LEC-06 Test functions, Distributions and Distributional Derivative

Def Test function $D(\Omega)$:

Ω: bd, non-empty set in $\mathbb{R}^n$

$D(\Omega) = \{ f \in C_c^\infty(\Omega),\ supp(f) \subset K \subset\subset \Omega \}$

$K \subset\subset \Omega$ < $K \subseteq \Omega$; K is compact!

We say: $\{\varphi_n\} \subseteq D(\Omega)$ converges to φ

if
1. $supp(\varphi_n) \subseteq K \subset\subset \Omega, \ \forall n \in \mathbb{N}$
2. $\varphi_n \xrightarrow{u} \varphi$ in Ω
3. $D^\alpha \varphi_n \xrightarrow{u} D^\alpha \varphi$ in Ω.

Def: Distributions $D'(\Omega)$:

$D'(\Omega) = \{$ linear functionals on $D(\Omega)$, cont. in sense:

- if $\varphi_n \to \varphi$ in $D(\Omega)$, then $\langle f, \varphi_n \rangle \to \langle f, \varphi \rangle$ $\forall f \in D'(\Omega) \}$

Ex: $\delta_0(x)$ is not a function!

$\delta_0(x) \notin L^p((-1,1))$, $\forall p \in [1,\infty]$.

$\langle \delta_0(x), 1 \rangle = \int \delta_0(x) \cdot 1 \, dx = 1 \neq \|\delta_0\|_{L^1}$

$\delta_0(x) \in D'(\Omega)$.

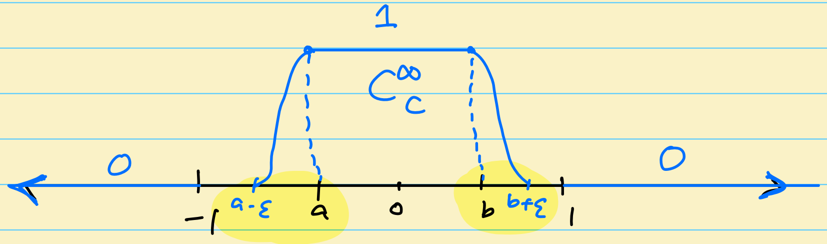
- $\langle \delta_0(x), \phi(x) \rangle = \phi(0)$, $\forall \phi \in D(\Omega)$. ✓
- $\langle \delta_0(x), c_1\varphi_1 + c_2\varphi_2 \rangle = c_1\varphi_1(0) + c_2\varphi_2(0)$ linear!
- $\varphi_n \to \varphi$ in $D(\Omega)$

$\langle \delta_0(x), \varphi_n(x) \rangle = \varphi_n(0) \longrightarrow \varphi(0) = \langle \delta_0(x), \varphi(x) \rangle$

Ex: $\{\phi_\varepsilon(x)\} \subseteq D(\Omega)$

$\phi_\varepsilon \to \delta_0(x) \notin D(\Omega)$ (but $\delta(x) \in D'(\Omega)$)

$$\phi_\varepsilon(x) = \begin{cases} 1 & x \in [a,b] \subseteq [-1,1] \\ 0 & x \in [-1,1]^c \end{cases}, \in D(\Omega)$$



as $\varepsilon \to 0$

$\phi_\varepsilon \not\to D(\Omega)$

Lemma: $f \in D'(\Omega)$, $\forall K \subset\subset \Omega$, $\exists\, C_K$ and

$n = n(K) \in \mathbb{N}$ s.t.

$$|\langle f, \phi \rangle| \leq C_K \sum_{|\alpha| \leq n(K)} \max_{x \in K} |D^\alpha \phi|$$

$\phi \in D(\Omega)$

$supp(\phi) \subseteq K$

if $\exists\, n \in \mathbb{N}$, independent of K

$\Downarrow$

smallest possible n is called order of distribution.

Ex: $f \in C^0(\bar{\Omega}) \subseteq D'(\Omega)$, Ω: bd

$\langle f, \phi \rangle = \int f \phi\, dx$, $\forall \phi \in D(\Omega)$

- linear + well-defined.

$\cdot \quad \varphi_n \to \varphi$ in $D(\Omega)$

$$\langle f,\varphi_n\rangle = \int f\varphi_n\,dx \xrightarrow{L.D.C.T} \int f\varphi\,dx = \langle f,\varphi\rangle$$

$$\cdot \ |\langle f,\varphi\rangle| = \left|\int_\Omega f\varphi\,dx\right| \leq |\Omega| \max_{x\in\bar\Omega}|f| \max_{x\in\bar\Omega}|\varphi|$$

$$\leq K \max_{x\in\bar\Omega}|D^0\varphi|$$

order zero!

Ex: $f = \delta_0(x) \in D'(\Omega)$

$$|\langle \delta_0(x),\varphi\rangle| = |\varphi(0)| \leq \max_{x\in K}|D^0\varphi|$$

Ex: $f \in D'(\Omega)$

$$\langle f,\varphi\rangle = -\varphi'(0)$$

$$|\langle f,\varphi\rangle| = |\varphi'(0)| \leq \max_{x\in K}|D^1\varphi| \leftarrow \text{Order: } 1.$$

Def: Convergence in $D'(\Omega)$:

$\{f_n\} \in D'(\Omega)$, we say $f_n \to f \in D'(\Omega)$

if $\forall \varphi \in D(\Omega)$

$$\langle f_n,\varphi\rangle \longrightarrow \langle f,\varphi\rangle \quad \text{(weak-* Convergence)}$$

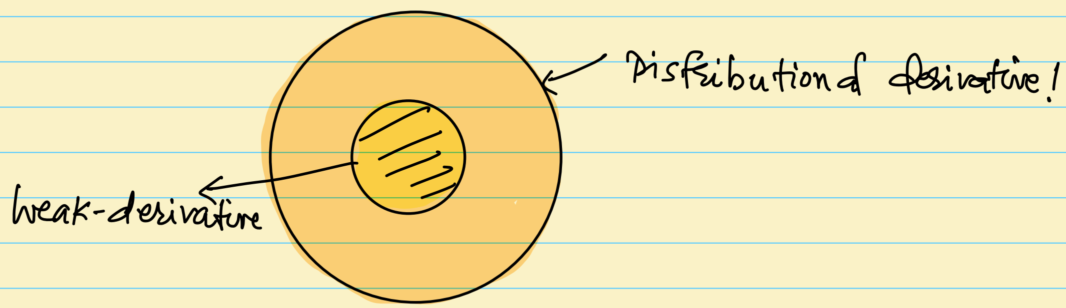
Def: <u>Distributional Derivative:</u>

$$f \in \mathcal{D}'(\Omega), \quad \Omega \subseteq \mathbb{R}^n$$

(*) $$\left\langle \frac{\partial f}{\partial x_i}, \varphi \right\rangle = - \left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle, \; \forall \varphi \in \mathcal{D}(\Omega)$$

first order derivative of f in the sense of distribution!

Rm: If $f \in L^1$ and weak derivative of f exists
$\Rightarrow$ its the derivative of f in the sense of distribution!



In general,

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle, \; \forall \varphi \in \mathcal{D}(\Omega)$$

Rm: $\{f_n\} \in \mathcal{D}'(\Omega)$ s.t. $f_n \longrightarrow f$ in $\mathcal{D}'(\Omega)$
then $D^\alpha f_n$ exists.

$\forall \varphi \in \mathcal{D}(\Omega)$

$$\langle D^\alpha f_n, \varphi \rangle = (-1)^{|\alpha|} \langle f_n, D^\alpha \varphi \rangle$$

$$= (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle \qquad [f_n \to f \text{ in } \mathcal{D}'(\Omega)]$$

$$= (-1)^{|\alpha|} (-1)^{|\alpha|} \langle D^\alpha f, \varphi \rangle = \langle D^\alpha f, \varphi \rangle.$$

(*) If $f_n \to f$ in $\mathcal{D}'(\Omega)$ $\Rightarrow$ $D^\alpha f_n \to D^\alpha f$ in $\mathcal{D}'(\Omega)$.

<u>Ex:</u> $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ $\notin L^1(\mathbb{R})$

$\in \mathcal{D}'(\Omega)$

$\forall \phi \in \mathcal{D}(\Omega)$

$$\left\langle H(x), \frac{\partial \phi}{\partial x} \right\rangle = \int_{-\infty}^{\infty} H(x) \frac{\partial \phi}{\partial x} dx$$

$$= \int_{0}^{\infty} \frac{\partial \phi}{\partial x} dx$$

$$= \phi(x) \Big|_{0}^{\infty}$$

$$= 0 - \phi(0)$$

$$= - \langle \delta_0(x), \phi(x) \rangle$$

$$\boxed{\frac{dH}{dx} = \delta_0(x) \in \mathcal{D}'(\Omega)}$$