Lec-02

Basic Notions in Measure Theory:

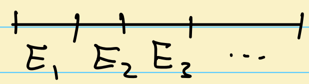
Def: $P(\mathbb{R})$: set of all subsets of $\mathbb{R}$.

Measure:

A map $\mu : P(\mathbb{R}) \longrightarrow [0, \infty)$ is called a measure if:

(i) $\mu(I) =$ length of $I \equiv \ell(I)$, $\forall I$: intervals.

(ii) $\{E_n\}$ of disjoint sets:



$$\mu\left(\bigcup_n E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

(iii) μ: translation invariant

$$\mu(E) = \mu(E + y)$$

Def: Outer measure:

Outer measure μ^* is defined on E:

$$\mu^*(E) = \inf_{E \subseteq \bigcup_n I_n} \sum_{n=1}^{\infty} \ell(I_n)$$

Def: E is Lebesgue measurable if

$$\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A) \quad \forall A \subseteq \mathbb{R}$$

Thm: The outer measure μ^* is a measure on B = all Lebesgue measurable sets.

(a) $E \in B \Rightarrow E^c \in B$

(b) $\{E_i\} \in B \Rightarrow \bigcup_i E_i \in B$

Def: $E \in B$, we say $\mu(E) = 0$ if $\forall \varepsilon > 0$, $\exists \{A_i\}$ open sets;

$E \subseteq \bigcup A_i$ and $\mu(\bigcup A_i) < \varepsilon$

$\Downarrow$

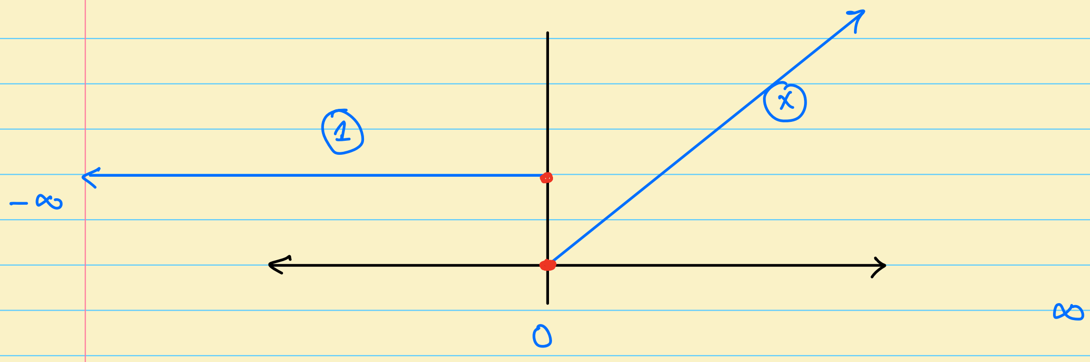
A property holds μ-a.e. if the set where this property doesn't hold; E, has $\mu(E) = 0$.

(*) $\mu([a,b]) = l([a,b]) = b - a$

(*) $\mu(\{a\}) = 0$

(*) Ex:

$$f(x) = \begin{cases} x & : x \in [0,\infty) \\ 1 & : x \in (-\infty, 0) \end{cases}$$



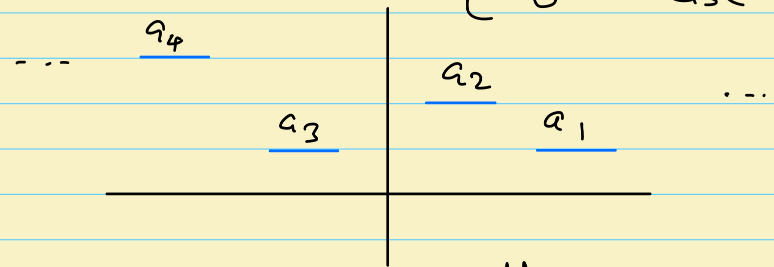
at $x = 0$ f is not continuous

$\mu(\{0\}) = 0$

$$\left\{ \begin{array}{l} \text{Cont. on } \mathbb{R} \setminus \{0\} \\ \text{Cont. a.e.} \end{array} \right.$$

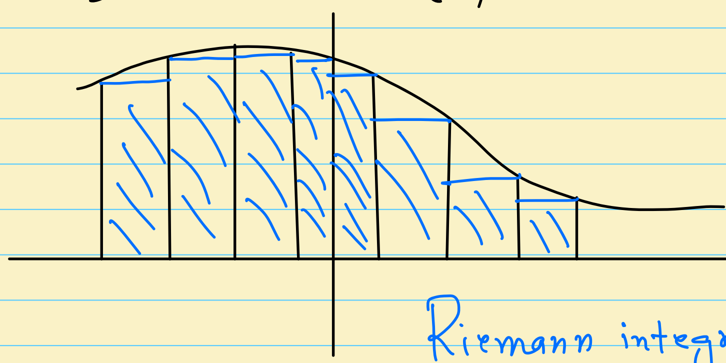
Def: A function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is $\mathcal{B}$ - measurable if
$$f^{-1}(\text{open set}) \in \mathcal{B}$$

Def: f : simple if $f = \begin{cases} a_i \;; & x \in E_i \,, \quad i = 1, 2, \cdots, N \\ 0 \;: & \text{else} \end{cases}$



$$\Downarrow$$

f : simple
$$\int f \, d\mu = \sum_{i=1}^{N} a_i \, \mu(E_i)$$



Riemann integral.

Def: f : measurable function ; the $\{f_n\}$ simple function $f_n \to f$, μ - a.e.

Then we say f is integrable
$$\lim_{n \to \infty} \int f_n \, d\mu = C < \infty$$
and it is independent of choice of $\{f_n\}$ and
$$\int f \, d\mu := \lim_{n \to \infty} \int f_n \, d\mu$$

($\int f\,dx$: μ : Lebesgue measure)

(*) Lebesgue Dominated Convergence Theorem (LDCT)

$f_i \to f$, f_i : integrable

g : integrable

$|f_i| \leq g$, then

f is integrable and

$$\int f_i \, d\mu \xrightarrow[i\to\infty]{} \int f\,d\mu$$

(*) Monotone Convergence Thm (MCT) :

$\{f_i\}$: integrable $\quad 0 \leq f_i \nearrow f$ a.e.

If $\int f_i \, d\mu \leq c < \infty$, then

$$\int f_i \, d\mu \xrightarrow[i\to\infty]{} \int f\,d\mu \leq c$$

↑ integrable

(*) Fatou's Lemma

$\{f_i\}$ integrable, $f_i \geq 0$

and $\int f_i \, d\mu \leq c < \infty$

and $\int (\liminf f_i)\, d\mu < \infty$

then

$$\int (\liminf f_i)\, d\mu \leq \liminf \int f_i \, d\mu.$$

(*) Fubini's theorem:

If $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, measurable

such that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x,y)| \, dx \, dy < \infty$$

then:

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(x,y) \, dx \, dy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y) \, dx \right) dy$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y) \, dy \right) dx$$