LEC-05

Sobolev Spaces $W^{1,p}$:

$$W^{1,p}(\Omega) := \{ f : f \in L^p(\Omega),\ Df \text{ exists},\ Df \in L^p(\Omega) \}$$

$1 \leq p \leq \infty$

$$D_i f \in L^p(\Omega) \quad , \quad 1 \leq i \leq n$$

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \sum_{i=1}^{n} \|D_i f\|_{L^p(\Omega)}$$

Fact: $L^p(\Omega)$, $W^{1,p}(\Omega)$: are Banach spaces.

Def: $f \in L^1(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, we say

$$g = D^\alpha f \quad , \quad \alpha = \langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$$

αth order weak derivative of f $\qquad |\alpha| = \sum_{i=1}^{n} \alpha_i$

if $\forall\ \phi \in C_c^\infty(\Omega)$

$$\int_\Omega f \boxed{\frac{\partial^\alpha}{\partial x^\alpha}} \phi \, dx = (-1)^{|\alpha|} \int_\Omega g\, \phi \, dx$$

$$\left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \right) \phi$$

Def

$$W^{n,p} := \{ f : f \in L^p(\Omega),\ D^\alpha f \in L^p(\Omega),\ |\alpha| \leq n \}$$

$$\|f\|_{W^{n,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \sum_{|\alpha| \leq n} \|D^\alpha f\|_{L^p(\Omega)}$$

$W^{n,p}(\Omega)$ is a Banach Space.

(*) **Product Rule:** $\frac{1}{p}+\frac{1}{p'}=1$

$f \in W^{1,p}(\Omega)$, $g \in W^{1,p'}(\Omega) \Rightarrow fg \in W^{1,1}(\Omega)$.

and

$$D(fg) = \underbrace{Df}_{\in L^p} \cdot \underbrace{g}_{L^{p'}} + \underbrace{Dg}_{\in L^{p'}} \cdot \underbrace{f}_{\in L^p}$$

(*) **Chain Rule:**

$f \in C^1(\Omega)$, f': bounded, $g \in W^{1,p}(\Omega)$

then:

$$\underbrace{D(f \circ g)}_{\in L^1} = (f' \circ g) \cdot Dg \in L^p$$

$$\in W^{1,p}(\Omega)$$

(*) **F.T.O.C:**

(a) If $g \in L^1((a,b))$, $f(x) = \int_a^x g(s)\,ds$, then

$f \in W^{1,1}((a,b))$

(b) If $f \in W^{1,1}((a,b))$ and $g = Df$ then:

$\boxed{f(x)} = c + \int_a^x g(s)\,ds$ for some $c \in \mathbb{R}$.

for each $f \in W^{1,1}((a,b))$ $\exists$ a representation of f that is continuous.