LEC-04 WEAK DIFFERENTIABILITY

Def: $\Omega \subseteq \mathbb{R}^n$, $f \in L^1(\Omega)$ $\left(\int_\Omega |f|^1 \, dx < \infty \right)$.

We say f is weakly differentiable if $\exists$ $\{g_1, \dots, g_n\} \in L^1(\Omega)$ s.t.

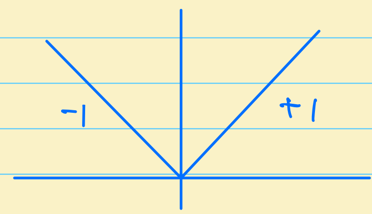
$\forall \phi \in C_c^\infty(\Omega)$:

$$\int_\Omega f \frac{\partial \phi}{\partial x_i} \, dx = - \int_\Omega g_i \phi \, dx$$

$D_i f = g_i$ weak - derivative of f

$Df = \langle D_1 f, \dots, D_n f \rangle = \langle g_1, g_2, \dots, g_n \rangle$

Ex: $f(x) = |x| \in L^1([-1,1])$



$$\int_{-1}^{1} f \phi' \, dx = \int_{-1}^{1} |x| \phi' \, dx$$

$$= \int_{-1}^{0} (-x) \phi' \, dx + \int_0^1 x \phi' \, dx$$

$\phi \in C_c^\infty([-1,1])$

$\boxed{\phi(-1) = \phi(1) = 0}$

IBP

$$= \cancel{-x\phi\Big|_{-1}^{0}} - \int_{-1}^{0} (-1)\phi \, dx + \cancel{x\phi\Big|_0^1} - \int_0^1 1 \phi \, dx$$

$$= - \int_{-1}^{0} (-1) \phi \, dx - \int_0^1 1 \phi \, dx$$

$$= - \int_{-1}^{1} g \phi \, dx$$

$$g = \begin{cases} -1, & -1 \leq x < 0 \\ c, & x = 0 \\ 1, & 0 < x \leq 1 \end{cases}$$

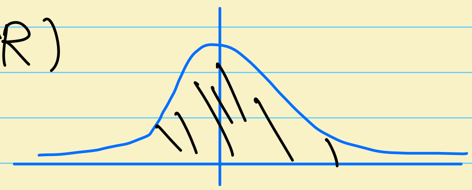
$\in L^1([-1,1])$

Ex: $\underline{\underline{2}}$ $f(x) = \text{sgn}(x)$ on $[-1,1]$

Find g, $\forall\ \phi \in C_c^\infty([-1,1])$:

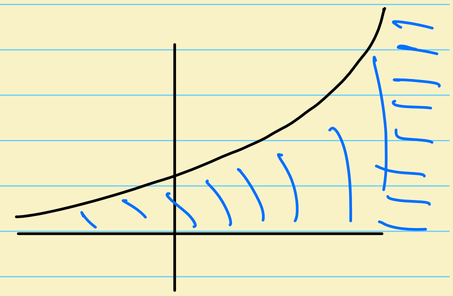
$$\int_{-1}^{1} f \phi' \, dx = -\int_{-1}^{1} g \phi \, dx, \quad g \in L^1([-1,1])$$

(i) $f_1(x) = e^{-x^2} \in L^1(\mathbb{R})$



$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} < \infty$$

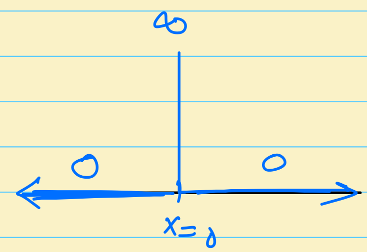
(ii) $f_2(x) = e^{x} \notin L^1(\mathbb{R})$



$$\int_{-\infty}^{\infty} |e^x|^1 dx = e^x \Big|_{-\infty}^{\infty} = \infty$$

$f_2 \in L^1((-\infty, 0])$

(*) <u>Delta functional</u>:



$$\underline{\delta(x)} = \begin{cases} 0, & x \neq 0 \\ \text{"}\infty\text{"}, & x = 0 \end{cases}$$

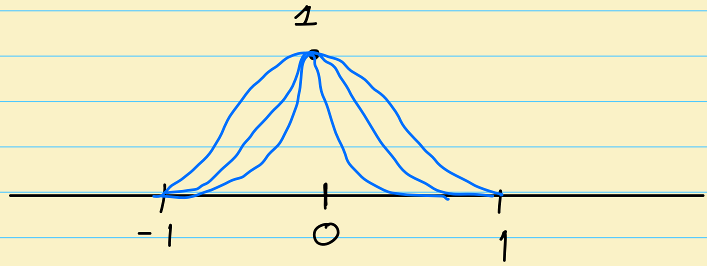
→ ✓ (i) $\int_{-\infty}^{\infty} |\delta(x)|^1 dx = 1 < \infty$ (?)

✓ (ii) $\langle \delta(x), f(x) \rangle = \int_{-\infty}^{\infty} \delta(x) f(x) \, dx = f(0)$

(iii) $\delta \notin L^1([-1,1])$ (Claim)

Proof: Suppose $\delta \in L^1([-1,1])$

Take $\phi_j \in C_c^\infty([-1,1])$, $\phi_j(0) = 1$, $0 \leq \phi_j \leq 1$



$$f_j = \delta(x)\,\phi_j(x), \qquad |f_j| \leq |\underbrace{\delta(x)}_{>0}|\,|\underbrace{\phi_j(x)}_{\leq 1}|$$

$$f_j \xrightarrow{a.e.} 0 \qquad \leq \delta(x) \text{ integrable}$$

"LDCT"

$$1 = \phi_j(0) = \int \delta(x)\,\phi_j(x)\,dx = \int f_j\,\cancel{dx} \xrightarrow[j\to\infty]{} \int 0\,dx = 0$$

$$1 = 0 \qquad \text{Contradiction!}$$

Sol: of Ex 2:

$$\int_{-1}^{1} \mathrm{sgn}(x)\,\phi'(x)\,dx = \int_{-1}^{0} (-1)\,\phi'(x)\,dx + \int_0^1 1\,\phi'(x)\,dx$$

$$= -[\phi(0) - \cancel{\phi(-1)}^{\,0}] + [\cancel{\phi(1)}^{\,0} - \phi(0)]$$

$$= -\phi(0) - \phi(0) = -\boxed{2\phi(0)}$$

$$= -\int_{-1}^{1} 2\delta(x)\,\phi(x)\,dx, \qquad 2\delta(x) \in L^1([-1,1])$$

$$\boxed{g = 2\delta(x)}$$