Lec-01

Major Results in Functional Analysis

X : Topological vector space (TVS) over $\mathbb{R}$ ($\mathbb{C}$)

Def: A norm on TVS is a mapping
$\|\cdot\| : X \longrightarrow \mathbb{R}$, such that

(i) $\|x\| \geq 0$, $\forall x \in X$

(ii) $\|\alpha x\| = |\alpha| \|x\|$, $\alpha \in \mathbb{R}$, $x \in X$

(iii) $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$

Def: Normed vector space (NLS) is
a TVS with a norm $\|\cdot\|$. $(X, \|\cdot\|)$ – linear

Def: Given X : NLS, define a mapping
$\rho : X \times X \longrightarrow \mathbb{R}$, $\rho(x,y) = \|x-y\|$
$\rightarrow$ metric $\rightarrow$ open balls $\rightarrow$ open sets.

Def: Banach Space: Complete NLS. $(X, \|\cdot\|)$
(BS)
Contains all its limit points.

if $\{x_n\} \subseteq X$, $x_n \to x \Rightarrow x \in X$

- We say BS is separable if it contains a countable dense set.

$\mathbb{R} \supseteq \mathbb{Q} \longrightarrow$ dense in $\mathbb{R}$
$\mathbb{Q}$ is countable.

Ex: ✓ $X = C^0([-1,1]) = \{ f : [-1,1] \to \mathbb{R}, f \text{ is cont.} \}$

$$\|f\| = \max_{x \in [-1,1]} |f(x)|$$

$$\{f_n\} \subseteq X, \quad f_n \xrightarrow[n\to\infty]{} f$$

$$\Leftrightarrow \|f_n - f\| \xrightarrow[n\to\infty]{} 0$$

$$\Leftrightarrow \max_{x\in[-1,1]} \|f_n(x) - f(x)\| \xrightarrow[n\to\infty]{} 0$$

$$\Leftrightarrow f_n \xrightarrow{u} f$$

$\Rightarrow$ f : Continous $\Rightarrow f \in X \Rightarrow X$ is a B.S.

Ex: ✗ $X = \{ p : [-1,1] \longrightarrow \mathbb{R}, \ p \text{ is a polynomial} \}$

$\{p_n\}$, $p_n \xrightarrow{u} f$ is f a polynomial ?

$$p_0 = 1, \quad p_1 = 1+x, \quad p_2 = 1 + x + \frac{x^2}{2!}$$

$$p_n = \sum_{i=1}^{n} \frac{x^i}{i!}$$

$$\lim_{n\to\infty} p_n = \sum_{i=1}^{\infty} \frac{x^i}{i!} = e^x \notin X$$

Ex: $\ell^p_{1 \le p < \infty} = \{ \{a\}_{n=1}^{\infty} = (a_1, a_2, \dots) : \sum_{i=1}^{\infty} |a_i| < \infty \}$

$$\| \{a\}_{n=1}^{\infty} \|_{\ell^p} = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p}$$

ℓ^p is BS

$$\{a_j\}_{n=1}^{\infty} \longrightarrow \{\tilde{a}\}_{n=1}^{\infty}$$

$$\sum_{n=1}^{\infty} |\tilde{a}|^p \leq \underbrace{\sum_{n=1}^{\infty} |\tilde{a} - a_N|^p}_{< \varepsilon} + \underbrace{\sum_{n=1}^{\infty} |a_N|^p}_{< \infty}, \quad N \in \mathbb{N}$$

$< \infty$

$\Rightarrow \{\tilde{a}\}_{n=1}^{\infty} \in X \quad \Rightarrow \quad \ell^p$ is a BS $(1 \leq p < \infty)$.

Def: X, Y : NLS. $T : X \to Y$ is called a contraction map if $\exists\ 0 \leq \theta < 1$ s.t.

$$\|Tx - Ty\|_Y \leq \theta \|x - y\|_X, \quad x, y \in X$$

Def: **Banach fixed Point Theorem:**

X : B.S, $T : X \circlearrowleft$ a contraction map,

$f(x) = x$

then T has a unique fixed point $x^* \in X$

$$Tx^* = x^*$$

Furthermore, $\forall\ y \in X$

$$\lim_{k \to \infty} T^{(k)} y = \lim_{k \to \infty} T(T - - - T(T(y))) = x^*, \quad \forall\ y \in X$$

Def: X, Y NLS. $T : X \to Y$ is bounded if

$$\|T\| := \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X} < \infty$$

$$\|Tx\|_Y \leq C_0 \|x\|_X, \quad C_0 \geq \|T\|$$

Thm: X: NLS X: finite dimensional, then:

T linear $\Leftrightarrow$ T continuous

X: NLS X: infinite dimensional, then:

T linear + bounded $\Leftrightarrow$ T continuous

Def: Dual Spaces:

X: NLS, we define the dual space of X, X^* is the set of all linear + bounded functionals on X

$$X^* = \{ f: X \to \mathbb{R} \,;\, f: \text{linear} + \text{bounded} \}$$

$$f \in X^*, \quad \|f\|_{X^*} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|_X}$$

$$\forall x \in X, \quad f(x) = \underbrace{\langle f, x \rangle}_{\text{action of } f \text{ on } x.} \in \mathbb{R}$$

Thm: X^* is a Banach space.

Def: X, Y are BS, $T: X \to Y$ (linear), the adjoint operator

$T^* = Y^* \longrightarrow X^*$ such that:

$\forall \boxed{g \in Y^*}$, $x \in X$:

$$\langle \underbrace{T^* g}_{X^*}, \underbrace{x}_{X} \rangle = \langle \underbrace{g}_{Y^*}, \underbrace{Tx}_{Y} \rangle$$

Ex: $\Delta : C^2_{per} \longrightarrow C^0_{per}$

$\langle f, g \rangle = \int fg \, dx$

$u, v \in C^2_{per}$

$\langle \Delta u, v \rangle = \int \Delta u \, v \, dx = \int u \, \Delta v \, dx = \langle u, \Delta v \rangle$

Δ^*

$\Delta : C^2_{per}$ is a self-adjoint operator

$$\Delta^* = \Delta .$$

Def: X : BS, X^* : dual space

(a) Strong convergence:

$\{x_n\} \subseteq X$, we say $x_n \to x$ (strongly)

if $\quad \|x_n - x\|_X \xrightarrow[n\to\infty]{} 0$

$$\not\Uparrow \quad \Downarrow$$

$$\|x_n\|_X \longrightarrow \|x\|_X$$

(b) Weak - Convergence

We say $x_n \xrightarrow{w} x$, if

$\forall y \in X^*$: $\langle y, x_n \rangle \xrightarrow{n\to\infty} \langle y, x \rangle$

$y(x_n) \xrightarrow{n\to\infty} y(x)$

(c) **Weak-* Convergence**

$\{y_n\} \in X^*$, we say $y_n \xrightarrow[n\to\infty]{w^*} y$ if:

$\forall x \in X \quad \langle y_n, x\rangle \xrightarrow{n\to\infty} \langle y, x\rangle$

$$y_n(x) \xrightarrow{n\to\infty} y(x)$$

Def: X: linear space, a map $(\cdot,\cdot): X \times X \to \mathbb{R}$ is a inner product if:

1. $(x,y) = (y,x) \quad \forall\ x, y \in X$
2. $(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 (x_1, y) + \lambda_2 (x_2, y)$ $\quad \forall\ \lambda_1, \lambda_2 \in \mathbb{R}$
3. $(x,x) \geq 0$

Def: X: LS with a scalar product is called an inner product space (IPS) with respect to $\|\cdot\|$, $\forall x \in X \quad \|\cdot\| = \sqrt{(x,x)}$

* Complete IPS are called Hilbert space.

We have:

(1) $|(x,y)| \leq \|x\|\,\|y\|$

(2) $\|x+y\| \leq \|x\| + \|y\|$

(3) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$