Theory of Partial Differential Equations
Homework-1

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1. Prove Hölder's and Minkowski's inequalities for L^p spaces.

2. - Let Ω be a bounded subset of $\mathbb{R}^n$, Show that $L^p(\Omega) \subset L^q(\Omega)$, for $p > q$.
   - Show by example that $L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$.

3. Let Ω be a bounded open set in $\mathbb{R}^n, p > q \geq 1$. Prove that $L^p(\Omega)$ is continuously embedded in $L^q(\Omega)$. By this we mean that $L^p(\Omega) \subset L^q(\Omega)$, and if $u_n \to u$ in $L^p(\Omega)$ then $u_n \to u$ in $L^q(\Omega)$, and is usually denoted as $L^p(\Omega) \hookrightarrow L^q(\Omega)$.

4. Consider $f_n(x) = 1 + \sin(nx)$ on $(-1, 1)$. Find the following:
   - the pointwise limit in $(-1, 1)$,
   - the weak limit in L^1,
   - is the convergence in part (b) strong in L^1 ? (d) what is the limit of $\|f_n\|_{L^1}$ in part (b)?

5. Let $\{\phi_k\}$ be a sequence of mollifiers in $\mathbb{R}^n$ and let $f \in L^p(\mathbb{R}^n)$, give a complete proof of the following:
   - $\phi_k * f \longrightarrow f$ in $L^p(\mathbb{R}^n)$.
   - $\|\phi_k * f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$, for any k.

6. Let $f(x, y) = u(x) + v(y)$. Prove that the weak derivative $D_x f$ exists if and only if the weak derivative $D_x u$ exists and $D_x f = D_x u$.