANSWERS

1. $-1, \dfrac{1 \pm \sqrt{3}i}{2}, -2 \pm \sqrt{3}i$

2. $-1, \dfrac{-1 \pm \sqrt{3}i}{2}, \dfrac{-1 \pm \sqrt{3}i}{2}$

3. $-1, 2, 1/2, -3, -1/3$

4. $2, 1/2, \dfrac{3 \pm \sqrt{5}}{2}, \dfrac{-1 \pm \sqrt{3}i}{2}$

5. $\dfrac{1 \pm \sqrt{3}i}{2}, \dfrac{1 \pm \sqrt{3}i}{2}$

6. $1, 1, -2, -1/2$

7. $2 \pm \sqrt{3}, 3 \pm 3\sqrt{2}$

8. $\dfrac{-7 \pm 3\sqrt{5}}{2}, \dfrac{1 \pm \sqrt{3}i}{2}$

9. $1, 2, 1/2, -3, -1/3$

10. $1, \dfrac{1 \pm \sqrt{3}i}{2}, \dfrac{-1 \pm \sqrt{3}i}{2}$

11. $\pm 1, \dfrac{-1 \pm \sqrt{3}i}{2}, \dfrac{-1 \pm \sqrt{3}i}{2}$

12. $\pm 1, 2, 1/2, 3, 1/3$

13. $\pm 1, -3, -1/3, \dfrac{3 \pm \sqrt{5}}{2}$

14. $\dfrac{\sqrt{5} + 3 \pm \sqrt{-10 + 2\sqrt{5}}}{4}, \dfrac{-\sqrt{5} + 3 \pm \sqrt{-10 + 2\sqrt{5}}}{4}$

15. $\dfrac{9 \pm \sqrt{21}}{2} \; \dfrac{1 \pm \sqrt{5}}{2}$

CHAPTER 3

Solution to Numerical Algebraic and Transcendental Equations

3.1 INTRODUCTION

We have seen that an expression of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

where a's are constants ($a_0 \neq 0$) and n is a positive integer, is called a *polynomial* in x of degree n, and the equation $f(x) = 0$ is called an *algebraic* equation of degree n. If $f(x)$ contains some other functions like exponential, trigonometric, logarithmic etc., then $f(x) = 0$ is called a *transcendental* equation. For example,

$$x^3 - 3x + 6 = 0, \quad x^5 - 7x^4 + 3x^2 + 36x - 7 = 0$$

are algebraic equations of third and fifth degree, whereas $x^2 - 3\cos x + 1 = 0$, $xe^x - 2 = 0$, $x \log_{10} x = 1.2$ etc., are transcendental equations. In both the cases, if the coefficients are pure numbers, they are called *numerical* equations.

In this chapter, we will solve the numerical, algebraic and transcendental equations. For the algebraic equations of degree two or three or four, methods are available to solve them. But, the need often arises to solve higher degree or transcendental equations for which no direct method exists. Such equations can best be solved by approximate methods. Before we proceed to solve such equations, let us recall the fundamental theorem on roots of $f(x) = 0$ in $a \leq x \leq b$.

THEOREM: If $f(x)$ is continuous in a closed interval $[a, b]$ and $f(a), f(b)$ are of opposite signs, then the equation $f(x) = 0$ will have atleast one real root between a and b.

Proof: This theorem can be verified easily by graphical method. If we draw the graph of $y = f(x)$ in $[a, b]$, where $f(a)$ and $f(b)$ are of opposite signs, then the graph must cut the x-axis atleast once [Fig. 3.1(a)] and always at an odd number of times [Fig. 3.1(b)]. Hence at $C, f(x) = 0$. So there is a root of $f(x) = 0$ in $[a, b]$.

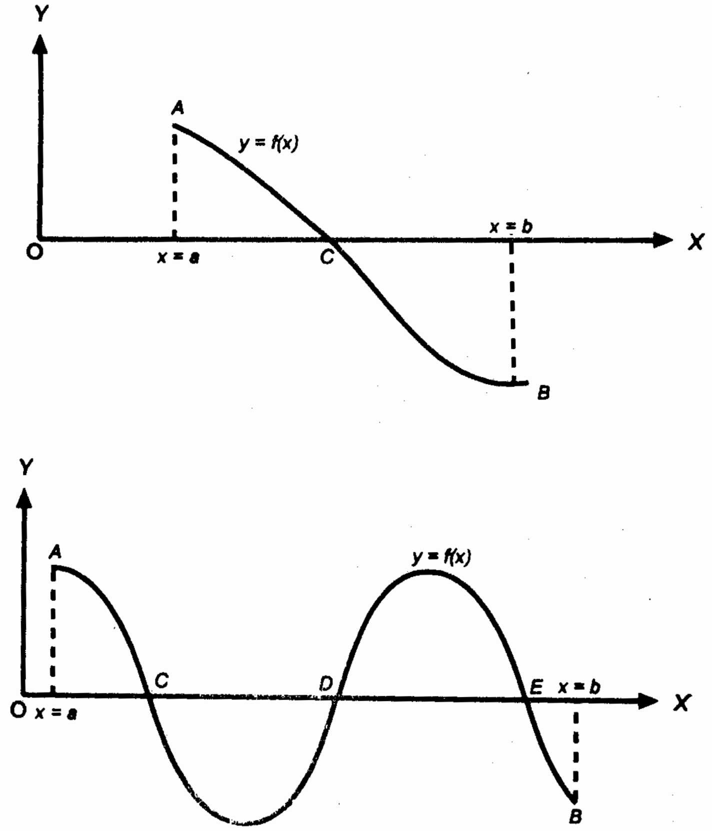

Fig. 3.1

3.2 BISECTION METHOD

Let the function $f(x)$ be continuous between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then there is a root of $f(x) = 0$, lying between a and b. Let the first approximation be $x_0 = \frac{1}{2}(a + b)$ (i.e. average of the ends of the range).

Now if $f(x_1) = 0$ then x_1 is a root of $f(x) = 0$. Otherwise, the root will lie between a and x_1 or x_1 and b depending upon whether $f(x_0)$ is positive or negative.

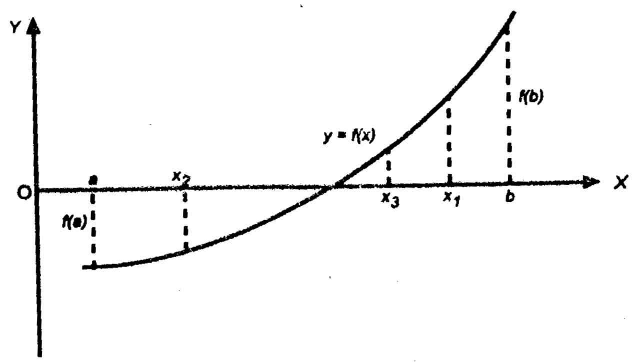

Fig. 3.2

Then, as before, we bisect the interval and continue the process till the root is found to the desired accuracy. In Fig. 3.2, $f(x_1)$ is positive; therefore, the root lies in between a and x_0. The second approximation to the root now is $x_2 = ½ (a + x_1)$. If $f(x_1)$ is negative as shown in the figure then the root lies in between x_2 and x_1, and the third approximation to the root is $x_3 = (x_2 + x_1)/2$ and so on.

This method is simple but slowly convergent. It is also called as *Bolzano method* or *Interval halving method.*

Example 3.1 Find a root of the equation $x^3 - x - 11 = 0$ correct to four decimals using bisection method. *(Gulbarga B.E., 1993)*

Solution Let $f(x) = x^3 - x - 11$.

Since $f(2) = -5 < 0$ and $f(3) = 13 > 0$, a root lies in between 2 and 3. Hence, the first approximation to the root is

$$x_1 = (2 + 3)/2 = 2.5$$

Now $f(2.5) = (2.5)^3 - 2.5 - 11 = 2.215$ (+)ve.
Therefore, the root lies between 2 and 2.5 $(= x_1)$.

Thus the second approximation to the root is

$x_2 = (2 + 2.5)/2 = 2.25$
Now $f(2.25) = (2.25)^3 - 2.25 - 11 = -1.859375$ (–)ve
Therefore, the root lies between x_1 and x_2.

Thus the third approximation to the root is

$x_3 = (x_1 + x_2)/2 = (2.5 + 2.25)/2 = 2.375$
Now $f(2.375) = (2.375)^3 - 2.375 - 11 = 0.0214843$ (+) ve
Therefore, the root lies in between x_2 and x_3.

Thus the fourth approximation to the root is

$x_4 = (x_2 + x_3)/2 = (2.25 + 2.375)/2 = 2.3125$
$f(2.3125) = (2.3125)^3 - 2.3125 - 11 = -0.9460449$ (–) ve

which means that the root lies in between x_3 and x_4.

Thus the fifth approximation to the root is

$x_5 = x_3 + x_4/2 = (2.375 + 2.3125)/2 = 2.34375$
$f(2.34375) = (2.34375)^3 - 2.34375 - 11 = -0.4691467$ (–) ve

Therefore, the root lies in between x_3 and x_5.

Thus the sixth approximation to the root is

$x_6 = (x_3 + x_5)/2 = (2.375 + 2.34375)/2 = 2.359375$
$f(2.359375) = (2.359375)^3 - 2.359375 - 11 = -0.2255592$ (–) ve

Therefore, the root lies in between x_3 and x_6.

Thus the seventh approximation to the root is

$x_7 = (x_3 + x_6)/2 = (2.375 + 2.359375)/2 = 2.3671875$
$f(2.3671875) = (2.3671875)^3 - (2.3671875) - 11$
$= -0.1024708$ (–)ve

which means that the root lies in between x_3 and x_7.

Thus the eighth approximation to the root is

$x_8 = (x_3 + x_7)/2 = (2.375 + 2.3671875)/2 = 2.3710938$
$f(2.3710938) = (2.3710938)^3 - (2.3710938) - 11$
$= -0.040601$ (–)ve

Therefore, the root lies in between x_3 and x_8.

Thus the ninth approximation to the root is

$x_9 = (x_3 + x_8)/2 = (2.375 + 2.3710938)/2 = 2.3730469$
$f(2.3730469) = (2.3730469)^3 - (2.3730469) - 11$
$= -9.585468 \times 10^{-3}$ (–)ve

Therefore, the root lies in between x_3 and x_9.

Thus the tenth approximation to the root is

$x_{10} = x_3 + x_9/2 = (2.375 + 2.3730469)/2 = 2.3740235$
$f(2.3740235) = (2.3740235)^3 - (2.3740235) - 11$
$= 5.942661 \times 10^{-3}$ (+)ve

Therefore, the root lies in between x_9 and x_{10}.

Thus the eleventh approximation to the root is

$x_{11} = (x_9 + x_{10})/2 = (2.3730469 + 2.3740235)/2 = 2.3135352$
$f(2.3735352) = (2.3735352)^3 - 2.3735352 - 11$
$= -1.823101 \times 10^{-3}$ (–)ve

Therefore, the root lies in between x_{10} and x_{11}.

Thus the twelfth approximation to the root is

$$\begin{aligned} x_{12} = (x_{10} + x_{11})/2 &= (2.3740235 + 2.3735352)/2 = 2.3737794 \\ f(2.3737794) &= (2.3737794)^3 - 2.3737794 - 11 \\ &= 2.059952 \times 10^{-3} \text{ (+)ve} \end{aligned}$$

Therefore, the root lies in between x_{11} and x_{12}.

Thus the thirteenth approximation to the root is

$$\begin{aligned} x_{13} = (x_{11} + x_{12})/2 &= (2.3735352 + 2.3737794)/2 = 2.3736573 \\ f(2.3736573) &= (2.3736573)^3 - 2.3736573 - 11 \\ &= 1.18915 \times 10^{-4} \text{ (+)ve} \end{aligned}$$

Therefore, the root lies in between x_{11} and x_{13}.

Thus the fourteenth approximation to the root is

$$\begin{aligned} x_{14} = (x_{11} + x_{13})/2 &= (2.3735352 + 2.3736573)/2 = 2.3735963 \\ f(2.3735963) &= (2.3735963)^3 - 2.3735963 - 11 \\ &= -8.51921 \times 10^{-4} \text{ (–)ve} \end{aligned}$$

Therefore, the root lies in between x_{13} and x_{14}.

Thus the fifteenth approximation to the root is

$x_{15} = (x_{13} + x_{14})/2 = (2.3736573 + 2.3735963)/2 = 2.3736268$ and
$f(2.3736268) = -3.66112 \times 10^{-4}$ (–)ve.

Therefore, from x_{14} and x_{15} we can see that $f(x_{14})$ and $f(x_{15})$ are nearly equal to zero. Hence the root correct to four decimal places is = 2.3736.

Example 3.2 Using bisection method, find the negative root of

$$x^3 - x + 11 = 0.$$

Solution Let $f(x) = x^3 - x + 11$

$$\therefore \qquad f(-x) = -x^3 + x + 11$$

The negative root of $f(x) = 0$ is the positive root of $f(-x) = 0$. Therefore, we will find the positive root of $f(-x) = 0$.

$$\phi(x) = x^3 - x - 11 = 0$$

Proceeding as explained in Example 3.1, we get $x = 2.3736$ and hence, the negative root is -2.3736.

3.3 METHOD OF SUCCESSIVE APPROXIMATION

This method is also known as *Iteration method.*

Let $f(x) = 0$ [Eqn (2.1)] be the given equation whose roots are to be determined. This equation can be written in the form

$$x = \phi(x) \tag{3.1}$$

Let $x = x_0$ be an initial approximation to the actual root, say, α of Eqn (3.1). Then the first approximation is $x_1 = \phi(x_0)$ and the successive approximations are $x_2 = \phi(x_1)$, $x_3 = \phi(x_2)$, $x_4 \ldots$, $x_n = \phi(x_{n-1})$.

If the sequence of approximate roots, $x_1, x_2, \ldots, x_n$, converges to α, it is taken as the root of the equation $f(x) = 0$.

For convergence purpose the initial approximation x_0 is to be done carefully. The choice of x_0 is determined according to the following theorem.

Theorem : If α be a root of $f(x) = 0$ which is equivalent to $x = \phi(x)$, I, be any interval containing the point $x = \alpha$ and $|\phi'(x)| < 1 \; \forall \; x \in I$, then the sequence of approximations $x_0, x_1, x_2, \ldots, x_n$ will converge to the root α provided the initial approximation x_0 is chosen in I.

Proof : Since α is a root of $x = \phi(x)$, we have

$$\alpha = \phi(\alpha)$$

If x_{n-1} and x_n be two successive approximations to α, we have

$$x_n = \phi(x_{n-1})$$

$$\therefore \qquad x_n - \alpha = \phi(x_{n-1}) - \phi(\alpha) \tag{3.2}$$

By Mean Value theorem,

$$[\phi(x_{n-1}) - \phi(\alpha)]/[x_{n-1} - \alpha] = \phi'(\xi), \text{ where } x_{n-1} < \xi < \alpha$$

Hence, Eqn (3.2) becomes

$$x_n - \alpha = (x_{n-1} - \alpha)\,\phi'(\xi) \tag{3.3}$$

Let k be the maximum absolute value of $\phi'(x)$ over the interval I. Then from Eqn (3.3),

$$|x_n - \alpha| \le k\,|x_{n-1} - \alpha| \tag{3.4}$$

Similarly, $\quad |x_{n-1} - \alpha| \le k\,|x_{n-2} - \alpha|$

$$\therefore \qquad |x_n - \alpha| \le k^2\,|x_{n-2} - \alpha|$$

Proceeding on,

$$|x_n - \alpha| \le k^n\,|x_0 - \alpha| \tag{3.5}$$

Now, if $k < 1$ over the entire interval, as n increases the RHS of Eqn (3.5) becomes small and therefore,

$$\underset{n\to\infty}{Lt}\,|x_n - \alpha| = 0, \quad \text{i.e. } \underset{n\to\infty}{Lt}\,x_n = \alpha$$

That is, the sequence of approximations converges to α if $k < 1$,

i.e. $\qquad |\phi'(x)| < 1 \; \forall \; x \in I.$

Note 1) Smaller the value of $\phi'(x)$, more rapid will be the convergence.
2) This method of iteration is particularly useful for finding the real roots

of an equation given in the form of an infinite series.

3) From Eqn (3.4), we have $|x_n - \alpha| \le k|x_{n-1} - \alpha|$, $k < 1$ which shows that the iteration method is linearly convergent.

Example 3.3 Find a real root of the equation $x^3 + x^2 - 1 = 0$ by iteration method.

Solution Let $f(x) = x^3 + x^2 - 1$

Now $f(0) = -1$ is negative and $f(1) = 1$ is positive. Hence a real root lies between 0 and 1.

Now, $x^3 + x^2 - 1 = 0$ can be written as

$$x = 1/\sqrt{(x+1)} = \phi(x)$$

$$\therefore \quad \phi'(x) = -\tfrac{1}{2}\,[1/(x+1)^{3/2}]$$

Clearly, $|\phi'(0)| = \tfrac{1}{2} < 1$ and $|\phi'(1)| = 1/2^{5/2} < 1$

i.e. $\quad |\phi'(x)| < 1$ for all x in $(0,1)$

Hence the iterative method can be applied. Let $x_0 = 0.65$ be the initial approximation, then

$$\begin{aligned}
x_1 &= \phi(x_0) &&= 1/\sqrt{(1+x_0)} = 1/\sqrt{1.65} = 0.7784989\\
x_2 &= 1/\sqrt{(1+x_1)} &&= 1/\sqrt{1.7784989} = 0.7498479\\
x_3 &= 1/\sqrt{(1+x_2)} &&= 1/\sqrt{1.7498479} = 0.7559617\\
x_4 &= 1/\sqrt{(1+x_3)} &&= 1/\sqrt{1.7559617} = 0.7546446\\
x_5 &= 1/\sqrt{(1+x_4)} &&= 1/\sqrt{1.7546446} = 0.7549278\\
x_6 &= 1/\sqrt{(1+x_5)} &&= 1/\sqrt{1.7549278} = 0.7548668\\
x_7 &= 1/\sqrt{(1+x_6)} &&= 1/\sqrt{1.7548668} = 0.7548799\\
x_8 &= 1/\sqrt{(1+x_7)} &&= 1/\sqrt{1.7548799} = 0.7548771\\
x_9 &= 1/\sqrt{(1+x_8)} &&= 1/\sqrt{1.7548771} = 0.7548777\\
x_{10} &= 1/\sqrt{(1+x_9)} &&= 1/\sqrt{1.7548777} = 0.7548776\\
x_{11} &= 1/\sqrt{(1+x_{10})} &&= 1/\sqrt{1.7548776} = 0.7548776
\end{aligned}$$

Hence the root is 0.7548776.

Example 3.4 Find a real root of the equation $\cos x = 3x - 1$ correct to seven decimal places by the method of successive approximation.

(Gulbarga B.E., 1993)

Solution Let $f(x) = \cos x - 3x + 1 = 0$

$$f(0) = -1 \text{ (–)ve and } f(\pi/2) = -3(\pi/2) - 1 \text{ (–)ve}$$

Therefore, a real root lies in between 0 and $\pi/2$.

Rewriting the equation as

$$x = 1/3\,(\cos x + 1) = \phi(x)$$

we have $\phi'(x) = -1/3 \sin x$

Now $|\,\phi'(x)\,| = |\,1/3 \sin x\,| < 1$ for all x in $(0, \pi/2)$.

Hence, the iteration method can be applied.

Let $x_0 = 0.5$ be the initial approximation, then

$$x_1 = \phi(x_0) = 1/3\,[\cos(0.5) + 1] = 0.6258608$$

$$x_2 = 1/3\,[\cos(0.6258608) + 1] = 0.6034863$$

$$x_3 = 1/3\,[\cos(0.6034863) + 1] = 0.6077873$$

$$x_4 = 1/3\,[\cos(0.6077873) + 1] = 0.6069711$$

$$x_5 = 1/3\,[\cos(0.6069711) + 1] = 0.6071264$$

$$x_6 = 1/3\,[\cos(0.6071264) + 1] = 0.6070969$$

$$x_7 = 1/3\,[\cos(0.6070969) + 1] = 0.6071025$$

$$x_8 = 1/3\,[\cos(0.6071025) + 1] = 0.6071014$$

$$x_9 = 1/3\,[\cos(0.6071014) + 1] = 0.6071016$$

$$x_{10} = 1/3\,[\cos(0.6071016) + 1] = 0.6071016$$

Hence the root is 0.6071016.

Example 3.5 Find the smallest root of the equation

$$x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \cdots = 0.4431135$$

Solution Writing the given equation as

$$x = 0.4431135 + \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{42} - \frac{x^9}{216} + \frac{x^{11}}{1320} + \cdots = \phi(x)$$

Neglecting powers of x^3 and higher powers of x in the above equation, we get $x = 0.4431135$ (approximately). To solve it let the initial approximation be $x_0 = 0.44$.

Then $x_1 = \phi(x_0)$

$$= 0.4431135 + \frac{(0.44)^3}{3} - \frac{(0.44)^5}{10} + \frac{(0.44)^7}{42} - \frac{(0.44)^9}{216} + \frac{(0.44)^{11}}{1320} - \ldots$$

$$= 0.4699$$

Similarly,

$$x_2 = \phi(x_1) = \phi(0.4699) = 0.4755$$

$$x_3 = \phi(x_2) = \phi(0.4755) = 0.47664$$

$$x_4 = \phi(x_3) = \phi(0.47664) = 0.47686$$
$$x_5 = \phi(x_4) = \phi(0.47686) = 0.47690$$

The values of x_4 and x_5 indicate that $x = 0.4769$, correct to four decimal places.

3.4 METHOD OF FALSE POSITION

This method, also known as *regula falsi method*, is the oldest method of finding the real root of an equation $f(x) = 0$ and is somewhat similar to the bisection method.

Consider the equation $f(x) = 0$. Let a and b $(a < b)$ be two values of x such that $f(a)$ and $f(b)$ are of opposite signs. Then the graph of $y = f(x)$ crosses the x-axis at some point between a and b (see Fig.3.3).

Therefore, the equation of the chord joining the two points $A\ [a, f(a)]$ and $B\ [b, f(b)]$ is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a) \tag{3.6}$$

Now in the interval, (a, b), the graph of the function can be considered as a straight line. So the intersection of the line given by Eqn (3.6) with the x-axis will give an approximate value of the root. Putting $y = 0$ in Eqn (3.6), we get

$$-f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or
$$x = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

Hence, the first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} \tag{3.7}$$

Now, if $f(x_1)$ and $f(a)$ are of opposite sign then the root lies in between a and x_1. So we replace b by x_1 in Eqn (3.7) and get the next approximation x_2.

But if $f(x_1)$ and $f(a)$ are of the same sign then $f(x_1)$ and $f(b)$ will be of opposite signs and therefore, the root lies in between x_1 and b. Hence, we replace a by x_1 in Eqn (3.7) and get the next approximation x_2. The process is to be repeated till the root is found to the desired accuracy.

The geometrical interpretation of the method in as follows:

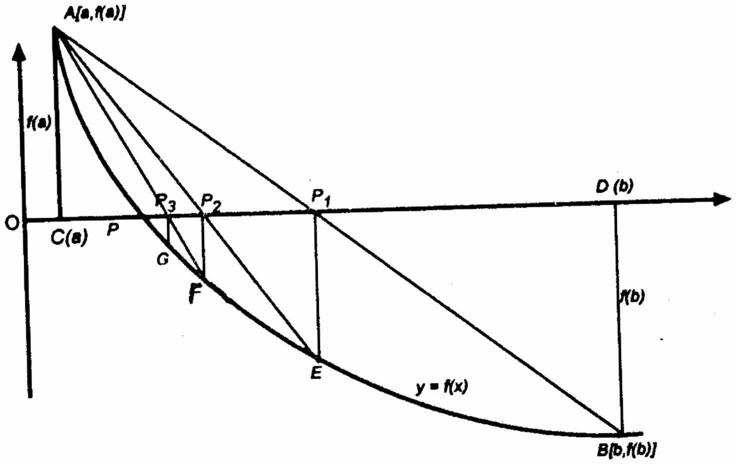

Fig. 3.3

In Fig. 3.3, curve $y = f(x)$ meets the x-axis at P_1. Therefore, $OP = x$ is the actual root of $f(x) = 0$. Chord AB meets the x-axis at P_1. Therefore $OP_1 = x_1$ is the first approximation. Now $f(a)$ and $f(x_1)$ are of opposite signs. So applying the method again in (a, x_1) we get $OP_2 = x_2$ as the second approximation to the root. Proceeding in this way, we get the root of desired accuracy. We can see that points $P_1, P_2, P_3, \ldots$ on chords $AB, AE, AF, \ldots$ tend to coincide with P, the point where the curve meets the x-axis, i.e. original root.

Example 3.6 Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.

(*M.U, B.E., 1992, B.U, B.E., 1993*)

Solution Given : $f(x) = x^3 - 2x - 5$; $a = 2$, $b = 3$

Now $f(a) = f(2) = 2^3 - 2(2) - 5 = -1$ (–)ve

and $f(b) = f(3) = 3^3 - 2(3) - 5 = +16$ (+)ve

Therefore, the root of $f(x) = 0$ lies in between 2 and 3.

The first approximation of the root is x_1 and is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2(16) - 3(-1)}{16 - (-1)}$$

$$= 35/17 = 2.0588 \text{ (approximately)}$$

Now $f(x_1) = f(2.0588) = (2.0588)^3 - 2(2.0588) - 5$

$= -0.391$ (–)ve

Therefore, the root of $f(x) = 0$ lies in between $x_1 = 2.0588$ and $b = 3$.

The second approximation to the root is given by

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)} = \frac{(2.0588)(16) - 3(-0.391)}{16 - (-0.391)}$$

$$= \frac{34.1138}{16.391} = 2.08125$$

$$f(x_2) = f(2.08125) = (2.08125)^3 - 2(2.08125) - 5$$
$$= -0.147 \text{ (–)ve.}$$

which means that the root of $f(x) = 0$ lies in between $x_2 = 2.08125$ and $b = 3$.

The third approximation to the root is given by

$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)} = \frac{(2.08125)(16) - 3(-0.147)}{16 - (-0.147)}$$

$$= \frac{33.741}{16.147} = 2.0896$$

$$f(x_3) = f(2.0896) = (2.0896)^3 - 2(2.0896) - 5$$
$$= -0.0551 \text{ (–)ve}$$

Therefore, the root of $f(x) = 0$ lies in between $x_3 = 2.0896$ and $b = 3$.

The fourth approximation to the root is given by

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)} = \frac{(2.0896)(16) - 3(-0.0551)}{16 - (-0.0551)}$$

$$= \frac{33.5989}{16.0551} = 2.0927$$

$$f(x_4) = f(2.0927) = (2.0927)^3 - 2(2.0927) - 5$$
$$= -0.0206 \text{ (–)ve}$$

Therefore, the root of $f(x) = 0$ lies in between $x_4 = 2.0927$ and $b = 3$.

The fifth approximation to the root is given by

$$x_5 = \frac{x_4 f(b) - b f(x_4)}{f(b) - f(x_4)} = \frac{(2.0927)(16) - 3(-0.0206)}{16 - (-0.0206)}$$

$$= \frac{33.545}{16.0206} = 2.0939$$

Now $f(x_5) = f(2.0939) = -0.00726$ (–)ve

which implies that the root lies in between $x_5 = 2.0939$ and $b = 3$.

The sixth approximation to the root is given by

$$x_6 = \frac{x_5 f(b) - b f(x_5)}{f(b) - f(x_5)} = \frac{(2.0939)(16) - 3(-0.00726)}{16 - (-0.00726)}$$

$$= \frac{33.52418}{16.00726} = 2.0943$$

Now $f(x_6) = f(2.0943) = -0.0028$

Therefore, the root lies in between $x_6 = 2.0943$ and $b = 3$.

The seventh approximation to the root is given by

$$x_7 = \frac{x_6 f(b) - b f(x_6)}{f(b) - f(x_6)} = \frac{(2.0943)(16) - 3(-0.0028)}{16 - (-0.0028)}$$

$$= \frac{33.5172}{16.0028} = 2.0944$$

Therefore, the root is 2.094, correct to three decimal places.

Example 3.7 Find the root of $xe^x = 3$ by regula falsi method correct to three decimal places.

Solution Given : $f(x) = xe^x - 3$

Now $f(1) = e - 3 = -0.28172$ (–) ve

and $f(1.5) = 1.5\, e^{1.5} - 3 = 3.72253$ (+)ve

Therefore, the root lies in between 1 and 1.5. Let $a = 1$ and $b = 1.5$, then $f(a) = -0.28172$ and $f(b) = 3.72253$.

The first approximation to the root is given by

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{1(3.72253) - 1.5(-0.28172)}{3.72253 - (-0.28172)}$$

$$= \frac{4.14511}{4.00425} = 1.035$$

Now $f(x_1) = f(1.035) = 1.035\, e^{1.035} - 3$

$= -0.0864$ (–)ve

Therefore, the root of $f(x) = 0$ lies in between $x_1 = 1.035$ and $b = 1.5$.

The second approximation to the root is given by

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)} = \frac{(1.035)(3.72253) - 1.5(-0.0286485)}{3.72253 - (-0.00286485)}$$

$$= \frac{3.98242}{3.80893} = 1.045$$

Now $f(x_2) = f(1.045) = 1.045\, e^{1.045} - 3 = -0.0286485$ (–)ve which implies that the root lies in between $x_2 = 1.045$ and $b = 1.5$.

The third approximation to the root is given by

$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)} = \frac{(1.045)(3.72253) - 1.5(-0.0286485)}{3.72253 - (-0.00286485)}$$

$$= \frac{3.9330166}{3.7511785} = 1.048$$

Now $f(x_3) = f(1.048) = -0.0111652$ (–)ve

Therefore, the root lies in between $x_3 = 1.048$ and $b = 1.5$.

The fourth approximation to the root is given by

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)} = \frac{(1.048)(3.72253) - 1.5(-0.0111652)}{3.72253 - (-0.0111652)}$$

$$= \frac{3.9179594}{3.7336953} = 1.049$$

Now $f(x_4) = f(1.049) = -5.320155 \times 10^{-3}$ (–)ve

Therefore, the root lies in between $x_4 = 1.049$ and $b = 1.5$.

The fifth approximation to the root is given by

$$x_5 = \frac{x_4 f(b) - b f(x_4)}{f(b) - f(x_4)} = \frac{3.9129142}{3.7278502} = 1.0496$$

Now $f(x_5) = f(1.0496) = -1.808903 \times 10^{-3}$ which means that the root lies in between $x_5 = 1.0496$ and $b = 1.5$.

The sixth approximation to the root is given by

$$x_6 = \frac{x_5 f(b) - b f(x_5)}{f(b) - f(x_5)} = \frac{3.9098808}{3.7243389} = 1.0498$$

Therefore, from x_5 and x_6, the root is 1.05.

EXERCISE 3.1

1. Find a root of the following equations correct to three decimal places, using the Bisection method.

   (i) $x^3 - x^2 + x - 7 = 0$

   (ii) $x^3 - 2x - 5 = 0$

   (iii) $x^3 - 3x - 5 = 0$ *(Bangalore, B.E., 1989)*

   (iv) $x^3 - 4x - 9 = 0$ *(Mysore, B.E., 1987)*

   (v) $x^4 - x - 10 = 0$ *(S. Gujarat B.E., 1990)*

   (vi) $x - \cos x = 0$ *(B.U, B.E., 1995)*

   (vii) $3x - e^x = 0$

   (viii) $3x = \sqrt{(1 + \sin x)}$

   (ix) $x \log_{10} x - 1.2 = 0$

2. Using Bisection method find the negative root of $x^3 - 4x + 9 = 0$, correct to three decimal places.

3. Find a root of the following equations correct to three decimal places, using Iteration method.

(i) $x^3 + x^2 - 100 = 0$ (ii) $x = \frac{1}{2} + \sin x$
(iii) $3x - 6 = \log_{10} x$ (iv) $xe^x - \cos x = 0$
(v) $\sin x = e^x - 3x$ (vi) $2x - 7 - \log_{10} x = 0$

(vii) $1 - x + \dfrac{x^2}{(2!)^2} - \dfrac{x^3}{(3!)^2} + \dfrac{x^4}{(4!)^2} - \dfrac{x^5}{(5!)^2} + \ldots = 0$

4. Find a negative root of $x^3 - 2x + 5 = 0$, correct to three decimal places, using Successive Approximation method.

5. Find a root of the following equations correct to four decimal places using the method of False Position (*regula false* method).

(i) $x^3 - 4x - 9 = 0$ (ii) $x^3 + 2x^2 + 10x - 20 = 0$
(iii) $x^3 - 4x - 1 = 0$ (iv) $x^6 - x^4 - x^3 - 1 = 0$
(v) $xe^x = 2$ (vi) $e^x \sin x = 1$
(vii) $x = \cos x$ (viii) $x \tan x = -1$ in (2.5, 3)
(ix) $x \log_{10} x = 1.2$

ANSWERS

1. (i) 2.105 (ii) 2.095 (iii) 2.280 (iv) 2.706 (v) 1.813
(vi) 0.739 (vii) 0.619 (viii) 0.392 (ix) 2.740
2. −2.706
3. (i) 4.331 (ii) 1.497 (iii) 2.108 (iv) 0.518 (v) 0.360
(vi) 3.789 (vii) 1.445
4. −2.095
5. (i) 2.7065 (ii) 1.3688 (iii) 0.2541 (iv) 1.7365 (v) 0.8526
(vi) 0.5885 (vii) 0.7391 (viii) 2.7981 (ix) 2.7406

3.5 NEWTON'S ITERATION METHOD

This method, also known as *Newton–Raphson method* and is a particular form of the iteration method discussed in Section 3.3. When an approximate value of a root of an equation is given, a better and closer approximation to the root can be found using this method.

It can be derived as follows :

Let x_0 be an approximation of a root of the given equation $f(x) = 0$, which may be algebraic or transcendental.

Let $x_0 + h$ be the exact value or the better approximation of the corresponding root, h being a small quantity. Then $f(x_0 + h) = 0$.

Expanding it by Taylor's theorem, we get

$$f(x_0 + h) = f(x_0) + hf'x_0 + h^2/2!\,f''(x_0) + \ldots = 0$$

Since h is small, we can neglect second, third and higher degree terms in h and thus we get

$$f(x_0) + hf'(x_0) = 0$$

or $$h = -\frac{f(x_0)}{f'(x_0)};\ f'(x_0) \neq 0$$

Hence, $$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now substituing x_1 for x_0 and x_2 for x_1 the next better approximations are given by $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ and $x_3 = x_2 - \dfrac{f(x_n)}{f'(x_n)}$

Proceeding in the same way n times, we get the general formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for } n = 0, 1, 2, \ldots \tag{3.8}$$

which is known as Newton--Raphson formula.

3.6 GEOMETRICAL INTERPRETATION

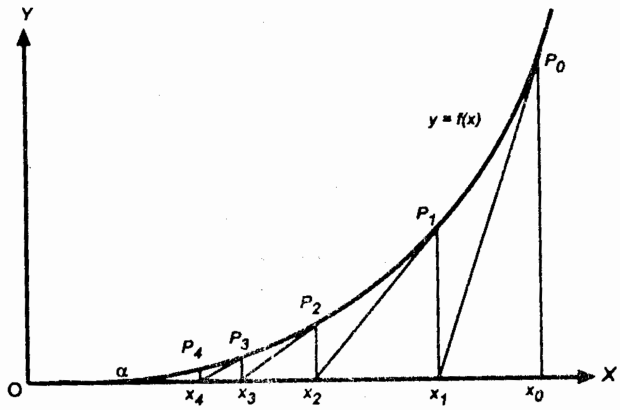

Fig. 3.4

Let the curve $f(x) = 0$ meet the x-axis at $x = \alpha$. It means that α is the original root of $f(x) = 0$. Let x_0 be the point near the root α of the equation $f(x) = 0$ (Fig. 3.4). Then, the equation of the tangent at P_0 $[x_0, f(x_0)]$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

which cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

This is the first approximation to the root α. If $P_1\ [x_1, f(x_1)]$ is the point corresponding to x_1 on the curve then the tangent at P_1 is

$$y - f(x_1) = f'(x_1)\,(x - x_1)$$

which cuts the x-axis again at x_2.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

This is the second approximation to the root α. Repeating this process we approach to the root α with better approximations quite rapidly.

Note :

1. When $f'(x)$ is very large, i.e. when the slope is large, then h will be small (as assumed) and hence, the root can be calculated in even less time.
2. If we choose the initial approximation x_0 close to the root then we get the root of the equation very quickly.
3. The process will evidently fail if $f'(x) = 0$ is in the neighbourhood of the root. In such cases the *regula falsi* method should be used.
4. If the initial approximation, to the root is not given, choose two values of x, say, a and b, such that $f(a)$ and $f(b)$ are of opposite signs. If $|f(a)| < |f(b)|$ then take a as the initial approximation.
5. Newton–Raphson method is also referred to as the *method of tangents*.

3.7 CONVERGENCE OF NEWTON–RAPHSON METHOD

In this section, we will see the condition for convergence of Newton–Raphson method. Newton–Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is an iteration method where

$$x_{n+1} = \phi(x_n)\ ;\ \phi(x_n) = x_n - \frac{f(x)}{f'(x)}$$

In general, $x = \phi(x)$, where $\phi(x) = x - \dfrac{f(x_n)}{f'(x_n)}$.

We know that the iteration method converges if

$$|\phi'(x)| < 1, \quad \text{i.e.}\ \left|1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

or
$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

i.e.
$$|f(x)\, f''(x)| < [f'(x)]^2$$

The interval containing the root α of $f(x) = 0$ should be selected in which the above is satisfied.

3.8 RATE OF CONVERGENCE OF NEWTON–RAPHSON METHOD

Let x_n and x_{n+1} be two successive approximations to the actual root α of $f(x) = 0$. If ε_n and ε_{n+1} are the corresponding errors, we have

$$x_n - \alpha = \varepsilon_n \text{ and } x_{n+1} - \alpha = \varepsilon_{n+1}$$

$$\therefore \varepsilon_{n+1} - \varepsilon_n = x_{n+1} - x_n$$

$$= x_{n+1} + x_n$$

$$= \frac{f(x_n)}{f'(x_n)} \quad \text{(using Newton–Raphson formula)}$$

$$= \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$$

$$= -\frac{f(\alpha) + \varepsilon_n f'(\alpha) + 1/2[\varepsilon_n^2 f''(\alpha)] + \cdots}{f'(\alpha) + \varepsilon_n f''(\alpha) + 1/2[\varepsilon_n^2 f'''(\alpha)] + \cdots}$$

(by Taylor's Theorem)

$$= -\frac{\varepsilon_n f'(\alpha) + 1/2[\varepsilon_n^2 f''(\alpha)] + \cdots}{f'(\alpha) + \varepsilon_n^2 f''(\alpha) + \cdots} \quad (\because f(\alpha) = 0)$$

$$\therefore \varepsilon_{n+1} = \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + 1/2[\varepsilon_n^2 f''(\alpha)]}{f'(\alpha) + \varepsilon_n f''(\alpha)}$$

(by omitting derivatives of order higher than two)

$$= \frac{1/2\varepsilon_n^2 f''(\alpha)}{f'(\alpha) + \varepsilon_n f''(\alpha)}$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)}\left[1 + \frac{\varepsilon_n f''(\alpha)}{f'(\alpha)}\right]^{-1} = \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)}$$

$$\left[\because \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)} \text{ is a small quantity, we neglect it}\right]$$

$$= k\,\varepsilon_n^2, \text{ where } k = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

which shows that at any stage, the subsequent error is proportional to the square of the previous error. Hence, Newton–Raphson method has a quadratic convergence. In other words, its order of convergence is two.

Example 3.8 Find an iterative formula to find $\sqrt{N}$, where N is a positive number and hence, find $\sqrt{12}$ correct to four decimal places.

(*M.U, B.E., 1992*)

Solution Let $x = \sqrt{N}$ $\therefore$ $x^2 - N = 0$

If $f(x) = x^2 - N$ then $f'(x) = 2x$

Now, from Newton–Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$\therefore x_{n+1} = ½\,[x_n + (N/x_n)] \qquad \text{(i)}$$

Eqn (i) is the required iterative formula.

Putting $N = 12$ in $f(x)$, we have $f(x) = x^2 - 12$.

Now, $f(3) < 0$ and $f(4) > 0$. Therefore, the root lies in between 3 and 4.

Let the initial approximation x_0 be 3.1.

Then, from Eqn (i) the first approximation to the root

$$x_1 = ½\,[x_0 + 12/x_0] = ½\,[3.1 + 12/3.1] = 3.4854839$$

The second approximation is

$$x_2 = ½\left[x_1 + \frac{12}{x_1}\right] = ½\left[3.4854839 + \frac{12}{3.4854839}\right] = 3.4641672$$

The third approximation is

$$x_3 = ½\left[3.4641672 + \frac{12}{3.4641672}\right] = 3.4641016$$

The fourth approximation is

$$x_4 = 1/2\left[3.4641016 + \frac{12}{3.4641016}\right] = 3.4641016$$

Thus, the value of $\sqrt{12}$ correct to four decimals is 3.4641.

Example 3.9 Solve $x^3 + 2x^2 + 10x - 20 = 0$ by Newton–Raphson method.

Solution Let $f(x) = x^3 + 2x^2 + 10x - 20$

$\therefore \quad f'(x) = 3x^2 + 4x + 10$

From Eqn (3.8)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \left[\frac{x_n^3 + 2x_n^2 + 10x_n - 20}{3x_n^2 + 4x_n + 10}\right]$$

$$= \frac{2(x_n^3 + x_n^2 + 10)}{3x_n^2 + 4x_n + 10} \qquad \text{(i)}$$

Now we can see that $f(1) = -7 < 0$ and $f(2) = 16 > 0$.
Therefore, the root lies in between 1 and 2.
Let $x_0 = 1.2$ be the initial approximation ($\because f(1.2) < 0$).
Putting $n = 0$ in Eqn (i), first approximation x_1 is given by

$$x_1 = \frac{2(x_0^3 + x_0^2 + 10)}{3x_0^2 + 4x_0 + 10} = \frac{2[(1.2)^3 + (1.2)^2 + 10]}{3(1.2)^2 + 4(1.2) + 10}$$

$$= \frac{26.336}{19.12} = 1.3774059$$

The second approximation x_2 is

$$x_2 = \frac{2(x_1^3 + x_1^2 + 10)}{3x_1^2 + 4x_1 + 10}$$

$$= \frac{2[(1.3774059)^3 + (1.3774059)^2 + 10]}{3(1.3774059)^2 + 4(1.3774059) + 10}$$

$$= \frac{29.021052}{21.201364} = 1.3688295$$

The third approximation x_3 is given by

$$x_3 = \frac{2(x_2^3 + x_2^2 + 10)}{3x_2^2 + 4x_2 + 10}$$

$$= \frac{2[(1.3688295)^3 + (1.3688295)^2 + 10]}{3(1.3688295)^2 + 4(1.3688295) + 10}$$

$$= \frac{28.876924}{21.0964} = 1.3688081$$

The fourth approximation x_4 (to the root) is given by

$$x_4 = \frac{2(x_3^3 + x_3^2 + 10)}{3x_3^2 + 4x_3 + 10}$$

$$= \frac{2[(1.3688081)^3 + (1.3688081)^2 + 10]}{3(1.3688081)^2 + 4(1.3688081) + 10}$$

$$= \frac{28.876567}{21.09614} = 1.3688081$$

Hence the root is 1.3688081.

Example 3.10 Using Newton–Raphson method, find the root of the equation $x\log_{10}x = 1.2$. *(M.U, B.E., 1991, 1994, 1995)*

Solution Let $f(x) = x\log_{10}x - 1.2$.

$$f'(x) = \log_{10}x + x(\log_{10}e/x) \qquad \left[\because \frac{d}{dx}\log_a x = \frac{1}{x}\log_a e\right]$$

$= \log_{10}x + 0.4343$

From Newton–Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.4343}$$

$$\therefore\ x_{n+1} = \frac{0.4343x_n + 1.2}{\log_{10} x_n + 0.4343} \qquad \text{(i)}$$

Now $f(2.5) = -0.2051499 < 0$ and $f(3) = 0.2313637 > 0$. Therefore, the real root of $f(x)$ lies in (2.5, 3). Let $x_0 = 2.7$ be the initial approximation. Putting $n = 0$ in Eqn (i), the first approximation x_1 is given by

$$x_1 = \frac{0.4343x_0 + 1.2}{\log_{10} x_0 + 0.4343} = \frac{0.4343(2.7) + 1.2}{\log_{10} 2.7 + 0.4343}$$

$$= \frac{2.37261}{0.8656637} = 2.7407986$$

The second approximation x_2 is

$$x_2 = \frac{0.4343x_1 + 1.2}{\log_{10} x_1 + 0.4343} = \frac{0.4343(2.7407986) + 1.2}{\log_{10}(2.7407986) + 0.4343}$$

$$= \frac{2.3903288}{0.8721771} = 2.7406461$$

Similarly, the third approximation is

$$x_3 = \frac{0.4343x_2 + 1.2}{\log_{10} x_2 + 0.4343} = \frac{0.4343\,(2.7406961) + 1.2}{\log_{10}(2.7406461) + 0.4343}$$

$$= \frac{2.3902626}{0.8721529} = 2.7406461$$

Hence, the root is 2.7406461.

Example 3.11 Solve $\sin x = 1 + x^3$ using Newton–Raphson method.

(M.U, B.E., 1992)

Solution Let $f(x) = \sin x - 1 - x^3$ $\therefore f'(x) = \cos x - 3x^2$

Then, from Newton–Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin x_n - 1 - x_n^3}{\cos x_n - 3x_n^2}$$

$$\therefore x_{n+1} = \frac{x_n \cos x_n - \sin x_n - 2x_n^3 + 1}{\cos x_n - 3x_n^2} \qquad \text{(i)}$$

Now $f(-1) = \sin(-1) - 1 - (-1)^3 = -0.8414709 < 0$ and $f(-2) = \sin(-2) - 1 - (-2)^3 = 6.0907026 > 0$ which means that the root lies in between -1 and -2. Let $x_0 = -1.1$ be the initial approximation. Then, by putting $n = 0, 1, 2, \ldots$ in Eqn (i), we obtain the successive approximations as

$$x_1 = \frac{x_0 \cos x_0 - \sin x_0 - 2x_0^3 + 1}{\cos x_0 - 3x_0^2} = \frac{4.0542516}{-3.1764039} = -1.2763653$$

$$x_2 = \frac{5.7452469}{-4.5971297} = -1.2497465$$

$$x_3 = \frac{5.4584049}{-4.370036} = -1.2490526$$

$$x_4 = \frac{5.4510835}{-4.364176} = -1.2490522$$

$$x_5 = \frac{5.4510786}{-4.3641722} = -1.2490521$$

$$x_6 = \frac{5.4510785}{-4.3641721} = -1.2490521 \; 5$$

Hence the root is average of x_5 and x_6, i.e. -1.24905215.

3.9 HORNER'S METHOD

This the best method of finding a real root of a numerical polynomial equation. The method of working is as follows :

Let a positive root of $f(x) = 0$ lie in between α and $\alpha + 1$, where α is an integer. Then the value of the root be $\alpha.\, d_1\, d_2\, d_3 \ldots$, where α is the integral part and $d_1, d_2, d_3, \ldots$ are the digits in their decimal part.

Finding d_1 First diminish the roots of $f(x) = 0$ by α so that the roots of the transformed equation lie between 0 and 1. That is, the root of the transformed equation is $0.\, d_1\, d_2\, d_3 \ldots$.

Now multiply the roots of the transformed equation by 10 so that the root of the new equation is $d_1 \,.\, d_2\, d_3 \ldots$.

Thus, the first figure after the decimal place is d_1.

Again, diminish the root by d_1 and multiply the roots of the resulting equation by 10 so that the root is $d_2 \,.\, d_3 \ldots$; that is, the second figure after the decimal place is d_2.

Continue the process to obtain the root to any desired degree of accuracy, digit by digit.

Example 3.12 Using Horner's method, find the root of $x^3 + 9x^2 - 18 = 0$, correct to two decimal places.

Solution Let $f(x) = x^3 + 9x^2 - 18$

Then $f(1) = 1 + 9 - 18 = (-)$ ve and $f(2) = 8 + 36 - 18 = (+)$ve

That is, $f(1)$ and $f(2)$ are of opposite signs. Hence, $f(x) = 0$ has a root between 1 and 2.

Therefore, the integral part of the root of $f(x) = 0$ is 1. Now diminish the roots of the equation by 1 :

$$
\begin{array}{c|cccc}
1 & 1 & 9 & 0 & -18 \\
 & 0 & 1 & 10 & 10 \\
\hline
1 & 1 & 10 & 10 & \boxed{-8} \\
 & 0 & 1 & 11 & \\
\hline
1 & 1 & 11 & \boxed{21} & \\
 & 0 & 1 & & \\
\hline
 & 1 & \boxed{12} & &
\end{array}
$$

Thus the transformed equation is

$$x^3 + 12x^2 + 21x - 8 = 0.$$

This equation has a root between 0 and 1. Multiply the roots of this equation by 10.

So now the new equation is

$$f_1(x) = x^3 + 120x^2 + 2100x - 8000 = 0$$

We can see that $f_1(3) < 0$ and $f_1(4) > 0$.

Therefore, the root of $f_1(x) = 0$ lies in between 3 and 4. Hence, the first figure after the decimal place is 3.

Now diminish the roots of $f_1(x) = 0$ by 3 :

3	1	120	2100	–8000
	0	3	369	7407
3	1	123	2469	–593
	0	3	378	
3	1	126	2847	
	0	3		
	3	129		

The transformed equation is

$$3x^3 + 129x^2 + 2847x - 593 = 0$$

whose root lies in between 0 and 1. Multiplying the roots of this equation by 10, we get the new equation as

$$f_2(x) = 3x^3 + 1290x^2 + 284700x - 593000 = 0$$

We can easily see that the root of $f_2(x)$ lies in between 2 and 3 since $f_2(2) < 0$ and $f_3(3) > 0$.

Therefore, the second figure after the decimal place is 2. Now diminish the roots of $f_2(x) = 0$ by 2 :

2	3	1290	284700	–593000
	0	6	2592	574584
2	3	1296	287292	–18416
	0	6	2604	
2	3	1302	289896	
	0	6		
	3	1308		

The transformed equation is

$$3x^3 + 1308x^2 + 289896x - 18416 = 0$$

and its root lies in between 0 and 1. Multiplying the roots of this equation by 10, we get the new equation as

$$f_3(x) = 3x^3 + 13080x^2 + 28989600x - 18416000 = 0$$

It can be seen that $f_3(0) < 0$ and $f_3(1) > 0$;
that is, the root of $f_3(x) = 0$ lies in between 0 and 1.

The third figure after the decimal is 0. We can stop with this, as we require the root correct to two decimal places.

Hence the root of $f(x) = 0$, correct to two decimal places, is 1.32.

3.10 DESCARTE'S RULE OF SIGNS

The equation $f(x) = 0$ cannot have more positive roots than the changes of signs in $f(x)$, and more negative roots than the changes of signs in $f(-x)$.

For example, consider the following equation :

$$f(x) = x^9 - 8x^3 + 7x - 9 = 0$$

Signs of $f(x)$ are $+, -, +, -$.

Clearly, $f(x)$ has three changes of signs ; that is + to – , – to + and + to – . Thus, $f(x)$ cannot have more than three positive roots.

Also, $$f(-x) = -x^9 + 8x^3 - 7x - 9$$

that is, $f(-x)$ has two changes of signs, from – to + and + to – . Hence $f(x)$ cannot have more than two negative roots. Since $f(x) = 0$ is of ninth degree, it follows that the remaining four roots may be imaginary.

In general, if an equation of n th degree has at the most p positive roots and q negative roots, then it follows that the equation has at least $n - (p + q)$ imaginary roots.

3.11 GRAEFFE'S ROOT – SQUARING METHOD

This method has a great advantage over other methods in that it does not require prior information about the approximate values etc. of the the roots. But, it is applicable to polynomial equations only and is capable of giving all the roots. Let us now see the case of a polynomial equation having real and distinct roots.

Consider the following polynomial equation :

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0 \tag{3.9}$$

Separating the even and odd powers of x and squaring, we get

$$(x^n + a_2 x^{n-2} + a_4 x^{n-4} + \ldots)^2 = (a_1 x^{n-1} + a_3 x^{n-3} + a_5 x^{n-5} + \ldots)^2$$

Putting $x^2 = y$ and simplifying, we get the new equation,

$$y^n + b_1 y^{n-1} + b_2 y^{n-2} + \cdots + b_{n-1} y + b_n = 0 \tag{3.10}$$

where $$b_1 = -a_1^2 + 2a_2$$
$$b_2 = a_2^2 - 2a_1a_3 + 2a_4$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$b_n = (-1)^n a_n^2 \tag{3.11}$$

If $p_1, p_2, \ldots, p_n$ be the roots of Eqn (3.9), then the roots of Eqn (3.10) are $p_1^2, p_2^2, \ldots, p_n^2$. Let us suppose that after m squarings, the new transformed equation is

$$z^n + \lambda_1 z^{n-1} + \cdots + \lambda_{n-1} z + \lambda_n = 0 \tag{3.12}$$

whose roots are $q_1, q_2, \ldots q_n$, such that $q_i = p_i^{2m}$ and $i = 1, 2, \ldots, n$.

Assuming the order of magnitude of the roots as

$|p_1| > |p_2| > \ldots > |p_n|$, we have

$|q_1| >> |q_2| >> \ldots >> |q_n|$, where >> stands for 'much greater than'.

Thus, $$\frac{|q_2|}{|q_1|} = \frac{q_2}{q_1}, \ldots, \frac{|q_n|}{|q_{n-1}|} = \frac{q_n}{q_{n-1}} \tag{3.13}$$

Also, q_i, being an even power of p_i, is always positive. Now from Eqn (3.12), we have

$$\Sigma q_1 = -\lambda_1 \Rightarrow \lambda_1 = -q_1(1 + q_2/q_1 + q_3/q_1 + \cdots)$$
$$\Sigma q_1q_2 = \lambda_2 \Rightarrow \lambda_2 = q_1q_2(1 + q_3/q_1 + \ldots)$$
$$\Sigma q_1q_2q_3 = -\lambda_3 \Rightarrow \lambda_3 = q_1q_2q_3(1 + q_4/q_1 + \cdots)$$
$$\ldots \quad \ldots \quad \ldots$$
$$q_1q_2q_3\ldots q_n = (-1)^n\lambda_n \Rightarrow \lambda_n = (-1)^n q_1q_2q_3\ldots q_n$$

Hence, using Eqn (3.13) we find

$$q_1 \approx -\lambda_1;\ q_2 \approx -\lambda_2/\lambda_1;\ q_3 \approx -\lambda_3/\lambda_2, \ldots, q_n \approx -\lambda_n/\lambda_{n-1}$$

But $q_i = p_i^{2m}$

$$\therefore\ p_i = (q_i)^{1/2m} = (-\lambda_i/\lambda_{i-1})^{1/2m} \tag{3.14}$$

We can thus determine $p_1, p_2, \ldots, p_n$, i.e., the roots of the equation Eqn (3.9).

Case 1: Double root

If the magnitude of λ_i is half the square of the magnitude of the corresponding co-efficient in the previous equation after a few squarings, then it implies that p_i is a double root of Eqn (3.9). It can be determined as follows:

$$q_i = -\lambda_i/\lambda_{i-1} \text{ and } q_{i+1} = -\lambda_{i+1}/\lambda_i$$

$$\therefore\ q_i\,q_{i+1} \approx q_i^2 \approx \left|\frac{\lambda_{i+1}}{\lambda_{i-1}}\right|$$

i.e. $$p_i^{2m} = q_i^2 = |\lambda_{i+1}/\lambda_{i-1}| \tag{3.15}$$

which gives the magnitude of the double root, and substituting in Eqn (3.9), we can find the sign.

Case 2 : Complex roots

If p_r and p_{r+1} from a complex pair $\rho_r\ e^{\pm i\phi_r}$, then the coefficient of x^{n-r} in successive squarings would vary both in magnitude and sign by an amount $2\rho_r^m \cos m\phi_r$. For sufficiently large ρ_r and ϕ_r it can be determined by

$$\rho r^{2(2^m)} \approx \frac{\lambda_{r+1}}{\lambda_{r-1}};\ 2\rho_r^{2^m}\ \cos 2^m\phi_r = -(\lambda_r/\lambda_{r-1}) \tag{3.16}$$

If there is only one pair of complex roots, say, $\rho_r\ e^{\pm i\phi_r}, = \xi_r + i\eta_r$ then ξ_r is given by

$$p_1 + p_2 + \cdots + p_{r-1} + 2\xi_r + p_{r+2} + \cdots + p_n = -a_1 \tag{3.17}$$

and
$$\eta_r = \sqrt{\rho_r^2 - \xi_r^2} \tag{3.18}$$

If there are two pairs of complex roots, say,

$\rho_r e^{\pm i\phi_r} = \xi_r \pm i\eta_r$ and $\rho_s\ e^{\pm i\phi_s} = \xi_s \pm i\eta_s$

where
$$p_1 + p_2 + \cdots + p_{r-1} + 2\xi_r + p_{r+2} + \cdots$$
$$+ p_{s-1} + 2\xi_s + p_{s+2} + \cdots + p_n = -a_1 \tag{3.19}$$

$$2\left(\frac{\xi_r}{\rho_r^2} + \frac{\xi_s}{\rho_s^2}\right) = -\left[\frac{a_{n-1}}{n} + \frac{1}{a_1} + \ldots + \frac{1}{a_n}\right] \tag{3.20}$$

and
$$\eta_r = \sqrt{\rho_r^2 - \xi_r^2}\ ;\ \eta_s = \sqrt{\rho_s^2 - \xi_s^2} \tag{3.21}$$

Example 3.13 Apply Graeffe's root squaring method to solve the equation $x^3 - 8x^2 + 17x - 10 = 0$. *(M.U, B.E., 1996)*

Solution

Here,
$$f(x) = x^3 - 8x^2 + 17x - 10 = 0 \tag{i}$$

Clearly, $f(x)$ has three changes, i.e. from + to – , – to + and + to –. Hence, from Descartes rule of signs, $f(x)$ may have three positive roots.

Rewriting Eqn (i) as

$$x(x^2 + 17) = (8x^2 + 10) \tag{ii}$$

and squaring on both sides, and putting $x^2 = y$, we get

$$y(y + 17)^2 = (8y + 10)^2$$

or
$$y^3 + 34y^2 + 289 = 64y^2 + 160y + 100$$

or
$$y(y^2 + 129) = (30y^2 + 100) \tag{iii}$$

Squaring again and putting $y^2 = z$, we get

$$z(z + 129)^2 = (30z + 100)^2$$

or
$$z^3 + 258z^2 + 16641z = 900z^2 + 6000z + 10000$$

or $$z(z^2 + 16641) = (642z^2 + 10000) \quad \text{(iv)}$$

Squaring again and putting $z^2 = u$, we get

$$u(u + 16641)^2 = (642u + 10000)^2$$

$$u^3 + 33282u^2 + 276922881u = 412164u^2 + 12840000u + 10^8$$

or $$u^3 - 378882u^2 + 264082u - 10^8 = 0 \quad \text{(v)}$$

If the roots of Eqn (i) are p_1, p_2, p_3 and those of Eqn (v) are q_1, q_2, q_3, then

$$p_1 = (q_1)^{1/8} = (-\lambda_1)^{1/8} = (378882)^{1/8} = 4.9809593 \cong 5$$

$$p_2 = (q_2)^{1/8} = (-\lambda_2/\lambda_1)^{1/8} = \left[\frac{264082}{378882}\right]^{1/8} = 0.9558821 \cong 1$$

$$p_3 = (q_3)^{1/8} = (-\lambda_3/\lambda_2)^{1/8} = \left[\frac{10^8}{264082}\right]^{1/8} = 2.1003064 \cong 2$$

Now $f(5) = 0, f(1) = 0, f(2) = 0$

Hence, the roots are 5, 1, 2.

Example 3.14 Solve the equation $x^3 - 6x^2 + 11x - 6 = 0$ by Graeffe's root squaring method. *(M.U, B.E., 1997)*

Solution

Here, $$f(x) = x^3 - 6x^2 + 11x - 6 \quad \text{(i)}$$

Clearly, $f(x)$ has three changes i.e., from + to – , – to + and + to –. Hence from Descartes rule of signs $f(x)$ may have three positive roots.

Rewriting (i) as

$$x(x^2 + 11) = 6(x^2 + 1) \quad \text{(ii)}$$

and squaring on both sides, and putting $x^2 = y$, we get

$$y(y + 11)^2 = 36(y + 1)^2$$

or $$y^3 - 14y^2 + 49y - 36 = 0$$

or $$y(y^2 + 49) = (14y^2 + 36) \quad \text{(iii)}$$

Squaring again and putting $y^2 = z$, we get

$$z(z + 49)^2 = (14z + 36)^2$$

or $$z^3 - 98z^2 + 1393z - 1296 = 0$$

or $$z(z^2 + 1393) = (98z^2 + 1296) \quad \text{(iv)}$$

Squaring again and putting $z^2 = u$, we get

$$u(u + 1393)^2 = (98u + 1296)^2$$

or $$u^3 - 6818u^2 + 1686433u - 1679616 = 0 \quad \text{(v)}$$

If the roots of Eqn (i) are p_1, p_2, p_3 and those of Eqn (v) are q_1, q_2, q_3, then

$p_1 = (q_1)^{1/8} = (-\lambda_1)^{1/8} = (6818)^{1/8} = 3.0144433 \approx 3$
$p_2 = (q_2)^{1/8} = (-\lambda_2/\lambda_1)^{1/8} = (1686433/6818)^{1/8} = 1.9914253 \approx 2$
$p_3 = (q_3)^{1/8} = (-\lambda_3/\lambda_2)^{1/8} = (1679616/1686433)^{1/8} = 0.99949 \approx 1$
Now $f(3) = 0, f(2) = 0$ and $f(1) = 0$
$\therefore$ The roots are 3, 2 and 1

Example 3.15 Find all the roots of the equation $x^4 - 3x + 1 = 0$ using Graeffe's method.

Solution

Here, $$f(x) = x^4 - 3x + 1 = 0 \tag{i}$$

Now $f(x)$ has two changes in sign, i.e. + to – and – to +. Therefore, it may have two positive real roots.

Again, $f(-x) = x^4 + 3x + 1$. Since there is no change in sign in $f(-x)$, there is no negative root. But, $f(x)$, being of degree four, will have four roots of which two are real positive and the remaining two will be complex.

Rewriting Eqn (i) as $x^4 + 1 = 3x$
and, squaring and putting $x^2 = y$, we have $(y^2 + 1)^2 = 9y$

Squaring again and putting $y^2 = z$,
$$(z + 1)^4 = 81z$$
i.e. $$z^4 + 4z^3 + 6z^2 - 77z + 1 = 0 \tag{ii}$$
or $$z^4 + 6z^2 + 1 = -z(4z^2 - 77)$$
Squaring once again and putting $z^2 = u$, we get
$$(u^2 + 6u + 1)^2 = u(4u - 77)^2$$
or $$u^4 - 4u^3 + 654u^2 - 5917u + 1 = 0 \tag{iii}$$

If p_1, p_2, p_3, p_4 are the roots of Eqn (i) and q_1, q_2, q_3, q_4 are the roots of Eqn (iii) then

$$p_1 = (q_1)^{1/8} = (-\lambda_1)^{1/8} = (4)^{1/8} = 1.1892071$$

$$p_2 = (q_2)^{1/8} = \left[-\frac{\lambda_2}{\lambda_1}\right]^{1/8} = \left[\frac{654}{4}\right]^{1/8} = 1.8909921$$

$$p_3 = (q_3)^{1/8} = \left[-\frac{\lambda_3}{\lambda_2}\right]^{1/8} = \left[\frac{5917}{654}\right]^{1/8} = 1.3169384$$

$$p_4 = (q_4)^{1/8} = \left[-\frac{\lambda_4}{\lambda_3}\right]^{1/8} = \left[\frac{1}{5917}\right]^{1/8} = 0.3376659$$

From Eqns (ii) and (iii), we observe that the magnitudes of the coefficients λ_1 and λ_4 have become constant, which implies p_1, and p_4 are the real roots, and p_2 and p_3 are complex roots. Let these complex roots be

$$\rho_2\, e^{\pm i\varphi_2} = \xi_2 \pm i\eta_2$$

From Eqn (iii) its magnitude is given by

$$\rho_2^{2(2^3)} = \frac{\lambda_3}{\lambda_1} = \frac{5917}{4} \quad \therefore\ \rho_2 = 1.5780749$$

Also, from Eqn (i), sum of roots is = 0, i.e. $p_1 + 2\xi_2 + p_4 = 0$

$$\therefore \qquad \xi_2 = -1/2\,(p_1 + p_4) = -0.7634365$$

and $$\eta_2 = \sqrt{\rho_2^2 - \xi_2^2} = \sqrt{1.9074851} = 1.3811173$$

Hence, the four roots are : 1.1892071, 0.3376659,
-0.734365 and $\pm 1.3811173\, i$.

EXERCISE 3.2

Using Newton–Raphson method, find a root correct to three decimal places of the following :

1. $x^3 - 3x^2 + 7x - 8 = 0$ (*M.U, B.E., 1992*)
2. $x^3 - 3x - 5 = 0$ (*Kerala B.Tech 1989*)
3. $x^3 - 5x + 3 = 0$ (*Gulbarga B.E, 1993*)
4. $x^4 - x - 10 = 0$
5. $x^4 - x - 13 = 0$
6. $e^x = 1 + 2x$
7. $xe^x - \cos x = 0$ (*Gujarat B.E., 1990; B.U, B.E., 1995*)
8. $e^x \sin x = 1$
9. $x^x = 1000$
10. $3x - 1 = \cos x$ (*B.R, B.E., 1993*)
11. $\sin x = 1 - x$
12. $x^2 + 4 \sin x = 0$
13. $2x \tan x = 1$
14. $x(1 - \log_e^{\,x}) = 0.5$ (*M.U, B.E., 1987*)
15. $3x - e^x + \sin x = 0$
16. $x \sin x + \cos x = 0$ near $x = \pi$ (*Karnataka B.E., 1993*)

17. Find the interative formulae for finding $1/\sqrt{N}$, $\sqrt[3]{N}$, $\sqrt[4]{N}$ where N is a positive real number, using Newton's method. Hence evaluate $1/\sqrt{17}$, $\sqrt[3]{10}$, $\sqrt[4]{25}$.

18. Find a negative root of the following equations using Newton's method.

    (i) $x^3 - x^2 + x + 100 = 0$ (ii) $x^3 - 21x + 3500 = 0$

19. Find by Horner's method the root of the following equations correct to three decimal places.

    (i) $x^3 + 3x^2 - 12x - 11 = 0$ (ii) $x^3 + x^2 + x - 100 = 0$

    (iii) $x^3 - 6x - 13 = 0$ (iv) $x^3 - 3x + 1 = 0$

    (v) $x^3 - 30 = 0$ (vi) $x^4 + x^3 - 4x^2 - 16 = 0$

20. A sphere of pine wood, 2 metres in diameter, floating in water sinks to the depth of h metre, given by the equation $h^3 - 3h^2 + 2.5 = 0$. Find h correct to two decimal places using Horner's method.

21. Find a negative root of $x^3 - 2x + 5 = 0$ correct to two decimal places using Horner's method.

22. Find all the roots of the following equations by Graeffe's method squaring thrice.

    (i) $x^3 - 4x^2 + 5x - 2 = 0$ (ii) $x^3 - 2x^2 - 5x + 6 = 0$

    (iii) $x^3 - 5x^2 - 17x + 20 = 0$ *(M.U, B.E., 1991)*

    (iv) $x^3 - 9x^2 + 18x - 6 = 0$

    (v) $x^3 - x - 1 = 0$

ANSWERS

1. 1.674	2. 2.279	3. 1.834
4. 1.856	5. 1.961	6. 1.256
7. 0.518	8. 0.589	9. 3.592
10. 0.607	11. 0.511	12. −1.934
13. 0.653	14. 0.187	15. 0.360
16. 2.798	17. 0.24246, 2.15466, 2.236	
18. (i) −4.264	(ii) −16.56	
19. (i) 2.769	(ii) 4.264	(iii) 3.177
(iv) 1.532	(v) 3.107	(vi) 2.231
20. 1.17	21. −2.094	
22. (i) 2,1,1	(ii) 3, −2, 1	
(iii) 7.018, −2.974, 0.958	(iv) 6.3, 2.3, 0.4	
(v) 1.3247, −0.6624, ± 0.5622i		

CHAPTER 4

Simultaneous Linear Algebraic Equations

4.1 INTRODUCTION

Simultaneous linear algebraic equations are very common in various fields of Engineering and Science. We use matrix inversion method or Cramer's rule to solve these equations in general. But these methods prove to be tedious when the system of equations contain a large number of unknowns. To solve such equations there are other numerical methods, which are particularly suited for computer operations. These are of two types: *direct* and *iterative*.

Gauss elimination method, Gauss–Jordan method, Triangularisation method and Crout's method are *direct* methods whereas Gauss–Jacobi method, Gauss–Seidel iterative method and Relaxation method are *iterative* methods. These methods are explained in the following sections.

4.2 GAUSS ELIMINATION METHOD

In this method, the unknowns are eliminated successively by transforming the given system into an equivalent system with upper triangular coefficient matrix (i.e. a matrix in which all the elements below the principal diagonal are zero) by means of elementary row operations, from which the unknowns are found by back substitution. Here, we shall explain it by considering a system of three equations in three unknowns.

Consider the system

$$a_1x + b_1y + c_1z = d_1 \tag{4.1a}$$
$$a_2x + b_2y + c_2z = d_2 \tag{4.1b}$$
$$a_3x + b_3y + c_3z = d_3 \tag{4.1c}$$

where x, y, z are unknowns. The system in matrix form is $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Consider the argumented matrix $[A|B]$

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \tag{4.2}$$

Now Eqn (4.2) is to be reduced to an upper triangular matrix. Let $a_1 \neq 0$. Then

$$\begin{array}{l} R_2 \to R_2 - \dfrac{a_2}{a_1} R_1 \\ R_3 \to R_3 - \dfrac{a_3}{a_1} R_1 \end{array} \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array}\right] \tag{4.3}$$

Here, a_1 is called the first pivot and $b_2', c_2', d_2', b_3', c_3', d_3'$ are transformed elements.

Now take b_2' as the pivot ($b'_2 \neq 0$). Then

$$R_3 \to R_3 - \frac{b_3'}{b_2'} R_2 \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right] \tag{4.4}$$

Now, if $c_3'' \neq 0$, from Eqn (4.4), the given system of linear equations is equivalent to $a_1x + b_1y + c_1z = d_1$

$$\begin{aligned} b_2' + c_2'z &= d_2' \\ d_3'z &= d_3'' \end{aligned}$$

Using back substitution, $z = \dfrac{d_3''}{c_3''}$

$$y = \frac{1}{b_2'c_3''}\{d_2'\,c_3'' - c_2'\,d_3''\} \text{ and}$$

$$x = \frac{1}{a_1\,b_2'\,c_3''}\{d_1\,b_2'\,c_3'' - b_1\,d_2'\,c_3'' + b_1\,c_2'\,d_3'' - b_2'\,c_1 d_3''\}$$

Note:

1. This method fails if any one of the pivots a_1, b_2', or c_3'' becomes zero. In such cases, by interchanging the rows we can get the non-zero pivots.
2. *Partial pivoting*: From the first column of Eqn (4.2) [called the pivot column if $a_i \neq 0$, $i = 1, 2, 3$], select the component with the largest absolute value. This is called the pivot. Then at the second stage, i.e. from the second column of Eqn (4.3), select once again the component with the largest absolute value as the pivot. Continue this process. This procedure is called partial pivoting [refer to Example 1.1].
3. *Complete pivoting*: If we are not interested in the elimination of x, y, z in a particular order, then we can choose at each stage the numerically largest coefficient of the entire coefficient matrix. This requires an interchange of equations and also an interchange of position of the variables.

Example 4.1 Solve the system of equations $3x + y - z = 3$, $2x - 8y + z = -5$, $x - 2y + 9z = 8$ using Gauss elimination method.

(M.U, B.E., 1992)

Solution The given system is equivalent to
$$\underset{A}{\begin{bmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}}$$

$\therefore$ The argumented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array}\right]$$

Now we will make A as upper triangular choosing '3' as pivot,

$$\begin{array}{l} R_2 \to R_2 - \frac{2}{3}R_1 \\ R_3 \to R_3 - \frac{1}{3}R_1 \end{array} \sim \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & -7 \\ 0 & \frac{-7}{3} & \frac{28}{3} & 7 \end{array}\right]$$

Now choosing $-\frac{26}{3}$ as the pivot from the second column,

$$R_3 \to R_3 - \frac{7}{26}R_2 \sim \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & -\frac{26}{3} & \frac{5}{3} & -7 \\ 0 & 0 & \frac{693}{78} & \frac{231}{26} \end{array}\right]$$

From this we get, $3x + y - z = 3$,

$$-\frac{26}{3}y + \frac{5}{3}z = -7 \text{ and } \frac{693}{78}z = \frac{231}{26}$$

Now by back substitution, $z = 1$,

$$-\frac{26}{3}y = -7 - \frac{5}{3}z = -7 - \frac{5}{3} = -\frac{26}{3} \quad \therefore y = 1$$

and $$x = \frac{1}{3}[3 - y + z] = \frac{1}{3}[3 - 1 + 1] = 1$$

$$\therefore x = 1,\ y = 1, z = 1$$

Example 4.2 Solve the system of equations,
$28x + 4y - z = 32$, $x + 3y + 10z = 24$ and $2x + 17y + 4z = 35$ by Gauss elimination method. *(M.U, 1990,1992)*

Solution The given system is equivalent to

$$\begin{bmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 24 \\ 35 \end{bmatrix}$$

$$A \qquad X = B$$

The argumented matrix is

$$[A/B] = \left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{array}\right]$$

Now we will make A as upper triangular choosing 28 as pivot.

$$\begin{array}{l} R_2 \to R_2 - \frac{R_1}{28} \\ R_3 \to R_3 - \frac{R_1}{14} \end{array} \sim \left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 0 & \frac{80}{28} & \frac{281}{28} & \frac{640}{28} \\ 0 & \frac{234}{14} & \frac{57}{14} & \frac{458}{14} \end{array}\right]$$

Now the pivot is $\dfrac{234}{14}$

$$\therefore\ R(2,3) \sim \left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 0 & \dfrac{234}{14} & \dfrac{57}{14} & \dfrac{458}{14} \\ 0 & \dfrac{80}{28} & \dfrac{281}{28} & \dfrac{640}{28} \end{array}\right]$$

$$R_3 \to R_3 - \frac{20}{117} R_2 \sim \left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 0 & \dfrac{234}{14} & \dfrac{57}{14} & \dfrac{458}{14} \\ 0 & 0 & \dfrac{30597}{1638} & \dfrac{56560}{1638} \end{array}\right]$$

From this, we get

$$28x + 4y - z = 32$$

$$234y + 57z = 458$$

and $$30597z = 56560$$

Now by back substitution, we get

$$\therefore \qquad z = \frac{56560}{30597} = 1.8485472$$

$$y = \frac{458 - 57z}{234} = 1.5069778$$

and $$x = \frac{-4y + z + 32}{28} = 0.9935941$$

Example 4.3 Using Gauss elimination method, solve the system

$$3.15\,x - 1.96\,y + 3.85\,z = 12.95$$

$$2.13\,x + 5.12\,y - 2.89\,z = -8.61$$

$$5.92\,x + 3.05\,y + 2.15\,z = 6.88 \qquad (M.K.U, 1981)$$

Solution The given system is equivalent to

$$\begin{bmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12.95 \\ -8.61 \\ 6.88 \end{bmatrix}$$

$$A \qquad X = B$$

$$\therefore [A/B] = \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 2.13 & 5.12 & -2.89 & -8.61 \\ 5.92 & 3.05 & 2.15 & 6.88 \end{array}\right]$$

Now we will make A as upper triangular, choosing 3.15 as pivot

$$\begin{array}{l} R_2 \to R_2 - \dfrac{2.13}{3.15} R_1 \\ R_3 \to R_3 - \dfrac{5.92}{3.15} R_1 \end{array} \sim \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3667 \\ 0 & 6.7335 & -5.0855 & -17.4578 \end{array}\right]$$

Choosing 6.4453 as pivot

$$R_3 \to R_3 - \frac{6.7335}{6.4453} R_2 \sim \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3667 \\ 0 & 0 & 0.6534 & 0.6853 \end{array}\right]$$

From this, we get

$$3.15\,x - 1.96\,y + 3.85\,z = 12.95$$

$$6.4453\,y - 5.4933\,z = -17.3667$$

$$0.6534\,z = 0.6853$$

By back substitution

$$z = \frac{0.6853}{0.6534} = 1.0488215$$

$$y = \frac{5.4933z - 17.3667}{6.4453} = -1.8005692$$

and $$x = \frac{1.96y - 3.85z + 12.95}{3.15} = 1.708864$$

Example 4.4 Solve the system of equations,

$x_1 + x_2 + x_3 + x_4 = 2,$ $\quad x_1 + x_2 + 3x_3 - 2x_4 = -6$

$2x_1 + 3x_2 - x_3 + 2x_4 = 7,$ $\quad x_1 + 2x_2 + x_3 - x_4 = -2$

by Gauss elimination method. *(M.U., B.E., 1986)*

Solution The given system in matrix form is

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -2 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 1 & -1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} 2 \\ -6 \\ 7 \\ -2 \end{bmatrix}}$$

The argumented matrix $[A|B]$ is

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & -2 & -6 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 2 & 1 & -1 & -2 \end{array}\right]$$

Choose 1 of first column as pivot. Then

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 1 & 0 & -2 & -4 \end{array}\right]$$

Since the element in 2nd row, 2nd column is zero, interchange 2nd and 3rd rows to get pivot element 1. That is,

$$R\,(2, 3) \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & 0 & -2 & -4 \end{array}\right]$$

$$R_4 \to R_4 - R_2 \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 0 & 3 & -2 & -7 \end{array}\right]$$

Now the pivot is 2, therefore,

$$R_4 \to R_4 - \frac{3}{2}R_3 \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 0 & 0 & \frac{5}{2} & 5 \end{array}\right]$$

From this we get,

$$x_1 + x_2 + x_3 + x_4 = 2 \qquad \text{(i)}$$

$$x_2 - 3x_3 = 3 \qquad \text{(ii)}$$

$$2x_3 - 3x_4 = -8 \qquad \text{(iii)}$$

$$\left(\frac{5}{2}\right) x_4 = 5 \qquad \text{(iv)}$$

Now, from Eqn (iv), $x_4 = 2$

from Eqn (iii),

$$x_3 = \frac{1}{2}(-8 + 3x_4) = \frac{1}{2}(-8 + 6) = -1$$

from Eqn (ii),

$$x_2 = 3 + 3x_3 = 3 - 3 = 0$$

and from Eqn (i),

$$x_1 = 2 - x_2 - x_3 - x_4 = 2 - 0 - (-1) - 2 = 1$$

$\therefore x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 2$

4.3 GAUSS–JORDAN METHOD

This method is a modified form of Gauss elimination method. In this method, the coefficient matrix A of $AX = B$ is reduced to a diagonal matrix or unit matrix by making all the elements above and below to the principal diagonal of A as zero. The labour of back substitution is saved here even though it involves additional computations.

Example 4.5 Solve the equations
$10x + y + z = 12$, $2x + 10y + z = 13$ and
$x + y + 5z = 7$ by Gauss–Jordan method. *(M.U, 1991)*

Solution The given system in matrix form is

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

$$A \quad X \quad = \quad B$$

$$\therefore [A/B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

$$R_1 \to R_1 - 9R_3 \sim \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 26 & 89 & 115 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

$$R_2 \to -(R_2 - 3R_3) \sim \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 1 & 58 & 59 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + 8R_2 \\ R_3 \to R_3 - 9R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \\ 0 & 0 & -473 & -473 \end{array}\right]$$

$$\begin{array}{l} R_3 \to -\dfrac{1}{473} R_3 \\ R_1 \to R_1 - 420R_3 \\ R_2 \to R_2 - 58R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore$ The system AX = B reduces to the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

i.e., $x = y = z = 1$

Example 4.6 Solve the equations
$10x_1 + x_2 + x_3 = 12$, $x_1 + 10x_2 - x_3 = 10$ and
$x_1 - 2x_2 + 10x_3 = 9$ by Gauss–Jordan method. *(M.U. 1997)*

Solution The matrix form of the given systems is

$$\begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 9 \end{bmatrix}$$

$$A \qquad\qquad X \;=\; B$$

$$[A/B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{array}\right]$$

$$R_1 \to R_1 - 9R_2 \sim \begin{bmatrix} 1 & -89 & 10 & -78 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{bmatrix}$$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 99 & -11 & 88 \\ 0 & 87 & 0 & 87 \end{array}\right]$$

$$\begin{array}{l} R_2 \to \dfrac{R_2}{99} \\ R_3 \to \dfrac{R_3}{87} \end{array} \sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 9 & -1 & 8 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 8R_3 \sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + 89R_2 \\ R_3 \to R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -79 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + 79R_3 \\ R_2 \to R_2 + R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore$ The system $AX = B$ reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow x_1 = x_2 = x_3 = 1$

Example 4.7 Solve the following equations by Gauss–Jordan method.

$x + 2y + z - w = -2;$ $\qquad 2x + 3y - z + 2w = 7$

$x + y + 3z - 2w = -6;$ $\qquad x + y + z + w = 2$

(*M.U, B.E., 1986*)

Solution The given system in matrix form is

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & -1 & 2 \\ 1 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -6 \\ 2 \end{bmatrix}$$

$$A \qquad X = B$$

The argumented matrix is

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 1 & 3 & -2 & -6 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$\begin{array}{l} R_2 \to -(R_2 - 2R_1) \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 0 & 1 & 3 & -4 & 11 \\ 0 & -1 & 2 & -1 & -4 \\ 0 & -1 & 0 & 2 & 4 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to \frac{1}{5}(R_3 + R_2) \\ R_4 \to R_4 + R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 7 & 20 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 3 & -2 & -7 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + 5R_3 \\ R_2 \to R_2 - 3R_3 \\ R_4 \to R_4 - 3R_3 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 - 2R_4 \\ R_2 \to R_2 + R_4 \\ R_3 \to R_3 + R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$\therefore$ The system $AX = B$ reduces to the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

i.e. $x = 1, y = 0, z = -1$ and $w = 2$.

4.4 INVERSE OF A MATRIX USING GAUSS ELIMINATION METHOD

Let A be a square matrix of order 3, $|A| \neq 0$ and X be its inverse. Then we know that $AX = I$, where I is a unit matrix of order 3.

$$\therefore \text{ If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

then

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4.5)$$

This equation is equivalent to the three equations given below

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (4.6)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (4.7)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (4.8)$$

Each of the systems given in Eqns (4.6)–(4.8) can be solved for

$$\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}, \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}, \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix}$$

using Gauss elimination method. The solution set of each system will be the corresponding column of the inverse matrix X. Since the coefficient matrix A is the same in all the three systems, we can solve them simultaneously by considering $[A|I]$.

Example 4.8 Find the inverse of

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$ using Gauss elimination method.

Solution We have the argumented system as

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - \left(\dfrac{3}{2}\right) R_1 \\ R_3 \to R_3 - \left(\dfrac{1}{2}\right) R_1 \end{array} \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \dfrac{1}{2} & \dfrac{3}{2} & \dfrac{-3}{2} & 1 & 0 \\ 0 & \dfrac{7}{2} & \dfrac{17}{2} & \dfrac{-1}{2} & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 - 7R_2 \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \dfrac{1}{2} & \dfrac{3}{2} & -\dfrac{3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right] \qquad \text{(i)}$$

Now if

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

is the inverse of the given matrix, then the system (i) is equivalent to the following three systems:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 10 \end{bmatrix} \qquad \text{(ii)}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix} \qquad \text{(iii)}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(iv)}$$

By back substitution, the three systems of equations, i.e. (ii)–(iv) give the three columns of the inverse matrix.

$$\therefore \quad x_{11} = -3; \qquad x_{21} = 12; \qquad x_{31} = -5$$

$$x_{12} = \frac{5}{2}; \qquad x_{22} = -\frac{17}{2}; \quad x_{23} = \frac{7}{2}$$

$$x_{13} = -\frac{1}{2}; \qquad x_{23} = \frac{3}{2}; \qquad x_{33} = -\frac{1}{2}$$

and the inverse matrix is

$$= \frac{1}{2}\begin{bmatrix} -6 & 5 & -1 \\ 24 & -17 & 3 \\ -10 & 7 & -1 \end{bmatrix}$$

Example 4.9 Find the inverse of the matrix

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$ using Gauss elimination method. *(M.U 1997)*

Solution We have the argumented system

$$[A/I] = \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - \frac{1}{2}R_1 \\ R_3 \to R_3 - \frac{1}{4}R_1 \end{array} \sim \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -2 & \frac{-1}{2} & 1 & 0 \\ 0 & \frac{-9}{4} & \frac{3}{2} & \frac{-1}{4} & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 + \frac{9}{10}R_2 \sim \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -2 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -\frac{3}{10} & -\frac{14}{20} & \frac{9}{10} & 1 \end{array}\right] \tag{i}$$

Now if $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ is the inverse of the given matrix, then the system (1) is equivalent to three systems

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{14}{20} \end{bmatrix} \tag{ii}$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{9}{10} \end{bmatrix} \tag{iii}$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(iv)}$$

By back substitution, the three systems i.e., (ii), (iii) and (iv) give the three columns of the inverse matrix. i.e.,

$$x_{11} = \frac{-4}{3}, \qquad x_{21} = \frac{5}{3}, \qquad x_{31} = \frac{7}{3}$$

$$x_{12} = 2, \qquad x_{22} = -2, \qquad x_{23} = -3$$

$$x_{31} = \frac{7}{3}, \qquad x_{11} = -\frac{8}{3}, \qquad x_{33} = -\frac{10}{3}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{-4}{3} & 2 & \frac{7}{3} \\ \frac{5}{3} & -2 & \frac{-8}{3} \\ \frac{7}{3} & -3 & \frac{-10}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -4 & 6 & 7 \\ 5 & -6 & -8 \\ 7 & -9 & -10 \end{bmatrix}$$

EXERCISE 4.1

Solve the following equations by Gauss elimination method

1. $3x + 4y - z = 8,\ -2x + y + z = 3,\ x + 2y - z = 2$
2. $x - y + z = 1,\ -3x + 2y - 3z = -6,\ 2x - 5y + 4z = 5$
3. $10x + y + z = 12,\ 2x + 10y + z = 13,\ 2x + 2y + 10z = 14$
   (M.U, B.E., 1991)
4. $2x - y + 2z = 2,\ x + 10y - 3z = 5,\ x - y - z = 3$
   (Ranchi, B. Tech, 1987)
5. $10x_1 + x_2 + x_3 = 18.141, x_1 + x_2 + 10x_3 = 38.139, x_1 + 10x_2 + x_3 = 28.140$
   (M.U, B.E., 1991)
6. $x + y + z = 6.6,\ x - y + z = 2.2,\ x + 2y + 3z = 15.2$
   (North Bengal B Tech, 1987)
7. $2x + 4y + 2z = 15,\ 2x + y + 2z = -5,\ 4x + y - 2z = 0$
   (M.U, B.E., 1989)

8. $2x + y + z = 10,\ 3x + 2y + 3z = 18,\ x + 4y + 9z = 16$
(*Bangalore, B.E., 1990*)

9. $2x_1 + 2x_2 + x_3 = 12,\ 3x_1 + 2x_2 + 2x_3 = 8,\ 5x_1 + 10x_2 - 8x_3 = 10$
(*S.Gujarat, B.E., 1990*)

10. $x + 2y - 12z + 8w = 27,\ 5x + 4y + 7z - 2w = 4,$
$-3x + 7y + 9z + 5w = 11,\ 6x - 12y - 8z + 3w = 49$
(*M.U, B.E., 1987*)

Solve the following equations by Gauss - Jordan method

11. $2x - 3y + z = -1,\ x + 4y + 5z = 25,\ 3x - 4y + z = 2$
(*M.U. B.E., 1993*)

12. $2x + y + z = 12,\ 3x + 2y + 3z = 24,\ x + 4y + 9z = 34$

13. $10x + y + z = 12,\ x + 10y + z = 12,\ x + y + 10z = 12$
(*Bhopal B.E., 1991*)

14. $x + 2y + z = 8,\ 2x + 3y + 4z = 20,\ 4x + 3y + 2z = 16$
(*Punjab, B.E., 1987*)

15. $4x - y - z = -7,\ x - 5y + z = -10,\ x + 2y + 6z = 9$

16. $2x + 2y - z + t = 4,\ 4x + 3y - z + 2t = 6,$
$8x + 5y - 3z + 4t = 12,\ 3x + 3y - 2z + 2t = 6$

17. $5x_1 + x_2 + x_3 + x_4 = 4\ ;\ x_1 + 7x_2 + x_3 + 4x_4 = 2$
$x_1 + x_2 + 6x_3 + x_4 = -5\ ;\ x_1 + x_2 + x_3 + x_4 = -6$

Find the inverse of the following matrices using Gauss elimination method.

18. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ 19. $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ 20. $\begin{bmatrix} 3 & -1 & 10 & 2 \\ 5 & 1 & 20 & 3 \\ 9 & 7 & 39 & 4 \\ 1 & -2 & 2 & 1 \end{bmatrix}$

ANSWERS

1. 1, 2, 3
2. − 2, 3, 6
3. 1, 1, 1
4. 2, 0, − 1
5. 1.234, 2.348, 3.455
6. 1.2, 2.2, 3.2
7. − 3.0556, 6.6667, − 2.778
8. 7, − 9, 5

9. $-12.75, 14.375, 8.75$

10. $3.0016, -1.999, 0.9998, 5.0001$

11. $8.7, 5.7, -1.3$

12. $7, -9, 7$

13. $1, 1, 1$

14. $1, 2, 3$

15. $-1, 2, 1$

16. $1, 1, -1, -1$

17. $1, 2, -1, -2$

18. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

19. $\frac{1}{5}\begin{bmatrix} 2 & 2 & -3 \\ -2 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 7 & -3 & 0 & -5 \\ 8 & 10 & -2 & 11 \\ -5 & 0 & 1 & 6 \\ 19 & 5 & -6 & -28 \end{bmatrix}$

4.5 METHOD OF FACTORISATION OR TRIANGULARISATION

Consider the following system of equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These equations can be written in matrix form as

$$AX = B \tag{4.9}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

In this method we use the fact that the square matrix A can be factorised into the form LU, where L is a unit lower triangular matrix and U, an upper triangular matrix if all the minors of A are non-singular.

Let

$$A = LU \tag{4.10}$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Now Eqn (4.9) becomes

$$LUX = B \tag{4.11}$$

Setting

$$UX = Y \tag{4.12}$$

Eqn (4.11) implies

$$LY = B \tag{4.13}$$

i.e.

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

giving

$$y = b_1,$$

$$l_{21} y_1 + y_2 = b_2$$

and

$$l_{31} y_1 + l_{32} y_2 + y_3 = b_3.$$

From these, y_1, y_2 and y_3 can be solved by forward substitution. Now, from Eqn (4.12) we have

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

i.e. $u_{11} x_1 + u_{12} x_2 + u_{13} x_3 = y_1$; $\quad u_{22} x_2 + u_{23} x_3 = y_2$

and $u_{33} x_3 = y_3$.

Solving these, we get, by back substitution, x_1, x_2, x_3.

Now L and U can be found from $LU = A$, i.e

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating the corresponding coefficients, we get

$$u_{11} = a_{11}, \; u_{12} = a_{12}, \; u_{13} = a_{13}$$

$$l_{21}\, u_{11} = a_{21} \text{ or } l_{21} = \frac{a_{21}}{a_{11}}; \; l_{31}\, u_{11} = a_{31} \text{ or } l_{31} = \frac{a_{21}}{a_{11}}$$

$$l_{21}\, u_{12} + u_{22} = a_{22} \text{ or } u_{22} = \frac{1}{a_{11}} (a_{11}\, a_{22} - a_{21}\, a_{12})$$

$$l_{21}\, u_{13} + u_{23} = a_{23} \text{ or } u_{23} = \frac{1}{a_{11}}(a_{11}\, a_{23} - a_{21}\, a_{13})$$

$$l_{31}\, u_{12} + l_{32}\, u_{22} = a_{32} \text{ or } l_{32} = \frac{1}{u_{22}}(a_{32} - l_{31}\, u_{12})$$

and $l_{31}\, u_{13} + l_{32}\, u_{23} + u_{33} = a_{33}$ or $u_{33} = a_{33} - l_{31}\, u_{13} - l_{32}\, u_{23}$

Thus, L and U are known.

Example 4.10 Solve the following system by the method of triangularisation:

$$2x - 3y + 10z = 3,\ -x + 4y + 2z = 20,\ 5x + 2y + z = -12$$

Solution The given system is $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Let $LU = A$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

$u_{11} = 2,\ u_{12} = -3,\ u_{13} = 10.$

$l_{21}u_{11} = -1, \qquad \therefore l_{21} = -\frac{1}{2},$

$l_{31}u_{11} = 5, \qquad \therefore l_{31} = \frac{5}{2}$

$l_{21}u_{12} + u_{22} = 4, \qquad \therefore u_{22} = 4 - l_{21}u_{12} = 4 - (-\frac{1}{2})(-3) = \frac{5}{2}.$

$l_{21}u_{13} + u_{23} = 2, \qquad \therefore u_{23} = 2 - l_{12}u_{13} = 2 - (-\frac{1}{2})(10) = 7$

$l_{31}u_{12} + l_{32}u_{22} = 5, \qquad \therefore\ l_{32} = \frac{1}{u_{22}}[5 - l_{31}u_{12}] = \frac{19}{5}$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1, \qquad \therefore\ u_{33} = 1 - l_{31}u_{13} - l_{32}u_{23} = -\frac{253}{5}$

$$\therefore \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & -\frac{253}{5} \end{bmatrix}$$

Let $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then $LY = B$,

i.e.
$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix} \qquad \text{(i)}$$

and
$$\begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & -\frac{253}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \text{(ii)}$$

Now Eqn (i) implies

$y_1 = 3; \qquad -\frac{1}{2}y_1 + y_2 = 20; \qquad \frac{5}{2}y_1 + \frac{19}{5}y_2 + y_3 = -12$

$\therefore\ y_1 = 3, \qquad y_1 = \frac{43}{2} \qquad y_3 = -\frac{506}{5}$

$$2x-3y+10z=3;\qquad \frac{5}{2}y+7z=\frac{43}{2},\qquad -\frac{253}{5}z=-\frac{506}{3}$$

Solving these, we get, by back substitution, $x=-4$, $y=3$ and $z=2$.

Example 4.11 Solve the following system by the method of factorisation
$x+3y+8z=4$, $x+4y+3z=-2$ and
$x+3y+4z=1$ *(M.U, 1991)*

Solution The given system is $AX=B$, where

$$A=\begin{bmatrix}1&3&8\\1&4&3\\1&3&4\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}\text{ and } B=\begin{bmatrix}4\\-2\\1\end{bmatrix}$$

Let $LU=A$ where

$$L=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\text{ and } U=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$$

$$\therefore\ \begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}1&3&8\\1&4&3\\1&3&4\end{bmatrix}$$

$\Rightarrow u_{11}=1,\ u_{12}=3,\ u_{13}=8$

$l_{21}u_{11}=1\Rightarrow l_{21}=1;$ $\qquad l_{31}u_{11}=1\Rightarrow l_{31}=1$

$l_{21}u_{12}+u_{22}=4$ $\qquad \Rightarrow u_{22}=4-l_{21}u_{12}=4-1(3)=1$

$l_{21}u_{13}+u_{23}=3$ $\qquad \Rightarrow u_{23}=3-l_{21}u_{13}=3-1(8)=-5$

$l_{31}u_{12}+l_{32}u_{22}=3$ $\qquad \Rightarrow l_{32}=\dfrac{1}{u_{22}}[3-l_{31}u_{12}]=0$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=4$

$\therefore\ u_{33}=4-l_{31}u_{13}-l_{32}u_{23}$

$=4-1.8-0.(-5)=-4$

$$\therefore L=\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}\text{ and } U=\begin{bmatrix}1&3&8\\0&1&-5\\0&0&-4\end{bmatrix}$$

Let $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then $LY = B$

$$\therefore \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad \text{(i)}$$

and
$$\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \text{(ii)}$$

Now (i) $\Rightarrow y_1 = 4, y_2 = -2$

and $y_1 + y_3 = 1 \Rightarrow y_3 = 1 - y_1 = 1 - 4 = -3$

From (ii) $\quad x + 3y + 8z = 4$

$$y - 5z = -2$$

$$-4z = -3$$

By back substitution,

$$z = \frac{3}{4}$$

$$y = -4 + 5z = -2 + 5\left(\frac{3}{4}\right) = \frac{7}{4}$$

and
$$x = 4 - 3y - 8z = 4 - 3\left(\frac{7}{4}\right) - 8\left(\frac{3}{4}\right) = \frac{29}{4}$$

$$\therefore x = -\frac{29}{4}, y = \frac{7}{4}, z = \frac{3}{4}$$

4.6 CROUT'S METHOD

This method is superior to the Gauss elimination method because it requires less calculation. It is based on the fact that every square matrix A can be expressed as the product of a lower triangular matrix and a unit upper triangular matrix. Let

$$AX = B \qquad (4.12)$$

be the given system [Ref 4.5] and let

$$A = LU \qquad (4.13)$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 1 & 1 \end{bmatrix}$$

Then Eqn (4.12) becomes

$$LUX = B \tag{4.14}$$

Let

$$UX = Y \tag{4.15}$$

so that Eqn (4.14) becomes

$$LY = B \tag{4.16}$$

i.e.

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\therefore\ l_{11} = y_1 = b_1; \qquad l_{21}\,y_1 + l_{22}\,y_2 = b_2; \qquad l_{31}\,y_1 + l_{32}\,y_2 + l_{33}\,y_3 = b_3$

By forward substitution, y_1, y_2 and y_3 can be found out if L is known. From Eqn (4.15),

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4.17}$$

i.e. $x_1 + u_{12}\,x_2 + u_{13}\,x_3 = y_1; \qquad x_2 + u_{23}x_3 = y_2; \qquad x_3 = y_3$

By back substitution, x_1, x_2 and x_3 can be found out if U is known.
Now L and U can be found from $LU = A$.

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

or

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating the corresponding elements on both sides, we get

$l_{11} = a_{11},\ l_{21} = a_{21},\ l_{31} = a_{31}$

$l_{11}\,u_{12} = a_{12} \qquad \therefore u_{12} = a_{12}/a_{11}$

$l_{11}\,u_{13} = a_{13} \qquad \therefore u_{13} = a_{13}/a_{11}$.

$l_{21}\, u_{12} + l_{22} = a_{22}$ $\quad\therefore l_{22} = a_{22} - l_{21}u_{12}$

$l_{31}\, u_{12} + l_{32} = a_{32}$ $\quad\therefore l_{32} = a_{32} - l_{31}u_{12}$

$l_{21}\, u_{13} + l_{22}\, u_{23} = a_{23}$ $\quad\therefore u_{32} = \{a_{23} - l_{21}u_{13}\}/u_{23}$

$l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33} = a_{33}$ $\quad\therefore l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$

Thus all the 12 unknowns are determined and the solution, as shown already, is obtained from Eqn (4.17).

Crout, during the above decomposition of the coefficient matrix A, devised a technique to determine the 12 unknowns systematically.

Computation Scheme by Crout's Method

The argumented matrix of the system given by Eqn (4.12) ($AX = B$) is

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

The matrix of 12 unknowns, so called *derived matrix or auxiliary matrix*, is

$$\begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}$$

and is to be calculated as follows.

Step 1 The *first column* of the derived matrix (DM) is identical with the first column of $[A|B]$.

Step 2 The *first row* to the right of the first column of the DM is obtained by dividing the corresponding element in $[A|B]$ by the leading diagonal element of that row.

Step 3 *Remaining second column of DM.*

$$\because l_{22} = a_{22} - l_{21}u_{12}\,;\; l_{32} = a_{32} - l_{31}u_{12}$$

{Each element on or below the diagonal} = {(Corresponding element in $[A|B]$) – (The product of the first element in that row and in that column)}

Step 4 *Remaining elements of second row of DM.*

Each element = {(Corresponding element in $[A|B]$) – (the product of the first element in that row and in that column divided by leading diagonal element in that row)

i.e. $u_{23} = \dfrac{a_{23} - l_{21}u_{13}}{l_{22}}$; $\quad y_2 = \dfrac{b_2 - l_{21}y_1}{l_{22}}$

Step 5 *Remaining elements of third column of DM.*

Each element = {(Corresponding element in $[A|B]$) – (sum of the inner products of the previously calculated elements in the same row and column)}

i.e. $l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$

Step 6 *Remaining elements of third row of DM.*

Each element = {[Corresponding element in $[A|B]$) – Sum of the inner products of the previously calculated elements in the same row and column] ÷ [The leading diagonal element in that row]}

i.e. $y_3 = \dfrac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}}$

Now the matrices L, U, and Y can be written and hence, X can be obtained from $UX = Y$.

Note: The above procedure holds good for any order square matrix A.

Example 4.12 Solve the system

$2x + y + 4z = 12, \quad 8x + 3y + 2z = 20, \; 4x + 11y + z = 33$

by Crout's method. *(M.U, B.E., 1991)*

Solution Argumented matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 23 \end{array}\right]$$

Let the derived matrix (DM)

$$= \begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}$$

(1) Elements of the first column of DM are $l_{11} = 2, l_{21} = 8, l_{31} = 4$

(2) Elements of the first row to the right of the first column are

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{1}{2}; u_{13} = \frac{a_{13}}{l_{11}} = \frac{y}{2} = 2; y_1 = \frac{b_1}{l_{11}} = \frac{12}{2} = 6$$

(3) Elements of the remaining second column are

$$l_{22} = a_{22} - u_{12}\, l_{21} = (-3) - \frac{1}{2}\,(8) = -7$$

$$l_{32} = a_{32} - u_{12}\, l_{31} = 11 - \frac{1}{2}\,(4) = 9$$

(4) Elements of the remaining second row are

$$u_{23} = \frac{a_{23} - u_{13}l_{21}}{l_{22}} = -\frac{1}{7}\{2-(-2)\,(8)\} = 2$$

$$y_2 = \frac{b_2 - l_{21}y_1}{l_{22}} = \frac{20-(8)(6)}{-7} = 4$$

(5) For the remaining third column,

$$l_{33} = a_{33} - (l_{31}\, u_{13} + l_{32}\, u_{23})$$
$$= -1 - (4)(2) - (9)(2) = -27$$

(6) For the remaining third row,

$$y_3 = \frac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}} = \frac{33-(4)(6)-(9)(4)}{-27} = -1$$

$$\therefore\ DM = \begin{bmatrix} 2 & \frac{1}{2} & 2 & 6 \\ 8 & -7 & 2 & 4 \\ 4 & 9 & -27 & 1 \end{bmatrix}$$

Now the solution is obtained from the system $UX = Y$,

i.e.
$$\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

which is equivalent to

$$x+\frac{1}{2}y+2z=6;\ y+z=4,\ z=1$$

By back substitution, $x=3, y=2$ and $z=1$

Example 4.13 Solve the following system by Crout's method.

$x+y+z=3,\ 2x-y+3z=16,\ 3x+y-z=-3$ (*M.U, 1997, M.K.U, 1981*)

Solution Argumented matrix of the system is

$$[A/B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 16 \\ 3 & 1 & -1 & -3 \end{array}\right]$$

Let the derived matrix $(DM)=\left[\begin{array}{ccc|c} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{array}\right]$

1. Elements of the first column of DM are $l_{11}=1, l_{21}=2, l_{31}=3$
2. Elements of the first row to the right of first column

$$u_{12}=\frac{a_{12}}{l_{11}}=\frac{1}{1}=1\ ;\ u_{13}=\frac{a_{13}}{l_{11}}=\frac{1}{1}=1\ ;\ y_1=\frac{b_1}{l_{11}}=\frac{3}{1}=3$$

3. Elements of the remaining second column

$$l_{22}=a_{22}-u_{12}\,l_{21}=-1-1(2)=-3$$
$$l_{32}=a_{32}-u_{12}\,l_{31}=1-1(3)=-2$$

4. Elements of the remaining second row

$$u_{23}=\frac{a_{23}-u_{13}l_{21}}{l_{22}}=\frac{3-1(2)}{-3}=-\frac{1}{3}$$

$$y_2=\frac{b_2-l_2y_1}{l_{22}}=\frac{16-3(2)}{-3}=-\frac{10}{3}$$

5. Remaining third column

$$l_{33}=a_{33}-(l_{31}\,u_{13}+l_{32}\,u_{23})$$
$$=-1-1(3)-(-\frac{1}{3})(-2)=-\frac{14}{3}$$

6. Remaining third row

$$y_3 = \frac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}} = \frac{-3 - 3.3 - (-2) - (10/3)}{-(14/3)} = 4$$

$$\therefore DM = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -1/3 & -10/3 \\ 3 & -2 & -14/3 & 4 \end{bmatrix}$$

Now the solution is got from the system UX = Y

i.e.
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10/3 \\ 4 \end{bmatrix}$$

$\therefore$
$$x + y + z = 3$$
$$y - z/3 = -(10/3)$$
$$z = 4$$

By back substitution,

$z = 4$, $y = -2$ and $x = 1$

Example 4.14 Solve the following system by Crout's method.

$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6;$
$-6x_1 - 8x_2 + x_3 + 4x_4 = 5;$
$3x_1 - x_2 + 4x_3 + 11x_4 = 2;$
$5x_1 - 9x_2 + 2x_3 + 4x_4 = 7.$

Solution Argumented matrix of the given system is

$$[A|B] = \begin{bmatrix} 10 & -7 & 3 & 5 & 6 \\ -6 & 8 & -1 & -4 & 5 \\ 3 & 1 & 4 & 11 & 2 \\ 5 & -9 & -2 & 4 & 7 \end{bmatrix}$$

The derived matrix is,

$$DM = \begin{bmatrix} l_{11} & u_{12} & u_{13} & u_{14} & y_1 \\ l_{21} & l_{22} & u_{23} & u_{24} & y_2 \\ l_{31} & l_{32} & l_{33} & u_{34} & y_3 \\ l_{41} & l_{42} & l_{43} & l_{44} & y_4 \end{bmatrix}$$

(1) Elements of the first column of DM.

$$l_{11} = 10, l_{21} = -6, l_{31} = 3, l_{41} = 5$$

(2) Elements of the first row right to the first column:

$$u_{12} = -\frac{7}{10}, u_{13} = \frac{3}{10}, u_{14} = \frac{5}{10}, y_1 = \frac{6}{10}$$

(3) Remaining second column:

$$l_{22} = a_{22} - l_{21}\, u_{12} = 8 - (-6)\,(-\frac{7}{10}) = \frac{38}{10}$$

$$l_{32} = a_{32} - l_{31}\, u_{12} = 1 - (3)\,(-\frac{7}{10}) = \frac{31}{10}$$

$$l_{42} = a_{42} - l_{41}\, u_{12} = -9 - (5)\,(-\frac{7}{10}) = -\frac{55}{10}$$

(3) Remaining second row:

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} = \frac{-1-(-6)\left(\frac{3}{10}\right)}{\left(\frac{38}{10}\right)} = \frac{8}{38}$$

$$u_{24} = \frac{a_{14} - l_{21}u_{14}}{l_{22}} = \frac{-4-(-6)\left(\frac{5}{10}\right)}{\left(\frac{38}{10}\right)} = -\frac{10}{38}$$

$$y_2 = \frac{b_2 - l_{21}y_1}{l_{22}} = \frac{5-(-6)\left(\frac{6}{10}\right)}{\left(\frac{38}{10}\right)} = \frac{86}{38}$$

(5) Remaining third column:

$$l_{33} = a_{33} - (l_{31}\, u_{13} + l_{32}\, u_{23})$$

$$= 4 - \{\, (3)\,(\frac{3}{10}) + (\frac{31}{10})\,(\frac{8}{38})\} = \frac{93}{38}$$

$$l_{43} = a_{43} - (l_{41}\, u_{13} + l_{42}\, u_{23})$$

$$= -2 - \{(5)\,(\frac{3}{10}) - (\frac{55}{10})\,(\frac{8}{38})\} = -\frac{89}{38}$$

(6) **Remaining third row:**

$$u_{34} = \frac{a_{34} - (l_{31}u_{14} + l_{32}u_{24})}{l_{33}} = \frac{11 - \left\{3\left(\frac{5}{10}\right) + \left(\frac{31}{10}\right)\left(-\frac{10}{38}\right)\right\}}{\left(\frac{93}{38}\right)}$$

$$= \frac{392}{93}$$

$$y_3 = \frac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}} = \frac{2 - \left\{3\left(\frac{6}{10}\right) + \left(\frac{31}{10}\right)\left(\frac{86}{38}\right)\right\}}{\left(\frac{93}{38}\right)}$$

$$= -\frac{259}{93}$$

(7) **Remaining fourth column:**

$$l_{44} = a_{44} - (l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34})$$

$$= 4 - \left\{5\left(\frac{5}{10}\right) + \left(\frac{55}{10}\right)\left(\frac{10}{38}\right) - \left(\frac{89}{38}\right)\left(\frac{392}{93}\right)\right\}$$

$$= \frac{35074}{3534}$$

(8) **Remaining fourth row:**

$$y_4 = \frac{b_4 - \{l_{41}y_1 + l_{42}y_2 + l_{43}y_3\}}{l_{44}}$$

$$= \frac{7 - \left\{5\left(\frac{6}{10}\right) - \left(\frac{55}{10}\right)\left(\frac{86}{38}\right) + \left(\frac{89}{38}\right)\left(\frac{259}{93}\right)\right\}}{\left(\frac{35074}{3534}\right)} = 1.$$

Now the solution to the given system is obtained by $UX = Y$, i.e.

$$\begin{bmatrix} 1 & -\frac{7}{10} & \frac{3}{10} & \frac{5}{10} \\ 0 & 1 & \frac{8}{38} & -\frac{10}{38} \\ 0 & 0 & 1 & \frac{392}{93} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{6}{10} \\ \frac{86}{38} \\ -\frac{259}{93} \\ 1 \end{bmatrix}$$

$$x_1 - \frac{7}{10}x_2 + \frac{3}{10}x_3 + \frac{5}{10}x_4 = \frac{6}{10}$$

$$x_2 + \frac{8}{38}x_3 - \frac{10}{38}x_4 = \frac{86}{38}$$

$$x_3 + \frac{392}{93}x_4 = -\frac{259}{93}$$

$$x_4 = 1$$

$\therefore$ By back substitution, we get
$x_4 = 1$, $x_3 = -7$, $x_2 = 4$, and $x_1 = 5$,

4.7 INVERSE OF MATRIX BY CROUT'S METHOD

We have seen that by crout's method the square matrix A can be decomposed into $A = LU$, where L is the lower triangular matrix and U, the unit upper triangular matrix.

$$\therefore \qquad A^{-1} = (LU)^{-1} = U^{-1} L^{-1} \tag{4.18}$$

where U^{-1}, L^{-1} are also unit upper triangular, lower triangular matrices.

Since L and U are known, their inverses i.e. L^{-1} and U^{-1}, can be determined using $LL^{-1} = I$ and $UU^{-1} = I$ and hence, A^{-1} from Eqn (4.18).

Example 4.15 Find the inverse of $\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ by Crout's method.

Solution Let $A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$

and the derived matrix be $\begin{bmatrix} l_{11} & u_{12} & u_{13} \\ l_{21} & l_{22} & u_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

Then $l_{11} = 2$, $l_{21} = 2$, $l_{31} = -1$

$u_{12} = -1$, $u_{13} = 2$, $l_{22} = 5$, $l_{32} = 0$, $u_{23} = -\frac{2}{5}$ and $l_{33} = 1$ [using Crout's scheme].

$$\therefore \qquad L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Let $$L^{-1} = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \quad \text{and } U^{-1} = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Using $LL^{-1} = I$, we get

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $2x_{11} = 1$; $\quad 2x_{11} + 5x_{21} = 0$; $\quad 5x_{22} = 1$;
$-x_{11} + x_{31} = 0$; $\quad x_{32} = 0$; $\quad x_{33} = 1$

$\Leftrightarrow$ $x_{11} = \frac{1}{2}$, $\quad x_{21} = -\frac{1}{5}$, $\quad x_{31} = \frac{1}{2}$, $\quad x_{22} = \frac{1}{5}$, $\quad x_{32} = 0$, $\quad x_{33} = 1$

Similarly, using $UU^{-1} = I$, i.e.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We get $y_{12} = 1, y_{13} = -8/5; y_{23} = 2/5$

$$\therefore L^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \text{ and } U^{-1} = \begin{bmatrix} 1 & 1 & -\frac{8}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$$

EXERCISE 4.2

Solve the following equations by Factorisation (or Triangularisation) method.

1. $3x + y + 2z = 16;\ 2x - 6y + 8z = 24;\ 5x + 4y - 3z = 2$
2. $3x + 2y + 7z = 32;\ 2x + 3y + z = 40;\ 3x + 4y - z = 56$
3. $10x + y + z = 12;\ 2x + 10y + z = 13;\ x + y + 5z = 7$ *(M.U, B.E., 1991)*
4. $28x + 4y - z = 32;\ x + 3y + 10z = 24;\ 2x + 17y + 4z = 35$
5. $2x - y + z = 0.3; -4x + 3y - 2z = -1.4;\ 3x - 8y + 3z = 0.1$
6. $10x + 7y + 8z + 7w = 32;\ 7x + 5y + 6z + 5w = 23,$
   $8x + 6y + 10z + 9w = 33;\ 7x + 5y + 9z + 10w = 31$

Solve the following equations by Crout's method

7. $x + 3y + 8z = 4, x + 4y + 3z = -2;\ x + 3y + 4z = 1$ *(M.U, B.E., 1991)*
8. $2x - 6y + 8z = 24, 5x + 4y - 3z = 2;\ 3x + y + 2z = 16$ *(M.U, B.E., 1993)*
9. $10x + y + 2z = 13; 3x + 10y + z = 14, 2x + 3y + 10z = 15$ *(Madurai, B.E., 1987)*
10. $9x - 2y + z = 50,\ x + 5y - 3z = 18,\ -2x + 2y + 7z = 19$
11. $10x + y + z = 12,\ 2x + 10y + z = 13,\ 2x + 2y + 10z = 14$ *(Coimbatore, B.Tech., 1988)*

12. $3x + y + 2z = 3,\ 2x - 3y - z = -3,\ x - 2y + z = 4$

13. $10x_1 + 9x_2 + 6x_3 + x_4 = 26,\ 11x_1 + 6x_2 - x_3 + 2x_4 = 18$
$x_1 - 7x_2 + 3x_3 + 6x_4 = 3,\ 7x_1 + x_2 + x_3 + x_4 = 10$

14. $5x + y + z + w = 4,\ x + 7y + z + 4w = 12,$
$x + y + 6z + w = -5,\ x + y + z + 4w = -6$

Find the inverse of the following matrices using Crout's method.

(15) $\begin{bmatrix} -2 & 4 & 8 \\ -4 & 18 & -16 \\ -6 & 2 & -20 \end{bmatrix}$ (16) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

(17) $\begin{bmatrix} 13 & 14 & 6 & 4 \\ 8 & -1 & 13 & 9 \\ 6 & 7 & 3 & 2 \\ 9 & 5 & 16 & 11 \end{bmatrix}$

ANSWERS

(1) 1, 3, 5
(2) 7, 9, − 1
(3) 1, 1, 1
(4) 0.996, 1.5070, 1.8485
(5) 1.6, − 0.8, − 3.71
(6) 1, 1, 1, 1
(7) $\frac{19}{4}, -\frac{9}{4}, \frac{3}{4}$
(8) 1, 3, 5
(9) 1, 1, 1
(10) 6.13, 4.31, 3.23
(11) 1, 1, 1
(12) 1, 2, − 1
(13) 1, 1, 1, 1
(14) 1, 2, − 1, − 2

(15) $\frac{1}{190}\begin{bmatrix} -41 & 12 & -26 \\ 2 & 11 & -8 \\ 12.5 & -2.5 & -2.5 \end{bmatrix}$ (16) $\frac{1}{12}\begin{bmatrix} -5 & 3 & 4 \\ -7 & 3 & -8 \\ 1 & -3 & 4 \end{bmatrix}$

(17) $\begin{bmatrix} 1 & 0 & -2 & 0 \\ -5 & 1 & 11 & -1 \\ 287 & -67 & -630 & 65 \\ -416 & 97 & 913 & -94 \end{bmatrix}$

4.8 ITERATIVE METHODS

So far we have studied some direct methods which yield, after a certain amount of fixed computation, the solution to simultaneous linear equations. Now we shall discuss the *iterative* or *indirect methods*. In these methods we start from an approximation to the true solution and, if convergent, derive a sequence of closer approximations. We repeat the cycle of computations till the required accuracy is obtained.

But, the method of iteration is not applicable to all systems of equations. For this, each equation of the system must contain one large coefficient (much larger than the others in that equation) and the large coefficient must be attached to a different unknown in that equation.

In other words, the solution to a system of linear equations will exist by iterative procedure *if the absolute value of the largest coefficient is greater than the sum of the absolute values of all remaining coefficients in each equation* (condition for convergence).

Now let us see some of such methods in detail.

4.9 JACOBI METHOD OF ITERATION

This method is also known as Gauss–Jacobi method. Here, consider the following system of equations.

$$a_1x + b_1y + c_1z = d_1 \tag{4.19a}$$
$$a_2x + b_2y + c_2z = d_2 \tag{4.19b}$$
$$a_3x + b_3y + c_3z = d_3 \tag{4.19c}$$

Let $\quad |a_1| > |b_1| + |c_1|; \quad |b_2| > |a_2| + |c_2|; \quad |c_3| > |a_3| + |b_3|$

That is, in each equation the coefficients of the diagonal terms are large. Hence the system (4.19) is ready for iteration. Solving for x, y and z, respectively, we get

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z) \tag{4.20a}$$

$$y = \frac{1}{b_2}(d_2 - a_1x - c_2z) \tag{4.20b}$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \tag{4.20c}$$

Let x_0, y_0 and z_0 be the initial approximations of the unknowns x, y and z. Substituting these on RHS of Eqns (4.20) the first approximations are given by

$$x_1 = \frac{1}{a_1}\,(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}\,(d_2 - a_3 x_0 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}\,(d_3 - a_3 x_0 - b_3 y_0)$$

Substituting the values x_1, y_1 and z_1 in the RHS of Eqn (4.20), the second approximations are given by

$$x_2 = \frac{1}{a_1}\,(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}\,(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}\,(d_3 - a_3 x_1 - b_3 y_1)$$

Proceeding in the same way, if x_r, y_r, z_r are the rth iterates then

$$x_{r+1} = \frac{1}{a_1}\,(d_1 - b_1 y_r - c_1 z_r)$$

$$y_{r+1} = \frac{1}{b_2}\,(d_2 - a_2 x_r - c_2 z_r)$$

$$z_{r+1} = \frac{1}{c_3}\,(d_3 - a_3 x_r - b_3 y_r)$$

The process is continued till convergency is secured.

Note: In the absence of any better estimates, the initial approximations are taken as $x_0 = 0, y_0 = 0, z_0 = 0$

Example 4.16 Solve by Jacobi iteration method the system $8x - 3y + 2z = 20$; $6x + 3y + 12z = 35$; and $4x + 11y - z = 33$.

(Bangalore, B.E., 1994, M.U, B.E., 1981)

Solution Consider the given system as

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 35$$

so that the diagonal elements are dominant in the coefficient matrix. Now we write the equations in the form

$$x = \frac{1}{8}(20 + 3y - 2z) \tag{ia}$$

$$y = \frac{1}{11}(33 - 4x + z) \tag{ib}$$

$$z = \frac{1}{12}(35 - 6x - 3y) \tag{ic}$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.
Substituting these on RHS of Eqns (i), we get

First approximation as

$$x_1 = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$

$$y_1 = \frac{1}{11}[33 - 4(0) + 0] = 3$$

$$z_1 = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.9166667$$

Second approximation:

Substituting x_1, y_1, z_1 on RHS of Eqns (i), we get

$$x_2 = \frac{1}{8}[20 + 3(3) - 2(2.9166667)] = 2.895833$$

$$y_2 = \frac{1}{11}[33 - 4(2.5) + 2.9166667] = 2.3560606$$

$$z_2 = \frac{1}{12}[35 - 6(2.5) - 3(3)] = 0.9166666$$

Third approximation:

Substituting x_2, y_2, z_2 on RHS of Eqns (i), we get

$$x_3 = \frac{1}{8}[20 + 3(2.3560606) - 2(0.9166666)] = 3.1543561$$

$$y_3 = \frac{1}{11}[33 - 4(2.8958333) + 0.9166666] = 2.030303$$

$$z_3 = \frac{1}{12}[35 - 6(2.8958333) - 3(2.3560606)] = 0.8797348$$

Fourth approximation:

Substituting x_3, y_3, z_3 on RHS of Eqns (i), we get

$$x_4 = \frac{1}{8}[20 + 3(2.030303) - 2(0.8797348)] = 3.0419299$$

$$y_4 = \frac{1}{11}[33 - 4(3.1543561) + 0.8797348] = 1.9329373$$

$$z_4 = \frac{1}{12}[35 - 6(3.1543561) - 3(2.030303)] = 0.8319128$$

Fifth approximation:

Substituting x_4, y_4, z_4 on RHS of Eqns (i), we get

$$x_5 = \frac{1}{8}[20 + 3(1.9329373) - 2(0.8319128)] = 3.0168733$$

$$y_5 = \frac{1}{11}[33 - 4(3.0414299) + 0.8319128] = 1.9696539$$

$$z_5 = \frac{1}{12}[35 - 6(3.0414299) - 3(1.9329373)] = 0.9127173$$

Sixth approximation:

Substituting x_5, y_5, z_5 on RHS of Eqns (i), we get

$$x_6 = \frac{1}{8}[20 + 3(1.9696539) - 2(0.9127173)] = 3.0104409$$

$$y_6 = \frac{1}{11}[33 - 4(3.0168733) + 0.9127173] = 1.9859295$$

$$z_6 = \frac{1}{12}[35 - 6(3.0168733) - 3(1.9696539)] = 0.9158165$$

Seventh approximation:

Substituting x_6, y_6, z_6 on RHS of Eqns (i), we get

$$x_7 = \frac{1}{8}[20 + 3(1.9859295) - 2(0.9158165)] = 3.0157694$$

$$y_7 = \frac{1}{11}[33 - 4(3.0104409) + 0.9158165] = 1.9885503$$

$$z_7 = \frac{1}{12}[35 - 6(3.0104409) - 3(1.9859295)] = 0.9149638$$

Eighth approximation:

Substituting x_7, y_7, z_7 on RHS of Eqns (i), we get

$$x_8 = \frac{1}{8}[20 + 3(1.9885503) - 2(0.9149638)] = 3.0169654$$

$$y_8 = \frac{1}{11}[33 - 4(3.0157694) + 0.9149638] = 1.9865351$$

$$z_8 = \frac{1}{12}[35 - 6(3.0157694) - 3(1.9885503)] = 0.9116443$$

Ninth approximation:

Substituting x_8, y_8, z_8 on RHS of Eqns (i), we get

$$x_9 = \frac{1}{8}[20 + 3(1.9865351) - 2(0.9116443)] = 3.0170396$$

$$y_9 = \frac{1}{11}[33 - 4(3.0169654) + 0.9116443] = 1.9857984$$

$$z_9 = \frac{1}{12}[35 - 6(3.0169654) - 3(1.9865351)] = 0.9115501$$

Tenth approximation:

Substituting x_9, y_9, z_9 on RHS of Eqns (i), we get

$$x_{10} = \frac{1}{8}[20 + 3(1.9857984) - 2(0.9115501)] = 3.0167869$$

$$y_{10} = \frac{1}{11}[33 - 4(3.0170396) + 0.9115501] = 1.9857629$$

$$z_{10} = \frac{1}{12}[35 - 6(3.0170396) - 3(1.9857984)] = 0.9116972$$

Eleventh approximation:

Substituting x_{10}, y_{10}, z_{10} on RHS of Eqns (i), we get

$$x_{11} = \frac{1}{8}[20 + 3(1.9857629) - 2(0.9116972)] = 3.0167368$$

$$y_{11} = \frac{1}{11}[33 - 4(3.0167869) + 0.9116972] = 1.9858681$$

$$z_{11} = \frac{1}{12}[35 - 6(3.0167869) - 3(1.9857629)] = 0.9118326$$

Twelfth approximation:

Substituting x_{11}, y_{11}, z_{11} on RHS of Eqns (i), we get

$$x_{12} = \frac{1}{8}[20 + 3(1.9858681) - 2(0.91183226)] = 3.0167424$$

$$y_{12} = \frac{1}{11}[33 - 4(3.0167368) + 0.9118326] = 1.9858987$$

$$z_{12} = \frac{1}{12}[35 - 6(3.0167368) - 3(1.9858681)] = 0.9118312$$

$\therefore$ From the 11th and 12th approximations, the values of x, y, z are same correct to four decimal places. Stopping at this stage, we get $x = 3.0167$, $y = 1.9858$, $z = 0.9118$.

Example 4.17 Solve the following system of equations by Jacobi iteration method:

$3x + 4y + 15z = 54.8,\ x + 12y + 3z = 39.66$

and $10x + y - 2z = 7.74$ *(M.U, 1990)*

Solution The coefficient matrix of the given system is not diagonally dominant. Hence we rearrange the equations, as follows, such that the elements in the coefficient matrix are diagonally dominant

$$10x + y - 2z = 7.74$$
$$x + 12y + 3z = 39.66$$
$$3x + 4y + 15z = 54.8$$

Now we write the equations in the form

$$x = \frac{1}{10}(7.74 - y + 2z) \quad \text{(ia)}$$

$$y = \frac{1}{12}(39.66 - x - 3z) \quad \text{(ib)}$$

$$z = \frac{1}{15}(54.8 - 3x - 4y) \quad \text{(ic)}$$

We start from an approximation $x_0 = y_0 = z_0 = 0$

Substituting these on RHS of (i), we get

First approximation as

$$x_1 = \frac{1}{10}[7.74 - 0 + 2(0)] = 0.774$$

$$y_1 = \frac{1}{12}[39.66 - 0 + 3(0)] = 1.1383333$$

$$z_1 = \frac{1}{15}[54.8 - 3(0) + 4(0)] = 3.6533333$$

Second approximation

$$x_2 = \frac{1}{10}[7.74 - 1.1383333 + 2(3.6533333)] = 1.3908333$$

$$y_2 = \frac{1}{12}[39.66 - 0.774 - 3(3.6533333)] = 2.3271667$$

$$z_2 = \frac{1}{15}[54.8 - 3(0.774) + 4(1.1383333)] = 3.1949778$$

Third approximation

$$x_3 = \frac{1}{10}[7.74 - 2.3271667 + 2(3.1949778)] = 1.1802789$$

$$y_3 = \frac{1}{12}[39.66 - 1.3908333 - 3(3.1949778)] = 2.3903528$$

$$z_3 = \frac{1}{15}[54.8 - 3(1.3908333) - 4(2.3271667)] = 2.7545889$$

Fourth approximation

$$x_4 = \frac{1}{10}[7.74 - 2.3903528 + 2(2.7545889)] = 1.0858825$$

$$y_4 = \frac{1}{12}[39.66 - 1.1802789 - 3(2.7545889)] = 2.5179962$$

$$z_4 = \frac{1}{15}[54.8 - 3(1.1802789) - 4(2.3903528)] = 2.7798501$$

Fifth approximation

$$x_5 = \frac{1}{10}[7.74 - 2.5179962 + 2(2.7798501)] = 1.0781704$$

$$y_5 = \frac{1}{12}[39.66 - 1.0858825 - 3(2.7798501)] = 2.5195473$$

$$z_5 = \frac{1}{15}[54:8 - 3(1.0858825) - 4(2.5179962)] = 2.7646912$$

Sixth approximation

$$x_6 = \frac{1}{10}[7.74 - 2.5195473 + 2(2.7646912)] = 1.0749835$$

$$y_6 = \frac{1}{12}[39.66 - 1.0781704 - 3(2.7646912)] = 2.5239797$$

$$z_6 = \frac{1}{15}[54.8 - 3(1.0781704) + 4(2.5195473)] = 2.76582$$

Seventh approximation

$$x_7 = \frac{1}{10}[7.74 - 2.5239797 + 2(2.76582)] = 1.074766$$

$$y_7 = \frac{1}{12}[39.66 - 1.0749835 - 3(2.76582)] = 2.523963$$

$$z_7 = \frac{1}{15}[54.8 - 3(1.0749835) + 4(2.5239797)] = 2.7652754$$

$\therefore$ From the sixth and seventh approximations

$x = 1.075$, $y = 2.524$ and $z = 2.765$ correct to three decimals.

4.10 GAUSS–SEIDEL ITERATION METHOD

This is a modification of Gauss–Jacobi method. As before, the system of the linear equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

is written as

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \quad (4.21a)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \quad (4.21b)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \quad (4.21c)$$

and we start with the initial approximation x_0, y_0, z_0. Substituting y_0 and z_0 in Eqn (4.21a), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

Now substituting $x = x_1$, $z = z_0$ in Eqn (4.21b), we get

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

Substituting $x = x_1$, $y = y_1$, in Eqn (4.21c), we get

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

This process is continued till the values of x, y, z are obtained to the desired degree of accuracy. The general algorithm is as follows:

If x_k, y_k, z_k are the kth iterates, then

$$x_{k+1} = \frac{1}{a_1}(d_1 - b_1 y_k - c_1 z_k)$$

$$y_{k+1} = \frac{1}{b_2}(d_2 - a_2 x_{k+1} - c_2 z_k)$$

and
$$z_{k+1} = \frac{1}{c_3}(d_3 - a_3 x_{k+1} - b_3 y_{k+1})$$

Since the current values of the unknowns at each stage of iteration are used in proceeding to the next stage of iteration, this method is more rapid in convergence than Gauss–Jacobi method.

The rate of convergence of Gauss–Seidel method is roughly twice that of Gauss–Jacobi and the condition of convergence is same as we saw earlier in Section 4.8.

Note: Gauss–Seidel iteration method converges only for special systems of equations. In general, the round of errors will be small in iteration methods. Moreover, these are self-correcting methods; that is, any error made in computation will be corrected in the subsequent iteration.

Example 4.18 Solve the equations given in Example 4.9 by Gauss–Seidel iteration method. *(B.U, B.E., 1994)*

Solution From the given equations, we have

$$x = \frac{1}{8}(20 + 3y - 2z) \quad \text{(i)}$$

$$y = \frac{1}{11}(33 - 4x + z) \quad \text{(ii)}$$

$$z = \frac{1}{12}(35 - 6x - 3y) \quad \text{(iii)}$$

Putting $y = 0$, $z = 0$ in RHS of (i), we get $x = \frac{20}{8} = 2.5$

Putting $x = 2.5$, $z = 0$ in RHS of (ii), we get

$$y = \frac{1}{11}[33 - 4(2.5)] = 2.0909091$$

Putting $x = 2.5$, $y = 2.0909091$ in RHS of (iii), we get

$$z_1 = \frac{1}{12}[35 - 6(2.5) - 3(2.0909091)] = 1.1439394$$

For the *second approximation,*

$$x_2 = \frac{1}{8}[20 + 3y_1 - 2z_1]$$

$$= \frac{1}{8}[20 + 3(2.0909091) - 2(1.1439394)] = 2.9981061$$

$$y_2 = \frac{1}{11}[33 - 4x_2 + z_1]$$

$$= \frac{1}{11}[33 - 4(2.9981061) + 1.1439394] = 2.0137741$$

$$z_2 = \frac{1}{12}[35 - 6x_2 - 3y_2]$$

$$= \frac{1}{12}[35 - 6(2.9981061) - 3(2.0137741)] = 0.9141701$$

Third approximation:

$$x_3 = \frac{1}{8}[20 + 3(2.0137741) - 2(0.9141701)] = 3.0266228$$

$$y_3 = \frac{1}{11}[33 - 4(3.0266228) + 0.9141701)] = 1.9825163$$

$$z_3 = \frac{1}{12}[35 - 6(3.0266228) - 3(1.9825163)] = 0.9077262$$

Fourth approximation:

$$x_4 = \frac{1}{8}[20 + 3(1.9825163) - 2(0.9077262)] = 3.0165121$$

$$y_4 = \frac{1}{11}[33 - 4(3.0165121) + 0.9077262)] = 1.9856071$$

$$z_4 = \frac{1}{12}[35 - 6(3.0165121) - 3(1.9856071)] = 0.9120088$$

Fifth approximation:

$$x_5 = \frac{1}{8}[20 + 3(1.9856071) - 2(0.9120088)] = 3.0166005$$

$$y_5 = \frac{1}{11}[33 - 4(3.0166005) + 0.9120088] = 1.9859643$$

$$z_5 = \frac{1}{12}[35 - 6(3.0166005) - 3(1.9859643)] = 0.9118753$$

Sixth approximation:

$$x_6 = \frac{1}{8}[20 + 3(1.9859643) - 2(0.9118753)] = 3.0167678$$

$$y_6 = \frac{1}{11}[33 - 4(3.0167678) + 0.9118753] = 1.9858913$$

$$z_6 = \frac{1}{12}[35 - 6(3.0167678) - 3(1.9858913)] = 0.9118099$$

Seventh approximation:

$$x_7 = \frac{1}{8}[20 + 3(1.9858913) - 2(0.9118099)] = 3.0167568$$

$$y_7 = \frac{1}{11}[33 - 4(3.0167568) + 0.9118099] = 1.9858894$$

$$z_7 = \frac{1}{12}[35 - 6(3.0167568) - 3(1.9858894)] = 0.9118159$$

Since at the sixth and seventh approximations, the values of x, y, z are the same, correct to four decimal places, we can stop the iteration process.

$\therefore x = 3.0167, \quad y = 1.9858, \quad z = 0.9118$

We find that 12 iterations are necessary in Gauss–Jacobi method to get the same accuracy as achieved by 7 iterations in Gauss–Seidel method.

Example 4.19 Solve by Gauss–Seidel method, the following system of equations

$28x + 4y - z = 32, \; x + 3y + 10z = 24,$

and $2x + 17y + 4z = 35$ *(B.U. 1997, M.U. 1991)*

Solution The coefficient matrix of the given system is not diagonally dominant. Hence, we rearrange the equations as follows, such that the elements in the coefficient matrix are diagonally dominant.

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

Hence we can apply Gauss–Seidal iteration method.

From the above equations

$$x = \frac{1}{28}[32 - 4y + z] \quad \text{(i)}$$

$$y = \frac{1}{17}[35 - 2x - 4z] \quad \text{(ii)}$$

$$z = \frac{1}{10}[24 - x - 3y] \quad \text{(iii)}$$

First approximation

Putting $y = z = 0$ in (i), we get

$$x_1 = \frac{1}{28}(32) = 1.1428571$$

Putting $x = 1.1428571, \; z = 0$ in (ii), we get

$$y_1 = \frac{1}{17}[35 - 2(1.1428571)] = 1.9243697$$

Putting $x = 1.1428571, \; y = 1.9243697$ in (iii), we get

$$z_1 = \frac{1}{10}[24 - 1.1428571 - 3(1.9243697)] = 1.7084034$$

Second approximation

$$x_2 = \frac{1}{28}[32 - 4(1.9243697) + 1.7084034] = 0.9289615$$

$$y_2 = \frac{1}{17}[35 - 2(0.9289615) - 4(1.7084034)] = 1.5475567$$

$$z_2 = \frac{1}{10}[24 - 0.9289615 - 3(1.5475567)] = 1.8428368$$

Third approximation

$$x_3 = \frac{1}{28}[32 - 4(1.5475567) + 1.8428368] = 0.9875932$$

$$y_3 = \frac{1}{17}[35 - 2(0.9875932) - 4(1.8428368)] = 1.5090274$$

$$z_3 = \frac{1}{10}[24 - 0.9875932 - 3(1.5090274)] = 1.8485325$$

Fourth approximation

$$x_4 = \frac{1}{28}[32 - 4(1.5090274) + 1.8485325] = 0.9933008$$

$$y_4 = \frac{1}{17}[35 - 2(0.9933008)] = 4(1.8485325)] = 1.5070158$$

$$z_4 = \frac{1}{10}[24 - 0.9933008 - 3(1.5070158)] = 1.8485652$$

Fifth approximation

$$x_5 = \frac{1}{28}[32 - 4(1.5070158) + 1.8485652] = 0.9935893$$

$$y_5 = \frac{1}{17}[35 - 2(0.9935893) - 4(1.8485652)] = 1.5069741$$

$$z_5 = \frac{1}{10}[24 - 0.9935893 - 3(1.5069741)] = 1.8485488$$

Sixth approximation

$$x_6 = \frac{1}{28}[32 - 4(1.5069741) + 1.8485488] = 0.9935947$$

$$y_6 = \frac{1}{17}[35 - 2(0.9935947) - 4(1.8485488)] = 1.5069774$$

$$z_6 = \frac{1}{10}[24 - 0.9935947 - 3(1.5069774)] = 1.8485473$$

$\because$ the values of x, y, z in the fourth and fifth iteration are same upto four decimals, we stop the process here.

Hence $x = 0.9936,\ y = 1.5069,\ z = 1.8485$

Example 4.20 Using Gauss–Seidel iteration method, solve the system of equations.

$$10x - 2y - z - w = 3; -2x + 10y - z - w = 15;$$
$$-x - y + 10z - 2w = 27; -x - y - 2z + 10w = -9$$

(*M.U, B.E., 1987*)

Solution The coefficient matrix of the given system is diagonally dominant. Hence we can apply Gauss–Seidel iteration method.

From the given equations, we can write

$$x = \frac{1}{10}[3 + 2y + z + w] \quad \text{(i)}$$

$$y = \frac{1}{10}[15 + 2x + z + w] \quad \text{(ii)}$$

$$z = \frac{1}{10}[27 + x + y + 2w] \quad \text{(iii)}$$

$$w = \frac{1}{10}[-9 + x + y + 2z] \quad \text{(iv)}$$

First approximation:

Putting $y = z = w = 0$ on RHS of (i), we get $x_1 = \dfrac{3}{10} = 0.3$

Putting $x = 0.3,\ z = w = 0$ on RHS of (ii),

we get $y_1 = \dfrac{1}{10}[15 + 2(0.3)] = 1.56$

Putting $x = 0.3,\ y = 1.56,\ w = 0$ on RHS of (iii),

we get $z_1 = \dfrac{1}{10}[27 + 0.3 + 1.56] = 2.886$

Putting $x = 0.3,\ y = 1.56,\ z = 2.886$ on RHS of (iv),

we get $w_1 = \dfrac{1}{10}[-9 + 0.3 + 1.56 + 2(2.886)] = -0.1368$

Second approximation:

$$x_2 = \frac{1}{10}[3 + 2(1.56) + 2.886 - 0.1368] = 0.88692$$

$$y_2 = \frac{1}{10}[15 + 2(0.88692) + 2.886 - 0.1368] = 1.952304$$

$$z_2 = \frac{1}{10}[27 + 0.88692 + 1.952304 + 2(-0.1368] = 2.9565624$$

$$w_2 = \frac{1}{10}[-9 + 0.88692 + 1.952304 + 2(2.9565624)] = -0.0247651$$

Third approximation:

$$x_3 = \frac{1}{10}[3 + 2(1.952304) + 2.9565624 - 0.0247651] = 0.9836405$$

$$y_3 = \frac{1}{10}[15 + 2(0.9836405) + 2.9565624 - 0.0247651] = 1.9899087$$

$$z_3 = \frac{1}{10}[27 + 0.9836405 + 1.9899087 + 2(-0.0247651)] = 2.9924019$$

$$w_3 = \frac{1}{10}[-9 + 0.983405 + 1.9899087 + 2(2.9924019)] = -0.0041647$$

Fourth approximation:

$$x_4 = \frac{1}{10}[3 + 2(1.9899087) + 2.9924019 - 0.0041647] = 0.9968054$$

$$y_4 = \frac{1}{10}[15 + 2(0.9968054) + 2.9924019 - 0.0041647] = 1.9981848$$

$$z_4 = \frac{1}{10}[27 + 0.9968054 + 1.9981848 + 2(-0.0041647)] = 2.9986661$$

$$w_4 = \frac{1}{10}[-9 + 0.9968054 + 1.9981848 + 2(2.9986661)] = -0.0007677$$

Fifth approximation:

$$x_5 = \frac{1}{10}[3 + 2(1.9981848) + 2.9986661 - 0.0007677] = 0.9994268$$

$$y_5 = \frac{1}{10}[15 + 2(0.9994268) + 2.998666 - 0.0007677] = 1.9996752$$

$$z_5 = \frac{1}{10}[27 + 0.9994268 + 1.9996752 + 2(-0.0007677)] = 2.9997567$$

$$w_5 = \frac{1}{10}[-9 + 0.9994268 + 1.9996752 + 2(2.9997567)] = -0.0001384$$

Sixth approximation:

$$x_6 = \frac{1}{10}[3 + 2(1.9996752) + 2.9997567 - 0.0001384] = 0.9998968$$

$$y_6 = \frac{1}{10}[15 + 2(0.9998968) + 2.9997567 - 0.0001384] = 1.9999412$$

$$z_6 = \frac{1}{10}[27 + 0.9998968 + 1.9999412 + 2(-0.0001384)] = 2.9999561$$

$$w_6 = \frac{1}{10}[-9 + 0.9998968 + 1.9999412 + 2(2.9999561)] = -0.0002498$$

Seventh approximation:

$$x_7 = \frac{1}{10}[3 + 2(1.9999412) + 2.9999561 - 0.0002498] = 0.9999588$$

$$y_7 = \frac{1}{10}[15 + 2(0.9999588) + 2.9999561 - 0.0002498] = 1.9999624$$

$$z_7 = \frac{1}{10}[27 + 0.9999588 + 1.9999264 + 2(-0.0002498)] = 2.9999422$$

$$w_7 = \frac{1}{10}[-9 + 0.9999588 + 1.9999624 + 2(2.9999422)] = -0.0001945$$

Now, from sixth and seventh approximations the values of x, y, z and w correct to four decimal places are

$$x = 0.9999 \qquad y = 1.9999 \qquad z = 2.9999 \qquad w = 0.0002$$

4.11 RELAXATION METHOD

Consider the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

We define the residuals r_1, r_2, r_3 by the relations

$$r_1 = d_1 - a_1x - b_1y - c_1z \tag{4.22a}$$
$$r_2 = d_2 - a_2x - b_2y - c_2z \tag{4.22b}$$
$$r_3 = d_3 - a_3x - b_3y - c_3z \tag{4.22c}$$

To start with, we assume $x = y = z = 0$ and calculate the initial residuals. Then these residuals are reduced step by step giving increments to the variables. If we can find x, y, z such that residuals $r_1 = r_2 = r_3 = 0$ then those values of x, y, z are the exact values. Otherwise, we liquidate the residuals smaller and smaller and finally negligible to get better approximate values of x, y, z. For this purpose we construct an *operation table* as shown below.

x	y	z	r_1	r_2	r_3
1	0	0	$-a_1$	$-a_2$	$-a_3$
0	1	0	$-b_1$	$-b_2$	$-b_3$
0	0	1	$-c_1$	$-c_2$	$-c_3$

From Eqns (4.22), we can see that if x is increased by 1, keeping y and z constant, r_1, r_2 and r_3 decrease by a_1, a_2 and a_3, respectively.

This is shown in the above table along with the effects on the residuals when y and z are given unit increments. It can be noted that the operation table consists of the unit matrix I and transpose of the coefficient matrix.

At each step, the numerically largest residual is reduced almost to zero. To reduce a particular residual, the value of the corresponding variable is changed, i.e. to reduce, say, r_2 by α, y should be increased by $\frac{\alpha}{b_2}$. When all the residuals have been reduced to almost zero, then the increments in x, y and z are added separately to give the desired solution.

Note: After finding x, y and z, substitute them in Eqns (4.22), and check whether the residuals are negligible or not. If not, then there is some mistake and the entire process should be rechecked.

Convergency of Relaxation Method

This method can be applied successfully only if the diagonal elements of the coefficient matrix dominate the other coefficients in the corresponding row, i.e. if in Eqns (4.22)

$$|a_1| \geq |b_1| + |c_1|$$
$$|b_2| \geq |a_2| + |c_2|$$
and
$$|c_3| \geq |a_3| + |b_3|$$

with strict inequality for atleast one row.

Example 4.21 Solve by relaxation method, the equations
$10x - 2y - 2z = 6$; $-x - 10y - 2z = 7$; $-x - y + 10z = 8$

(M.U, B.E., 1991)

Solution The residuals r_1, r_2 and r_3 are given by

$$r_1 = 6 - 10x + 2y + 2z$$
$$r_2 = 7 + x - 10y + 2z$$
$$r_3 = 8 + x + y - 10z$$

The operation table is as follows.

x	y	z	r_1	r_2	r_3		
1	0	0	−10	1	1	←	L_1
0	1	0	2	−10	1	←	L_2
0	0	1	2	2	−10	←	L_3

The relaxation table is as follows.

x	y	z	r_1	r_2	r_3		
0	0	0	6	7	8	←	L_4
0	0	1	8	9	−2	←	$L_5 = L_4 + L_3$
0	1	0	10	−1	−1	←	$L_6 = L_5 + L_2$
1	0	0	0	0	0	←	$L_7 = L_6 + L_1$

Explanation:

(i) In line L_4, the largest residual is 8. To reduce it, we give an increment of $\frac{8}{c_3} = \frac{8}{10} = 0.8 \simeq 1$. The resulting residuals are obtained by $L_4 + (1)\, L_3$, i.e.line L_5.

(ii) In line L_5, the largest residual is 9.

$$\therefore \text{ increment} = \frac{9}{b_2} = \frac{9}{10} = 0.9 \simeq 1$$

The resulting residuals $(= L_6) = L_5 + 1 \,.\, L_2$

(iii) In line L_6, the largest residual is 10.

$$\therefore \text{ increment} = \frac{10}{a_1} = \frac{10}{10} = 1$$

The resulting residuals ($= L_7$) $= L_6 + 1.\, L_3$, which are all zeros.

$\Rightarrow$ Exact solution is arrived and it is $x = 1, y = 1, z = 1$

Example 4.22 Solve by relaxation method, the equations
$9x - y + 2z = 9,\ x + 10y - 2z = 15,\ 2x - 2y - 13z = -17$

(M.U, B.E., 1996)

Solution The residuals r_1, r_2, r_3 are given by
$r_1 = 9 - 9x + y - 2z$; $r_2 = 15 - x - 10y + 2z$; $r_3 = -17 - 2x + 2y + 13z$ (i)

Operation table

x	y	z	r_1	r_2	r_3	
1	0	0	–9	–1	–2	$\leftarrow$ L_1
0	1	0	1	–10	2	$\leftarrow$ L_2
0	0	1	–2	2	13	$\leftarrow$ L_3

Relaxation table

x	y	z	r_1	r_2	r_3	
0	0	0	9	15	–17	$\leftarrow L_4$
0	0	1	7	17	–4	$\leftarrow L_5 = L_4 + 1.\, L_3$
0	1	0	8	7	–2	$\leftarrow L_6 = L_5 + 1.\, L_2$
0.89	0	0	–0.01	6.11	–3.78	$\leftarrow L_7 = L_6 + 0.89L_1$
0	0.61	0	0.6	0.01	–2.56	$\leftarrow L_8 = L_7 + 0.61L_3$
0	0	0.19	0.22	0.39	–0.09	$\leftarrow L_9 = L_8 + 0.1L_3$
0	0.039	0	0.259	0	–0.012	$\leftarrow L_{10} = L_9 + 0.039L_2$
0.028	0	0	0.007	–0.028	–0.068	$\leftarrow L_{11} = L_{10} + 0.028L_1$
0	0	0.00523	–0.00346	–1.01754	–0.00001	$\leftarrow L_{12} = L_{11} + 0.00523L_3$

Thus, $x = 0.89 + 0.028 = 0.918$; $y = 1 + 0.61 + 0.039 = 1.649$ and
$z = 1 + 0.19 + 0.00523 = 1.19523$.

Now substituting the values of x, y, z in (i), we get

$r_1 = 9 - 9\,(0.918) + 1.649 - 2(1.19523) = -\,0.00346$

$r_2 = 15 - 0.918 - 10\,(1.649) + 2\,(1.19523) = -\,0.1754$

$r_3 = -\,17 - 2\,(0.918) + 2\,(1.649) + 13\,(1.19523) = -\,0.00001$

which are in agreement with the final residuals in the table.

Example 4.23 Solve by relaxation method, the equations
$10x - 2y + z = 12$, $x + 9y - z = 10$ and $2x - y + 11z = 20$

(*M.U, 1997*)

Solution The residuals r_1, r_2, r_3 are given by

$$r_1 = 12 - 10x + 2y - z$$
$$r_2 = 10 - x - 9y + z$$
$$r_3 = 20 - 2x + y - 11z$$

The operation table is

x	y	z	r_1	r_2	r_3	
1	0	0	−10	−1	−2	$\leftarrow L_1$
0	1	0	2	−9	1	$\leftarrow L_2$
0	0	1	−1	1	−11	$\leftarrow L_3$

The relation table is

x	y	z	r_1	r_2	r_3	
0	0	0	12	10	20	$\leftarrow L_4$
0	0	1.8	10.2	11.8	0.8	$\leftarrow L_5 = L_4 + 1.8\ L_3$
0	1.31	0	12.82	0.01	1.51	$\leftarrow L_6 = L_5 + 1.31\ L_2$
1.28	0	0	0.02	−1.27	−1.05	$\leftarrow L_7 = L_6 + 1.28\ L_1$
0.02	0	0	0	−1.272	−1.054	$\leftarrow L_8 = L_7 + 0.002\ L_1$
0	0	−0.1	0.1	−1.372	0.046	$\leftarrow L_9 = L_8 - 0.1\ L_3$
0.01	0	0	0	1.382	0.026	$\leftarrow L_{10} = L_9 + 0.01\ L_1$
0	0	0.0023	−0.0023	−1.3797	0.0007	$\leftarrow L_{11} = L_{10} + 0.0023$

Thus $x = 1.28 + 0.02 + 0.01 = 1.31$; $y = 1.31$ and
$z = 1.8 - 0.1 + 0.0023 = 1.7023$.

EXERCISE 4.3

Solve the following system of linear equations by (i) Gauss and (ii) Gauss Seidel iteration method.

1. $2x + y + z = 4$, $x + 2y + z = 4$; $x + y + 2z = 4$
2. $8x + y + z = 8$; $2x + 4y + z = 4$; $x + 3y + 5z = 5$
3. $5x + 2y + z = 12$, $x + 4y + 2z = 15$, $x + 2y + 5z = 20$

4. $9x + 2y + 4z = 20, x + 10y + 4z = 6, 2x - 4y + 10z = -15$
5. $54x + y + z = 110, 2x + 15y + 6z = 72, -x + 6y + 27z = 85$ (*M.U, B.E., 1993*)
6. $28x - 4y - z = 32, x + 3y + 10z = 24, 2x + 17y + 4z = 35$
7. $5x - y + z = 10, 2x + 4y = 12, x + y + 5z = -1$ (*Bangalore, B.E., 1990*)
8. $10x_1 - 5x_2 - 2x_3 = 3, 4x_1 - 10x_2 + 3x_3 = -3, x_1 + 6x_2 + 10x_3 = -3$
9. $10x + 2y + z = 9, 2x + 20y - 2z = -44, -2x + 3y + 10z = 22$
10. $10x_1 + 7x_2 + 8x_3 + 7x_4 = 32, 7x_1 + 5x_2 + 6x_3 + 5x_4 = 23;$
    $8x_1 + 6x_2 + 10x_3 + 9x_4 = 33, 7x_1 + 5x_2 + 9x_3 + 10x_4 = 31$

Solve by relaxation method the following equations:

11. $9x + 2y + z = 50, x + 5y - 3z = 18, -2x + 2y + 7z = 19$
12. $3x + 9y - 2z = 11, 4x + 2y + 13z = 24, 4x - 4y + 3z = -8$ (*M.U, B.E., 1993*)
13. $4.215x - 1.212y + 1.105z = 3.216$
    $-2.120x + 3.505y - 1.632z = 1.247$
    $1.122x - 1.313y + 3.986z = 2.112$
14. $10x - 2y - 3z = 305, -2x + 10y - 2z = 154, -2x - y + 10z = 120$
15. $8x_1 + x_2 + x_3 + x_4 = 14; 2x_1 + 10x_2 + 3x_3 + x_4 = -8$
    $x_1 - 2x_2 - 20x_3 + 3x_4 = 111,\ 3x_1 + 2x_2 + 2x_3 + 19x_4 = 53$

ANSWERS

1. $x = 1, y = 1, z = 1$
2. $x = 0.876, y = 0.919, z = 0.574$
3. $x = 0.996, y = 1.95, z = 3.16$
4. $x = 2.733, y = 0.986, z = -1.652$
5. $x = 1.926, y = 3.573, z = 2.425$
6. $x = 0.994, y = 1.507, z = 1.849$
7. $x = 2.556, y = 1.722, z = -1.055$
8. $x_1 = 0.342, x_2 = 0.285, x_3 = -0.505$
9. $x = 1.013, y = -1.996, z = 3.001$
10. $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1$
11. $x = 6.13, y = 4.31, z = 3.23$
12. $x = 1.35, y = 2.103, z = 2.845$
13. $x = 0.943, y = 1.239, z = 0.673$
14. $x = 32, y = 26, z = 21$
15. $x_1 = 2, x_2 = 0, x_3 = -5, x_4 = 3$

CHAPTER 5

Finite Differences

5.1 INTRODUCTION

The calculus of finite differences plays an important role in Numerical methods. It deals with the variations in a function when the independent variable changes by finite jumps which may be equal or unequal. In contrast, the infinitesimal calculus deals with the relationships that exist between the values assumed by the function, whenever the independent variable changes continuously in a given interval. In this chapter, we shall study the variations in a function due to the changes in the independent variable by equal intervals.

5.2 FINITE DIFFERENCES

Let $y=f(x)$ be a discrete function. If $x_0, x_0+h, x_0+2h, \ldots, x_0+nh$ are the successive values of x, where two consecutive values differ by a quantity h, then the corresponding values of y are $y_0, y_1, y_2, \ldots, y_n$. The value of the independent variable x is usually called the argument and the corresponding functional value is known as the entry. The arguments and entries can be shown in a tabular form as follows:

Argument x	x_0	$x_1 = x_0+h$	$x_2 = x_0+2h$	...	$x_n = x_0+nh$
Entry $y=f(x)$	$y_0 = f(x_0)$	$y_1 = f(x_0+h)$	$y_2 = f(x_0+2h)$	...	$y_n = f(x_0+nh)$

To determine the values of $f(x)$ or $f'(x)$ etc., for some intermediate arguments, the following three types of differences are found useful:

(i) Forward differences
(ii) Backward differences and
(iii) Central differences

5.3 FORWARD DIFFERENCES

If we subtract from each value of y (except y_0) the preceding value of y, we get $y_1 - y_0$, $y_2 - y_1, \ldots, y_n - y_{n-1}$ respectively, known as the first differences of y. These results which may be denoted Δy_0, $\Delta y_1, \ldots, \Delta y_n$

i.e. $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$

where Δ is a symbol representing an operation of forward difference, are called *first forward differences*. Thus, the first forward differences are given by

$$\Delta y_i = y_{i+1} - y_i,\ i = 0, 1, 2, \ldots, n.$$

Now, the second forward differences are defined as the differences of the first differences, that is,

$$\begin{aligned}
\Delta^2 y_0 &= \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 \\
&= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0 \\
\Delta^2 y_1 &= \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1 = y_3 - 2y_2 + y_1 \\
\cdots & \quad \cdots \quad \cdots \\
\Delta^2 y_n &= \Delta y_{n+1} - \Delta y_n = y_{n+2} - 2y_{n+1} + y_n
\end{aligned}$$

Here, Δ^2 is called *second forward difference operator*.

Similarly, the third forward differences are :

$$\begin{aligned}
\Delta^3 y_0 &= \Delta(\Delta^2 y_0) = \Delta^2 y_1 - \Delta^2 y_0 = \Delta(\Delta y_1) - \Delta(\Delta y_0) \\
&= \Delta(y_2 - y_1) - \Delta(y_1 - y_0) = \Delta y_2 - 2\Delta y_1 + \Delta y_0 \\
&= (y_3 - y_2) - 2(y_2 - y_1) + y_1 - y_0 \\
&= y_3 - 3y_2 + 3y_1 - y_0 \\
\Delta^3 y_1 &= \Delta^2 y_2 - \Delta^2 y_1 = y_4 - 3y_3 + 3y_2 - y_1 \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\Delta^3 y_n &= \Delta^2 y_{n+1} - \Delta^2 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n
\end{aligned}$$

In general, the nth differences are defined as

$$\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k \qquad (5.1)$$

In function notation, the forward differences are as written below:

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
\Delta^2 f(x) &= f(x+2h) - 2f(x+h) + f(x) \\
\Delta^3 f(x) &= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)
\end{aligned}$$

and so on, where h is the interval of differencing.

The forward differences are usually arranged in a tabular form in the following manner.

x argument	$y=f(x)$ entry	1st difference	2nd difference	3rd difference	4th difference	5th difference
x_0	$y_0=f(x_0)$					
		Δy_0				
$x_1=x_0+h$	$y_1=f(x_1)$		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2=x_0+2h$	$y_2=f(x_2)$		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3=x_0+3h$	$y_3=f(x_3)$		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4=x_0+4h$	$y_4=f(x_4)$		$\Delta^2 y_3$			
		Δy_4				
$x_5=x_0+5h$	$y_5=f(x_5)$					

The first term in the table y_0 is called the *leading term* and the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ are called *leading differences*. It can be seen that the differences $\Delta^k y_i$ with a subscript 'i' lie along the diagonal sloping downwards; that is, forward with respect to the direction of x. The above difference table is known as *Forward difference table* or *Diagonal difference table.*

Properties of Δ The operator 'Δ' satisfies the following properties:

(i) $\Delta\,[f(x) \pm g(x)] = \Delta f(x) \pm \Delta\, g(x)$, i.e. Δ is linear

(ii) $\Delta\,[\alpha f(x)] = \alpha\,\Delta f(x)$, α being a constant

(iii) $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^n \Delta^m f(x)$, where m and n are positive integers

(iv) $\Delta\,[f(x).\, g(x)] \neq f(x).\, \Delta g(x)$

Observation **1** We can express any higher order forward difference of y_0 in terms of the entries $y_0, y_1, y_2, \ldots, y_n$.

From
$$\Delta y_0 = y_1 - y_0$$
$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

and so on, we can see that the coefficients of the entries on the RHS are binomial coefficients. Therefore, in general,

$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - \cdots + (-)^n y_0 \tag{5.2}$$

Observation 2 We can express any value of y in terms of leading entry y_0.

We know that $y_1 - y_0 = \Delta y_0$

$$\therefore \qquad y_1 = y_0 + \Delta y_0 = (1 + \Delta)\, y_0 \qquad (5.3)$$

Now, $y_2 = y_1 + \Delta y_1 = (1 + \Delta)\, y_1 = (1 + \Delta)^2 y_0$ [using Eqn (5.3)]

Similarly, $y_3 = (1 + \Delta)^3 y_0$ and so on. In general,

$$y_n = (1 + \Delta)^n y_0 = y_0 + {}^nC_1 \Delta y_0 + {}^nC_2 \Delta^2 y_0 + \cdots + \Delta^n y_0 \qquad (5.4)$$

5.4 BACKWARD DIFFERENCES

The differences $y_1 - y_0,\ y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively, are called the ***first backward differences***, where ∇ is the backward difference operator called ***nabla operator***.

$$\therefore \qquad \nabla y_1 = y_1 - y_0,\quad \nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}$$

Now the second backward differences are defined as the differences of the first backward differences, i.e.

$$\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1$$
$$= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2 = y_3 - 2y_2 + y_1 \text{ and so on.}$$

In general,

$$\nabla^n y_k = \nabla^{n-1} y_k - \nabla^{n-1} y_{k-1} \qquad (5.5)$$

In function notation, these are written as

$$\nabla f(x) = f(x) - f(x - h)$$
$$\nabla f(x + h) = f(x + h) - f(x)$$
$$\nabla^2 f(x + 2h) = f(x + 2h) - 2f(x + h) + f(x)$$
$$\nabla^3 f(x + 3h) = f(x + 3h) - 3f(x + 2h) + 3f(x + h) - f(x)$$

and so on, where h is the interval of differencing.

These backward differences are arranged in a tabular form in the following manner. In this table, the difference $\nabla^k y_i$ with a fixed subscript 'i' lies along the diagonal sloping upwards; that is, backwards with respect to the direction of increasing argument x.

x argument	$y=f(x)$ entry	1st difference	2nd difference	3rd difference	4th difference	5th difference
x_0	y_0					
		∇y_1				
$x_1=x_0+h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\Delta^3 y_3$		
$x_2=x_0+2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		Δy_3		$\Delta^3 y_4$		$\Delta^5 y_5$
$x_3=x_0+3h$	y_3		$\nabla^2 y_4$		$\Delta^4 y_5$	
		Δy_4		$\Delta^3 y_5$		
$x_4=x_0+4h$	y_4		$\Delta^2 y_5$			
		Δy_5				
$x_5=x_0+5h$	y_5					

Properties of ∇

(i) $\nabla[f(x) \pm g(x)] = \nabla f(x) \pm \nabla g(x)$, i.e ∇ is a linear operator.

(ii) $\nabla[\alpha f(x)] = \alpha \nabla f(x)$, α being a constant.

(iii) $\nabla^m \nabla^n f(x) = \nabla^{m+n} f(x)$, m and n being positive integers.

(iv) $\nabla[f(x)\, g(x)] \neq [\nabla f(x)].\, g(x)$.

Observation We can express any value of y in terms of y_n and the backward differences ∇y_n, $\nabla^2 y_n$, etc.

By definition, $\quad y_n - y_{n-1} = \nabla y_n$

or
$$y_{n-1} = y_n - \nabla y_n = (1-\nabla) y_n \tag{5.6}$$

Now, $y_{n-2} = y_{n-1} - \nabla y_{n-1} = (1-\nabla) y_{n-1} = (1-\nabla)^2 y_n$ [using Eqn (5.6)]

Similarly, $y_{n-3} = (1-\nabla)^3 y_n$ and so on.

In general,
$$y_{n-k} = (1-\nabla)^k y_n \tag{5.7}$$

$$\therefore \quad y_{n-k} = y_n - {}^kC_1 \nabla y_n + {}^kC_2 \nabla^2 y_n - \cdots + (-1)^k \nabla^k y_n \tag{5.8}$$

5.5 CENTRAL DIFFERENCES

Sometimes, it is more convenient to employ another system of differences known as *central differences*. In this system the symbol δ is used instead of Δ and is known as *central difference operator*. The subscript of δy for any difference is the average of the subscripts of the two members of the difference.

$$\therefore \quad y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2},\ y_3 - y_2 = \delta y_{5/2},\ \ldots$$

For higher order differences, we have

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1,\ \delta y_{5/2} - \delta y_{1/2} = \delta^2 y_2, \ldots, \delta^2 y_2 - \delta^2 y_1 = \delta^2 y_{3/2},$$

and so on.

The central differences are tabulated below.

x argument	$y = f(x)$ entry	1st difference	2nd difference	3rd difference	4th difference	5th difference
x_0	y_0					
		$\delta y_{1/2}$				
$x_1 = x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_2 = x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_3 = x_0 + 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
$x_4 = x_0 + 4h$	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
$x_5 = x_0 + 5h$	y_5					

We can see from the table that central differences on the same horizontal line have the same suffix. Also, all odd differences have a fractional suffix, and the even differences have integer suffix.

Note

1. From all the three tables, we can see that only the notation changes, not the differences. For examples,

$$y_1 - y_0 = \Delta y_0 = \Delta y_1 = \delta y_{1/2}$$

2. If we write $y = f(x)$ as $y = f_x$ or $y = y_x$ then the entries corresponding to $x,\ x+h,\ x+2h, \ldots$ are $y_x,\ y_{x+h},\ y_{x+2h}, \ldots$, respectively, and $\Delta y_x = y_{x+h} - y_x$, $\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$ and so on.
   Similarly, $\nabla y_x = y_x - y_{x-h}$,
   $\delta y_x = y_{x+½h} - y_{x-½h}$ and so on.

5.6 DIFFERENCES OF POLYNOMIAL

If y_x is a polynomial of nth degree, its nth differences are constant and the $(n+1)$th differences are zero.

Let $y_x = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ be the polynomial of nth degree, where $a_0, a_1, \ldots, a_n$ are constants and $a_0 \neq 0$. Let h be the interval of differencing. Then,

$$y_{x+h} = a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n$$

Now, $\Delta y_x = y_{x+h} - y_x$

$$= a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \cdots + a_{n-1}[(x+h) - x]$$
$$= a_0 nh\,x^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + k'x + l'$$

where $b', c', \ldots, k', l'$ are constants involving h but not x. Thus, the first difference of a polynomial of the nth degree is another polynomial of the of degree $(n-1)$. Consider now

$$\Delta^2 y_x = \Delta(\Delta y_x) = \Delta y_{x+h} - \Delta y_x$$
$$= a_0\, nh[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - x^{n-2}] + \cdots + k'(x+h-x)$$
$$= a_0\, n(n-1)h^2 x^{n-2} + b''\, x^{n-3} + c''x^{n-4} + \cdots + k''$$

The second difference, $\Delta^2 y_x$, is thus a polynomial of degree $(n-2)$.

Proceeding in this manner we obtain the result

$$\Delta^n y_x = a_0\, n(n-1)\,(n-2)\ldots 2.1\, h^n\, x^{n-x}$$

$a_0\, n!\, h^n$ = a constant independent of x.

Therefore, the nth difference is constant. Hence, $\Delta^{n+1} y_x = 0$, $\Delta^{n+2} y_x = 0$, and so on.

Note The converse of above is true. That is, if the nth differences of a function tabulated at equally spaced intervals are constant, the function is a polynomial of degree n.

5.7 FACTORIAL NOTATION OR FACTORIAL POLYNOMIAL

Consider the continued product,

$$x(x-h)\,(x-2h)\ldots[x-(r-1)h]$$

containing r factors of which x is the first one and the successive factors are decreased by a constant difference h. This is known as *factorial polynomial* and is denoted $x^{(r)}$.

$$\therefore\; x^{(r)} = x(x-h)\,(x-2h)\ldots[x-(r-1)h]$$

Hence, $x^{(1)} = x$, $x^{(2)} = x(x-h)$, $x^{(3)} = x\,(x-h)(x-3h)$ and so on.

Differences of $x^{(r)}$

$$\Delta\, x^{(r)} = [x+h]^{(r)} - x^r$$
$$= (x+h)\,x\,(x-h)\ldots[x-(r-2)h] - x(x-h)\ldots[x-(r-1)h]$$
$$= x(x-h)\ldots[x-(r-1)h]\{x+h-x-(r+1)h\}$$
$$= x^{(r-1)}\, rh \;=\; rh\, x^{(r-1)}$$

Similarly,

$$\Delta^2 x^{(r)} = \Delta[\Delta x^{(r)}] = \Delta[rh\, x^{(r-1)}]$$
$$= rh\,(r-1)h\, x^{(r-2)} = r(r-1)h^2\, x^{(r-2)}$$
$$\Delta^3 x^{(r)} = r\,(r-1)(r-2)\, h^3\, x^{(r-3)}$$

Proceeding on, we get

$$\Delta^r x^{(r)} = r\,(r-1)(r-2)\cdots 1.\, h^r\, x^{(r-r)} = r!\, h^r$$

Note

1. The result of differencing $x^{(r)}$ is analogous to that of differentiating x^r.
2. In particular, if $h = 1$ then
   $\Delta\, x^{(r)} = r\, x^{(r-1)}$, $\Delta^2 x^{(r)} = r\,(r-1)\, x^{(r-2)}, \cdots, \Delta^r x^{(r)} = r\,!$

5.8 RECIPROCAL FACTORIAL

The reciprocal factorial function, $x^{(-r)}$, is defined as

$$x^{(-r)} = \frac{1}{(x+h)(x+2h)\ldots(x+rh)}$$

where r is a positive integer.

Differences of $x^{(-r)}$

(i) $\Delta\, x^{(-r)} = (x+h)^{(-r)} - x^{(-r)}$

$$= \frac{1}{(x+2h)(x+3h)\ldots[x+(r+1)h]} - \frac{1}{(x+h)(x+2h)\ldots(x+rh)}$$

$$= \frac{(x+h)-[x+(r+1)h]}{(x+h)(x+2h)\ldots[x+(r+1)h]}$$

$$= \frac{-rh}{(x+h)(x+2h)\ldots[x+(r+1)h]} = (-r)h\, x^{[-(r+1)]}$$

(ii) $\Delta^2 x^{(-r)} = \Delta\,(\Delta\, x^{(-r)}) = \Delta\,[-rh\, x^{[-(r+1)]}]$

$$= (-r)h\,[-(r+1)]h\, x^{[-(r+2)]}$$
$$= (-1)^2\, r\,(r+1)h^2\, x^{[-(r+2)]}$$

Proceeding on, we get

$$\Delta^k x^{(-r)} = (-1)^k\, r(r+1)\ldots(r+k-1)\, x^{[-(r+k)]}$$

5.9 POLYNOMIAL IN FACTORIAL NOTATION

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$ be the polynomial of nth degree to be expressed in factorial notation. Since $x^{(n)}$, $x^{(n-1)}$, $x^{(n-2)}$ etc., are, respectively, polynomials of nth degree, $(n-1)$th degree, $(n-2)$th degree etc. we can express $f(x)$ as

$$f(x) = A_0 + A_1 x^{(1)} + A_2 x^{(2)} + A_3 x^{(3)} + \cdots + A_n x^{(n)} \tag{5.9}$$

where $A_0, A_1, A_2, A_3, \ldots, A_n$, are to be determined.

Now,

$$\begin{aligned}
\Delta f(x) &= A_1 + 2A_2 x^{(1)} + 3A_3 x^{(2)} + \cdots + nA_n x^{(n-1)} \\
\Delta^2 f(x) &= 2A_2 + 6A_3 x^{(1)} + \cdots + n(n-1)A_n x^{(n-2)} \\
\Delta^3 f(x) &= 6A_3 + \cdots + n(n-1)(n-2)A_n x^{(n-3)} \\
\cdots &\quad \cdots \qquad \cdots \\
\Delta^n f(x) &= n(n-1)(n-2)\ldots 2.1.A_n x^{(0)} = A_n(n!)
\end{aligned}$$

Putting $x = 0$ in the above equations, we get

$$f(0) = A_0;\ \Delta f(0) = A_1;\ \Delta^2 f(0) = 2A_2 \Rightarrow A_2 = \frac{1}{2!}\Delta^2 f(0)$$

$$\Delta^3 f(0) = 6A_3 \Rightarrow A_3 = \frac{1}{3!}\Delta^3 f(0);$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\Delta^n f(0) = A_n(n!);\ \Rightarrow A_n = 1/n!\ \Delta^n f(0)$$

Substituting the above values in Eqn (5.9), we get

$$f(x) = f(0) + \Delta f(0)\,x^{(1)} + \frac{1}{2!}\Delta^2 f(0)\,x^{(2)} + \frac{1}{3!}\Delta^3 f(0)\,x^{(3)}$$

$$+ \cdots + \frac{1}{n!}\Delta^n f(0)\,x^{(n)} \tag{5.10}$$

Thus, any polynomial of degree n can be expressed as a factorial polynomial of the same degree and vice versa.

Note: We have some other methods to express a polynomial in factorial notation. They are best illustrated through examples.

Example 5.1 Construct a forward difference table from the following data:

x	0	1	2	3	4
y_x	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 y_1$, y_x and y_5.

Solution The forward difference table is as given below:

x	y_x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$y_0 = 1$				
		$\Delta y_0 = 0.5$			
$x_1 = 1$	$y_1 = 1.5$		$\Delta^2 y_0 = 0.2$		
		$\Delta y_1 = 0.7$		$\Delta^3 y_0 = 0$	
$x_2 = 2$	$y_2 = 2.2$		$\Delta^2 y_1 = 0.2$		$\Delta^4 y_0 = 0.4$
		$\Delta y_2 = 0.9$		$\Delta^3 y_1 = 0.4$	
$x_3 = 3$	$y_3 = 3.1$		$\Delta^2 y_2 = 0.6$		
		$\Delta y_3 = 1.5$			
$x_4 = 4$	$y_4 = 4.6$				

Now, $\Delta^3 y_1 = y_4 - 3y_3 + 3y_2 - y_1$

$$= 4.6 - 3(3.1) + 3(2.2) - 1.5 = 0.4$$

Again, from Observation 2 of Section 5.3, we have

$$y_x = y_0 + {}^xC_1 \Delta y_0 + {}^xC_2 \Delta^2 y_0 + {}^xC_3 \Delta^3 y_0 + {}^xC_4 \Delta^4 y_0$$

$$= 1 + x(0.5) + \frac{1}{2} x(x-1)(0.2) + \frac{1}{3!} x(x-1)(x-2)(0)$$

$$+ \frac{1}{4!} x(x-1)(x-2)(x-3)(0.4)$$

$$= 1 + \frac{1}{2} x + \frac{1}{10} (x^2 - x) + \frac{1}{60} (x^4 - 6x^3 + 11x^2 - 6x)$$

$$\therefore \quad y_x = \frac{1}{60} (x^4 - 6x^3 + 17x^2 + 18x + 60)$$

$$\therefore \quad y_5 = \frac{1}{60} [5^4 - 6(5)^3 + 17(5)^2 + 18(5) + 60] = 7.5$$

Example 5.2 Evaluate (i) $\Delta \cos x$, (ii) $\Delta \log f(x)$, (iii) $\Delta^2 \sin(px + q)$, (iv) $\Delta \tan^{-1} x$, and (v) $\Delta^n e^{ax+b}$

Solution Let h be the interval of differencing.

(i) $\Delta \cos x = \cos(x + h) - \cos x$

$$= -2 \sin\left(x + \frac{h}{2}\right) \sin\frac{h}{2}$$

(ii) $\Delta \log f(x) = \log f(x+h) - \log f(x)$

$$= \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$

$$= \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

(iii) $\Delta \sin(px+q) = \sin[p(x+h)+q] - \sin(px+q)$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$

$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\therefore\ \Delta^2\sin(px+q) = 2\sin\frac{ph}{2}\,\Delta\left[\sin\left(px+q+\frac{1}{2}(\pi+ph)\right)\right]$$

$$= \left(2\sin\frac{ph}{2}\right)^2 \sin\left(px+q+2.\frac{1}{2}(\pi+ph)\right)$$

(iv) $\Delta \tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x$

$$= \tan^{-1}\left[\frac{x+h-x}{1+x(x+h)}\right] = \tan^{-1}\frac{h}{1+x(x+h)}$$

(v) $\Delta e^{(ax+b)} = e^{a(x+h)+b} - e^{ax+b}$

$= e^{(ax+b)}(e^{ah}-1)$

$\Delta^2 e^{(ax+b)} = \Delta[\Delta(e^{ax+b})] = \Delta[(e^{ab}-1)\,e^{ax+b}]$

$= (e^{ah}-1)^2\Delta(e^{ax+h})$

$= (e^{ah}-1)^2 e^{ax+h}$ $\quad [\because e^{ah}-1$ is a constant$]$

Proceeding on, we get, $\Delta^n(e^{ax+b}) = (e^{ah}-1)^n e^{ax+b}$

Example 5.3 Evaluate (i) $\Delta[f(x)\,g(x)]$ and (ii) $\Delta\left[\dfrac{f(x)}{g(x)}\right]$

Solution Let h be the interval of differencing.

(i) $\Delta[f(x)\,g(x)] = f(x+h)\,g(x+h) - f(x)\,g(x)$

$= f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)$

$= f(x+h)\,[g(x+h)-g(x)] + g(x)\,[f(x+h)-f(x)]$

$= f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)$

(ii) $$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - g(x+h)f(x)}{g(x+h)g(x)}$$

$$= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)}$$

$$= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)]}{g(x+h)g(x)}$$

Example 5.4 If $y = (3x+1)(3x+4)\ldots(3x+22)$, prove that
$\Delta^4 y = 136080\,(3x+13)(3x+16)(3x+19)(3x+22)$

Solution The given equation $y = (3x+1)(3x+4)\ldots(3x+22)$ contains eight factors:

$$\therefore \quad y = 3^8\left(x+\frac{1}{3}\right)\left(x+\frac{4}{3}\right)\ldots\left(x+\frac{22}{3}\right)$$

$$= 3^8\left(x+\frac{22}{3}\right)^{(8)}$$

$$\therefore \quad \Delta y = 3^8.\,8\left(x+\frac{22}{3}\right)^{(7)}$$

$$\Delta^2 y = 3^8.\,8\,.\,7\left(x+\frac{22}{3}\right)^{(6)}$$

$$\Delta^3 y = 3^8.\,8\,.\,7\,.\,6\left(x+\frac{22}{3}\right)^{(5)}$$

and $$\Delta^4 y = 3^8.\,8\,.\,7.\,6\,.\,5\left(x+\frac{22}{3}\right)^{(4)}$$

$$\therefore \Delta^4 y = 11022480\left(x+\frac{22}{3}\right)\left(x+\frac{22}{3}-1\right)\left(x+\frac{22}{3}-2\right)\left(x+\frac{22}{3}-3\right)$$

$$= 136080\,(3x+22)\,(3x+19)(3x+16)(3x+13)$$

Example 5.5 Find $\Delta^2 \left[\dfrac{1}{x(x+3)(x+6)}\right]$

Solution $y = \dfrac{1}{x(x+3)(x+6)} = (x-3)^{(-3)}$, where $h = 3$

$$\Delta y = (-3)(3)(x-3)^{(-4)}$$

$$\Delta^2 y = (-3)(-4)(3)^2(x-3)^{(-5)}$$

$$= \left[\frac{108}{x(x+3)(x+6)(x+9)(x+12)}\right]$$

Example 5.6 Evaluate $\Delta^{10}(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)$

(B.R, B.E., 1996)

Solution $\Delta^{10}(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)$
$= \Delta^{10}\,[abcd\,x^{10} + \text{terms involving lesser degree}]$
$= abcd\,\Delta^{10}(x^{10}) = abcd\,(10!)$

Example 5.7 Represent the function $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ and its successive differences in factorial notation in which the interval of differencing is one. *(M.U, B.E., 1997)*

Solution

Method 1: The values of $f(x)$ at $x = 0, 1, 2, 3$ and 4 are 9, 10, 37, 54 and 49. The forward difference table is as follows.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	$9 = f(0)$				
		$1 = \Delta f(0)$			
1	10		$26 = \Delta^2 f(0)$		
		27		$-36 = \Delta^3 f(0)$	
2	37		-10		$24 = \Delta^4 f(0)$
		17		-12	
3	54		-22		
		-5			
4	49				

Therefore,

$$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$$

in factorial form is

$$f(x) \;= f(0) + \frac{\Delta f(0)}{1!}x^{(1)} + \frac{\Delta^2 f(0)}{2!}x^{(2)} + \frac{\Delta^3 f(0)}{3!}x^{(3)} + \frac{\Delta^4 f(0)}{4!}x^{(4)}$$

(See Section 5.9)

$$= 9 + \frac{1}{1!}x^{(1)} + \frac{26}{2!}x^{(2)} + \frac{-36}{3!}x^{(3)} + \frac{24}{4!}x^{(4)}$$

$$= 9 + x^{(1)} + 13x^{(2)} - 6x^{(3)} + x^{(4)}$$

Now, $\Delta f(x) = 1 + 26x^{(1)} - 18x^{(2)} + 4x^{(3)}$

$\Delta^2 f(x) = 26 - 36x^{(1)} + 12x^{(2)}$

$\Delta^3 f(x) = -36 + 24x^{(1)}$

$\Delta^4 f(x) = 24$

Method 2 : Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$

$$= ax(x-1)(x-2)(x-3) + bx(x-1)(x-2) + cx(x-1) + dx + e \qquad \text{(i)}$$

Putting $x = 0$ in Eqn (i), we get $e = 9$.

Putting $x = 1$ in Eqn (i), we get $d + e = 10$. $\therefore d = 1$.

Putting $x = 2$ in Eqn (i), we get $2c + 2d + e = 37$ $\therefore c = 13$

Now comparing the coefficients of x^4 and x^3 on both sides of Eqn (i), we get

$$a = 1; -6a + b = -12 \;\therefore b = -6$$

Hence, Eqn (i) becomes

$$f(x) = x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9$$

Method 3 (Synthetic division method): Dividing the given polynomial successively by x, $x-1$, $x-2$ and $x-3$, we have

0	1	–12	42	–30	9
	0	0	0	0	0
1	1	–12	42	–30	$\boxed{9}$
	0	1	–11	31	
2	1	–11	31	$\boxed{1}$	
	0	2	–18		
3	1	–9	$\boxed{13}$		
	0	3			
	1	$\boxed{-6}$			

Now the required equation is

$$y_x = x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9.$$

Note Unless otherwise stated, the interval of differencing may be taken as 1.

Example 5.8 Find the second difference of the polynomial

$$x^4 - 12x^3 + 42x^2 - 30x + 9$$

with interval of differencing $h = 2$.

Solution

Synthetic division method : Dividing the given polynomial successively by x, $x-2$, $x-4$ and $x-6$, we have

0	1	−12	42	−30	9
	0	0	0	0	0
2	1	−12	42	−30	$\boxed{9}$
	0	2	−20	44	
4	1	−10	22	$\boxed{14}$	
	0	4	−24		
6	1	−6	$\boxed{-2}$		
	0	6			
	1	$\boxed{0}$			

$\therefore$ $$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$$

In factorial notation

$f(x) = x^{(4)} - 2x^{(2)} + 14x^{(1)} + 9$

$\Delta f(x) = 4.2x^{(3)} - 2.2.2x^{(1)} + 14.2.1$ (using $\Delta x^{r} = rh\,x^{(r-1)}$)

$\Delta^2 f(x) = 4(2)^2\, 3x^{(2)} - 2{\cdot}2(2)^2$

$= 48x\,(x-2) - 16 \quad = 48x^2 - 96x - 16$

5.10 ERROR PROPAGATION IN DIFFERENCE TABLE

Let $y_0, y_1, \ldots, y_8$ be the true values of a function and suppose in the value y_5 there is an error ε. Then the effect of this error in the differences is as shown in the table given the next page.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1 + \varepsilon$
		Δy_3		$\Delta^3 y_2 + \varepsilon$	
x_4	y_4		$\Delta^2 y_3 + \varepsilon$		$\Delta^4 y_2 - 4\varepsilon$
		$\Delta y_4 + \varepsilon$		$\Delta^3 y_3 - 3\varepsilon$	
x_5	$y_5 + \varepsilon$		$\Delta^2 y_4 - 2\varepsilon$		$\Delta^4 y_3 + 6\varepsilon$
		$\Delta y_5 - \varepsilon$		$\Delta^3 y_4 + 3\varepsilon$	
x_6	y_6		$\Delta^2 y_5 + \varepsilon$		$\Delta^4 y_4 - 4\varepsilon$
		Δy_6		$\Delta^3 y_5 - \varepsilon$	
x_7	y_7		$\Delta^2 y_6$		$\Delta^4 y_5 + \varepsilon$
		Δy_7		$\Delta^3 y_6$	
x_8	y_8		$\Delta^2 y_7$		
		Δy_8			
x_9	y_9				

This table shows that

(i) the effect of an error increases with the order of differences.

(ii) the errors in any column are given by the binomial coefficients of $(1 - \varepsilon)^n$. Thus, in the third column the errors are ε, -3ε, 3ε and $-\varepsilon$, and in the fourth column the errors are ε, -4ε, 6ε, -4ε, ε and so on.

(iii) the algebraic sum of the errors in any difference column is zero.

(iv) the maximum error in each column exists opposite to the entry containing the error i.e. y_5.

These facts enable us to detect errors in a difference table.

Observation The sum of the entries in any column of differences is the difference between the last entry and the first entry in the previous column. Consider the sum of the entries in the third column:

$$\Delta^3 y_0 + \Delta^3 y_1 + \Delta^3 y_2 + \Delta^3 y_3 + \Delta^3 y_4 + \Delta^3 y_5 + \Delta^3 y_6$$
$$= (\Delta^2 y_1 - \Delta^2 y_0) + (\Delta^2 y_2 - \Delta^2 y_1) + (\Delta^2 y_3 - \Delta^2 y_2) + \cdots + (\Delta^2 y_7 - \Delta^2 y_6)$$
$$= \Delta^2 y_7 - \Delta^2 y_0$$

Example 5.9 In the following table, one value of y is incorrect and that y is a cubic polynomial in x.

x	0	1	2	3	4	5	6	7
y	25	21	18	18	27	45	76	123

Construct a difference table for y and use it to locate and correct the wrong value.

Solution Since y is a cubic polynomial in x, the difference of third order i.e. $\Delta^3 y$ must be constant.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	25			
		−4		
1	21		1	
		−3		2
2	18		3	
		0		6
3	18		9	
		9		0
4	27		9	
		18		4
5	45		13	
		31		3
6	76		16	
		47		
7	123			

The sum of the third differences = 15

$\therefore$ each entry must be 15/5 = 3

Hence, there are errors in the first four entries. They can be written as

$$2 = 3 + (-1),\quad 6 = 3 - 3(-1),\quad 0 = 3 + 3(-1),\quad 4 = 3 - (-1)$$

$$\therefore \varepsilon = -1$$

Thus, the correct entry corresponding to $x = 3$ in $y = 18 - \varepsilon$ $= 18 - (-1) = 19$.

Example 5.10 Assuming that the following values of y_x belong to a polynomial of degree four, compute the next two values.

x	2	4	6	8	10	12	14
y	2	3	5	8	9	–	–

Solution

x	y_x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	$2 = y_0$				
		1			
4	$3 = y_1$		1		
		2		0	
6	$5 = y_2$		1		-3
		3		-3	
8	$8 = y_3$		-2		-3
		1		$\Delta^3 y_2 = -6$	
10	$9 = y_4$		$\Delta^2 y_3 = -8$		-3
		$\Delta y_4 = -7$		$\Delta^3 y_3 = -9$	
12	$y_5 = 2$		$\Delta^2 y_4 = -11$		
		$\Delta y_5 = -24$			
14	$y_6 = -22$				

Since the values of y belongs to a polynomial of degree four, the fourth differences must be constant. Then from the table,

$$
\begin{array}{llll}
 & \Delta^3 y_2 - (-3) = -3 & \therefore \Delta^3 y_2 & = -6 \\
 & \Delta^3 y_3 - \Delta^3 y_2 = -3 & \Rightarrow \Delta^3 y_3 & = -3 + \Delta^3 y_2 = -9 \\
 & \Delta^2 y_3 - (-2) = \Delta^3 y_2 & \Rightarrow \Delta^2 y_3 & = \Delta^3 y_2 - 2 = -8 \\
 & \Delta^2 y_4 - \Delta^2 y_3 = \Delta^3 y_3 & \Rightarrow \Delta^2 y_4 & = \Delta^3 y_3 + \Delta^2 y_3 = -9 - 8 = -17 \\
 & \Delta y_4 - 1 = \Delta^2 y_3 & \Rightarrow \Delta y_4 & = 1 + \Delta^2 y_3 = 1 - 8 = -7 \\
 & \Delta y_5 - \Delta y_4 = \Delta^2 y_4 & \Rightarrow \Delta y_5 & = \Delta y_4 + \Delta^2 y_4 = -7 - 17 = -24 \\
 & y_5 - y_4 = \Delta y_4 & \Rightarrow y_5 & = y_4 + \Delta y_4 = 9 - 7 = 2 \\
\text{and} & y_6 - y_5 = \Delta y_5 & \Rightarrow y_6 & = y_5 + \Delta y_5 = 2 - 24 = -22
\end{array}
$$

Hence the missing entries corresponding to $x = 12, 14$ are 2 and -22.

5.11 OTHER DIFFERENCE OPERATORS

So far we have studied the operators Δ, ∇ and δ. Now we shall introduce other operators like E, μ, D etc. which also play a vital role in numerical methods.

Shift Operator E

If h is the interval of differencing in the argument x then the operator E is defined as

$$E f(x) = f(x + h).$$

It is also called *translation operator* due to the reason that it results the next value of the function.

$$E^2 f(x) = E[Ef(x)] = Ef(x+h) = f(x+2h)$$

Similarly, $E^3 f(x) = f(x+3h)$, $E^4 f(x) = f(x+4h)$ and $E^n f(x) = f(x+nh)$.

If y_x is the function $f(x)$ then $E y_x = y_{x+h} \ldots$ and $E^n y_x = y_{x+nh}$ the inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x-h)$$

Similarly, $E^{-n} f(x) = f(x-nh)$.

In general,

$$E^n f(x) = f(x+nh) \text{ for any real } n. \tag{5.11}$$

Note: If $y_0, y_1, \ldots, y_n$ are the consecutive values of y_x then

$$E y_0 = y_1,\ E^2 y_3 = y_5, \ldots, E^k y_n = y_{n+k}$$

Averaging Operator μ

The averaging operator μ is defined by

$$\mu f(x) = ½[f(x+h/2) + f(x-h/2)]$$

i.e.

$$\mu y_x = ½[y_{(x+h/2)} + y_{(x-h/2)}] \tag{5.12}$$

Differential Operator D

The differential operator D is defined as $Df(x) = \dfrac{d}{dx} f(x)$

In general,

$$D^n f(x) = \frac{d^n}{dx^n} f(x) \tag{5.13}$$

Unit Operator 1

The unit operator 1 is defined as $1 \cdot f(x) = f(x)$

Note: All the above operators are linear and obey index laws.

Relation between Δ and E

$$\Delta f(x) = f(x+h) - f(x)$$

$$= Ef(x) - f(x) = (E-1) f(x)$$

$$\Delta = E - 1 \text{ or } E = 1 + \Delta$$

Relation between E and ∇

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1} f(x) = (1 - E^{-1}) f(x)$$

$\therefore\ \nabla = 1 - E^{-1}$ or $E^{-1} = 1 - \nabla$

$\therefore\ E = (1 - \nabla)^{-1}$ $[\because (E^{-1})^{-1} = E]$

Relation between E and δ

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$
$$= E^{1/2} f(x) - E^{-1/2} f(x)$$
$$= (E^{1/2} - E^{-1/2}) f(x)$$

$\therefore\ \delta = E^{1/2} - E^{1/2}$

Again, $\delta = E^{1/2}(1 - E^{-1}) = E^{1/2}\nabla$

and $\delta = E^{-1/2}(E - 1) = E^{-1/2}\Delta$

$\therefore\ \delta = E^{1/2}\nabla = E^{-1/2}\nabla$

Relation between E and μ

$$\mu f(x) = \tfrac{1}{2}[f(x + h/2) + f(x - h/2)]$$
$$= \tfrac{1}{2}[E^{1/2} f(x) + E^{-1/2} f(x)]$$
$$= \tfrac{1}{2}(E^{1/2} + E^{-1/2}) f(x)$$

$\therefore\ \mu = \tfrac{1}{2}(E^{1/2} + E^{-1/2})$

Relation of D with other Operators

We know that $Df(x) = \dfrac{d}{dx} f(x) = f'(x)$ etc.

By Taylor's series

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

or $Ef(x) = f(x) + hDf(x) + \dfrac{h^2}{2!} D^2 f(x) + \dfrac{h^3}{3!} D^3 f(x) + \ldots$

$$= \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \ldots\right] f(x)$$
$$= e^{hD} f(x)$$

$\therefore\ E = e^{hD}$

Taking logarithms on both sides,we get

$$hD = \log E = \log(1+\Delta)$$

$$\therefore \qquad D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2!} + \frac{\Delta^3}{3!} - \frac{\Delta^4}{4!} + \ldots\right] \qquad (5.16)$$

Also, $\nabla = 1 - E^{-1}$, $\therefore E^{-1} = 1 - \nabla$

i.e. $e^{-hD} = 1 - \nabla$

Taking logarithm on both sides,

$$-hD = \log(1-\nabla)$$

$$\therefore \qquad D = \frac{1}{h}\left[\nabla - \frac{\nabla^2}{2!} + \frac{\nabla^3}{3!} - \frac{\nabla^4}{4!} + \ldots\right] \qquad (5.17)$$

Again, $\sin(hD) = \dfrac{e^{hD} - e^{-hD}}{2} = \dfrac{E - E^{-1}}{2}$

$$= \left[\frac{E^{1/2} + E^{-1/2}}{2}\right]\left[E^{1/2} + e^{-1/2}\right] = \mu\delta$$

$\therefore \qquad hD = \sin^{-1}(\mu\delta)$

or

$$D = \frac{1}{h}\sin^{-1}(\mu\delta) \qquad (5.18)$$

Example 5.11 Prove the following results:

(i) $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$ *(M.U, B.E., 1996)*

(ii) $\Delta + \nabla = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$ *(B.R, B.E., 1996)*

(iii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$ *(Madurai B.E., 1998)*

(iv) $1 + \mu^2\delta^2 = (1 + \tfrac{1}{2}\delta^2)^2$

(v) $\Delta = \tfrac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$ *(M.U, B.E., 1997, Madurai B.E., 1988)*

(vi) $\mu^{-1} = 1 - \dfrac{1}{8}\delta^2 + \dfrac{3}{128}\delta^4 - \dfrac{5}{1024}\delta^6 + \ldots$

Solution (i) We have,

$$\begin{aligned}
\Delta\nabla f(x) &= \Delta[\nabla f(x)] = \Delta[f(x) - f(x-h)] \\
&= \Delta f(x) - \Delta f(x-h) \\
&= [f(x+h) - f(x)] - [f(x) - f(x-h)] \\
&= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)
\end{aligned}$$

$\therefore \quad \Delta\nabla = \Delta - \nabla$

Similarly, $\nabla\Delta f(x) = \nabla[\Delta f(x)] = \nabla[f(x+h) - f(x)]$

$= \nabla f(x+h) - \nabla f(x)$

$= [f(x+h) - f(x)] - [f(x) - f(x-h)]$

$= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)$

$\therefore \quad \nabla\Delta = \Delta - \nabla$

Again, $\delta^2 f(x) = [E^{1/2} - E^{-1/2}]^2 f(x)$

$= (E + E^{-1} - 2) f(x)$

$= f(x+h) + f(x-h) - 2f(x)$

$= f(x+h) - f(x)] - [f(x) - f(x-h)]$

$= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)$

$\therefore \quad \delta^2 = \Delta - \nabla$

Hence, $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$

(ii) RHS $= \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta} = \dfrac{\Delta^2 - \nabla^2}{\nabla\Delta}$

$$= \frac{(\Delta + \nabla)(\Delta - \nabla)}{(\Delta - \nabla)} = \Delta + \nabla = \text{LHS}$$

(iii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2}) E^{1/2}$

$= E + 1 = 1 + \Delta + 1 = 2 + \Delta$

(iv) $1 + \mu^2\delta^2 = 1 + \left[\dfrac{E^{1/2} + E^{-1/2}}{2}\right]^2 \left[E^{1/2} - E^{-1/2}\right]^2$

$$= 1 + \left[\frac{E - E^{-1}}{2}\right]^2$$

$$= \frac{4 + (E - E^{-1})^2}{4} = \left[\frac{E + E^{-1}}{2}\right]^2 \qquad \text{(i)}$$

Now, $\left[1 + \dfrac{1}{2}\delta^2\right]^2 = \left[1 + \dfrac{1}{2}\left[E^{1/2} + E^{-1/2}\right]\right]^2$

$$= \left[1 + \frac{1}{2}\left[E + E^{-1} - 2\right]\right]^2 = \left[\frac{E + E^{-1}}{2}\right]^2 \qquad \text{(ii)}$$

Hence, from Eqns (i) and (ii), we have

$$1 + \mu^2\delta^2 = \left[1 + \frac{1}{2}\delta^2\right]^2$$

(v) $$\text{RHS} = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$$

$$= \frac{1}{2}\delta\left[\delta + \sqrt{4 + \delta^2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{4 + (E^{1/2} - E^{-1/2})^2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + (E^{1/2} + E^{-1/2})\right]$$

$$= \frac{1}{2}(E^{1/2} - E^{-1/2})(2E^{1/2})$$

$$= E - 1 = \Delta = \text{LHS}$$

(vi) By definition, we have

$$\mu^2 = \left[\frac{1}{2}(E^{1/2} + E^{-1/2})\right]^2$$

$$= \frac{1}{4}\left[(E^{1/2} - E^{-1/2})^2 + 4\right]$$

$$= \frac{1}{4}(\delta^2 + 4) = \frac{\delta^2}{4} + 1$$

$$\therefore\ \mu = \left[1 + \frac{\delta^2}{4}\right]^{1/2} \quad \text{or} \quad \mu^{-1} = \left[1 + \frac{\delta^2}{4}\right]^{-1/2}$$

$$\therefore \mu^{-1} = 1 - \frac{1}{2}\frac{\delta^2}{4} + \frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left[\frac{\delta^2}{4}\right]^2 - \frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left[\frac{\delta^2}{4}\right]^3 + \cdots$$

$$= 1 - \frac{1}{8}\delta^2 + \frac{3}{128}\delta^4 - \frac{5}{1024}\delta^6 + \cdots$$

Example 5.12 Prove the following :

(i) $\left(\frac{\Delta^2}{E}\right)u_x \neq \frac{\Delta^2 u_x}{Eu_x}$ (*M.U, B.E., 1996*)

(ii) $\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{E(e^x)}{\Delta^2 e^x} = e^x$ (*Kerala, B.Tech, 1990*)

Solution

(i)

$$\begin{aligned} \text{LHS} &= E^{-1}(\Delta^2 u_x) = E^{-1}(E-1)^2 u_x \\ &= (E - 2 + E^{-1})u_x \\ &= u_{x+h} - 2u_x + u_{x-h} \end{aligned}$$

$$\text{RHS} = \frac{(E-1)^2 u_x}{U_{x+h}} = \frac{(E^2 - 2E + 1)u_x}{u_{x+h}}$$

$$= \frac{u_{x+h}^{2} - 2u_{x+h} + u_x}{u_{x+h}}$$

$\therefore$ LHS $\neq$ RHS

(ii)

$$\text{LHS} = E^{-1}(\Delta^2 e^x)\frac{e^{x+h}}{e^x(e^h-1)^2}$$

$$= E^{-1}[e^x(e^h-1)^2]\frac{e^h}{(e^h-1)^2} = E^{-1} e^x . e^h$$

$$= e^{x-h} e^h = e^x = \text{RHS}$$

Example 5.13 Using the method of seperation of symbols, prove that

(i) $u_1 x + u_2 x^2 + u_3 x^3 + \cdots = \frac{x}{1-x}u_1 + \left(\frac{x}{1-x}\right)^2 \Delta u_1 + \left(\frac{x}{1-x}\right)^3 \Delta^2 u_1 + \cdots$

(*Kerala, B.Tech., 1985*)

(ii) $u_0 + u_1 + u_2 + \cdots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \cdots + \Delta^n u_0.$

(*Rourkela, B.Tech., 1985*)

Solution

(i) We know that $u_{x+h} = E^h u_x$

$$\begin{aligned} \therefore \text{LHS} &= xu_1 + x^2 E u_1 + x^3 E^2 u_1 + \cdots \\ &= x[1 + xE + x^2E^2 + \cdots]u_1 \\ &= x\left(\frac{1}{1-xE}\right)u_1 = x\left(\frac{1}{1-x(1+\Delta)}\right)u_1 \end{aligned}$$

$$= x\left(\frac{1}{1-x-x\Delta}\right)u_1 = \frac{x}{1-x}\left[1-\frac{x\Delta}{1-x}\right]^{-1}u_1$$

$$= \frac{x}{1-x}\left[1+\frac{x\Delta}{1-x}+\frac{x^2\Delta^2}{(1-x)^2}+\cdots\right]u_1$$

$$= \frac{x}{(1-x)}u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^3}{(1-x)^3}\Delta^2 u_1 + \cdots = \text{RHS}$$

(ii) $\text{LHS} = u_0 + u_1 + u_2 + \cdots + u_n$

$$= u_0 + Eu_0 + E^2u_0 + \cdots + E^n u_0$$

$$= (1 + E + E^2 + \cdots + E^n)u_0$$

$$= \left[\frac{E^{n+1}-1}{E-1}\right]u_0 = \left[\frac{(1+\Delta)^{n+1}-1}{\Delta}\right]u_0$$

$$= \frac{1}{\Delta}[(1 + {}^{n+1}C_1\Delta + {}^{n+1}C_2\Delta^2 + {}^{n+1}C_3\Delta^3 + \cdots + \Delta^{n+1} - 1]\, u_0$$

$$= \frac{1}{\Delta}[{}^{n+1}C_1\Delta + {}^{n+1}C_2\Delta^2 + {}^{n+1}C_3\Delta^3 + \cdots + \Delta^{n+1}]\, u_0$$

$$= {}^{n+1}C_1 u_0 + {}^{n+1}C_2\Delta^2 u_0 + {}^{n+1}C_3\Delta^2 u_0 + \cdots + \Delta^n u_0 = \text{RHS}$$

5.12 SUMMATION OF SERIES

One of the very important applications of the calculus of finite differences is to obtain a formula for summing a series of upto n terms. The method is best illustrated by the following examples.

Example 5.14 Find the sum to n terms of the series $u_0 + u_1 + u_2 + \cdots + u_n$, where the general term u_x is the first difference of another function. Using this concept, find the sum to n terms of the following series.

(i) $1.3.5 + 2.4.6 + 3.5.7 + \cdots$

(ii) $\dfrac{1}{2.3.4} + \dfrac{1}{3.4.5} + \dfrac{1}{4.5.6} + \cdots$

Solution It is given that the general term u_x of the given series

$$u_1 + u_2 + u_3 + \cdots + u_n$$

is the first difference of another function.

Let $u_x = \Delta y_x = y_{x+1} - y_x$ ($\because h = 1$)

$$\therefore \sum_{x=1}^{n} u_x = u_1 + u_2 + u_3 + \cdots + u_n$$

$$= \Delta y_1 + \Delta y_2 + \Delta y_3 + \cdots + \Delta y_n$$
$$= (y_2 - y_1) + (y_3 - y_2) + (y_4 - y_3) + \cdots + (y_{n+1} - y_n)$$
$$= (y_{n+1} - y_1) = [y_x]_1^{n+1}$$

$$= [\Delta^{-1} u_x]_1^{n+1} \quad (\because y_x = \Delta^{-1} u_x \text{, where } \Delta^{-1} = 1/\Delta)$$

If the general term u_x of the series can be expressed in factorial notation, then $\Delta^{-1} u_x$ can be found easily by integration and hence, the sum of the series.

(i) Consider the series

$$1.3.5 + 2.4.6 + 3.5.7 + \cdots$$

General term, $u_x = x(x+2)(x+4) = x^3 + 6x^2 + 8x$.

In factorial notation, $u_x = x^{(3)} + 9x^{(2)} + 15x^{(1)} = \Delta y_x$

$$\therefore \quad y_x = \Delta^{-1} u_x = \frac{1}{4}x^{(4)} + \frac{9}{3}x^{(3)} + \frac{15}{2}x^{(2)}$$

Hence, $\sum_{x=1}^{n} u_x\,[\Delta^{-1}u_x]_1^{n+1} = \left[\frac{1}{4}x^{(4)} + 3x^{(3)} + \frac{15}{2}x^{(2)}\right]_1^{n+1}$

$$= \left[\frac{1}{4}(n+1)^{(4)} + 3(n+1)^{(3)} + \frac{15}{2}(n+1)^{(2)}\right]$$
$$- \left[\frac{1}{4}(1)^{(4)} + 3(1)^{(3)} + \frac{15}{2}(1)^{(2)}\right]$$

$$= \frac{1}{4}(n+1)(n)(n-1)(n-2) + 3(n+1)(n)(n-1)$$
$$+ \frac{15}{2}(n+1)n - 0$$

$$= \frac{(n+1)n}{4}\{n^2 + 9n + 20\}$$

$$= \frac{n(n+1)(n+4)(n+5)}{4}$$

(ii) Consider the series

$$\frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6} + \cdots$$

Here, $u_x = \dfrac{1}{(x+1)(x+2)(x+3)} = x^{(-3)}$ $\qquad (\because h = 1)$

But $u_x = \Delta y_x \qquad \therefore y_x = \Delta^{-1} u_x = -\dfrac{1}{2} x^{(-2)}$

Hence, $\displaystyle\sum_{x=1}^{n} u_x = \left[\Delta^{-1} u_x\right]_1^{n+1} = \left[-\frac{1}{2}x^{(-2)}\right]_1^{n+1}$

$$= \left[-\frac{1}{2}\frac{1}{(x+1)(x+2)}\right]_1^{n+1}$$

$$= -\frac{1}{2}\left[\frac{1}{(n+2)(n+3)} - \frac{1}{2.3}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{(n+2)(n+3)} - \frac{1}{6}\right]$$

Example 5.15 Prove that $\displaystyle\sum_{r=1}^{n} u_r = {}^nC_1 u_1 + {}^nC_2 \Delta u_1 + {}^nC_3 \Delta^2 u_1 + \cdots + \Delta^{n-1} u_1$.

Hence or otherwise find the sum $\displaystyle\sum_1^n r^2$.

Solution $\displaystyle\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \cdots + u_n$

$$= u_1 + Eu_1 + E^2 u_1 + \cdots + E^{n-1} u_1$$
$$= (1 + E + E^2 + \cdots + E^{n-1}) u_1$$

$$= \left[\frac{E^n - 1}{E - 1}\right] u_1 = \left[\frac{(1+\Delta)^n - 1}{1 + \Delta - 1}\right] u_1$$

$$= \frac{1}{\Delta}[(1 + {}^nC_1\Delta + {}^nC_2\Delta^2 + \cdots + \Delta^n) - 1]u_1$$

$$= {}^nC_1 u_1 + {}^nC_2 \Delta u_1 + \cdots + \Delta^{n-1} u_1 \qquad \text{(i)}$$

To find the sum

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

Let us first construct the difference table for the coefficients of the series given in Eqn (i) taking $u_1 = 1$, $u_2 = 4$, $u_3 = 9$, $u_4 = 16$ and $u_5 = 25$.

u	Δu	$\Delta^2 u$	$\Delta^3 u$
$1 = u_1$			
	3		
$4 = u_2$		2	
	5		0
$9 = u_3$		2	
	7		0
$16 = u_4$		2	
	9		
$25 = u_5$			

Here, $u_r = r^2$ is a polynomial of second degree. Hence, the third and higher order differences will be zero.

$$\therefore \qquad \Delta u_1 = 3, \; \Delta^2 u_2 = 2$$

Hence from Eqn (i),

$$\sum_{r=1}^{n} r^2 = {}^nC_1(1) + {}^nC_2(3) + {}^nC_3(2)$$

$$= n + \frac{n(n-1)}{2!}(3) + \frac{n(n-1)(n-2)}{3!}(2)$$

$$= \frac{n(2n^2+3n+1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

Example 5.16 Prove Montmort's theorem

$$u_0 + u_1x + u_2x^2 + \cdots = \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta^2 u_0}{(1-x)^3} + \cdots$$

and hence find the sum to infinity of the series $1.2 + 2.3x + 3.4x^2 + \cdots\cdots$

Solution Consider $u_0 + u_1x + u_2x^2 + \cdots\cdots$

$$= u_0 + xEu_0 + x^2E^2u_0 + \cdots\cdots$$

$$= (1 + xE + x^2E^2 + \cdots\cdots)\, u_0$$

$$= \left[\frac{1}{1-xE}\right] u_0 = \left[\frac{1}{1-x(1+\Delta)}\right] u_0$$

$$= \left[\frac{1}{1-x-x\Delta}\right] u_0 = \frac{1}{(1-x)\left[1-\dfrac{x\Delta}{(1-x)}\right]} u_0$$

$$= \frac{1}{(1-x)}\left[1-\frac{x\Delta}{1-x}\right]^{-1} u_0$$

$$= \frac{1}{(1-x)}\,[1+\frac{x\Delta}{1-x}+\frac{x^2\Delta^2}{(1-x)^2}+\ldots]u_0$$

$$= \frac{u_0}{1-x}+\frac{x\,\Delta\,u_0}{(1-x)^2}+\frac{x^2\,\Delta^2\,u_0}{(1-x)^3}+\ldots \qquad \text{(i)}$$

= R.H.S

To find the sum of the series

$$1.2 + 2.3x + 3.4x^2 + \cdots$$

Let us construct the difference table for the coefficients of the series given in Eqn (i) by taking $u_0 = 2$, $u_1 = 6$, $u_2 = 12$, $u_3 = 20$ and $u_4 = 30$.

u	Δu	$\Delta^2 u$	$\Delta^3 u$
2			
	4		
6		2	
	6		0
12		2	
	8		0
20		2	
	10		
30			

Hence, from Eqn (i), the sum of the series

$$= \frac{u_0}{1-x}+\frac{x}{(1-x)^2}\,\Delta\,u_0+\frac{x^2}{(1-x)^3}\,\Delta^2\,u_0$$

$$= \frac{2}{(1-x)}+\frac{4x}{(1-x)^2}+\frac{2x^2}{(1-x)^3}$$

$$= \frac{2(1-x)^2+4x(1-x)+2x^2}{(1-x)^3} = \frac{2}{(1-x)^3}$$

Example 5.17 Show that

$$u_0+\frac{u_1x}{1!}+\frac{u_2x^2}{2!}+\cdots = e^x(u_0+x\Delta u_0+\frac{x^2}{2!}\Delta^2u_0+\cdots)$$

and hence sum to infinity the series

$$1+\frac{2^3}{1!}x+\frac{3^3}{2!}x^2+\frac{4^3}{3!}x^3+\cdots$$

Solution Consider $u_0+\frac{u_1 x}{1!}+\frac{u_2 x^2}{2!}+\cdots$

$$= u_0+\frac{x}{1!}Eu_0+\frac{x^2}{2!}E^2u_0+\cdots$$

$$= [1+\frac{xE}{1!}+\frac{x^2E^2}{2!}+\cdots]\,u_0$$

$$= (e^{xE})u_0 = [e^{x(1+\Delta)}]\,u_0 = e^x.e^{x\Delta}\,u_0$$

$$= e^x\,[\,1+x\Delta+\frac{x^2\Delta^2}{2!}+\frac{x^3\Delta^3}{3!}+\cdots]\,u_0$$

$$= e^x\,[\,u_0+x\,\Delta\,u_0+\frac{x^2}{2!}\Delta^2u_0+\frac{x^3}{3!}\Delta^3u_0+\cdots \qquad \text{(i)}$$

= R.H.S

To find the sum of the series

$$1+\frac{2^3}{1}x+\frac{3^3}{2!}x^2+\frac{4^3}{3!}x^3+\cdots$$

let us construct the difference table for the coefficients of the series given in Eqn (i) by taking $u_0 = 1$, $u_1 = 8$, $u_2 = 27$, $u_3 = 64$, $u_4 = 125$, and $u_5 =$ 216.

u	Δu	Δ^2u	Δ^3u	Δ^4u
1				
	7			
8		12		
	19		6	
27		18		0
	37		6	
64		24		0
	61		6	
125		30		
	91			
216				

From the table, $u_0 = 1$, $\Delta u_0 = 7$, $\Delta^2u_0 = 12$, $\Delta^3u_0 = 6$, $\Delta^4u_0 = 0$.

Hence, from Eqn (i), sum of the series is

$$= e^x [u_0 + x\,\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0]$$

$$= e^x [1 + 7x + \frac{12x^2}{2!} + \frac{6x^3}{3!}]$$

$$= e^x (1 + 7x + 6x^2 + x^3)$$

EXERCISE 5.1

1. Tabulate the forward differences for the given data:

x	1	2	3	4	5	6	7	8	9
y	1	8	27	64	125	216	343	512	729

2. Form a table of backward differences of the function

$$f(x) = x^3 - 3x^2 - 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5.$$

3. Form the difference table of $f_x = x^4 - 5x^3 + 6x^2 + x - 2$ for the values of $x = -3, -2, -1, 0, 1, 2, 3$. Extend the table in both directions to give f_4, f_{-5}, f_4, f_5.

4. Show that

   (i) $y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$

   (ii) $\nabla^2 y_8 = y_8 - 2y_7 + y_6$ (iii) $\delta^2 y_5 = y_6 - 2y_5 + y_4$

5. If $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 2000$, $y_4 = 100$ show that $\Delta^4 y_0 = -7459$.

6. If the interval of differencing is unity, prove that

   (i) $\Delta \sin x = 2\sin\frac{1}{2}\cos(x + \frac{1}{2})$

   (ii) $\Delta f(x) = \dfrac{-\Delta f(x)}{f(x)f(x+1)}$

   (iii) $\Delta \tan^{-1}\left(\dfrac{n-1}{n}\right) = \tan{-1}\,\dfrac{1}{2n^2}$

   (iv) $\Delta \dfrac{2^x}{x!} = \dfrac{2^x(1-x)}{(x+1)!}$

(v) $\Delta[x(x+1)(x+2)(x+3)] = 4(x+1)(x+2)(x+3)$

(vi) $\Delta^2\left[\dfrac{5x+12}{x^2+5x+6}\right] = \dfrac{10x+32}{(x+2)(x+3)(x+4)(x+5)}$

(vii) $\Delta^n e^x = (e-1)^n e^x$

(viii) $\Delta^n(1/x) = \dfrac{(-1)^n n!}{x(x+1)x+2)\ldots(x+n)}$

7. If h is the interval of differencing, prove that

(i) $\Delta^2\cos 2x = -4\sin^2 h\cos 2(x+h)$ (*Kerala B.E., 1989, M.U, B.E., 1996*)

(ii) $\Delta^3 a^{cx+d} = (a^{ch}-1)^3 a^{cx+d}$

(iii) $\Delta^n\sin(ax+b) = 2\sin(ah/2)^n \sin\left(ax+b+\dfrac{nah+n\pi}{2}\right)$

8. Show that

(i) $\Delta^3[(1-x)(1-2x)(1-3x)] = -36$ if $h=1$.

(ii) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)] = 24\times 2^{10}\times 10!$ if $h=2$.

9. Find the seventh term of the sequence 2, 9, 28, 65, 126, . . . and also find the general term.

10. Evaluate $\Delta^2 f(x)$ if $f(x)$ is

(i) $\dfrac{1}{x(x+4)x+8)}$ (ii) $\dfrac{1}{(3x+1)(3x+4)(3x+7)}$

11. Find $\Delta^3 f(x)$ if $f(x)$ is $(3x+1)(3x+4)(3x+7)\ldots(3x+19)$

12. Express the following in factorial notation.

(i) $f(x) = 2x^3 - 3x^2 + 3x - 10$

(ii) $f(x) = x^3 - 2x^2 + x - 1$

(iii) $f(x) = 3x^4 - 4x^3 + 6x^2 + 2x + 1$

(iv) $f(x) = x^4 - 3x^3 - 5x^2 + 6x - 7$ and get their successive forward differences.

13. Obtain the function whose first difference is $x^3 + 3x^2 + 5x + 12$.

14. Express the following in factorial notation taking $h = 2$ and find their differences of second order.

(i) $f(x) = 7x^4 + 12x^3 - 6x^2 + 5x - 3$

(ii) $f(x) = x^3 - 3x^2 + 5x + 7$

15. Prepare a forward difference table for values (x_i, y_i), i = 1, 2, 3, . . 7. Indicate the propogation error ε introduced in the tabulated value of y_4.

16. The value of a polynomial of degree 5 are tabulated below. If $f(3)$ is known to be an error, find its correct value.

x	0	1	2	3	4	5	6
$f(x)$	1	2	33	254	1025	3126	7777

17. A polynomial function is given by the following table:

x	0	1	2	3	4	5	6
$f(x)$	0	3	14	39	84	155	258

Form a difference table and explain how the correctness of the arithmetic may be checked.

18. Find y_6 if $y_0 = 9, y_1 = 18, y_2 = 20, y_3 = 24$ and the third differences are constant.

19. Assuming that the following values of y_x belong to a polynomial of degre 4, compute the next three values.

x	0	1	2	3	4	5	6	7
$f(x)$	1	–1	1	–1	1	–	–	–

20. Find the missing term in the following table:

x	1	2	3	4	5	6	7
$f(x)$	2	4	8	–	32	64	128

21. Find and correct a single error in y in the following table:

x	0	1	2	3	4	5	6	7
$f(x)$	0	0	1	6	24	60	120	210

22. With the usual notations prove that

(i) $E\Delta = \Delta E$

(ii) $E\nabla = \nabla E = \Delta$

(iii) $E = (\Delta / \delta)^2$ *(M.U., B.E., 1996)*

(iv) $\nabla = 1 - (1 + \nabla)^{-1}$

(v) $\Delta = \delta E^{1/2}$; $\nabla = \delta E^{-1/2}$

(vi) $(1 + \Delta)(1 - \nabla) = 1$

(vii) $\Delta^2 = (1+\Delta)\delta^2$ (viii) $\mu\,\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta$

(ix) $E^{\frac{1}{2}} = \mu + \frac{1}{2}\delta;\ E^{-\frac{1}{2}} = \mu - \frac{1}{2}\delta$ (x) $\delta = \Delta(1+\Delta)^{-\frac{1}{2}} = \nabla(1-\nabla)^{-\frac{1}{2}}$

(xi) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$ (xii) $\mu = \dfrac{2+\Delta}{2\sqrt{1+\Delta}}$

(xiii) $\dfrac{\Delta^2}{E^2} = E^{-2} - 2E^{-1} + 1$ (xiv) $\mu^2 = 1 + \dfrac{1}{4}\delta^2$

(*Coimbatore B.E., 1985*)

(xv) $E = \sum_{i=o}^{\infty} \nabla_i$

(xvi) $\nabla^2 = h^2D^2 - h^3D^3 + \dfrac{7}{12}h^4D^4 + \cdots$ (*Madurai, B.E, 1989*)

(xvii) $\dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta} = E - E^{-1}$ (*M.U, B.E, 1997*)

(xviii) $(1+\Delta)(1-\nabla) = 1$ (*M.U, B.E, 1997*)

23) Show the following

(i) $\nabla^3 y_2 = \nabla^3 y_5$ (ii) $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$

(iii) $(\Delta + \nabla)^2 (x^2 + x) = 8$ (iv) $(\Delta^2 E^{-1}) x^3 = 6x$

(v) $\dfrac{\Delta^2}{E}\sin(x+h) + \dfrac{\Delta^2 \sin(x+h)}{E\sin(x+h)} = 2(\text{co } h - 1)[\sin(x+h) + 1]$

(vi) $\Delta f_k^2 = (f_k + f_{k+1})\Delta f_k$

(vii) $\dfrac{\Delta^2 x^2}{E(x+\log x)} = \dfrac{2}{x+1+\log(x+1)}$

24) Use the method of separation of symbols to prove that

$(u_1 - u_0) - x(u_2 - u_1) + x^2(u_3 - u_2) - \cdots$

$$= \frac{\Delta u_0}{1+x} - x\frac{\Delta^2 u_0}{(1+x)^2} + x^2\frac{\Delta^3 u_1}{(1+x)^3} - \cdots$$

25) $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \cdots + \Delta^n u_{x-n}$

26) $y_x = y_n - {}^{n-x}C_1 \Delta y_{n-1} + {}^{n-x}C_2 \Delta^2 y_{n-2} - \cdots + (-1)^{n-x}\Delta^{n-x} y_x$

27) $u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \cdots = e^x \left(u_0 + x\,\Delta\, u_0 + \frac{x^2}{2!}\,\Delta^2 u_0 + \cdots\right)$

28) $u_0 + {}^xC_1\,\Delta u_1 + {}^xC_2\,\Delta^2 u_2 + \cdots = u_x + {}^xC_1\,\Delta^2 u_{x-1} + {}^xC_2\,\Delta^4 u_{x-2} + \cdots$

29) Sum the series to n terms:

(i) 1.2.3 + 2.3.4 + 3.4.5 + $\cdots$

(ii) 4.5.6. + 5.6.7 + 6.7.8 + $\cdots$

(iii) 2.5 + 5.8 + 8.11 + $\cdots$

30) Using the method of finite differences, find the sum to n terms of the series whose nth term is $n(n-1)(n-2)$.

31) Using the method of finite differences, find the sum of the first
(i) n squares and (ii) n cubes.

32) Sum the series using the identity of Example 5.15.

(i) $5 + \frac{4x}{1!} + \frac{5x^2}{2!} + \frac{14x^3}{3!} + \frac{37x^4}{4!} + \cdots$

(ii) $1 + \frac{4x}{1!} + \frac{10x^2}{2!} + \frac{20x^3}{3!} + \frac{35x^4}{4!} + \cdots$

33) Using Montmort's theorem, sum the series
$1.3 + 3.5x + 5.7x^2 + 7.9x^3 + \cdots$

ANSWERS

9) 344, $(n+1)^3 + 1$

10) (i) $\Delta^2 f(x) = \frac{192}{x(x+4)(x+8)(x+12)(x+16)}$

(ii) $\Delta^2 f(x) = \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$

11) $\Delta^3 f(x) = 459270\,(3x+19)(3x+16)(3x+13)(3x+10)$

12) (i) $f(x) = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10$

(ii) $f(x) = x^{(3)} + x^{(2)} - 1$;

(iii) $f(x) = 3x^{(4)} + 14x^{(3)} + 15x^{(2)} + 7x^{(1)} + 1$;

(iv) $f(x) = x^{(4)} + 9x^{(3)} + 11x^{(2)} + 5x^{(1)} - 7$

13) $f(x) = \frac{1}{4}x^{(4)} + 2x^{(3)} + \frac{9}{2}x^{(2)} + 12x^{(1)}$ + constant

14) (i) $f(x) = 7x^{(4)} + 96x^{(3)} + 262x^{(2)} + 97x^{(1)} - 3$

$\Delta f(x) = 56x^{(3)} + 576x^{(2)} + 1048x^{(1)} + 194$

$\Delta^2 f(x) = 336x^{(2)} + 2304x^{(1)} + 2096$

(ii) $f(x) = x^{(3)} + 3x^{(2)} + 3x^{(1)} + 7$

$\Delta f(x) = 6x^{(2)} + 12x^{(1)} + 6$

$\Delta^2 f(x) = 24x^{(1)} + 24$

15) $f(x) = 244$, error $= -10$

18) $y_6 = 138$

19) 31, 129, 351

20) 16.1

21) Error is at $x = 2$; $y(2) = 0$

29) (i) $\frac{1}{4}n(n+1)\ (n+2)\ (n+4)$

(ii) $\frac{1}{4}[\,(n+6)\,(n+5)\,(n+4)\,(n+3) - 360]$

(iii) $n\,(3n^2 + 6n + 1)$

30) $\frac{1}{4}(n+1)\,(n)(n-1)\,(n-2)$

31) (i) $\frac{n(n+1)(2n+1)}{6}$ (ii) $\frac{n^2(n+1)^2}{4}$

32) (i) $e^x(x^3 + x^2 - x + 5)$ (ii) $e^x\left(1+3x+\frac{3x^2}{2}+\frac{x^3}{6}\right)$

33) $\frac{3+6x-x^2}{(1-x)^3}$.

CHAPTER 6

Interpolation with Equal Intervals

6.1 INTRODUCTION

Interpolation is a technique of obtaining the value of a function for any intermediate values of the independent variable, i.e. argument within an interval, when the values of the arguments are given. Suppose we are given the following values of $y = f(x)$ for a set of values of x.

x (argument)	:	x_0	x_1	x_2	$\ldots$	x_n
y_x	:	y_0	y_1	y_2	$\ldots$	y_n

Then the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Theile defines it as '*the art of reading between the lines of a table.*'

The process of finding the value of a function outside the given range of arguments is called *extrapolation*. However, the term *interpolation* is applied to both processes.

If the form of the function $f(x)$ is known, we can find $f(x)$ for any value of x by simple substitution. But in most practical problems that occur in Engineering and Science, the form of the function $f(x)$ is unknown and it is very difficult to determine its exact form with the help of a tabulated set of values (x_i, y_i). In such cases we replace $f(x)$ by a simple function $\phi(x)$ called *interpolating function* or *smoothing function*, which assumes the same values as those of $f(x)$ and from which other values may be computed to the desired degree of accuracy.

If $\phi(x)$ is a polynomial then it is called *interpolating polynomial* and the process is known as *polynomial interpolation.* If $\phi(x)$ is a finite trigonometric series the process is called *trigonometric interpolation.*

Usually, polynomial interpolation is preferred due to the reason that they are free from singularities, and easy to manipulate, differentiate and integrate.

Eventhough there are other methods like graphical method and method of curve fitting, in this chapter we will study polynomial interpolation using the calculus of finite differences by deriving two important interpolation formulae, which are used often in all the fields, by means of forward and backward differences of a function.

6.2 GREGORY–NEWTON FORWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a function which takes the values $y_0, y_1, \ldots, y_n$ for $(n+1)$ values, $x_0, x_1, \ldots, x_n$, of the independent variable x (argument). Let these values of x be equidistant, i.e. $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$ and let $y(x)$ be the polynomial in x of nth degree, such that $y_i = f(x_i)$, $i = 0, 1, \ldots, n$.

$$\begin{aligned}\therefore \quad y(x) &= A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) \\ &\quad + A_3(x-x_0)(x-x_1)(x-x_2) + \cdots \\ &\quad + A_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \end{aligned} \tag{6.1}$$

Putting $x = x_0, x_1, \ldots, x_n$ successively in Eqn (6.1), we get

$$y_0 = A_0,\ y_1 = A_0 + A_1(x_1 - x_0)$$

$$y_2 = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_0)(x_2 - x_1)$$

and so on. From these,

$$A_0 = y_0$$

$$A_1 = \frac{y_1 - A_0}{x_1 - x_0} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$A_2 = \frac{y_2 - A_0 - A_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{1}{2!h^2}\Delta^2 y_0$$

$[\because x_2 - x_0 = (x_2 - x_1) + (x_1 - x_0) = 2h]$

Similarly, $$A_3 = \frac{1}{3!h^3}\Delta^3 y_0$$

and so on. Putting these values in Eqn (6.1), we get

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1)$$

$$+ \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \cdots \tag{6.2}$$

Putting $\frac{x - x_0}{h} = p$, i.e. $x = x_0 + ph$, where p is a real number, Eqn (6.2) takes the form

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \cdots + \frac{p(p-1)\ldots[p-(n-1)]}{n!}\Delta^n y_0 \tag{6.3}$$

where $y_p = y(x_0 + ph)$. Eqn (6.3) is known as *Gregory–Newton forward interpolation formula.*

Aliter We can also derive the above, using difference operators, E and Δ.

For any real p, we know that

$$y_p = y(x_0 + ph) = E^p y(x_0) = E^p y_0$$

$$\therefore\ y_p = E^p y_0 = (1 + \Delta)^p y_0$$

$$= [1 + {}^pC_1\Delta + {}^pC_2\Delta^2 + {}^pC_3\Delta^3 + \cdots]\, y_0 \text{ (using Binomial theorem)}$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \tag{6.4}$$

If $y(x)$ is a polynomial of nth degree then

$$\Delta^{n+k} y_0 = 0, k = 1, 2, \ldots$$

Hence, Eqn (6.4) becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \cdots + \frac{p(p-1)\ldots[p-(n-1)]}{n!}\Delta^n y_0$$

which is same as Eqn (6.3).

Note :

(i) This is called forward interpolation formula due to the fact that this formula contains values of the tabulated function from y_0 onward to the right and none to the left of this value. This formula is used mainly to interpolating the values of y near the begining of a set of tabulated values and to extrapolating y a little to the left of y_0.

The first two terms of Eqn (6.3) will give the linear interpolation while the first three terms a parabolic interpolation, and so on.

(ii) Gregory–Newton forward interpolation formula can also be written as

$$y_p = y_0 + p^{(1)}\Delta y_0 + \frac{1}{2!}p^{(2)}\Delta^2 y_0 + \frac{1}{3!}p^{(3)}\Delta^3 y_0 + \cdots + \frac{1}{n!}p^{(n)}\Delta^n y_0$$

where $p(n) = p(p-1)\ldots(p-n+1)$

6.3 GREGORY–NEWTON BACKWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a function which takes the values $y_0, y_1, \ldots, y_n$ for the $(n+1)$ values of $x_0, x_1, \ldots, x_n$ of the independent variable x. Let $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$ and $y(x)$ be the polynomial in x of nth degree, such that $y_i = f(x_i)$, $i = 0, 1, \ldots, n$. Suppose it is required to evaluate $y(x)$ near the end of the table of values, then we can assume that

$$\begin{aligned} y(x) = {} & A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) \\ & + A_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \cdots \\ & + A_n(x - x_n)(x - x_{n-1})\cdots(x - x_1) \end{aligned} \tag{6.5}$$

Putting $x = x_n, x_{n-1}, \ldots, x_0$ successively in Eqn (6.5), we get

$$A_0 = y(x_n) = y_n$$

$$y(x_{n-1}) = y_{n-1} = A_0 + A_1(x_{n-1} - x_n)$$

$$y_{n-2} = A_0 + A_1(x_{n-2} - x_n) + A_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

and so on. These equations give

$$A_0 = y_n$$

$$A_1 = \frac{y_{n-1} - A_0}{x_{n-1} - x_n} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{\nabla y_n}{h}$$

$$A_2 = \frac{y_{n-2} - A_0 - A_1(x_{n-2} - x_n)}{(x_{n-2} - x_n)(x_{n-2} - x_{n-1})}$$

$$= \frac{y_{n-2} - y_n - (y_n - y_{n-1})(-2)}{(-2h)(-h)}\frac{1}{h} = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2!h^2}$$

Similarly, $A_3 = \dfrac{1}{3!h^3}\nabla^3 y_n$ and so on. Putting these values in Eqn (6.5) we get

$$y(x) = y_n + \frac{1}{h}\nabla y_n(x - x_n) + \frac{1}{2!h^2}\nabla^2 y_n(x - x_n)(x - x_{n-1}) + \frac{1}{3!h^3}\nabla^3 y_n$$
$$(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \cdots \tag{6.6}$$

Let $\frac{x - x_n}{h} = p$, i.e. $x = x_n + ph$,

where p is a real number. Then Eqn (6.6) takes the form

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$
$$+ \frac{p(p+1)(p+2)\ldots(p+n-1)}{n!}\nabla^n y_n \tag{6.7}$$

where $y_p = y(x_n + ph)$, Eqn (6.7) is known as *Gregory–Newton backward interpolation formula.*

Aliter We can also derive the above formula using the difference operators.

For any real p, we have

$$y_p = y(x_n + ph) = E^p y(x_n) = (1 - \nabla)^{-p} y_n$$
$$[\because E = (1 - \nabla)^{-1}]$$

Using Binomial theorem,

$$y_p = \{1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots\}y_n$$
$$= y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

Note: Since the formula involves the backward differences, it is called backward interpolation formula and is used to interpolate the values of y near to the end of a set of tabular values. This may also be used to extrapolate the values of y a little to the right of y_n.

6.4 ERROR IN POLYNOMIAL INTERPOLATION

If $y = f(x)$ is the exact curve and $y = y_p$ is the interpolating polynomial curve, then the error in polynomial interpolation is given by

$$\text{Error} = f(x) - y_p = \frac{(x - x_0)(x - x_1)\ldots\ldots.(x - x_n)}{(n+1)!} f^{n+1}(c) \tag{6.8}$$

for any x, where $x_0 < x < x_n$ and $x_0 < c < x_n$.

The error in Newton's forward interpolation formula is given by

$$f(x) - y_n = \frac{p(p-1)(p-2).......(p-n)}{(n+1)!} \Delta^{n+1} f(c)$$

where $p = \dfrac{x - x_0}{h}$.

The error in Newton's backward interpolation formula is given by

$$f(x) - y_n = \frac{p(p+1)(p+2).......(p+n)}{(n+1)!} h^{n+1} y^{n+1}(c)$$

where $p = \dfrac{x - x_n}{h}$.

Example 6.1 The following data give I, the indicated HP and V, the speed in knots developed by a ship.

V	8	10	12	14	16
I	1000	1900	3250	5400	8950

Find I when $V = 9$, using Newton's forward interpolation formula.

Solution We note that $V = 9$ is near the beginning of the table. Hence, to get the corresponding I, we use Newton's forward interpolation formula. The forward differences are calculated and tabulated as below.

V	I	ΔI	$\Delta^2 I$	$\Delta^3 I$	$\Delta^4 I$
8	**1000**				
		900			
10	1900		**450**		
		1350		**350**	
12	3250		800		**250**
		2150		600	
14	5400		1400		
		3550			
16	8950				

Here, $V_0 = 8$, $I_0 = 1000$, $\Delta I_0 = 900$, $\Delta^2 I_0 = 450$, $\Delta^3 I_0 = 350$, $\Delta^4 I_0 = 250$. Hence, the interpolation polynomial will be of degree 4. That is,

$$I = I_0 + p\,\Delta I_0 + \frac{p(p-1)}{2!}\Delta^2 I_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 I_0$$
$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 I_0$$

Let I_p be the value of I when $V = 9$

Then $p = \dfrac{V - V_0}{h} = \dfrac{9-8}{2} = \dfrac{1}{2} = 0.5$

$$\therefore\ I_p = 1000 + (0.5)(900) + \frac{0.5(0.5-1)}{2!}(450)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)}{3!}(350) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(250)$$

$$= 1000 + 450 - 56.25 + 21.875 - 9.765625$$

$$= 1405.8594$$

Example 6.2 The amount A of a substance remaining in a reacting system after an interval of time t in a certain chemical experiment is tabulated below:

t (min)	2	5	8	11
A (gm)	94.8	87.9	81.3	75.1

Obtain the value of A where $t = 9$ using Newton's backward interpolation formula.

Solution Since the value $t = 9$ is near the end of the table, to get the corresponding value of t we use Newton's backward interpolation formula. The backward differences are calculated and tabulated below.

t	A	∇A	$\nabla^2 A$	$\nabla^3 A$
2	94.8			
		–6.9		
5	87.9		0.3	
		–6.6		0.1
8	81.3		0.4	
		–6.2		
11	75.1			

Here, $t_n = 11$, $A_n = 75.1$, $\nabla A_n = -6.2$, $\nabla^2 A_n = 0.4$, $\nabla^3 A_n = 0.1$.
Hence, the interpolation polynomial will be of degree 3. That is,

$$A = A_n + p\nabla A_n + \frac{p(p+1)}{2!}\nabla^2 A_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 A_n$$

Let A_p be the value of A when $t = 9$.

Then $p = \dfrac{t - t_n}{h} = \dfrac{9 - 11}{3} = -\dfrac{2}{3}$

$$\therefore A_p = 75.1 + \left(-\frac{2}{3}\right)(-6.2) + \frac{1}{2!}\left(-\frac{2}{3}\right)\left(-\frac{2}{3}+1\right)(0.4)$$

$$+\frac{1}{3!}\left(-\frac{2}{3}\right)\left(-\frac{2}{3}+1\right)\left(-\frac{2}{3}+2\right)(0.1)$$

$$= 75.1 + 4.1333333 - 0.0444444 - 4.9382716 \times 10^{-3}$$
$$= 79.183951$$

Example 6.3 From the following data, estimate the number of persons having income in between (i) 1000–1700 and (ii) 3500–4000.

Income	below 500	500–1000	1000–2000	2000–3000	3000–4000
No. of persons	6000	4250	3600	1500	650

(M.U, B.E., 1991)

Solution First we prepare the cumulative frequency table as follows:

Person's income less than x:	1000	2000	3000	4000
No. of persons y_x:	10250	13850	15350	16000

Now the difference table is as follows:

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$
1000	10250			
		3600		
2000	13850		–2100	
		1500		1250
3000	15350		–850	
		650		
4000	16000			

(i) Here, we have to find the number of persons having income in between 1000–1700.
Taking $x_0 = 1000$, $x = 1700$, we have

$$p = \frac{x - x_0}{h} = \frac{1700 - 1000}{1000} = 0.7.$$

Let $y_p = y_{1700}$.

Using Newton's forward interpolation formula, we get

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$\therefore\ y_{1700} = 10250 + (0.7)(3600) + \frac{1}{2!}(0.7)(0.7-1)(-2100)$$

$$+ \frac{1}{3!}(0.7)(0.7-1)(0.7-2)(1250)$$

$$= 10250 + 2520 + 220.5 + 56.875$$

$$= 13047.375$$

$\therefore$ The number of persons having income less than 1700 is 13047.375, i.e. 13047

But the number of persons having income less than 1000 is 10250.

$\therefore$ The number of persons having income in between 1000–1700 is $13047 - 10250 = 2797$.

(ii) Here, we have to find the number of persons having income in between 3500–4000.

Taking $x_n = 4000$, $x = 3500$, we have

$$p = \frac{x - x_n}{h} = \frac{3500 - 4000}{1000} = -0.5.$$

Let $y_p = y_{3500}$. Using Newton's backward interpolation formula, we get

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

where $y_n = 16000$, $\nabla y_n = 650$, $\nabla^2 y_n = -850$, $\nabla^3 y_n = 1250$

$$\therefore\ y_{3500} = 16000 + (-0.5)(650) + \frac{1}{2!}(-0.5)(-0.5+1)(-850)$$

$$+ \frac{1}{3!}(-0.5)(-0.5+1)(-0.5+2)(1250)$$

$$= 1600 - 325 + 106.25 - 78.125$$

$$= 15703.125$$

$\therefore$ The number of persons having income less than 3500 is 15703.125, i.e. 15703.

But the number of persons having income less than 4000 is 16000.

$\therefore$ No. of persons whose income in between 3500–4000 is $16000 - 15703 = 297$.

Example 6.4 Find a polynomial which takes the following values

x	1	3	5	7	9	11
y_x	3	14	19	21	23	28

and hence compute y_x at $x = 2, 12$.

Solution The difference table is as follows:

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
1	3				
		11			
3	14		–6		
		5		3	
5	19		–3		0
		2		3	
7	21		0		0
		2		3	
9	23		3		
		5			
11	28				

Take $x_0 = 1,\ y_0 = 3,\ p = \dfrac{x-1}{2}$

Using Newton's forward interpolation fomula, we get

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$= 3 + \frac{1}{2}(x-1)(11) + \frac{1}{2!}\frac{1}{2}(x-1)\left\{\frac{1}{2}(x-1)-1\right\}(-6)$$

$$+ \frac{1}{3!}\frac{1}{2}(x-1)\left\{\frac{1}{2}(x-1)-1\right\}\left\{\frac{1}{2}(x-1)-2\right\}(3)$$

$$= 3 + 11/2(x-1) - 3/4(x^2 - 4x + 3) + 1/16(x^3 - 9x^2 + 23x - 15)$$

$$= 1/16\,(x^3 - 21x^2 + 159x - 91) \qquad \text{(i)}$$

Again take $x_n = 11,\ y_n = 28,\ p = \dfrac{x-11}{2}$.

Using Newton's backward interpolation formula,

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$= 28 + \frac{5}{2}(x-11) + \frac{1}{2!}\frac{1}{2^2}(x-11)(x-9)(3)$$

$$+ \frac{1}{3!}\frac{1}{2^3}(x-11)(x-9)(x-7)(3)$$

$$= 28 + \frac{5}{2}(x-11) + \frac{1}{16}(x-11)(x-9)(x-1)$$

$$= \frac{1}{16}(x^3 - 21x^2 + 159x - 91) \qquad \text{(ii)}$$

which is the same as Eqn (i) representing the data. So we can use any one of the formula to find the polynomial.

$$\therefore \qquad y_n = \frac{1}{16}(x^3 - 21x^2 + 159x - 91)$$

Now $$y_2 = \frac{1}{16}(2^3 - 21.2^2 + 159.2 - 91) = 9.4375$$

and $$y_{12} = \frac{1}{16}[(12)^3 - 21(12)^2 + 159(12) - 91] = 32.5625$$

Example 6.5 Find a cubic polynomial which takes the following set of values

(0, 1), (1, 2), (2, 1) and (3, 10). *(M.U, B.E., 1997)*

Solution The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2		–2	
		–1		12
2	1		10	
		9		
3	10			

we take $x_0 = 0$ and $p = \frac{x - x_0}{h} = \frac{x-0}{1} = x$

$\therefore$ Using Newton's forward interpolation forumulae, we get

$$f(x) = f(0) + x\,\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(0)$$

$$= 1 + x \,.\, 1 + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1$$

which is the required polynomial.

Example 6.6 Given $\sum_{1}^{10} f(x) = 500426$, $\sum_{4}^{10} f(x) = 329240$

$\sum_{7}^{10} f(x) = 175212$ and $f(10) = 40365$, find $f(1)$. (*Karnataka, B.E., 1989*)

Solution Here, we are given the cumulative function $f(x)$ and

$$F(1) = \sum_{1}^{10} f(x) = 500426, \qquad F(4) = \sum_{4}^{10} f(x) = 329240,$$

$$F(7) = \sum_{7}^{10} f(x) = 175212, \qquad \text{and } F(10) = f(10) = 40365.$$

Here is the difference table.

x	$F(x)$	$\Delta F(x)$	$\Delta^2 F(x)$	$\Delta^3 F(x)$
1	500426			
		−171186		
4	329240		17158	
		− 154028		2023
7	175212		19181	
		− 134847		
10	40365			

Now we shall find $F(2) = \sum_{2}^{10} f(x)$.

Taking $x_0 + ph = 2$, we get

$$p = \frac{x_0 - 2}{h} = \frac{1-2}{3} = \frac{1}{3}$$

∴ By Newton's forward interpolation formula, we get

$$F(2) = F(1) + p\Delta F(1) + \frac{1}{2!} p(p-1)\,\Delta^2 F(1)$$

$$+ \frac{1}{3!} p(p-1)(p-2)\,\Delta^3 F(1)$$

$$= 500426 + 1/3\,(-171186) + 1/2\,(1/3)\,(-2/3)\,(17158) + 1/6\,(1/3)(-2/3)(-5/3)\,(2023)$$
$$= 500426 - 57062 - 1906.4444 + 124.8765$$
$$= 441582.432$$

$$\text{Hence, } f(2) = F(1) - F(2) = \sum_{1}^{10} f(x) - \sum_{2}^{10} f(x)$$
$$= 500426 - 441582.432$$
$$= 58843.568$$

Example 6.7 The following table gives the values of density of saturated water for various temperatures of saturated steam.

Temp° C(= T)	100	150	200	250	300
Density hg/m³(= d)	958	917	865	799	712

Find by interpolation, the densities when the temperatures are 130°C and 275°C respectively. (*M.U, B.E., 1997*)

Solution We note that $T = 130°$ is near the beginning of the table and $T = 275°$ is near the end of the table. So for the former, we use Newton's Forward interpolation formula and for the later, we use Newton's Backward interpolation formula. The difference table is as given below:

T	d	Δd	$\Delta^2 d$	$\Delta^3 d$	$\Delta^4 d$
100	958				
		−41			
150	917		−11		
		−52		−3	
200	865		−14		−4
		−66		−7	
250	799		−21		
		−87			
300	712				

Here $T_0 = 100$, $d_0 = 958$, $\Delta d_0 = -41$, $= \Delta^2 d_0 - 11$, $\Delta^3 d_0 = -3$ and $\Delta^4 d_0 = -4$.

Hence the interpolation polynomial will be of degree 4.

Let d_u be the value of d at $T = 130°$

$$\therefore \qquad u = \frac{T - T_0}{h} = \frac{130 - 100}{50} = 0.6$$

$$\therefore\ d_u = d_0 + u\,\Delta\,d_0 + \frac{u(u-1)}{2!}\,\Delta^2 d_0 + \frac{u(u-1)(u-2)}{3!}\,\Delta^3 d_0$$

$$+\ \frac{u(u-1)(u-2)(u-3)}{4!}\,\Delta^4 d_0$$

$$= 958 + (0.6)(-41) + \frac{(0.6)(0.6-1)}{2!}\,(-11)$$

$$+\ \frac{(0.6)(0.6-1)(0.6-2)}{3!}\,(-3) + \frac{(0.6)(0.6-1)(0.6-2)(0.6-3)}{4!}\,(-4)$$

$$= 958 - 24.6 + 1.32 - 0.168 + 0.1344$$

$$= 934.6864 \simeq 935$$

Again $T_n = 300$, $d_n = 712$, $\nabla d_n = -87$, $\nabla^2 d_n = -21$, $\nabla^3 d_n = -7$ and $\nabla^4 d_n = -4$.

Let d_v be the value of d at $T = 275°C$

$$\therefore\ v = \frac{T - T_n}{h} = \frac{275-300}{50} = -0.5$$

$$\therefore\ d_v = d_n + v\nabla d_n + \frac{v(v+1)}{2!}\,\nabla^2 d_n + \frac{v(v+1)(v+2)}{3!}\,\nabla^3 d_n$$

$$+\ \frac{v(v+1)(v+2)(v+3)}{4!}\,\nabla^4 d_n$$

$$= 712 + (-0.5)(-87) + \frac{(-0.5)(-0.5+1)}{2!}\,(-21)$$

$$+\ \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}\,(-7)$$

$$+\ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}\,(-4)$$

$$= 712 + 43.5 + 2.625 + 0.4375 + 0.15625$$

$$= 758.71875 \simeq 759$$

Example 6.8 The following are data from the steam table:

tempC°(t)	140	150	160	170	180
Pressure kgf/cm²(P)	3.685	4.854	6.302	8.076	10.225

Using Newton's formula, find the pressure of the steam for temperatures 142° and 175°. *(M.U, B.E., 1996)*

Solution We note that $t = 142°$ is near the beginning of the table and $t = 175°$ is near the end of the table. So for the former we use Newton's Forward interpolation formula and for the later, we use Newton's Backward interpolation formula. The differences are calculated and tabulated below.

t	P	ΔP	$\Delta^2 P$	$\Delta^3 P$	$\Delta^4 P$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

Here $t_0 = 140$, $P_0 = 3.685$, $\Delta P_0 = 1.169$, $\Delta^2 P_0 = 0.279$, $\Delta^3 P_0 = 0.047$ and $\Delta^4 P_0 = 0.002$

Hence the interpolation polynomial will be of degree 4.

Let P_u be the value of P when $t = 142°$

$$\therefore\ U = \frac{t - t_0}{h} = \frac{142 - 140}{10} = 0.2$$

$$\text{i.e., } P_u = P_0 + u\,\Delta P_0 + \frac{u(u-1)}{2!}\Delta^2 P_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 P_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 P_0$$

$$\therefore\ P_u = 3.685 + (0.2)(1.169) + \frac{(0.2)(0.2-1)}{2!}(0.279)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)}{3!}(0.047) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{4!}(0.002)$$

$= 3.685 + 0.2338 - 0.02332 + 0.002256 - 0.0000672$

$= 3.8986688 \simeq 3.899$

Again $t_n = 180°$

$\therefore\ P_n = 10.225$, $\nabla P_n = 2.149$, $\nabla^2 P_n = 0.375$, $\nabla^3 P_n = 0.049$, and $\nabla^4 P_n = 0.002$

Let P_v be the value of P at $t = 175°$.

$$v = \frac{t - t_n}{h} = \frac{175 - 180}{10} = -0.5$$

$$P_v = P_n + v\,\nabla P_n + \frac{v(v+1)}{2!}\nabla^2 P_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 P_n$$

$$+ \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 P_n$$

$$= 10.225 + (-0.5)(2.149) + \frac{(-0.5)(-0.5+1)}{2!}(0.375)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(0.049)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(0.002)$$

$$= 10.225 - 1.0745 - 0.046875 - 0.0030625 - 0.000078125$$
$$= 9.10048438 \simeq 9.1005$$

Example 6.9 Given $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$ and $\sin 60° = 0.8660$, find $\sin 52°$ using Newton's interpolation formula. Estimate the error. *(Punjab B.E, 1987)*

Solution Let $y = \sin x$ be the function. We construct the following difference table:

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		**– 0.0057**	
		0.0532		**– 0.0007**
55°	0.8192		– 0.0064	
		0.0468		
60°	0.8660			

Here, $x_0 = 45$, $y_0 = 0.7071$, $\Delta y_0 = 0.0589$ $\Delta^2 y_0 = -0.0057$ and $\Delta^3 y_0 = -0.0007$

Using Newton's forward interpolation formula,

$$y = y_0 + p\Delta y_0 + \frac{1}{2!}p(p-1)\Delta^2 y_0 + \frac{1}{3!}p(p-1)(p-2)\Delta^3 y_0$$

where $p = \dfrac{x - x_0}{h}$. Let y_p be the value of y at $x = 52°$

$\therefore p = (52 - 45)/5 = 7/5 = 1.4$

$$\therefore\ y_{52} = 0.7071 + (1.4)(0.0589) + \frac{1}{2}(1.4)(1.4-1)(-0.0057)$$

$$+\frac{1}{6}(1.4)(1.4-1)(1.4-2)(-0.0007)$$

$$= 0.7071 + 0.08246 - 0.001596 + 0.0000392$$

$$= 0.7880032$$

$$\therefore\ \sin 52^\circ = 0.7880032$$

$$\text{Error} = \frac{p(p-1)\ldots(p-n)}{3!}\Delta^{n+1}y_{(c)} \qquad \text{[Eqn (6.8)]}$$

$$= \frac{1.4(1.4-1)(1.4-2)}{6}\Delta^3 y_{(c)} \qquad \text{[by taking } n = 2\text{]}$$

$$= \frac{1.4(0.4)(-0.6)}{6}(-0.0007) = 0.0000392$$

6.5 EQUIDISTANT TERMS WITH ONE OR MORE MISSING VALUES

When one or more of the values of the function $y = f(x)$ corresponding to the equidistant values of x are missing, we can find these missing values using finite difference operators E and Δ. The method is best illustrated by the following solved example.

Example 6.10 Find the missing value in the following table.

x	16	18	20	22	24	26
y	43	89	–	155	268	388

Solution Since five values are given it is possible to express y as a polynomial of fourth degree.

Hence, the fifth differences of y are zeros.

Taking the origin for x at 16, from the given data we have

$y_0 = 43, y_1 = 89, y_3 = 155, y_4 = 268, y_5 = 388$ and we have to find y_2. We know that

$\Delta^5 y_x = 0$ for all values of x

$$\therefore \qquad \Delta^5 y_0 = 0, \text{ i.e. } (E-1)^5 y_0 = 0$$

$$\text{i.e.} \qquad (E^5 - {}^5C_1 E^4 + {}^5C_2 E^3 - {}^5C_3 E^2 + {}^5C_4 E - 1)\,y_0 = 0$$

or $\qquad (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\,y_0 = 0$

i.e. $\qquad E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5E y_0 - y_0 = 0$

or $\qquad y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

Substituting the given values,

$$388 - 5(268) + 10(155) - 10y2 + 5(89) - 43 = 0$$

$\therefore$ $$y_2 = 100$$

Example 6.11 Find the missing values in the following table of values of x and y:

x :	0	1	2	3	4	5	6
y :	-4	-2	-	-	220	546	1148

(M.U, B.E., 1997)

Solution There being given five values and two missing figures, we may have

$$\Delta^5 y_0 = 0 \text{ and } \Delta^6 y_0 = 0$$

Let $y_0 = -4,\ y_1 = -2, y_4 = 220,\ y_5 = 546, y_6 = 1148$ and we have to find out y_2, y_3.

$$\Delta^5 y_0 = (E-1)^5 y_0 = 0$$

i.e., $\qquad E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5Ey_0 - y_0 = 0$

or $\qquad y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

Substituting the values, we get

$$546 - 5(220) + 10y_3 - 10y_2 + 5(-2) - (-4) = 0$$

or $$10y_3 - 10y_2 - 560 = 0$$

or $$y_3 - y_2 = 56 \qquad \text{(i)}$$

Again $$\Delta^6 y_0 = (E-1)^6 y_0 = 0$$

i.e., $\qquad E^6 y_0 - 6E^5 y_0 + 15E^4 y_0 - 20E^3 y_0 + 15E^2 y_0 - 6Ey_0 + y_0 = 0$

or $$y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$$

or $$1148 - 6(546) + 15(220) - 20y_3 + 15y_2 - 6(-2) - 4 = 0$$

or $$-20y_3 + 15y_2 + 1180 = 0$$

or $$-4y_3 + 3y_2 = -236 \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$y_2 = 12 \text{ and } y_3 = 68$$

EXERCISE 6.1

1. From the following data find y at $x = 43$ using Newton's forward – interpolation formula.

x	40	50	60	70	80	90
y	184	204	226	250	176	304

2. The population of a town in decennial census was as given below. Estimate the population for the year 1895.

Years (x)	1891	1901	1911	1921	1931
Population (y) in thousands	46	66	81	93	101

3. Using Newton's forward interpolation formula find the value of $f(1.6)$ if

x	1	1.4	1.8	2.2
y	3.49	4.82	5.96	6.5

(*Bangalore, B.E., 1989*)

4. The following data gives the melting point of an alloy of lead and zinc, where t°C is the temperature and p is the percentage of lead in the alloy.

p	40	50	60	70	80	90
t	184	204	226	250	276	304

Using Newton's backward interpolation formula, find the melting point of the alloy containing 84% of lead.

5. The area A of a circle of diameter d is given for the following values

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105.

6. The following are the numbers of deaths in four successive ten-year age groups. Find the number of deaths at 45–50 and 50–55 age groups.

Age group	25–35	35–45	45–55	55–65
Death	13229	18139	24225	31496

7. From the following table, find y when $x = 1.85$ and $x = 2.4$ using Newton's interpolation formula.

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$	5.474	6.050	6.686	7.389	8.166	9.025	9.974

8. Estimate the values of $f(22)$ and $f(42)$ from the following data :

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

(*Gulbarga B.E., 1993*)

9. Find a polynomial which takes the following values :

x	4	6	8	10
y	1	3	8	16

Hence calculate y at $x = 5$. (*M.U, B.E., 1989*)

10. Using Newton's backward interpolation formula, find the polynomial of degree four passing through (1, 1), (2, – 1), (3, 1), (4, – 1) and (5, 1). (*Karnataka, B.E., 1989*)

11. Obtain the estimate of the missing figure in the following table :

x	1	2	3	4	5
y	2	5	7	–	32

12. Interpolate the missing values in the following table of rice cultivation:

Year x	1911	1912	1913	1914	1915	1916	1917	1918	1919
Acres y (in millions)	76.6	78.2	–	77.7	78.7	–	80.6	77.6	78.6

13. Given $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 200, y_4 = 100$. Find $\Delta^4 y_0$ without forming the difference table. (*M.U, B.E., 1989*)

14. If $u_{-1} = 10, u_1 = 8, u_2 = 10, u_4 = 50$, find u_0 and u_3. (*M.U, B.E., 1993*)

15. If y_x is the value of y at x for which the fifth differences are constant and $y_1 + y_7 = -784, y_2 + y_6 = 686, \; y_3 + y_5 = 1088$, find y_4.

16. Determine the maximum step size that can be used in the tabulation of $f(x) = e^x$ in [0, 1], so that the error in linear interpolation be less than 5×10^{-4}.

17. Given sin 25° = 0.42262, sin 26° = 0.43837, sin 27° = 0.45399, sin 28° = 0.46947, sin 29° = 0.48481 and sin 30° = 0.5. Using Newton's interpolation formula find sin 28° 24′. Estimate the error.

ANSWERS

1. 189.70
2. 54.8528
3. 5.54
4. 287 (nearly)
5. 8666
6. 11278, 12947
7. 6.36, 11.02
8. 352, 219
9. $y = \frac{1}{8}(3x^2 - 22x + 48)$, 1.625
10. $y = \frac{1}{3}(2x^4 - 24x^3 + 100x^2 - 168x + 93)$
11. 14
12. $y_2 = 78.34$, $y_5 = 80.59$
13. -259
14. $u_0 = 10$, $u_3 = 22$
15. 571
16. $h = 0.3836$
17. 0.47562, -0.01714

CHAPTER 7

Central Difference Interpolation Formulae

7.1 INTRODUCTION

Newton's forward and backward Interpolation formulae which have been discussed in the previous chapter are best suited for interpolation near the beginning and the end of a difference table. But, to interpolate near the middle (centre) of a difference table, the following central difference interpolation formulae are most suitable.

Let $y = f(x)$ be the functional relation between x and y. If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$ and $x_0 + 2h$, and the corresponding values of y are y_{-2}, y_{-1}, y_0, y_1 and y_2 then we can write the difference table in the two notations · · · as follows using the operator $\Delta = \delta E^{1/2}$, i.e. $\delta = \Delta E^{1/2}$, known as central difference table.

x	y	First difference	Second difference	Third difference	Fourth difference
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2}\ (= \delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2}\ (= \delta^2 y_{-1})$		
		$\Delta y_{-1}\ (= \delta y_{-1/2})$		$\Delta^3 y_{-2}\ (= \delta^3 y_{-1/2})$	
x_0 ······	y_0	·······	$\Delta^2 y_{-1}\ (= \delta^2 y_0)$	·······	$\Delta^4 y_{-2}\ (= \delta^4 y_0)$
		$\Delta y_0\ (= \delta y_{1/2})$		$\Delta^3 y_{-1}\ (= \delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0\ (= \delta^2 y_1)$		
		$\Delta y_1\ (= \delta y_{3/2})$			
$x_0 + 2h$	y_2				

7.2 GAUSS'S FORWARD INTERPOLATION FORMULA

The Gregory–Newton forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \quad (7.1)$$

where $p = \dfrac{x - x_0}{h}$.

We have,

$$\Delta^2 y_0 = \Delta^2 E_{-1} = \Delta^2(1+\Delta)y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1} \quad (7.2)$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \quad (7.3)$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \quad (7.4)$$

Similarly,

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2} \quad (7.5)$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.} \quad (7.6)$$

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0 \cdots$ from Eqns (7.2)–(7.6), in Eqn (7.1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \left\{\frac{p(p-1)}{2!} + \frac{p(p-1)(p-2)}{3!}\right\}\Delta^3 y_{-1}$$

$$+ \left\{\frac{p(p-1)(p-2)}{3!} + \frac{p(p-1)(p-2)(p-3)}{4!}\right\}\Delta^4 y{-1} + \cdots$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \frac{(p+1)p(p-1)(p-2)(p-3)}{5!}\Delta^5 y_{-1}$$

$$+ \cdots$$

$$= y_0 + p\Delta y_0 + {}^pC_2\,\Delta^2 y_{-1} + {}^{(p+1)}C_3\,\Delta^3 y_{-1} + {}^{(p+1)}C_4\{\Delta^4 y_{-2} + \Delta^5 y_{-2}\}$$

$$+ {}^{(p+1)}C_5\{\Delta^5 y_{-2} + \Delta^6 y_{-2}\} + \cdots \quad \text{[using Eqn (7.6) etc.]}$$

$$= y_0 + {}^pC_1\,\Delta y_0 + {}^pC_2\,\Delta^2 y_{-1} + {}^{(p+1)}C_3\,\Delta^3 y_{-1} + {}^{(p+1)}C_4\,\Delta^4 y_{-2}$$

$$+ \{{}^{(p+1)}C_5 + {}^{(p+1)}C_5\}\,\Delta^5 y_{-2} + \cdots$$

$$\therefore y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$$+\ldots \tag{7.7}$$

which is called *Gauss's forward interpolation formula.*

In the central differences notation, this formula will be

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2}$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 y_0$$

$$+\frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\delta^5 y_{1/2} + \cdots \tag{7.8}$$

Note: This formula, Eqn (7.7), involves odd differences below the central line ($x = x_0$) and even differences on the line as shown below.

$$\begin{array}{ccccccccccccc}
y_0 & \cdots & & \cdots & \Delta^2 y_{-1} & \cdots & & \cdots & \Delta^4 y_{-2} & \cdots & & \cdots & \Delta^6 y_{-3} \; \cdots \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
& & \Delta y_0 & & & & \Delta^3 y_{-1} & & & & \Delta^5 y_{-2} & & \Delta^7 y_{-3}
\end{array}$$

It is used to interpolate the values of y for $0 < p < 1$.

7.3 GAUSS'S BACKWARD INTERPOLATION FORMULA

Gregory–Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \tag{7.9}$$

where $p = \dfrac{x - x_0}{h}$.

We have,

$$\Delta y_0 = \Delta E y_{-1} = \Delta(1+\Delta) y_{-1} = \Delta y_{-1} + \Delta^2 y_{-1} \tag{7.10}$$

Similarly,

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \tag{7.11}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \text{ etc.} \tag{7.12}$$

Also,

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2} \tag{7.13}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.} \tag{7.14}$$

Substituting for Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0 \ldots$ from Eqns (7.10) – (7.12) in Eqn (7.9), we get

$$y_p = y_0 + p\,(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}$$

$$(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

$$= y_0 + p\,\Delta y_{-1} + \left\{p + \frac{p(p-1)}{2!}\right\}\Delta^2 y_{-1} + \left\{\frac{p(p-1)}{2!} + \frac{p(p-1)(p-2)}{3!}\right\}$$

$$\Delta^3 y_{-1} + \left\{\frac{p(p-1)(p-2)}{3!} + \frac{p(p-1)(p-2)(p-3)}{4!}\right\}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + p\,\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + p\,\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}$$

$$\{\Delta^3 y_{-2} + \Delta^4 y_{-2}\} + \frac{(p+1)p(p-1)(p-2)}{4!}\{\Delta^4 y_{-2} + \Delta^5 y_{-2}\}$$

$+ \cdots$ [using Eqns (7.13) and (7.14)]

$$\therefore\; y_p = y_0 + p\,\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \tag{7.15}$$

Eqn (7.15) is called *Gauss's backward interpolation formula.*
In central differences notation, this formula can be written as

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{-1/2}$$

$$+ \frac{(p+2)(p+1)p(p+1)}{4!}\delta^4 y_0 + \cdots \tag{7.16}$$

Note : Eqn (7.15) involves odd differences above the central line and even differences on the central line as shown below:

$$\begin{array}{ccccccccccc}
 & & \Delta y_{-1} & & & & \Delta^3 y_{-1} & & & & \Delta^5 y_{-3} \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
y_0 & \ldots & & & \Delta^2 y_{-1} & \ldots & & & \Delta^4 y_{-2} & \ldots & & & \Delta^6 y_{-3} \ldots \text{central line}
\end{array}$$

It is useful when $-1 < p < 0$.

7.4 STIRLING'S FORMULA

Taking the mean of Gauss's forward interpolation and backward interpolation formulae, that is, Eqns (7.7) and (7.15), we get the following equation:

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{1}{2}\left[\frac{p(p-1)}{2!} + \frac{(p+1)p}{2!}\right]\Delta^2 y_{-1}$$

$$+ \frac{(p+1)p(p-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \frac{1}{2}\left[\frac{(p+1)p(p-1)(p-2)}{4!} + \frac{(p+2)(p+1)p(p-1)}{4!}\right]\Delta^4 y_{-2} + \cdots$$

or $$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \tag{7.17}$$

The above equation is called *Stirling's formula.*

In the central differences notation, Eqn (7.17) takes the form

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0$$

$$+ \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \cdots \tag{7.18}$$

where $\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\delta y_0$

$$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\delta^3 y_0 \text{ etc.}$$

Note: This formula involves the means of the odd differences just above and below the central line and even differences on the central line as shown below:

$$y_0 \cdots \begin{bmatrix}\Delta y_{-1}\\ \Delta y_0\end{bmatrix} \cdots \Delta^2 y_{-1} \cdots \begin{bmatrix}\Delta^3 y_{-2}\\ \Delta^3 y_{-1}\end{bmatrix} \cdots \Delta^4 y_{-2} \cdots \begin{bmatrix}\Delta^5 y_{-3}\\ \Delta^5 y_{-2}\end{bmatrix} \cdots \Delta\, 6y_{-3} \cdots$$

This is useful when $-\frac{1}{2} < p < \frac{1}{2}$ and a good estimate when $-\frac{1}{4} < p < \frac{1}{4}$.

7.5 BESSEL'S FORMULA

Once again, consider Gauss's forward interpolation fomula given by Eqn (7.7). We know that

$$y_1 - y_0 = \Delta y_0 \quad \therefore\ y_0 = y_1 - \Delta y_0 \tag{7.19}$$

$$\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1} \quad \therefore\ \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1} \tag{7.20}$$

similarly,

$$\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2} \text{ etc.} \tag{7.21}$$

So, Eqn (7.7) can be written as

$$y_p = \left(\frac{y_0}{2} + \frac{y_0}{2}\right) + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1}$$

$$+ \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \ldots \tag{7.22}$$

Substituting Eqns (7.19) and (7.20) in Eqn (7.22), we get

$$y_p = \frac{y_0}{2} + \frac{1}{2}(y_1 - \Delta y_0) + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}$$

$$(\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$= \frac{1}{2}(y_0 + y_1) + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^2 y_0)$$

$$+ \frac{p(p-1)}{2!}\left(-\frac{1}{2} + \frac{p+1}{3!}\right)\Delta^3 y_{-1} + \cdots$$

$$= \frac{1}{2}(y_0 + y_1) + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \cdots \tag{7.23}$$

Eqn (7.23) is known as *Bessel's formula.*

In central differences notation, Eqn (7.23) becomes

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\mu\delta^2 y_{1/2} + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\delta^3 y_{1/2}$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\mu\delta^4 y_{1/2} + \cdots \quad (7.24)$$

where $\qquad \frac{1}{2}(\Delta^2 y_{-1} + \Delta^2 y_0) = \mu\delta^2 y_{1/2}$

$\frac{1}{2}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) = \mu\delta^4 y_{1/2}$ etc.

Note

1. This formula involves odd differences below the central line and means of the even differences on and below the central line as shown below:

$$\text{Central line}\cdots\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}\cdots \Delta y_0 \cdots \begin{bmatrix} \Delta^2 y_{-1} \\ \Delta^2 y_0 \end{bmatrix}\cdots \Delta^3 y_{-1} \cdots \begin{bmatrix} \Delta^2 y_{-1} \\ \Delta^2 y_{-2} \end{bmatrix}\cdots$$

2. If $p = \frac{1}{2}$, the coefficients of the all odd differences are zero. Hence, we have

$$y_{1/2} = \frac{1}{2}(y_0 + y_1) - \frac{1}{8}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) \frac{3}{128}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)$$

$$-\frac{5}{1024}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \cdots$$

which is the special case of Bessel's formula and is known as the formula for *interpolating to halves*. It is best suited to compute the values of the function midway between two given values.

7.6 LAPLACE–EVERETT'S FORMULA

Consider the following equations:

$$\Delta y_0 = y_1 - y_0,\ \Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1},\ \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}.\ \text{etc.}$$

Substituing these in Eqn (7.7), i.e. Gauss's forward interpolation formula, we get the following equation:

$$y_p = y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!}\,\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\,(\Delta^2 y_0 - \Delta^2 y_{-1})$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\,\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}$$

$$(\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \cdots$$

$$= \quad (1-p)\,y_0 + py_1 + \left[\frac{p(p-1)}{2!} - \frac{(p+1)p(p-1)}{3!}\right]\Delta^2 y_{-1}$$

$$+\frac{(p+1)p(p-1)}{3!}\,\Delta^2 y_0 + \frac{(p+1)p(p-1)(p-2)}{4!}\left[1 - \frac{(p+2)}{5!}\right]\Delta^4 y_{-2}$$

$$+\frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\,\Delta^4 y_{-1} + \cdots$$

$$= \quad (1-p)\,y_0 + py_1 - \frac{p(p-1)(p-2)}{3!}\,\Delta^2 y_{-1}$$

$$+\frac{(p+1)p(p-1)}{3!}\,\Delta^2 y_0 - \frac{(p+1)p(p-1)(p-2)(p-3)}{5!}\,\Delta^4 y_{-2}$$

$$+\frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\,\Delta^4 y_{-1} + \cdots$$

or $$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\,\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\,\Delta^4 y_{-2} + \cdots$$

$$+py_1 + \frac{p(p^2-1^2)}{3!}\,\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\,\Delta^4 y_{-1} + \cdots \quad (7.25)$$

where $p = 1 - q$. This is known as *Laplace–Everett's formula.*

Note Laplace–Everett's formula involves only even differences on and below the central line. It is convenient especially when using tables in which only even order differences are tabulated. This can be used when $0 < p < 1$. Accurate results are obtained when $p \le 3/4$.

7.7 RELATION BETWEEN BESSEL'S AND LAPLACE–EVERETT'S FORMULAE

There is a close relationship between Bessel's formula and Laplace–Everett's formula, and one can be deduced from the other by a suitable rearrangement. Starting with Bessel's formula,

$$y_p = \frac{1}{2}(y_0 + y_1) + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \cdots$$

$$y_p = \frac{1}{2}(y_0 + y_1) + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1} + \cdots \tag{7.26}$$

[keeping only upto 3rd differences]

Now, putting $\Delta y_0 = y_1 - y_0$, $\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$ in Eqn (7.26) we get,

$$y_p = \frac{1}{2}(y_0 + y_1) + \left(p - \frac{1}{2}\right)(y_1 - y_0) + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1}) + \cdots$$

$$= (1-p)\,y_0 + py_1 + \frac{(p+1)p(p-1)}{3!}\Delta^2 y_0$$

$$- \frac{p(p-1)(p-1)}{3!}\Delta^2 y_{-1} + \cdots$$

$$= qy_0 + \frac{(q+1)q(q-1)}{3!}\Delta^2 y_{-1} + \cdots + py_1$$

$$+ \frac{(p+1)p(p-1)}{3!}\Delta^2 y_0 + \cdots \tag{7.27}$$

which is Laplace–Everett's formula upto second defferences. Thus, *Bessel's formula truncated after third differences is equivalent to Laplace–Everette's formula truncated after second differnces.* We can also start from Laplace–Everett's formula and obtain Bessel's formula.

7.8 ADVANTAGES OF CENTRAL DIFFERENCE INTERPOLATION FORMULAE

The coefficients in the central difference formulae are smaller and converge faster than those in Gregory–Newton's formulae. The coefficients in the Stirling's formula decrease more rapidly than those of the Bessel's formula. And, in Bessel's formula the coefficients decrease more rapidly than those of Gregory–Newton's formulae. The right choice of an interpolation formula, however, depends on the position of the value to be interpolated in the given data. The following rules will be useful in selecting the interpolation formulae.

1. If interpolation is required near the beginning of the table then use Gregory–Newton forward interpolation formula.
2. If interpolation is required near the end of the table then use Gregory–Newton backward interpolation formula.
3. To interpolate near the centre of the table, use eigher Stirling's or Bessel's or Laplace–Everett's formula.

   Stirling's formula is to be preferred if $-\frac{1}{4} < p < \frac{1}{4}$.

   Bessel's or Everett's formulae are better suited for the condition $\frac{1}{4} < p < \frac{3}{4}$.

Example 7.1 Using Gauss's forward interpolation formula, find the value of log 337.5 from the following table :

x	310	320	330	340	350	360
$y_x = \log x$	2.4914	2.5051	2.5185	2.5315	2.5441	2.5563

Solution Let $x_0 = 330$, then $p = \dfrac{x-330}{10}$ since $h = 10$. Now the central difference table is as follows:

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
310	–2	2.4914					
			0.0138				
320	–1	2.5052		–0.0005			
			0.0133		0.0002		
330	0	**2.5185**		**–0.0003**		**–0.0003**	
			0.0130		**–0.0001**		**0.0004**
340	1	2.5315		–0.0004		0.0001	
			0.0126		0.0000		
350	2	2.5441		–0.0004			
			0.0122				
360	3	2.5563					

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p+2)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p+2)}{5!}\Delta^5 y_{-2} + \cdots$$

From the central difference table we get the following values:

$$x = 337.5,\ p = \frac{337.5 - 330}{10} = 0.75,\ y_0 = 2.5185$$

$\Delta y_0 = 0.0130$, $\Delta^2 y_{-1} = -0.0003$, $\Delta^3 y_{-1} = -0.0001$, $\Delta^4 y_{-2} = -0.0003$, $\Delta^5 y_{-2} = 0.0004$

$$\therefore\ y_{0.75} = 2.5185 + (0.75)(0.0130) + \frac{(0.75)(0.75-1)}{2!}(-0.0003) +$$

$$+ \frac{(0.75+1)(0.75)(0.75-1)}{3!}(-0.0001)$$

$$+ \frac{(0.75+1)(0.75)(0.75-1)(0.75-2)}{4!}(-0.0003)$$

$$+ \frac{(0.75+2)(0.75+1)(0.75)(0.75-1)(0.75-2)}{5!}(0.0004)$$

$$= 2.5185 + 9.75\times 10^{-3} + 2.8125\times 10^{-5} + 5.46875\times 10^{-6}$$

$$- 5.1269531\times 10^{-6} + 3.7597656\times 10^{-6}$$

$$= 2.5282822$$

$\therefore \log 337.5 = 2.5283$

Example 7.2 Interpolate by means of Gauss's backward interpolation formula, the sales of a concern for the year 1976, given that

Year	1940	1950	1960	1970	1980	1990
Sales (in lakhs of Rs)	17	20	27	32	36	38

Solution Taking 1970 as the origin and $h = 10$ years as one unit, the sales is to be found for $p = \dfrac{x - 1970}{10}$. The central difference table will be as follows:

x	p	y	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1940	–3	17					
			3				
1950	–2	20		4			
			7		–6		
1960	–1	27		–2		7	
			5		–1		–9
1970	0	32		–1		–2	
			4		–1		
1980	1	36		–2			
			2				
1990	2	38					

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-3}$$

$$+ \cdots$$

where $x = 1976$, $p = \dfrac{1976-1970}{10} = 0.6$, $y_0 = 32$, $\Delta y_{-1} = 5$, $\Delta^2 y_{-1} = -1$,
$\Delta^3 y_{-2} = 1$, $\Delta^4 y_{-2} = -2$, $\Delta^5 y_{-3} = -9$

$$\therefore y_{0.6} = 32 + (0.6)(5) + \frac{(0.6+1)(0.6)}{2!}(-1) + \frac{(0.6+1)(0.6)(0.6-1)}{3!}(1)$$

$$+\frac{(0.6+2)(0.6+1)(0.6)(0.6-1)}{4!}(-2)$$

$$+\frac{(0.6+2)(0.6+1)(0.6)(0.6-1)(0.6-2)}{5!}(-9)$$

$$= 32 + 3 - 4.8 - 0.064 + 0.0832 - 0.104832$$

$$= 30.114368$$

Therefore, the sales in the year 1976 is Rs 30.114368 lakhs.

Example 7.3 Use Stirling's formula to find y_{35} given that $y_{10} = 600$, $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$ *(Mysore B.E., 1987)*

Solution Take $x_0 = 30$, $h = 10$ $\therefore p = \dfrac{x-30}{10}$

Now the central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
10	−2	600				
			−88			
20	−1	512		15		
			−73		−35	
30	0	439		−20		145
			−93		10	
40	1	346		−10		
			−103			
50	2	243				

Thus, at $x = 35$, $p = \dfrac{35-30}{10} = 0.5$, $y_0 = 439$, $\Delta y_0 = -93$, $\Delta y_{-1} = -73$,
$\Delta^2 y_{-1} = -20$, $\Delta^3 y_{-1} = 10$, $\Delta^3 y_{-2} = -35$ and $\Delta^4 y_{-2} = 145$

Substituting the above values in Eqn (7.17) (Stirling's formula), we get

$$y_{0.5} = 435 + (0.5)\left[\frac{(-93)+(-73)}{2}\right] + \frac{(0.5)^2}{2!}(-20)$$

$$+ \frac{(0.5)[(0.5)^2-1]}{3!}\left[\frac{10+(-35)}{2}\right] + \frac{(0.5)^2[(0.5)^2-1]}{4!}(145)$$

$$= 435 - 41.5 - 2.5 + 0.78125 - 1.1328125$$

$$= 390.64844$$

$$\therefore y_{35} \simeq 390.648$$

Example 7.4 Apply Bessel's formula to obtain y_{25} given that $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$. *(Mysore B.E., 1987)*

Solution Taking $x_0 = 24$, $p = \dfrac{x-24}{4}$, where $h = 4$. The central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$
20	−1	2854			
			308		
24	0	3162		74	
			382		−8
28	1	3544		66	
			448		
32	2	3992			

From the table we get the following values:

$$x = 25, \quad p = \frac{25-24}{4} = 0.25, \quad y_0 = 3162, \quad \Delta y_0 = 382$$

$$\Delta^2 y_{-1} = 74, \; \Delta^2 y_0 = 66, \; \Delta^3 y_{-1} = -8.$$

Putting the above values in Eqn (7.23) (Bessel's formula) we get the following equation.

$$y_{0.25} = 3162 + (0.25)(382) + \frac{(0.25)(0.25-1)}{2!}\left\{\frac{74+66}{2}\right\}$$

$$+ \frac{(0.25-0.5)(0.25)(0.25-1)}{3!}(-8)$$

$$= 3162 + 95.5 - 0.65625 - 0.0625 = 3256.7813$$

$$\therefore y_{25} = 3256.7813$$

Example 7.5 Use Laplace–Everett's formula to obtain $f(1.15)$ given that $f(1) = 1.000, f(1.10) = 1.049, f(1.20) = 1.096 f(1.30) = 1.140$

Solution Let $x_0 = 1.10, h = 0.1, p = \frac{x - x_0}{h} = \frac{x-1.10}{0.1}$

The central difference table is

x	p	$y_p = f(x)$	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$
1	–1	1.000			
			0.49		
1.10	0	1.049		–0.002	
			0.047		– 0.001
1.20	1	1.096		–0.003	
			0.044		
1.30	2	1.140			

Laplace–Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \cdots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \cdots$$

Here, $p = \frac{1.15-1.10}{0.1} = 0.5$ since $x = 1.15$

$\therefore q = 1-p = 1-0.5 = 0.5, \; y_0 = 1.049, \; y_1 = 1.096, \; \Delta^2 y_{-1} = 0.002,$ $\Delta^2 y_0 = -0.003$

$$\therefore y_{0.5} = (0.5)(1.049) + \frac{(0.5)(0.25-1)}{3!}(-0.002) + (0.5)(1.096)$$

$$+ \frac{(0.5)(0.25-1)}{3!}(0.003)$$

$$= 0.5245 + 1.25 \times 10^{-4} + 0.548 + 1.875 \times 10^{-4}$$

$$= 1.0728125$$

$$\therefore f(1.15) = 1.0728125$$

EXERCISE 7.1

1. The values of annuities for certain ages are given for the following ages. Find the annuity at age 27½ using Gauss's forward interpolation formula.

Age	25	26	27	28	29
Annuity	16.195	15.919	15.630	15.326	15.006

2. Using Gauss's forward interpolation formula, find y at $x = 1.7489$ given that

x	1.72	1.73	1.74	1.75	1.76	1.77	1.78
y	0.1791	0.1773	0.1775	0.1738	0.1720	0.1703	0.1686

3. Find $\sqrt{12516}$ using Gauss's backward interpolation foumula given that $\sqrt{12500} = 111.8033$, $\sqrt{12510} = 111.8481$, $\sqrt{12520} = 111.8928$ and $\sqrt{12530} = 111.9374$.

4. Find sin 45° using Gauss's backward interpolation formula given that sin 20° = 0.342, sin 30° = 0.502, sin 40° = 0.642, sin 50° = 0.766, sin 60°= 0.866, sin 70° = 0.939, sin 80° = 0.984.

5. Employ Bessel's formula to find the value of y at $x = 1.95$ given that

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
y	2.979	3.144	3.283	3.391	3.463	3.997	4.491

6. Use Bessel's formula to find the value of y when $x = 3.75$ given the table:

x	2.5	3.0	3.5	4.0	4.5	5.0
y	24.145	22.043	20.225	18.644	17.262	16.047

7. Given cos (0.8050) = 0.6931, cos (0.8055) = 0.6928, cos (0.8060) = 0.6924, cos (0.8065) = 0.6920, cos (0.8070) = 0.6917, cos (0.8075) = 0.6913, and cos (0.8080) = 0.6909, find cos (0.806595) using Stirling's formula.

8. Apply Stirling's formula to find a polynomial of degree four which takes

x	1	2	3	4	5
y	1	−1	1	−1	1

9. Use Laplace–Everett's formula to find log 337.5 given that log 310 = 2.4913, log 320 = 2.5051, log 330 = 2.5185, log 340 = 2.5315, log 350 = 2.5441 and log 360 = 2.5563.

10. Find y at $x = 34$ using Laplace – Everett's formula given the table.

x	20	25	30	35	40
y	11.4699	12.7834	13.7648	14.4982	15.0463

11. The following table gives the values of the probability integral $f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx$ for certain values of x. Find the value of their integral at $x = 0.5437$ using (i) Stirling's formula, (ii) Bessel's formula, and (iii) Everett's formula.

x	0.51	0.52	0.53	0.54	0.55
y	0.5292437	0.5378987	0.5464641	0.5549392	0.5633233

x	0.56	0.57
y	0.5716157	0.5798158

ANSWERS

1. 15.480
2. 0.1739
3. 111.8749
4. 0.707
5. 3.347
6. 19.4074
7. 0.6919
8. $\frac{1}{2}\{2(x-3)^4 - 8(x-3)^2 + 3\}$
9. 2.5283
10. 14.3684
11. 0.55805196, 0.55805196, 0.55805195.

CHAPTER 8

Interpolation with Unequal Intervals

8.1 INTRODUCTION

So far we have studied the application of interpolation formulae where the arguments are equally spaced. In this chapter, we will study interpolating the values when the arguments are not equally spaced by introducing the concept of divided differences.

Let $f(x_0)$, $f(x_1)$, $f(x_2)$, . . . , $f(x_n)$ be the entry values of the function $y = f(x)$ at the arguments $x_0, x_1, x_2, \ldots, x_n$ which are not equally spaced (i.e., interval of differencing is not constant).

Divided difference can now be defined as the difference between two successive values of the entry divided by the difference between the corresponding values of the argument. Therefore, the first divided difference of $f(x)$ for the arguments x_0 and x_1 is defined as

$\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$ and is denoted $f(x_0,x_1)$ or $[x_0,x_1]$ or $\underset{x_1}{\Delta} f(x_0)$.

i.e.
$$f(x_0,x_1) = [x_0,x_1] = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1)-f(x_0)}{x_1-x_0} \tag{8.1}$$

Similarly,

$$f(x_1,x_2) = [x_1,x_2] = \underset{x_2}{\Delta} f(x_1) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$f(x_2,x_3) = [x_2,x_3] = \underset{x_3}{\Delta} f(x_2) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

and so on.

The second divided difference of $f(x)$ for three arguments, x_0, x_1 and x_2 is defined as

$$f(x_0,x_1,x_2) = [x_0,x_1,x_2] = \frac{f(x_1,x_2) - f(x_0,x_1)}{x_2 - x_0} \tag{8.2}$$

which is also denoted as $\underset{x_1,x_2}{\Delta^2} f(x)$.

The third divided difference of $f(x)$ for four arguments, x_0, x_1, x_2 and x_3 is defined as

$$f(x_0, x_1, x_2, x_3) = [x_0, x_1, x_2, x_3] = \frac{f(x_1,x_2,x_3) - f(x_0,x_1,x_2)}{x_3 - x_0} \tag{8.3}$$

Similarly, we can define higher divided differences.

Thus, to define a first divided difference we need the functional values corresponding to two arguments, to define a second divided difference we need two first divided differences of two arguments, one of which is common, and so on. The quantities in Eqns (8.1)–(8.3) are called first order, second order and third order divided differences, respectively. The divided difference table is as follows:

Argument x	Entry $f(x)$	1st divided difference $\Delta f(x)$	2nd divided difference $\Delta^2 f(x)$	3rd divided difference $\Delta^3 f(x)$
x_0	$f(x_0)$			
		$f(x_0,x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$
x_3	$f(x_3)$		$f(x_2, x_3, x_4)$	
		$f(x_3, x_4)$		
x_4	$f(x_4)$			

Note: Here, in any difference column the value of the divided difference between two adjecent values is obtained by dividing the difference between the two adjecent values immediately in the previous column by the difference between the arguments against the two entries on the two diagonals passing through the divided difference.

Example 8.1 Construct a divided difference table for the following data

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	0
7	294		$\frac{202-97}{10-5}=21$		
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
10	900		$\frac{310-202}{11-7}=27$		0
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

Example 8.2 If a, b, c, d are the arguments of $f(x) = \frac{1}{x}$, show that

$$f(a, b, c, d) = -\frac{1}{abcd}.$$

Solution Given $f(x) = \frac{1}{x}$

$$\therefore f(a, b) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = \frac{1}{ab}$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c-a}$$

$$= \frac{\left(-\frac{1}{bc}\right) - \left(-\frac{1}{ab}\right)}{c-a} = \frac{1}{abc}$$

$$f(a, b, c, d) = \frac{f(b,c,d) - f(a,b,c)}{d-a}$$

$$= \frac{\frac{1}{bcd} - \frac{1}{abc}}{d-a} = - \frac{1}{abcd}$$

8.2 PROPERTIES OF DIVIDED DIFFERENCES

1. The divided differences are symmetrical in all their arguments, i.e. the value of any divided difference is independent of the order of the arguments.

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_1, x_0)$$

Also, $$f(x_0, x_1) = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \tag{8.4}$$

which shows that $f(x_0, x_1)$ is symmetric with respect to the arguments x_0, x_1.

$$\text{Now } f(x_0, x_1, x_2) = \frac{f(x_1,x_2) - f(x_0,x_1)}{x_2 - x_1}$$

$$\text{RHS} = \frac{1}{x_2 - x_0}\left[\left\{\frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1}\right\} - \left\{\frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}\right\}\right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \tag{8.5}$$

Eqn (8.5) shows that $f(x_0, x_1, x_2)$ is symmetric with respect to the arguments x_0, x_1, x_2.

Similarly, it can be shown that

$$f(x_0, x_1, x_2, \ldots, x_n) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

$$+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)}+\cdots$$

$$+\frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}$$

That is, $f(x_0, x_1, x_2, \ldots x_n)$ is symmetric with respect to the arguments $x_0, x_1, x_2 \ldots x_n$.

2. The divided difference (of any order) of the sum or difference of two functions is equal to the sum or difference of the corresponding separate divided differences; that is the operators Δ, Δ^2, . . . etc. are linear.

Let $f(x)$ and $g(x)$ be two functions with arguments x_0, x_1. Then,

$$\begin{aligned}\Delta[f(x)\pm g(x)] &= \frac{[f(x_1)\pm g(x_1)]-[f(x_0)\pm g(x_0)]}{x_1-x_0}\\ &= \frac{[f(x_1)-f(x_0)]}{x_1-x_0}\pm\frac{[g(x_0)-g(x_0)]}{x_1-x_0}\\ &= \Delta f(x)\pm\Delta g(x)\end{aligned}$$

Similarly, we can prove it for any higher order difference, i.e.

$$\Delta^r[f(x)\pm g(x)] = \Delta^r f(x)\pm\Delta^r g(x)]$$

3. The divided difference of the product of a constant and a function is equal to the product of the constant and the divided difference of the function, i.e. $\Delta\{cf(x)\} = c\Delta f(x)$.

$$\text{By definition, } \Delta\{cf(x)\} = \frac{cf(x_1)-cf(x_0)}{x_1-x_0}$$

$$= c\left\{\frac{f(x_1)-f(x_0)}{x_1-x_0}\right\} = c\Delta f(x)$$

This is also true for any higher order difference.

4. The nth divided differences of a polynomial of nth degree are constants.

Let $f(x) = x^n$ be the polynomial, where n is a positive integer. Then,

$$f(x_0, x_1) = \frac{x_1^n - x_0^n}{x_1 - x_0} = x_1^{n-1} + x_0 x_1^{n-2} + x_0^2 x_1^{n-3} + \cdots + x_0^{n-1}$$

RHS is a polynomial function of $(n-1)$th degree, symmetrical in x_0 and x_1

with leading coefficient 1.

Again, $f(x_0, x_1, x_2) = \dfrac{[f(x_1,x_2) - f(x_0,x_1)]}{x_2 - x_0}$

$$\text{RHS} = \frac{[x_2^{n-1} + x_1 x_2^{n-2} + \cdots + x_1^{n-1}] - [x_0^{n-1} + x_1 x_0^{n-2} + \cdots + x_1^{n-1}]}{[x_2 - x_1]}$$

$$= \frac{x_2^{n-1} - x_0^{n-2}}{x_2 - x_0} + \frac{x_1(x_2^{n-2} - x_0^{n-2})}{x_2 - x_0} + \cdots + \frac{x_1^{n-2}(x_2 - x_0)}{x_2 - x_0}$$

$$= (x_2^{n-2} + x_0 x_2^{n-3} + \cdots + x_0^{n-2}) + x_1(x_2^{n-3} + x_0 . x_2^{n-4} + \cdots + x_0^{n-3}) + \cdots + x_1^{n-2}$$

= a polynomial of $(n-2)$th degree, symmetrical in x_0, x_1, and x_2 with leading coefficient 1.

Proceeding in this way, the pth order divided difference of x^n will be a polynomial of $(n-p)$th degree, symmetrical in $x_0, x_1, x_2, \ldots x_p$ with leading coefficient 1.

Hence, the nth order divided difference of x^n will be a polynomial of $(n-n)$th degree, i.e. of order 0 symmetrical in $x_0, x_1, x_2, \ldots x_n$ with leading coefficient 1.

i.e. it will be a constant = 1 $\therefore\ \Delta^n x^n = 1$ and $\Delta^{n+i} x^n = 0$ for $i = 1, 2, \ldots$

Hence $\quad \Delta^n \{a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n\}$
$\quad = a_0 . 1 + a_1 . 0 + a_2 . 0 + \cdots + 0 = a_0.$

8.3 RELATION BETWEEN DIVIDED DIFFERENCES AND FORWARD DIFFERENCES

Let the arguments $x_0, x_1, x_2, \ldots, x_n$ be equally spaced.
i.e. $x_{i+1} - x_i = h$ (say) for $i = 0, 1, 2, \ldots, n-1$

Then $f(x_0, x_1) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{\Delta f(x_0)}{h}$

$$f(x_0, x_1, x_2) = \frac{f(x_1,x_2) - f(x_0,x_1)}{x_2 - x_0} = \frac{\dfrac{\Delta f(x_1)}{h} - \dfrac{\Delta f(x_0)}{h}}{2h} = \frac{1}{2h^2}\Delta^2 f(x_0)$$

Similarly, $f(x_0, x_1, x_2, x_3) = \dfrac{1}{3!h^3}\Delta^3 f(x_0)$

$$\begin{array}{cccc} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{array}$$

$$f(x_0, x_1, x_2, \dots x_n) = \frac{1}{n!h^n}\Delta^n f(x_0)$$

8.4 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$ be the values of $f(x)$ corresponding to the non-equally spaced arguments $x_0, x_1, x_2, \dots x_n$. From the definition of divided differences, we have

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\therefore \quad f(x) = f(x_0) + (x - x_0) f(x, x_0) \tag{8.6}$$

Again, $$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\therefore \quad f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$$

Substituting $f(x, x_0)$ in Eqn (8.6), we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x, x_0, x_1) \tag{8.7}$$

Also, $$f(x, x_0, x_1, x_2) = \frac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$$

$$\therefore \quad f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2)$$

Substituting this in Eqn (8.7), we get

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x, x_0, x_1, x_2) \end{aligned} \tag{8.8}$$

Proceeding in the same manner, we get

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \\ &\dots \dots \dots \dots \dots \\ &+ (x - x_0)(x - x_1) \dots (x - x_{n-1}) f(x_0, x_1, x_2, \dots, x_n) \\ &+ (x - x_0)(x - x_1) \dots (x - x_n) f(x, x_0, x_1, \dots, x_n) \end{aligned} \tag{8.9}$$

If $f(x)$ is a polynomial of nth degree, then the $(n+1)$th divided difference will be zero. That is, $f(x, x_0, x_1, \dots, x_n) = 0$.

Hence Eqn (8.9) becomes

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) +$$
$$\dots \dots \dots \dots$$
$$+ (x - x_0)(x - x_1) \dots (x - x_{n-1}) f(x_0, x_1, x_2, \dots, x_n) \quad (8.10)$$

Eqn (8.10) is the *Newton's interpolation formula for unequal intervals* which, in general, is also called *Newton's divided difference interpolation formula for unequal intervals.*

Deduction From the above, we can obtain *Gregory–Newton forward interpolation formula with equal intervals.*

As stated in Section 8.3, if the arguments $x_0, x_1, \dots, x_n$ are equally spaced, then

$$f(x_0, x_1) = \frac{1}{h} \Delta f(x_0); \qquad f(x_0, x_1, x_2) = \frac{1}{2! h^2} \Delta^2 f(x_0)$$

$$f(x_0, x_1, x_2, x_3) = \frac{1}{3! h^3} \Delta^3 f(x_0)$$

$$\dots \dots \dots \dots$$

$$f(x_0, x_1, x_2, \dots, x_n) = \frac{1}{n! h^n} \Delta^n f(x_0)$$

Setting $x - x_0 = ph$, we get

$$x - x_1 = (x - x_0) - (x_1 - x_0) = ph - h = (p - 1) h$$
$$(x - x_2) = (p - 2) h \text{ and so on.}$$

Substituting these values in Eqn (8.10), we get

$$f(x) = f(x_0 + ph) = f(x_0) + ph \frac{\Delta f(x_0)}{h} + \frac{ph(p-1)h}{2! h^2} \Delta^2 f(x_0)$$

$$+ \frac{ph(p-1)h(p-2)h}{3! h^3} \Delta^3 f(x_0) + \cdots$$

$$+ \frac{ph(p-1)h \dots \{p-(n-1)\}h}{n! h^n} \Delta^n f(x_0)$$

$$= f(x_0) + p \Delta f(x_0) + \frac{p(p-1)}{2!} \Delta^2 f(x_0) \frac{p(p-1)(p-2)}{3!} \Delta^3 f(x_0)$$

$$+ \dots + \frac{p(p-1) \dots [p-(n-1)]}{n!} \Delta^n f(x_0) \quad (8.11)$$

which is *Newton's forward interpolation formula.*

Example 8.3 Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$ $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$. Find by using Newton's divided difference formula, the value of $\log_{10} 656$. *(Kerala 1989)*

Solution The divided difference table is

x	$f(x) = \log_{10} x$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
654	2.8156			
		$\frac{2.8182 - 2.8156}{658 - 654}$ $= 6.5 \times 10^{-4}$		
658	2.8182		$\frac{(7 - 6.5) \times 10^{-4}}{659 - 654}$ $= 1 \times 10^{-5}$	
		$\frac{2.8189 - 2.8182}{659 - 658}$ $= 7 \times 10^{-4}$		$\frac{(-1.6 - 1) \times 10^{-5}}{661 - 654}$ $= -\cancel{1.3} \times 10^{-6}$ 3.7
659	2.8189		$\frac{(6.5 - 7) \times 10^{-4}}{661 - 659}$ $= 1.6 \times 10^{-5}$	
		$\frac{2.8202 - 2.8189}{661 - 659}$ $= 6.5 \times 10^{-4}$		
661	2.8202			

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

Here, $x_0 = 654, f(x_0) = 2.8156$, $x_1 = 658$, $x_2 = 659$,

$f(x_0, x_1) = \Delta f(x_0) = 0.00065, f(x_0, x_1, x_2) = \Delta^2 f(x_0) = 0.00001$
and $f(x_0, x_1, x_2, x_3) = -0.000013$, and we require $f(656)$

$\therefore$ At $x = 656$

$$\begin{aligned} f(656) &= 2.8156 + (656 - 654)(0.00065) \\ &\quad + (656 - 654)(656 - 658)(0.00001) \\ &\quad + (656 - 654)(656 - 658)(656 - 659)(-0.000013) \\ &= 2.8156 + 0.0013 - 0.00004 - 0.000156 \\ &= 2.816704 \end{aligned}$$

$\therefore \log_{10} 656 = 2.8167$

Example 8.4 Find a polynomial satisfied by the following table:

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Solution The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		$\frac{33-1245}{-1+4}$ $=-404$			
-1	33		$\frac{-28+404}{0+4}$ $=94$		
		$\frac{5-33}{0+1}$ $=-28$		$\frac{10-94}{2+4}$ $=-14$	
0	5		$\frac{2+28}{2+1}$ $=10$		$\frac{13+14}{5+4}$ $=3$
		$\frac{9-5}{2-0}$ $=2$		$\frac{88-10}{5+1}$ $=13$	
2	9		$\frac{442-2}{5-0}$ $=88$		
		$\frac{1335-9}{5-2}$ $=442$			
5	1335				

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 f(x_0) \qquad \text{(i)}$$

Here, $x_0 = -4, x_1 = -1, x_2 = 0, x_3 = 2$

$f(x_0) = 1245$, $\Delta f(x_0) = -404$, $\Delta^2 f(x_0) = 94$, $\Delta^3 f(x_0) = -14$, $\Delta^4 f(x_0) = 3$

Substituting these in Eqn (i), we get

$$f(x) = 1245 - (x + 4)(-404) + (x + 4)(x + 1)(94)$$
$$+ (x + 4)(x + 1)\,x\,(-14) + (x + 4)(x + 1)\,x\,(x - 2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

Example 8.5 Using the following table find $f(x)$ as a polynomial in powers of $(x - 6)$.

x	-1	0	2	3	7	10
$f(x)$	-11	1	1	1	141	561

Solution The divided difference table is as given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-11			
		$\frac{1+11}{0+1}=12$		
0	1		$\frac{0-12}{2+1}=-4$	
		$\frac{1-1}{2-0}=0$		$\frac{0+4}{3+1}=1$
2	1		0	
		$\frac{1-1}{3-1}=0$		$\frac{7+0}{7-0}=1$
3	1		$\frac{35-0}{7-2}=7$	
		$\frac{141-1}{7-3}=35$		$\frac{15-7}{10-2}=1$
7	141		$\frac{140-35}{10-3}=15$	
		$\frac{561-141}{10-7}=140$		$\frac{a-15}{6-3}=1$
10	561		$\frac{b-140}{6-7}=a$	
………	………	………	………	………
		$\frac{c-561}{6-10}=b$		$\frac{a'-a}{6-7}=1$
6	c		$\frac{b'-b}{6-10}=a'$	
		b'		$\frac{a''-a'}{6-10}=1$
6	c		a''	
		b''		
6	c			

Since the third differences are constant, the polynomial $f(x)$ will be of degree 3. Hence the argument 6 should appear three times successively in the table. The extended table has been shown below the dotted line.

Now, from the table,

$$\frac{a-15}{6-3} = 1 \Rightarrow a = 18$$

$$\frac{a'-a}{6-7} = 1 \Rightarrow a' = a - 1 = 17$$

$$\frac{a''-a'}{6-10} = 1 \Rightarrow a'' = a' - 4 = 13$$

$$\frac{b-140}{6-7} = a \Rightarrow b = 140 - a = 122$$

$$\frac{b'-b}{6-10} = a' \Rightarrow b' = b - 4a' = 54$$

and $$\frac{c-561}{6-10} \Rightarrow c = 561 - 4b = 73$$

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0)$$

Taking $x_0 = x_1 = x_2 = x_3 = 6$, we get

$$f(x) = f(6) + (x - 6)\,b' + (x - 6)^2\,a'' + (x - 6)^3\,(1)$$
$$= 73 + 54(x - 6) + 13\,(x - 6)^2 + (x - 6)^3$$

EXERCISE 8.1

1. If $f(x) = x^{-2}$ show that $f(a, b, c, d) = -\dfrac{(abc + bcd + acd + abd)}{a^2b^2c^2d^2}$
2. If $f(x) = x^3$ show that $f(a^3, b^3, c^3) = a + b + c$.
3. If $f(x) = x^{-1}$ show that $f(x_0, x_1, \ldots, x_n) = \dfrac{(-1)^n}{x_0, x_1, \ldots, x_n}$
4. If $f(x) = x^3 - 9x^2 + 17x + 6$, compute $f(-1, 1, 2, 3)$.
5. The following table gives some relation between steam pressure and temperature. Find the pressure at 372.1°

Temp.°C	361°	367°	378°	387°	399°
Pressure (kgf/cm²)	154.9	167.9	191	212.5	244.2

6. Using Newton's divided difference formula, evaluate $f(8)$ and $f(15)$ given that

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

7. The observed values of a function are 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the argument, respectively. What is the best estimate for the value of the function at position 6.

8. Fit a polynomial of third degree to the following data using Newton's divided difference method.

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	9

9. If $f(0) = -18, f(1) = 0, f(3) = 0, f(5) = -248, f(6) = 0, f(9) = 13104$, find $f(x)$.

10. From the following table, obtain $f(x)$ as a polynomial in powers of $(x-5)$ using Newton's method.

x	0	2	3	4	5	6
$f(x)$	4	26	58	112	466	922

ANSWERS

4. 1 5. 177.84 6. 448, 3150 7. 147

8. $f(x) = x^3 - 9x^2 + 21x + 1$ 9. $f(x) = x^5 - 9x^4 + 18x^3 - x^2 + 9x - 18$

10. $f(x) = 194 + 98(x-5) + 17(x-5)^2 + (x-5)^3$

8.5 LAGRANGE'S INTERPOLATION FORMULA

Let $y = f(x)$ be a function which takes the $(n+1)$ values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$. Now $f(x)$ can be represented as a polynomial of nth degree in x.

Let this polynomial be of the form

$$\begin{aligned} y = f(x) = {} & a_0(x-x_1)(x-x_2)\ldots(x-x_n) \\ & + a_1(x-x_0)(x-x_2)\ldots(x-x_n) \\ & + a_2(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n) \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \\ & + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \end{aligned} \tag{8.12}$$

Putting $x = x_0, y = y_0$ in Eqn (8.12), we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

$$\therefore \qquad a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

Again putting $x = x_1$ and $y = y_1$ in Eqn (8.12), we get

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}$$

Proceeding on the same lines, we get

$$a_3 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_n)}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2, \ldots a_n$ in Eqn (8.12), we get.

$$\begin{aligned} y = f(x) = {} & \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} y_0 \\ & + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_2 - x_n)} y_1 \\ & + \ldots \quad \ldots \quad \ldots \quad \ldots \\ & + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} y_n \end{aligned} \tag{8.13}$$

This is known as *Lagrange's interpolation formula.*

Note

1. This formula can be used irrespective of whether the values $x_0, x_1, x_2, \ldots, x_n$ are equally spaced or not.
2. It is simple and easy to remember but its application is not speedy.
3. The main drawback of it is that if another interpolation value is inserted, then the interpolation coefficients are required to be recalculated.

Corollary: Dividing both sides of Eqn (8.13) $(x-x_0)(x-x_1)\ldots(x-x_n)$, we get

$$\frac{f(x)}{(x-x_0)(x-x_1)\ldots(x-x_n)} =$$

$$\frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\frac{1}{x-x_0}$$

$$+\frac{y_1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\frac{1}{x-x_1}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$+\frac{y_n}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\frac{1}{x-x_n}$$

Example 8.6 Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the values of x and y are given as below:

x	5	6	9	11
y	12	13	14	16

(*Gulbarga 1993*)

Solution Here, $x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11, y_0 = 12, y_1 = 13, y_2 = 14$ and $y_3 = 16$.By Lagrange's formula, i.e. Eqn (8.13),

$$y = f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+\frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

Putting $x = 10$ in the above equation we get

$$y = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= 2 - 4.3333333 + 11.666667 + 5.3333333$$

$$= 14.666667$$

Example 8.7 Use Lagrange's formula to find the form of $f(x)$, given:

x	0	2	3	6
$f(x)$	648	704	729	792

(*Bangalore 1987*)

Solution Here, $x_0 = 0, x_1 = 2, x_2 = 3, x_4 = 6$,
$y_0 = 648, y_1 = 704, y_2 = 729$ and $y_3 = 792$

By Lagrange's formula, i.e. Eqn (8.13),

$$y = f(x) = \frac{(x-2)(x-3)(x-6)}{(0-2)(0-3)(0-6)}(648) + \frac{(x-0)(x-3)(x-6)}{(2-0)(2-3)(2-6)}(704)$$

$$+ \frac{(x-0)(x-2)(x-6)}{(3-0)(3-2)(3-6)}(729) + \frac{(x-0)(x-2)(x-3)}{(6-0)(6-2)(6-3)}(792)$$

$$\therefore f(x) = (x^3 - 11x^2 + 36x - 36)(-18) + (x^3 - 9x^2 + 18x)(88)$$
$$+ (x^3 - 8x^2 + 12x)(-81) + (x^3 - 5x^2 + 6x)(11)$$

$$= -x^2 + 30x + 648$$

Example 8.8 By means of Lagrange's formula, prove that

$$y_1 = y_3 - 0.3\,(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5}) \text{ (approximately).}$$

Solution Here, for the given y_{-5}, y_{-3}, y_3, y_5, we have to obtain y_1. We can have the table as follows:

x	-5	-3	3	5
y	y_{-5}	y_{-3}	y_3	y_5

Therefore, by Lagrange's interpolation formula,

$$y(x) = \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} + \frac{(x+5)(x-3)(x-5)}{(-3+5)(-3-3)(-3-5)} y_{-3}$$

$$+ \frac{(x+5)(x+3)(x-5)}{(3+5)(3+3)(3-5)} y_3 + \frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)} y_5$$

Now y_1 means that y at x is 1.

$$\therefore \quad y_1 = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} y_{-5} + \frac{(6)(-2)(-4)}{2(-6)(-8)} y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)} y_{-3}$$

$$+ \frac{(6)(4)(-2)}{(10)(8)(2)} y_5$$

$$= 0.2\, y_{-5} + 0.5\, y_{-3} + y_3 - 0.3\, y_5$$

$$\therefore \quad y_1 = y_3 - 0.3\,(y_5 - y_{-3}) + 0.2\,(y_{-3} - y_{-5})$$

Example 8.9 Prove that the Lagrange's formula can be put in the form

$$P_n(x) = \sum_{r=0}^{n} \frac{\phi(x) f(x_r)}{(x - x_r)\phi'(x_r)}$$

where $\phi(x) = \prod_{r=0}^{n} (x - x_r)$

Solution We can write Lagrange's interpolation formula as

$$P_n(x) = \sum_{r=0}^{n} \frac{(x - x_0)(x - x_1)\ldots(x - x_n)}{(x_r - x_0)(x_r - x_1)\ldots(x_r - x_n)} f(x_r)$$

$$= \sum_{r=0}^{n} \frac{\phi(x) f(x_r)}{(x - x_r)} \left\{ \frac{1}{(x_r - x_0)(x_r - x_1)\ldots(x_r - x_n)} \right\}$$

$$\text{Now } \phi(x) = \prod_{r=0}^{n} (x - x_r)$$

$$\therefore \phi_n'(x_r) = [\phi(x)]_{x = x_r}$$

$$= (x_r - x_0)(x_r - x_1)\ldots(x_r - x_{r-1})(x_r - x_{r+1})\ldots(x_r - x_n)$$

$$\therefore P_n(x) = \sum_{r=0}^{n} \frac{\phi(x) f(x_r)}{(x - x_r)\phi'(x_r)}$$

8.6 INVERSE INTERPOLATION

So far, given a set of values of x and y, we were required to find the value of y corresponding to a value of x. Sometimes, we may require to find the value of x corresponding to a certain value of y. The process of finding such a value of x is called *inverse interpolation*. In this section, we shall study two methods of inverse interpolation:

(1) *Lagrange's method* and

(2) *Iterative method.*

We apply Lagrange's method when the arguments (x_i) are unequally spaced and the Iterative method when the arguments are equally spaced.

8.7 LAGRANGE'S METHOD

Choosing y as an independent variable and x as a dependent variable in Lagrange's interpolation formula, i.e. Eqn (8.13),

$$x = f(y) = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)}x_1$$

$$+\frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_2$$

$$+\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$+\frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n \qquad (8.14)$$

which is used for inverse interpolation.

Example 8.10 The following table gives the values of x and y:

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

Find the value of x corresponding to $y = 13.6$

Solution Here, $y_0 = 15.9, y_1 = 14.9, y_2 = 14.1, y_3 = 13.3, y_4 = 12.5$, $x_0 = 30, x_1 = 35, x_2 = 40, x_3 = 45$ and $x_4 = 12.5$.Taking $y = 13.6$, Eqn (8.14) gives

$$x = \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)}(30)$$

$$+\frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)}(35)$$

$$+\frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)}(40)$$

$$+\frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)}(45)$$

$$+\frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)}(50)$$

$$= 0.4044117 - 4.3237305 + 21.41276 + 27.79541 - 2.1470014$$

$$= 43.14185 \cong 43.1$$

8.8 ITERATIVE METHOD

Let us consider Newton's forward interpolation formula:

$$y_x = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0$$

$$+ \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 y_0 + \cdots$$

From this we get

$$x = \frac{1}{\Delta y_0}\Big[y_x - y_0 - \frac{x(x-1)}{2!}\Delta^2 y_0 - \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0$$

$$- \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 y_0 - \ldots\ldots\ldots\Big] \quad (8.15)$$

Neglecting the second and higher differences, we get a first approximation for x and this we write as

$$x^{(1)} = \frac{1}{\Delta y_0}(y_x - y_0) \quad (8.16)$$

Substituting this in RHS of Eqn (8.15), the second approximation is given by

$$x^{(2)} = \frac{1}{\Delta y_0}\Big[y_x - y_0 - \frac{x^{(1)}(x^{(1)}-1)}{2!}\Delta^2 y_0 - \frac{x^{(1)}(x^{(1)}-1)(x^{(1)}-2)}{3!}\Delta^3 y_0$$

$$- \frac{x^{(1)}(x^{(1)}-1)(x^{(1)}-2)(x^{(1)}-3)}{4!}\Delta^4 y_0 + \cdots\Big] \tag{8.17}$$

The third approximation is obtained by substituting $x = x^{(2)}$ in RHS of Eqn (8.15), i.e.,

$$x^{(3)} = \frac{1}{\Delta y_0}\Big[y_x - y_0 - \frac{x^{(2)}(x^{(2)}-1)}{2!}\Delta^2 y_0 - \frac{x^{(2)}(x^{(2)}-1)(x^{(2)}-2)}{3!}\Delta^3 y_0$$

$$- \frac{x^{(2)}(x^{(2)}-1)(x^{(2)}-2)(x^{(2)}-3)}{4!}\Delta^4 y_0 - \cdots\Big] \tag{8.18}$$

Similarly, further approximations can be obtained and the process is to be continued until two successive approximations of x agree with each other to the required accuracy. If the final approximation is x^*, then the required $x = x_0 + x^*h$.

Note

1. This method can be applied equally well by starting with any other interpolation formula.
2. It is a powerful iterative procedure to find the roots of an equation to a good degree of accuracy.

Example 8.11 From the following data

x	1.8	2.0	2.2	2.4	2.6
y	2.9	3.6	4.4	5.5	6.7

find x when $y = 5$, using iterative method.

Solution The forward difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.8	**2.9**				
		0.7			
2.0	3.6		**0.1**		
		0.8		**0.2**	
2.2	4.4		0.3		**−0.4**
		1.1		−0.2	
2.4	5.5		0.1		
		1.2			
2.6	6.1				

Here, $y_x = 5$, $y_0 = 2.9$, $\Delta y_0 = 0.7$, $\Delta^2 y_0 = 0.1$, $\Delta^3 y_0 = 0.2$ and $\Delta^4 y_0 = -0.4$. Let us consider Newton's forward interpolation formula [see Section 8.8].

Now, from Eqn (8.16), the first approximation is

$$x^{(1)} = \frac{1}{\Delta y_0}(y_x - y_0) = \frac{1}{0.7}(5-2.9) = 3$$

Using Eqn (8.17), the second approximation is given by

$$x^{(2)} = \frac{1}{0.7}[5 - 2.9 - 0.3 - 0.2] = 2.2857143$$

Proceeding on [as explained in Section 8.8], we get

$x^{(4)} = 2.4724147$; $x^{(5)} = 2.6364724$
$x^{(6)} = 2.5372869$; $x^{(7)} = 2.5985062$
$x^{(8)} = 2.5611579$; $x^{(9)} = 2.5841148$
$x^{(10)} = 2.5700665$; $x^{(11)} = 2.5786872$
$x^{(12)} = 2.573406$; $x^{(13)} = 2.5766447$
$x^{(14)} = 2.5746598$; $x^{(15)} = 2.5758768$

$\therefore x^* = 2.575$, correct to three decimal places.

Hence, the value of x corresponding to $y_x = 5 = x_0 + x^*h$

$= 1.8 + (2.575)(0.2) = 2.315$

Example 8.12 Find the real root of the equation $x^3 + x - 3 = 0$ which lies in between 1.2 and 1.3.

Solution The difference table is as given below:

x	u	$y(=x^3+x-3)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	–2	–1				
			0.431			
1.1	–1	–0.569		0.066		
			0.497		0.006	
1.2	0	–0.072		0.072		0
			0.569		0.006	
1.3	1	0.497		0.078		
			0.647			
1.4	2	1.144				

In the table, $u = \dfrac{x - 1.2}{0.1}$, where $x_0 = 1.2$.

Now using Stirling's formula,

$$y_x = y_0 + x\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{x^2}{2!}\Delta^2 y_{-1} + \frac{x(x^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

we get

$$0 = -\,0.0720 + x\left[\frac{0.569 + 0.497}{2}\right] + \frac{x^2}{2}(0.072)$$

$$+\ \frac{x(x^2-1)}{6}\left[\frac{0.006 + 0.006}{2}\right] \qquad (\because y = 0)$$

or $\qquad 0 = -\,0.072 + 0.532\,x + 0.036\,x^2 + 0.001x^3 \qquad$ (i)

Now Eqn (i) can be written as

$$x = \frac{1}{0.532}[0.072 - 0.036x^2 - 0.001x^3] \qquad \text{(ii)}$$

The first approximation is given by

$$x^{(1)} = \frac{0.072}{0.532} = 0.1353383$$

Putting $x = x^{(1)} = 0.1353383$ on RHS of Eqn (ii), the second approximation is given by

$$x^{(2)} = \frac{1}{0.532}[0.072 - 6.5939284 \times 10^{-4} - 2.4789205 \times 10^{-6}]$$
$$= 0.1340942 \cong 0.1341$$

$\therefore$ Desired root = $1.2 + (0.1341)(0.1) = 1.21341$

EXERCISE 8.2

1. Given that $\log_{10}300 = 2.4771$, $\log_{10}304 = 2.4829$, $\log_{10}305 = 2.4843$ and $\log_{10}307 = 2.4871$, find by using Lagrange's formula, the value of $\log_{10} 310$. *(Karnataka 1993)*
2. Given the values $f(14) = 68.7$, $f(17) = 64$, $f(31) = 44$ and $f(35) = 39.1$, find $f(27)$ using Lagrange's formula.
3. Given: $u_1 = 22$, $u_2 = 30$, $u_4 = 82$, $u_7 = 106$ and $u_8 = 206$. Find u_6 using Lagrange's interpolation formula.
4. The amount A of a substance remaining in a reacting system after an interval of time t in a certain chemical experiment is given by the following data:

t	2	5	8	14
A	94.8	87.9	81.3	68.7

Find the value of A at $t = 11$

5. The following table gives the viscosity of an oil as a function of temperature. Use Lagrange's formula to find the viscosity of oil at a temperature of 140°.

T°	110	130	160	190
Viscosity	10.8	8.1	5.5	4.8

6. The following are the measurements of t made on a curve recorded on an oscillograph representing a change in the conditions of electric current i.

t	1.2	2.0	2.5	3.0
i	1.36	0.58	0.34	0.20

Find the value of i at $t = 1.6$.

7. Using a polynomial of third degree, complete the record given below of the export of a certain commodity during five years.

Year	1917	1918	1919	1920	1921
Export (in tons)	443	384	–	397	467

8. The following data give the percentage of criminals for different age groups:

Age (less than x)	25	30	40	50
% of criminals	52	67.3	84.1	94.4

Using Lagrange's formula, find the percentage of criminals under the age of 35.

9. If $y_0, y_1, y_2, \ldots, y_6$ are the consecutive terms of a series, then using Lagrange's formula prove that $y_3 = 0.05\,(y_0 + y_6) - 0.3\,(y_1 + y_5) + 0.75\,(y_2 + y_4)$.

10. Given that $f(-1) = -2, f(0) = -1, f(2) = 1, f(3) = 4$, fit a polynomial of third degree.

11. Determine $f(x)$ as a polynomial in x for the following data:

x	–4	–1	0	2	5
$f(x)$	1245	33	5	9	1335

12. Find a polynomial of fifth degree from the following data:

x	0	1	3	5	6	9
$f(x)$	−18	0	0	−248	0	13104

(M.U. 1991)

13. Apply Lagrange's formula inversely to obtain the root of $f(x) = 0$, given that $f(30) = -30, f(34) = -13$ $f(38) = 3$ and $f(42) = 18$.

(M.U. B.E, 1993)

14. Given that $f(0) = 16.35, f(5) = 14.88, f(10) = 13.59$ and $f(15) = 12.46$, find x when $f(x) = 14$.

15. Find x when $f(x) = 0.163$, given that

x	80	82	84	86	88
$f(x)$	0.134	0.154	0.176	0.200	0.221

16. Obtain the value of t when $A = 85$ in Problem 4. *(Madurai B.E, 1983)*

17. The following table gives the values of the probability integral $P(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$. For what value of x, $P(x) = 0.5$?

x	0.45	0.46	0.47	0.48	0.49	0.50
$P(x)$	0.4755	0.4847	0.4937	0.5027	0.5117	0.5205

18. Given that $f(10) = 1754, f(15) = 2648, f(20) = 3564$, find the value of x when $f(x) = 3000$ by iterative method.

19. Given $\cos hx = 1.285$, find x by iterative method using the following data:

x	0.736	0.737	0.738	0.739	0.740	0.741
$\cos hx$	1.28330	1.28410	1.28490	1.28572	1.23652	1.28733

20. Solve the equation

(i) $x^3 - 6x - 11 = 0$ $(3<x<4)$ and

(ii) $x = \frac{1}{2} + \sin x$ by iterative method.

ANSWERS

1. 2.4786
2. 49.3
3. 83.515
4. 74.9
5. 7.03
6. 0.8908
7. 369
8. 77.4

10. $f(x) = \frac{1}{6}(x^3 - x^2 + 4x - 6)$
11. $f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$
12. $f(x) = x^5 - 5x^4 + 6x^3 - x^2 + 5x - 6$
13. 37.23
14. 8.34
15. 82.8
16. 6.5928
17. 0.477
18. 16.9
19. 0.73811
20. (i) 3.092 (ii) 1.4973

CHAPTER 9

Numerical Differentiation and Integration

9.1 NUMERICAL DIFFERENTIATION

Consider a set of values (x_i, y_i) of a function. The process of computing the derivative or derivatives of that function at some values of x from the given set of values is called *Numerical differentiation*. This may be done by first approximating the function by a suitable interpolation formula and then differentiating it as many times as desired.

If the values of x are equispaced and the derivative is required *near the beginning* of the table, we employ *Gregory–Newton forward interpolation formula*. If it is required *near the end* of the table, we use *Gregory–Newton backward interpolation formula*. For the values *near the middle* of the table, the derivative is calculated by means of *central difference interpolation formulae*.

If the values of x are not equispaced, we use, *Newton's divided difference interpolation formula* or *Lagrange's interpolation formula* to get the derivative value.

9.2 DERIVATIVES USING NEWTON'S FORWARD DIFFERENCE FORMULA

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \quad (9.1)$$

where $p = \dfrac{x - x_0}{h}$

Differentiating both sides of Eqn (9.1) with respect to p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \cdots \tag{9.2}$$

Now $\quad \dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx} = \dfrac{dy}{dp}\cdot\dfrac{1}{h} \qquad \left(\because \dfrac{dp}{dx} = \dfrac{1}{h}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 \right.$$

$$\left. + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \cdots\right] \tag{9.3}$$

At $x = x_0$, $p = 0$. Hence, putting $p = 0$ in Eqn (9.3), we get

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \tag{9.4}$$

Differentiating Eqn (9.3) again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h}\times\frac{d}{dp}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \cdots\right] \tag{9.5}$$

Putting $p = 0$ in Eqn (9.5), we get

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right] \tag{9.6}$$

Similarly,

$$\left.\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right] \tag{9.7}$$

and so on.

Aliter: We know that $1 + \Delta = E = e^{hD}$, where

$$hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots$$

$$\therefore D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 \cdots\right]$$

$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right]^2$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots\right]$$

$$\text{and } D^3 = \frac{1}{h^3}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \cdots\right]$$

Now applying these identities to y_0,

$$Dy_0 = \left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$D^2 y_0 = \left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{2}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

and

$$D^3 y_0 = \left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0 - \cdots\right]$$

which are same as Eqns (9.4)–(9.7), respectively.

9.3 DERIVATIVES USING NEWTON'S BACKWARD DIFFERENCE FORMULA

Newton's backward interpolation formula is

$$y = y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots \tag{9.8}$$

$$\text{where } p = \frac{x - x_n}{h} \tag{9.9}$$

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{dy}{dp}\cdot\frac{1}{n} \qquad \left[\because \frac{dp}{dx} = \frac{1}{n}\right]$$

$$= \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots\right] \tag{9.10}$$

At $x = x_n$, $p = 0$. Hence, putting $p = 0$ in Eqn (9.10),

$$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right] \tag{9.11}$$

Again, differentiating Eqn (9.10) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\cdot\frac{dp}{dx} = \frac{1}{h}\cdot\frac{d}{dp}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2}[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots] \quad (9.12)$$

Putting $p = 0$, we get

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots] \quad (9.13)$$

Similarly,

$$\left.\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots] \quad (9.14)$$

and so on.

Aliter: We know that $1 - \nabla = E^{-1} = e^{-hD}$

$$-hD = \log(1-\nabla) = -[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots]$$

$$\therefore D = \frac{1}{h}[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 \cdots]$$

$$D^2 = \frac{1}{h^2}[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots]^2$$

$$= \frac{1}{h^2}[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \cdots]$$

Similarly,

$$D^3 = \frac{1}{h^3}[\nabla^3 + \frac{3}{2}\nabla^4 + \frac{7}{4}\nabla^5 + \cdots]$$

Applying these identities to y_n, we get

$$Dy_n = \left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots]$$

$$D^2 y_n = \left.\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots]$$

and

$$D^3y_n = \left.\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}\left[\nabla^3y_n + \frac{3}{2}\nabla^4y_n + \frac{7}{4}\nabla^5y_n + \cdots\right]$$

Which are same as Eqns (9.11)–(9.14), respectively.

9.4 DERIVATIVES USING STIRLING'S FORMULA

Stirling's formula is

$$y = y_0 + \frac{p}{1!}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$\left. + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots\right] \tag{9.15}$$

$$\text{where } p = \frac{x - x_0}{h} \tag{9.16}$$

Now $\quad \dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx} = \dfrac{dy}{dp}\cdot\dfrac{1}{h} \qquad \left[\because \dfrac{dp}{dx} = 1\right]$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2}\right\} + p\Delta^2 y_{-1} + \frac{3(p^2-1)}{6}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\}\right.$$

$$\left. + \frac{(2p^3 - p)}{12}\Delta^4 y_{-2} + \cdots\right] \tag{9.17}$$

At $x = x_0$, $p = 0$. Hence, putting $p = 0$, we get

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2}\right\} - \frac{1}{6}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\}\right.$$

$$\left. + \frac{1}{30}\left\{\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right\} + \cdots\right] \tag{9.18}$$

Differentiating Eqn (9.17) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + p\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \frac{6p^2 - 1}{12}\Delta^4 y_{-2} + \cdots\right]$$

$$\therefore \left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots] \tag{9.19}$$

Similarly,

$$\left.\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}[\frac{1}{2}\{\Delta^3 y_{-1} + \Delta^3 y_{-2}\} + \cdots] \tag{9.20}$$

and so on.

In the same manner, we can use any other interpolation formula for computing the derivatives.

Note: Numerical differentiation should be performed only if it is clear from the tabulated values that differences of some order are constant. Otherwise, the method will involve errors of considerable magnitude and they go on increasing significantly as the derivatives of higher orders are computed. This is due to the fact that the original function $f(x)$ and the approximating function $\phi(x)$ may not differ much at the data points but $f'(x) - \phi'(x)$ may be large.

9.5 MAXIMA AND MINIMA OF TABULATED FUNCTION

Differentiating Newton's forward interpolation formula (Eqn (9.1)) with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{h}[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \cdots] \tag{9.21}$$

We know that the maximum and minimum values of a function $y = f(x)$ can be found by equating dy/dx to zero and solving for x.

$\therefore$ From Eqn (9.21) $dy/dx = 0$

$$\Rightarrow \qquad \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots = 0$$

Hence, by keeping only upto the third difference, we have

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 = 0$$

Solving this for p, by substituting Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ (which we get from the difference table), we get x as $x_0 + ph$, at which y is a maximum or minimum.

Example 9.1 Find the first, second and third derivatives of $f(x)$ at $x = 1.5$ if

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.000	13.625	24.000	38.875	59.000

Solution We have to find the derivative at the point $x = 1.5$ which is at the beginning of the given data. Therefore, we use here the derivatives of Newton's forward interpolation formula. The forward difference table is as follows:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
		3.625			
2.0	7.000		**3.000**		
		6.625		**0.750**	
2.5	13.625		3.750		0
		10.375		0.750	
3.0	24.000		4.500		0
		14.875		0.750	
3.5	38.875		5.250		
		20.125			
4.0	59.000				

Here, $x_0 = 1.5$, $y_0 = 3.375$, $\Delta y_0 = 3.625$, $\Delta^2 y_0 = 3$, $\Delta^3 y_0 = 0.75$ and $h = 0.5$

Now, from Eqn (9.4), we have

$$\left.\frac{dy}{dx}\right]_{x=x_0} = f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 - \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\therefore f'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3) + \frac{1}{3}(0.75)\right] = 4.75$$

From Eqn (9.6), we have

$$\left.\frac{d^2 y}{dx^2}\right]_{x=x_0} = f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

$$\therefore \quad f''(1.5) = \frac{1}{(0.5)^2}\,[3 - 0.75] = 9$$

Again from Eqn (9.7), we have

$$\left.\frac{d^3y}{dx^3}\right]_{x=x_0} = f'''(x_0) = \frac{1}{h^3}[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots]$$

$$\therefore \qquad f'''(1.5) = \frac{1}{(0.5)^3}(0.75) = 6$$

Example 9.2 The population of a certain town (as obtained from census data) is shown in the following table.

Year	1951	1961	1971	1981	1991
Population (in thousands)	19.96	36.65	58.81	77.21	94.61

Find the rate of growth of the population in the year 1981.

Solution Here, we have to find the derivative at 1981 which is near the end of the table. Hence, we use the derivative of Newton's backward difference formula. The table of differences is as follows:

x (year)	y (population)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1951	19.96				
		16.69			
1961	36.65		5.47		
		22.16		−9.23	
1971	58.81		−3.76		**11.99**
		18.40		**2.76**	
1981	77.21		**−1**		
		17.40			
1991	94.61				

Hence, $h = 10$, $x_n = 1991$, $\nabla y_n = 17.4$, $\nabla^2 y_n = -1$, $\nabla^3 y_n = 2.76$ and $\nabla^4 y_n = 11.99$

We know from Eqn (9.10) that

$$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n$$

$$+ \frac{2p^3+9p^2+11p+3}{12}\nabla^4 y_0 + \cdots] \qquad (1)$$

Now, we have to find out the rate of growth of the population in the year 1981,

i.e $\left.\dfrac{dy}{dx}\right]_{x=1981}$ i.e. $x_n + ph = 1981 \therefore p = \dfrac{1981-1991}{10} = -1$

$\therefore$ Putting $p=-1$, $h=10$ and the values of ∇y_n, $\nabla^2 y_n$, $\nabla^3 y_n$ and $\nabla^4 y_n$ in Eqn (1), we get

$$y'(1981) = \frac{1}{10}[17.4 + \frac{2(-1)+1}{2}(-1) + \frac{3(-1)^2+6(-1)+2}{6}(2.76)$$

$$+ \frac{2(-1)^3+9(-1)^2+11(-1)+3}{12}(11.99)]$$

$$= 1/10\,[17.4 + 0.5 - 0.46 - 0.9991666]$$
$$= 1.6440833$$

$\therefore$ The rate of growth of the population in the year 1981 is 1.6440833.

Example 9.3 Obtain the value of $f'(90)$ using Stirling's formula to the following data:

x	60	75	90	105	120
$f(x)$	28.2	38.2	43.2	40.9	37.7

Also find the maximum value of the function from the data.

Solution Since $x = 90$ is in the middle of the table, we use central difference formula and in particular, Stirling's formula.

The central difference table is as given below.

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
60	28.2				
		10			
75	38.2		−5		
		5		**−2.3**	
90	**43.2**		**−7.3**		**8.7**
		−2.3		**6.4**	
105	40.9		−0.9		
		−3.2			
120	37.7				

Here, $x_0 = 90$, $y_0 = 43.2$, $\Delta y_0 = -2.3$, $\Delta y_{-1} = 5$, $\Delta^3 y_{-1} = -2.3$, $\Delta^3 y_{-2} = 6.4$ and $h = 15$.

Now, from Eqn (9.18),

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\left\{\frac{\Delta y_0 + \Delta y_{-1}}{2}\right\} - \frac{1}{6}\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\} + \cdots\right]$$

$$\therefore f'(90) = \frac{1}{15}\left[\left\{\frac{-2.3+5}{2}\right\} - \frac{1}{6}\left\{\frac{-2.3+6.4}{2}\right\}\right]$$

$$= 1/15\,[\,1.35 - 0.3416666] = 0.0672222.$$

To find the maximum value of the tabular function:

By Stirling's formula,

$$y = y(x_0 + ph) = y_0 + \frac{p}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{p^2}{2!}\Delta^2 y_{-1}$$

$$+ \frac{p(p^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p(p^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots$$

Substituting the values from the table, we get, after simplification,

$y = 43.2 + 1.35\,p - 3.65\,p^2 + 0.3417\,(p^3 - p)$

or $y = 0.3417\,p^3 - 3.65\,p^2 + 1.0083\,p + 43.2$

If y is maximum, $dy/dp = 0$

i.e. $1.0251\,p^2 - 7.3\,p + 1.0083 = 0$

$$\therefore p = \frac{7.3 \pm \sqrt{(1.3)^2 - 4(1.0251)(1.0083)}}{2(1.0251)} = 6.9803 \text{ or } 0.1409$$

$p = 6.9803$ is out of range. $\therefore\ p = 0.1409$

Hence, $x = x_0 + ph = 90 + 15\,(0.1409) = 92.1135$

and maximum of y

$= 0.3417\,(0.1409)^3 - 3.65\,(0.1409)^2 + 1.0083\,(0.1409) + 43.2$

$= 43.27$

Example 9.4 Using Bessel's formula, find the derivative of $f(x)$ at $x = 3.5$ from the following table.

x	3.47	3.48	3.49	3.50	3.51	3.52	3.53
$f(x)$	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Solution The central difference table is as follows:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
3.47	0.193						
		0.002					
3.48	0.195		0.001				
		0.003		–0.001			
3.49	0.198		0.000		0.000		
		0.003		–0.001		0.003	
3.50	0.201		–0.001		0.003		–0.010
		0.002		0.002		–0.007	
3.51	0.203		0.001		–0.004		
		0.003		–0.002			
3.52	0.206		–0.001				
		0.002					
3.53	0.208						

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right]$$

$$+ \frac{\left(p-\frac{1}{2}\right)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)(p-3)}{6!}\left[\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right] + \cdots \qquad \text{(i)}$$

where $p = \dfrac{x - x_0}{h}$ · Differentiating Eqn (i) with respect to x, we get

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\cdot\frac{dy}{dp} \qquad \left[\because \frac{dp}{dx} = \frac{1}{h}\right]$$

Now, $\left.\dfrac{dy}{dx}\right]_{x=x_0} = \dfrac{1}{h}\cdot\left[\dfrac{dy}{dp}\right]_{p=0}$

$$= \frac{1}{h}[\Delta y_0 - \frac{1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1}$$

$$+ \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})]$$

Substituting values from the table in above, we get

$$\left.\frac{dy}{dx}\right]_{x=3.5} = f'(3.5) = \frac{1}{0.01}[0.02 - \frac{1}{4}(-0.001 + 0.001) + \frac{1}{12}(0.002)$$

$$+ \frac{1}{24}(-0.004 + 0.003) + \frac{1}{120}(-0.00)$$

$$- \frac{1}{240}(-0.010 + 0)]$$

$$= [0.02 - 0 + 0.01666 - 0.04166 + 0.00583 + 0.04166] = 0.22249$$

Example 9.5 Given the following data, find the maximum value of y

x	-1	1	2	3
y	-21	15	12	3

Solution Since the arguments are not equispaced, we will form the divided difference table as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

Using Newton's divided difference formula, we get

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0$$

$$= -21 + (x + 1)(18) + (x + 1)(x - 1)(-7) + (x + 1)(x - 1)(x - 2)(1)$$

$$= x^3 - 9x^2 + 17x + 6$$

Now for maximum $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 18x + 17 = 0$

$$x = \frac{18 \pm \sqrt{(-18)^2 - 4(3)(17)}}{2(3)} = 4.8257 \text{ or } 1.1743$$

$x = 4.8257$ is out of range $\therefore x = 1.1743$ is the value giving maximum of y.

$\therefore$ Max of y (at $x = 1.1743$) $= (1.1743)^3 - 9(1.1743)^2 + 17(1.1743) + 6$
$= 15.171612$

EXERCISE 9.1

1. Find the first and second derivatives of the function tabulated below at the point $x = 19$

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0	0.128	0.544	1.296	2.432	4.00

(*Madras 1991*)

2. The following data gives corresponding values of pressure and specific volume of super-heated steam.

V	2	4	6	8	10
P	105	42.07	25.3	16.7	13

(i) Find the rate of change of pressure with respect to volume when $V = 2$.

(ii) Find the rate of change of volume with respect to pressure when $P = 105$.

3. Find $y'(0)$ and $y''(0)$ from the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

4. From the values in the table given below, find the value of sec 31° using numerical differentiation.

θ °	31	32	33	34
$\tan\theta$	0.6008	0.6249	0.6494	0.6745

5. The table given below reveals velocity V of a body during time 't' specified. Find its acceleration at $t = 1.1$

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

6. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of time t in seconds.

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.122	0.493	0.123	2.022	3.200	4.61

Find the angular velocity and angular acceleration at $t = 0.6$.

7. From the following table of values of x and y, find y' (1.25) and y'' (1.25).

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

8. Obtain the value of f' (0.04) using Bessel's formula given the table below:

x	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

9. Use Stirling's formula to compute f' (0.5) from the following data:

x	0.35	0.40	0.45	0.50	0.55	0.60	0.65
$f(x)$	1.521	1.506	1.488	1.467	1.444	1.418	1.389

10. A slider in a machine moves along a fixed straight rod. Its distance x (cm) along the rod is given below for various values of time t seconds. Find the veocity of the slider and its acceleration when $t = 0.3$.

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.13	31.62	32.87	33.64	33.95	33.81	33.24

11. For the following pairs of values of x and y, find numerically the first derivative at $x = 4$.

x	1	2	4	8	10
y	0	1	5	21	27

12. Find the value of f' (7.60) from the following table using Gauss's formula.

x	7.47	7.48	7.49	7.50	7.51	7.52	7.53
$f(x)$	0.193	0.195	0.198	0.201	0.203	0.206	0.208

13. Find the maximum and minimum values of the function from the following table:

x	0	1	2	3	4	5
$f(x)$	0	0.25	0	2.25	16.00	56.25

14. From the table below, for what value of x, y is maximum. Also find this value of y.

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.262	0.250	0.224

15. Given the following data, find the maximum value of y.

x	0	2	3	4	7	9
y	4	26	58	112	466	922

ANSWERS

1. 0.63, 6.6
2. –52.4, –0.01908
3. –27.9, 117.67
4. 1.17
5. 44.917
6. 3.82 rad/sec 6.75 rad/sec^2
7. 0.44733, 0.158332
8. 0.25625
9. –0.44
10. 5.33, –45.6
11. 2.8326
12. 0.223
13. Maximum value = 0.25 at $x = 1$; Minimum value 0 at $x = 0$ or 2
14. Minimum value = 0.2628 at $x = 5.6875$
15. No maximum or minimum value

9.6 NUMERICAL INTEGRATION

The process of computing $\int_a^b y\,dx$, where $y = f(x)$ is given by a set of tabulated values $[x_i, y_i]$, $i = 0, 1, 2, \ldots, n$, $a = x_0$ and $b = x_n$ is called *numerical integration*. Since $y = f(x)$ is a single variable function, the process in general, is known as *quadrature*. Like that of numerical differentiation, here also we replace $f(x)$ by an interpolation formula and integrate it in between the given limits. In this way, we can derive quadrature formulae for approximate integration of a function defined by a set of numerical values.

9.7 GENERAL QUADRATURE FORMULA

In this section, we will derive a general quadrature formula for equidistant ordinates.

Let $I = \int_a^b y\,dx$, where $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x_0, x_1, x_2, \ldots, x_n$. Let us divide the interval (a,b) into n equal parts of width h, so that

$$a = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh = b$$

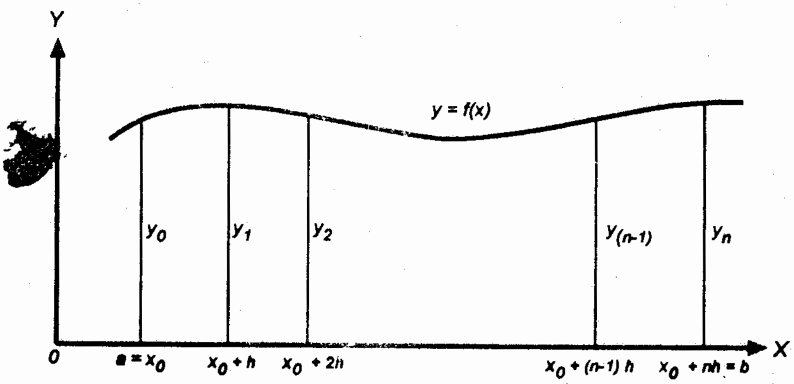

Fig. 9.1

Then $I = \int_{x_0}^{x_0+nh} f(x)\,dx$

Putting $x = x_0 + ph$, so that $dx = h\,dp$, in above, we get

$$I = h \int_0^n f(x_0 + ph)\,dp = h \int_0^n y_p\,dp$$

Now replacing y_p by Newton's forward interpolation formula, we get

$$I = h\int_0^n [y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0$$

$$+\frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 y_0 + \cdots]$$

Now integrating it term by term, we get, after substituting the limits, as

$$\int_{x_0}^{x_0+nh} f(x)\,dx = h\,[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left\{\frac{n^3}{3} - \frac{n^2}{2}\right\}\Delta^2 y_0$$

$$+\frac{1}{6}\left\{\frac{n^4}{4} - n^3 + n^2\right\}\Delta^3 y_0$$

$$+\frac{1}{24}\left\{\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - \frac{3n^2}{1}\right\}\Delta^4 y_0$$

$$+\frac{1}{120}\left\{\frac{n^6}{6} - 2n^5 + \frac{35n^4}{4} - \frac{50n^3}{3} + 12n^2\right\}\Delta^5 y_0$$

$$+\frac{1}{720}\left\{\frac{n^7}{7} - \frac{15n^6}{6} + 17n^5 - \frac{225n^4}{4} + \frac{274n^3}{3} - 60n^2\right\}\Delta^6 y_0 + \cdots] \quad (9.22)$$

Eqn (9.22) is known as *Newton–Cote's quadrature formula*, which is a general quadrature formula for equidistant ordinates. In the following sections, we deduce important quadrature formula from this equation, taking $n = 1, 2, 3, \ldots$

9.8 TRAPEZOIDAL RULE

Putting $n = 1$ in Eqn. 9.22 and neglecting second and higher order differences, we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\,[y_0 + \frac{1}{2}\Delta y_0]$$

$$= h\,[y_0 + \frac{1}{2}(y_1 - y_0)] = \frac{h}{2}(y_0 + y_1)$$

Similarly,

$$\int_{x_0+h}^{x_0+2h} f(x)dx = \frac{h}{2}[y_1 + y_2]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}[y_{n-1} + y_n]$$

Adding these n integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})] \qquad (9.23)$$

This rule is known as Trapezoidal rule.

9.9 SIMPSON'S 1/3 RULE

Here taking $n = 2$ in Eqn (9.22), and neglecting third and higher order differences, we get

$$\int_{x_0}^{x_0+2h} f(x)dx = h\,[2y_0 + 2\Delta y_0 + \frac{1}{2}(\frac{8}{3} - 2)\Delta^2 y_0]$$

$$= h\,[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

where n is even.

Adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4\,(y_1 + y_3 + \cdots + y_{n-1})$$
$$+ 2(y_2 + y_4 + \cdots + y_{n-2})] \qquad (9.24)$$

This is known as Simpson's 1/3 rule.

9.10 SIMPSON'S 3/8 RULE

Putting $n = 3$ in Eqn (9.22), and neglecting all differences above the third order, we get

$$\int_{x_0}^{x_0+3h} f(x)dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(\frac{27}{3} - \frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4} - 27 + 9\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

... ...

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

(if n is a multiple of 3)

Adding all these integrals, where n is a multiple of 3, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3})] \quad (9.25)$$

Eqn (9.25) is known as Simpson's 3/8 rule.

9.11 WEDDLE'S RULE

Here, taking $n = 6$ in Eqn (9.22), and neglecting all differences above the sixth order, we get

$$\int_{x_0}^{x_0+6h} f(x)dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right]$$

Replacing the coefficient of $\Delta^6 y_0$ by $\frac{42}{140}$ (the error made will be negligible) and the differences in terms of y's, we get

$$\int_{x_0}^{x_0+6h} f(x)dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Similarly,

$$\int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\int_{x_0+(n-6)h}^{x_0+nh} = \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

(if n is a multiple of 6)

Adding all these integrals and rearranging them in an order, we have (if n is a multiple of 6),

$$\begin{aligned}\int_{x_0}^{x_0+nh} f(x)dx &= \frac{3h}{10}[(y_0 + y_n) + (5y_1 + y_2 + 6y_3 + y_4 + 5y_5) \\ &\quad + (2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11}) + \cdots \\ &\quad + (2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1})] \quad (9.26)\end{aligned}$$

Eqn (9.26) is known as Weddle's rule.

Note

1. In Trapezoidal rule, $f(x)$ is a linear function of x, i.e of the form $f(x) = ax + b$. It is the simplest rule but least accurate.
2. In Simpson's 1/3 rule, $f(x)$ is a polynomial of second degree, i.e. $f(x) = ax^2 + bx + c$. To apply this rule, the number of intervals n must be even; in otherwords, the number of ordinates must be odd.
3. In Simpson's 3/8 rule, $f(x)$ is a polynomial of third degree, i.e. $f(x) = ax^3 + bx^2 + cx + d$. To apply this rule, the number of intervals n must be a multiple of 3.
4. In Weddle's rule, $f(x)$ is a polynomial of degree six, and is applicable only if the number of intervals n is a multiple of 6. It is more accurate than all the above formulae but it requires atleast seven consecutive values of the function.

9.12 ERRORS IN QUADRATURE FORMULAE

If y_p is a polynomial representing the function $y = f(x)$ in the interval $[a, b]$, then error in the Quadrature formulae is given by

$$E = \int_a^b f(x)\, dx - \int_a^b y_p\, dx \tag{9.27}$$

9.12.1 Error in Trapezoidal Rule

Expanding $y - f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series, we get

$$y = y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!}\, y_0'' + \cdots \tag{9.28}$$

where $y_0' = [y'(x)]_{x=x_0}$ and so on.

$$\therefore \int_{x_0}^{x_1} y\, dx = \int_{x_0}^{x_0+h} [y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \cdots]\, dx$$

$$= y_0 h + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \tag{9.29}$$

Now, area of the first trapezium in the interval $[x_0, x_1]$

$$= A_1 = \frac{h}{2}(y_0 + y_1) \tag{9.30}$$

Putting $x = x_0 + h$, $y = y_1$ in Eqn (9.28),

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots \tag{9.31}$$

$\therefore$ From Eqns (9.30) and (9.31), we get

$$A_1 = \frac{h}{2}[y_0 + y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots]$$

$$= hy_0 + \frac{h^2}{2} y_0' + \frac{h^3}{2(2!)} y_0'' + \cdots \tag{9.32}$$

Subtracting Eqn (9.32) from Eqn (9.29) gives the error in $[x_0, x_1]$ which is

$$= \int_{x_0}^{x_1} y\,dx - A_1 = \left[\frac{1}{3!} - \frac{1}{2(2!)}\right] h^3 y_0'' + \cdots$$

$$= -\frac{h^3}{12} y_0'' + \cdots$$

$\therefore$ The principal part of the error in the interval $[x_0, x_1]$ is $-\frac{h^3}{12} y_0'' + \cdots$ (by neglecting other terms).

Similarly, the error in $[x_0, x_2]$ is $-\frac{h^3}{12} y_1'' + \cdots$

and in $[x_{n-1}, x_n]$ is $-\frac{h^3}{12} y_{n-1}''$.

Hence, the total error is

$$E = -\frac{h^3}{12} [y_0'' + y_1'' + \cdots + y_{n-1}'']$$

Let $y''(\xi)$, $a < \xi < b$ be the maximum of $|y_0''|, |y_1''|, \ldots, |y_{n-1}''|$, then we have

$$E < -\frac{nh^3}{12} y''(\xi) = -\frac{(b-a)h^2}{12} y''(\xi) \quad [\because b - a = nh]$$

Hence, the error in the trapezoidal rule is of the order h^2.

9.12.2 Error in Simpson's 1/3 Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series, we get

$$y = y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \cdots \tag{9.33}$$

$$\therefore \int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_2 + 2h} [y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \cdots]\,dx$$

$$= 2hy_0 + 2h^2 y_0' + \frac{8h^3}{3!} y_0'' + \frac{16h^4}{4!} y_0''' + \frac{32h^5}{5!} y_0^{\text{iv}} + \cdots \tag{9.34}$$

Now, A_1 = area of the curve in the interval $[x_0, x_2]$. By Simpson's 1/3 rule

$$A_1 = \frac{h}{3}[y_0 + 4y_1 + y_2] \tag{9.35}$$

Putting $x = x_1 = x_0 + h$, $y = y_1$ in Eqn (9.33), we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \tag{9.36}$$

Putting $x = x_2 = x_0 + 2h$, $y = y_2$ in Eqn (9.33), we get

$$y_2 = y_0 + 2hy_0' + \frac{4h^2}{2}y_0'' + \frac{8h^3}{3!}y_0''' + \cdots \tag{9.37}$$

Substituting Eqn (9.36) and (9.37) in Eqn (9.35), we get

$$A_1 = 2hy_0 + 2h^2y_0' + \frac{4h^3}{3}y_0'' + \frac{2h^4}{3}y_0''' + \frac{5h^5}{18}y_0^{iv} + \cdots \tag{9.38}$$

Now the error in interval $[x_0, x_2]$ is given by

$$\int_{x_0}^{x_2} yd - A_1 = \left(\frac{4}{15} - \frac{5}{18}\right)h^5 y_0^{iv} + \cdots$$

So by neglecting the terms of order $h^6, h^7, \ldots$, the principal part of the error is $-\frac{h^5}{90}y_0^{iv}$.

Similarly, in interval $[x_2, x_4]$ the principal part of the error is $-\frac{h^5}{90}y_2^{iv}$ and so on.

Hence total principal error is

$$E = -\frac{h^5}{90}[y_0^{iv} + y_2^{iv} + \ldots + y_{2(n-1)}^{iv}]$$

Let $y^{iv}(\xi)$ be the maximum of $|y_0^{iv}|, |y_2^{iv}|, \ldots, |y^{iv}_{2n-2}|$.
Then we have,

$$E < -\frac{h^5}{90}y^{iv}(\xi) = \frac{-(b-a)h^4}{180}y^{iv}(\xi).$$

That is, Simpson's 1/3 rule has an error of the order of h^4.

9.12.3 Errors in Simpson's 3/8 Rule and Weddle's Rule

Proceeding as above, the principal part of the error:

(i) for Simpson's 3/8 rule is $-\dfrac{3h^5}{80}y^{iv}$ in the interval $[x_0, x_3]$

(ii) for Weddle's rule is $-\dfrac{h^7}{140}y^{vi}$ in the interval $[x_0, x_6]$.

Example 9.6 Evaluate $\int_0^{10} \dfrac{dx}{1+x^2}$ by using

(i) Trapezoidal rule,
(ii) Simpson's 1/3 rule,
(iii) Simpson's 3/8 rule, and
(iv) Weddle's rule. Compare the results with the actual value.

Solution Taking $h = 1$, divide the whole range of the integration [0, 10] into ten equal parts. The values of the integrand for each point of sub-division are given below:

x	$0 = x_0$	$1 = x_1$	$2 = x_2$	$3 = x_3$	$4 = x_4$	$5 = x_5$
$y = \dfrac{1}{1+x^2}$	$1 = y_0$	$0.5 = y_1$	$0.2 = y_2$	$0.1 = y_3$	$0.0588235 = y_4$	$0.0384615 = y_5$
x	$6 = x_6$	$7 = x_7$	$8 = x_8$	$9 = x_9$	$10 = x_{10}$	
$y = \dfrac{1}{1+x^2}$	$0.027027 = y_6$	$0.02 = y_7$	$0.0153846 = y_8$	$0.0121951 = y_9$	$9.9009901\times10^{-3} = y_{10}$	

(i) By Trapezoidal rule:

$$\int_0^{10} \frac{1}{1+x^2}dx = \frac{h}{2}\left[(y_0+y_{10})+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)\right]$$

$$= \frac{1}{2}[(1 + 9.9009901 \times 10^{-3}) + 2(0.5 + 0.2 + 0.1 + 0.0588235 + 0.0384615 + 0.027027 + 0.02 + 0.0153846 + 0.0121951)]$$

$$= 1.4768422$$

(ii) By Simpson's 1/3 rule:

$$\int_0^{10} \frac{1}{1+x^2}dx = \frac{h}{3}[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)$$
$$+2(y_2+y_4+y_6+y_8)]$$
$$= \frac{1}{3}[(1+9.9009901\times 10^{-3})+4(0.5+0.1$$
$$+0.0384615+0.02+0.0121951)$$
$$+2(0.2+0.0588235+0.027027+0.0153846)]$$
$$= 1.4316659$$

(iii) By Simpson's 3/8 rule:

$$\int_0^{10} \frac{1}{1+x^2}dx = \frac{3h}{8}[(y_0+y_{10})+3(y_1+y_2+y_4+y_5+y_7+y_8)$$
$$+2(y_3+y_6+y_9)]$$
$$= \frac{3}{8}[(1+9.9009901\times 10^{-3})+3(0.5+0.2+0.0588235$$
$$+0.0384615+0.02+0.0153846)$$
$$+2(0.1+0.027027+0.0121951)]$$
$$= 1.4198828$$

(iv) By Weddle's rule:

$$\int_0^{10} \frac{1}{1+x^2}dx = \frac{3h}{10}[(y_0+y_{10})+5y_1+y_2+6y_3+y_4+5y_5+2y_6+5y_7$$
$$+y_8+6y_9)$$
$$= \frac{3}{10}[(1+9.9009901\times 10^{-3})+5(0.5)+0.2+6(0.1)$$
$$+0.0588235+5(0.0384615)+2(0.027027)$$
$$+5(0.02)+0.0153846)+6(0.0121951)]$$
$$= 1.4410924$$

Now $\displaystyle\int_0^{10} \frac{1}{1+x^2}dx = \tan^{-1}x\Big]_0^{10} = \tan^{-1}10 = 1.4711277$

which shows that the value of the integral found by Weddle's rule is the nearest to the actual value than others.

Example 9.7 The velocity v of a particle at distance s from a point on its path is given by the following table :

s (ft)	0	10	20	30	40	50	60
v [ft/s]	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule.

Solution We know that the rate of displacement is velocity, i.e. $ds/dt = v$, Therefore, the time taken to travel 60 ft is given by

$$t = \int_0^{60} \frac{1}{v}\, ds = \int_0^{60} y\, dx$$

where $s = x$ and $y = 1/v$. The table is as given below:

$x (= s)$	0	10	20	30
$y = 1/v$	0.0212765 $= y_0$	0.0172413 $= y_1$	0.015625 $= y_2$	0.0153846 $= y_3$
$x (= s)$	40	50	60	
$y = 1/v$	0.0163934 $= y_4$	0.0192307 $= y_5$	0.0263157 $= y_6$	

By Simpson's 1/3 rule :

$$\int_0^{60} y\, dx = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{10}{3}[(0.0212765 + 0.0263157) + 2(0.015625 + 0.0163934) + 4(0.0172413 + 0.0153846 + 0.0192307)]$$

$$= 1.063518$$

Hence, the time taken to travel 60 ft is 1.064s.

By Simpson's 3/8 rule :

$$\int_0^{60} y\, dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3)$$

$$= \frac{30}{8}[(0.0212765 + 0.0263157) + (0.0172413 + 0.015625 + 0.0163934 + 0.0192307) + 2(0.0153846)]$$

$$= 1.0643723$$

By this method also the time taken to travel 60 ft is 1.064s.

Example 9.8 A curve is drawn to pass through the following points :

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bound by the curve, x-axis and lines $x = 1, x = 4$. Also find the volume of solid generated by revolving this area using Weddle's rule.

Solution The area bound by the curve, x-axis and lines $x = 1$, $x = 4$ is given by

$$A = \int_1^4 y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6]$$

$$= 3\frac{(0.5)}{10}[2 + 5(2.4) + 2.7 + 6(2.8) + 3 + 5(2.6) + 2(2.1)]$$

$$= 8.055 \text{ sq. units}$$

Volume of the solid generated,

$$V = \pi \int_1^4 y^2\,dx = \pi\frac{3h}{10}[y_0^2 + 5y_1^2 + y_2^2 + 6y_3^2 + y_4^2 + 5y_5^2 + 2y_6^2]$$

$$= \pi\, 3\frac{(0.5)}{10}[4 + 5(2.4)^2 + (2.7)^2 + 6(2.8)^2 + 9 + 5(2.6)^2 + 2(2.1)^2]$$

$$= \pi \times 20.8125 = 65.384397 \text{ cubic units.}$$

9.13 ROMBERG'S METHOD

For an interval of size h, the error in the trapezoidal rule is

$$= -\frac{(b-a)h^2}{12}y''(\xi),\ a < \xi < b$$

$$= Ch^2$$

where $C = \dfrac{-(b-a)}{12}y''(\xi)$ may be chosen as a constant if $y''(\xi)$ is reasonably constant.

Suppose, we evaluate $I = \int_a^b y\,dx$ by the trapezoidal rule with two different subintervals h_1 and h_2. Let I_1, I_2 be the approximations with errors E_1 and E_2, respectively.

Then $I = I_1 + E_1 = I_1 + Ch_1^2$
also $I = I_2 + E_2 = I_2 + Ch_2^2$

$$\therefore I_1 + Ch_1^2 = I_2 + Ch_2^2 \text{ or } C = \frac{I_1 - I_2}{h_2^2 - h_1^2}$$

$$\therefore I = I_1 + \left(\frac{I_1 - I_2}{h_2^2 - h_1^2}\right) h_1^2 = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$$

which will be a better approximation to I than I_1 or I_2.

To evaluate systematically, we take $h_1 = h$ and $h_2 = \frac{1}{2}h$. Then

$$I = \frac{I_1\left(\frac{h}{2}\right)^2 - I_2 h^2}{\left(\frac{h}{2}\right)^2 - h^2} = \frac{4I_2 - I_1}{3}$$

$$= I_2 + \frac{1}{3}(I_2 - I_1) \tag{9.39}$$

We obtained this result by applying Trapezoidal rule twice. By applying the rule several times, every time halving h, we get a sequence of results $A_1, A_2, A_3, A_4, \ldots$ in which the error is reduced by $\frac{1}{4}$ every time.

We apply the formula [Eqn (9.39)] again to each pair of A's, i.e. A_1, A_2; A_2, A_3, etc., to get improved results B_1, B_2, B_3 etc.

Again applying Eqn (9.39) to the pairs B_1, B_2; B_2, B_3, etc., we get still better results C_1, C_2 etc., We continue this process until two successive values are close to each other. This method is called *Richardson's deferred approach to the limit* and its systematic improvement is called *Romberg integration* or *Romberg's method.*

Example 9.9 Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using Romberg's method, correct to four decimal places. Hence find an approximate value of π.

Solution By taking $h = 0.5, 0.25, 0.125$, respectively, let us evaluate the given integral using Trapezoidal rule.

(i) When $h = 0.5$

x	0	0.5	1
y	1	0.8	0.5

$$I = \frac{0.5}{2}\left[(1 + 0.5) + 2(0.8)\right] = 0.775$$

(ii) When $h = 0.25$

x	0	0.25	0.5	0.75	1
y	1	0.9412	0.8	0.64	0.5

$$I = \frac{0.25}{2}\left[(1 + 0.5) + 2(0.9412 + 0.8 + 0.64)\right]$$

$$= 0.7828$$

(iii) When $h = 0.125$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

$$I = \frac{0.125}{2}\left[(1 + 0.5) + 2(0.9846 + 0.9412 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5664)\right]$$

$$= 0.78475$$

Now we have three values for the given definite integral.
Let $I_1 = 0.775$, $I_2 = 0.7828$, $I_3 = 0.78471$.

Applying the formula $I = I_2 + \frac{1}{2} + \frac{1}{3}(I_2 - I_1)$ to the pairs I_1, I_2 and I_2, I_3 we get new values, say,

$$I_1^* = 0.7828 + \frac{1}{3}(0.7828 - 0.775) = 0.7854$$

and $I_2^* = 0.78475 + \frac{1}{3}(0.78475 - 0.7828) = 0.7854$

Since these two are the same, we conclude that the value of the integral $= 0.7854$.

$$\text{i.e.} \quad \int_0^1 \frac{1}{1+x^2}\,dx = 0.7854 \qquad \text{(i)}$$

Now $\int_0^1 \frac{1}{1+x^2}\, dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$

$\therefore$ from (i), $\pi/4 = 0.7854$ or $\pi \cong 3.1416$

EXERCISE 9.2

1. Evaluate $\int_0^2 y\,dx$ from the following table using Trapezoidal rule.

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1.21	1.37	1.46	1.59	1.67	2.31	2.91	3.83	4.01	4.79	5.31

2. Find an approximate value of $\log_e 5$ by calculating to four decimal places by Simpson's 1/3 rule the integral $\int_0^5 \frac{dx}{4x+5}$, dividing the range into 10 equal parts.

3. Apply Simpson's 3/8 rule to evaluate $\int_0^2 \frac{dx}{1+x^3}$ to two decimal places by dividing the range into eight equal parts.

4. Evaluate $\int_0^{10} e^x\, dx$ by Weddle's rule given that $e^0 = 1$, $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$, $e^5 = 148.41$, $e^6 = 403.43$, $e^7 = 1096.63$, $e^8 = 2980.96$, $e^9 = 8103.08$, $e^{10} = 22026.47$

5. Evaluate $\int_0^{\frac{\pi}{2}} \sin x\, dx$ by (i) Trapezoidal rule, (ii) Simpson's rule using 11 ordinates. Also estimate the errors by finding the value of the integral.

6. Calculate the value of the following integrals by (i) Trapezoidal rule, (ii) Simpson's 1/3 rule, (iii) Simpson's 3/8 rule, and (iv) Weddle's rule. After finding the true value of the integral, compare the errors in the four cases

   (i) $\int_4^{5.2} \log x\, dx$ (ii) $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx$

7. A river is 80 feet wide. Depth d in feet at a distance of x feet from one bank is given by the following table

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross-section.

8. Find the approximate distance travelled by a train between 11.50 a.m. and 12.30 p.m. from the following data using Simpson's 1/3 rule.

time	11.50 a.m.	12.00	12.10 p.m.	12.20 p.m.	12.30 p.m.
Speed m.p.h.	24.2	35.0	41.3	42.8	39.2

9. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's 1/3 rule, find the velocity and height of the rocket at $t = 80$.

t (s)	0	10	20	30	40	50	60	70	80
a (m/s^2)	30	31.63	33.64	35.47	37.75	40.33	43.25	46.69	50.67

10. When a train is moving at 30 miles an hour, steam is shut off and brakes are applied. The speed of the train in miles per hour after t seconds is given by

t	0	5	10	15	20	25	30	35	40
v	30	24	19.5	16	13.6	11.7	10.0	8.5	7.0

Determine how far the train has moved in the 40 seconds.

11. The speed of an electric train at various times after leaving one station until it stops at the next station are given in the following table:

Speed in m.p.h	0	13	33	39.5	40	40	36	15	0
Time in min	0	0.5	1	1.5	2	2.5	3	3.25	3.5

Find the distance between the two stations.

12. A solid of revolution is formed by rotating about the x-axis, the area between x-axis and lines $x = 0$ and $x = 1$, and a curve through the points with the following coordinates.

x	0	0.25	0.50	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's 1/3 rule.

13. A curve passes through the points (1, 0.2), (2, 0.7), (3, 1), (4, 1.3), (5, 1.5), (6, 1.7), (7, 1.9), (8, 2.1), (9, 2.3). Using Weddle's rule, estimate the volume generated by revolving the area between the curve x axis and the ordinates $x = 1$ and $x = 9$ about the x axis.

14. The table below gives the velocity v of a moving particle at time t. Find the distance covered by the particle in 12 s and also the acceleration at $t = 2$ s.

t	0	2	4	6	8	10	12
v	4	6	16	34	60	94	136

15. Estimate the length of the arc of the curve $3y = x^3$ from (0, 0) to (1, 3) using Simpson's 1/3 rule taking eight sub-intervals.

16. Apply Romberg's method to evaluate $\int_4^{5.2} \log x \, dx$ given that $\log_e 4 = 1.3863$, $\log_e 4.2 = 1.4351$, $\log_e 4.4 = 1.4816$, $\log_e 4.6 = 1.526$, $\log_e 4.8 = 1.5686$, $\log_e 5 = 1.6094$, $\log_e 5.2 = 1.6486$.

17. Evaluate $\int_0^1 \frac{dx}{1+x}$ correct to three decimal places by Trapezoidal rule with $h = 0.5, 0.25,$ and 0.125. Use Romberg's method to get an accurate value for the definite integral. Hence find the value of $\log_e 2$.

18. A reservoir discharging water through sluices at a depth h feet below the water surface, has a surface area A for various values of h as given below.

h (ft)	10	11	12	13	14
A (in sq ft)	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by

$dh/dt = \dfrac{-48\sqrt{h}}{A}$. Estimate the time taken for the water level to fall from 14 to 10 feet above the sluices.

ANSWERS

1. 5.44
2. 1.6101
3. 0.8687
4. 24256.53
5. 0.9981, 1.0006; 0.0019, –0.0006
6. (i) 1.8276551, 1.8278472, 1.827847, 1.8278475
   errors: 0.0001924, 0.0000003, 0.0000005, 0.0000001
   (ii) 4.05617, 4.05106, 4.05116, 4.05098
   errors: –0.00522, –0.00011, –0.00021, –0.00003
7. 710 sq. ft
8. 25.4 miles
9. 30.87 m/s, h = 112.75 km
10. 296.7 yards
11. 5/3 miles
12. 2.8192
13. 59.68 cu. units
14. 532 m, 3 m/s^2
15. 1.0893 units
16. 1.8278
17. 0.708, 0.697, 0.694; 0.6931
18. 29 min(approx.)

CHAPTER 10

Difference Equations

10.1 INTRODUCTION

An equation that consists of an independent variable x, a dependent variable y_x [$= y(x)$] and one or several differences of the dependent variable, y_x, as $\Delta y_x, \Delta^2 y_x, \ldots, \Delta^n y_x$ is called a difference equation. That is, it is a functional relationship of the form:

$$F(x, y, \Delta y_x, \Delta^2 y_x, \ldots, \Delta^n y_x) = 0$$

Examples

(i) $\Delta^3 y_x + 3\Delta^2 y_x - 6\Delta y_x + y_x = 3x + 2$

(ii) $\Delta^2 y_x + 3\Delta y_x - 7y_x = 0$

Using $\Delta^k = (E - 1)^k$ and noting that $E^h y_x = y_{x+h}$ under the assumption that the interval of differencing is one, we can express the differences $\Delta y_x, \Delta^2 y_x$ etc. in terms of successive values of y_x.

$$\therefore \quad \Delta y_x = (E-1)y_x = Ey_x - y_x = y_{x+1} - y_x$$
$$\Delta^2 y_x = (E-1)^2 y_x = (E^2 - 2E + 1)y_x = y_{x+2} - 2y_{x+1} + y_x$$
$$\Delta^3 y_x = (E-1)^3 y_x = y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x$$

Hence, the difference equation (i) can be written as

$$y_{x+3} - 9y_{x+1} + 9y_x = 3x + 2$$

This equation can also be written as

$$y(x+3) - 9y(x+1) + 9y(x) = 3x + 2$$

or
$$(E^3 - 9E + 9)y_x = 3x + 2 \qquad (10.1)$$

Similarly, the difference equation (ii) can be written as

$$y_{x+2} - y_{x+1} - 9y_x = 0$$

or
$$(E^2 + E - 9)y_x = 0 \qquad (10.2)$$

Note: The study of *difference equations* is analogous to the study of *differential equations*. In most of the physical situations, the interval of differencing, h, is one. Hence we take $h = 1$ and proceed unless otherwise specifically mentioned.

10.2 ORDER OF DIFFERENCE EQUATION

The order of a difference equation written in a form free from Δs, is the difference between the highest and the lowest subscripts of the ys (or arguments of y).

Thus, the order of Eqn (10.1) is $(x + 3) - x = 3$ and for Eqn (10.2), it is $(x + 2) - x = 2$

10.3 DEGREE OF DIFFERENCE EQUATION

The degree of a difference equation written in a form free from Δs is the highest power of the ys.

Thus, for Eqns (10.1) and (10.2) the degree is one.

For the equation of the form

$y^2_{x+1}\, y^3_{x+2} - y_{x+1} y_x - y^2_x = x$, the order is 2 and the degree is 3.

10.4 SOLUTION TO DIFFERENCE EQUATION

A solution to a difference equation is any function which satisfies it. A general solution to a difference equation of order n involves n arbitrary constants.

A particular solution to a difference equation is obtained from the general solution by giving particular values to the constants.

For example, $y_x = A.2^x + B.3^x$ is the general solution to $y_{x+2} - 5y_{x+1} + 6y_x = 0$, while $y_x = 2^x$ or $y_x = 3^x$ or $y_x = 5(2^x) + 8(3^x)$ are particular solutions.

10.5 FORMATION OF DIFFERENCE EQUATIONS

The procedure is best illustrated by the following examples.

Example 10.1 Form the difference equation corresponding to the family of curves:

$$y = ax^2 + bx - 3$$

Solution In the given equation,

$$y_x = ax^2 + bx - 3 \qquad \text{(i)}$$

a and b are arbitrary constants to be eliminated.

$$\therefore\ y_{x+1} = a(x+1)^2 + b(x+1) - 3$$
$$y_{x+2} = a(x+2)^2 + b(x+2) - 3$$

Now,

$$\Delta y_x = y_{x+1} - y_x = (2x+1)a + b \qquad \text{(ii)}$$
$$\Delta^2 y_x = y_{x+2} + 2y_{x+1} + y_x = 2a$$

or

$$a = \frac{1}{2}\Delta^2 y_x \qquad \text{(iii)}$$

Therefore, from Eqn(ii)

$$b = \Delta y_x - \frac{1}{2}(2x+1)\,\Delta^2 y_x \qquad \text{(iv)}$$

Eliminating a, b from Eqns (i), (iii) and (iv), we get

$$y_x = \left(\frac{1}{2}\Delta^2 y_x\right)x^2 + \left(\Delta y_x - \frac{1}{2}[2x+1]\Delta^2 y_x\right)x - 3$$

or

$$(x+1)x\,\Delta^2 y_x - 2x\,\Delta y_x + 2\,y_x + 6 = 0$$

or

$$(x^2+x)y_{x+2} - 2\,(x^2+2x)y_{x+1} + (x^2+3x+2)y_x + 6 = 0$$

which is the required difference equation.

Example 10.2 Form the difference equation given that $y_n = A\,3^n + B\,5^n$, where A and B are arbitrary constants.

Solution Given $y_n = A3^n + B5^n$ (i)

$$\therefore\qquad y_{n+1} = A3^{n+1} + B5^{n+1} = 3\,A3^n + 5\,B5^n \qquad \text{(ii)}$$

and

$$y_{n+2} = A3^{n+2} + B5^{n+2} = 9\,A3^n + 25\,B5^n \qquad \text{(iii)}$$

Eliminating A and B from Eqns (i) – (iii), we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 3 & 5 \\ y_{n+2} & 9 & 25 \end{vmatrix} = 0$$

or $y_{n+2} - 8\,y_{n+1} + 15\,y_n = 0$ which is the required difference equation.

10.6 LINEAR DIFFERENCE EQUATIONS

A linear difference equation is one in which y_x, y_{x+1}, y_{x+2} etc. occur in the first degree only and are not multiplied together. The general form is

$$a_0 y_{x+n} + a_1 y_{x+n-1} + a_2 y_{x+n-2} + \cdots + a_n y_x = f(x) \quad (10.3)$$

$$\text{i.e., } [a_0 E^n + a_1 E^{n-1} + a_2 E^{n-2} + \cdots + a_n] y_x = f(x) \quad (10.4)$$

where $a_0, a_1, a_2, \ldots, a_n$ and $f(x)$ are known as functions of x.

Eqn (10.3) or (10.4) can be written as

$$\phi(E) y_x = f(x) \quad (10.5)$$

where $\phi(E)$ is a polynomial expression in E and is known as *non-homogeneous linear equation*. If $f(x) = 0$ in Eqn (10.5), then

$$\phi(E) y_x = 0 \quad (10.6)$$

and is known as *homogeneous linear equation* corresponding to Eqn (10.5).

Analogous to properties of linear differential equations, we can easily prove the following properties.

1. If $y_x = f_1(x)$ is a solution to Eqn (10.6), then $y_x = c_1 f_1(x)$ is also a solution to Eqn (10.6), where c_1 is any constant.
2. If $y_x = f_1(x), y_x = f_2(x), \ldots, y_x = f_n(x)$ are n independent solutions to Eqn (10.6), then its general solution is $y_x = \sum_{i=1}^{n} c_i f_i(x)$, where c_i are constants.
3. If $y_x = u_x$ is a particular solution to the non-homogeneous Eqn (10.5), then

   $y_x = \sum_{i=1}^{n} c_i f_i(x) + u_x$ is general solution to Eqn (10.5).

   Here, $\sum_{i=1}^{n} c_i f_i(x)$ is called the *complementary function* (CF) of Eqn (10.5) [or general solution to Eqn (10.6)] and u_x is called the *particular integral* (PI) of Eqn (10.5).

   Therefore, y_x = CF + PI is the general solution of Eqn (10.5).

10.7 LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

These equations are of the form

$$a_0 y_{x+n} + a_1 y_{x+n-1} + \cdots + a_{n-1} y_{x+1} + a_n y_x = 0$$

$$\text{i.e., } (a_0 E^n + a_1 E^{n-1} + \cdots + a_{n-1} E + a_n) y_x = 0 \quad (10.7)$$

where $a_0, a_1, \ldots a_{n-1}, a_n$ are constants. Eqn (10.7) can be put in the form $\phi(E) y_x = 0$, where $\phi(E) = a_0 E^n + a_1 E^{n-1} + \cdots + a_{n-1} E + a_n$, which is a polynomial of nth degree in E. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n distinct and real roots of $\phi(E) = 0$ then in factor form,

$$\phi(E) = a_0 (E - \alpha_1)(E - \alpha_2) \ldots (E - \alpha_n) = 0$$

Hence Eqn (10.7), i.e $\phi(E)\, y_x = 0$ is

$$(E-\alpha_1)(E-\alpha_2)\dots(E-\alpha_n)\, y_x = 0 \tag{10.8}$$

If ux is the solution of $(E-\alpha_1)\, y_x = 0$ then it will satisfy $(E-\alpha_2)\dots(E-\alpha_n)(E-\alpha_1)\, y_x = 0$ also. Hence, the complete solution to Eqn (10.7) is made up of the solutions to the component equations $(E-\alpha_1)\, y_x = 0$, $(E-\alpha_2)\, y_x = 0, \dots, (E-\alpha_n)\, y_x = 0$.

Now, consider $(E-\alpha_1)\, y_x = 0$, i.e. $y_{x+1} - \alpha_1 y_x = 0$

or $$\frac{y_{x+1}}{\alpha_1^{x+1}} - \frac{y_x}{\alpha_1^x} = 0 \quad \text{or} \quad \Delta\left(\frac{y_x}{\alpha_1^x}\right) = 0$$

$y_x = c_1\, \alpha_1^x$ is the solution.

Similarly, $c_2\, \alpha_2^x, \dots, c_n\, \alpha_n^x$ are the solutions of the other component equations.

Hence the general solution to Eqn (10.7), i.e., $\phi(E)\, y_x = 0$ is

$$y_x = c_1\, \alpha_1^x + c_2\, \alpha_2^x + \dots + c_n\, \alpha_n^x.$$

Here, $\phi(E) = 0$ is called the *auxiliary equation* (AE).

Case 1 When the auxiliary equation $\phi(E) = 0$ has repeated real roots, say, $\alpha_2 = \alpha_1$ and the others are real and distinct.

Let $\alpha_2 = \alpha_1 = \alpha$ and $\alpha_3, \dots, \alpha_n$ be real and distinct.

Then $y_x = A\alpha^x + c_3\alpha_3^x + \dots + c_n\alpha_n^x$, which is not a general solution to Eqn (10.7) due to the reason that it contains only $(n-1)$ arbitrary constants.

Now consider $(E-\alpha)^2\, y_x = 0$

i.e. $$y_{x+2} - 2\alpha\, y_{x+1} + \alpha^2 y_x = 0 \tag{10.9}$$

To solve it, assume

$$y_x = Z_x \alpha^x \tag{10.10}$$

where Z_x is some function of x. Substituting in Eqn (10.9), we get

$$Z_{x+2}\alpha^{x+2} - 2\alpha\, Z_{x+1}\alpha^{x+1} + \alpha^2 Z_x \alpha^x = 0$$

or $(Z_{x+2} - 2\, Z_{x+1} + Z_x) = 0$ (since $\alpha^{x+2} \neq 0$)

i.e. $(E-1)^2\, Z_x = 0$ or $\Delta^2 Z_x = 0$

$\Rightarrow Z_x = c_1 + c_2 x$

$\therefore$ From Eqn (10.10), $y_x = (c_1 + c_2 x)\alpha^x$

and hence, the general solution to Eqn (10.7) is

$$y_x = (c_1 + c_2 x)\alpha^x + c_3\alpha_3^{\,x} + \dots + c_n\alpha_n^{\,x} \tag{10.11}$$

This can be extended. If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and the other roots are real and distinct, then

$$y_x = (c_1 + c_2 x + c_3 x^2)\alpha^x + c_4\alpha_4^{\,x} + \dots + c_n\alpha_n^{\,x} \tag{10.12}$$

and if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$ (all roots are equal) then the general solution to Eqn (10.7) is

$$y_x = (c_1 + c_2x + c_3x^2 + \cdots + c_nx^{n-1})\,\alpha^x \tag{10.13}$$

Case 2 When the auxiliary equation $\phi(E) = 0$ has imaginary roots.

Let $\alpha_1 = a + ib$; $\alpha_2 = a - ib$ and the other roots be real and distinct. Then, the general solution to $\phi(E)\,y_x = 0$ is

$$y_x = c_1(a+ib)^x + c_2(a-ib)^x + c_3\alpha_3^x + \cdots + c_n\alpha_n^x \tag{10.14}$$

Now let $a + ib = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$

Then $(a+ib)^x = r^x(\cos x\theta + i\sin x\theta)$

also $(a-ib)^x = r^x(\cos x\theta - i\sin x\theta)$ by using *DeMoivres theorem.* Substituting in Eqn. (10.14), we get after simplification the general solution to Eqn (10.7) as

$$y_x = (A\cos x\theta + B\sin x\theta)r^x + c_3\alpha_3^x + \cdots + c_n\alpha_n^x \tag{10.15}$$

where $A = c_1 + c_2$ and $B = c_1 - c_2$

Case 3 When the auxiliary equation $\phi(E) = 0$ has repeated imaginary roots.

Suppose $a + ib$ and $a - ib$ are repeated twice. Then, the general solution to Eqn (10.7) is

$$y_x = [(c_1 + c_2x)\cos x\,\theta + (c_3 + c_4x)\sin x\,\theta]r^x + c_5\alpha_5^x + \cdots + c_n\alpha_n^x \tag{10.16}$$

Example 10.3 Solve the difference equation

$$y_{x+3} - 3\,y_{x+2} - 10\,y_{x+1} + 24\,y_x = 0$$

Solution The given equation in symbolic form is

$$(E^3 - 3E^2 - 10E + 24)\,y_x = 0$$

Its auxiliary equation is $E^3 - 3E^2 - 10E + 24 = 0$

or $(E-2)(E+3)(E-4) = 0 \quad \therefore E = 2, -3, 4$

Thus, the general solution is

$$y_x = c_1 2^x + c_2(-3)^x + c_3 4^x$$

Example 10.4 Solve the difference equation

$$u_{n+4} - 8\,u_{n+3} + 18\,u_{n+2} - 27\,u_n = 0$$

Solution The given equation in symbolic form is

$$(E^4 - 8E^3 + 18E^2 - 27)\,u_n = 0$$

Its AE is $E^4 - 8E^3 + 18E^2 - 27 = 0$

or $(E+1)(E-3)^3 = 0 \quad \therefore E = -1, 3, 3, 3$

Thus, the general solution is
$u_n = c_1(-1)^n + (c_2 + c_3 n + c_4 n^2)\, 3^n$

Example 10.5 Solve $y_{x+1} - 2y_x \cos\alpha + y_{x-1} = 0$

Solution The given equation in symbolic form is

$$(E^2 - 2E\cos\alpha + 1)\, y_{x-1} = 0$$

and its AE is $E^2 - 2E\cos\alpha + 1 = 0$

$$\therefore\ E = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2} = \cos\alpha \pm i\sin\alpha$$

$\therefore\ r = 1$ and $\theta = \alpha$

Thus, the solution is

$y_{x-1} = \{c_1\cos\alpha(x-1) + c_2\sin\alpha(x-1)\}\,(1)^{x-1}$

or $y_x = c_1\cos\alpha x + c_2\sin\alpha x$

Example 10.6 Solve $y_{n+4} - 4y_{n+3} + 8y_{n+2} - 8y_{n+1} + 4y_n = 0$

Solution The symbolic form of the given equation is

$$(E^4 - 4E^3 + 8E^2 - 8E + 4)\, y_n = 0$$

Its AE is $E^4 - 4E^3 + 8E^2 - 8E + 4 = 0$

or $(E^2 - 2E + 2)^2 = 0 \Rightarrow E = 1 \pm i,\ 1 \pm i$

Here, $r = \sqrt{2}$, $\theta = \pi/4$

$\therefore$ The complete solution is

$$y_n = \left\{(c_1 + c_2 n)\cos\frac{n\pi}{4} + (c_3 + c_4 n)\sin\frac{n\pi}{4}\right\}(\sqrt{2})^n$$

Example 10.7 Form the Fibonacci difference equation and solve it.

Solution The Fibonacci numbers are members of an interesting sequence in which each number is equal to the sum of the previous two numbers. The integers 0, 1, 1, 2, 3, 5, 8, 13, 21, . . . are said to form a Fibonacci sequence.

Let u_n be the nth term of the sequence.

Then $u_n = u_{n-1} + u_{n-2}$ if $n > 2$

or $u_{n+2} - u_{n+1} - u_n = 0$ if $n > 0$

and in symbolic form, $(E^2 - E - 1)u_n = 0$

AE is $E^2 - E - 1 = 0 \Rightarrow E = \dfrac{1 \pm \sqrt{5}}{2}$

$$\therefore\quad u_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \text{ if } n > 0 \qquad \text{(i)}$$

Now $$u_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 0 \qquad \text{(ii)}$$

and $$u_2 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1 \qquad \text{(iii)}$$

Solving (ii) and (iii), we get

$$c_1 = \frac{5-\sqrt{5}}{10} \text{ and } c_2 = \frac{5+\sqrt{5}}{10}$$

Hence,

$$u_n = \left[\frac{5-\sqrt{5}}{10}\right]\left[\frac{1+\sqrt{5}}{2}\right]^n + \left[\frac{5+\sqrt{5}}{10}\right]\left[\frac{1-\sqrt{5}}{2}\right]^n$$

is the required solution.

10.8 NON-HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

These are equations of the form

$$\phi(E)\, y_x = f(x) \qquad (10.17)$$

where $\phi(E) = a_0 E^n + a_1 E^{n-1} + \cdots + a_{n-1}E + a_n$
and the complete solution is given by

$y_x = \text{CF} + u_x$. Here, CF is the solution of $\phi(E)y_x = 0$ and the particular integral u_x is some function of x, such that $\phi(E)u_x = f(x)$.

Operating on both sides by $1/\phi(E)$, it results

$u_x = \dfrac{1}{\phi(E)} f(x)$ which represents the symbolic notation for the particular integral (PI). Let us now see the evaluation of particular integrals in certain cases.

Case 1 When $f(x) = a^x$, where a is a constant.

Then,
$$\begin{aligned}\phi(E)a^x &= (a_0E^n + a_1 E^{n-1} + \cdots + a_{n-1}E + a_n)a^x \\ &= (a_0a^{x+n} + a_1a^{x+n-1} + \cdots + a_{n-1}a^{x+1} + a_na^x) \\ &= (a_0a^n + a_1a^{n-1} + \cdots + a_{n-1}a + a_n)\, a^x \\ &= \phi(a)a^x\end{aligned}$$

$$\therefore \quad \frac{1}{\phi(E)}\phi(E)a^x = \frac{1}{\phi(E)}\phi(a)a^x \qquad (10.18)$$

or $\dfrac{1}{\phi(E)}\, a^x = \dfrac{1}{\phi(a)}\, a^x$ provided $\phi(a) \neq 0$.

Case of failure: If $\phi(a) = 0$ then $\phi(E)$ will have any one of the following factors:

$$(E - a) \text{ or } (E - a)^2 \text{ etc., or } (E - a)^n$$

Now let $$\frac{1}{E-a}\, a^x = u_x$$

$\therefore$ $$u_{x+1} - a u_x = a^x \text{ or } a^{-(x+1)} u_{x+1} - a^{-x} u_x = a^{-1}$$

or $\Delta(a^{-x} u_x) = a^{-1}$ which gives $(a^{-x} u_x) = a^{-1}x$

$\therefore$ $$ux = x\, a^{x-1}, \text{ i.e. } \frac{1}{E-a}\, a^x = x\, a^{x-1}$$

Similarly,

$$\frac{1}{(E-a)^2}\, a^x = \frac{x(x-1)}{2!} a^{x-2}$$

and in general $$\frac{1}{(E-a)^n} a^x = \frac{x(x-1)\ldots(x-n+1)}{n!}\, a^{x-n}$$

Case 2 When $f(x) = a^x F(x)$, where $F(x)$ is some function of x.

Noting that $E^n a^x F(x) = a^{x+n} F(x+n) = a^x a^n E^n F(x)$,

we have $\phi(E)\ \{a^x F(x)\}$

$$= a^x (a_0 a^n E^n + a_1 a^{n-1} E^{n-1} + \cdots + a_{n-1} aE + a_n) F(x)$$
$$= a^x\, \phi(aE)\, F(x).$$

Therefore, the inverse result is

$$\frac{1}{\phi(E)}\left\{a^x F(x)\right\} = a^x \frac{1}{\phi(E)} F(x)$$

Case 3 When $f(x)$ is a polynomial in x of degree m (say).

In this case, $\dfrac{1}{\phi(E)} f(x) = \dfrac{1}{\phi(1+\Delta)} f(x) = [\phi(1+\Delta)]^{-1} f(x)$

Here, we expand $[\phi(1+\Delta)]^{-1}$ in ascending powers of Δ and operate on $f(x)$.

Example 10.8 Solve $y_{n+2} - 4y_{n+1} + 4y_n = 3^n + 2^n + 4$.

Solution The given equation in symbolic form is

$$(E^2 - 4E + 4)\, y_n = 3^n + 2^n + 4 \qquad \text{(i)}$$

AE is $E^2 - 4E + 4 = 0$,

i.e. $(E-2)^2 = 0 \therefore E = 2, 2$

$\therefore$ $\text{CF} = (C_1 + C_2 n)\, 2^n$

$$\text{PI for } 3^n = \frac{1}{E^2-4E+4}3^n = \frac{1}{(E-2)^2}3^n = \frac{1}{(3-2)^2} = 3^n$$

$$\text{PI for } 2^n = \frac{1}{(E-2)^2}2^n = \frac{n(n-1)}{2!}\,2^{n-2}$$

$$\text{PI for } 4 = \frac{1}{(E-2)^2}\,4 = 4\,\frac{1}{(E-2)^2}\,1^n = 4$$

Hence, the general solution to Eqn (i)

$y_n = (C_1 + C_2 n)\, 2^n + 3^n + n(n-1)2^{n-3} + 4.$

Example 10.9 Solve $y_{n+2} - 4\, y_{n+1} + 3y_n = 3^n + 1.$

Solution The given equation in symbolic form is

$$(E^2 - 4E + 3)\, y_n = 3^n + 1 \qquad \text{(i)}$$

AE is $E^2 - 4E + 3 = 0$, i.e. $(E-1)(E-3) = 0$, $\therefore E = 1, 3$

$\therefore$ $\text{CF} = C_1 (1)^n + C_2\, 3^n = C_1 + C_2\, 3^n$

$$\text{PI} = \frac{1}{E^2-4E+3}(3^n+1) = \frac{1}{(E-1)(E-3)}(3^n+1)$$

$$= \frac{1}{(E-1)(E-3)}3^n + \frac{1}{(E-1)(E-3)}(1)^n$$

$$= \frac{1}{(3-1)}\;\frac{1}{(E-3)}\,3^n + \frac{1}{(1-3)}\cdot\frac{1}{(E-1)}(1)^n$$

$$= \frac{1}{2}n\, 3^{n-1} - \frac{1}{2}n\,(1)^{n-1} = \frac{1}{2}n\,(3^{n-1}-1)$$

Hence, the general solution to Eqn (i) is

$$y_n = C_1 + C_2\, 3^n + \frac{1}{2}n\,(3^{n-1}-1)$$

Example 10.10 Solve $u(x+2) - 4\, u(x) = 9x^2$

Solution The symbolic form of the given equation is

$$(E^2 - 4)\, u_x = 9x^2 \qquad \text{(i)}$$

AE is $E^2 - 4 = 0 \;\therefore\; E = \pm 2$
and CF $= C_1 2^x + C_2(-2)^x$

$$\text{PI} = \frac{1}{E^2-4}(9x^2) = 9\frac{1}{(1+\Delta)^2-4}(x^2)$$

$$= 9\frac{1}{\Delta^2+2\Delta-3}(x^2) = -3\left[1-\frac{2\Delta+\Delta^2}{3}\right]^{-1} x^2$$

$$= -3\left[1+\left\{\frac{2\Delta+\Delta^2}{3}\right\}+\left\{\frac{2\Delta+\Delta^2}{3}\right\}^2+\cdots\right]x^2$$

$$= -3\left[1+\frac{2\Delta}{3}+\frac{7\Delta^2}{9}\right]x^2$$

[by neglecting $\Delta^3, \Delta^4 \ldots$ etc.]

$$= -3\left[x^2+\frac{2}{3}\Delta x^2+\frac{7}{9}\Delta^2(x^2)\right] \qquad \text{(ii)}$$

Now $\Delta(x^2) = (x+1)^2 - x^2 = 2x + 1$
$\Delta^2(x^2) = [2(x+1)+1] - [2x+1] = 2$
Substituting in Eqn (ii), we get

$$\text{PI} = -3\left[x^2+\frac{2}{3}(2x+1)+\frac{7}{9}(2)\right]$$

$$= -3\left(x^2+\frac{4x}{3}+\frac{20}{9}\right)$$

Hence, the general solution to Eqn (i) is given by

$$u_x = C_1 2^x + C_2(-2)^x - 3x^2 - 4x - 20/3$$

Alternative Method: The PI can be determined by the method of undetermined coefficients.

Let $u_x = Ax^2 + Bx + C$. Substituting in Eqn (i), we get
$A(x+2)^2 + B(x+2) + C - 4(Ax^2 + Bx + C) = 9x^2$
i.e. $-3Ax^2 + x(4A - 3B) + (4A + 2B - 3C) = 9x^2$
Equating the coefficients,
$-3A = 9\,;\; 4A - 3B = 0\,;\; 4A + 2B - 3C = 0$

which gives $A = -3$, $B = -4$, $C = -\dfrac{20}{3}$

Hence, the PI $= -3x^2 - 4x - \dfrac{20}{3}$

Example 10.11 Solve $\Delta u_x + \Delta^2 u_x = \sin x$.

Solution The given equation in symbolic form is

$$\{(E-1)+(E-1)^2\}u_x = \sin x \qquad \text{(i)}$$

or $$(E^2 - E)\, u_x = \sin x$$

or $$(E-1)\, u_{x+1} = \sin x \quad \text{(this is of order 1)}$$

$\therefore$ $$\text{AE is } E-1=0, \text{ i.e. } E=1$$

and $$\text{CF} = C_1(1)^x = C_1$$

$$\text{PI} = \frac{1}{E-1}\sin x = \text{IP of } \frac{1}{E-1}e^{ix}$$

$$= \text{IP of } \frac{1}{E-1}\left(e^i\right)^x$$

$$= \text{IP of } \frac{e^{ix}}{e^i-1} = \text{IP of } \frac{e^{ix}(e^{-i}-1)}{(e^i-1)(e^{-i}-1)}$$

$$= \text{IP of } \frac{e^{i(x-1)}-e^{ix}}{1-(e^i+e^{-i})+1}$$

$$= \frac{\sin(x-1)-\sin x}{2(1-\cos 1)}$$

Hence, the general solution to Eqn (i) is

$$u_{x+1} = C_1 + \frac{\sin(x-1)-\sin x}{2(1-\cos 1)}$$

$$\text{or } u_x = C_1 + \frac{\sin(x-2)-\sin(x-1)}{2(1-\cos 1)}$$

Example 10.12 Solve $y_{n+2} - 7y_{n+1} - 8y_n = (n^2-n)\, 2^n$

Solution The given equation in symbolic form is

$$(E^2 - 7E - 8)y_n = (n^2-n)2^n \qquad \text{(i)}$$

AE is $E^2 - 7E - 8 = 0 \;\therefore\; E = 8, -1$

$CF = C_1 8^n + C_2(-1)^n$

$$\text{PI} = \frac{1}{E^2-7E-8}(n^2-n)2^n$$

$$= 2^n \frac{1}{(2E)^2 - 7(2E) - 8}(n^2-n)$$

$$= 2^n \frac{1}{4E^2 - 14E - 8}(n^2 - n)$$

$$= 2^{n-1} \frac{1}{2(1+\Delta)^2 - 7(1+\Delta) - 8}(n^2 - n)$$

$$= 2^{n-1} \frac{1}{2\Delta^2 - 3\Delta - 9}(n^2 - n)$$

$$= -\frac{2^{n-1}}{9}\left[1 - \frac{2\Delta^2 - 3\Delta}{9}\right]^{-1}(n^2 - n)$$

$$= -\frac{2^{n-1}}{9}\left[1 + \frac{2\Delta^2 - 3\Delta}{9} + \left(\frac{2\Delta^2 - 3\Delta}{9}\right)^2 + \cdots\right](n^2 - n)$$

$$= -\frac{2^{n-1}}{9}\left[1 + \frac{2}{9}\Delta^2 - \frac{\Delta}{9} + \frac{\Delta^2}{9}\right](n^2 - n)$$

[$\therefore$ From third difference, the value = 0]

$$= -\frac{2^{n-1}}{9}\left[1 - \frac{\Delta}{9} + \frac{\Delta^2}{9}\right](n^2 - n) \quad \text{(ii)}$$

Now $\Delta(n^2 - n) = \{[(n+1)^2 - (n+1)] - (n^2 - n)\} = 2n$

$\Delta^2(n^2 - n) = \Delta[\Delta(n^2 - n)] = \Delta(2n)$

$= 2(n+1) - 2n = 2$

Substituting in Eqn (ii), we get

$$\text{PI} = -\frac{2^{n-1}}{9}\left[n^2 - n - \frac{1}{3}(2n) + \frac{1}{3}(2)\right]$$

$$= -\frac{2^{n-1}}{27}(3n^2 - 5n + 2)$$

Hence, the general solution to Eqn (i) is

$$y_n = C_1 8^n + C_2(-1)^n - \frac{2^{n-1}}{27}(3n^2 - 5n + 2)$$

EXERCISE 10.1

Form the difference equations by eliminating arbitrary constants.

1. $y = C_1 3^x + C_2 8^x$
2. $y = C_1 x^2 + C_2 x + 9$
3. $y = (C_1 + C_2 n)(-2)^n$
4. $y = C_1 x^2 + C_2 x + C_3$

Solve the following difference equations.

5. $y_{n+3} - 2y_{n+2} - y_{n+1} + 2y_n = 0$
6. $y_{x+4} + y_{x+3} - 13y_{x+2} - y_{x+1} + 12y_x = 0$
7. $y_{n+2} + 2y_{n+1} + 4y_n = 0$
8. $\Delta^3 u_n - 5\Delta u_n + 4u_n = 0$
9. $u_{k+4} + 6u_{k+3} + 9u_{k+2} - 4u_{k+1} - 12u_k = 0$
10. $y_{x+4} - 9y_{x+3} + 30y_{x+2} - 44y_{x+1} + 24y_x = 0$
11. $y_{x+2} - y_{x+1} + y_x = 0$, given $y_0 = 1$ and $y_1 = \dfrac{1+\sqrt{3}}{2}$
12. $u_{x+4} - 5u_{x+2} + 8u_{x+1} - 4u_x = 0$, given $u_0 = 3$, $u_1 = 2$, and $u_4 = 22$
13. If y_k satisfies the difference equation $y_{k+1} - \alpha y_k + y_{k-1} = 0$, $k = 1, 2, 3$ and the end conditions $y_0 = y_4 = 0$, show that non-trivial solution exists when $\alpha = 0, \pm\sqrt{2}$.
14. If y_n satisfies $y_{n+1} - 2y_n \cos\alpha + y_{n-1} = 0$ for $n = 1, 2, \ldots$ and if $y_0 = 0$, $y_1 = 0$, find y_2, y_3, y_4.

Solve the following difference equations.

15. $y_{x+2} - 6y_{x+1} + 8y_x = 4^x$
16. $y_{n+2} - 3y_{n+1} + 2y_n = 5^n + 2^n$
17. $u_{x+2} - 4u_{x+1} + 4u_x = 3.2^x + 5.4^x$
18. $u_{n+2} - 4u_{n+1} + 3u_n = 2^n + 3^n + 7$
19. $y_{x+2} - 5y_{x+1} + 6y_x = x^2 + x + 1$
20. $\Delta^2 u_x + 2\Delta u_x + u_x = 3x + 2$
21. $\Delta u_x + \Delta^2 u_x = \cos x$
22. $u(x+2) - 7u(x+1) + 12u(x) = \cos x$
23. $u_{n+2} - 7u_{n+1} - 8u_n = 2^n n^2$
24. $u_{x+2} - 2u_{x+1} + u_x = 2^x x^2$
25. $2u_{n+2} + 5u_{n+1} + 2u_n = 2^n + n^2$

ANSWERS

1. $y_{x+2} - 11y_{x+1} + 24y_x = 0$
2. $x(1+x)y_{x+2} - 2x(x+2)y_{x+1} + (x^2 + 3x + 2)y_x + 9 = 0$
3. $y_{n+2} + 4y_{n+1} + 4y_n = 0$

4. $y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x = 0$
5. $y_n = C_1 + C_2(-1)^n + C_3 2^n$
6. $y_x = C_1 + C_2(-1)^x + C_3 3^x + C_4(-4)^x$
7. $y_n = \left\{C_1 \cos\frac{2n\pi}{3} + C_2 \sin\frac{2n\pi}{3}\right\} 2^n$
8. $u_n = C_1\, 2^n + C_2\left[\frac{1+\sqrt{17}}{2}\right]^n + C_3\left[\frac{1-\sqrt{17}}{2}\right]^n$
9. $u_k = C_1 + C_2\,(-3)^k + (C_3 + C_4 k)\,(-2)^k$
10. $y_x = (C_1 + C_2 x + C_3 x^2)2^x + C_4(-3)^x$
11. $y_x = \cos\frac{n\pi}{3} + \sin\frac{n\pi}{3}$
12. $y_x = 6 + (-3 + n)2^n$
14. $y_2 = 2\cos\alpha$; $y_3 = 4\cos^2\alpha - 1$; $y_4 = 8\cos^3\alpha - 4\cos\alpha$
15. $y_x = C_1 2^x + C_2 4^x + \frac{x}{8} 4^x$
16. $y_n = C_1 + C_2\, 2^n + \frac{5^n}{12} - n\, 2^{n-1}$
17. $u_x = (C_1 + C_2\, x)2^x + 3x(x-1)\, 2^{x-3} + 5 \,.\, 4^{x-1}$
18. $u_n = C_1 + C_2\, 3^n - 2^n + \frac{n}{2} 3^{n-1} - \frac{7n}{2}$
19. $y_x = C_1 2^x + C_2 3^x + \frac{1}{4}(2x^2 + 8x + 15)$
20. $u_x = 3x - 4$
21. $u_x = C_1 + \frac{\cos(x-2) - \cos(x-1)}{2(1-\cos 1)}$
22. $u_x = C_1 4^x + C_2 3^x + \frac{\cos(x-2) - 7\cos(x-1) + 12\cos x}{24\cos 2 - 182\cos 1 + 194}$
23. $u_n = C_1 8^n + C_2(-1)^n - \frac{2^{n-1}}{9}\left(n^2 - \frac{2n}{3} + \frac{1}{3}\right)$
24. $u_x = C_1 + C_2 x + 2^x(x^2 - 8x + 20)$
25. $u_n = C_1(-2)^n + C_2\left(-\frac{1}{2}\right)^n + \frac{2n}{20} + \frac{1}{9}\left(n^2 - 2n + \frac{5}{9}\right)$

CHAPTER 11

Numerical Solution to Ordinary Differential Equations

11.1 INTRODUCTION

Many problems in Engineering and Science can be formulated into ordinary differential equations satisfying certain given conditions. If these conditions are prescribed for one point only, then the differential equation together with the conditions is known as an *initial value problem.* If the conditions are prescribed for two or more points then the problem is termed as *boundary value problem.* Analytically, the solution to an ordinary differential equation in which x is the independent variable and y is the dependent variable means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution is known as the *closed* or *finite* form of solution. But the analytical methods are applicable only to a select class of differential equations. Since the differential equations appearing in the fields of Engineering and Science are due to physical and natural phenomena, they do not belong to any of the above class and hence, cannot have a closed solution. In such cases, we try to approximate a particular solution to the differential equation, i.e. we find numerical values of $y_1, y_2, y_3, \ldots$ corresponding to given numerical values of independent variable values $x_1, x_2, x_3, \ldots$, so that the ordered pairs (x_1,y_1), (x_2,y_2), (x_3,y_3), . . . satisfy approximately a pre-assigned particular solution. A solution of this type is called *pointwise solution.*

Let us consider the first order differential equation $dy/dx = f(x,y)$ given $y(x_0) = y_0$. Let $y = f(x)$ be the exact solution (smooth curve in Fig.11.1) and $y_1, y_2, y_3, \ldots$ be the pointwise solutions at $x = x_1, x_2, x_3, \ldots$, using a suitable recursive formula (dotted curve in the figure). Computation of these approximate values is known as *Numerical solution* to the differential equation. The difference between the computed value y_i and the true value $f(x_i)$, say, ε_i, is termed as *truncation error* at $x = x_i$. There are many numerical methods for the approximate solution to ordinary differential equations given an initial condition. The most important and frequently used procedures are discussed in this chapter.

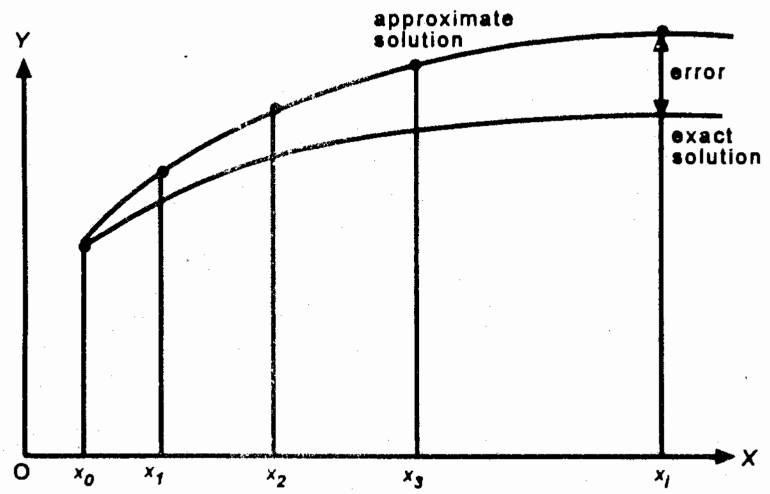

Fig. 11.1

11.2 POWER SERIES SOLUTION

Consider the differential equation

$$\frac{dy}{dx} = y' = f(x, y) \tag{11.1}$$

subject to the condition

$$y(x_0) = y_0 \tag{11.2}$$

This is an initial value problem. Now we can expand the solution to Eqn (11.1), i.e., $y(x)$ in the neighbourhood of $x = x_0$ in power series as

$$y(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \cdots \tag{11.3}$$

where $A_0, A_1, A_2, \ldots$ are constants to be determined such that Eqn (11.3) satisfies Eqn (11.1) subject to Eqn (11.2).

To expand $y(x)$ in power series, generally, ***Taylor's theorem*** or ***Maclaurin's theorem*** is used. If the boundary condition is at x_0 ($\neq 0$), then we use *Taylor's series* to expand $y(x)$ about $x = x_0$ and it is

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \cdots \qquad (11.4)$$

If the boundary condition is at $x_0 = 0$, then we use *Maclaurin's series* to expand $y(x)$ about $x = 0$ and it is

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \cdots \qquad (11.5)$$

The method is best illustrated by the following examples.

Example 11.1 Evaluate the solution to the differential equation $y' = (1 + x)\, xy^2$ subject to $y(0) = 1$ by taking five terms in Maclaurin's series for $x = 0(0.1)\, 0.4$.

Compare the values with the exact solution.

Solution Maclaurin's series is

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0) + \cdots \qquad \text{(i)}$$

The given equation is $y' = (1 + x)\, xy^2$; $y(0) = 1$

$$\therefore\; y'' = 2xyy' + y^2 + 2xy^2 + 2yy'x^2$$

$$y''' = 4yy' + 8xyy' + 2y^2 + 2xy'^2 + 2xyy'' + 2x^2yy'' + 2x^2y'^2$$

$$y^{iv} = 2yy'^2 + 6y'^2 + 6xy'y'' + 6yy'' + 2xyy''' + 12yy' +$$

$$12xyy'' + 12xy'^2 + 6x^2y'y'' + 2x^2yy'''$$

$$\therefore\; y'(0) = 0\,,\; y''(0) = 1\,,\; y'''(0) = 2 \text{ and } y^{iv}(0) = 6$$

Putting these values in Eqn (i), we get

$$y(x) = 1 + \frac{x^2}{2}(1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(6)$$

$$= 1 + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \qquad \text{(ii)}$$

Putting $x = 0.1, 0.2, 0.3$, and 0.4 successively in Eqn (ii), we have

$$y_1 = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{4} = 1.0053583$$

$$y_2 = 1 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{4} = 1.0230667$$

$$y_3 = 1 + \frac{(0.3)^2}{2} + \frac{(0.3)^3}{3} + \frac{(0.3)^4}{4} = 1.056025$$

$$y_4 = 1 + \frac{(0.4)^2}{2} + \frac{(0.4)^3}{3} + \frac{(0.4)^4}{4} = 1.1077333$$

Exact solution : Given $\dfrac{dy}{dx} = (x + x^2)y^2$

i.e. $$\frac{dy}{y^2} = (x + x^2)dx$$

On integrating ,

$$-\frac{1}{y} = \left(\frac{x^2}{2} + \frac{x^3}{3}\right) + C \qquad \text{(iii)}$$

Using the condition $y(0) = 1$, i.e $y = 1$ at $x = 0$ in Eqn (iii),
we get $C = -1$

$\therefore$ Eqn (iii) gives $-1/y = \left(\dfrac{x^2}{2} + \dfrac{x^3}{3}\right) - 1$

$$\text{or } y = 6(6 - 3x^2 - 2x^3) \qquad \text{(iv)}$$

Now putting $x = 0.1, 0.2, 0.3$ and 0.4 successively in Eqn (iv) we get $y_1 = 1.0053619$, $y_2 = 1.0231924$, $y_3 = 1.0570825$ and $y_4 = 1.1127596$.
Now the numerical and exact solutions to y for corresponding values of x in tabular form is

x	app. value of x (a)	Exact value of x (b)	Error (a) – (b)
0	1	1	0
0.1	1.0053583	1.0053619	-3.6×10^{-6}
0.2	1.0230667	1.0231924	-1.257×10^{-4}
0.3	1.056025	1.0570825	-1.0575×10^{-3}
0.4	1.1077333	1.1127596	-5.0263×10^{-3}

11.3 POINTWISE METHODS

In the above example, we have determined the approximate values of y corresponding to $x_1, x_2, x_3, \ldots$ by directly substituting in the function $y(x)$ expanded in series about $x = x_0$. But these approximations are valid only in a small neighbourhood covering the point x_0. For greater precision, it is better to find out y_1 and then use it to find y_2. Then use y_2 to find out y_3 and so on. In general, y_n will be found by using predetermined ordinates. These type of methods of solution, i.e. where a step is taken only after carrying out all the previous steps, are known as *step by step methods* or *pointwise methods.*

11.4 SOLUTION BY TAYLOR'S SERIES

Let $$y' = f(x, y) \ ; \ y(x_0) = y_0 \qquad (11.6)$$

be the differential equation to which the numerical solution is required. Expanding $y(x)$ about $x = x_0$ by Taylor's series,

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

$$= y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots$$

Putting $x = x_1 = x_0 + h$, we have

$$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \qquad (11.7)$$

Here, y_0', y_0'', y_0''', ... can be found using Eqn (11.6) and its successive differentiations at $x = x_0$. The series in Eqn (11.7) can be truncated at any stage if h is small. Now, having obtained y_1, we can calculate y_1', y_1'', y_1''' ... from Eqn (11.6) at $x_1 = x_0 + h$.

Now expanding $y(x)$ by Taylor's series about $x = x_1$, we get

$$y_2 = y_1 + \frac{h}{1!} y' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

Proceeding on, we get

$$y_n = y_{n-1} + \frac{h}{1!} y_{n-1}' + \frac{h^2}{2!} y_{n-1}'' + \frac{h^3}{3!} y_{n-1}''' + \cdots \qquad (11.8)$$

where $$y_{n-1}^{r} = \left[\frac{d^r}{dx^r}(y_{n-1})\right]_{(x_{n-1}, y_{n-1})}$$

By taking sufficient number of terms in the above series, the value of y_n can be obtained without much error.

If we retain the terms upto h^n on the RHS of Eqn (11.8), the error will be proportional to the $(n + 1)$th power of step size, i.e. h^{n+1} and Taylor's algorithm is said to be of nth order. By including more number of terms on the RHS of Eqn (11.8), the error can be reduced further.

Example 11.2 Using Taylor's series method, solve $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$ at $x = 0.1, 0.2, 0.3$ and 0.4. Compare the values with the exact solution.

Solution Given : $y' = x^2 - y \ ; \ y(0) = 1$

$\therefore\ x_0 = 0,\ y_0 = 1,\ h = 0.1,\ x_1 = 0.1, x_2 = 0.2,\ x_3 = 0.3$ and $x_4 = 0.4$

Now, $\quad y' = x^2 - y \qquad\qquad \therefore\ y_0' = x_0^2 - y_0 = -1$

$\qquad\quad y'' = 2x - y' \qquad\qquad \therefore\ y_0'' = 2x_0 - y_0' = 1$

$\qquad\quad y''' = 2 - y'' \qquad\qquad \therefore\ y_0''' = 2 - y_0'' = 1$

and $\quad y^{iv} = -y''' \qquad\qquad \therefore\ y_0^{iv} = -y_0''' = -1$

By Taylor's series,

$$y_1 = y_0 + h\,y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots$$

$$\therefore\quad y_1 = y(0.1)$$

$$= 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(-1) + \cdots$$

$$= 1 - 0.1 + 0.005 + 0.0001667 - 0.0000417 + \cdots$$

$$= 0.905125$$

Now, y_1'	$= x_1^2 - y_1$	$= (0.1)^2 - 0.905125$	$= -0.895125$
y_1''	$= 2x_1 - y_1'$	$= 2(0.1) - (-0.895125)$	$= 1.095125$
y_1'''	$= 2 - y_1''$	$= 2 - 1.095125$	$= 0.904875$
y_1^{iv}	$= -y_1'''$	$= -0.904875$	

and $$y_2 = y(0.2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \cdots$$

$$\therefore\ y_2 = 0.905125 + (0.1)(-0.895125) + \frac{(0.1)^2}{2!}(1.095125)$$

$$+ \frac{(0.1)^3}{3!}(0.904875) + \frac{(0.1)^4}{4!}(-0.904875) + \cdots$$

$$= 0.8212352$$

Now, y_2'	$= x_2^2 - y_2$	$= (0.2)^2 - 0.8212352$	$= -0.7812352$
y_2''	$= 2x_2 - y_2'$	$= 2(0.2) - (-0.7812352)$	$= 1.1812352$
y_2'''	$= 2 - y_2''$	$= 2 - 1.1812352$	$= 0.8187648$
y_2^{iv}	$= -y_2'''$	$= -0.8187648$	

and $$y_3 = y(0.3) = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \frac{h^4}{4!}y_2^{iv} + \ldots$$

$$\therefore\ y_3 = 0.8212352 + (0.1)(-0.7812352) + \frac{(0.1)^2}{2!}(1.1812352)$$

$$+ \frac{(0.1)^3}{3!}(0.8187648) + \frac{(0.1)^4}{4!}(-0.8187648) + \ldots$$

$$= 0.7491509$$

Now, $y_3' = x_3^2 - y_3 = (0.3)^2 - 0.7491509 = -0.6591509$

$y_3'' = 2x_3 - y_3' = 0.6 - (-0.6591509) = 1.2591509$

$y_3''' = 2 - y_3'' = 0.740849$

$y_3^{iv} = -y_3''' = -0.740849$

and $y_4 = y(0.4) = y_3 + \frac{h}{1!}y_3' + \frac{h^2}{2!}y_3'' + \frac{h^3}{3!}y_3''' + \frac{y^4}{4!}y_3^{iv} + \cdots$

$$\therefore\ y_4 = 0.7491509 + (0.1)(-0.6591509) + \frac{(0.1)^2}{2!}(1.2591509) + \frac{(0.1)^3}{3!}(0.740849) + \frac{(0.1)^4}{4!}(-0.740849) + \cdots$$

$$= 0.6896519$$

Exact solution : Given : $\frac{dy}{dx} = x^2 - y$

or $\frac{dy}{dx} + y = x^2$ (a linear equation)

$\therefore$ Solution is $y\, e^{\int dx} = \int (\int dx) x^2\, dx + c$

or $y\, e^x = \int e^x x^2\, dx + c$

or $y\, e^x = (x^2 - 2x + 2)e^x + c$

or $y = [(x-1)^2 + 1] + ce^{-x}$ (i)

Given that $y(0) = 1 \Rightarrow 1 = 2 + c$ or $c = -1$

Hence, the exact solution is

$y = [(x-1)^2 + 1] - e^{-x}$

$\therefore y_1 = y(0.1) = (0.1 - 1)^2 + 1 - e^{-0.1} = 0.9051625$

$y_2 = y(0.2) = 0.8212692, y_3 = 0.7491817$ and $y_4 = 0.6896799$.

Now we can see that the values coincides upto four decimals and the error is very less.

11.5 TAYLOR'S SERIES METHOD FOR SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \tag{11.9}$$

and

$$\frac{dz}{dx} = g(x, y, z) \tag{11.10}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by Taylor's series method as illustrated below:

If h be the step size, then $y_1 = y(x_0 + h)$, and $z_1 = z(x_0 + h)$.

Now, Taylor's algorithm for Eqns (11.9) and (11.10) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \tag{11.11}$$

and
$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots \tag{11.12}$$

Differentiating Eqns (11.9) and (11.10) successively, we get y'', y''', . . . ,and z_0'', z_0''', . . . etc. So the values y_0'', y_0''', . . .and z_0'', z_0''', . . . are known. Substituting these in Eqns. (11.11) and (11.12), we obtain y_1, z_1 for the next step.

Similarly, we have

$$y_1 = y_0 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \tag{11.13}$$

and
$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots \tag{11.14}$$

Since y_1, z_1 are known, y_1', y_1'', y_1''' . . .and z_1', z_1'', $z_1''' + \cdots$ can be calculated. Substituting in Eqns (11.13) and (11.14), we get y_2 and z_2.

Proceeding in the same manner, we get other values of y, step by step.

Example 11.3 Solve $\dfrac{dy}{dx} = x + z$, $\dfrac{dz}{dx} = x - y^2$ with $y(0) = 2$, $z(0) = 1$ to get $y(0.1)$, $y(0.2)$, $z(0.1)$ and $z(0.2)$, approximately, by Taylor's algorithm.

Solution: Given that $y' = x + z$; $z' = x - y^2$ with the initial conditions $y(0) = 2$ and $z(0) = 1$.

Now, $y' = x + z$ $\qquad z' = x - y^2$

$y'' = 1 + z'$ $\qquad z'' = 1 - 2yy'$

$y''' = z''$ etc., $\qquad z''' = -2[yy'' + y'^2]$ etc.

Now by Taylor's series for y_1 and z_1, we have

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \tag{i}$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots \tag{ii}$$

Here, $y_0 = 2$, $x_0 = 0$, $z_0 = 1$, $h = 0.1$ and

$$\begin{aligned} y_0' &= x_0 + z_0 = 1 & z_0' &= x_0 - y_0^2 = -4 \\ y_0'' &= 1 + z_0' = 1 - 4 = -3 & z_0'' &= 1 - 2y_0y_0' = 1 - 2(2)(1) = -3 \\ y_0''' &= z_0'' = -3 & z_0''' &= -2[y_0y_0'' + y_0'^2] \\ & & &= -2[2(-3) + (1)^2] = 10 \end{aligned}$$

Substituting these values in (i) and (ii), we get

$$\begin{aligned} y_1 &= 2 + (0.1)(1) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(-3) + \cdots \\ &= 2 + 0.1 - 0.015 - 0.0005 = 2.0845 \text{ (approx.)} \end{aligned}$$

$$\begin{aligned} z_1 &= 1 + (0.1)(-4) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(10) + \cdots \\ &= 1 - 0.4 - 0.015 + 0.001667 = 0.5867 \text{ (approx.)} \end{aligned}$$

$$\therefore y(0.1) = 2.0845 \text{ and } z(0.1) = 0.5867$$

To find out $y(0.2)$ and $z(0.2)$, Taylor's algorithm is

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \quad \text{(iii)}$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots \quad \text{(iv)}$$

Here, $x_1 = 0.1$, $y_1 = 2.0845$ and $z_1 = 0.5867$ and

$$\begin{aligned} y_1' &= x_1 + z_1 = 0.6867 & z_1' &= x_1 - y_1^2 = -4.2451403 \\ y_1'' &= 1 + z_1' = -3.2451403 & z_1'' &= 1 - 2y_1y_1' = -1.8628523 \\ y_1''' &= z_1'' = -1.8628523 & z_1''' &= -2[y_1y_1'' + y'^2] \\ & & &= 12.585876 \end{aligned}$$

Substuting these values in (iii) and (iv), we get

$$\begin{aligned} y_2 &= 2.0845 + (0.1)(0.6867) + \frac{1}{2!}(0.1)^2(-3.2451403) \\ &\quad + \frac{1}{3!}(0.1)^3(-1.8628523) + \cdots \\ &= 2.1366338 \text{ (approx.)} \end{aligned}$$

$$\begin{aligned} z_2 &= 0.5867 + (0.1)(-4.2451403) + \frac{(0.1)^2}{2!}(-1.8628523) \\ &\quad + \frac{(0.1)^3}{3!}(12.585876) + \cdots \\ &= 0.1549693 \text{ (approx.)} \end{aligned}$$

$$\therefore y(0.2) = 2.1366338 \text{ and } z(0.2) = 0.1549693$$

11.6 TAYLOR SERIES METHOD FOR HIGHER ORDER DIFFERENTIAL EQUATIONS

Taylor series method can be used to solve numerically second and higher order differential equations. The procedure for solving second order differential equation is illustrated below.

Let
$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

i.e.
$$y'' = f(x, y, y') \tag{11.15}$$

be the differential equation to be solved numerically subject to the initial conditions

$$y(x_0) = y_0 \tag{11.16}$$

and
$$y'(x_0) = y_0' \tag{11.17}$$

where y_0 and y_0' are given constants. Now put

$$y' = p \tag{11.18}$$

so that $y'' = p'$ and hence Eqn (11.15) becomes

$$p' = f(x, y, p) \tag{11.19}$$

The initial conditions, i.e. Eqns (11.16) and (11.17) becomes

$$y(x_0) = y_0 \tag{11.20}$$

and
$$p(x_0) = p_0 \tag{11.21}$$

where $y_0' = p_0$.

Hence, we have to solve the two first order differential equations (11.18) and (11.19) subject to the conditions (11.20) and (11.21). The Taylor algorithm for Eqn (11.19) is

$$p_1 = p_0 + hp_0' + \frac{h^2}{2!}p_0'' + \frac{h^3}{3!}p_0''' + \cdots \tag{11.22}$$

Here, h is the step-size and p_1 is the approximate value of p at $x = x_1 = x_0 + h$.

The Taylor algorithm for Eqn (11.18) is

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$= y_0 + hp_0 + \frac{h^2}{2!}p_0' + \frac{h^3}{3!}p_0'' + \cdots \tag{11.23}$$

Now equation (11.19) gives p'. Differentiating it successively, we get p'', p''', . . . etc. and hence, the values of p_0', p_0'', p_0''', . . . can be calculated. Substituting them in Eqns (11.22) and (11.23), we get p_1, and y_1. Similarly, for the next interval, we have the algorithms

$$p_2 = p_1 + h p_1' + \frac{h^2}{2!} p_1'' + \frac{h^3}{3!} p_1''' + \cdots \tag{11.24}$$

and
$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

$$= y_1 + h p_1 + \frac{h^2}{2!} p_1' + \frac{h^3}{3!} p_1'' + \cdots \tag{11.25}$$

Knowing p_1 and y_1, we can calculate $p_1', p_1'', \ldots$ at (x_1, y_1) and hence, p_2 and y_2 from Eqns (11.24) and (11.25).

Proceeding on the same lines, we can calculate the other values of y step by step.

The above procedure can be extended to solve higher order differential equations numerically, when sufficient initial conditions are given. But in this chapter, we shall restrict to second order only.

Example 11.4 Evaluate by means of Taylor's series expansion, the following problem at $x = 0.1, 0.2$ to four significant figures.
$y'' - x(y')^2 + y^2 = 0$; $y(0) = 1$, $y'(0) = 0$

Solution Putting $y' = p$, the given equation reduces to

$$p' - xp^2 + y^2 = 0$$

$$\text{i.e. } p' = xp^2 - y^2 \tag{i}$$

The initial conditions are

$$y_0 = y(0) = 1 \; ; \; p_0 = y_0' = 0 \tag{ii}$$

Now to solve Eqn (i), given $p_0 = p(0) = 0$, we have

$$p_1 = p_0 + h p_0' + \frac{h^2}{2!} p_0'' + \frac{h^3}{3!} p_0''' + \cdots \tag{iii}$$

From (i), we have

$$\begin{aligned} p' &= xp^2 - y^2 & \qquad y'' &= p' \\ p'' &= p^2 + 2xpp' - 2yy' & \qquad y''' &= p'' \\ p''' &= 2pp' + 2[xpp'' + x(p')^2 + pp'] & \qquad y^{iv} &= p''' \\ &\quad - 2[yy'' + (y')^2] & & \end{aligned}$$

$$\begin{aligned} \therefore \quad p_0' &= x_0 p_0^2 - y_0^2 = 0(0)^2 - (1)^2 = -1 \\ p_0'' &= p_0^2 + 2x_0 p_0 p_0' - 2y_0 y_0' \\ &= (0)^2 + 2(0)(0)(-1) - 2(1)(0) = 0 \\ \text{and} \quad p_0''' &= 2. \end{aligned}$$

Substituting these in (iii), we get

$$p_1 = 0 + (0.1)(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \cdots$$

$$= -0.0997$$

By Taylor's series,

$$y_1 = y(0.1) = y_0 + h y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots$$

$$= 1 + (0.1)p_0 + \frac{(0.1)^2}{2!}(p_0') + \frac{(0.1)^3}{3!}(p_0'') + \frac{(0.1)^4}{4!}(p_0''') + \cdots$$

$$= 1 + (0.1)(0) + \frac{0.01}{2}(-1) + \frac{0.001}{6}(0) + \frac{0.0001}{24}(2) + \cdots$$

$$= 0.9950083 \cong 0.995$$

Now $y_2 = y(0.2) = y_1 + h y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$

$$= y_1 + h p_1 + \frac{h^2}{2!}p_1' + \frac{h^3}{3!}p_1'' + \cdots \qquad \text{(iv)}$$

Here, $y_1 = 0.995, p_1 = -0.0997,$

$p_1' = x_1 p_1^2 - y_1^2 = (0.1)(-0.0997)^2 - (0.995)^2$

$= -0.9890309$

$p_1'' = p_1^2 + 2x_1 p_1 p_1' - 2y_1 y_1'$

$= -0.1687416.$ 0.22799

Substituting in Eqn (iv), we get

$$y_2 = 0.995 + \frac{(0.1)}{1!}(-0.0997) + \frac{(0.1)^2}{2!}(-0.9890309)$$

$$+ \frac{(0.1)^3}{3!}(-0.1687416) + \cdots$$

$$= 0.9801129 \cong 0.9801$$

$\therefore \qquad y(0.1) = 0.9950$ and $y(0.2) = 0.9801$

11.7 PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{11.26}$$

subject to $y(x_0) = y_0$. This equation can be written as

$$dy = f(x, y)\, dx$$

Integrating between the limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx$$

or
$$y - y_0 = \int_{x_0}^{x} f(x, y)dx$$

or
$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \tag{11.27}$$

which is an integral equation and can be solved by successive approximation or iteration. Now by Picard's method, for first approximation y_1, we replace y by y_0 in $f(x, y)$ in the RHS of Eqn (11.27),

i.e.
$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx \tag{11.28}$$

For second approximation y_2, we replace y by y_1 in $f(x, y)$ on the RHS of Eqn (11.27),

i.e.
$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx \tag{11.29}$$

... ...

For nth approximation,

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

The process is to be stopped when the two values of y, viz. y_n and y_{n-1}, are same to the desired degree of accuracy.

Note : This method is applicable only to a limited class of equations in which the successive integrations can be performed easily.

Example 11.5 Use Picard's method to approximate the value of y when $x = 0.1, 0.2, 0.3, 0.4$ and 0.5, given that $y = 1$ at $x = 0$ and $y = 1 + xy$, correct to three decimal places.

Solution Given : $\dfrac{dy}{dx} = 1 + xy$; $y(0) = 1$

$\therefore$ $f(x, y) = 1 + xy$; $y_0 = 1, x_0 = 0$

First approximation :

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx = 1 + \int_0^x (1 + xy_0)dx$$

$$= 1 + \int_0^x (1 + x)dx = 1 + x + \frac{x^2}{2} \qquad \text{(i)}$$

Second approximation :

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx = 1 + \int_0^x (1 + xy_1)dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2}\right)\right\}dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} \qquad \text{(ii)}$$

Third approximation :

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx = 1 + \int_0^x (1 + xy_2)dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right\}dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} \qquad \text{(iii)}$$

Fourth approximation :

$$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3)dx = 1 + \int_0^x (1 + xy_3)dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}\right)\right\}dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} + \frac{x^7}{105} + \frac{x^8}{384} \tag{iv}$$

Now at $x = 0.1$,

$y_1 = 1.105,\ y_2 = 1.1053458,\ y_3 = 1.1053465, y_4 = y_3$

at $x = 0.2$,

$y_1 = 1.22,\ y_2 = 1.2228667,\ y_3 = 1.2228894,\ y_4 = 1.2228895$

at $x = 0.3$,

$y_1 = 1.345,\ y_2 = 1.35550125,\ y_3 = 1.3551897,\ y_4 = 1.355192$

at $x = 0.4$,

$y_1 = 1.48,\ y_2 = 1.5045333;\ y_3 = 1.5053013,\ y_4 = 1.5053186$

and at $x = 0.5$,

$y_1 = 1.625,\ y_2 = 1.6744792,\ y_3 = 1.6768881,\ y_4 = 1.6769727$

$\therefore$ correct to three decimal places,

$y(0.1) = 1.105,\ y(0.2) = 1.223,\ y(0.3) = 1.355, y(0.4) = 1.505$ and $y(0.5) = 1.677$.

Example 11.6 Use Picard's method to approximate the value of y when $x = 0.1$ given that $y = 1$ when $x = 0$ and $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$.

Solution Given : $\dfrac{dy}{dx} = f(x, y) = \dfrac{y-x}{y+x}, y_0 = 1, x_0 = 0$

First approximation :

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx = y_0 + \int_{x_0}^{x} \frac{y_0 - x}{y_0 + x}\,dx$$

$$= 1 + \int_0^x \frac{1-x}{1+x}\,dx = 1 + \int_0^x \left[\frac{2}{1+x} - 1\right] dx$$

$$= 1 + \left[2\log(1+x) - x\right]_0^x = 1 + 2\log(1+x) - x$$

$$\therefore y_1 = 1 - x + 2\log(1+x) \tag{i}$$

Second approximation :

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx = y_0 + \int_{x_0}^{x} \left[\frac{y_1 - x}{y_1 + x}\right] dx$$

$$= 1 + \int_0^x \left\{ \frac{1 - x + 2\log(1+x) - x}{1 - x + 2\log(1+x) + x} \right\} dx$$

$$= 1 + 2 \int_0^x \frac{x}{1 + 2\log(1+x)} dx$$

which is quite difficult to integrate.

$\therefore$ We use only first approximation. By putting $x = 0.1$ in (i), we get

$y_1 = 1 - 0.1 + 2\log(1 + 0.1) = 0.9828$

11.8 PICARD'S METHOD FOR SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

Let $\frac{dy}{dx} = f(x, y, z)$ and $\frac{dz}{dx} = \phi(x, y, z)$ be the simultaneous differential equations with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$.

Picard's method gives

$$\left. \begin{aligned} y_1 &= y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\,dx \\ z_1 &= z_0 + \int_{x_0}^{x} \phi(x, y_0\, z_0)\,dx \end{aligned} \right\} \tag{11.30}$$

$$\left. \begin{aligned} y_2 &= y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\,dx \\ z_2 &= z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\,dx \end{aligned} \right\} \tag{11.31}$$

and so on as successive approximations.

Example 11.7 Approximate y and z at $x = 0.1$ using Picard's method for the solution to the equations $\frac{dy}{dx} = z$, $\frac{dz}{dx} = x^3(y + z)$, given that $y(0) = 1$ and $z(0) = \frac{1}{2}$

Solution Here, $x_0 = 0$, $y_0 = 1$, $z_0 = \frac{1}{2}$

and $\dfrac{dy}{dx} = f(x, y, z) = z$

$$\frac{dz}{dx} = \phi(x, y, z) = x^3(y + z)$$

$$\therefore\ y = y_0 + \int_{x_0}^{x} f(x,y,z)dx \qquad \text{and } z = z_0 + \int_{x_0}^{x} \phi(x,y,z)dx$$

First approximation :

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0,z_0)dx = 1 + \int_0^x \left(\frac{1}{2}\right)dx = 1 + \frac{x}{2}$$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x,y_0,z_0)dx = \frac{1}{2} + \int_0^x x^3\left(1 + \frac{1}{2}\right)dx = \frac{1}{2} + \frac{3x^4}{8}$$

Second approximation :

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1,z_1)dx$$

$$= 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8}\right)dx = 1 + \frac{x}{2} + \frac{3x^5}{40}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x,y_1,z_1)dx$$

$$= \frac{1}{2} + \int_0^x x^3\left(1 + \frac{x}{2} + \frac{1}{2} + \frac{3x^4}{8}\right)dx$$

$$= \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}$$

Third approximation :

$$y_3 = y_0 + \int_{x_0}^{x} f(x,y_2,z_2)dx$$

$$= 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right)dx$$

$$= 1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{x^6}{60} + \frac{x^9}{192}$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx$$

$$= \frac{1}{2} + \int_0^x x^3\left(1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right) dx$$

$$= \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64} + \frac{7x^9}{360} + \frac{x^{12}}{256}$$

and so on. When $x = 0.1$,

$y_1 = 1.05$; $y_2 = 1.500008$; $y_3 = 1.500008$;

$z_1 = 0.5000375$; $z_2 = 0.5000385$; $z_3 = 0.5000385$;

Hence, $y(0.1) = 1.05$ and $z(0.1) = 0.5$.

11.9 PICARD'S METHOD FOR SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$$

By putting $\frac{dy}{dx} = z$, it can be reduced to two first order simultaneous differential equations :

$$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx} = f(x, y, z).$$

These can be solved as explained above.

Example 11.8 Use Picard's method to approximate y when $x = 0.1$ given that $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ and $y = 0.5$, $\frac{dy}{dx} = 0.1$ when $x = 0$.

Solution Let $\frac{dy}{dx} = z$ so that $\frac{d^2y}{dx^2} = \frac{dz}{dx}$.

Thus the given equation reduces to

$$\frac{dz}{dx} + 2xz + y = 0\ ;\ y(0) = 0.5\ ,\ z(0) = 0.1$$

Now the equations to be solved are

$$\frac{dy}{dx} = f(x, y, z) = z$$

and $$\frac{dz}{dx} = \phi(x, y, z) = -(2xz + y)$$

with the conditions $y_0 = 0.5,\ z_0 = 0.1,\ \text{at } x_0 = 0$

$$\therefore y = y_0 + \int_{x_0}^{x} f(x, y, z)\,dx$$

$$= 0.5 + \int_0^x z\,dx$$

$$z = z_0 + \int_{x_0}^{x} \phi(x, y, z)\,dx$$

$$= 0.1 - \int_0^x (2xz + y)\,dx$$

First approximation :

$$y_1 = 0.5 + \int_0^x z_0\,dx = 0.5 + \int_0^x (0.1)\,dx$$

$$= 0.5 + (0.1)\,x$$

$$z_1 = 0.1 - \int_0^x (2xz_0 + y_0)\,dx = 0.1 - \int_0^x (0.2x + 0.5)\,dx$$

$$= 0.1 - 0.5x - (0.1)x^2$$

Second approximation :

$$y_2 = 0.5 + \int_0^x z_1\,dx$$

$$= 0.5 + \int_0^x (0.1 - 0.5x - 0.1x^2)\,dx$$

$$= 0.5 + 0.1\,x - \frac{0.5x^2}{2} - \frac{0.1x^3}{3}$$

$$z_2 = 0.1 - \int_0^x (2xz_1 + y_1)\,dx$$

$$= 0.1 - \int_0^x 2x(0.1 - 0.5x - 0.1x^2) + (0.5 + 0.1x)\,dx$$

$$= 0.1 - 0.5x - \frac{0.3x^2}{2} + \frac{x^3}{3} + \frac{0.2x^4}{4}$$

Third approximation :

$$y_3 = 0.5 + \int_0^x z_2\,dx = 0.5 + 0.1x - \frac{0.5x^2}{2} - \frac{0.1x^3}{2} + \frac{x^4}{12} + \frac{0.1x^5}{10}$$

$$z_3 = 0.1 - \int_0^x (2xz_2 + y_2)\,dx$$

$$= 0.1 - 0.5x + \frac{0.3x^2}{2} - \frac{2.5x^3}{6} + 0.2\,x4 + \frac{2x^5}{15} + \frac{0.1x^6}{6}$$

Now at $x = 0.1$,
$y_1 = 0.51$, $y_2 = 0.50746667$, $y_3 = 0.50745933$
$\therefore$ $y(0.1) = 0.5075$ correct to four decimals.

EXERCISE 11.1

1. Using first four terms of the Maclaurin's series find y at $x = 0.1(0.1)(0.6)$ given that $2y' = (1 + x)\,y^2$, $y(0) = 1$. Compare the values with the exact solution.
2. Find the first six terms of the power series solution of $y' = \sin x + y^2$ which passes through the point (0, 1).
3. Given $y' = 3x + \frac{y}{2}$ and $y(0) = 1$, find by Taylor's series $y(0.1)$ and $y(0.2)$.
4. Using Taylor's series method solve $y' = xy + y^2$, $y(0) = 1$ at $x = 0.1$, 0.2, 0.3.
5. Solve by Taylor's series method of third order, the problem $y' = (x^3 + xy^2)\,e^{-x}$, $y(0) = 1$ to find y for $x = 0.1, 0.2, 0.3$.

6. Employ Taylor's method to obtain the approximate value of y at $x = 0.2$ for $y' = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with exact solution.

7. Solve $y' = y^2 + x$, $y(0) = 1$ using Taylor's series method to compute $y(0.1)$ and $y(0.2)$.

8. Solve $\dfrac{dy}{dx} = z - x$, $\dfrac{dz}{dx} = y + x$ with $y(0) = 1, z(0) = 1$ to get $y(0.1)$ and $z(0.1)$, using Taylor's method.

9. Given $\dfrac{dx}{dt} - ty - 1 = 0$ and $\dfrac{dy}{dt} + tx = 0$, $t = 0, x = 0, y = 1$, evaluate $x(0.1), y(0.1), x(0.2)$ and $y(0.2)$.

10. Using Taylor's series method, obtain the values of y at $x = (0.1)(0.1)$ 0.3 to four significant figures if y satisfies the equation $\dfrac{d^2y}{dx^2} + xy = 0$ given that $\dfrac{dy}{dx} = \dfrac{1}{2}$ and $y = 1$ when $x = 0$.

11. Evaluate the integral of the following problem to four significant figures at $x = 1.1(0.1)\,1.3$ using Taylor's series expansion.

$$\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} - x^3 = 0;\ \left.\frac{dy}{dx}\right]_{x=1} = 1\ ;\ y(1) = 1$$

12. Using Picard's method find $y(0.2)$ given that $y' = x - y$; $y(0) = 1$.

13. Using Picard's method obtain a solution upto the fifth approximation to the equation $y' = y + x$, such that $y(0) = 1$. Check your answer by finding the exact particular solution. Also find $y(0.1)$ and $y(0.2)$.

14. Using Picard's method find $y(0.2)$ and $y(0.4)$ given that $y' = 1 + y^2$ and $y(0) = 0$.

15. Use Picard's method to approximate the value of y when $x = 0.1$ given that $y(0) = 1$ and $y' = 3x + y^2$.

16. Using Picard's method find the approximate values of y and z corresponding to $x = 0.1$ given that $y(0) = 2$, $z(0) = 1$ and $\dfrac{dy}{dx} = x + z$, $\dfrac{dz}{dx} = x - y^2$.

17. Using Picards method obtain the second approximation to the solution to $y'' - x^3y' - x^3y = 0$ so that $y(0) = 1, y'(0) = 0.5$.

ANSWERS

1.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
Approx. value of y	1	1.055375	1.123	1.205125	1.304	1.421875	1.561
Exact value of y	1	1.055	1.124	1.209	1.316	1.455	1.64

2. $y = 1 + x + \dfrac{3x^2}{2} + \dfrac{4x^3}{3} + \dfrac{11x^4}{8} + \dfrac{23x^5}{15} + \cdots$
3. $y(0.1) = 1.0665$; $y(0.2) = 1.167196$
4. $y(0.1) = 1.1167, y(0.2) = 1.2767, y(0.3) = 1.5023$
5. $y(0.1) = 1.0047, y(0.2) = 1.01812, y(0.3) = 1.03995$
6. $y(0.2) = 0.811$, exact value of $y(0.2) = 0.8112$
7. $y(0.1) = 1.1164$, $y(0.2) = 1.2725$
8. $y(0.1) = 1.1003$, $z(0.1) = 1.1102$
9. $x(0.1) = 0.105$, $y(0.1) = 0.9997$
   $x(0.2) = 0.21998$, $y(0.2) = 0.9972$
10. $y(0.1) = 1.050$, $y(0.2) = 1.099$, $y(0.3) = 1.145$
11. $y(1.1) = 1.100, y(0.2) = 1.201$, $y(0.3) = 1.306$
12. $y(0.2) = 0.837$
13. $y(0.1) = 1.1103$; $y(0.2) = 1.2428$
14. $y(0.2) = 0.2027$, $y(0.4) = 0.4227$
15. $y(0.1) = 1.127$
16. $y(0.1) = 2.0845$; $z(0.1) = 0.5867$
17. $y_2 = 1 + \dfrac{1}{2}x + \dfrac{3}{40}x^5$

11.10 EULER'S METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{11.32}$$

where $y(x_0) = y_0$

Suppose that we wish to find successively $y_1, y_2, \ldots, y_m$, where y_m is the value of y corresponding to $x = x_m$, where $x_m = x_0 + mh$, $m = 1, 2, \ldots$, h being small. Here, we use the property that in a small interval, a curve is nearly a straight line.

Thus, in the interval x_0 to x_1 of x, we approximate the curve by the tangent at the point (x_0, y_0)

Therefore, the equation of the tangent at (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

$$= f(x_0, y_0)(x - x_0) \quad \text{[from Eqn (11.32)]}$$

or $y = y_0 + (x - x_0) f(x_0, y_0)$

Hence, the value of y corresponding to $x = x_1$ is

$$y_1 = y_0 + (x_1 - x_0) f(x_0, y_0)$$

or
$$y_1 = y_0 + h f(x_0, y_0) \tag{11.33}$$

Since the curve is approximated by the tangent in $[x_0, x_1]$, Eqn (11.33) gives the approximated value of y_1.

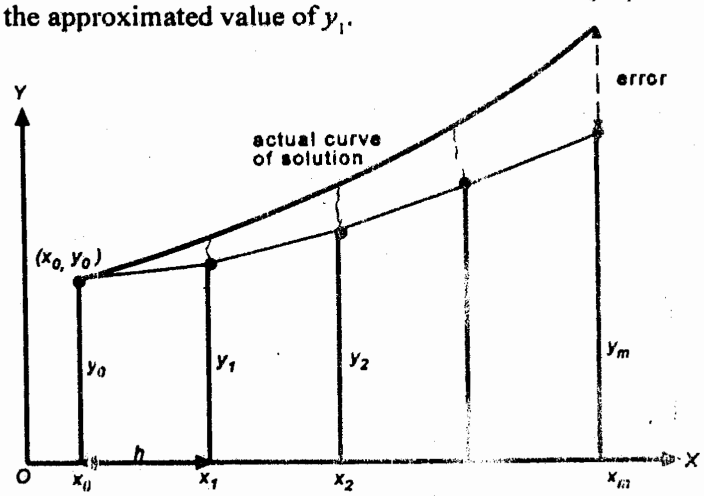

Fig 11.2

Similarly, approximating the curve in the next interval $[x_1, x_2]$ by a line through (x_1, y_1) with slope $f(x_1, y_1)$, we get

$$y_2 = y_1 + hf(x_1, y_1) \tag{11.34}$$

Proceeding on, in general it can be shown that

$$y_{m+1} = y_m + hf(x_m, y_m) \tag{11.35}$$

Remarks In Euler's method, the actual curve of a solution is approximated by a sequence of short lines as shown in Fig 11.2 is possible that the sequence of lines may deviate from the curve of solution significantly. The process is very slow and to obtain it with reasonable accuracy using Euler's method, we have to take h very small. An improvement over this method is discussed in the following section.

11.11 IMPROVED EULER'S METHOD

Here, we consider a line passing through $A(x_0, y_0)$ whose slope is the average of the slopes at $A\ (x_0, y_0)$ and $P(x_1, y_1^{(1)})$ such that $y^1_1 = y_0 + hf(x_0, y_0)$.

In Fig. 11.3, let AL_1 be the tangent to the curve at $A(x_0, y_0)$ and PL_2 be the line through $P(x_1, y_1^{(1)})$ having the slope $f(x_1, y_1^{(1)})$. Now PM is the line having slope

$$\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

that is, average of two slopes $f(x_0, y_0)$ and $f(x_1, y_1^{(1)})$.

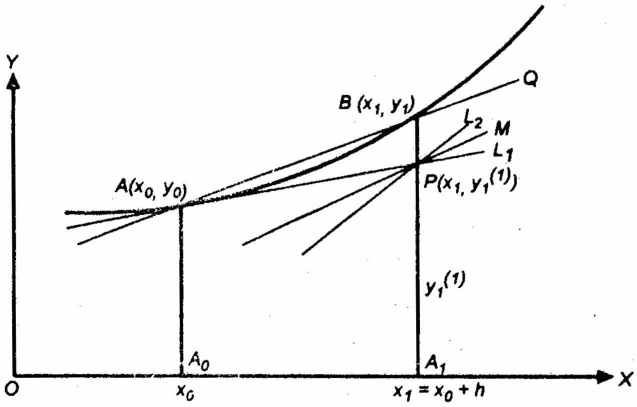

Fig 11.3

Line AQ through (x_0, y_0) and parallel to PM is used to approximate the curve. Then, ordinate of point E will give the value of y_1.

Therefore, equation to ABQ is

$$y - y_0 = (x - x_0)\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\} \qquad (11.36)$$

As we are assuming that $A_1B = y_1$, coordinates of B will be (x_1, y_1). This point will lie on AQ.

$$\therefore\ y_1 - y_0 = (x_1 - x_0)\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

or $$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

$$= y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0)\} \qquad (11.37)$$

In general, we have the formula

$$y_{m+1} = y_m + \frac{h}{2} \{f(x_m, y_m) + f(x_m + h, y_m + hf(x_m, y_m))\} \qquad (11.38)$$

where $x_m - x_{m-1} = h$.

11.12 MODIFIED EULER'S METHOD

In this method the curve in the interval (x_0, x_1), where $x_1 = x_0 + h$, is approximated by the line through (x_0, y_0) with slope

$$f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)) \qquad (1)$$

that is, the slope at the middle point whose abscissa is the average of x_0 and x_1, i.e. $x_0 + \frac{h}{2}$.

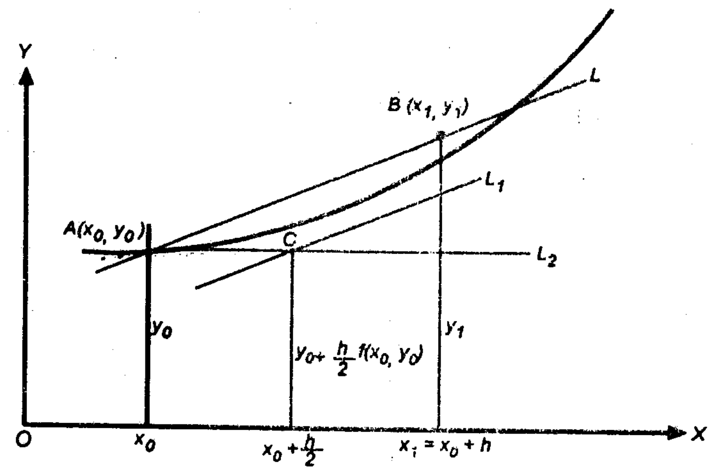

Fig 11.4

Geometrically, line L through (x_0, y_0) which is parallel to L_1, a line through $(x_0 + \frac{h}{2}, \frac{h}{2} f(x_0, y_0))$, with the slope (1) approximates the curve in the interval $[x_0, x_1]$. The ordinate at $x = x_1$, meeting the line L at B, will give the value of y_1.

The equation for line L is

$$y - y_0 = (x - x_0) \{f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))\}$$

Putting $x = x_1$, we get

$$y_1 = y_0 + (x_1 - x_0)\{f(x_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(x_0, y_0)\}$$

$$= y_0 + hf(x_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(x_0, y_0)) \qquad (11.39)$$

Proceeding in the same way, it can be shown that,

$$y_{m+1} = y_m + hf(x_m + \frac{h}{2},\ y_m + \frac{h}{2} f(x_m, y_m)) \qquad (11.40)$$

Example 11.9 Solve $\frac{dy}{dx} = 1 - y, y(0) = 0$ in the range $0 \le x \le 0.3$ using (i) Euler's method (ii) improved Euler's method, and (iii) modified Euler's method by choosing $h = 0.1$. Compare the answers with exact solution.

Solution Given $\frac{dy}{dx} = 1 - y$ and $y(0) = 0$ and $h = 0.1$. (i)

Now we have to find out the solutions at $x = 0.1$, 0.2 and 0.3.

(i) Euler's method: The algorithm is,

if $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

then, $y_{m+1} = y_m + hf(x_m, y_m)$ (ii)

Here, $f(x, y) = 1 - y$, $h = 0.1$; $x_0 = 0$, $y_0 = 0$

$\therefore$ From Eqn (ii), $y_{m+1} = y_m + (0.1)(1 - y_m)$

or $y_{m+1} = 0.1 + 0.9\,y_m$ (iii)

Putting $m = 0, 1, 2$ successively, we get

$y_1 = 0.1 + 0.9\,y_0 = 0.1 + (0.9)(0) = 0.1$

$y_2 = 0.1 + 0.9\,y_1 = 0.1 + (0.9)(0.1) = 0.19$

$y_3 = 0.1 + 0.9\,y_2 = 0.1 + (0.9)(0.19) = 0.271$

$\therefore y(0.1) = 0.1$; $y(0.2) = 0.19$; $y(0.3) = 0.271$

(ii) Improved Euler's method: Here, the formula is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f(x_{m+1}, y^{(1)}_{m+1})\}$$

where, $f(x_{m+1}, y^{(1)}_{m+1}) = f\{x_m + h, y_m + hf(x_m, y_m)\}$

Here, $f(x, y) = 1 - y$, $\therefore f(x_m, y_m) = 1 - y_m$

$$\therefore f(x_{m+1}, y^{(1)}_{m+1}) = 1 - \{y_m + hf(x_m, y_m)\}$$
$$= 1 - y_m - h(1 - y_m)$$
$$= (1 - h)(1 - y_m)$$

Substituting it in Eqn (iv), we get

$$y_{m+1} = y_m + \frac{h}{2}\{(1-y_m) + (1-h)(1-y_m)\}$$

$$= y_m + \frac{1}{2}h(2-h)(1-y_m)$$

$$= y_m + 0.095(1-y_m) \quad [\because h = 0.1]$$

$$y_{m+1} = 0.095 + 0.095\, y_m \qquad \text{(v)}$$

Putting $m = 0, 1, 2$ successively in Eqn (v), we get

$y_1 = 0.095 + 0.905\, y_0 = 0.095 + (0.905)(0) = 0.095$

$y_2 = 0.095 + 0.905\, y_1 = 0.095 + (0.905)(0.095) = 0.180975$

$y_3 = 0.095 + 0.905\, y_2 = 0.095 + (0.905)(0.180905) = 0.2587823$

(iv) Modified Euler's method : The formula is,

$$y_{m+1} = y_m + hf\{x_m + \frac{h}{2}, y_m + \frac{h}{2} f(x_m, y_m)\}$$

$$= y_m + h\{1 - [y_m + \frac{h}{2} f(x_m, y_m)]\}$$

$$= y_m + h\{1 - y_m - \frac{h}{2}(1-y_m)\}$$

$$= y_m + h\{1 - \frac{h}{2})(1-y_m)\}$$

$$= y_m + 0.095(1-y_m) \quad [\because \text{h} = 0.1]$$

$$= 0.095 + 0.905\, y_m$$

which is identical to Eqn (vi). Putting $m = 0, 1, 2$ successively in Eqn (vi), we get

$$y_1 = 0.095, y_2 = 0.180975, y_3 = 0.2587823$$

Exact solution : We have

$$\frac{dy}{dx} = 1 - y \text{ or } \frac{dy}{1-y} = dx$$

On integrating, we get

$$-\log(1-y) + \log C = x \text{ or } \frac{C}{1-y} = e^x$$

or $$e^x(1-y) = C \qquad \text{(vii)}$$

But $y = 0$ at $x = 0$. Therefore, from Eqn (vii), we get $C = 1$ and hence

$$e^x(1-y) = 1 \text{ or } y = 1 - e^{-x} \qquad \text{(viii)}$$

$\therefore \quad y_1 = y(0.1) = 1 - e^{-0.1} = 0.0951625$

$\quad\;\; y_2 = y(0.2) = 1 - e^{-0.2} = 0.1812692$

and $\; y_3 = y(0.3) = 1 - e^{-0.3} = 0.2591817$

Now compare the results in the following table.

x	Euler's method	Improved Euler's method	Modified Euler's method	Exact solution
0	0	0	0	0
0.1	0.1	0.095	0.095	0.0951625
0.2	0.19	0.180975	0.180975	0.1812692
0.3	0.271	0.2587823	0.2587823	0.2591817

Example 11.10 Solve $\dfrac{dy}{dx} = y - \dfrac{2x}{y}$, $y(0) = 1$ in the range $0 \le x \le 0.2$ using (i) Euler's method (ii) improved Euler's method, and (iii) modified Euler's method. Take $h = 0.1$.

Solution Given

$$\frac{dy}{dx} = y - \frac{2x}{y},\; y(0) = 1,\; h = 0.1.$$

Now we have to find out the solutions at $x = 0.1$ and $x = 0.2$.

(i) Euler's method: The algorithm is,

$$\text{if } \frac{dy}{dx} = f(x, y),\; y(x_0) = y_0$$

$$\text{then } y_{m+1} = y_m + hf(x_m, y_m) \qquad \text{(i)}$$

Putting $m = 0$, we get

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h\left(y_0 \frac{2x_0}{y_0}\right)$$

$$= 1 + (0.1)\left[1 - \frac{2(0)}{1}\right] = 1.1 \qquad [\because x_0 = 0, y_0 = 1]$$

Putting $m = 1$ in Eqn (i), we get

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + 0.1\left(y_1 - \frac{2x_1}{y_1}\right)$$

$$= 1.1 + (0.1)\left[1.1 - \frac{2(0.1)}{1.1}\right] \qquad [\because x_1 = 0.1, y_1 = 1.1]$$

$$= 1.1918182$$

(ii) Improved Euler's method : Here,

$$y_{m+1} = y_m + \frac{h}{2}[f(x_m, y_m) + f(x_{m+1}, y^{(1)}_{m+1})] \qquad \text{(ii)}$$

where $y^{(1)}_{m+1} = y_m + h f(x_m, y_m)$

Putting $m = 0$ in Eqn (ii), we get

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y^{(1)}_1)] \qquad \text{(iii)}$$

where $y^{(1)}_1 \quad = y_0 + h f(x_0, y_0)$

$$= 1 + (0.1)\left[1 - \frac{2(0)}{1}\right] = 1.1$$

$\therefore$ From Eqn (iii),

$$y_1 \quad = y_0 + \frac{h}{2}\left[\left(y_0 - \frac{2x_0}{y_0}\right) + \left(y_1^{(1)} - \frac{2x_1}{y_1^{(1)}}\right)\right]$$

$$= 1 + \frac{0.1}{2}\left[\left\{1 - \frac{2(0)}{1}\right\} + \left\{1.1 - \frac{2(0.1)}{1.1}\right\}\right]$$

$$= 1.0959091$$

Putting $m = 1$ in Eqn (ii), we get

$$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)}) \qquad \text{(iv)}$$

where $y^{(1)}_2 = y_1 + hf(x_1, y_1)$

Now, $f(x_1, y_1) = y_1 - \dfrac{2x_1}{y_1}$

$$= 1.0959091 - \frac{2(0.1)}{1.0959091}$$

$$= 0.9134122$$

$y^{(1)}_2 = y_1 + hf(x_1, y_1) = 1.0959091 + (0.1)\,(0.9134122)$
$= 1.1872503$

$f(x_2, y^{(1)}_2) = [y^{(1)}_2 - 2x_2/y^{(1)}_2]$

$$= 1.1872503 - \frac{2(0.2)}{1.1872503}$$

$$= 0.8503373$$

Substituting all the requisites in Eqn (iv), we get

$$y_2 = 1.0959091 + \frac{0.1}{2}[0.913422 + 0.8503373]$$
$$= 1.1840966$$

(iii) Modified Euler's method : The formula is,

$$y_{m+1} = y_m + hf\{x_m + \frac{h}{2}, y_m + \frac{h}{2}f(x_m, y_m)\} \quad \text{(v)}$$

Putting $m = 0$ in the above Equation, we get

$$y_1 = y_0 + hf\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\} \quad \text{(vi)}$$

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1$$

$$y_0 + \frac{h}{2}f(x_0, y_0) = 1 + \frac{0.1}{2}(1) = 1.05 \; ; x_0 + \frac{h}{2} = 0.05$$

$$\therefore f\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\} = f\{0.05, 1.05\}$$

$$= 1.05 - \frac{2(0.05)}{1.05} = 0.9547619$$

Hence, from Eqn (vi), we get
$y_1 = 1 + (0.1)(0.09547619) = 1.0954762$
Putting $m = 1$ in Eqn (v), we get

$$y_2 = y_1 + hf\{x_1 + \frac{h}{2}, y_1 + \frac{h}{2}f(x_1, y_1)\} \quad \text{(vii)}$$

$$x_1 + \frac{h}{2} = 0.1 + \frac{0.1}{2} = 0.15$$

$$f(x_1, y_1) = y_1 - \frac{2x_1}{y_1} = 1.0954762 - \frac{2(0.1)}{1.0954762}$$
$$= 0.9129071$$

$$y_1 + \frac{h}{2}f(x_1, y_1) = 1.0954762 + \frac{0.1}{2}(0.9129071)$$
$$= 1.1411216$$

$$\therefore \quad f\{x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\} = f\{0.15, 1.1411216\}$$

$$= 1.1411216 - \frac{2(0.15)}{1.1411216} = 0.8782223$$

Now substituting the requisites in Eqn (vii), we get

$y_2 = 1.0954762 + (0.1)(0.8782223) = 1.1832984$

The values obtained are shown below in a tabular form.

x	Euler's method	Improved Euler's method	Modified Euler's method
0	1	1	1
0.1	1.1	1.0959091	1.0954762
0.2	1.1918182	1.1840966	1.1832984

EXERCISE 11.2

1. Use Euler's method and Improved Euler's method to approximate y when $x = 0.1$, given that

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \quad y(0) = 1 \text{ taking } h = 0.2.$$

2. Solve $y' = 3x^2 + y$ in $0 \le x \le 1$ by Euler's method taking $h = 0.1$ given that $y(0) = 4$.
3. Solve $y' = x + y$, $y(0)$ choosing the step length 0.2 for $y(1.2)$ by Euler's method.
4. Using Euler's method solve $y' = x + y$ in $0 \le x \le 1$ with $h = 0.1$, if $y(0) = 1$. Find the exact value of y at $x = 1$, using analytical method.
5. Using Euler's method find $y(0.6)$ of $y' = 1 - 2xy$, given that $y(0) = 0$ taking $h = 0.2$.
6. Solve $y' = -y$; $y(0) = 1$ by (i) Euler's method for $y(0.04)$ and (ii) Modified Euler's method for $y(0.6)$.
7. Solve $y' = x + y + xy$, $y(0) = 1$ for $y(0.1)$ taking $h = 0.025$, using Euler's method.
8. Given that $y' = \log(x + y)$ with $y(0) = 1$. Use (i) Improved Euler's method to find $y(0.2)$, $y(0.5)$, (ii) Modified Euler's method to find $y(0.2)$.

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9. Use Euler's method and its Modified form to obtain $y(0.2)$, $y(0.4)$ and $y(0.6)$ correct to three decimal places given that $y' = y - x^2$, $y(0) = 1$.

10. Use Euler's modified method to get $y(0.25)$ given that $y' = 2xy$, $y(0) = 1$.

11. Using Improved Euler's method, solve
$y' = x + |\sqrt{y}|$, $y(0) = 1$ in the range $0 \le x \le 0.6$ taking $h = 0.2$.

12. Given that $y' = 2 + \sqrt{(xy)}$ and $y(1) = 1$. Find $y(2)$ in steps of 0.2 using Improved Euler's method.

13. Given $y^{(1)} = x^2 + y^2$, $y(0) = 1$, determine $y(0.1)$ and $y(0.2)$ by Modified Euler's method.

14. Solve $y^{(1)} = y + e^x$, $y(0) = 0$ for $y(0.2)$, $y(0.4)$ by Improved Euler's method.

15. Solve $y^{(1)} = y + x^2$, $y(0) = 1$ for $y(0.02)$, $y(0.04)$ and $y(0.06)$ using Euler's Modified method.

ANSWERS

1. 1.0928, 1.0932
2. 4.4, 4.843, 5.3393, 5.90023, 6.538253, 7.2670783, 8.1017861, 9.0589647, 9.1039647, 10.257361
3. 1.1831808
4. 1.1, 1.22, 1.362, 1.5282, 1.7210, 1.9431, 2.1974, 2.4871, 2.8158, 3.1873 ; exact solution = 3.4366
5. 0.4748
6. 0.9603 ; 0.551368
7. 1.1448
8. 1.0082, 1.0490 ; 1.0095
9. 1.2, 1.432, 1.686 ; 1.218, 1.467, 1.737
10. 1.0625
11. 1.2309, 1.5253, 1.8851
12. 5.051
13. 1.1105, 1.25026
14. 0.24214, 0.59116
15. 1.0202, 1.0408, 1.0619

11.13 RUNGE'S METHOD

Let the differential equation be

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

Let h be the width of the equispaced values of x.

Then the first increment in y is obtained by the following set of formulae:

$$k = f(x_0, y_0) \tag{11.41}$$

$$= \text{slope at the beginning of the interval } (x_0, y_0)$$

$$k' = f(x_0 + h,\ y_0 + kh) \text{ (say)} \tag{11.42}$$

$$k'' = f(x_0 + h,\ y_0 + k'h) \tag{11.43}$$

$$= \text{slope at the beginning of the interval } (x_0, y_0)$$

Also, slope at the middle point of interval (x_0, y_0) is

$$k_1 = f(x_0 + \frac{h}{2},\ y_0 + k\frac{h}{2}) \tag{11.44}$$

Now the increment in the value of y in the first interval is given by

$$\Delta y = \frac{h}{6}(k + 4k_1 + k'') \tag{11.45}$$

which can be easily obtained by the Simpson's rule,

$$\Delta y = \int_{x_0}^{x_0+\Delta x} \frac{dy}{dx}\, dx$$

The increment in y in the second interval (x_1, y_1) is obtained by the following formulae :

$$k = f(x_1, y_1)$$
$$k' = f(x_1 + h,\ y_1 + kh)$$
$$k'' = f(x_1 + h,\ y_1 + k'h)$$
$$k_1 = f(x_1 + \frac{h}{2},\ y_1 + k\frac{h}{2})$$

and $$\Delta y = \frac{h}{6}(k + 4k' + k'')$$

In the same way the increments in y can be obtained very easily in the succeeding intervals.

Working method Given $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$

find out $k_1 = hf(x_0, y_0)$

$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$
$$k_3 = hf(x_0 + h,\ y_0 + k_2)$$

$$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

Then $k = \dfrac{1}{6}(k_1 + 4k_4 + k_3)$

$\therefore$ Value of $y = y_0 + k$

Example 11.11 Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $y' = x + y$ and $y(0) = 1$.

Solution Here,

$$f(x, y) = y' = x + y, x_0 = 0, y_0 = 1, h = 0.2$$

$$\therefore \quad f(x_0, y_0) = 0 + 1 = 1$$

$k_1 = hf(x_0, y_0) = (0.2)(1) = 0.2$

$k_2 = hf(x_0 + h, y_0 + k_1) = hf(0.2, 1.2)$
$= (0.2)(0.2 + 1.2) = 0.28$

$k_3 = hf(x_0 + h, y_0 + k_2) = hf(0.2, 1.28)$
$= (0.2)(0.2 + 1.28) = 0.296$

$$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf(0.1, 1.1)$$

$= (0.2)(0.1 + 1.1) = 0.24$

Then $k = \dfrac{1}{6}(k_1 + 4k_4 + k_3)$

$$= \frac{1}{6}[0.2 + 4(0.24) + 0.296] = 0.2426666$$

$\therefore \quad y_1 = y(0.2) = y_0 + k = 1.2426666$

11.14 RUNGE–KUTTA METHODS

We have seen that solving differential equations numerically using Taylor's series method to determine higher order derivatives is a lengthy process. To overcome this there is a class of methods known as Runge–Kutta methods, which do not require the calculations of higher order derivatives and give greater accuracy. These methods agree with Taylor's series solution upto the term h^r, where r differs from method to method and is known as the order of that method.

First order Runge–Kutta method We have seen that Euler's method (section 11.10) gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \quad [\because y' = f(x, y)] \tag{11.46}$$

Now $y_1 = y(x_0 + h)$. Expanding it by Taylor's series, we get,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots$$

This implies that Euler's method agrees with Taylor's series solution upto term in h.

Hence, Runge–Kutta method of first order is the Euler's method only.

Runge–Kutta method of second order The Improved Euler's method (section 11.11) gives

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h,\ y_1^{(1)})] \tag{11.47}$$

where $y_1^{(1)} = y_0 + hf(x_0, y_0)$. That is,

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f\{x_0 + h,\ y_0 + hf(x_0, y_0)\}] \tag{11.48}$$

Now, $$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \tag{11.49}$$

(by Taylor's series)

Now expanding $f\{x_0 + h,\ y_0 + hf(x_0, y_0)\}$ by Taylor's series for a function of two variables

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)}$$

$$1 + hf(x_0, y_0) + \left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} + \text{terms containing second}$$

and higher powers of h]

$$= y_0 + \frac{1}{2}[2hf(x_0, y_0) + h^2 \left\{\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} + f(x_0, y_0)\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)}\right\}$$

$$+ \text{terms containing } h^3 \text{ onwards (or } O(h^3))]$$

$$= y_0 + hf(x_0, y_0) + \frac{h^2}{2} f'(x_0, y_0) + O(h^3)$$

$$\left[\because f'(x_0,y_0) - \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} + f(x_0,y_0)\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)}\right]$$

$$= y_0 + hy_c' + \frac{h^2}{2!}y_0'' + O(h^3) \tag{11.50}$$

Eqns (11.49) and (11.50) imply that the Improved Euler's method agrees with the Taylor's series solution upto the term in h^2. Therefore, the Improved Euler's method is the Runge–Kutta method of second order.

The algorithm to Runge–Kutta method of second order is

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

and

$$y_1 = y_0 + k$$

11.15 HIGHER ORDER RUNGE–KUTTA METHODS

In this section, we will study the formula for third and fourth order Runge–Kutta methods. Their derivations being tedious and unrequired, the formulae have not been derived here.

Runge–Kutta method of third order: It is defined by the following equations:

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$

$$k_3 = hf(x_0 + h,\ y_0 + k_2)$$

$$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$

and

$$y_1 = y_0 + K$$

We can see that this is identical to Runge's method.

Runge–Kutta method of fourth order: It is most commonly known as Runge–Kutta method and the working procedure is as follows. Consider the following equations.

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0\ .$$

To compute y_1, calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, \; y_0 + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then $$y_1 = y_0 + k \text{ and } x_1 = x_0 + h$$

The increment in y in second interval is computed in a similar manner by means of the formulae

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, \; y_1 + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then $$y_2 = y_1 + k \text{ and } x_2 = x_1 + h$$

and so on for succeeding intervals.

You can notice that the only change in the formulae for the different intervals is in the values of x and y. Thus, to find k in ith interval, we should have substituted x_{i-1} and y_{i-1} in the expressions for k_1, k_2, k_3 and k_4.

Example 11.12 Given $y' = x^2 - y$, $y(0) = 1$, find $y(0.1)$, $y(0.2)$ using Runge–Kutta methods of (i) second order, (ii) third order and (iii) fourth order.

Solution Given

$$y' = f(x, y) = x^2 - y, \; x_0 = 0, \; y_0 = 1$$

$\therefore$ $$f(x_0, y_0) = -1.$$

Let $h = 0.1$

Runge–Kutta method of 2nd order : Here,

$$k_1 = hf(x_0, y_0) = (0.1)(-1) = -0.1$$
$$k_2 = hf(x_0 + h,\ y_0 + k_1) = hf(0.1, 0.9)$$
$$= (0.1)\,[(0.1)^2 - 0.9] = -0.089$$

$$\therefore \quad k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[(-0.1) + (0.089)] = -0.0945$$

$$\therefore \quad y_1 = y\,(0.1) = y_0 + k = 1 - 0.0945 = 0.9055$$

Again, taking $x_1 = 0.1$, $y_1 = 0.9055$ in place of (x_0, y_0) and repeating the process,

$$k_1 = hf(x_1, y_1) = h(x_1^2 - y_1)$$
$$= (0.1)\,[(0.1)^2 - 0.9055] = -0.08955$$
$$k_2 = hf[x_1 + h,\ y_1 + k_1] = hf(0.2, 0.81595)$$
$$= (0.1)\,[(0.2)^2 - 0.81595] = -0.077595$$

$$\therefore k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[(-0.08955 - 0.077595] = -0.0835725$$

$$\therefore y_2 = y\,(0.2) = y_1 + k = 0.9055 - 0.0835725$$
$$= 0.8219275$$

*Runge–Kutta method of 3rd order :*Here,

$$k_1 = hf(x_0, y_0) = -0.1$$
$$k_2 = hf(x_0 + h,\ y_0 + k_1) = -0.089$$
$$k_3 = hf(x_0 + h,\ y_0 + k_2) = hf(0.1, 0.911)$$
$$= (0.1)\,[(0.1)^2 - 0.911] = -0.0901$$
$$k_4 = hf(x_0 + h/2,\ y_0 + k_1/2) = hf(0.05, 0.95)$$
$$= (0.1)\,[(0.05)^2 - 0.95] = -0.09475$$

$$\therefore\ k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$

$$= \frac{1}{6}[(-0.1) + 4\,(-0.09475) + (-0.0901\ ] = -0.09485$$

$$\therefore y_1 = y\,(0.1) = 0.90515$$

Taking $x_1 = 0.1$, $y_1 = 0.90515$, $h = 0.1$ in place of (x_0, y_0) and repeating the process, we get

$$k_1 = hf(x_1, y_1) = (0.1)\,[(0.1)^2 - 0.90515] = -0.089515$$
$$k_2 = hf(x_1 + h,\ y_1 + k_1) = hf(0.2, 0.815635)$$
$$= (0.1)\,[(0.2)^2 - 0.815635] = -0.0775635$$
$$k_3 = hf(x_1 + h,\ y_1 + k_2) = hf(0.2, 0.8275865)$$
$$= (0.1)\,[(0.2)^2 - 0.8275865] = -0.0787586$$

$$k_4 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = hf(0.15,\ 0.8603925)$$

$$= (0.1)\,[(0.15)^2 - 0.8603925] = -\,0.0837892$$

$$\therefore k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$

$$= \frac{1}{6}[(-\,0.089515) + 4\,(-\,0.0837892) - 0.0787586] = -\,0.0839051$$

$$\therefore y_2 = y\,(0.2) = y_1 + k = 0.90515 - 0.0839051 = 0.8212449$$

Runge–Kutta method of 4th order : Here,

$$k_1 = hf(x_0, y_0) = -\,0.1$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = (0.1)f[0.05,\ 0.95]$$

$$= (0.1)\,[(0.05)^2 - 0.95] = 0.09475$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = hf(0.05,\ 0.952625)$$

$$= (0.1)\,[(0.05)^2 - 0.952625] = -\,0.0950125$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = hf[0.1,\ 0.9049875]$$

$$= (0.1)\,[(0.1)^2 - 0.9049875] = -\,0.0894987$$

$$\text{Now, } k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[-\,0.1 + 2(-\,0.09475) + 2(-\,0.0950125) - 0.0894987]$$

$$= -0.0948372$$

$$\therefore\ y_1 = y\,(0.1) = y_0 + k = 1 - 0.0948372 = 0.9051627$$

Taking $x_1 = 0.1$, $y_1 = 0.9051627$ in place of x_0, y_0 and repeating the process, we get

$$k_1 = hf(x_1, y_1) = hf\ (0.1,\ 0.9051627)$$

$$= (0.1)\,[(0.1)^2 - 0.9051627] = -\,0.0895162$$

$$k_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right) = hf\ (0.15,\ 0.8604046)$$

$$= (0.1)\,[(0.15)^2 - 0.8604046] = \quad 0.0837904$$

$$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = hf(0.15,\ 0.8632674)$$

$$= (0.1)\,[(0.15)^2 - 0.8632674] = -\,0.0840767$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3) = hf(0.2, 0.8210859)$$
$$= (0.1)\,[(0.2)^2 - 0.8210859] = -\,0.0781085$$

$$k = \frac{1}{6}\,(k_1 + 2k_2 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}\,[-0.0895162 + 2(-0.0837904) + 2(-0.0840767) - 0.0781085]$$

$= -\,0.0838931$

$\therefore\ y_2 = y\,(0.2) = y_1 + k = 0.9051627 - 0.0838931$
$= 0.08212695$

Example 11.13 Using Runge–Kutta method of fourth order, solve for $y(0.1)$, $y(0.2)$ and $y(0.3)$ given that $y' = xy + y^2$, $y(0) = 1$.

Solution Here,

$$y' = f(x, y) = xy + y^2,\ x_0 = 0,\ y_0 = 1,\ h = 0.1$$

$k_1 = hf(x_0, y_0) = (0.1)\,[(0.1) + (1)^2] = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf[0.05, 1.05]$$

$= (0.1)\,[(0.05)(1.05)^2 + (1.05)^2] = 0.1155$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf(0.05, 1.05775)$$

$= (0.1)\,[(0.05)(1.05775) + (1.05775)^2] = 0.1171723$

$k_4 = hf(x_0 + h,\ y_0 + k_3) = hf[0.1, 1.1171723]$
$= (0.1)\,[(0.1)(1.1171723) + (1.1171723)^2] = 0.1359791$

$$k = \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}\,[0.1 + 2(0.1155) + 2(0.1171723) + 0.1359791]$$

$= 0.1168873$

$y_1 = y_{(0.1)} = y_0 + k = 1 + 0.1168873 = 1.1168873$

Now, taking $x_1 = 0.1$, $y_1 = 1.1168873$ in place of (x_0, y_0) and repeating the process,

$k_1 = hf(x_1, y_1) = (0.1)\,[(0.1)(1.1168873) + (1.1168873)^2]$
$= 0.1359125$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = hf[0.15, 1.1848436]$$

$= (0.1)\,[(0.15)(1.1848436) + (1.1848436)^2] = 0.158158$

$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = hf(0.15, 1.1959663)$

$= (0.1)\ [(0.15)(1.1959663) + (1.1959663)^2] = 0.1609730$

$k_4 = hf(x_1 + h,\ y_1 + k_3) = hf[0.2, 1.2778603]$

$= (0.1)\ [(0.2)(1.2778603) + (1.2778603)^2] = 0.1888499$

$k = \frac{1}{6}\ (k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}\ [0.1359125 + 2(0.158158) + 2(0.1609730) + 0.1888499]$

$= 0.160504$

$y_2 = y(0.2) = y_1 + k = 1.1168873 + 0.160504 = 1.2773914$

Taking $x_2 = 0.2$, $y_2 = 1.2773914$ in place of (x_1, y_1) and repeating the process, we get

$k_1 = hf(x_2, y_2) = (0.1)\ [(0.2)(1.2773914) + (1.2773914)^2]$

$= 0.1887207$

$k_2 = hf\left(x_2+\frac{h}{2}, y_2+\frac{k_1}{2}\right) = hf[0.25, 1.3717517]$

$= (0.1)\ [(0.25)(1.3717517) + (1.3717517)^2] = 0.222464$

$k_3 = hf\left(x_2+\frac{h}{2}, y_2+\frac{k_2}{2}\right) = hf(0.25, 1.3886234)$

$= (0.1)\ [(0.25)(1.3886234) + (1.3886234)^2] = 0.227543$

$k_4 = hf(x_2 + h,\ y_2 + k_3) = (0.1)f[0.3, 1.5049345]$

$= (0.1)\ [(0.3)(1.5049345) + (1.5049345)^2] = 0.2716308$

$k = \frac{1}{6}\ (k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}\ [0.1887207 + 2(0.222464) + 2(0.227543) + 0.2716308]$

$= 0.2267275$

$\therefore y_3 = y(0.3) = y_2 + k = 1.2773914 + 0.2267275 = 1.504119$

$\therefore y(0.1) = 0.11689, y(0.2) = 1.27739\ ; y(0.3) = 1.50412$

11.16 RUNGE–KUTTA METHODS FOR SIMULTANEOUS FIRST ORDER EQUATIONS

Consider the simultaneous equations

$$\frac{dy}{dx} = f_1(x, y, z)\ ; \quad \frac{dx}{dz} = f_2(x, y, z)$$

with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Now starting from (x_0, y_0, z_0) the increment k and l in y and z are given by the following formulae :

$$k_1 = hf_1\,(x_0, y_0, z_0)\ ; \ l_1 = hf_2(x_0, y_0, z_0)$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right);$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right);$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf_1\,(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)\ ;$$

$$l_4 = hf_2(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$k = \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$$

$$l = \frac{1}{6}\,(l_1 + 2l_2 + 2l_3 + l_4)$$

Hence, $\quad y_1 = y_0 + k,\ z_1 = z_0 + l$

To compute y_2, z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

If we consider the second order Runge–Kutta method, then

$$k_1 = hf_1\,(x_0, y_0, z_0)\ ; \quad l_1 = hf_2(x_0, y_0, z_0)$$

$$k_2 = hf_1\,(x_0 + h,\ y_0 + k_1,\ z_0 + l_1)\ ; \ l_2 = hf_2(x_0 + h,\ y_0 + k_1,\ z_0 + l_1)$$

$$k = \frac{1}{2}(k_1 + k_2); \qquad l_1 = \frac{1}{2}(l_1 + l_2)$$

$$\therefore\ y_1 = y_0 + k \quad \text{and} \quad z_1 = z_0 + k$$

Example 11.14 Solve $\frac{dy}{dx} = yz + x$; $\frac{dz}{dx} = xz + y$ given that $y(0) = 1$; $z(0) = -1$ for $y(0.2)$, $z(0.2)$.

Solution Here, $f_1(x, y, z) = yz + x$,

$f_2(x, y, z) = xz + y$, $x_0 = 0$, $y_0 = 1$, $z_0 = -1$.

Let $h = 0.1$.

$$k_1 = hf_1(x_0, y_0, z_0) = (0.1)[(1)(-1) + 0] = -0.1$$

$$l_1 = hf_2(x_0, y_0, z_0) = (0.1)[(0)(-1) + 1] = 0.1$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_1(0.05, 0.95, -0.95)$$

$$= (0.1)[(0.95)(-0.95) + 0.05] = -0.08525$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_2(0.05, 0.95, -0.95)$$

$$= (0.1)[(0.05)(-0.95) + 0.95] = 0.09025$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = hf_1(0.05, 0.957375, -0.954875)$$

$$= (0.1)[(0.957375)(-0.954875) + 0.05] = -0.0864173$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = hf_2(0.05, 0.957375, -0.954875)$$

$$= (0.1)[(0.05)(-0.954875) + 0.957375] = -0.0909631$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3) = hf_1(0.1, 0.9135827, -0.9090369)$$

$$= (0.1)[(0.9135827)(-0.9090369) + 0.1] = -0.073048$$

$$l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3) = hf_2(0.1, 0.9135827, -0.9090369)$$

$$= (0.1)[(0.1)(-0.9090369) + 0.9135827] = 0.822679$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + + k_4)$$

$$= \frac{1}{6}[0.1 + 2(-0.08525) + 2(-0.0864173) - 0.073048]$$

$$= -0.0860637$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= \frac{1}{6}[0.1 + 2(0.09025) + 2(0.0909631) - 0.0822679]$$

$$= -0.0907823$$

$\therefore\ y_1 = y(0.1) = y_0 + k = 1 - 0.0860637 = 0.9139363$

$z_1 = z(0.1) = z_0 + l = -1 + 0.0907823 = -0.9092176$

Now replacing $x_1 = 0.1$, $y_1 = 0.9139363$ and $z_1 = -0.9092176$ and repeating the process,

$k_1 = hf_1(x_1, y_1, z_1) = h(y_1 z_1 + x_1) = -0.0730966$

$l_1 = hf_2(x_1, y_1, z_1) = h(x_1 z_1 + y_1) = -0.08230145$

$k_2 = hf_1(x_1 + h/2,\ y_1 + k_1/2,\ z_1 + l_1/2) = hf_1(0.15, 0.877388, -0.8680669)$
$= (0.1)[(0.877388)(-0.8680669) + 0.15] = -0.0611631$

$l_2 = hf_2(x_1 + h/2,\ y_1 + k_1/2,\ z_1 + l_1/2) = hf_2(0.15, 0.877388, -0.8680669)$
$= (0.1)[(0.15)(-0.8680669) + 0.877388] = 0.0747177$

$k_3 = hf_1(x_1 + h/2,\ y_1 + k_2/2,\ z_1 + l_2/2) = hf_1(0.15, 0.8833547, -0.8718587)$
$= (0.1)[(0.8833547)(-0.8718587) + 0.15] = -0.062016$

$l_3 = hf_2(x_1 + h/2,\ y_1 + k_2/2,\ z_1 + l_2/2)$
$= hf_2(0.15, 0.8833547, -0.8718587)$
$= (0.1)[(0.15)(-0.8718587) + 0.8833547] = 0.0750851$

$k_4 = hf_1(x_1 + h,\ y_1 + k_3,\ z_1 + l_3) = hf_1(0.2, 0.8519203, -0.8341324)$
$= (0.1)[(0.8519203)(-0.8341324) + 0.2] = -0.0510614$

$l_4 = hf_2(x_1 + h,\ y_1 + k_3,\ z_1 + l_3) = hf_2(0.2, 0.8519203, -0.8341324)$
$= (0.1)[(0.2)(-0.8341324) + 0.8519203] = 0.0685093$

$k = 1/6[k_1 + 2k_2 + 2k_3 + k_4]$
$= 1/6[-0.0730966 + 2(-0.0611631) + 2(-0.062016) - 0.0510614]$
$= -0.0617527$

$l = 1/6[l_1 + 2l_2 + 2l_3 + l_4]$
$= 1/6[0.08230145 + 2(-0.0747177) + 2(0.0750851) + 0.0685093]$
$= 0.0750693$

$\therefore\ y_2 = y(0.2) = y_1 + k = 0.9139363 - 0.0617527 = 0.8521836$

$\therefore\ z_2 = z(0.2) = z_1 + l = -0.9092176 + 0.0750693 = -0.8341482$

11.17 RUNGE–KUTTA METHOD FOR SECOND ORDER DIFFERENTIAL EQUATION

Consider the second order differential equation,

$$\frac{d^2y}{dx^2} = \phi\left[x, y, \frac{dy}{dx}\right];\ y(x_0) = y_0;\ y'x_0 = y_0' \qquad (11.51)$$

Let $\dfrac{dy}{dx} = z$ then $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

Substituting Eqn (11.51), we get

$$\frac{dz}{dx} = \phi[x, y, z]\ ;\ y(x_0) = y_0\ ;\ z(x_0) = z_0$$

$\therefore$ The problem reduces to solving the simultaneous equations:

$$\frac{dy}{dx} = z = f_1(x, y, z) \text{ and } dz/dx = z = f_2(x, y, z)$$

subject to $y(x_0) = y_0$; $z(x_0) = z_0$ and this can be solved as shown in the previous section.

Example 11.15 Solve $y'' = xy' - y$; $y(0) = 3$, $y'(0) = 0$ to approximate $y(0.1)$.

Solution Given

$$y'' = xy' - y\ ;\ y(0) = 3,\ y'(0) = 0 \tag{i}$$

Let $y' = z$ then $y'' = z'$

$\therefore$ Eqn(i) reduces to

$y' = z = f_1(x, y, z)$

$z' = xz - y = f_2(x, y, z)$

subject to $y(0) = 3$ and $z(0) = 0$, i.e. $x_0 = 0$, $y_0 = 3$, $z_0 = 0$

Now,

$k_1 = hf_1(x_0, y_0, z_0) = h(z_0) = (0.1)(0) = 0$

$l_1 = hf_2(x_0, y_0, z_0) = h(x_0 z_0 - y_0) = -0.3$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_1(0.05, 3, -0.15)$$

$= (0.1)(-0.15) = -0.015$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = hf_2(0.05, 3, -0.15)$$

$= (0.1)[(0.05)(-0.15) - 3] = 0.30075$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = hf_1(0.05, 2.9925, -0.150375)$$

$= (0.1)(0.150375) = -0.0150375$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = hf_2(0.05, 2.9925, -0.150375)$$

$= (0.1)[(0.05)(-0.150375) - 2.9925] = -0.3000018$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3) = hf_1(0.1, 2.9849625, -0.3000018)$

$= (0.1)(-0.3000018) = -0.0300001$

$$l_4 = hf_2(x_0 + h,\ y_0 + k_3,\ z_0 + l_3) = hf_2(0.1, 2.9849625, -0.3000018)$$
$$= (0.1)\,[(0.1)(-0.3000018) - 2.9849625] = -0.3014962$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0 + 2(-0.015) + 2(-0.0150375) - 0.0300001]$$

$$= -0.0150125$$

$$l = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

$$= \frac{1}{6}[-0.3 + 2(-0.30075) + 2(-0.3000018) - 0.3014962]$$

$$= -0.3004999$$

$$\therefore y_1 = y(0.1) = y_0 + k = 3 - 0.0150125 = 2.9849875$$
$$z_1 = z(0.1) = z_0 + l = 0 - 0.3004999 = -0.3004999$$

EXERCISE 11.3

1. Solve $y' = x - y$ given that $y = 0.4$ at $x = 1$ for $y(1.6)$ using Runge's method.
2. Using Runge's method, find y at $x = 1.1$ given

   $$\frac{dy}{dx} = 3x + y^2,\ y(1) = 1.2$$
3. Evaluate $y(0.8)$ using Runge's method given

   $y' = \sqrt{x+y}$; $y = 0.41$ at $x = 0.4$
4. Using second order Runge–Kutta method, find y at $x = 0.1$, 0.2 and 0.3 given $2y' = (1 + x)y^2$; $y(0) = 1$.
5. Find $y(1.2)$ by Runge–Kutta method of fourth order given $y' = x^2 + y^2$; $y(1) = 1.5$. Take $h = 0.1$.
6. If $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$; $y(1) = 0$, solve for y at $x = 1.2$, 1.4 using Runge–Kutta method of fourth order.
7. Using Runge–Kutta method of fourth order, find y at $x = 1.1$, 1.2 given that

   $2y' = 2x^3 + y$; $y(1) = 2$.

8. Find y at $x = 0.1, 0.2$ using fourth order Runge–Kutta algorithm given that
$y' - yx^2 = 0$; $y(0) = 1$.

9. Use Runge–Kutta method to evaluate y at $x = 0.2, 0.4, 0.6$ given that
$\frac{dy}{dx} - xy = 1$; $y(0) = 2$.

10. Using Runge–Kutta method of fourth order, find $y(0.1)$, $y(0.2)$ given that
$\frac{dy}{dx} - y = -x$; $y(0) = 2$.

11. Solve $10y' = x^2 + y^2$, $y(0) = 1$ to evaluate $y(0.2)$ and $y(0.4)$ by fourth order R–K algorithm.

12. Given $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$; $x_0 = 0$, $y_0 = 1$, $h = 0.2$, find y_1 and y_2 using Runge–Kutta method.

13. Solve $\frac{dy}{dx} = \frac{1}{x+y}$ for $x = 0.5$ to z using R–K method with $x_0 = 0$, $y_0 = 1$ (take $h = 0.5$).

14. Use Runge–Kutta method of order four to find y at $x = 0.1, 0.2$ given that
$x(dy + dx) = y(dx - dy)$; $y(0) = 1$.

15. Solve $y' = x + y$, $y(0) = 1$ to find y at $x = 0.1, 0.2, 0.3$ using R–K method.

16. Solve the following for $y(0.1)$, $y(0.2)$ using Runge–Kutta algorithms of (i) second order, (ii) third order and (iii) fourth order.

(a) $\frac{dy}{dx} + y = 0$; $y(0) = 1$

(b) $\frac{dy}{dx} + 2y = x$; $y(0) = 1$

17. Use second order Runge–Kutta algorithm to solve $\frac{dy}{dx} + xz = 0$;
$\frac{dy}{dx} - y^2 = 0$ at $x = 0.2, 0.4$ given that $y = 1, z = 1$ at $x = 0$.

18. Solve $\frac{dy}{dx} = 1 + xz$, $\frac{dz}{dx} = -xy$ for $x = 0.3, 0.6, 0.9$ given that $y = 0$, $z = 1$ at $x = 0$ by R–K method.

19. Solve $\dfrac{dy}{dx} = x + z$, $\dfrac{dz}{dx} = x - y^2$ for $y(0.1)$, $z(0.1)$ given that $y(0) = 2$, $z(0) = 1$ by Runge – Kutta method.
20. Solve $y' = x + z$, $z' = x - y$ for $x = 0.1, 0.2$ given that $y = 0$, $z = 1$ at $x = 0$ by Runge–Kutta method.
21. Solve $\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} - 2xy = 0$ given that $y(0) = 1$, $y'(0) = 0$ for $y(0.1)$ using Runge–Kutta method.
22. Use Runge–Kutta method to solve $y'' - xy + 4y = 0$; $y(0) = 3$; $y'(0) = 0$ at $x = 0.1$.
23. Apply R–K algorithm to find y at $x = 0.1$ given $\dfrac{d^2y}{dx^2} = y^3$; $y(0) = 10$; $y'(0) = 5$.
24. Solve $\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - y = 0$, $y(0) = 1, y'(0) = 0$ to find $y(0.2)$, $y'(0.2)$ using Runge–Kutta method.
25. Evaluate y(0.2) by R–K method given that $y'' - xy'^2 + y^2 = 0$; $y(0) = 1, y'(0) = 0$.

ANSWERS

1. 0.8176
2. 1.7278
3. 0.8481
4. 1.0552, 1.1230, 1.2073
5. 2.5505
6. 0.1402, 0.2705
7. 2.2213, 2.4914
8. 1.0053, 1.0227
9. 2.243, 2.589, 2.072
10. 2.20517, 2.42139
11. 1.0207, 1.038
12. $y_1 = y(0.2) = 1.19598$; $y_2 = y(0.4) = 1.3751$
13. 1.3571, 1.5837, 1.7555, 1.8956
14. 1.0911, 1.1678,
15. 1.1103, 1.2428, 1.3997
16. (a) 0.905, 0.81901 ; 0.91, 0.82337 ; 0.90484, 0.81873

    (b) 0.825, 0.6905 ; 0.8234, 0.6878 ; 0.82342, 0.6879
17. 0.978, 1.2 ; 0.9003 , 1.382

18. 0.3448, 0.99 ; 0.7738, 0.9121 ; 1.255, 0.66806
19. 2.0845, 0.586
20. 0.1050, 0.9998 ; 0.2199, 0.9986
21. 1.005334
22. 2.9399
23. 17.42
24. 0.9802, – 0.196
25. 0.9801

11.18 PREDICTOR–CORRECTOR METHODS

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y)\ ;\ y(x_0) = y_0 \tag{11.52}$$

To solve the above equation, we use Euler's formula

$$y_{i+1} = y_i + hf'(x_i, y_i), = 0, 1, 2, \ldots \tag{11.53}$$

$$y_{i+1} = y_i + \frac{1}{2}\, h\{f(x_i, y_i) + f(x_{i+1}, y_{i+1})\} \tag{11.54}$$

The value of y_{i+1} is first determined using Eqn (11.53), and is substituted on RHS of Eqn (11.54) giving a better approximation of y_{i+1}. Again, this value y_{i+1} is substituted Eqn (11.55) in to calculate a still better approximation of y_{i+1}. The process is repeated till two consecutive values of y_{i+1} are equal upto the desired accuracy. This method of refining an initially crude estimate of y_{i+1} by means of a more accurate formula is known as *Predictor–Corrector method.* In the above, Eqn (11.53) is called *Predictor* and Eqn (11.54) is called *Corrector* of y_{i+1}.

In the following sections we will study two such methods, namely, (i) Milne's method, and (ii) Adam's–Bashforth method.

11.19 MILNE'S METHOD

Here, the equation to be solved numerically is

$$\frac{dy}{dx} = f(x, y)\ ;\ y(x_0) = y_0$$

The value $y_0 = y(x_0)$ being given, we calculate $y_1 = y(x_0 + h) = y(x_1)$; $y_2 = y(x_0 + 2h) = y(x_2)$; $y_3 = y(x_0 + 3h) = y(x_3) \ldots$
where h is a suitably chosen spacing.

By Newton's forward interpolation formula, we have

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\,\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\,\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\,\Delta^4 y_0 + \cdots$$

where $x = x_0 + uh$.

For $y = y'$ the above gives

$$y' = y'_0 + u\Delta y_0' + \frac{u(u-1)}{2!}\,\Delta^2 y_0' + \frac{u(u-1)(u-2)}{3!}\,\Delta^3 y_0' + \frac{u(u-1)(u-2)(u-3)}{4!}\,\Delta^4 y'_0 + \cdots \qquad (11.55)$$

Then to find $y_4 = y(x_0 + 4h)$, we integrate Eqn (11.55) with respect to x_0 over the interval $x_0 + 4h$ to , i.e.

$$\int_{x_0}^{x_0+4h}[y']dx = \int_{0}^{x_0+4h}\left\{y_0' + u\Delta y_0' + \frac{1}{2}(u^2-u)\Delta^2 y_0' + \frac{1}{6}(u^3-3u^2+2u)\Delta^3 y_0' + \frac{1}{24}(u^4-6u^3+11u^2-6u)\Delta^4 y_0' + \cdots\right\}dx$$

$$\text{or}\quad [y]_{x_0}^{x_0+4h} = h\int_0^4\left\{y_0' + u\Delta y_0' + \frac{1}{2}(u^2-u)\Delta^2 y_0' + \frac{1}{6}(u^3-3u^2+2u)\Delta^3 y_0' + \frac{1}{24}(u^4-6u^3+11u^2-6u)\Delta^4 y_0' + \cdots\right\}du$$

$$[\because x = x_0 + uh \Rightarrow dx = h\,du]$$

This gives, after simplification on both sides,

$$y_4 - y_0 = h\left[4y_0' + 8\Delta y_0' + \frac{20}{3}\Delta^2 y_0' - \frac{8}{3}\Delta^3 y_0' + \frac{14}{45}\Delta^4 y_0' + \cdots\right]$$

Now using $\Delta^n = (1 - E)^n$, $n = 1, 2, 3$ and simplyfying the above, we get

$$y_4 - y_0 = h\left[4y_0' + 8(y_1' - y_0') + \frac{20}{3}(y_2' - 2y_1' + y_0') + \right.$$

$$\left.\frac{8}{3}(y_3' - 3y_2' + 3y_1' - y_0') + \frac{14}{45}\Delta^4 y_0' + \cdots\right] \quad (11.56)$$

$$\text{or } y^4 = y^0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \quad (11.57)$$

[by considering only differences upto third order]

Hence, the error in Eqn (11.57) is $= \frac{14}{45}\Delta^4 y_0' + \cdots$ and this can be proved to be $= \frac{14}{45} y^5(\xi)$, where ξ lies in between x_0 and x_4. Hence, Eqn (11.57) can be written as

$$y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') + \frac{14h^5}{45} y^5(\xi) \quad (11.58)$$

$\therefore x_0, x_1, x_2, x_3, x_4$ are any five consecutive values of x, the above, in general, can be written as

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n') + \frac{14h^5}{45} y^5(\xi_1) \quad (11.59)$$

where ξ_1 lies between x_{n-3} and x_{n+1}. Eqn (11.59) is known as *Milne's predictor formula.*

To get *Milne's corrector formula*, integrate Eqn (11.55) with respect to x over the interval x_0 to $x_0 + 2h$. Then we have

$$\int_{x_0}^{x_0+2h}[y']dx = h\int_0^2\left\{y_0' + u\Delta y_0' + \frac{1}{2}(u^2 - u)\Delta^2 y_0'\right.$$

$$+ \frac{1}{6}(u^3 - 3u^2 + 2u)\Delta^3 y_0'$$

$$\left.+ \frac{1}{24}(u^4 - 6u^3 + 11u^2 - 6u)\Delta^4 y_0' + \cdots\right\}du$$

$$\text{or } y_2 - y_0 = h\left[2y_0' + 2\Delta y_0' - \frac{1}{3}\Delta^2 y_0' - \frac{4}{15}\cdot\frac{1}{24}\Delta y_0' + \cdots\right]$$

$$= h\left[2y_0' + 2(y_1' - y_0') + \frac{1}{3}(y_2' - 2y_1' + y_0')\right.$$

$$\left. - \frac{h}{90}\Delta^4 y_0' + \cdots\right] \qquad [\text{using } \Delta^n = (E-1)^n, n = 1, 2, 3]$$

or
$$y_2 = y_0 + \frac{h}{3}(y_0' - 4y_1' + y_2') \tag{11.60}$$

Considering only differences upto third order, Eqn (11.60) gives

$$y_2 = y_0 + \frac{h}{3}(y_0' - 4y_1' + y_2') \tag{11.61}$$

$\therefore$ Error $= -\dfrac{h}{90}\Delta^4 y_0' + \cdots$ and this can be proved to be $= -\dfrac{h^5}{90} y^5(\xi)$,

where $x_0 < \xi < x_2$.

$\therefore$ Eqn (11.61) can be written as

$$y_2 = y_0 + \frac{h}{3}(y'_0 - 4y'_1 + y'_2) - \frac{h^5}{90} y^5(\xi) \tag{11.62}$$

Since x_0, x_1, x_2 are any three consecutive values of x, the above can be written in general as

$$y_{n+1} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}') - \frac{h^5}{90} y^5(\xi_2) \tag{11.63}$$

where ξ_2 lies in between x_{n-1} and x_{n+1}. Eqn (11.63) is called as *Milne's corrector formula.*

Note : This method requires at least four values prior to the required value. If the initial four values are not given, we can obtain them by using Picard's method or Taylor's series method or Euler's method or Runge–Kutta method.

Example 11.16 Given $\dfrac{dy}{dx} = 1/x + y$, $y(0) = 2$, $y(0.2) = 2.0933$, $y(0.4) = 2.1755$, $y(0.6) = 2.2493$, find $y(0.8)$ using Milne's method.

Solution In the usual notation, Milne's predictor formula is

$$y_{n+1,p} = y_{n-2} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n') \tag{i}$$

where $y_{n+1,p}$ denotes the predicted value at y_{n+1}.

In the given problem, $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$ $h = 0.2$, $y_0 = 2$, $y_1 = 2.0933$, $y_2 = 2.1755$, $y_3 = 2.2493$

and $y' = \dfrac{1}{x+y}$

Putting $n = 3$ in Eqn (i), the predictor is

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad \text{(ii)}$$

Now $y_1' = \dfrac{1}{x_1 + y_1} = \dfrac{1}{0.2 + 2.0933} = 0.4360528$

$y_2' = \dfrac{1}{x_2 + y_2} = \dfrac{1}{0.4 + 2.1755} = 0.3882741$

$y_3' = \dfrac{1}{x_3 + y_3} = \dfrac{1}{0.6 + 2.2493} = 0.3509633$

Substituting in Eqn (ii), we get

$y_{4,p} = 2 + 4(0.2)/3\ [2(0.4360528) - 0.3882741 + 2(0.3509633)]$

$= 2.3162022$ (iii)

Now, Milne's corrector formula in general form is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}') \qquad \text{(iv)}$$

where $y_{n+1,c}$ denotes the corrected value of y_{n+1}.

Putting $n = 3$ in above, we get

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \qquad \text{(v)}$$

From Eqn (iii), $y_{4,p} = 2.3162022$ and $x_4 = 0.8$.

$y_4' = \dfrac{1}{x_4 + y_{4,p}} = \dfrac{1}{0.8 + 2.3162022} = 0.3209034$

Hence, from Eqn (v),

$y_{4,c} = 2.1755 + \dfrac{0.2}{3}[0.3882741 + 4(0.3509633) + 0.3209034]$

$= 2.3163687$

$\therefore y(0.8) = y_4 = 2.3164$ corrected to four decimals.

Example 11.17 Solve $\dfrac{dy}{dx} = (x + y)y$, $y(0) = 1$ using Milne's Predictor – Corrector method for $y(0.4)$. The values for $x = 0.1$, 0.2 and 0.3 should be obtained by Runge–Kutta method of fourth order.

Solution Given $y' = (x + y)\, y$; $y(0) = 1$.

Now the values of $y(0.1)$, $y(0.2)$, $y(0.3)$ using Runge–Kutta method of fourth order are

$y(0.1) = 1.11689$, $y(0.2) = 1.27739$ and

$y(0.3) = 1.50412$, [refer to Ex. 11.13]

$\therefore x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$

$y_0 = 1, y_1 = 1.11689,\ y_2 = 1.27739,\ y_3 = 1.50412$

Milne's predictor formula is

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad \text{(i)}$$

Now, $y' = (x + y)\, y$

$\therefore \quad y_1' = (x_1 + y_1)\, y_1 = (0.1 + 1.11689)(1.11689) = 1.3591323$

$y_2' = (x_2 + y_2)\, y_2 = (0.2 + 1.27739)(1.27739) = 1.8872032$

$y_3' = (x_3 + y_3)\, y_3 = (0.3 + 1.50412)(1.50412) = 2.713613$

Substituting in Eqn (i), we get

$$y_{4,p} = 1 + \frac{4(0.1)}{3}[2(1.3591323) - 1.8872032 + 2(2.713613)]$$

$$= 1.8344383 \qquad \text{(ii)}$$

Now Milne's corrector formula is

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' - 4y_3' + 2y_4') \qquad \text{(iii)}$$

From Eqn (ii), $y_{4,p} = 1.8344383$, $x_4 = 0.4$

$\therefore\ y_4' = (x_4 + y_{4,p})\, y_{4,p} = (0.4 + 1.8344383)(1.8344383)$

$= 4.0989392$

Hence, from Eqn (iii),

$$y_{4,c} = 1.27739 + \frac{0.1}{3}[1.8872032 + 4(2.713613) + 4.0989392]$$

$$= 1.8387431$$

$\therefore\ y_4 = y(0.4) = 1.83874$ correct to five decimals.

11.20 ADAMS–BASHFORTH METHOD

Given

$$\frac{dy}{dx} = f(x, y) \qquad (11.64)$$

and $y(x_0) = y_0$

we compute $y_{-1} = y(x_0 - h),\ y_{-2} = y(x_0 - 2h),\ y_{-3} = y(x_0 - 3h), \ldots$

Now integrating Eqn (11.64), on both sides with respect to x in $[x_0, x_0 + h]$, we get

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)\,dx \tag{11.65}$$

Replacing $f(x, y)$ by Newton's backward interpolation formula, we get

$$y_1 = y_0 + h\int_0^1 \Big\{ f_0 + u\nabla f_0 + \frac{1}{2}(u^2 + u)\nabla^2 f_0$$

$$\frac{1}{6}(u^3 + 3u^2 + 2u)\nabla^3 f_0 + \cdots \Big\} du$$

$[\therefore x = x_0 + uh, dx = h\,du]$

$$= y_0 + h\left\{ f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + \cdots \right\} \tag{11.66}$$

Neglecting the fourth order and higher order differences and using $\nabla f_0 = f_0 - f_{-1}$, $\nabla^2 f_0 = f_0 - 2f_{-1} + f_{-2}$, $\nabla^3 f_0 = f_0 - 3f_{-1} + 3f_{-2} - f_{-3}$, in Eqn (11.66), we get, after simplification,

$$y_1 = y_0 + \frac{h}{24}\{55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\}$$

which is known *Adams–Bashforth predictor formula* and is denoted generally as

$$y_{n+1,p} = y_n + \frac{h}{24}\{55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\}$$

$$\text{or} \quad y_{n+1,p} = y_n + \frac{h}{24}\{55 y_n' - 59 y_{n-1}' + 37 y_{n-2}' - 9 y_{n-3}'\} \tag{11.67}$$

Having found y_1, we find $f_1 = f(x_0 + h, y_1)$

Then to find a better value of y_1, we derive a corrector formula by substituting *Newton's backward interpolation formula* at f_1 in place of $f(x, y)$ in Eqn (11.65), i.e.

$$y_1 = y_0 + \int_{x_0}^{x_0+h} \Big\{ f_1 + u\nabla f_1 + \frac{1}{2!}u(u+1)\nabla^2 f_1$$

$$+ \frac{1}{3!}u(u+1)(u+2)\nabla^3 f_1 + \cdots \Big\} dx$$

$$= y_0 + h \int_{-1}^{0} \left\{ f_1 + u\nabla f_1 + \frac{1}{2}(u^2 + u)\nabla^2 f_1 \right.$$

$$\left. + \frac{1}{3}(u^3 + 3u^2 + 2u)\nabla^3 f_1 + \cdots \right\} du$$

$$[\because x = x_1 + uh, dx = hdu]$$

$$= y_0 + h\left\{ f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^3 f_1 \ldots \right\} \qquad (11.68)$$

Neglecting the fourth order and higher order differences and using

$$\nabla f_1 = f_1 - f_0, \ \nabla^2 f_1 = f_1 - 2f_0 + f_{-1}, \ \nabla^3 f_1 = f_1 - 3f_0 + 3f_{-1} - f_2$$

in Eqn (11.68), we get, after simplification,

$$y_1 = y_0 + \frac{h}{24}[9f_1 + 19f_0 - 5f_{-1} + f_{-2}]$$

which is known as *Adams–Bashforth corrector formula* and is denoted , generally,

$$y_{n+1,c} = y_n + \frac{h}{24}\{9f_{n+1} - 19f_n - 5f_{n-1} + f_{n-2}\}$$

or $$y_{n+1,c} = y_n + \frac{h}{24}\{9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\} \qquad (11.69)$$

Note : Here also we require at least four values of y prior to the required value of y.

Example 11.18 Using Adams–Bashforth method, find $y(1.4)$ given $y' = x^2(1 + y)$, $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$ and $y(1.3) = 1.979$.

Solution Given

$y' = x^2(1 + y)$, $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $y_0 = 1$,
$y_1 = 1.233$, $y_2 = 1.548$, $y_3 = 1.979$, $h = 0.1$.

Adams–Bashforth predictor formula is

$$y_{4,p} = y_3 + \frac{h}{24}(55y_3' - 59y_2' - 37y_1' - 9y_0') \qquad \text{(i)}$$

$y_0' = x_0^2(1 + y_0) = (1)^2[1 + 1] = 2$
$y_1' = x_1^2(1 + y_1) = (1.1)^2[1 + 1.233] = 2.70193$
$y_2' = x_2^2(1 + y_2) = (1.2)^2[1 + 1.548] = 3.66912$
$y_3' = x_3^2(1 + y_3) = (1.3)^2[1 + 1.979] = 5.03451$

$\therefore \ y_{4,p} = 1.979 + 0.1/24\ \{55(5.03451) - 59(3.66912)$
$+ 37(2.70193) - 9(2)\}$
$= 2.5722974$

Now $y_{4,p}' = x_4^2(1 + y_{4,p}) = (1.4)^2\{1 + 2.5722974\}$
$= 7.0017029$

The corrector formula is

$$y_{4,c} = y_3 + \frac{h}{24}\{9\,y_4' + 19\,y_3' - 5\,y_2' + y_1'\}$$

$$= 1.979 + \frac{0.1}{24}\{9(7.0017029) + 19(5.03451) - 5(3.66912) + 2.70193\}$$

$$= 2.5749473$$

$\therefore y(0.4) = 2.575$, correct to three decimal places.

Example 11.19 Find $y(0.1), y(0.2), y(0.3)$, from $y' = x^2 - y$; $y(0) = 1$ using Taylor's series method and hence obtain $y(0.4)$ using Adams–Bashforth method.

Solution Given $y' = x^2 - y$; $y(0) = 1$.

From Ex. 11.2 we have

$y(0.1) = 0.905125,\ y(0.2) = 0.8212352,\ y(0.3) = 0.7491509$

i.e. $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3,$

$y_0 = 1,\ y_1 = 0.905125,\ y_2 = 0.8212352, y_3 = 0.7491509$

Also, $y_0' = -1, y_1' = -0.895125,\ y_2' = -0.7812352$ and $y_3' = -0.6591509$

Adam's predictor formula is

$$y_{4,p} = y_3 + \frac{h}{24}\{55\,y_3' - 59\,y_2' + 37\,y_1' - 9\,y_0'\}$$

$$= 0.7491509 + \frac{0.1}{24}[55(-0.6591509) - 59(-0.7812352) + 37(-0.895125) - 9(-1)]$$

$$= 0.6896509$$

Now $y_{4,p}' = x_4^2 - y_{4,p} = (0.4)^2 - 0.6896507 = -0.5296507$

The corrector is,

$$y_{4,c} = y_3 + \frac{h}{24}\{9\,y_4' + 19\,y_3' - 5\,y_2' + y_1'\}$$

$$= 0.7491509 + \frac{0.1}{24}[9(-0.5296507) + 19(-0.6591509) - 5(-0.7812352) + (-0.895125)]$$

$$= 0.6892522$$

$\therefore y(0.4) = 0.6896522$

EXERCISE 11.4

1. If $y' = 2e^x - y, y(0) = 2, y(0.1) = 2.010, y(0.2) = 2.040\ y(0.3) = 2.090$, find $y(0.4)$, $y(0.5)$ correct to three decimal places applying Milne's Predictor–Corrector method.
2. Solve $y' = x^2 - y$ given that $y(0) = 1, y(0.1) = 0.9052,\ y(0.2) = 0.8213$, for $y(0.5)$. Here, use Milne's method by computing $y(0.3) = 1$, using Taylor's method.
3. Tabulate the solution to $y' = x + y$ with the initial condition

   (i) $y(0) = 0$ for $0.4 < x\ 1 \le 1.0,\ h = 0.1$

   (ii) $y(0) = 1$ for $0.1 \le x\ 1 \le 0.3,\ h = 0.05$

   using Milne's predictor – Corrector method.
4. Using Taylor's series method, solve $y' = xy + y^2$, $y(0) = 1$ at $x = 0.1$, 0.2, 0.3. Continue the solution at $x = 0.4$ by Milne's method.
5. Solve $y' = 1 + xy^2$, for $y(0.4)$ by Milne's method given that $y(0) = 1$, $y(0.1) = 1.105, y(0.2) = 1.223, y(0.3) = 1.355$.
6. Use Milne's method to compute $y(0.3)$ from $y' = x^2 + y^2$, $y(0) = 1$. Find the initial values $y(-0.1), y(0.1), y(0.2)$ from the Taylor's series.
7. Solve $y' = x^2 + y^2 - 2$, using Milne's method for $x = 0.3$ given that $y = 1$ at $x = 0$. Compute $y(-0.1)$, $y(0.1)\ y(0.2)$ using Runge–Kutta method of fourth order.
8. Given $y' + y = 1, y(0) = 0$, find $y(0.1)$ by using Euler's method, $y(0.2)$ by modified Euler's method, $y(0.3)$ by Improved Euler's method, and $y(0.4)$ by Milne's method.
9. Solve by Taylor's series of third order, the problem $y' = (x^3 + xy^2)e^{-x}$, $y(0) = 1$ to find y for $x = 0.1, 0.2, 0.3$. Continue the solution at $x = 0.4$ and $x = 0.5$ by Milne's method.
10. Using Adams–Bashforth predictor–corrector method, find $y(1.4)$ given that $x^2y' + xy = 1$; $y(1) = 1, y(1.1) = 0.996, y(1.2) = 0.986$, $y(1.3) = 0.972$.
11. Using Adams–Bashforth formulae, determine $y(0.4)$ given the equation $y' = 0.5\,xy$; $y(0) = 1$, $y(0.1) = 1.0025$, $y(0.2) = 1.0101$, $y(0.3) = 1.0228$.
12. Using Adams–Bashforth formulae, find $y(0.4)$, $y(0.5)$, if y satisfies

    $\frac{dy}{dx} = 3e^x + 2y$ with $y(0) = 0$. Compute y at $x = 0.1, 0.2, 0.3$ by means of Runge–Kutta method.

ANSWERS

1. 2.1621, 2.2546
2. 0.6435
3. (i) 0.0918, 0.1487, 0.2221, 0.3138, 0.4255, 0.5596, 0.7183
   (ii) 1, 1.0525, 1.1105, 1.2312, 1.2604, 1.3265
4. 1.8369
5. 1.45982
6. 1.4392
7. 0.61432
8. 0.3333
9. 1.0709, 1.1103
10. 0.94934
11. 1.0408
12. 2.2089, 3.20798

CHAPTER 12

Numerical Solution to Partial Differential Equations

12.1 INTRODUCTION

In Engineering, Science and many branches of Applied Mathematics (e.g. in Fluid dynamics, Boundary layer theory, Heat transfer, Quantum Mechanics etc.), we often come across partial differential equations. Only a few of them can be solved by analytical methods. So, in most cases, we go in for numerical methods to approximate solution. Of all the numerical methods available for the solution to partial differential equations, the method of finite differences is commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Then the given equation is changed into a system of linear equations which are solved by iterative procedures.

12.2 DIFFERENCE QUOTIENTS

A difference quotient is the quotient obtained by dividing the difference between two values of a function by the difference between the two corresponding values of the independent variable.

Thus, for a function $u(x)$ of a single variable, the difference coefficient is

$$\frac{u(x+h)-u(x)}{h},$$

whose limiting value is the derivative of $u(x)$ w.r.t. x, i.e. du/dx.

Here, $u(x, y)$ is a function of two independent variable x, y. Therefore, we have to consider the differences in both.

First, let us consider the differences in the x-direction.

Taylor's series for $u(x, y_0)$ about the point (x_0, y_0) is

$$u(x, y_0) = u(x_0, y_0) + (x - x_0) + u_x(x_0, y_0) + \frac{(x - x_0)^2}{2!} u_{xx}(\xi, y_0) \quad (12.1)$$

where $x_0 \leq \xi \leq x$. Let $x = x_0 + h$ in Eqn (12.1).

$$\therefore u(x_0 + h, y_0) = u(x_0, y_0) + hu_x(x_0, y_0) + \frac{h^2}{2!} u_{xx}(\xi, y_0)$$

or $$\frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} = u_x(x_0, y_0) + \frac{h}{2!} u_{xx}(\xi, y_0)$$

$$\therefore u_x(x_0, y_0) = \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} - \frac{h}{2} u_{xx}(\xi, y_0) \quad (12.2)$$

where $x_0 \leq \xi \leq x_0 + h$

Eqn (12.2) implies that, if we replace $u_x(x_0, y_0)$ by

$$\frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h}$$

the truncation error $= -\dfrac{h}{2} u_{xx}(\xi, y_0)$; $x_0 \leq \xi \leq x_0 + h$

Hence, we have the finite difference formula for the first partial derivative at (x_0, y_0) as

$$u_x(x_0, y_0) = \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} \quad (12.3)$$

which is called a *forward difference approximation* to $u_x(x_0, y_0)$.

Similarly, by putting $x = x_0 - h$ in Eqn (12.1) or by changing h to $-h$ in Eqn (12.3), we get

$$u_x(x_0, y_0) = \frac{u(x_0, y_0) - u(x_0 - h, y_0)}{h} \quad (12.4)$$

which is known as *backward approximation* to $u_x(x_0, y_0)$

Now, to obtain an approximation for u_{xx}, we use both forward and backward differences.

Using forward difference for u_x, we get

$$u_{xx}(x_0, y_0) = \frac{u_x(x_0 + h, y_0) - u_x(x_0, y_0)}{h} \quad (12.5)$$

Since there is a bias in the forward direction, we use backward differences for u_x to avoid this effect, i.e.

$$u_{xx}(x_0, y_0) = \frac{u(x_0, y_0) - u(x_0 - h, y_0)}{h} \quad (12.6)$$

Changing x_0 to $x_0 + h$ in Eqn (12.6), we get

$$u_x(x_0 + h, y_0) = \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} \tag{12.7}$$

Substituting Eqns (12.6) and (12.7) in Eqn (12.5),

$$u_{xx}(x_0, y_0) = \frac{u(x_0 + h, y_0) - 2u(x_0, y_0) + u(x_0 - h, y_0)}{h^2} \tag{12.8}$$

We can notice a symmetry in Eqn (12.8).

Truncation error

By Taylor's series, we have

$$u(x, y_0) = u(x_0, y_0) + (x - x_0)u_x(x_0, y_0) = \frac{(x - x_0)^2}{2!} u_{xx}(x, y_0)$$

$$+ \frac{(x - x_0)^3}{3!} u_{xxx}(x_0, y_0) + \frac{(x - x_0)^4}{4!} u_{xxxx}(\xi, y_0) \tag{12.9}$$

where $x_0 \leq \xi \leq \dot{x}_0$. By putting $x = x_0 + h$, Eqn (12.9) reduces to

$$u(x_0 + h) = u(x_0, y_0) + hu_x(x_0, y_0) + \frac{h^2}{2} u_{xx}(x_0, y_0) + \frac{h^3}{6} u_{xxx}(x_0, y_0)$$

$$+ \frac{h^4}{24} u_{xxxx}(\xi_1, y_0) \tag{12.10}$$

where $x_0 \leq \xi_1 \leq x_0 + h$.

Again putting $x = x_0 - h$ in Eqn (12.9) it reduces to

$$u(x_0 - h, y_0) = u(x_0, y_0) - hu_x(x_0, y_0) + \frac{h^2}{2} u_{xx}(x_0, y_0)$$

$$- \frac{h^3}{6} u_{xxx}(x_0, y_0) + \frac{h^4}{24} u_{xxxx}(\xi_2, y_0) \tag{12.11}$$

where $x_0 - h \leq \xi_2 \leq x_0$. Adding Eqns (12.10) and (10.11), we get

$$u(x_0 + h, y_0) + u(x_0 - h, y_0) = 2u(x_0, y_0) = h^2 u_{xx}(x_0, y_0)$$

$$+ \frac{h^4}{24} [\, u_{xxxx}(\xi_1, y_0) + u_{xxxx}(\xi_2, y_0) \,]$$

$$\therefore \quad \frac{u(x_0 + h, y_0) - 2u(x_0, y_0) + u(x_0 - h, y_0)}{h^2}$$

$$= u_{xx}(x_0, y_0) + \frac{h^2}{24} [\, u_{xxxx}(\xi_1, y_0) + u_{xxxx}(\xi_2, y_0) \,]$$

$$= u_{xx}(x_0, y_0) + \frac{h^2}{12} u_{xxxx}(\xi, y_0)$$

where $x_0 - h \le \xi \le x_0 + h$

$$u_{xx}(x_0, y_0) = \frac{u(x_0 + h, y_0) - 2u(x_0, y_0) + u(x_0 - h, y_0)}{h^2}$$

$$- \frac{h^2}{12} u_{xxxx}(\xi, y_0) \tag{12.12}$$

Eqn (12.12) implies that if we replace $u_{xx}(x_0, y_0)$ by

$$\frac{u(x_0 + h, y_0) - 2u(x_0, y_0) + u(x_0 - h, y_0)}{h^2},$$

the truncation error is $-\frac{h^2}{12} u_{xxxx}(\xi, y_0)$; where $x_0 - h \le \xi \le x_0 + h$

Proceeding on the same lines, we have the following formulae for the derivatives in y-direction taking the step size as k.

i) Forward difference formula:

$$u_y(x_0, y_0) = \frac{u(x_0, y_0 + k) - u(x_0, y_0)}{k} \tag{12.13}$$

ii) Backward difference formula:

$$u_y(x_0, y_0) = \frac{u(x_0, y_0) - u(x_0, y_0 - k)}{k} \tag{12.14}$$

iii) $$u_{yy} = \frac{u(x_0, y_0 + k) - 2u(x_0, y_0) + u(x_0, y_0 - k)}{k^2} \tag{12.15}$$

The truncation error in above will be $-\frac{k^2}{12} u_{yyyy}(x_0, \eta)$,

where $y_0 - k \le \eta \le y_0 + k$

12.3 GEOMETRICAL REPRESENTATION OF PARTIAL DIFFERENCE QUOTIENTS

The $x - y$ plane is divided into a series of rectangles of sides $\Delta x = h$ and $\Delta y = k$ by equidistant lines drawn parallel to the axis of coordinates.

Points (x, y), $(x + h, y)$, $(x + 2h, y)$, $(x - h, y)$, $(x - 2h, y)$, . . . , $(x, y - k)$, $(x, y - 2k)$ are shown in Fig 12.1

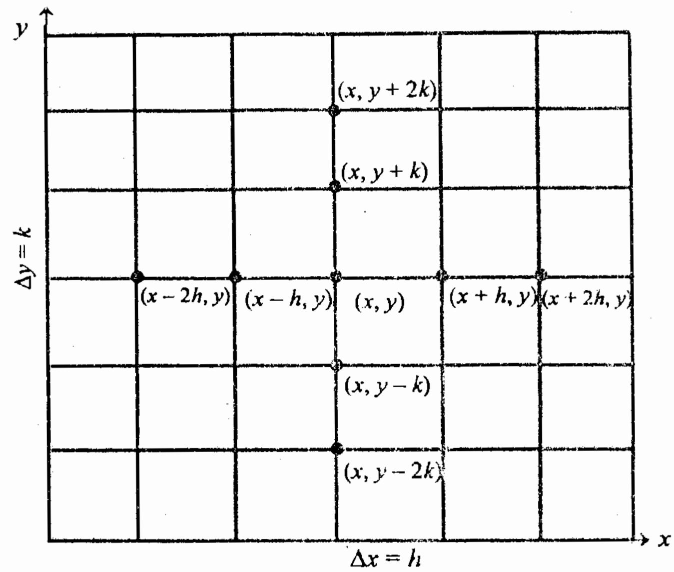


Fig. 12.1

We can interpret the above idea in a different notation by drawing two sets of parallel lines $x = ih$ and $y = jk$, $i, j = 0, 1, 2, \ldots$. The points of intersection of these family of lines are called *mesh points*, *lattice points* or *grid points*. Point (i, j) is called the *grid point* and is surrounded by the neighbouring points as shown in Fig 12.2.

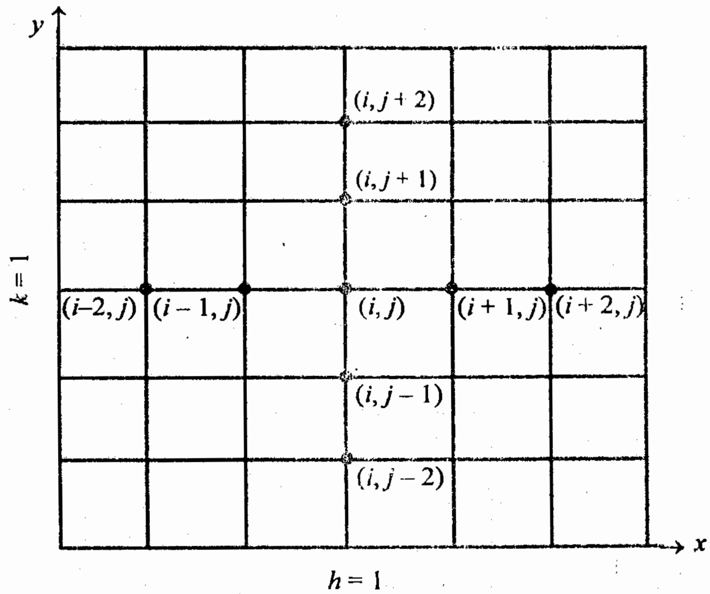


Fig. 12.2

If u is a function of two variables x and y, then the value of $u(x, y)$ at point (i, j) is denoted by $u_{i,j}$, since $u(x, y) = u(ih, jk)$
$= u_{i,j}$

With reference to the above figures, we can write

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} \quad \text{[forward difference]} \tag{12.16}$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} \quad \text{[backward difference]} \tag{12.17}$$

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} \quad \text{[forward difference]} \tag{12.18}$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} \quad \text{[backward difference]} \tag{12.19}$$

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \tag{12.20}$$

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} \tag{12.21}$$

We also have

$$u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \text{ and } u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

2.4 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

In this section, we will classify partial differential equations of the second order. The general linear partial differential equation of second order in two independent variables is of the form,

$$A\frac{\partial^2 u}{dx^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

$$\text{or } Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \tag{12.22}$$

where A, B, C, D, E and F are in general, functions of x and y.

Eqn (12.22) is said to be

(i) **Elliptic** at a point (x, y) in the plane if $B^2 - 4AC < 0$

(ii) **Parabolic** if $B^2 - 4AC = 0$

(iii) **Hyperbolic** if $B^2 - 4AC > 0$.

Note: It is possible for an equation to be of more than one type depending on the values of the coefficients.

For example, the equation $yu_{xx} + u_{yy} = 0$ is *elliptic* if $y > 0$, *parabolic* if $y < 0$.

But here, we are concerned with constant coefficients only.

Example 12.1 Classify the following equations

(i) $3u_{xx} + u_{xy} - 4u_{yy} + 3u_y = 0$

(ii) $x^2 u_{xx} + (a^2 - y^2) u_{yy} = 0;\ -\infty < x < \infty, -a < y < a$

(iii) $u_{xx} - 6u_{xy} + 9u_{yy} - 17u_y = 0$

Solution (i) Here, $A = 3, B = 1, C = -4$

$\therefore B^2 - 4AC = (1)^2 - 4(3)(-4) > 0$

$\therefore$ The given equation is hyperbolic.

(ii) Here, $A = x^2, B = 0, C = a^2 - y^2$

$\therefore B^2 - 4AC = -4x^2(a^2 - y^2) = 4x^2(a^2 - y^2)$

Now x^2 is always (+)ve for all $-\infty < x < \infty$

and $a^2 - y^2$ is (–)ve for all $-a < y < a$.

$\therefore B^2 - 4AC = 4(+\text{ ve})(-\text{ ve}) = (-)$ ve i.e. < 0

So, the given equation is elliptic.

(iii) Here, $A = 1, B = -6, C = 9$

$\therefore B^2 - 4AC = (-6)^2 - 4(1)(9) = 0$

So, the given equation is parabolic.

12.5 ELLIPTIC EQUATIONS

An important equation of the elliptic type is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \text{ i.e. } u_{xx} + u_{yy} = 0 \tag{12.23}$$

This equation is called *Laplace's equation.*

Replacing the derivatives by the corresponding difference expressions in Eqn (12.23), we get

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0$$

Taking a square mesh and putting $h = k$, we get from above

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}] \tag{12.24}$$

that is, the value of u at any interior mesh point is the arithmetic mean of its values at the four neighbouring mesh points to the left, right, below and above. This is called the *standard five point formula* (*SFPF*).

Instead of Eqn (12.24), we may also use the formula

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1}\right] \tag{12.25}$$

which shows that the value of $u_{i,j}$ is the arithmetic mean of its values at the four neighbouring diagonal mesh points. This is called the *diagonal five point formula* [*DFPF*].

The SFPF and DFPF are represented in Figs. 12.3 and 12.4 below.

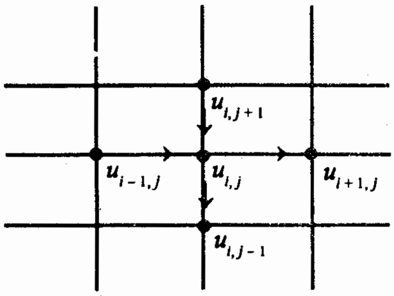


Fig. 12.3 *SFPF*

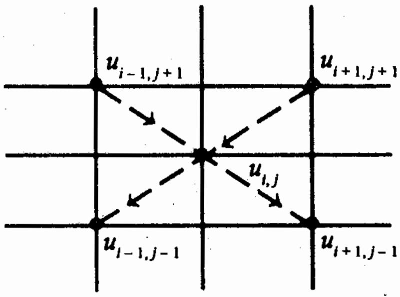


Fig. 12.4 *DFPF*

Note: The DFPF is valid since, we know that Laplace equation remains invariant when the coordinate axes are rotated through 45°. But the error in DFPF is four times the error in SFPF. Therefore, we prefer SFPF.

12.6 SOLUTION TO LAPLACE'S EQUATION BY LIEBMANN'S ITERATION PROCESS

Consider the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the given boundary conditions. For simplicity, we assume that function $u(x, y)$ is required over a rectangular region R with boundary C. Let R be divided into a network of small squares of side h. Let the values of $u(x, y)$ on boundary C be given by C_i and the interior mesh points and boundary points be as shown in Fig. 12.5

We know that the value of $u(x, y)$ satisfying the Laplacian equation can be replaced by either SFPF or DFPF. To start the iteration process, initially we find rough values at interior points and then improve it by iterative process, mostly using SFPF.

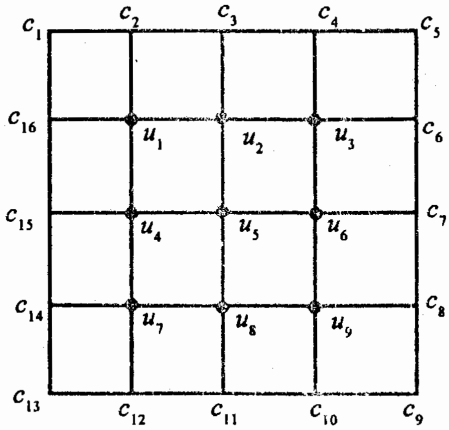


Fig. 12.5

We first find u_5, at the centre of the square by taking the average of four boundary values (SFPF).

$$\therefore\ u_5 = \frac{1}{4}\ (c_{15} + c_7 + c_3 + c_{11})$$

Next, we find the initial values at the centres of the four large inner squares using DFPF.

Thus, $$u_1 = \frac{1}{4}\ (u_5 + c_1 + c_3 + c_{15})$$

$$u_3 = \frac{1}{4}\ (u_5 + c_5 + c_3 + c_7)$$

$$u_7 = \frac{1}{4}\ (u_5 + c_{13} + c_{11} + c_{15})$$

$$u_9 = \frac{1}{4}\ (u_5 + c_9 + c_7 + c_{11})$$

The values at the remaining interior points are obtained by SFPF.

Thus, $$u_2 = \frac{1}{4}\ (u_1 + u_3 + c_3 + u_5)$$

$$u_4 = \frac{1}{4}\ (c_{15} + u_5 + u_1 + u_7)$$

$$u_6 = \frac{1}{4}(u_5 + c_7 + u_3 + u_9)$$

$$u_8 = \frac{1}{4}(u_7 + u_9 + u_5 + c_{11})$$

Now that we have got all the boundary values of u and rough values at every mesh (grid) point in the interior of the region R, we proceed with an iteration process to improve their accuracy. We start with u_1 and iterate it using the latest available values of the four adjacent points. Thus, we iterate all the mesh points systematically from left to right along successive rows. The iterative formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

Here, the superscript denotes iteration number. This is known as *Liebmann's iteration process.*

Example 12.2 Solve $u_{xx} + u_{yy} = 0$ in $0 \le x \le 4$, $0 \le y \le 4$, given that $u(0, y) = 0$; $u(4, y) = 81.2y$;

$u(x, 0) = \dfrac{x^2}{2}$ and $u(x, 4) = x^2$. Take $h = k = 1$ and obtain the result correct to one decimal.

Solution Let us divide the given region R, i.e. $0 \le x \le 4$, $0 \le y \le 4$ into 16 square meshes. The numerical values of the boundary, using the given analytical expression are calculated and exhibited in Fig. 12.6.

Let $u_1, u_2, u_3, \ldots, u_9$ be the values of u at the interior mesh points. Now the initial values of u's are calculated either by SFPF or DFPF as given below:

$$u_5^{(0)} = \frac{0 + 12 + 4 + 2}{4} = 4.5 \text{ (SFPF)}$$

$$u_1^{(0)} = \frac{0 + 4 + 0 + 4.5}{4} = 2.125 \text{ (DFPF)}$$

$$u_3^{(0)} = \frac{4.5 + 16 + 4 + 12}{4} = 9.125 \text{ (DFPF)}$$

$$u_7^{(0)} = \frac{0 + 2 + 0 + 4.5}{4} = 1.625 \text{ (DFPF)}$$

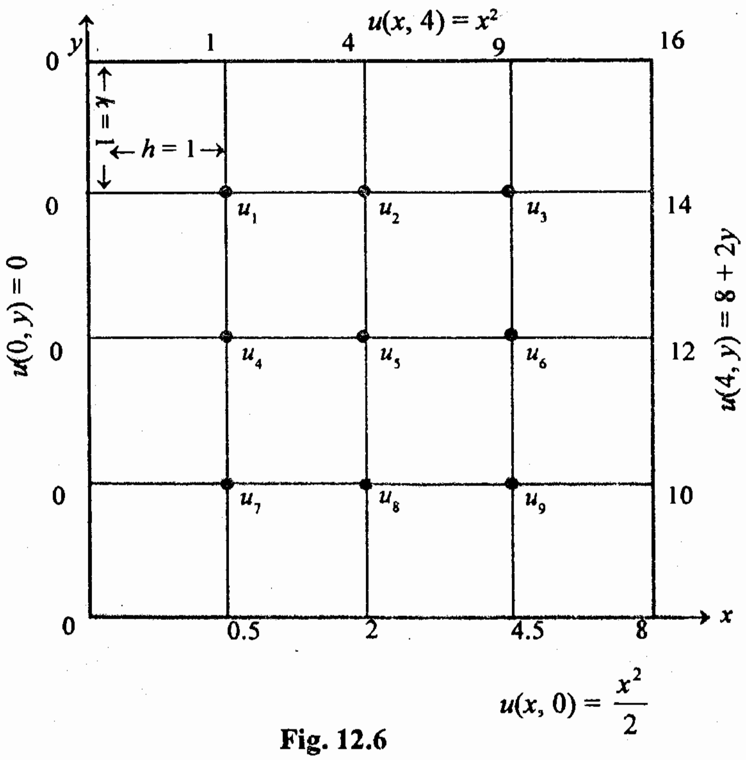


Fig. 12.6

$$u_9^{(0)} = \frac{4.5+8+2+12}{4} = 6.625 \text{ (DFPF)}$$

$$u_2^{(0)} = \frac{4+4.5+2.125+9.125}{4} = 4.9375 \text{ (SFPF)}$$

$$u_4^{(0)} = \frac{0+4.5+2.125+1.625}{4} = 2.0625 \text{ (SFPF)}$$

$$u_6^{(0)} = \frac{4.5+12+9.125+6.625}{4} = 8.0625 \text{ (SFPF)}$$

$$u_8^{(0)} = \frac{1.625+6.625+4.5+2}{4} = 3.6875 \text{ (SFPF)}$$

Now we use Liebmann's iteration formula, i.e.

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

to improve above results. When we use this formula at points $u_1, u_2, \ldots, u_9$, we get the following equations for iteration:

$$u_1^{(n+1)} = \frac{1}{4}[0 + u_2^{(n)} + u_4^{(n)} + 1]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_3^{(n)} + u_5^{(n)} + 4]$$

$$u_3^{(n+1)} = \frac{1}{4}[u_2^{(n+1)} + 14 + u_6^{(n)} + 9]$$

$$u_4^{(n+1)} = \frac{1}{4}[0 + u_5^{(n)} + u_7^{(n)} + u_1^{(n+1)}]$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_6^{(n)} + u_8^{(n)} + u_2^{(n+1)}]$$

$$u_6^{(n+1)} = \frac{1}{4}[u_5^{(n+1)} + 12 + u_9^{(n)} + u_3^{(n+1)}]$$

$$u_7^{(n+1)} = \frac{1}{4}[0 + u_8^{(n)} + 0.5 + u_4^{(n+1)}]$$

$$u_8^{(n+1)} = \frac{1}{4}[u_7^{(n+1)} + u_9^{(n)} + 2 + u_5^{(n+1)}]$$

$$u_9^{(n+1)} = \frac{1}{4}[u_8^{(n+1)} + 10 + 4.5 + u_6^{(n+1)}]$$

First iteration $(n = 0)$

$$u_1^{(1)} = \frac{1}{4}[0 + u_2^{(0)} + u_4^{(0)} + 1]$$

$$= \frac{1}{4}[0 + 4.9375 + 2.0625 + 1] = 2$$

$$u_2^{(1)} = \frac{1}{4}[u_1^{(1)} + u_3^{(0)} + u_5^{(0)} + 4]$$

$$= \frac{1}{4}[2 + 9.125 + 4.5 + 4] = 4.90625$$

$$u_3^{(1)} = \frac{1}{4}[u_2^{(1)} + 14 + u_6^{(0)} + 9]$$

$$= \frac{1}{4}[4.90625 + 14 + 8.0625 + 9] = 8.9921875$$

$$u_4^{(1)} = \frac{1}{4}[0 + u_5^{(0)} + u_7^{(0)} + u_1^{(1)}]$$

$$= \frac{1}{4}[0 + 4.5 + 1.625 + 2] = 2.03125$$

$$u_5^{(1)} = \frac{1}{4}[u_4^{(1)} + u_6^{(0)} + u_8^{(0)} + u_2^{(1)}]$$

$$= \frac{1}{4}[2.03125 + 8.0625 + 3.6875 + 4.90625] = 4.671875$$

$$u_6^{(1)} = \frac{1}{4}[u_5^{(1)} + 12 + u_9^{(0)} + u_3^{(1)}]$$

$$= \frac{1}{4}[4.671875 + 12 + 6.625 + 8.9921875] = 8.0722656$$

$$u_7^{(1)} = \frac{1}{4}[0 + u_8^{(0)} + 0.5 + u_4^{(1)}]$$

$$= \frac{1}{4}[0 + 3.6875 + 0.5 + 2.03125] = 1.5546875$$

$$u_8^{(1)} = \frac{1}{4}[u_7^{(1)} + u_9^{(0)} + 2 + u_5^{(1)}]$$

$$= \frac{1}{4}[1.5546875 + 6.625 + 2 + 4.671875] = 3.7128906$$

$$u_9^{(1)} = \frac{1}{4}[u_8^{(1)} + 10 + 4.5 + u_6^{(1)}]$$

$$= \frac{1}{4}[3.7128906 + 10 + 4.5 + 8.0722656] = 6.5712891$$

Second iteration ($n = 1$)

$$u_1^{(2)} = \frac{1}{4}[0 + u_2^{(1)} + u_4^{(1)} + 1]$$

$$= \frac{1}{4}\ [0 + 4.90625 + 2.03125 + 1] = 1.984375$$

$$u_2^{(2)} = \frac{1}{4}\ [u_1^{(2)} + u_3^{(1)} + u_5^{(1)} + 4]$$

$$= \frac{1}{4}\ [1.984375 + 8.9921875 + 4.671875 + 4] = 4.9121094$$

$$u_3^{(2)} = \frac{1}{4}\ [u_2^{(2)} + 14 + u_6^{(1)} + 9]$$

$$= \frac{1}{4}\ [23 + 4.9121094 + 8.0722656] = 8.9960937$$

$$u_4^{(2)} = \frac{1}{4}\ [0 + u_5^{(1)} + u_7^{(1)} + u_1^{(2)}]$$

$$= \frac{1}{4}\ [0 + 4.671875 + 1.5546875 + 1.984375] = 2.0527344$$

$$u_5^{(2)} = \frac{1}{4}\ [u_4^{(2)} + u_6^{(1)} + u_8^{(1)} + u_2^{(2)}]$$

$$= \frac{1}{4}\ [2.0527344 + 8.0722656 + 3.7128906 + 4.9121094]$$

$$= 4.6875$$

$$u_6^{(2)} = \frac{1}{4}\ [u_5^{(2)} + 12 + u_9^{(1)} + u_3^{(2)}]$$

$$= \frac{1}{4}\ [4.6875 + 12 + 6.5712891 + 8.9960937] = 8.063720$$

$$u_7^{(2)} = \frac{1}{4}\ [0 + u_8^{(1)} + 0.5 + u_4^{(2)}]$$

$$= \frac{1}{4}\ [0.5 + 3.7128906 + 2.0527344] = 1.5664063$$

$$u_8^{(2)} = \frac{1}{4}\ [u_7^{(2)} + u_9^{(1)} + 2 + u_5^{(2)}]$$

$$= \frac{1}{4}\ [1.5664063 + 6.5712891 + 2 + 4.6875] = 3.7062988$$

$$u_9^{(2)} = \frac{1}{4}[u_8^{(2)} + 10 + 4.5 + u_6^{(2)}]$$

$$= \frac{1}{4}[3.7062988 + 14.5 + 8.0637207] = 6.5675049$$

Third iteration $(n = 2)$

$$u_1^{(3)} = \frac{1}{4}[0 + u_2^{(2)} + u_4^{(2)} + 1]$$

$$= \frac{1}{4}[0 + 4.9121094 + 2.0527344 + 1] = 1.991211$$

$$u_2^{(3)} = \frac{1}{4}[u_1^{(3)} + u_3^{(2)} + u_5^{(2)} + 4]$$

$$= \frac{1}{4}[1.991211 + 8.9960931 + 4.6815 + 4] = 4.9181012$$

$$u_3^{(3)} = \frac{1}{4}[u_2^{(3)} + 14 + u_6^{(2)} + 9]$$

$$= \frac{1}{4}[4.9187012 + 14 + 8.0637207 + 9] = 8.9956055$$

$$u_4^{(3)} = \frac{1}{4}[0 + u_5^{(2)} + u_7^{(2)} + u_1^{(3)}]$$

$$= \frac{1}{4}[0 + 4.6875 + 1.5664063 + 1.991211] = 2.0612793$$

$$u_5^{(3)} = \frac{1}{4}[u_4^{(3)} + u_6^{(2)} + u_8^{(2)} + u_2^{(3)}]$$

$$= \frac{1}{4}[2.0612793 + 8.0637207 + 3.7062988 + 4.9187012]$$

$$= 4.6875$$

$$u_6^{(3)} = \frac{1}{4}[u_5^{(3)} + 12 + u_9^{(2)} + u_3^{(3)}]$$

$$= \frac{1}{4}[4.6875 + 12 + 6.5675049 + 8.9956055] = 8.0626526$$

$$u_7^{(3)} = \frac{1}{4}[0 + u_8^{(2)} + 0.5 + u_4^{(3)}]$$

$$= \frac{1}{4}[0 + 3.7062988 + 0.5 + 2.0612793] = 1.5668945$$

$$u_8^{(3)} = \frac{1}{4}[u_7^{(3)} + u_9^{(2)} + 2 + u_5^{(3)}]$$

$$= \frac{1}{4}[1.5668945 + 6.5675049 + 2 + 4.6875] = 3.7054749$$

$$u_9^{(3)} = \frac{1}{4}[u_8^{(3)} + 10 + 4.5 + u_6^{(3)}]$$

$$= \frac{1}{4}[3.7054749 + 10 + 4.5 + 8.0626526] = 6.56707319$$

$\therefore$ $u_1 \approx 1.99$; $u_2 \approx 4.91$; $u_3 \approx 8.99$; $u_4 \approx 2.06$
$u_5 \approx 4.69$; $u_6 \approx 8.06$; $u_7 \approx 1.57$; $u_8 \approx 3.71$;
and $u_9 \approx 6.57$

Example 12.3 Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown.

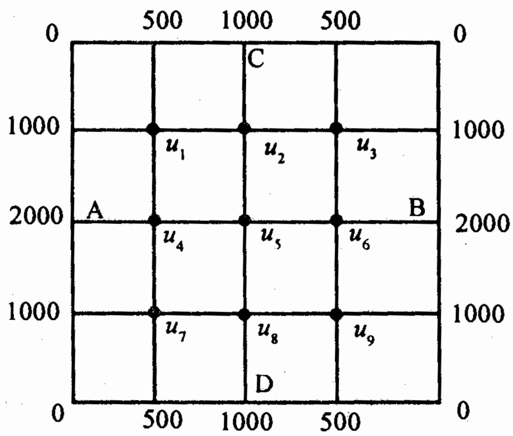


Fig. 12.7

Solution Let $u_1, u_2, \ldots, u_9$ be the values of x at the interior mesh points. From Fig 19.7, we can see that the boundary values of u are symmetrical about AB.

$$\therefore \quad u_7 = u_1; \qquad u_8 = u_2; \qquad u_9 = u_3$$

Also, the values of u being symmetrical about CD are

$$u_3 = u_1; \qquad u_6 = u_4; \; u_9 = u_7$$

$\therefore$ It is sufficient to find the values u_1, u_2, u_4 and u_5. The initial values of these are given below

$$u_5^{(0)} = \frac{1}{4}[2000 + 2000 + 1000 + 1000] = 1500 \text{ (SFPF)}$$

$$u_1^{(0)} = \frac{1}{4}[0 + 1500 + 1000 + 2000] = 1125 \text{ (DFPF)}$$

$$u_2^{(0)} = \frac{1}{4}[1125 + 1125 + 1000 + 1500] = 1187.5 \text{ (SFPF)}$$

$$u_4^{(0)} = \frac{1}{4}[2000 + 1500 + 1125 + 1125] = 1437.5 \text{ (SFPF)}$$

Now we carry out the iterations using the following formulae by SFPF:

$$u_1^{(n+1)} = \frac{1}{4}[1000 + u_2^{(n)} + 500 + u_4^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_1^{(n+1)} + 1000 + u_5^{(n)}]$$

$$u_4^{(n+1)} = \frac{1}{4}[2000 + u_5^{(n)} + u_1^{(n+1)} + u_1^{(n+1)}]$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_4^{(n+1)} + u_2^{(n+1)} + u_2^{(n+1)}]$$

First iteration ($n = 0$)

$$u_1^{(1)} = \frac{1}{4}[1000 + 1187.5 + 500 + 1437.5] = 1031.25$$

$$u_2^{(1)} = \frac{1}{4}[1031.25 + 1031.25 + 1000 + 1500] = 1140.625$$

$$u_4^{(1)} = \frac{1}{4}[2000 + 1500 + 1031.25 + 1031.25] = 1390.625$$

$$u_5^{(1)} = \frac{1}{4}[1390.625 + 1390.625 + 1140.625 + 1140.625] = 1265.625$$

Second iteration $(n = 1)$

$$u_1^{(2)} = \frac{1}{4}[1000 + 1140.625 + 500 + 1390.625] = 1007.8125$$

$$u_2^{(2)} = \frac{1}{4}[1007.8125 + 1007.8125 + 1000 + 1265.625] = 1070.3125$$

$$u_4^{(2)} = \frac{1}{4}[2000 + 1265.625 + 1007.8125 + 1007.8125] = 1320.3125$$

$$u_5^{(2)} = \frac{1}{4}[1320.3125 + 1320.3125 + 1070.3125 + 1070.3125] = 1195.3125$$

Third iteration $(n = 2)$

$$u_1^{(3)} = \frac{1}{4}[1000 + 1070.3125 + 500 + 1320.3125] = 972.65625$$

$$u_2^{(3)} = \frac{1}{4}[972.65625 + 972.65625 + 1000 + 1195.3125] = 1035.1563$$

$$u_4^{(3)} = \frac{1}{4}[2000 + 1195.3125 + 972.65625 + 972.65625] = 1285.1563$$

$$u_5^{(3)} = \frac{1}{4}[1285.1563 + 1285.1563 + 1035.1563 + 1035.1563] = 1160.1563$$

Fourth iteration $(n = 3)$

$$u_1^{(4)} = \frac{1}{4}[1000 + 1035.1563 + 500 + 1285.1563] = 955.07815$$

$$u_2^{(4)} = \frac{1}{4}[955.07815 + 955.07815 + 1000 + 1160.1563] = 1017.5782$$

$$u_4^{(4)} = \frac{1}{4}[2000 + 1160.1563 + 955.07815 + 955.07815] = 1267.5782$$

$$u_5^{(4)} = \frac{1}{4}[1267.5782 + 1267.5782 + 1017.5782 + 1017.5782] = 1142.5782$$

Fifth iteration ($n = 4$)

$$u_1^{(5)} = \frac{1}{4}[1000 + 1017.5782 + 500 + 1267.5782] = 946.2891$$

$$u_2^{(5)} = \frac{1}{4}[946.2891 + 946.2891 + 1000 + 1142.5782] = 1008.7891$$

$$u_4^{(5)} = \frac{1}{4}[2000 + 1142.5782 + 946.2891 + 946.2891] = 1258.7891$$

$$u_5^{(5)} = \frac{1}{4}[1258.7891 + 1258.7891 + 1008.7891 + 1008.7891]$$

$$= 1133.7891$$

Sixth iteration ($n = 5$)

$$u_1^{(6)} = \frac{1}{4}[1000 + 1008.7891 + 500 + 1258.7891] = 941.89455$$

$$u_2^{(6)} = \frac{1}{4}[941.89455 + 941.89455 + 1000 + 1133.7891] = 1004.3946$$

$$u_4^{(6)} = \frac{1}{4}[2000 + 1133.7891 + 941.89455 + 941.89455] = 1254.3946$$

$$u_5^{(6)} = \frac{1}{4}[1254.3946 + 1254.3946 + 1004.3946 + 1004.3946]$$

$$= 1129.3946$$

Seventh iteration ($n = 6$)

$$u_1^{(7)} = \frac{1}{4}[1000 + 1004.3946 + 500 + 1254.3946] = 939.6973$$

$$u_2^{(7)} = \frac{1}{4}[939.6973 + 939.6973 + 1000 + 1129.3946] = 1002.1973$$

$$u_4^{(7)} = \frac{1}{4}[2000 + 1129.3946 + 939.6973 + 939.6973] = 1252.1973$$

$$u_5^{(7)} = \frac{1}{4}[1252.1973 + 1252.1973 + 1002.1973 + 1002.1973]$$

$$= 1127.1973$$

Eighth iteration ($n = 7$)

$$u_1^{(8)} = \frac{1}{4}[1000 + 1002.1973 + 500 + 1252.1973] = 938.59865$$

$$u_2^{(8)} = \frac{1}{4}[938.59865 + 938.59865 + 1000 + 1127.1973] = 1001.0987$$

$$u_4^{(8)} = \frac{1}{4}[2000 + 1127.1973 + 938.59865 + 938.59865] = 1251.0987$$

$$u_5^{(8)} = \frac{1}{4}[1251.0987 + 1251.0987 + 1001.0987 + 1001.0987]$$

$$= 1126.0987$$

Ninth iteration ($n = 8$)

$$u_1^{(9)} = \frac{1}{4}[1000 + 1001.0987 + 500 + 1251.0987] = 938.04935.$$

$$u_2^{(9)} = \frac{1}{4}[938.04935 + 938.04935 + 1000 + 1126.0987] = 1000.5494$$

$$u_4^{(9)} = \frac{1}{4}[2000 + 1126.0987 + 938.04935 + 938.04935] = 1250.5494$$

$$u_5^{(9)} = \frac{1}{4}[1250.5494 + 1250.5494 + 1000.5494 + 1000.5494]$$

$$= 1125.5494$$

From the eighth and ninth iterations, we see that there is a negligible difference between the values.

$\therefore\ u_1 = 939,\ u_2 = 1001,\qquad u_4 = 1251,\qquad \text{and}\qquad u_5 = 1126$

$\Rightarrow u_3 = 939,\ u_6 = 1251,\qquad u_7 = 939,\qquad u_8 = 1001,\qquad u_9 = 939.$

Example 12.4 Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in the figure. Iterate until the maximum difference between two successive values at any point is less than 0.001.

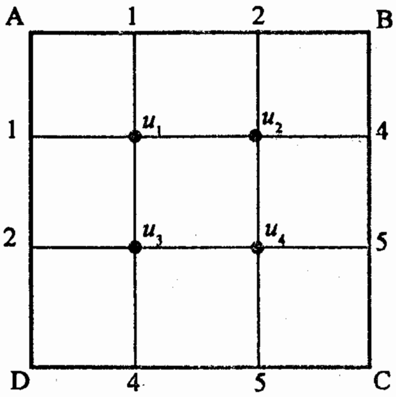


Fig. 12.8

Solution Figure 12.8 is symmetrical about line AC. $\therefore u_2 = u_3$. Let us assume $u_2 = 0$.

$$\therefore \quad u_1^{(0)} = \frac{1}{4}[1 + 0 + 0 + 1] = 0.5 \text{ (SFPF)}$$

$$u_4^{(0)} = \frac{1}{4}[0 + 0 + 5 + 5] = 2.5 \text{ (SFPF)}$$

$$u_2^{(0)} = \frac{1}{4}[0.5 + 2 + 4 + 2.5] = 2.25 \text{ (SFPF)}$$

Now using the Liebmann's iteration formula, we get the equation at u_1, u_2, u_4 as

$$u_1^{(n+1)} = \frac{1}{4}[1 + 1 + u_2^{(n)} + u_2^{(n)}] = \frac{1}{2}[1 + u_2^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + 2 + 4 + u_4^{(n)}] = \frac{1}{4}[6 + u_1^{(n+1)} + u_4^{(n)}]$$

$$u_4^{(n+1)} = \frac{1}{4}[u_2^{(n+1)} + u_2^{(n+1)} + 5 + 5] = \frac{1}{2}[u_2^{(n+1)} + 5]$$

***First iteration* ($n = 0$)**

$$u_1^{(1)} = \frac{1}{2}[1 + 2.25] = 1.625$$

$$u_2^{(1)} = \frac{1}{4}[6 + 1.625 + 2.5] = 2.53125$$

$$u_4^{(1)} = \frac{1}{2}[2.53125 + 5] = 3.765625$$

Second iteration ($n = 1$)

$$u_1^{(2)} = \frac{1}{2}[1 + 2.53125] = 1.765625$$

$$u_2^{(2)} = \frac{1}{4}[6 + 1.765625 + 3.765625] = 2.8828125$$

$$u_4^{(2)} = \frac{1}{2}[2.8828125 + 5] = 3.9414063$$

Third iteration ($n = 2$)

$$u_1^{(3)} = \frac{1}{2}[1 + 2.8828125] = 1.9414063$$

$$u_2^{(3)} = \frac{1}{4}[6 + 1.9414063 + 3.9414063] = 2.9707031$$

$$u_4^{(3)} = \frac{1}{2}[2.9707031 + 5] = 3.9853516$$

Fourth iteration ($n = 3$)

$$u_1^{(4)} = \frac{1}{2}[1 + 2.9707031] = 1.9853516$$

$$u_2^{(4)} = \frac{1}{4}[6 + 1.9853516 + 3.9853516] = 2.9926758$$

$$u_4^{(4)} = \frac{1}{2}[2.9926758 + 5] = 3.9963379$$

Fifth iteration ($n = 4$)

$$u_1^{(5)} = \frac{1}{2}[1 + 2.9926578] = 1.9963289$$

$$u_2^{(5)} = \frac{1}{4}[6 + 1.9963289 + 3.9963379] = 2.9981667$$

$$u_4^{(5)} = \frac{1}{4}[2.9981667 + 5] = 3.9990834$$

Sixth iteration ($n = 5$)

$$u_1^{(6)} = \frac{1}{2}[1 + 2.9981667] = 1.9990834$$

$$u_2^{(6)} = \frac{1}{4}[6 + 1.9990834 + 3.9990834] = 2.9995417$$

$$u_4^{(6)} = \frac{1}{2}[2.9995417 + 5] = 3.9997709$$

$\therefore \quad u_1 \approx 1.999, \qquad u_2 \approx 2.999, \qquad u_4 \approx 3.999$

12.7 POISSON'S EQUATION – ITS SOLUTION

The partial differential equation

$$\nabla^2 u = f(x, y) \text{ or } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \text{ or } u_{xx} + u_{yy} = f(x, y) \qquad (12.26)$$

where $f(x, y)$ is a given function of x and y is called the Poisson's equation. It is of elliptic type.

To solve the Poisson equation numerically, the derivatives in Eqn (12.26) are replaced by difference expressions at the points $x = ih$, $y = jk$ (here, $h = k$). Then we get

$$\frac{1}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] + \frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = f(ih, jh)$$

$$\text{or} \qquad u_{i-1,j} + u_{i+1,j} + u_{i,j-1}, u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) \qquad (12.27)$$

Applying the above formula at each mesh point, we get similar equations in the pivotal values i, j. These equations can be solved by iteration techniques.

Note: The error involved in Eqn (12.26) by difference method is of the order of $0(h^2)$.

Example 12.5 Solve the equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides $x = 0, y = 0, x = 3, y = 3$ with $u = 0$ on the boundary and mesh length = 1.

Solution

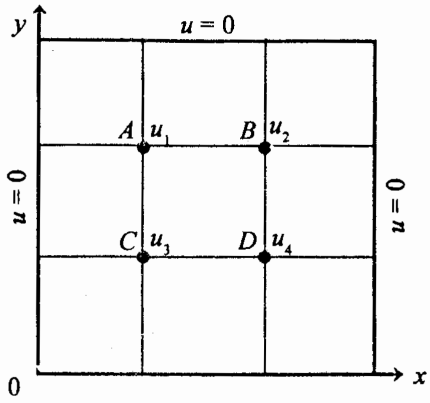


Fig. 12.9

The given differential equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10(x^2 + y^2 + 10) \qquad \text{(i)}$$

Let u_1, u_2, u_3, u_4 be the values of u at the four mesh points A, B, C and D as shown in Fig 12.9. Replacing LHS of Eqn (1) by finite difference expressions and putting $x = ih$, $y = jh$ on the RHS of it, we get

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \qquad \text{(ii)}$$

At A, i.e. for u_1, by putting $i = 1, j = 2$ in Eqn (ii), we get

$$0 + u_2 + u_3 + 0 + -4u_1 = -10\,(1 + 4 + 10)$$

$$\text{or } u_1 = \frac{1}{4}(u_2 + u_3 + 150) \qquad \text{(iii)}$$

At B, i.e. for u_2, by putting $i = 2, j = 2$ in Eqn (ii), we get

$$u_2 = \frac{1}{4}\,(u_1 + u_4 + 180) \qquad \text{(iv)}$$

At C, i.e. for u_3, by putting $i = 1, j = 1$ in Eqn (ii), we get

$$u_3 = \frac{1}{4}\,(u_1 + u_4 + 120) \qquad \text{(v)}$$

At D, i.e. for u_4, by putting $i = 2, j = 1$ in Eqn (ii), we get

$$u_2 = \frac{1}{4}\,(u_2 + u_3 + 150) \qquad \text{(iv)}$$

From Eqns (iii) and (vi) we can see that $u_4 = u_1$. So it is enough if we find u_1, u_2, and u_3.

Moreover, with $u_4 = u_1$, Eqns (iii)–(v) reduce to

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150)$$

$$u_2 = \frac{1}{2}(u_1 + 90)$$

$$u_3 = \frac{1}{2}(u_1 + 60)$$

Now let us solve these equations by Gauss–Seidal iteration method.

First iteration: We start the iteration by putting $u_2 = 0$, $u_3 = 0$

$$u_1^{(1)} = \frac{150}{4} = 37.5$$

$$u_2^{(1)} = \frac{1}{2}(37.5 + 90) = 63.75$$

$$u_3^{(1)} = \frac{1}{2}(37.5 + 60) = 48.75$$

Second iteration:

$$u^{(2)} = \frac{1}{4}(63.75 + 48.75 + 150) = 65.625$$

$$u_2^{(2)} = \frac{1}{2}(65.625 + 90) = 77.8125$$

$$u_3^{(2)} = \frac{1}{2}(65.625 + 60) = 62.8125$$

Third iteration:

$$u_1^{(3)} = \frac{1}{4}(77.8125 + 62.8125 + 150) = 72.65625$$

$$u_2^{(3)} = \frac{1}{2}(72.65625 + 90) = 81.328125$$

$$u_3^{(3)} = \frac{1}{2}(72.65625 + 60) = 66.32815$$

Fourth iteration:

$$u_1^{(4)} = \frac{1}{4}(81.328125 + 66.328125 + 150) = 74.414063$$

$$u_2^{(4)} = \frac{1}{2}(74.414063 + 90) = 82.207031$$

$$u_3^{(4)} = \frac{1}{2}(74.414063 + 60) = 67.207031$$

Fifth iteration:

$$u_1^{(5)} = \frac{1}{4}(82.207031 + 67.207031 + 150) = 74.853516$$

$$u_2^{(5)} = \frac{1}{2}(74.853516 + 90) = 82.426758$$

$$u_3^{(5)} = \frac{1}{2}(74.853516 + 60) = 67.426758$$

Sixth iteration:

$$u_1^{(6)} = \frac{1}{4}(82.426758 + 67.426758 + 150) = 74.963379$$

$$u_2^{(6)} = \frac{1}{2}(74.963379 + 90) = 82.481689$$

$$u_3^{(6)} = \frac{1}{2}(74.963379 + 60) = 67.481689$$

Since these values are the same as those of fifth iteration, we have, $u_1 = 75$, $u_2 = 82.5$ and $u_3 = 67.5$, $\therefore\ u_4 = 75$.

EXERCISE 12.1

1. Classify the following partial differential equations.
   (i) $u_{xx} - 2u_{xy} + u_{yy} + 3u_y - 4u_x = 3x - 2y$
   (ii) $(x + 1)\, u_{xx} - 2(x + 2)u_{xy} + (x + 3)u_{yy} = \cos(x - 2y)$
   (iii) $u_{xxx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x + y)$
2. Solve $u_{xx} - u_{yy} = 0$ over the square mesh of side four units satisfying the following boundary conditions:

(i) $u(0, y) = 0$ for $0 \le y \le 4$ (ii) $u(4, y) = 12 + y$ for $0 \le y \le 4$
(iii) $u(x, 0) = 3x$ for $0 \le x \le 4$ (iv) $u(x, 4) = x^2$ for $0 \le x \le 4$

3. Solve for the following square mesh with boundary conditions as shown in Fig. 12.10. Iterate until the maximum difference between two successive values at any grid point is less than 0.005.

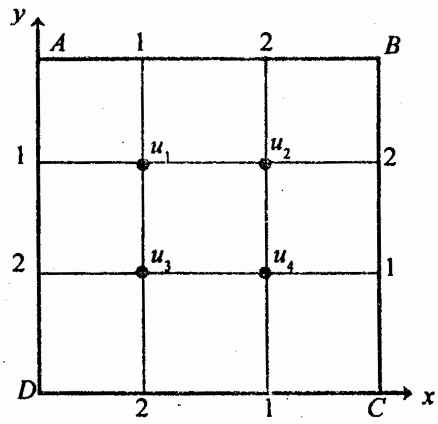


Fig. 12.10

4. Find the values of $u(x, y)$ satisfying the Laplace equation $\nabla^2 u = 0$, at the pivotal points of a square region, with boundary values as shown in (i) Fig.12.11 and (ii) Fig.12.12.

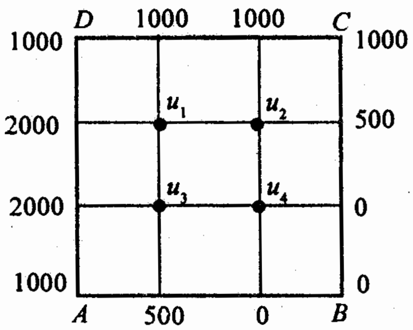


Fig. 12.11

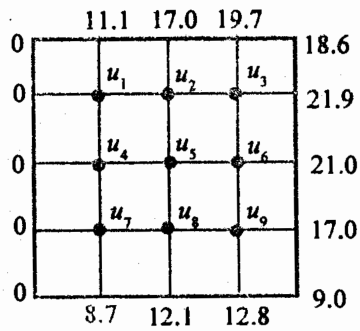


Fig. 12.12

5. Solve $u_{xx} + u_{yy} = 0$ for the following square meshes with boundary conditions as exhibited in Figures (i) 12.13 (ii)12.14 (iii) 12.15 (iv) 12.16 and (v) 12.17.

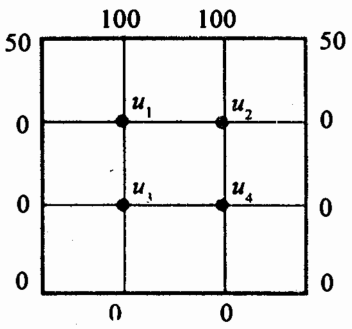


Fig. 12.13

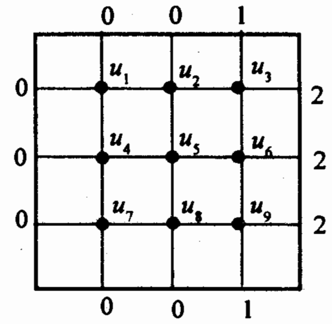


Fig. 12.14

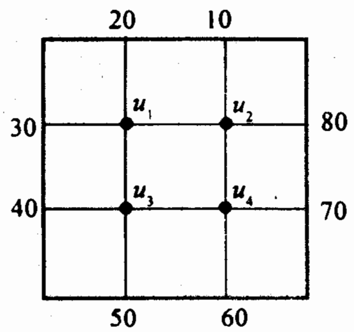


Fig. 12.15

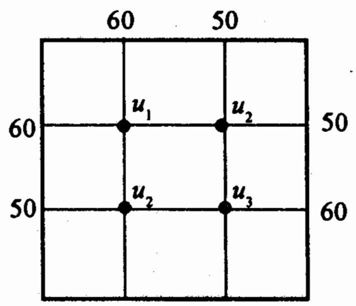


Fig. 12.16

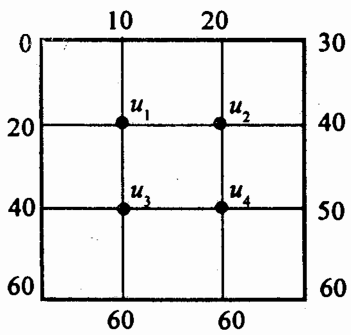


Fig. 12.17

6. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ for square mesh given $u = 0$ on the four boundaries dividing the square into 16 sub-squares of length one unit.

ANSWERS

1. (i) Parabolic (ii) Hyperbolic

   (iii) Elliptic in the region outside the ellipse $\frac{x^2}{4} + \frac{y^2}{4} = 1$;

   Parabolic on the ellipse; hyperbolic inside the ellipse
2. 2.37, 5.59, 9.87, 2.88, 6.13, 9.88, 3.01, 6.16, 9.51
3. 1.333, 1.667, 1.667. 1.333
4. (i) 1208.3, 791.7, 1041.7, 458.4
   (ii) 7.9, 13.7, 17.9, 6.6, 11.9, 16.3, 6.6, 11.2, 14.3
5. (i) 37.5, 37.5, 12.5, 12.5
   (ii) 0.1875, 0.5000, 1.1875, 0.2500, 0.6250, 1.2500
   (iii) 34.986, 44.993, 44.993, 54.996
   (iv) 56.601, 52.051, 56.025
   (v) 26.65, 33.33, 43.32, 46.66
6. –3, –2, –3, –2, –2, –2, –3, –2, –3

12.8 PARABOLIC EQUATIONS

The one dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \text{ where } \alpha^2 = \frac{k}{c\rho} \tag{12.28}$$

(c is the specific heat of the material, ρ is the density and k is the thermal conductivity) is a well-known example of a parabolic equation. The solution to it is a function of x and t i.e. $u(x, t)$. It is defined for values of x from $x = 0$ to $x = l$, and for values of time t from $t = 0$ to $t = \infty$. The solution is not defined in a closed form (as in the case of elliptic equations) but propagates in an open-ended region from initial values satisfying the prescribed boundary conditions (see Fig. 12.18).

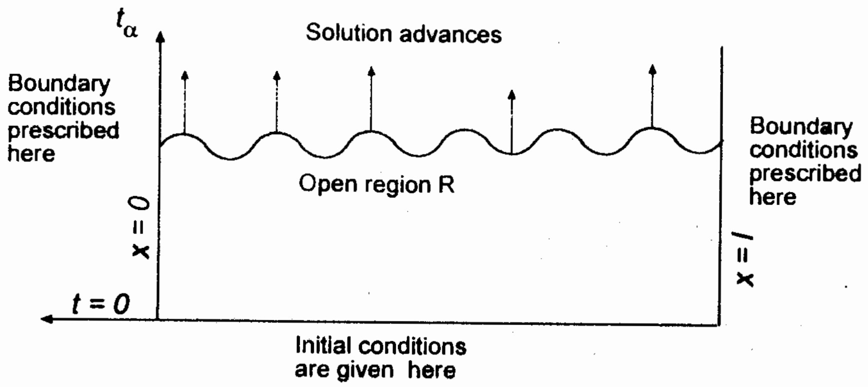


Fig. 12.18

12.8.1 Bender–Schmidt method

Solution to one dimensional heat equations can be obtained using Bender–Schmidt method. Here, consider the one dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; \alpha^2 = \frac{k}{c\rho} \tag{12.29}$$

This can be wirtten as

$$u_{xx} = au_t, \text{ where } a = \frac{1}{\alpha^2} \tag{12.30}$$

Now our aim is to solve Eqn (12.30) subject to the boundary conditions

$$u(0, t) = T_0 \tag{12.31}$$

$$u(l, t) = T_l \tag{12.32}$$

and the initial conditions

$$u(x, 0) = f(x) \tag{12.33}$$

by finite differences method. We select a spacing h for the variable x and a spacing k for the time variable t. We know that

$$u_{xx} = \frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

and $$u_t = \frac{1}{k}[u_{i,j+1} - u_{i,j}]$$

Substituting the above in Eqn (12.30) it becomes

$$\frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] = a\frac{1}{k}[u_{i,j+1} - u_{i,j}]$$

or $\quad u_{i,j+1} - u_{i,j} = \lambda\,[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$

where $\quad \lambda = \dfrac{k}{h^2}a.$

or $\quad u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j})$ (12.34)

The boundary conditions (12.31) and (12.32) can be put in difference form as

$$u_{0,j} = T_0 \qquad (12.35a)$$

and $\quad u_{n,j} = T_l$ (12.35b)

where $j = 1, 2, \ldots$ [here, $nh = l$] and the initial condition (12.33) as

$$u_{i,0} = f(ih),\ i = 1, 2, \ldots \qquad (12.36)$$

Eqn (12.34) gives the value of u at $x = ih$ at time t_{j+k} in terms of values of u at $x = (i-1)h$, ih and $(i+1)h$ at a time t_j.

$\because u(x, 0) = f(x)$, u is known at $t = 0$

Therefore, the recurrence relation (12.34) allows the evaluation of u at each pivotal point x_i at any t_j.

If h, k are chosen such that the coefficient of $u_{i,j}$ vanishes,

i.e. $1 - 2\lambda = 0$ or $\lambda = \dfrac{1}{2}$, then

Eqn (12.34) becomes

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}] \qquad (12.37)$$

and $\quad k = \dfrac{a}{2}h^2$ (12.38)

Eqn (12.37) implies that the value of u at $x = x_i$ at time t_{j+1} is equal to the average of the values of u at the surrounding points x_{i-1} and x_{i+1} at the previous time t_j. Eqn (12.37) is called *Bender–Schmidt recurrence equation.*

Example 12.6 Solve

$$\frac{\partial u}{\partial t} = \frac{1}{2}\cdot\frac{\partial^2 u}{\partial x^2}$$

with the conditions $u(0, t) = 0$, $u(4, t) = 0$, $u(x, 0) = x(4-x)$ taking $h = 1$ and employing Bender–Schmidt recurrence equation. Continue the solution through ten time steps.

Solution The general equation is $u_{xx} = au_t$

Here, $a = 2$, $h = 1$ $\quad \therefore \quad \lambda = \dfrac{k}{h^2} \qquad a = \dfrac{k}{2}$

Therefore, $\lambda = \dfrac{1}{2}$ and k should be equal to 1. Now using Bender–Schmidt recurrence relation, the values of u_{ij} are tabulated below (Fig. 12.19)

direction of x →

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0
6	0	0.375	0.5	0.375	0
7	0	0.25	0.375	0.25	0
8	0	0.1875	0.25	0.1875	0
9	0	0.125	0.1875	0.125	0
10	0	0.094	0.125	0.094	0

direction of t →

Fig. 12.19

Explanation: Range for x is $0 \le x \le 4$

$x = ih \quad \Rightarrow \quad x_i = ih = i \; (\because h = 1)$

$t = jk \quad \Rightarrow \quad t_i = jk = j \; (\because k = 1)$

Given $u(x, 0) = x(4 - x)$ or $u(i, 0) = i(4 - i)$

Now for $0 \le x \le 4$, i.e. $0 \le i \le 4$, we have $u(i, 0) = 0, 3, 4, 3, 0$. These are filled in the first row.

Given $u(0, t) = 0 \; \forall \, t$, i.e. $u(0, j) = 0 \; \forall \, j$. Hence the entries in the first column are zero.

Also, $u(4, t) = 0 \; \forall \, t$, i.e. $u(4, j) = 0 \; \forall \, j$. $\therefore$ The entries in the last column are zero.

The Bender–Schmidt's recurrence relation is

$$u_{i,j+1} = \frac{1}{2}\,[u_{i+1,j} + u_{i-1,j}] \qquad \text{(i)}$$

Putting $j = 0$ in Eqn (i), we get

$$u_{i,1} = \frac{1}{2}[u_{i+1,0} + u_{i-1,0}] \qquad \text{(ii)}$$

Putting $i = 1$ in Eqn (ii), we get

$$u_{1,1} = \frac{1}{2}[u_{2,0} + u_{0,0}] = \frac{1}{2}[4 + 0] = 2$$

Putting $i = 2$ in Eqn (ii), we get

$$u_{2,1} = \frac{1}{2}[u_{3,0} + u_{1,0}] = \frac{1}{2}[3 + 3] = 3$$

Putting $i = 3$ in Eqn (ii), we get

$$u_{3,1} = \frac{1}{2}[u_{4,0} + u_{2,0}] = \frac{1}{2}[0 + 4] = 2$$

Thus the second row is filled. Similarly, putting $j = 1, 2, 3, 4, 5, 6, 7, 8, 9$, the other rows are filled.

Example 12.7 Find the solution to $u_t = u_{xx}$ subject to $u(x, 0) = \sin \pi x$, $0 \le x \le 1$, $u(0, t) = u(1, t) = 0$ using Schmidt method.

Solution Given equation is $ut = u_{xx}$

$\therefore \quad a = 1$

Here, h and k are not given.

For λ to be $\frac{1}{2}$, we take $h = 0.2$ and $k = 0.02$ so that $k = \frac{1}{2} a^2h^2$. Range for x is $0 < x < 1$.

$$x = ih \Rightarrow x_i = ih = 0.2i$$
$$y = jk \Rightarrow y_j = jk = 0.02k$$

Given $u(x, 0) = \sin \pi x$ or $u(0.2i, 0) = \sin(0.2\pi i)$

Now, for $0 \le x \le 1$, i.e. $0 \le 0.2i \le 1$, we have $u(0.2i, 0) = 0, 0.5878, 0.9511, 0.9511, 0.5878, 0$

These are filled in the first row.

$$u(0, t) = 0 \ \forall t, \text{ i.e. } u(0, 0.02j) = 0 \ \forall j$$
$$u(1, t) = 0 \ \forall t, \text{ i.e. } u(1, 0.02j) = 0 \ \forall j$$

$\therefore$ The entries in the first and last columns are zero's. The remaining entries of the second row, third row etc., are calculated using Bender–Schmidt recurrence relation:

$$u_{i,j+1} = \frac{1}{2}\,[u_{i+1,j} + u_{i-1,j}]$$

and are exhibited in the following table [Fig. 12.20].

→ *direction of x*

↓ *direction of t*

j \ i	0	0.2	0.4	0.6	0.8	1.0
0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	0	0.4756	0.7695	0.7695	0.4756	0
0.04	0	0.3848	0.6225	0.6225	0.3848	0
0.06	0	0.3113	0.5036	0.5036	0.3113	0
0.08	0	0.2518	0.4074	0.4074	0.2518	0
0.1	0	0.2037	0.3296	0.3296	0.2037	0

Fig. 12.20

12.8.2 Crank–Nicholson method

In this section, we will derive *Crank–Nicholson difference method* to solve parabolic equations.

$$\text{Let } u_{xx} = au_t \tag{12.39}$$

be the partial differential equation to be solved subject to the conditions

$$u(0, t) = T_0 \tag{12.40}$$

$$u(l, t) = T_l \tag{12.41}$$

and

$$u(x, 0) = f(x) \tag{12.42}$$

We know that at point $u_{i,j}$, the finite difference approximation for u_{xx} is

$$u_{xx} = \frac{1}{h^2}\{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\} \tag{12.43}$$

At point $u_{i,j+1}$ the finite difference approximation for u_{xx} is

$$u_{xx} = \frac{1}{h^2}\{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\} \tag{12.44}$$

Average of Eqns (12.43) and (12.44) is

$$u_{xx} = \frac{1}{2h^2}\{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\} \tag{12.45}$$

For u_t, the forward difference approximation is

$$u_t = \frac{1}{k}\{u_{i,j+1} - u_{i,j}\} \tag{12.46}$$

Substituting Eqns (12.45) and (12.46) in Eqn (12.39), we get (after simplification)

$$\frac{\lambda}{2}\{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\} = \{u_{i,j+1} - u_{i,j}\}$$

(where $\lambda = \frac{k}{h^2}a$)

or
$$\frac{\lambda}{2}u_{i+1,j+1} - (\lambda+1)u_{i,j+1} + \frac{\lambda}{2}u_{i-1,j+1}$$
$$= -\frac{\lambda}{2}u_{i+1,j} + (\lambda-1)u_{i,j} - \frac{\lambda}{2}u_{i-1,j}$$

or
$$\lambda\{u_{i+1,j+1} + u_{i-1,j+1}\} - 2(\lambda+1)u_{i,j+1}$$
$$= 2(\lambda-1)u_{i,j} - \lambda\{u_{i+1,j} + u_{i-1,j}\} \tag{12.47}$$

Eqn (12.47) is called *Crank–Nicholson difference scheme or method.*
Note:(i) Choosing λ is very important. Very often a good choice of λ is $\lambda - 1$. In such a case, the Crank–Nicholson scheme becomes

$$u_{i,j+1} = \frac{1}{4}\{u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\} \tag{12.48}$$

Subject to $k = ah^2$

(ii) The Crank–Nicholson's formula is convergent for all values of λ.

Example 12.8 Using Crank–Nicholson's method, solve
$u_{xx} = 16u_t$, $0 < x < 1$, $t > 0$, given $u(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = 50t$.

Compute u for two steps in t direction taking $h = \frac{1}{4}$

Solution Here, $a = 16$, $h = \frac{1}{4}$

$$\therefore \qquad k = ah^2 = 16\left(\frac{1}{16}\right) = 1.$$

The Crank–Nicholson scheme is given by

$$u_{i,j+1} = \frac{1}{4}\,[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}] \qquad \text{(i)}$$

x increasing →

j \ i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	u_1	u_2	u_3	50
2	0	u_3	u_4	u_5	100

Fig. 12.21

Applying Eqn (i) at the mesh points u_1, u_2, u_3, we get

$$u_1 = \frac{1}{4}(0 + 0 + 0 + u_2) = \frac{1}{4}u_2 \qquad \text{(ii)}$$

$$u_2 = \frac{1}{4}(0 + 0 + u_1 + u_3) = \frac{1}{4}(u_1 + u_3) \qquad \text{(iii)}$$

$$u_3 = \frac{1}{4}(0 + 0 + u_2 + 50) = \frac{1}{4}(u_2 + 50) \qquad \text{(iv)}$$

Substituting (iv) and (ii) in (iii), we get

$$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}(u_2 + 50)\right]$$

or $\qquad 16u_2 = 2u_2 + 50 \;\therefore u_2 = 3.5714$

$u_1 = 0.89285$; $u_3 = 13.39285$

Applying Eqn (i) again at the mesh points u_4, u_5, u_6, we get

$$u_4 = \frac{1}{4}u_5 \qquad \text{(v)}$$

$$u_5 = \frac{1}{4}(u_4 + u_6) \qquad \text{(vi)}$$

$$u_6 = \frac{1}{4}(u_3 + 100) \qquad \text{(vii)}$$

On Solving, we get $u_4 = 1.7857$, $u_5 = 7.1429$ and $u_6 = 26.7857$

EXERCISE 12.2

1. Given $u_t = 25u_{xx}$; $u(0, t) = 0 = u(10, t)$; $u(x, 0) = \frac{1}{25}x(10-x)$.
   Choosing $h = 1$ and k suitably, find u_{ij} for $0 \le i \le 9$, $1 \le j \le 4$.
2. Solve the equation $u_{xx} = u_t$ with the conditions $u(0, t) = 0$, $u(x, 0) = x(1-x)$; $u(1, t) = 0$. Assume that the region between $x = 0$ and $x = 1$ is divided into 10 equal parts of $h = 0.1$. Tabulate u for $t = k, 2k, 3k$, choosing an appropriate value of k.
3. Find the values of $u(x, t)$ satisfying the parabolic equation $u_t = 4u_{xx}$ and the boundary conditions $u(0, t) = 0 = u(8, t)$ and

   $u(x, 0) = 4x - \frac{1}{2}x^2$ at the points $x = i$; $i = 0, 1, 2, \ldots, 7$ and $t = \frac{1}{8}j$;

   $j = 0, 1, 2, \ldots, 5$.
4. Solve $u_t = 5u_{xx}$ with $u(0, t) = 0$; $u(5, t) = 60$ and
   $u(x, 0) = 20x$ for $0 < x \le 3$
   $= 60$ for $3 < x \le 5$
   for five time steps having $h = 1$ by Schmidt method.
5. Compute u for one time step by Crank–Nicholson method if $ut = u_{xx}$; $0 < x < 5$, $t > 0$; $u(x, 0) = 20$; $u(0, t) = 0$ and $u(5, t) = 100$
6. Solve $u_t = u_{xx}$ subject to the conditions $u(x,0) = 0$; $u(0,t) = 0$ and

   $u(1,t)=1$. Compute u for $t = \frac{1}{8}$ in two steps, using Crank–Nicholson

   scheme.
7. Obtain the numerical solution to solve $u_t = u_{xx}$, $0 \le x \le 1$, $t \ge 0$, under the conditons that $u(0,t) = u(1,t) = 0$ and

$$u(x,0) = \begin{cases} 2x \; for \; 0 \le x \le \frac{1}{2} \\ 2(1-x) \; for \; \frac{1}{2} \le x \le \frac{1}{2} \end{cases}$$

ANSWERS

1.

$j \backslash i$	1	2	3	4	5	6	7	8	9
1	0.32	0.6	0.8	0.92	0.96	0.92	0.8	0.6	0.32
2	0.3	0.56	0.76	0.88	0.92	0.88	0.76	0.64	0.3
3	0.28	0.53	0.72	0.84	0.88	0.84	0.76	0.53	0.32
4	0.265	0.5	0.685	0.8	0.84	0.82	0.685	0.54	0.265

2.

j \ i	0	1	2	3	4	5	6	7	8	9	10
0	0	0.09	0.16	0.21	0.24	0.25	0.24	0.21	0.16	0.09	0
1	0	0.08	0.15	0.20	0.23	0.24	0.23	0.20	0.15	0.08	0
2	0	0.075	0.14	0.19	0.22	0.23	0.22	0.19	0.14	0.075	0
3	0	0.07	0.1325	0.18	0.21	0.22	0.21	0.18	0.1325	0.07	0

3.

j \ i	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

4.

j \ i	0	1	2	3	4	5
0	0	20	40	60	60	60
0.1	0	20	40	50	60	60
0.2	0	20	35	50	55	60
0.3	0	17.5	35	45	55	60
0.4	0	17.5	31.25	45	52.5	60
0.5	0	15.625	31.25	41.875	52.5	60

5.

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	9.80	20.19	30.72	59.92	100

6.

j \ i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
$\frac{1}{16}$	0	0.00116	0.004464	0.01674	$\frac{1}{16}$
$\frac{1}{8}$	0	0.005899	0.019132	0.052771	$\frac{1}{8}$

7.

j \ i	0	0.2	0.4	0.6	0.8	1.0	0.8	0.6	0.4	0.2	0
0.1	0	0.1936	0.3689	0.5400	0.6461	0.6921	0.6461	0.5400	0.3689	0.1936	0
0.02	2	0.1989	0.3956	0.5834	0.7381	0.7691	0.7381	0.5834	0.3956	0.1989	0

12.9 HYPERBOLIC EQUATIONS

Given below is a wave equation of one dimension,

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \text{ or } a^2 u_{xx} - u_{tt} = 0 \tag{12.49}$$

This is a hyperbolic equation.
We know that $B^2 - 4AC = 0 - 4(a^2)(-1) = 4a^2 > 0$

Solution by Method of Finite Differences

Eqn (12.49) is subject to the conditions

$$u(0, t) = 0 \tag{12.50}$$

$$u(l, t) = 0 \tag{12.51}$$

$$\text{and } u(x, 0) = f(x) \tag{12.52}$$

$$u_t(x, 0) = 0 \tag{12.53}$$

Substituting

$$u_{xx} = \frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

$$u_{tt} = \frac{1}{k^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1}]$$

in Eqn (12.49) where h and k are selected spacings for the variables x and t, we get

$$\frac{a}{h^2} (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - \frac{1}{k^2} (u_{i,j+1} - 2u_{i,j} - u_{i,j-1}) = 0$$

or $$\lambda^2 a^2 (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - (u_{i,j+1} - 2u_{i,j} - u_{i,j-1}) = 0$$

where $$\lambda = \frac{k}{h}$$

or $$u_{i,j+1} = 2(1 - \lambda^2 a^2) u_{i,j} + \lambda^2 a^2 (u_{i+1,j} + u_{i-1,j}) - u_{i,j-1} \tag{12.54}$$

The boundary conditions (12.50) and (12.51) can be put in the difference form as

$$u_{0,j} = 0 = u_{n,j}; j = 1, 2, 3 \ldots \tag{12.55}$$

(Here it means $nh = l$)

The initial condition (12.52) as

$$u_{i,0} = f(ih), i = 1, 2, \ldots \tag{12.56}$$

and (12.53) as

$$\frac{1}{k}\ [u_{i,\,j+1} - u_{i,\,0}] = 0 \text{ when } t = 0, \text{ i.e. } j = 0$$

$$\therefore u_{i,\,1} - u_{i,\,0} = 0 \tag{12.57}$$

$$\text{i.e. } u_{i,\,1} = u_{i,\,0} = f(i,\,h) \tag{12.58}$$

using Eqn (12.56).

Now Eqns (12.56) and (12.57) give the values of u on the first two rows $j = 0$ and $j = 1$. Putting $j = 1$ in Eqn (12.54), we get

$$u_{i,\,2} = 2(1 - \lambda^2 a^2)u_{i,\,1} + \lambda^2 a^2(u_{i+1,\,1} + u_{i-1,\,1}) - u_{i,\,0} \tag{12.59}$$

R.H.S of Eqn (12.59) involves the values of u on the first two rows j=0 and j = 1. These are known from the initial conditions (12.56) and (12.58). Hence $u_{i,\,2}$ is found explicitly.

Knowing $u_{i,\,2}$, we can calculate $u_{i,\,3}$ by putting j = 2 in (12.54) and so on. Thus, Eqn (12.54) is an explicit scheme for the solution to the given equation.

Note:

(i) If $k < h$ solution (12.54) is convergent.

(ii) The coefficient of $u_{i,j}$ in Eqn(12.50) will be zero if $\lambda^2 = \dfrac{1}{a^2}$, i.e.

$$k = \frac{h}{a}.$$

Then Eqn (12.54) takes the form

$$u_{i,\,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

That is, the value of u at $x = x_i$ at a time $t = t_j + k$

= Value of u at $x = x_{i-1}$ at previous time $t = t_j$
+ Value of u at $x = x_{i+1}$ at previous time $t = t_j$
– Value of u at $x = x_i$ at time $t = t_j - k$.

Example 12.9 Solve $u_{tt} = 4u_{xx}$ with the boundary conditions $u(0, t) = 0 = u(4, t)$, $u_t(x, 0)$ and $u(x, 0) = x(4 - x)$.

Solution Given equation is $u_{tt} = 4u_{xx}$

Here, $a^2 = 4$, i.e. $a = 2$. Taking $h = 1$, we get $k = \dfrac{h}{a} = \dfrac{1}{2} = 0.5$.

From the initial conditions,

$u(0, t) = 0 \Rightarrow u = 0$ along entire line $x = 0$
$u(4, t) = 0 \Rightarrow u = 0$ along entire line $x = 4$

In difference form, these are

$u_{0,j} = 0$ and $u_{4,j} = 0$ for all j

Now $u(x, 0) = x(4 - x) \Rightarrow u(0, 0) = 0, u(1, 0) = 3, u(2, 0) = 4,$

$u(3, 0) = 3, \; u(4, 0) = 0$

In difference notation,

$u_{i,0} = u(i, 0) = i(4 - i)$ for different i

Putting $i = 0, 1, 2, 3, 4$ we get

$u_{0,0} = 0, u_{1,0} = 3, u_{2,0} = 4, u_{3,0} = 3, u_{4,0} = 0$

Now the condition $u_t(x, 0) = 0$

$$\Rightarrow \frac{1}{k}[u_{i,j+1} - u_{i,j}] = 0 \text{ when } j = 0$$

$$\Rightarrow u_{i,1} = u_{i,0} \; \forall i$$

i.e. u on the first two rows are equal.

Now consider the recurrence relation

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

If we put $j = 1$, we get

$$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0}$$

Putting $i = 1, 2, 3\ldots$ successively, we get

$$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0} = 4 + 0 - 3 = 1$$
$$u_{2,2} = u_{3,1} + u_{1,1} - u_{2,0} = 3 + 3 - 4 = 2$$
$$u_{3,2} = u_{4,1} + u_{2,1} - u_{3,0} = 0 + 4 - 3 = 1$$

that is, the third row is filled in. In a similar way we can fill in the remaining rows as shown in the following table

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	3	4	3	0
2	0	1	2	1	0
3	0	–1	–2	–1	0
4	0	–3	–4	–3	0

Fig 12.22

EXERCISE 12.3

1. Evaluate the pivotal values for the following equation taking $h = 1$ and upto one half of the period of vibration.
   $16u_{xx} = u_{tt}$, given that $u(0, t) = u(5, t) = 0$
   $u(x, 0) = x^2(x - 5)$ and $u_t(x, 0) = 0$

2. Solve the hyperbolic partial differential equation (vibration of strings) for one half period of oscillation taking $h = 1$.
   $u_{tt} = 25u_{xx}$, $u(0, t) = u(5, t) = 0$; $u_t(x, 0) = 0$

$$u(x, 0) = \begin{cases} 2x \ for \ 0 \le x \le 2.5 \\ 10 - 2x \ for \ 2.5 \le x \le 5 \end{cases}$$

3. Solve $u_{tt} = u_{xx}$ upto $t = 0.5$ with spacing of 0.1 given that $u(0, t) = 0 = u(1, t)$; $u_t(x, 0) = 0$ and $u(x, 0) = 10 + x(1 - x)$

ANSWERS

1.

j \ i	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	4	12	18	16	0
2	0	8	10	10	2	0
3	0	6	6	–6	–6	0
4	0	–2	–10	–10	–8	0
5	0	–16	–18	–12	–4	0

2.

j \ i	0	1	2	3	4	5
0	0	2	4	4	2	0
0.2	0	2	4	4	2	0
0.4	0	2	2	2	2	0
0.6	0	0	0	0	0	0
0.8	0	–2	–2	–2	–2	0
1.0	0	–2	–4	–4	–2	0

3.

j \ i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0	10.09	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	0
0.1	0	10.09	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	0
0.2	0	0.07	10.14	10.19	10.22	10.23	10.22	10.19	10.17	0.7	0
0.3	0	0.05	0.1	10.15	10.18	10.19	10.18	10.15	0.1	0.05	0
0.4	0	0.03	0.06	0.09	10.12	10.13	10.12	0.09	0.06	0.03	0
0.5	0	0.01	0.02	0.03	0.04	10.05	0.04	0.03	0.02	0.01	0

12.10 SOLUTION TO PARTIAL DIFFERENTIAL EQUATIONS BY RELAXATION METHOD

We have already seen the method of relaxation in Chapter 4. Now we shall use the same technique in solving an elliptic equation. Let us consider a Laplace equation

$$u_{xx} + u_{yy} = 0 \tag{12.60}$$

This is to be solved in a square region of mesh size h, given boundary values.

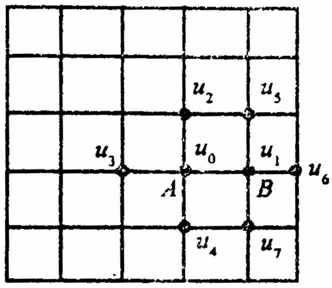


Fig 12.23

Let u_0 be the value of u at A and u_1, u_2, u_3, u_4 be the values of u at four adjacent points as shown in Fig 12.23.

Since $\dfrac{\partial^2 u_0}{\partial x^2} \approx \dfrac{1}{h^2}(u_1 + u_3 - 2u_0)$

and $\dfrac{\partial^2 u_0}{\partial y^2} \approx \dfrac{1}{h^2}(u_2 + u_4 - 2u_0)$

from $\nabla^2 u_0 = 0$, we have

$$u_1 + u_2 + u_3 + u_4 - 4u_0 \approx 0$$

$\Rightarrow$ There will be a residual at u_0.

Let r_0 be the residual at u_0(i.e. at A)

$$\therefore\ r_0 = u_1 + u_2 + u_3 + u_4 - 4u_0 \tag{12.61}$$

Similarly, the residuals at B and other points be

$$r_1 = u_0 + u_4 + u_5 + u_6 - 4\,u_1$$

and so on. The aim of the relaxation method is to reduce all the residuals to zero or to make them as small as possible. For this, we try to liquidate the values of u at the internal points.

It should be noted that when the values of the function u is changed at a mesh point, the values of the residuals at the adjacent interior points will also be changed.

Let us consider the effect on r_0, r_1, r_2 etc. When an increment of 1 is given to u_0, we can see from Eqn (12.61) that r_0 is changed by –4; i.e. r_1 increases by 1. Hence, when the value of function u is changed at a mesh point by 1 unit, the residual at that point is changed by –4 units, while the residuals at the adjacent interior points get increased each by the same amount, 1 unit. This is represented by the relaxation operator R_0 in Fig.12.24.

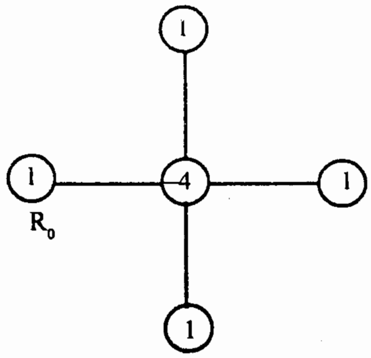


Fig. 12.24

If we apply R_0 at a point next to the boundary, one or more of the end points of R_0, will be chopped off because there are no residuals at the boundary.

At each step, we take mesh points with the numerically largest residual and apply to it, a suitable multiple of R_0 so as to liquidate the residual as far as possible.

Example 12.10 Solve the Laplace equation $u_{xx} + u_{yy} = 0$ inside a square region bounded by the lines $x = 0, x = 4, y = 0, y = 4$ given that $u = x^2y^2$ at the boundary, using relaxation technique.

Solution Choose $h = k = 1$. The boundary values using $u = x^2y^2$ are exhibited in Fig.12.25.

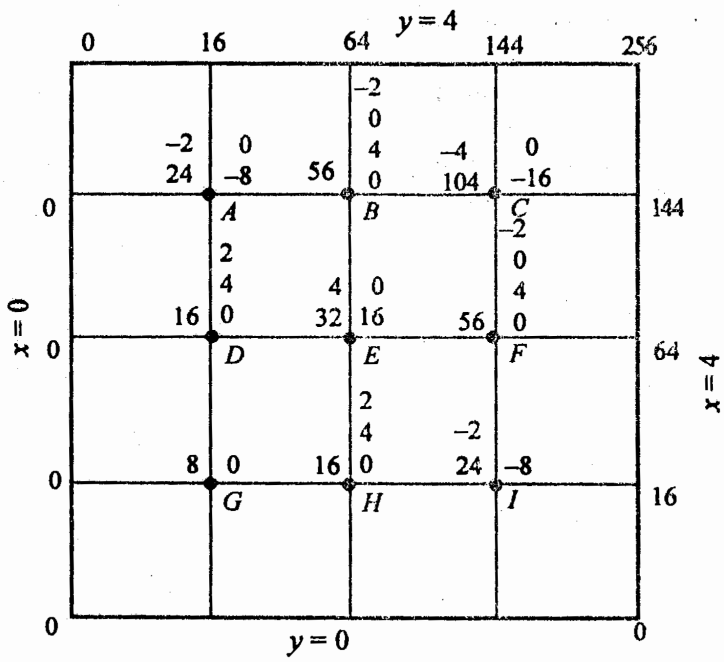


Fig. 12.25

Using either SFPF or DFPF the initial values of u at the nine mesh points can be calculated. They are 24, 56, 104, 16, 32, 56, 8, 16 and 24, and are shown in the figure on the left side of each point.

Residual at $A = 0 + 56 + 16 + 16 - 4(24) = -8$

Residual at $B = 64 + 32 + 24 + 104 - 4(56) = 0$

Similarly, the residuals at C, D, E, F, G, H and I are -16, 0, 16, 0, 0, 0 and -8, respectively.

Now the numerically largest residual is at (2, 2), i.e. at $E = 16$. To liquidate it, we apply $4R_0$ at E. Then the residual at this point will be increased by $4X - 4 = -16$ and so it will be zero. The value of u at this node is increased by 4. The residuals at the neighbouring nodes will increase by 4.

Next, the maximum residual is -16 at (3, 3), i.e. at C. To liquidate this, apply $-4R_0$ at (3, 3) so that the residual at this point becomes 0 and the value of u at this node increases by -4. The residuals at the neighbouring nodes also increase by -4.

Next, the numerically maximum residual is -8 at (1, 3), i.e. at A. To liquidate it, apply $-2R_0$ at (1, 3) so that the residual at this point is increased by $(-2)(-4) = 8$, and so it will be zero. The value of u at this node is increased by -1 and the residuals at the neighbouring points will change by -2.

Finally, the largest residual is -8 at $(3, 1)$, i.e. at I. Apply $-2R_0$, at $(3, 1)$ so that the residual becomes zero and the value of u at this node is increased by -2. The residuals at the neighbourig points change by -2. In Fig. 12.25, at each node, the right hand side top most value is the residual at that point, while the value of u at each grid point is the sum of the values on the left side of that point.

We stop the relaxation process here since the numerically largest residual is only 2. The final answer is shown in Fig. 12.26.

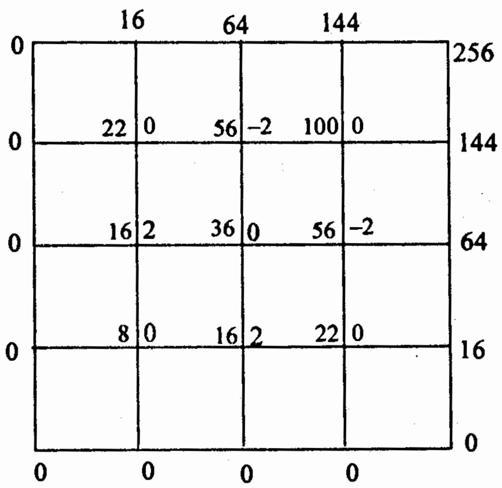


Fig. 12.26

EXERCISE 12.4

1. Given that $u(x, y)$ satisfies the equation $\nabla^2 u = 0$ and the boundary conditions are $u(0, y) = 0$, $u(4, y) = 8 + 2y$, $u(x, 0) = \frac{1}{2}x^2$ and $u(x, 4) = x^2$, find the values of $u(i, j)$; $i = 1, 2, 3$; $j = 1, 2, 3$ by relaxation method.

2. Solve by relaxation method, the Laplace equation $\nabla^2 u = 0$ in the following square region starting with the values $u_1 = u_2 = u_3 = u_4 = 1$

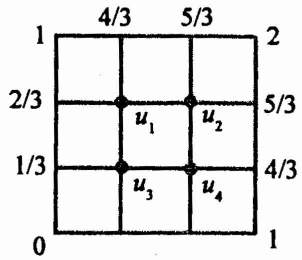


Fig 12.27

ANSWERS

1) $u_1 = 1.9$, $u_2 = 4.9$, $u_3 = 9.1$, $u_4 = 2.1$, $u_5 = 4.7$ $u_6 = 8.4$, $u_7 = 1.6$, $u_8 = 3.9$, $u_9 = 6.7$
2) $u_1 = 1$, $u_2 = 1.3$, $u_3 = 0.7$, $u_4 = 1$

8. The finite difference formula equivalent to Laplaces' equation $u_{xx} + u_{yy} = 0$ with $\Delta x = \Delta y = h$ is given by u_{ij} = ...

9. If the roots of the equation $27x^3 + 42x^2 - 28x - 8 = 0$ are in G.P, then one of the roots equals ....

10. The number of real roots of $x^4 + x^3 + x^2 + x + 1 = 0$ is ....

Choose the correct answer

11. To find the smallest positive root of $x^3 - x - 1 = 0$, by the method of simple iteration, the equation should be rewritten as

    (a) $x = x^3 - 1$; (b) $x = (x + 1)^{1/3}$;

    (c) $x = \dfrac{1}{x^2 - 1}$; (d) $x = \dfrac{x+1}{x^2}$.

12. If an approximate value of the root of the equation $x^x = 1000$ is 4.5, a better approximation of the root got by Newton-Raphson method is

    (a) 4.44 (b) 4.56
    (c) 5.17 (d) None of the above

13. If the roots $x^3 - px + q = 0$ are in A.P., then

    (a) $p = 0$ (b) $q = 0$
    (c) $p = q$ (d) $d + q = 0$

14. In solving equation $u_t = \alpha^2 u_{xx}$ by Crank-Nicholson method, to simplify method we take $\dfrac{(\Delta x)^2}{\alpha^2 K}$ as

    (a) $\dfrac{1}{2}$ (b) 2
    (c) 1 (d) 0

15. The area bounded by the curve passing through the points (0, 1), (1, 7), (2, 23), (3, 55) and (4, 109), the x-axis and the ordinates $x = 0$ and $x = 4$, as computed by Simpson's 1/3 rule is nearly equal to

    (a) 115 (b) 130 (c) 135 (d) 145

Short Questions

16. Define a residual.

17. What is the equation whose roots are $\pm 1, \pm i, 1 \pm i$?

18. Define μ and δ.

19. Under the what conditions are the group and block relaxations not necessary for solving a set of simultaneous algebraic equations?

20. Express $(\Delta^2 - 3\Delta + 2)$ in terms of operator E.

Part - B ($5 \times 12 = 60$ marks)

21. a) (i) Fit the least square straight lines y = a + bx to the following data:

x :	-5	-3	-1	0	1	2	3
y :	0.4	-0.1	-0.2	-0.3	-0.3	0.1	0.4

(ii) Solve $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, given that the sum of two of its roots is zero.

OR

b) (i) Find a and b so that $y = ab^x$ best fits the following data :

x :	0.2	0.3	0.4	0.5	0.6	0.7
y :	3.16	2.38	1.75	1.34	1.00	0.74

(ii) Solve the equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$.

22. a) (i) Given that $yy' = y^2 - 2x$, $y(0) = 1$, compute y(0.2) and y(0.4), using Runge-Kutta method of the fourth order with $h = 0.2$.

(ii) With step with $h = \frac{1}{2} = K$, Solve the equation $\nabla^2 u = -100$, $|x| < 1$, $|y| < 1$; $u(\pm 1, y) = u(x, \pm 1) = 0$.

OR

b) (i) Solve the equation $y' = x^2 + y^2$, $y(0) = 1$, by Taylor series (with the first five terms) method for $y(.1)$ and $y(.2)$.

(ii) Compute u for 4 time steps with $h = 0.25$, given that $u_t = u_{xx}$, $0 < x < 1$, $t > 0$; $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 100x(1 - x)$.

23. a) (i) If $f(-1) = -2, f(0) = -1, f(2) = 7, f(5) = 124$, compute $f(1)$

(ii) Compute $\int_0^4 e^x\,dx$ by Simpson's 1/3 rule with 10 subdivisions.

OR

b) (i) Compute $f'(1)$ using the data :

x :	10	15	2.0	2.5	3.0
y :	27.00	106.75	324.00	783.75	1621.00

(ii) Compute $\int_0^4 \frac{dx}{1+x^2}$ by Simpson's 1/3 rule with six subdivisions

24. a) Sum the series 1.2.3 + 2.3.4 + 3.4.5 + ... to n terms by finite integration.

x:	0	1	2	3	4
y:	1	3	9	–	81

b) (i) If $ux = x^3 + 3x^2 - 5x + 1$, find Δux, $\Delta^2 ux$, $\Delta^3 ux$ and $\Delta^{-1} ux$.

(ii) Apply Gauss's forward formula to find $y(3.75)$ from the table

x:	2.5	3.0	3.5	4.0	4.5	5.0
y:	24.14	22.04	20.22	18.64	17.26	16.04

25. a) Find the positive root of $x^3 - 2x - 5 = 0$ by the Regula–Falsi method.

OR

b) Find the inverse of $\begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$ by the Gauss elimination method.

Model Question Paper II

Time : Three hours Maximum : 100 marks

Answer All Questions

Part A (20 × 2 = 40 Marks)

State True or False

1. If a curve is fit to the points (x_i, y_i), i = 1, 2, . . . n by least square method, then the curve passes through the point $(\bar{x}, \bar{y})$,

$$\text{i.e.,}\left(\frac{\Sigma x_1}{n}, \frac{\Sigma y_1}{n}\right).$$

2. Newton–Raphson's iterative formula for $\frac{1}{N}$ is $x_n + 1 = x_n(2 - Nx_n)$

3. Iteration method is a self-correction method.

4. Crank-Nicholson's difference formula is used to solve wave equations.

5. Euler's improved formula $y_{m+1} = y_m = \frac{h}{2}(y_m' + y_{m+1}')$ can be treated as a corrector formula.

Fill in the blanks

6. $\Delta^m f_n = \nabla_m f_p$ where p is ....

7. If α, β, γ are the roots of $x^3 + x^2 + x + 1 = 0$, then the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, is ....

8. If $\alpha, \beta, \gamma, \delta, \xi$, are the roots of $ax^5 + bx^4 + cx + d = 0$, the equation whose roots are $-\alpha, -\beta, -\gamma, -\delta, -\xi$ is ....

9. When Gauss elimination method is used to solve $AX = B$, A is transferred in a .... matrix.

10. In terms of factorial powers, $\frac{1}{x^2 - 1}$ is ....

Choose the correct answer

11. If $\Delta y = 1 + 2x + 3x^3$, which of the following is not true?

    (a) $\Delta^2 y = 6x + 5$; (b) $\Delta^3 y = 6$;
    (c) $\Delta^4 y = 0$; (d) $y = x^2 + x^3$.

12. If $u_1 = 1$, $u_3 = 17$, $u_4 = 43$ and $u_5 = 89$, the value of u_2 is

    (a) 5 (b) 10 (c) 12 (d) 15.

13. $\frac{1}{\Delta} - \frac{1}{\nabla}$ is equal to

    (a) 1 (b) -1 (c) Δ (d) ∇

14. Simpson's rule of integration is exact for all polynomials of degree not exceeding

    (a) 2 (b) 3 (c) 4 (d) 5

15. For solving numerically, the hyperbolic equation $u_{tt} = c^2 u_{xx}$, the starting solution is provided by the boundary condition

(a) $u(0, t) = 0$ (b) $u(l, t) = 0$

(c) $u_t(x, 0) = 0$ (d) $u(x, 0) = f(x)$

Short Questions

16. What is the condition for the convergence of Gauss–Seidel iterative method?

17. Write down the normal equations to be used for finding a, b and c when fitting a parabola $y = ax^2 + bx + c$ by the method of least squares.

18. State a difference formula for solving wave equation.

19. Write down the normal equations to be used for finding a and b, when fitting a parabola $y = a + bx + cx^2$ by the method of least squres.

20. Prove the operator identity $E = \left[\frac{\Delta}{\delta}\right]^2$

Part - B (5 × 12 = 60 marks)

21. a) (i) Find the root of the equation $x^3 - 2x - 5 = 0$ lying between 2 and 3, correct to 3 places of decimals, using Regula Falsi method.

(ii) Using Gauss elimination method, find the inverse of the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix}$$

OR

b) (i) Compute the positive root of the equation $x - \cos x = 0$, correct to 2 places of decimals using the bisection method.

(ii) Solve the following set of equations, correct to 3 places of decimals, using relaxation method

$28x + 4y - z = 32$, $x + 3y + 10z = 24$, $2x + 17y + 4z = 35$.

22. a) Fit $y = ab^x$ by the method of least squares, to the data given below:

x:	0	1	2	3	4	5	6	7
y:	10	21	35	59	92	200	400	610

OR

b) Find the equation whose roots are the negative reciprocals of the roots of $x^4 + 7x^3 + 8x^2 + 9x + 10 = 0$.

23. a) (i) Given that $y' = y, y(0) = 1$, express $y(h)$ as a polynomial in h by Runge–Kutta method.

(ii) Obtain the five-point formula for solving Laplace equation.

OR

b) (i) Compute $y(0, 2)$ correct to 4 decimal places from the Taylor series solution of the equation $yy' = y^2 - 2x, y(0) = 1$.

(ii) Solve the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -100 \qquad |x| \le 1, \quad |y| = 1$$

given that $u = 0$ on the boundary of the square. Take $h = \frac{1}{2}$

24. a) Given the table

x:	14	17	31	35
y:	68.7	64.1	44.2	39.6

find $f(27)$.

(ii) Find $f'(5)$ if

x:	0	2	3	4	5	9
$f(x)$:	5	25	55	100	460	900

OR

b) (i) Solve: $u_{x+2} + 6u_{x+1} + 9u_x = 3^x + x.2^x + 7$.

(ii) Fit a polynomial of least degree to fit the data by Lagrange's formula

x:	0	1	3	4
y:	–4	1	29	52

25. a) Find the cubic polynomial which takes the following set of values (0, 1), (1, 2), (2, 1) and (3, 10).

OR

b) Find the missing values in the following table of values of x and y :

x :	0	1	2	3	4	5	6
y :	−4	−2	–	–	220	546	1148

Model Question Paper III

Time : Three hours | Maximum : 100 marks

Answer All Questions

Part A – (20 × 2 = 40 Marks)

State True or False

1. Gauss-Seidel iteration converges only if the coefficient matrix is diagonally dominant.
2. $\Delta^{10}\{(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\} = abcd.$
3. The third differences of a polynomial of degree 4 are zeros.
4. Liebmann's iteration process is used to solve two dimensional heat equation.
5. Milne's corrector formula is given by

$$y_{n+1} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 2y_n' + y_n' + y_{n+1}')$$

Fill in the blanks

6. The solution of the difference equation $Y_{n+1} - 2y_n = 1$, $y_0 = 1$ is $y_n =$ ....
7. Gauss-elimination and Gauss-Jordan are direct methods while .... and .... are iterative methods.
8. If we start with zero values for x, y, z while solving the equations $10x + y + z = 12$, $x + 10y + z = 12$, $x + y + 10z = 12$ by Gauss-Seidel iteration, the values for x, y, z after one iteration will be ....

9. The general solution of the difference equation $y_{n+2} - 4y_{n+1} + 4y_n = 0$ is ....

10. Newton–Raphson method is also known as the method of ....

Choose the correct answer

11. For solving numerically, the hyperbolic equation $u_{tt} = c^2 u_{xx}$, the starting solution is provided by the boundary condition

(a) $u(0, t) = 0$ (b) $u(l, t) = 0$
(c) $u_t(x, 0) = 0$ (d) $u(x, 0) = f(x)$.

12. If the roots of the equation $x^3 - 19x^2 + 114x - 216 = 0$ are in G.P., then the product of the two extreme roots equals

(a) 24 (b) 54 (c) 36 (d) 35

13. By evaluating $\int_0^1 \frac{dx}{1+x^2}$ by a numerical integration method, we can obtain an approximate value of

(a) $\log_e 2$ (b) π (c) e (d) $\log_{10} 2$

14. Simpson's rule of integration is exact for all polynomials of degree not exceeding

(a) 1 (b) 3 (c) 4 (d) 5

15. If an approximate value of the root of the equation $x^x = 1000$ is 4.5, a better approximation of the root got by Newton–Raphson method is

(a) 4.44 (b) 4.56 (c) 5.17 (d) None of the above

Short Questions

16. State Schmidt's explicit formula for solving heat equations.

17. State Lagrange's interpolation formula.

18. State the condition for convergence of Jacobi's iteration method, for solving a system of simultaneous algebraic equations.

19. Derive Newton's algorithm for finding the pth root of a number N.

20. Show that $y_n = 1 - \frac{2}{n}$ is a solution of the difference equation

$(n + 1)\, y_{n+1} + n y_n = 2n - 3.$

Part - B ($5 \times 12 = 60$ marks)

21. a) (i) Express $x^3 - 2x + 1$ in terms of factorial polynomials:

(ii) Obtain the missing terms in the following table:

x:	1	2	3	4	5
y:	0	7	–	63	124

OR

b) (i) Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{11}{18}$.

(ii) Find f (2.36) from the following table:

x:	1.6	1.8	2.0	2.2	2.4	2.6
y:	4.95	6.05	7.39	9.03	11.02	13.46

22. a) Find, to 4 decimals, by Newton's method, a root of $x^{\sin x} = 4$.

OR

b) Solve by relaxation method, to 2 decimals,
$4x_1 - 3x_2 + x_3 = .5$; $x_1 + 5x_2 - 4x_3 = .3$ $x_1 + 4x_2 - 6x_3 = -1.1$.

23. a) Fit a least square parabola $y = a + bx + cx^2$ to the data:

x:	0	1	2	3	4	5	6
y:	71	89	67	43	31	18	9

OR

b) Find the equation whose roots are 10 less than those of $x^4 + 4x^3 + x^2 + 7x - 12 = 0$.

24. a) (i) Apply Runge-Kutta method of fourth order to calculate $y(0.2)$ given $\frac{dy}{dx} = x + y, y(0) = 1$ taking $h = 0.1$

(ii) Solve $\nabla^2 u = 0$ in the region $0 \le x \le 4$, $\sigma \le y \le 4$ under the conditions $(h = 1, k = 1)$ $u(0, y) = 0$, $u(4, y) = 12 + y$, $0 \le y \le 4$ $u(x, 0) = 3x$, $u(x, 4) = x^2$, $0 \le x \le 4$.

OR

b) (i) Derive Bendu-Schmidt formula for one dimensional heat equation.

(ii) Evaluate the pivotal values of the equation $25u_{xx} = u_{tt}$ for one-half period oscillation given

$$u(0, t) = u(5, t) = 0$$
$$u(x, 0) = 2x, \ 0 \le x \le 2.5$$
$$= 10 - 2x, \ 2.5 \le x \le 5$$

25. a) The velocity $v\left(\frac{\text{km}}{\text{min}}\right)$ of a moped which starts from rest is given at fixed intervals of time 't' (members) as follows :

x :	2	4	6	8	10	12	14	16	18	20
y :	10	18	25	29	32	20	11	5	2	0

Estimate the distance covered in 20 minutes by Simpson's rule.

OR

b) (i) Solve : $y_{n+2} - 2y_{n+1} + y_n = n^2 2^n$.

(ii) Find $y'(0)$ from the following table :

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

Model Question Paper IV

Time : Three hours **Maximum : 100 marks**

Answer All Questions

Part A – (20 × 2 = 40 Marks)

State True or False

1. A non-linear relation between x and y cannot be transformed to a linear relation always.
2. Simple Relaxation method will succeed only when the coefficient matrix is diagonally dominant.
3. The number of real roots of an odd degree algebraic equation with real coefficient is odd.

4. For any root, the order of convergence of Newton–Raphson method is two.
5. To fit a straight line for the given data (x_r, y_r), $r = 1, 2, \ldots n$, by the method of least squares, the values of x_r must be equally spaced.

Fill in the blanks

6. In the equation with real coefficients, complex roots occur in ....
7. The difference equation which satisfies $y_x = A.2^x + B.3^x$ is ....
8. The compute $\nabla^3 u_5$, we require, apart from u_5, the value of ....
9. The general solution of the difference equation $y_{n+2} - 4y_{n+1} + 4y_n = 0$ is ....

Choose the correct answer

10. Which of the following formulae is a particular case of Ruge–kutta formula of the second order?

    (a) Taylor series formula
    (b) Picard's formula
    (c) Euler's modified formula
    (d) Milne's predictor corrector formula.

11. Which of the following is true?

    (a) $\Delta x^r = rx^{r-1}$
    (b) $\Delta x^{(r)} = rx^{(r-1)}$
    (c) $\Delta^n e^x = e^x$
    (d) $\Delta \sin x = \cos x$

12. $\frac{1}{\Delta} - \frac{1}{\nabla}$ is equal to

    (a) 1
    (b) –1
    (c) Δ
    (d) ∇

13. To find the smallest positive root of the equation $x^3 - x - 1 = 0$ by the method of simple iteration the equation should be rewritten as

    (a) $x = x^3 - 1$
    (b) $x = (x + 1)^{1/3}$
    (c) $x = \dfrac{1}{x^2 - 1}$
    (d) $x = \dfrac{x+1}{x^2}$.

Short Questions

14. If α, β, γ are the roots of $x^3 + 2x - 1 = 0$, what is the value of $\alpha^3 + \beta^3 + \gamma^3$?
15. State the condition for convergence of Gauss-Seidel method.
16. Define a difference quotient.
17. State Bessel's interpolation formula.
18. State Laplace-Everett interpolation formula.
19. State the Adams–Bashforth predictor–corrector formula.
20. Derive Picard's formula for the solution of the equation $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

Part - B (5 × 12 = 60 marks)

21. a) (i) Find $f(2)$, if $f(-1) = 2, f(0) = 1, f(1) = 0$ and $f(3) = -1$.
    (ii) Solve $y_{n+2} - 5y_{n+1} + 6y_n = 2^n + 1$.

OR

b) (i) Evaluate $\int_0^{0.8} e^{-x^2} dx$, by Romberg's method with h = 0.1 and 0.2.
   (ii) Solve $y_{n+2} - 2y_{n+1} + 2y_n = 2^n + n^2$.

22. a) Using Newton's forward interpolation formula, find $y(21)$ from the following tabulated values of the function.

x	20	23	26	29
y	.342	.3907	.4384	.4848

OR

b) Prove $\Delta\nabla = \Delta - \nabla = \delta^2$.

23. a) Solve by Newton's method, a root of $e^x - 4x = 0$.

OR

b) Solve the Gauss Jordan method
$10x_1 + x_2 + x_3 = 12;\ \ x_1 + 10x_2 - x_3 = 10;\ \ x_1 - 2x_2 + 10x_3 = 9.$

24. a) (i) Fit a second degree parabola to the following

$x = 0$	1	2	3	4
$y = 0$	1.8	1.3	2.5	6.3

OR

b) (i) Evaluate $\Delta^2(\cos 2x)$.

(ii) Explain the difference between $\left(\frac{\Delta^2}{E}\right)u_x$ and $\frac{\Delta^2 u_x}{EU_x}$ and find the values of these when $u_x = x^3$.

25. a) Given $\frac{dy}{dx} = \frac{(1+x^2)y^2}{2}$ and $y(0) = 1$, $y(0,\ 1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$ evaluate $y(0.4)$ by Milne's predictor corrector method.

OR

b) Solve $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in the figure below :

Model Question Paper V

Time : Three hours | Maximum : 100 marks

Answer All Questions

Part A – ($20 \times 2 = 40$ Marks)

Fill in the blank

1. Newton's forward differences interpolation formula is ....
2. $E - \Delta =$ ....
3. To get the simplest explicit difference formula for the parabolic equation $u_t = \alpha^2 u_{xx}$, we should take $\frac{\Delta x^2}{\alpha^2 \Delta t}$ = ...
4. The finite differences formula equivalent to Poisson's equation $u_{xx} + u_{yy} = -f(x, y)$ with $\Delta x = \Delta y = h$ is given by $u_{ij} =$ ....
5. The equation $yu_{xx} + u_{yy} = 0$ is hyperbolic in the region ....

Choose the correct answer

6. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + 1 = 0$, the equation whose roots are $-\frac{1}{\alpha}, -\frac{1}{\beta}, -\frac{1}{\gamma}$ is

   (a) $x^3 + qx^2 - px - 1 = 0$ (b) $x^3 + qx^2 - px + 1 = 0$
   (c) $x^3 - qx^2 + px - 1 = 0$ (d) $x^3 - qx^2 + px + 1 = 0$.

7. If $u_1 = 1$, $u_3 = 17$, $u_4 = 43$ and $u_4 = 89$, the value of u_2 is

   (a) 12 (b) 15 (c) 5 (d) 10

8. If α, β, γ are the roots of the equation $x^3 - 2x - 3 = 0$, then $\alpha^2 + \beta^2 + \gamma^2$ is

   (a) 2 (b) 3 (c) 4 (d) 1

9. If $\Delta y = 1 + 2x + 3x^2$, which of the following is not true?

   (a) $\Delta^2 y = 6x + 5$ (b) $\Delta^3 y = 6$
   (c) $\Delta^4 y = 0$ (d) $y = x^2 + x^3$.

Short Questions

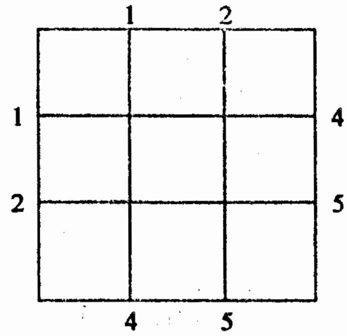
10. What is the solution of the difference equation $y_{n+2} - 2y_{n+1} - 24y_n = 0$?
11. State a difference formula for solving heat equation.
12. Write down the normal equations to be used for finding a and b, when fitting a straight line $y = ax + b$ by the method of moments.
13. State Simpson's one–third rule.
14. State Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rules of numerical integration.
15. Write down the general and the simplest forms of the difference equation corresponding to the hyperbolic equation $u_{tt} = c^2 u_{xx}$.
16. When should we use Newton's backward interpolation formula?
17. Write down the normal equations to fit a quadratic equation by least square method.
18. Write down the general form of normal equations.
19. What is the principle of least squares?
20. If α, β, γ are the roots of $x^3 + 2x - 1 = 0$, what is the value of $\alpha^3, \beta^3, \gamma^3$?

Part - B ($5 \times 12 = 60$ marks)

21. a) (i) Solve $y' = y - x^2$, $y(0) = 1$, by Picard's method, upto the third approximation.

(ii) Using Liebmann's method, solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

for the following square mesh with boundary values as shown in the fig. Iterate until the maximum difference between successive values at any point is less than 0.001



OR

b) Find the inverse of $\begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$ by the Gauss elimination method.

22. a) From the following table of values of x and y find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.05$.

x	1	1.05	1.1	1.15	1.2	1.25	1.3
y	1	1.025	1.049	1.072	1.095	1.118	1.14

OR

b) Derive Simpson's $\frac{1}{3}$ rule for numerical integration.

23. a) Using Lagrange interpolation, find $y(2)$ from the following data:

x:	0	1	3	4	5
y:	0	1	81	256	625

OR

b) State and prove Newton's backward difference interpolation formula.

24. a) The following are data from the steam table

temp C°	140	150	160	170	180
pressure kgf/cm²	3.685	4.854	6.302	8.076	10.225

Using Newton's formula, find the pressure of the steam for a temp of 142°.

OR

b) Dividing the range into 10 equal parts, find the approximate value of $\int_0^x \sin x\,dx$ by (i) Trapezoidal rule (ii) Simpson's rule.

25. a) Use the method of least squares to determine the constants a and b such that $y = ae^{bx}$ fits the following data :

x:	0.0	0.5	1.0	1.5	2.0	2.5
y:	0.10	0.45	2.15	9.15	40.35	180.75

OR

b) Prove that

(i) $\delta = E^{1/2} - E^{-1/2}$

(ii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$.

Model Question Paper VI

Time : Three hours | Maximum : 100 marks

Answer All Questions

Part A ($20 \times 2 = 40$ Marks)

Short Questions

1. Convert $y = \dfrac{x}{a+bx}$ to linear form.
2. Define roundoff error.
3. Write the distributive, commutative and Index laws of operator Δ.
4. Prove that $\dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta} = E - E^{-1}$.
5. Write the relation between E and Δ.
6. What is the condition for convergence of Gauss–Jacobi method of iteration?
7. When does relaxation method succeed in solving a set of simultaneous equations?
8. Write the difference table for

x	3	5	7	9
y	6	24	58	108

9. Classify the equation $U_{xx} + 2U_{xy} + U_{yy} = 0$.
10. Explain convergency of the relaxation method.
11. When Gauss–Seidel method converges?
12. Is the iteration method, a self correcting method always?
13. Write the standard five point formula to solve the Laplace equation $U_{xx} + U_{yy} = 0$.
14. State Milne's corrector formula.
15. Why is trapesoidal rule is so called?
16. State the finite differences scheme of $u_{xx} + u_{yy} = 0$.
17. State Schmidt's explicit formula for solving heat flow equations.

18. Write the Crank Nicholson difference scheme to solve $u_{xx} = au_t$ with $u(0, t) = T_0$, $u(l, t) = T_l$ and the initial condition as $u(x, 0) = f(x)$.

19. Give the Crank–Nicolson difference scheme to solve a parabolic equation.

20. Find the equation whose roots are the roots of $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ each decreased by 10.

Part - B (5 × 12 = 60 marks)

21. a) (i) Fit a parabola of the form $y = a + bx + cx^2$ to the following data by the method of least squares:

x:	1	2	3	4	5	6
y:	3.13	3.76	6.94	12.62	20.86	31.53

(ii) Solve the equation $6x^6 - 25x^5 + 31x^2 + 25x - 6 = 0$

OR

b) (i) Determine a and b so that $y = ae^{bx}$ best fits the following data:

x:	1	2	3	4
y:	7	11	17	27

(ii) Increase the roots of the equation $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ by 2 and hence solve it.

22. a) Solve $\dfrac{dy}{dx} = \dfrac{3x+y}{x+2y}$; $y(1) = 1$ at $x = 1.2, 1.4$ using Runge–Kutta fourth order method.

OR

b) Solve $u_{xx} + u_{yy} = 0$; $0 \le x, y \le 1$ with $u(0, y) = 10 = u(1, y)$ and $u(x, 0) = 20 = u(x, 1)$. Take $h = .25$ and apply Liebmann method to 3 decimal accuracy.

23. a) Solve $u_{n+2} + 3u_{n+1} + 2u_n = 2^n$.

OR

b) Derive Trapezoidal rule of numerical integration.

24. a) (i) Use Newton–Raphson method to find the roots of the following equation $e^x = 2x + 21$.

(ii) By Gauss–Seidal method, solve $4x_1 + x_2 + 2x_3 = 4$; $3x_1 + 5x_2 + x_3 = 7$ and $x_1 + x_2 + 3x_3 = 3$.

OR

b) (i) Solve $9x - y + 2z = 9$, $x + 10y - 2z = 15$, $2x - 2y - 13z = 17$ to three decimal places by relaxation method.

(ii) Solve $x^3 - 8x^2 + 17x - 10 = 0$ by Graeffe's method.

25. a) (i) The following table gives the values of density of saturated water for various temperature of saturated steam

Temp °C :	100	150	200	250	300
Density hg/m³:	958	917	865	799	712

Find by interpolation, the densities when the temperatures are 130°C and 275°C respectively.

(ii) Solve the difference equation
$y_{x+2} - 6y_{x+1} + 8y_x = 4^x$.

OR

The following table gives the velocity v of a particle at time 't'

t (seconds) :	0	2	4	6	8	10	12
v (metres/sec) :	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 seconds and also the acceleration at $t = 2$ secs.

Model Question Paper VII

Time : Three hours — Maximum : 100 marks

Answer All Questions

Part A – ($20 \times 2 = 40$ Marks)

Short Questions

1. What is the equation whose roots are ± 1, $\pm$ i, $1\pm$i ?
2. Find $\Delta f(x)$, if $f(x) = x^2 + 2x + 2$ and the interval differencing is unity.
3. Define a difference Quotient.

4. If α, β, ϕ are the roots of $x^3 + px + q = 0$ find the value of $\Sigma(\alpha-\beta)^2$.

5. If α, β, γ are the roots of $x^3 + 2x - 1 = 0$, what is the value of $\alpha^3 + \beta^3 + \gamma^3$?

6. Express $x^4 - 12x^3 + 42x^2 - 30x + 9$ in factorial notation.

7. Form the difference equation by eliminating a and b form the relation $y_x = a2^x + b(-2)^x$.

8. Explain the terms : Round off error, Truncation error.

9. Solve $Y_{n+2} - 6Y_{n+1} + 8Y_n = 0$?

10. How to reduce the number of iterations while finding the root of an equation by Regula–Falsi method?

11. Write the RK fourth order formula for solving $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

12. When is Newton's backward interpolation formula used?

13. Give the Crank Nicolson difference scheme formula to solve the equation $u_t = \alpha^2 u_{xx}$.

14. How many prior values are required to predict the next value in Adam's method?

15. State Milne's predictor formula.

16. What is the order of the error in Simpson's formula?

17. Classify the partial differential equation

$$x^2 \frac{\partial^2 u}{\partial x^2} - (y^2 - 1) \frac{\partial^2 u}{\partial y^2} = 0 - \infty < x < \infty \ ; \ -1 < y < 1$$

18. Write down the finite difference form of the equation $\nabla^2 u = f(x, y)$.

19. If α, β, γ are the roots of the equation $x^3 + px + q = 0$, find the value of $\Sigma \frac{1}{\alpha}$.

20. State the special advantage of Runge–Kutta method over Taylor series method.

Part - B (5 × 12 = 60 marks)

21. a) (i) Find the root of the equation $2x^3 + 3x - 10 = 0$ lying between 1 and 2, correct to two places of decimals using Horner's method.

(ii) Solve the following set of equations by Crout's reduction: $2x - 2y - 4z = 1$, $2x + 3y + 2z = 9$, $-x + y + z = 0.5$

OR

b) (i) Find the positive root of the equation $xe^x - \cos x = 0$, correct to 4 places of decimals, using Newton-Raphson method.

(ii) Solve the following set of equations by Gauss-Seidel iteration, correct to 3 places of decimals:
$10x - 5y - 2z = 3$, $4x - 10y + 3z = -3$, $x + 6y + 10z = -3$.

22. a) (i) Fit the least square straight line $y = a + bx$ to the data : $f(0) = 6, f(1) = 5, f(2) = 3, f(4) = 0, f(5) = -3$.

(ii) Solve the reciprocal equation
$4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$

OR

b) (i) Fit the least square parabola $y = a + bx + cx^2$ to the data : $f(-1) = -2, f(0) = 1, f(1) = 2, f(2) = 4$

(ii) Solve $x^3 - 19x^2 + 114x - 216 = 0$, given that the roots are in G.P.

23. a) Solve $y' + y = 0$; $y(1) = 1$ at $x = 1.8$ (finding with $h = .2$, the intermediate values by Taylor method) by Milne method.

OR

b) Explain the Crank Nicolson method of solving parabolic partial differential equation.

24. a) Using Euler's modified method, solve $\frac{dy}{dx} = x + y$ and $y(0) = 1$ at $x = 0.05$ to 0.20.

OR

b) Solve $y' = x^2 + y^2 - 2$ using Milne's predictor corrector method for $x = 0.3$ given the initial value $x = 0, y = 1$ the values of y for $x = -0.1$, 0.1 and 0.2 should be computed by Taylor series expansion.

25. a) (i) Find the root of the equation $x^3 - 2x - 5 = 0$ which lies between 2.0 and 2.1, correct to five decimal places using the method of false position.

(ii) Find all the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$ by Graeffe's root squaring method.

OR

b) (i) Using Crout's method solve the following system

$$x+y+z=3$$
$$2x-y+3z=16$$
$$3x+y-z=-3.$$

(ii) Solve, by using relaxation method

$$10x-2y+z=12$$
$$x+9y-z=10$$
$$2x-y+11z=20.$$

Model Question Paper VIII

Time : Three hours **Maximum : 100 marks**

Answer All Questions

Part A – (20 × 2 = 40 Marks)

Short Questions

1. Explain convergency of the relaxation method.
2. Form the difference equation by eliminating a and b from the relation $yx = a2^x + b\,(-2)^x$
3. Explain the terms: Round off error, truncation error.
4. Evaluate $\frac{\Delta^2}{E}x^3$.
5. Form the equation whose roots are the roots of the euqation $x^3 + 4x + 2 = 0$, multiplied by 2.
6. Prove $(1 + \Delta)(1 - \nabla) = 1$
7. What is the order of error in Simpson's 1/3 rule?
8. Solve $U_{x+2} - 6U_{x+1} + 9U_x = 0$.
9. How the accuracy can be increased in trapezoidal rule of evaluating a given definite integral
10. Explain Regula Falsi method of getting a root.

11. Show that Newton–Raphson formula to find $\sqrt{a}$ can be expressed in the form $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ $n = 0, 1, 2.$

12. Write Bessel's formula.

13. What is the disadvantage in Taylor series method?

14. Give the Newton's divided difference interpolation formula.

15. Write the merits and demerits of the Taylor method of solution.

16. Write the finite difference form of $\frac{\partial^2 u}{\partial t^2}$.

17. Write the difference scheme for solving the Laplace's equation.

18. What is the equation whose roots are $\pm 1, \pm i, 1 \pm i$?

19. Define μ and δ.

20. Write down the standard five point formula to solve the Laplace equation $\nabla^2 u = 0$.

Part - B ($5 \times 12 = 60$ marks)

21. a) (i) Show that (1) $\Delta - \nabla = \Delta\nabla$; (2) $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$.

(ii) Prove that $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$.

(iii) Find $f(x)$ at $x = 1.5$ from the following data:

x	:	0	1	2	3	4	5
$f(x)$	:	–1	0	7	26	63	124

OR

b) (i) Prove that $E = e^{hD}$.

(ii) If $C_n = 2^{n/2} \cos\frac{\pi n}{4}$, $S_n = 2^{n/2} \sin\frac{\pi n}{4}$, Prove that $\Delta C_n = S_n$ and $\Delta S_n = C_n$.

(iii) Obtain the missing terms in the following table:

x:	1	2	3	4	5	6	7	8
y:	10	18	–	74	135	–	353	522

22. a) (i) Compute the positive root of $x^4 - x - 10 = 0$ correct to 2 decimal places.

(ii) By Gauss elimination method, invert the matrix

$$\begin{pmatrix} 10 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

OR

b) (i) Compute the root of $4x = e^x$ near 2 correct to 2 decimal places.

(ii) Solve by Crout's method : $2x + 5y - z = 10,\ 8x - y + 3z = 12,$ $x + 3y + 6z = -1.$

23. a) (i) Fit a straight line to the data by the method of least squares:

x :	0	5	10	15	20
y :	7	10	15	21	25

(ii) Solve : $6x^5 + x^4 - 43x^3 - 43x^3 + x + 6 = 0$

OR

b) (i) Fit a curve $y = ae^{bx}$ to the data:

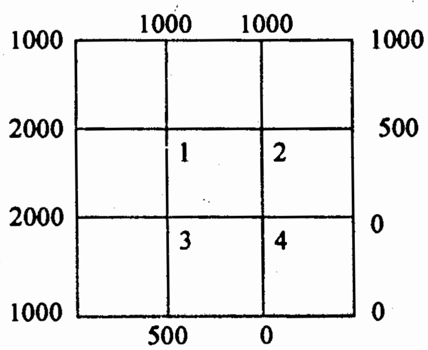
x :	0	2	4
y :	5.1	10	31.1

(ii) Diminish the roots of the equation

$$x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$$

by 1 and hence solve it.

24. a) Given the values of $u(x, y)$ on the boundary of the square given in the figure, evaluate the function $u(x, y)$ satisfying Laplace's equation $\nabla^2 u = 0$ at the pivotal points of this figure.



OR

b) Solve by Crank–Nicolson's method

$$\frac{\partial u}{\partial t} = \frac{1}{16}\frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1, t > 0$$

$u(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = 100t$.

Compute u for one step with $h = \frac{1}{4}$.

25. a) Apply the fourth order Runge–Kutta method to find an approximate value of y when $x = .2$, given that $y' = x + y$, $y(0) = 1$.

OR

b) Given $\frac{dy}{dx} = \frac{1}{2}(1 + x^2)\, y^2$ and the table

x:	0	.1	.2	.3
y:	1	1.06	1.12	1.21

evaluate y(.4) by Milne's method.

Model Question Paper IX

Time : Three hours | Maximum : 100 marks

Answer All Questions

Part A ($20 \times 2 = 40$ Marks)

Short Questions

1. State the condition for convergence of Jacobi's iteration method, for solving a system of simultaneous algebraic equation.
2. Form the equation with rational coefficients whose roots are $\sqrt{3}$, $1 + i$.
3. State Newton's formula on interpolation.
4. Find $\Delta^n e^x$.
5. When should we use Newton's backward interpolation formula?

6. What is Regula–Falsi method?

7. State Simpson's rule

8. Show that $y_n = 1 - \frac{2}{n}$ is a solution of the difference equation $(n+1)\, y_{n+1} + ny_n = 2n - 3$

9. Write the observation equations when the equation $y = ax + b$ is fit by the method of moments.

10. Write a sufficient condition to apply Jacobi's method to solve a system of equations?

11. While solving a polynomial equation by Graff's root squaring method, how many roots are obtained?

12. What is the relation between Bessel's and Everett's formulae?

13. Is Euler's modified formula, a particular case of second order Runge–Kutta method?

14. What is the Lagrange's formula to find 'y' if three sets of values $(x_0, y_0)(x_1, y_1)$ and (x_2, y_2) are given?

15. State the Adams–Bashforth predictor–corrector formula.

16. What is a predictor–corrector method of solving a differential equation?

17. Identify the equation $f_{xx} + 2f_{xy} + f_{yy} = 0$.

18. What is the classification of $f_x - f_{yy} = 0$?

19. When is Newton's backward interpolation formula used?

20. Give the Crank Nicolson difference scheme formula to solve the equation $u_t = \alpha^2 u_{xx}$.

Part - B ($5 \times 12 = 60$ marks)

21. a) (i) Find $f(1)$ from the following table

x:	–1	0	2	5	10
$f(x)$:	–2	–1	7	124	999

(ii) Solve the difference equation

$$y_{n+2} - 2y_{n+1} + 2y_n = \cos\frac{\pi n}{2}.$$

OR

b) (i) Compute the value of π by evaluating $\int_0^1 \frac{dx}{1+x^2}$, using Simpson's 1/3 rule with 10 divisions.

(ii) Solve $y_{n+2} - y_{n+1} - y_n = 0, y_0 = 0, y_1 = 1$.

22. a) (i) Compute the missing value :

x :	1	2	3	4	5
y :	0	7	26	–	124

(ii) Find the sum $\sum_{x=1}^{\infty} \frac{1}{x(x+2)}$

OR

b) (i) Express x^3 in factorial powers.

(ii) Compute the missing value :

x :	–1	0	1	2	3
f:	5	2	–	0	1

23. a) (i) Find the positive root of $x^3 + x - 1 = 0$ correct to 2 decimals by Horner's method.

(ii) Using Gauss-Seidel method, solve

$$28x + 4y - z = 35$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

OR

b) (i) Solve by Crout's method

$$2x - 6y + 8z = 24$$
$$3x + y + 2z = 6$$
$$5x + 4y - 3z = 2$$

(ii) Find the inverse of $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ by Gauss method.

24. a) By using the method of moments, fit a parabola to the following data:

x:	1	2	3	4
y:	0.30	0.64	1.32	5.40

OR

b) (i) Solve $x^3 - 19x^2 + 114x - 216 = 0$ given that the roots are in G.P.

(ii) Find the sum of the cubes of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$.

25. a) Find the solution of the parabolic equation $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0$ when $u(0, t) = u(4, t) = 0$, $u(x, 0) = x(4 - x)$, $0 < x < 4$. Assume $h = 1$. Find the values upto $t = 5$.

OR

b) Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, subject to

(i) $u(0, y) = 0$, for $0 \le y \le 4$

(ii) $u(4, y) = 12 + y$, $0 \le y \le 4$

(iii) $u(x, 0) = 3x$, for $0 \le x \le 4$

(iv) $u(x, 4) = x^2$, for $0 \le x \le 4$

by dividing the square into 16 square meshes of side 1.

Appendix A

The power method or iterative method for dominant eigen value of a square matrix

In many engineering problems, it is required to find out the numerically largest eigen value and the corresponding eigen vector. In such cases, the following iterative method, so called *power method*, is quite convenient.

Let $A = [a_{ij}]$ be the square matrix of order n. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigen values such that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq \lambda_n$. Then λ_1 is known as the largest or dominant value.

If we operate A repeatedly on a vector y_0 which is a linear combination of eigen vectors $X_1, X_2, \ldots, X_n$ of $\lambda_1, \lambda_2 \ldots \lambda_n$

$$\text{i.e. } y_0 = x_1 X_1 + x_2 X_2 + \cdots + x_n X_n \tag{A.1}$$

where $x_1, x_2, \ldots, x_n$ are scalars, then

$$Y_1 = AY_0 = x_1 AX_1 + x_2 AX_2 + \cdots + x_n AX_n$$

$$= x_1 \lambda_1 X_0 + x_2 \lambda_2 X_2 + \cdots + x_n \lambda_n X_n \tag{A.2}$$

$$[\because AX = \lambda X]$$

Similarly,

$$Y_2 = AY_1 = x_1 \lambda_1^2 X_1 + x_2 \lambda_2^2 X_2 + \ldots + x_n \lambda_n^2 X_n \tag{A.3}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$Y_k = AY_{k-1} = x_1 \lambda_l^k X_1 + x_2 \lambda_2^k X_2 + \cdots + x_n \lambda_n^k X_n$$

$$= \lambda_l^k [x_1 X_1 + x_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k X_2 + \cdots + x_n \left(\frac{\lambda_n}{\lambda_1}\right)^k X_n] \tag{A.4}$$

For large K, the vector

$$x_1X_1 + x_2\left(\frac{\lambda_2}{\lambda_1}\right)^k X_2 + \cdots + x_n\left(\frac{\lambda_n}{\lambda_1}\right)^k X_n$$

$$= x_1X_1 + \sum_{i=2}^{n} x_i\left(\frac{\lambda_i}{\lambda_1}\right)^k X_i$$

converges towards x_1X_1 i.e. eigen vector of λ_1

$$\because \lim_{k\to\infty}\left(\frac{\lambda_i}{\lambda_1}\right)^k \to 0 \text{ as } |\lambda_1| > |\lambda_i| \qquad \text{(A.5)}$$

$\therefore$ From Eqn (A.4), $\quad Y_k = x_1\lambda_1^k X_1 \qquad$ (A.6)

= a scalar multiple of eigen vector X_1

$\Rightarrow Y_k$ is also an eigen vector corresponding to the largest eigen value λ_1.

Practically, taking $Y_1 = AY_0 = \alpha_1 Z_1$

or $Z_1 = \dfrac{1}{\alpha_1} Y_1$, α_1 being the largest element of Y_1, the process is repeated by conveniently choosing Y_0 till we get

$$Y_{k+1} = AZ_k = A^{k+1} Y_0 \qquad \text{(A.7)}$$

Finally, $\quad Y_{k+1} = \alpha_{k+1} Z_{k+1}$

then α_{k+1} is the eigen value and Z_{k+1} is the eigen vector

Note: We can also determine the smallest eigen value by modifying the matrix form $(A - \lambda_1 I)$ which has the eigen values

$$\lambda_k' = \lambda_k - \lambda_1,\ k = 1, 2, \ldots, n \qquad \text{(A. 8)}$$

where λ_k are the eigen values of A.

Now for λ_k to be the smallest, λ_k' is the largest normed eigen values of $(A - \lambda, I)$. Again, if X_k' is the corresponding eigen vector, then

$$[A - \lambda_1 I]X_k' = (\lambda_k - \lambda_1)X_k' \Rightarrow AX_k' = \lambda_k X_k' \qquad \text{(A.9)}$$

that is, X_k' is also the eigen vector of A corresponding to the smallest eigen value of A.

Hence for evaluation of eigen values if we diminish (or increase) the eigen values of A by a constant, say, μ, i.e. if we set $\mu = A - \mu I$, where λ is an eigen value of A with eigen vector X such that $AX = \lambda X$ then

$$BX = (A - \mu I)X = AX - \mu IX$$

$$= \lambda X - \mu X = (\lambda - \mu)X$$

$\Rightarrow$ eigen values of $B \approx$ eigen values of A diminished by a constant μ with the same eigen vector.

Example Using Power method, find the largest eigen value and corresponding eigen vector of the matrix

$$\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ Also find the smallest eigen value.

Solution Let

$$Y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

be the initial eigen vector. Then, we have

$$Y_1 = AY_0 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = Z_1$$

$$\therefore \alpha_1 = 1, Z_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$Y_2 = AZ_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = 7Z_2$$

$$Y_3 = AZ_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 1.8 \\ 0 \end{bmatrix} = 3.4 \begin{bmatrix} 1 \\ 0.53 \\ 0 \end{bmatrix} = 3.4Z_3$$

$$Y_4 = AZ_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.53 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.18 \\ 2.06 \\ 0 \end{bmatrix} = 4.18 \begin{bmatrix} 1 \\ 0.49 \\ 0 \end{bmatrix} = 4.18Z_4$$

$$Y_5 = AZ_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.49 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.94 \\ 1.98 \\ 0 \end{bmatrix} = 3.94 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 3.94 Z_5$$

$$Y_6 = AZ_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4Z_6$$

$\because Z_5 = Z_6$, the largest eigen value is 4 and the corresponding vector is

$$\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

To find smallest eigen value

Consider

$$B = A - 4I = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Starting with $Y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Y_1 = BY_0 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -0.33 \\ 0 \end{bmatrix} = -3Z_1$$

$$Y_2 = BZ_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.33 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.98 \\ 1.66 \\ 0 \end{bmatrix} = -4.98 \begin{bmatrix} 1 \\ -0.33 \\ 0 \end{bmatrix}$$

$$= -4.98 Z_2$$

$$Y_3 = BZ_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.33 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.98 \\ 1.66 \\ 0 \end{bmatrix} = -4.98 \begin{bmatrix} 1 \\ -0.33 \\ 0 \end{bmatrix}$$

$$= -4.98Z_3$$

$\because$ $Z_2 = Z_3$, the largest eigen value of B is -4.98 or -5, approximately, and the corresponding eigen vector is

$$\begin{bmatrix} 1 \\ -0.33 \\ 0 \end{bmatrix}$$

Hence, the smallest eigen value is $-5 + 4 = -1$.

The third eigen value is obtained by $4 - 1 + \lambda_3$ = sum of principle diagonal elements of A(trace of A)

$$= 1 + 2 + 3$$

$$\therefore \quad \lambda_3 = 3$$

EXERCISE

Use power method to find the dominant eigen value of the following matrices. Also write the corresponding eigen vectors.

1. $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$

ANSWERS

1. $4.9, \begin{bmatrix} 1 \\ 0.49 \end{bmatrix}$

2. $7, \begin{bmatrix} 0.34 \\ 0.185 \\ 1 \end{bmatrix}$

3. $25.1821, \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$

4. $11.66 \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}$

Appendix B

Model Question Papers

Model Question Paper I

Time : Three hours Maximum : 100 marks

Answer All Questions

Part A $(20 \times 2 = 40$ Marks)

State True or False

1. The number of real roots of an odd degree algebraic equation with real coefficient is odd.
2. To fit a straight line for the given data (x_r, y_r), $r = 1, 2, \ldots n$, by the method of least squares, the values of x_r must be equally spaced.
3. By the method of least square, a curve of the form $y = ax^2 + bx + c$ is fitted to the data (x_1, y_1), $i = 1, 2, \ldots n$. This curve will pass through all the points.
4. For any root the order of convergence of Newton-Raphson method is two.
5. The nth divided differences of a polynomial of the nth degree are not constant.

Fill in the blanks

6. The sum of the roots of the equation $x^2 - |x| - 12 = 0$ is ....
7. The finite differences expression, in terms of central differences, of he second order partial derivative u_{xx} is ....

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