LEC-03 FUNCTION SPACES

Ex (i) $X = C^0(\Omega)$

$$\|f\|_{C^0(\Omega)} := \sup_{x \in \Omega} |f(x)|$$

(ii) $X = C^k(\Omega)$

$$= \{ D^\alpha f \in C^0(\Omega) \, , \, |\alpha| \leq k \}$$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\alpha = \langle \alpha_1 , \alpha_2 , \ldots , \alpha_n \rangle$$

$$D^\alpha = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_n}^{\alpha_n}$$

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

$$\|f\|_{C^k(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{C^0(\Omega)}$$

Ex: Hölder Spaces:

$C^{0,\gamma}(\Omega)$, $\gamma \in [0,1]$

$$C^{0,\gamma}(\Omega) = \{ f : \Omega \to \mathbb{R} \, ; \, |f(y) - f(x)| \leq C |x-y|^\gamma, \ \forall x, y \in \Omega \}$$

Hölder subnorm:

$$[f]_{C^{0,\gamma}} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(y) - f(x)|}{|x-y|^\gamma}$$

$$\|f\|_{C^{0,\gamma}} = \|f\|_{C^0(\Omega)} + [f]_{C^{0,\gamma}}$$

In general,

$$C^{k,\gamma}(\Omega) = \{ f : D^\alpha f \in C^{0,\gamma},\ |\alpha| \leq k \}$$

<u>Hölder Sub-norm:</u>

$$[f]_{C^{k,\gamma}(\Omega)} = \sum_{|\alpha| \leq k} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|D^\alpha f(y) - D^\alpha f(x)|}{|x-y|^\gamma}$$

$$\|f\|_{C^{k,\gamma}(\Omega)} = \|f\|_{C^k(\Omega)} + [f]_{C^{k,\gamma}(\Omega)}$$

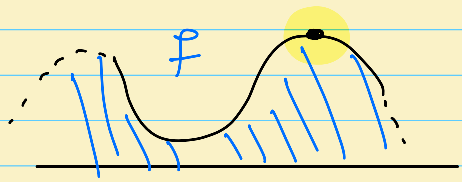
Fact: $C^0(\Omega)$, $C^k(\Omega)$, $C^{0,\gamma}(\Omega)$, $C^{k,\gamma}(\Omega)$: Banach spaces.

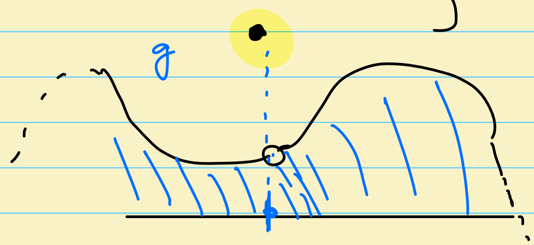
Ex: <u>L^p-spaces</u> $(1 \leq p \leq \infty)$

$1 \leq p < \infty$: $\mathcal{L}^p(E) = \{ f : E \to \mathbb{R} : \int_E |f|^p d\mu < \infty \}$

$p = \infty$: $\mathcal{L}^p(E) = \{ f : E \to \mathbb{R} : \operatorname{ess\,sup}_E |f| < \infty \}$

$$\operatorname{ess\,sup}_E |f| = \inf \{ K \geq 0 : |f| \leq K \text{ a.e.} \}$$





$f = g$ a.e. $\Rightarrow \int f \, d\mu = \int g \, d\mu$

$$L^p(E) = \mathcal{L}^p(E)/\sim \qquad 1 \leq p \leq \infty$$

$f \in L^p(E) \Rightarrow f$ is class of functions $\int |f|^p d\mu < \infty$

(*) The Dual spaces of $L^p(E)$:

$(L^p)^* =$ functionals on $L^p(E)$

one example:

$$T_g : L^p \longrightarrow \mathbb{R}$$

$$T_g f = \int \underset{\in L^p}{f} \underset{\in L^{p'}}{g} \, d\mu$$

$$\frac{1}{p} + \frac{1}{p'} = 1$$

$$\|T_g\| = \sup_{\substack{f \neq 0 \\ f \in L^p}} \frac{|T_g(f)|}{\|f\|_{L^p}} \overset{\text{Hölder's ineq.}}{\leq} \frac{\cancel{\|f\|_{L^p}} \|g\|_{L^{p'}}}{\cancel{\|f\|_{L^p}}} = \|g\|_{L^{p'}}$$

Thm: $J : L^{p'} \longrightarrow (L^p)^*$ $\quad 1 \leq p < \infty$;

$J(g) = T_g$

then J: onto and $\|J(g)\| = \|T_g\|$

	Dual space $(L^p)^*$	Separable	Reflexive
L^p, $1<p<\infty$	$L^{p'}$	Yes	Yes
L^1	L^∞	Yes	NO!
L^∞	Bigger than L^1	NO	NO!

(*) **Hölder's Inequality:**

$f \in L^p(E)$, $g \in L^{p'}(E)$, $\frac{1}{p} + \frac{1}{p'} = 1$, then

i) $fg \in L^1(E)$

ii) $\|fg\|_{L^1(E)} \leq \|f\|_{L^p(E)} \|g\|_{L^{p'}(E)}$

(*) **Young's inequality:**

$a, b \in \mathbb{R}$

$$\Rightarrow \quad ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

(*) **Minkowski's inequality:**

$$1 \leq p \leq \infty \Rightarrow \|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Recall:

$$u * v(x) = \int_{\mathbb{R}^n} u(x-y)\, v(y)\, dy$$

Proposition:

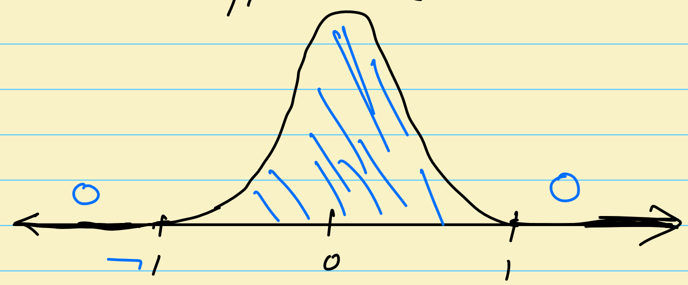
(i) Let $u, v \in L^1(\mathbb{R}^n)$, then $u * v \in L^1(\mathbb{R}^n)$ and $\|u * v\|_{L^1(\mathbb{R}^n)} \leq \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^1(\mathbb{R}^n)}$

(ii) Let $u \in L^1(\mathbb{R}^n)$, $v \in L^p(\mathbb{R}^n)$

$$\Rightarrow \|u * v\|_{L^p(\mathbb{R}^n)} \leq \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^p(\mathbb{R}^n)}$$

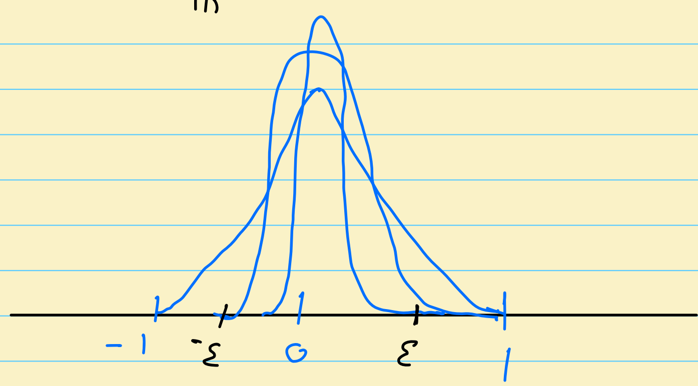
Def: Let $\phi \in C_c^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$

with support $B(0,1)$



$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\,\phi\left(\frac{x}{\varepsilon}\right)$$

$$\Rightarrow \begin{cases} \text{(i)} & \phi_\varepsilon(x) \in L^1 \\ \text{(ii)} & \int_{\mathbb{R}^n} \phi_\varepsilon(x)\,dx = 1 \end{cases}$$



$B(0,\varepsilon)$

$\{\phi_\varepsilon\}$: Called a sequence of mollifiers.

(*) $\quad \phi_\varepsilon(x) \longrightarrow 0 \quad$ as $\varepsilon \to 0$ a.e.