CSC 244/444

Assignment 1 Solutions

1. Human Knowledge and Reasoning

   (No 'gold standard' answer for these.)

2. Logical Syntax

   (i) No: this is not well-formed. The BNF production with '=' requires two terms on each side, but $P(A)$ is a formula, not a term.

   (ii) No: this is not well-formed. The BNF production with '=' requires two terms on each side, but $(P(A) \wedge P(A))$ is a formula, not a term.

   (iii) No: this is not well-formed. A predicate can only have terms as arguments, but $Hillary \wedge Bill$ is not a term ($Hillary \wedge Bill$ itself is not well-formed either, since binary connectives require two formulas on each side, and a formula cannot consist of a single term.

   (iv) Yes: this is well-formed.

   (v) Yes: this is well-formed. Although it seems weird to us, nothing about the BNF grammar says that nested quantifiers cannot use the same variable, syntactically speaking.

   (vi) No: this is not well-formed. The function 'f' does not have consistent arity, as both $f(A)$ and $f(A, B)$ show up.

   (vii) Yes: this is well-formed (although itś not what we would typically call a *sentence*, which is reserved for *closed* well-formed formulas).

   (viii) No: this is not well-formed. As per the BNF grammar, one of the nested formula expressions should be wrapped with outer parentheses to avoid ambiguity.

3. From English to Logic

   (a) (i)

$$\exists x.Ocean(x) \wedge Beneath(x, \text{surface-of}(Europa))$$

   (ii)

$$\forall x.Planet(x) \Rightarrow \exists y.Star(y) \wedge Orbits(x, y)$$

   (iii)

$$\forall x.Dromedary(x) \Rightarrow \exists y.Hump(y) \wedge \textit{Has-as-part}(x, y) \wedge \forall z.(Hump(z) \wedge \textit{Has-as-part}(x, z)) \Rightarrow z = y$$

(iv)

$$\begin{aligned}\forall x.Elephant(x) \Rightarrow \exists t1, t2.Tusk(t1) \wedge \textit{Has-as-part}(x, t1) \wedge \\ Tusk(t2) \wedge \textit{Has-as-part}(x, t2) \wedge \neg(t1 = t2) \wedge \\ \forall z.(Tusk(z) \wedge \textit{Has-as-part}(x, z)) \Rightarrow ((z = t1) \vee (z = t2))\end{aligned}$$

(v)

$$\begin{aligned}\forall g.\ \textit{Undir-Graph}(g) \Rightarrow \textit{Strongly-Connected}(g) \Leftrightarrow \\ \forall u, v.(\textit{Vertex-of}(u, g) \wedge \textit{Vertex-of}(v, g) \wedge \neg(u = v)) \Rightarrow \\ \exists e.\ \textit{Edge-of}(e, g) \wedge Joins(e, u, v)\end{aligned}$$

(vi)

$$\exists x, e.Person(x) \wedge \textit{Fly-to}(x, Mars, e) \wedge After(e, \textit{Sept-27-19})$$

(vii)

$$\begin{aligned}\forall x.Person(x) \Rightarrow \forall y.Ancestor(x, y) \Leftrightarrow \\ (Parent(x, y) \vee (\exists z.Parent(x, z) \wedge Ancestor(z, y)))\end{aligned}$$

(viii)

$$\exists x.Person(x) \wedge \forall y, e.Politician(y) \Rightarrow Fool(y, x, e)$$

$$\forall y, e.Politician(x) \Rightarrow \exists x.Person(y) \wedge Fool(y, x, e)$$

(b) a. The difficulty here is the quantifier "few", which does not have a well-defined FOL equivalent and cannot be written in terms of $\exists$ and $\forall$. Hence, we would require some sort of generalized quantifier to be able to fully capture the meaning of "few" (If you discussed this in some way, you got full credit). To do justice to "few", one approach is to modify the syntax of FOL to use restricted quantification: $[\mathfrak{F}x : Dog(x)].Vicious(x)$. Here, the formula following the colon is a *restriction* on the quantifier, and is intended to restrict x to only range over those elements of the domain which are in the extension of Dog. A first stab at the semantics for this example, then, might be:

$$\begin{aligned}&\vDash_M [\mathfrak{F}x : Dog(x)].Vicious(x)) \\ &\quad \text{iff } card(Dog^I \cap Vicious^I) < card(Dog^I - Vicious^I)\end{aligned}$$

That is, the cardinality of the set of individuals that are both Dog and Vicious is less than the cardinality of the set of individuals that are Dog and not Vicious. Note that this satisfaction condition is defined as a binary relation over the restrictor and the scope of the quantifier.

b. This can be done in standard FOL including a time-period as an argument to Visited (similar to (vi) in the previous section), and using 10 existentially quantified time-periods/events, including predicates that assert none of these time-periods are not self-identical to each other. However, such a formula would be rather ungainly. Ideally we would want to extend FOL using generalized quantifiers to allow quantification over cardinalities, e.g. $\exists_{=10} e.Visit(Jack, India, e)$.

c. Syntactically, representing this sentence in FOL seems to force us to nest a predicate inside a predicate (e.g. $Suspects(Mary, Loves(John, Mary))$, which is not well-formed. Semantically, 'Suspects' is really a modal operator, i.e. it may be the case that $\not\models Loves(John, Mary)$, yet simultaneously $\models Suspects(Mary, Loves(John, Mary))$ (ignoring for a moment the syntax issue). This type of sentence has lead to a branch of modal logic which represents belief/suspicion through use of a modal operator, e.g. $\mathcal{B}_{Mary} Loves(John, Mary)$, where the semantics of the modal operator are expressed in terms of "possible worlds" and accessibility relations between them.

d. One might naively express this sentence as $\forall x.Red(x) \wedge Hair(x) \Rightarrow$ *Copper-Colored*(x). However, this doesn't seem to capture the meaning of the sentence: the above FOL says that anything which is in both the set of things that are red and the set of things that are hair is copper-colored, but the sentence appears to be talking about hair which does not actually belong to the set of things that are red (hence "actually"). In this case, we would need to define an extension of FOL which allows *predicate modifiers*, where for instance the semantic interpretation of "Red" maps the interpretation of "Hair" (a set) to some other set (in this case a subset of the interpretation of "Hair").

e. A way of expressing this sentence might be $\exists x, e.Colloquium(x) \wedge Cancel(x, e) \wedge Before(e, Now1)$. However, the issue with this is that some individual in the domain ought not to be in the set denoted by Colloquium if it was cancelled (that is, a cancelled colloquium is not in fact a colloquium). So we would need some sort of intensional modifier, true of mental/imaginary entities, to be able to talk about entities like "the expected colloquium" for instance.

f. Expressing this as $\forall x.Mosquito(x) \Rightarrow Widespread(x)$ would be incorrect, since itś not the case that every individual mosquito is widespread. Rather, this sentence appears to be making a proposition about mosquitos as a collective class. One might address this by extending FOL with a "kind" operator, e.g. $Widespread(k : Mosquito(x)$, which takes a unary predicate such as Mosquito (in this case with a free variable, although the syntax is ad hoc) and maps it to a single individual representing the abstract kind denoted by the predicate.

g. Similarly to the previous example, the issue here is the impossibility of making propositions about *types of actions*, such as copying, in FOL. Ideally, we would want a "kind" operator for actions which maps some binary predicate to an individual in the domain, perhaps similarly to the previous example: $Forbidden(ka : Copy(x, y))$. Again, the syntax here is ad-hoc.

- $\models_M \neg Q(A) \lor \neg Q(C)$

  iff $\not\models_M Q(A)$ or $\not\models_M Q(C)$
  iff $T_I(A) \notin Q^I$ or $T_I(C) \notin Q^I$
  iff $one \notin \{one\}$ or $two \notin \{one\}$, which is true.

- $\models_M \forall x.P(x) \Rightarrow Q(x)$

  iff for all v.a.'s U and all $d \in \mathcal{D}$, $\models_M (P(x) \Rightarrow Q(x))[U_{x:d}]$
  iff for all $d \in \mathcal{D}$, $\not\models_M P(x)[U_{x:d}]$ or $\models_M Q(x)[U_{x:d}]$
  iff for all $d \in \mathcal{D}$, $T_{IU_{x:d}}(x) \notin P^I$ or $T_{IU_{x:d}}(x) \in Q^I$
  iff for all $d \in \mathcal{D}$, $d \notin \{one\}$ or $d \in \{one\}$, which is true.

(b) The smallest possible domain that can make each formula false has 2 elements: $\mathcal{D} = \{one, two\}$. $A^I = one$, $B^I = one$, $C^I = one$, $P^I = \{two\}$, $Q^I = \{one\}$.

- $\not\models_{M'} P(A)$

  iff $T_I(A) \notin P^I$
  iff $one \notin \{two\}$, which is true.

- $\not\models_{M'} \neg Q(B)$

  iff $\models_{M'} Q(B)$
  iff $T_I(B) \in Q^I$
  iff $one \in \{one\}$, which is true.

- $\not\models_{M'} \neg Q(A) \lor \neg Q(C)$

  iff $\models_{M'} Q(A)$ and $\models_{M'} Q(C)$
  iff $T_I(A) \in Q^I$ and $T_I(C) \in Q^I$
  iff $one \in \{one\}$ and $one \in \{one\}$, which is true.

- $\not\models_{M'} \forall x.P(x) \Rightarrow Q(x)$

  iff for all v.a.'s U and all $d \in \mathcal{D}$, $\not\models_{M'} (P(x) \Rightarrow Q(x))[U_{x:d}]$
  iff for some $d \in \mathcal{D}$, $\models_{M'} P(x)[U_{x:d}]$ and $\not\models_{M'} Q(x)[U_{x:d}]$
  iff for some $d \in \mathcal{D}$, $T_{IU_{x:d}}(x) \in P^I$ and $T_{IU_{x:d}}(x) \notin Q^I$
  iff for some $d \in \mathcal{D}$, $d \in \{two\}$ and $d \notin \{one\}$, which is true for $d = two$.

(c) One first-order language is defined by the vocabulary:

$$\Sigma = \{Circle, Triangle, \mathit{Contained\text{-}Within}, \mathit{Left\text{-}Of}, \mathit{Right\text{-}Of}, C1, C2, T1, T2\}$$

A FOL description of the picture in this language might be:

$Circle(C1) \wedge Circle(C2) \wedge Triangle(T1) \wedge Triangle(T2)$
$\wedge$ *Contained-Within*$(T1, C1) \wedge$ *Contained-Within*$(C2, T2)$
$\wedge$ *Left-Of*$(C1, T2) \wedge$ *Left-Of*$(T1, T2) \wedge$ *Left-Of*$(C1, C2) \wedge$ *Left-Of*$(T1, C2)$
$\wedge \forall x, y.$ *Left-Of*$(x, y) \Rightarrow$ *Right-Of*(y, x)

(d) $\models_M (\forall x \phi)[U]$

iff for all $d \in \mathcal{D},\ \models_M \phi[U_{x:d}]$ (by satisfaction conds for '$\forall$')
iff not for some $d \in \mathcal{D},\ \not\models_M \phi[U_{x:d}]$
iff not for some $d \in \mathcal{D},\ \models_M \neg\phi[U_{x:d}]$ (by satisfaction conds for '$\neg$')
iff not $\models_M (\exists x \neg\phi)[U]$ (by satisfaction conds for '$\exists$')
iff $\models_M (\neg\exists x \neg\phi)[U]$ (by satisfaction conds for '$\neg$')

(e) $\models_M \neg(\phi \Rightarrow \psi)[U]$

iff $\not\models_M (\phi \Rightarrow \psi)[U]$ (by satisfaction conds for '$\neg$')
iff $\models_M \phi[U]$ and $\not\models_M \psi[U]$ (by satisfaction conds for ' $\Rightarrow$ ')
iff $\models_M \phi[U]$ and $\models_M \neg\psi[U]$ (by satisfaction conds for '$\neg$')
iff $\models_M (\phi \wedge \neg\psi)[U]$ (by satisfaction conds for ' $\wedge$ ')

(f) $\models_M (x = A \wedge P(x))[U]$

iff $\models_M (x = A)[U]$ and $\models_M P(x)[U]$ (by satisfaction conds for ' $\wedge$ ')
iff $T_{IU}(x) = T_{IU}(A)$ and $T_{IU}(x) \in P^I$ (by satisfaction conds for equality/predicates)
iff $T_{IU}(x) = T_{IU}(A)$ and $T_{IU}(A) \in P^I$ (substitution of equals)
iff $\models_M (x = A)[U]$ and $\models_M P(A)[U]$ (by satisfaction conds for equality/predicates)
iff $\models_M (x = A \wedge P(A))[U]$ (by satisfaction conds for ' $\wedge$ ')

5. Validity and Entailment

(a) (i) Not valid. For a model M, $\models_M Thing(A)$ iff $A^I \in Thing^I$. Clearly, one can create a model where this is not true, such as one in which $Thing^I = \{\}$.

(ii) Valid. For a model M, $\models_M (Zod = Zod)$ iff $Zod^I = Zod^I$, which is clearly true regardless of which M is chosen.

(iii) Valid. For a model M, $\models_M \exists x.x = x$ iff for all v.a.'s U and for some $d \in \mathcal{D}$, $T_{IU_{x:d}}(x) = T_{IU_{x:d}}(x)$, which is true iff for some $d \in \mathcal{D}$, $d = d$. Clearly this is true regardless of which M is chosen.

(iv) Valid. For a model M, $\vDash_M \forall x.Rose(x) \Rightarrow Rose(x)$ iff for all v.a.'s U and for all $d \in \mathcal{D}$, $\vDash_M (Rose(x) \Rightarrow Rose(x))[U_{x:d}]$, which is true iff for all $d \in \mathcal{D}$, $\nvDash_M Rose(x)[U_{x:d}]$ or $\vDash_M Rose(x)[U_{x:d}]$. Clearly this is true regardless of which M is chosen.

(b) (i) This entailment holds. Suppose $\vDash_M \{P(A), A = B\}$. Then, by the truth conditions for each formula, we have $A^I \in P^I$ and $A^I = B^I$. Substituting equal terms, $B^I \in P^I$, therefore $\vDash_M P(B)$. Since every model of $\{P(A), A = B\}$ is a model of $P(B)$, $\{P(A), A = B\} \vDash P(B)$.

(ii) This entailment holds. Suppose $\vDash_M \forall x P(x)$. Then, for all v.a.'s U and for all $d \in \mathcal{D}$, $\vDash_M P(x)[U_{x:d}]$. This is true iff for all $d \in \mathcal{D}$, $T_{IU_{x:d}}(x) \in P^I$, iff for all $d \in \mathcal{D}$, $d \in P^I$. Since $A^I \in \mathcal{D}$, this means that $A^I \in P^I$. Therefore, by the satisfaction condition for predicates, $\vDash_M P(A)$. Hence, $\forall x P(x) \vDash P(A)$.

(iii) This entailment holds. Suppose $\vDash_M P(A)$. Then, $A^I \in P^I$. It follows that for some $d \in \mathcal{D}$, $d \in P^I$. So for all v.a.'s U and for some $d \in \mathcal{D}$, $T_{IUx:d}(x) \in P^I$. Therefore, by the satisfaction condition for existential quantifiers, $\vDash_M \exists x P(x)$. Hence, $P(A) \vDash \exists x P(x)$.

(iv) This entailment holds. We have already proven in Problem 4, part (d) that $\vDash_M (\forall x \phi)[U]$ iff $\vDash_M (\neg \exists x \neg \phi)[U]$. Since this proof shows that any model M of the former must also be a model of the latter, it follows trivially that $\forall x \phi \vDash \neg \exists x \neg \phi$.