General theory of localizations of rings and modules

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Abstract

The aim of the paper is to start to develop the most general theory of localizations/inversion. Several new concepts are introduced and studied.

Key Words: Localizable set, localization of a ring at a localizable set, Goldie's Theorem, absolute quotient ring of a ring, maximal localizable set, maximal left denominator set, the localization radical of a ring, maximal localization of a ring, localizable element, non-localizable element, completely localizable element, direct limit of rings, ultrafiler.

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1 Introduction

Throughout, all rings are associative, with a unit element 1, which is inherited by subrings, preserved by homomorphisms and which acts unitally on modules. Let K be a commutative ring with 1. In the paper, a ring R means a K-algebra with fixed ring homomorphism $\nu_R: K \to R$ where the image $\operatorname{im}(\nu_R)$ belongs to the centre Z(R) of R. A ring homomorphism $f: R \to R'$ is a K-homomorphism, that is $f\nu_R = \nu_{R'}$. Ring homomorphism means a K-homomorphism. For a ring R, let R^{\times} be its group of units and \mathfrak{n}_R be its prime radical.

Inversion of elements in a ring is an important and difficult operation that 'simplifies' the situation as a rule. Ore's method of localization is an example of a theory of one-sided fractions where by definition only the elements of a denominator set can be inverted and the result is a ring of one-sided fractions which always exsists, i.e. it is not equal to zero, [14, 9, 13]. In [1, Theorem 4.15], it is proven that the elements of an arbitrary (left and right) Ore set can be inverted and the result is also a ring of one-sided fractions (which always exists) but in general Ore set is not a denominator set. This was the starting point in [7] and [8] where the most general theory of one-sided fractions was presented. The goal of the paper is to consider the general situation and to start to develop the most general theory of localization/inversion where the result of localization/inversion of elements does not necessarily yields one-sided fractions.

Let $R\langle X_S \rangle$ be a ring freely generated by the ring R and a set $X_S = \{x_s \mid s \in S\}$ of free noncommutative indeterminates (indexed by the elements of the set S). Let I_S be the ideal of $R\langle X_S \rangle$ generated by the set $\{sx_s - 1, x_ss - 1 \mid s \in S\}$. The factor ring

$$R\langle S^{-1}\rangle := R\langle X_S\rangle/I_S. \tag{1}$$

is called the localization of R at S. Let $ass(S) = ass_R(S)$ be the kernel of the ring homomorphism

$$\sigma_S: R \to R\langle S^{-1} \rangle, \ r \mapsto r + I_S.$$
 (2)

The factor ring $\overline{R} := R/\mathrm{ass}_R(S)$ is a subring of $R\langle S^{-1}\rangle$ and the map $\pi_S : R \to \overline{R}, r \mapsto \overline{r} := r + \mathrm{ass}_R(S)$ is an epimorphism. The ideal $\mathrm{ass}_R(S)$ of R has a complex structure, its description is given in [7, Proposition 2.12] when

$$R\langle S^{-1}\rangle = \{\overline{s}^{-1}\overline{r} \mid s \in S, r \in R\}$$

is a ring of left fractions. There is an example of a domain R and a finite set S such that ass(S) = R, i.e. $R\langle S^{-1}\rangle = \{0\}$ [12, Exercises 9.5]. Proposition 2.10.(1) and its proof describe an explicit ideal $\mathfrak{a}(S)$ of R that is contained in $ass_R(S)$. The ideal $\mathfrak{a}(S)$ is the least ideal such that the elements of the set $\{s+\mathfrak{a}(S)\in R/\mathfrak{a}(S)\mid s\in S\}$ are non-zero-divisors in the factor ring $R/\mathfrak{a}(S)$, [7, Proposition 1.1]. The proof of Proposition 2.10 contains an explicit description of the ideal $\mathfrak{a}(S)$. If S is a left denominator set then the set

$$\operatorname{ass}_l(S) := \{ r \in R \, | \, sr = 0 \text{ for some } s \in S \}$$

is an ideal of R and $\operatorname{ass}_l(S) = \mathfrak{a}(S) = \operatorname{ass}_R(S)$. In general, for a multiplicative set S, $\operatorname{ass}_l(S) \subseteq \mathfrak{a}(S) \subseteq \operatorname{ass}_R(S)$ (in general, the inclusions are strict).

In Section 2, we recall Ore's method of localization. The aim of this section is to introduce several new concepts that are associated with localization of a ring at a set and to prove some general properties for them (Corollary 2.2, Lemma 2.6, Proposition 2.7, Proposition 2.9, Corollary 2.11): A subset S of R is called a *localizable set* if $R\langle S^{-1}\rangle \neq \{0\}$; L(R) is the set of all localizable sets of R and $L(R,\mathfrak{a}) := \{S \in L(R) | \operatorname{ass}_R(S) = \mathfrak{a}\}$; $\operatorname{Loc}(R)$ and $\operatorname{Loc}(R,\mathfrak{a})$ are the sets of R-isomorphism classes in the sets

$$\{R\langle S^{-1}\rangle \mid S \in L(R)\}\$$
and $\{R\langle S^{-1}\rangle \mid S \in L(R,\mathfrak{a})\},\$

respectively; $L(R, \mathfrak{a}, A) := \{ S \in L(R, \mathfrak{a}) \mid R \langle S^{-1} \rangle \simeq A, \text{ an } R\text{-isomorphism (necessarily unique)} \}.$ All the above sets are partially ordered sets.

Proposition 2.1 is the universal property of localization. Proposition 2.4 is a criterion for a subset of a ring to be a localizable set. Theorem 2.23 and Theorem 2.24 demonstrates some of the results of this section for Ore sets. Direct limits are essential in proving that the partially ordered sets above admit maximal elements. Several results on direct limits of rings are proven that are used later in the paper (Theorem 2.14, Corollary 2.15, and Lemma 2.16). For a ring R, the concept of the absolute quotient ring

$$Q_a(R) := \varinjlim_{S \in L(R)} R\langle S^{-1} \rangle$$

is introduced. Theorem 2.18 is a description of the ring $Q_a(R)$ in terms of maximals localization sets of R. Corollary 2.22 presents a sufficient condition for a ring R to have at least two maximal localization sets.

In Section 3, for a ring R, the following concepts are introduced: the maximal localizable sets, the localization radical Lrad(R), the set of localizable elements $\mathcal{L}L(R)$, the set of non-localizable elements $\mathcal{N}\mathcal{L}L(R)$, the set of completely localizable elements $\mathcal{C}L(R)$, and the complete localization

$$Q_c(R) := R \langle \mathcal{C}L(R)^{-1} \rangle$$

of R. These sets are invariant under the action of the automorphism group of R. Proposition 3.10 shows that there are tight connections between the sets $\mathcal{L}L(R)$, $\mathcal{N}\mathcal{L}L(R)$, and Lrad(R). For an arbitrary ring, Theorem 3.1 states that the set of maximal localizable set is a non-empty set and every localizable set is a subset of a maximal localizable set.

Proposition 3.5.(1), is an explicit description of the largest element $S(R, \mathfrak{a}, A)$ of the poset $(L(R, \mathfrak{a}, A), \subset)$. Corollary 3.6 is a criterion for a set $S \in L(R, \mathfrak{a}, A)$ to be the largest element

of the poset $L(R, \mathfrak{a}, A)$. Theorem 3.8 describes the sets $\max L(R)$ and $\max Loc(R)$, presents a bijection between them, and shows that $\max Loc(R) \neq \emptyset$. Similarly, for each ideal $\mathfrak{a} \in \operatorname{ass} L(R)$, Theorem 3.9 describes the sets $\max L(R, \mathfrak{a})$ and $\max Loc(R, \mathfrak{a})$, presents a bijection between them, and shows that they are nonempty sets. The \mathfrak{a} -absolute quotient ring

$$Q_a(R,\mathfrak{a}) := \varinjlim_{S \in \mathcal{L}(R,\mathfrak{a})} R\langle S^{-1} \rangle$$

is introduced for each ideal $\mathfrak{a} \in \operatorname{ass} L(R)$. Theorem 3.15 gives an explicit description of the ring $Q_a(R,\mathfrak{a})$, a criterion for $Q_a(R,\mathfrak{a}) \neq \{0\}$, and a criterion for $\mathfrak{a}_{a,\mathfrak{a}} = \mathfrak{a}$.

Theorem 3.17.(1) is an R-isomorphism criterion for localizations of the ring R and Theorem 3.17.(2) is a criterion for existing of an R-homomorphism between localizations of R. Proposition 3.14 contains some properties of the set of completely localizable elements CL(R) and the ideals $\mathfrak{c}_R := \operatorname{ass}_R(CL(R))$. Theorem 3.20 is a criterion for the ring $R\langle S^{-1}\rangle$ to be an R-isomorphic to a localization of R at a denominator set and it also describes all such denominator sets.

For a denominators set T of R, Proposition 3.21 describes all localizable sets S of the ring R such that the ring $R\langle S^{-1}\rangle$ is an R-isomorphic to the localization of R at T. For a localizable multiplicative set S of the ring R, Proposition 3.22 is a criterion for the set S to be a left denominator set of R.

In Section 4, for an arbitrary commutative ring R, descriptions of the following sets are obtained: the sets of localizable and non-localizable elements (Lemma 4.1), the maximal localizable sets, maximal localization rings, the set of completely localizable elements, and the ideal \mathfrak{c}_R (Theorem 4.2). Theorem 4.2 also describes relations between the ideals $\operatorname{Lrad}(R)$, \mathfrak{n}_R , and \mathfrak{c}_R .

For a commutative ring R with finitely many minimal primes and nilpotent prime radical (eg, R is a commutative Noetherian ring), Proposition 4.3 describes the spectrum $\operatorname{Spec}(Q_c(R))$ and the rings $Q_c(R)$ and $Q_c(R)/\mathfrak{n}_{Q_c(R)}$.

In Section 5, Proposition 5.1.(4) describes the maximal localizable sets of finite direct product of rings and their localizations. Proposition 5.1.(1) describes localizations of the direct product of rings via localizations of its components. For a direct product of simple rings

$$A = \prod_{i=1}^{s} A_i$$
 such that $\mathcal{L}L(A_i) = A_i^{\times}$ for $i = 1, \dots, s$,

Theorem 5.3 describes the sets $\max L(A)$, $\operatorname{ass}_A(S)$ for all $S \in \max L(A)$, the sets of localizable, non-localizable, and completely localizable elements of A. It also shows that $\operatorname{Lrad}(A) = 0$. Every semisimple Artinian ring satisfies the assumptions of Theorem 5.3, see Corollary 5.4 for detail. For a semiprime left Goldie ring, Lemma 5.8 describes all the maximal localizable sets that contain the set of regular elements of the ring. Let

$$D = \prod_{i \in I} D_i$$

be a direct product of division rings D_i where I is an arbitrary set. Proposition 5.9 describes all the localizable sets in the ring D and all the localizations of D. Theorem 5.11 describes the following sets: $\max L(D)$, $\max Loc(D)$, $\operatorname{Lrad}(D)$, $\operatorname{CL}(D)$, $\operatorname{Qc}(D)$, $\operatorname{LL}(D)$, and $\operatorname{NL}(D)$. Theorem 5.11.(2) shows that for every $\mathcal{S} \in \max L(D)$, the ring $D(\mathcal{S}^{-1})$ is a division ring.

In Section 6, basic properties of the localization of a module at a localizable set are considered. Proposition 6.1.(2) is a universal property of localization of a module. Theorem 6.2 is a criterion for the localization functor $M \to S^{-1}M$ to be an exact functor.

2 Localization of a ring at a set

At the beginning of the section we recall Ore's method of localization, i.e. the localization of a ring at a left or right denominator set. The aim of this section is to introduce several new concepts

that are associated with localization of a ring at a set and to prove some general properties for them (Corollary 2.2, Lemma 2.6, Proposition 2.7, Proposition 2.9, Corollary 2.11). Proposition 2.1 is the universal property of localization. Proposition 2.4 is a criterion for a subset of a ring to be a localizable set. Every Ore set is a localizable set, [1, Theorem 4.15]. Theorem 2.23 and Theorem 2.24 demonstrates some of the results of this section for Ore sets.

Direct limits play a prominent role in proving that certain natural posets (partially ordered sets) that are associated with localizations admit maximal elements. Several results on direct limits of rings are proven that are used later in the paper (Theorem 2.14, Corollary 2.15, and Lemma 2.16).

For a ring R, the concept of the absolute quotient ring $Q_a(R)$ is introduced. Theorem 2.18 is a description of the ring $Q_a(R)$ in terms of maximals localization sets of R. Corollary 2.22 presents a sufficient condition for a ring R to have at least two maximal localization sets.

Ore and denominator sets, localization of a ring at a denominator set. Let R be a ring. A subset S of R is called a *multiplicative set* if $SS \subseteq S$, $1 \in S$ and $0 \notin S$. A multiplicative subset S of R is called a *left Ore set* if it satisfies the *left Ore condition*: for each $r \in R$ and $s \in S$,

$$Sr \bigcap Rs \neq \emptyset.$$

Let $\operatorname{Ore}_l(R)$ be the set of all left Ore sets of R. For $S \in \operatorname{Ore}_l(R)$, $\operatorname{ass}_l(S) := \{r \in R \mid sr = 0 \text{ for some } s \in S\}$ is an ideal of the ring R.

A left Ore set S is called a *left denominator set* of the ring R if rs = 0 for some elements $r \in R$ and $s \in S$ implies tr = 0 for some element $t \in S$, i.e., $r \in \operatorname{ass}_l(S)$. Let $\operatorname{Den}_l(R)$ (resp., $\operatorname{Den}_l(R,\mathfrak{a})$) be the set of all left denominator sets of R (resp., such that $\operatorname{ass}_l(S) = \mathfrak{a}$). For $S \in \operatorname{Den}_l(R)$, let

$$S^{-1}R = \{s^{-1}r \mid s \in S, r \in R\}$$

be the left localization of the ring R at S (the left quotient ring of R at S). Let us stress that in Ore's method of localization one can localize precisely at left denominator sets. In a similar way, right Ore and right denominator sets are defined. Let $\operatorname{Ore}_r(R)$ and $\operatorname{Den}_r(R)$ be the set of all right Ore and right denominator sets of R, respectively. For $S \in \operatorname{Ore}_r(R)$, the set $\operatorname{ass}_r(S) := \{r \in R \mid rs = 0 \text{ for some } s \in S\}$ is an ideal of R. For $S \in \operatorname{Den}_r(R)$,

$$RS^{-1} = \{rs^{-1} \mid s \in S, r \in R\}$$

is the right localization of the ring R at S.

Given ring homomorphisms $\nu_A : R \to A$ and $\nu_B : R \to B$. A ring homomorphism $f : A \to B$ is called an R-homomorphism if $\nu_B = f\nu_A$. A left and right set is called an $Ore\ set$. Let Ore(R) and Den(R) be the set of all $Ore\ and\ denominator\ sets$ of R, respectively. For $S \in Den(R)$,

$$S^{-1}R \simeq RS^{-1}$$

(an R-isomorphism) is the localization of the ring R at S, and $ass(S) := ass_l(S) = ass_r(S)$.

The ring $R\langle S^{-1}\rangle$ and the ideal $\operatorname{ass}_R(S)$. Let R be a ring and S be a subset of R. Let $R\langle X_S\rangle$ be a ring freely generated by the ring R and a set $X_S=\{x_s\,|\,s\in S\}$ of free noncommutative indeterminates (indexed by the elements of the set S). Let I_S be the ideal of $R\langle X_S\rangle$ generated by the set $\{sx_s-1,x_ss-1\,|\,s\in S\}$ and

$$R\langle S^{-1}\rangle := R\langle X_S\rangle/I_S.$$
 (3)

The ring $R\langle S^{-1}\rangle$ is called the *localization of* R at S. Let $ass(S) = ass_R(S)$ be the kernel of the ring homomorphism

$$\sigma_S: R \to R\langle S^{-1} \rangle, \ r \mapsto r + I_S.$$
 (4)

The map $\pi_S: R \to \overline{R} := R/\operatorname{ass}_R(S), r \mapsto \overline{r} := r + \operatorname{ass}_R(S)$ is an epimorphism. The ideal $\operatorname{ass}_R(S)$ of R has a complex structure, its description is given in [7, Proposition 2.12] when $R\langle S^{-1}\rangle =$

 $\{\overline{s}^{-1}\overline{r} \mid s \in S, r \in R\}$ is a ring of left fractions. We identify the factor ring \overline{R} with its isomorphic copy in the ring $R(S^{-1})$ via the monomorphism

$$\overline{\sigma_S}: \overline{R} \to R\langle S^{-1} \rangle, \quad r + \operatorname{ass}_R(S) \mapsto r + I_S.$$
 (5)

Clearly, $\overline{S} := (S + \operatorname{ass}_R(S))/\operatorname{ass}_R(S) \subseteq \mathcal{C}_{R\langle S^{-1}\rangle}$. Corollary 2.2.(2) shows that the rings $R\langle S^{-1}\rangle$ and $\overline{R}\langle \overline{S}^{-1}\rangle$ are R-isomorphic. For $S = \emptyset$, $R\langle \emptyset^{-1}\rangle := R$ and $\operatorname{ass}_R(\emptyset) := 0$.

Definition. A subset S of a ring R is called a localizable set of R if $R\langle S^{-1}\rangle \neq \{0\}$. Let L(R) be the set of localizable sets of R and

$$\operatorname{ass} L(R) := \{ \operatorname{ass}_R(S) \mid S \in L(R) \}. \tag{6}$$

Proposition 2.1 is the universal property of localization.

Proposition 2.1 Let R be a ring, $S \in L(R)$, and $\sigma_S : R \to R\langle S^{-1} \rangle$, $r \mapsto r + \operatorname{ass}_R(S)$. Let $f : R \to A$ be a ring homomorphism such that $f(S) \subseteq A^{\times}$. The there is a unique R-homomorphism $f' : R\langle S^{-1} \rangle \to A$ such $f = f'\sigma_S$, i.e. the diagram below is commutative

$$\begin{array}{ccc} R & \stackrel{\sigma_S}{\rightarrow} & R\langle S^{-1}\rangle \\ & \stackrel{f}{\searrow} & \downarrow^{\exists ! \; f'} \\ & & A \end{array}$$

Proof. The unique R-homomorphism f' is defined by the rule

$$f': R\langle S^{-1}\rangle = R\langle X_S\rangle/I_X \to A, \ r+I_S \mapsto f(r), \ x_s+I_S \mapsto f(s)^{-1}$$

for all $r \in R$ and $s \in S$. \square

Corollary 2.2 Suppose that $S \in L(R, \mathfrak{a})$, \mathfrak{b} is an ideal of R such that $\mathfrak{b} \subseteq \mathfrak{a}$, $\overline{S} := \pi_{\mathfrak{b}}(S)$ where $\pi_{\mathfrak{b}} : R \to R/\mathfrak{b}$, $r \mapsto r + \mathfrak{b}$. Then

- 1. $\overline{S} \in L(R/\mathfrak{b}, \mathfrak{a}/\mathfrak{b})$ and $(R/\mathfrak{b})\langle \overline{S}^{-1} \rangle \simeq R\langle S^{-1} \rangle$, an R-isomorphism.
- 2. In particular, for $\mathfrak{a} = \mathfrak{b}$, $\overline{S} \in L(R/\mathfrak{a},0)$ and $(R/\mathfrak{a})\langle \overline{S}^{-1} \rangle \simeq R\langle S^{-1} \rangle$, an R-isomorphism.

Proof. 1. By the universal property of the localization, there is a unique R-homomorphism $R\langle S^{-1}\rangle \to (R/\mathfrak{b})\langle \overline{S}^{-1}\rangle$ (since $\mathfrak{b}\subseteq \mathfrak{a}$). Conversely, since $\mathfrak{b}\subseteq \mathfrak{a}$, there is a unique R-homomorphism $(R/\mathfrak{b})\langle \overline{S}^{-1}\rangle \to R\langle S^{-1}\rangle$, by the universal property of the localization, and the lemma follows.

2. Statement 2 is a particular case of statement 1 where $\mathfrak{b} = \mathfrak{a}$. \square

Corollary 2.3 is a criterion for two localizations of a ring R to be R-isomorphic.

Corollary 2.3 Let $S \in L(R, \mathfrak{a})$, $T \in L(R, \mathfrak{b})$, $\overline{S} = \pi(S)$ and $\overline{T} = \pi(T)$ where $\pi : R \to \overline{R} = R/\mathfrak{a}$, $r \mapsto r + \mathfrak{a}$. Then the rings $R\langle S^{-1} \rangle$ and $R\langle T^{-1} \rangle$ are R-isomorphic iff $\mathfrak{a} = \mathfrak{b}$, $\overline{S} \subseteq R\langle T^{-1} \rangle^{\times}$, and $\overline{T} \subseteq R\langle S^{-1} \rangle^{\times}$.

Proof. (\Rightarrow) The implication is obvious.

(\Leftarrow) Suppose that $\mathfrak{a} = \mathfrak{b}$, $\overline{S} \subseteq R\langle T^{-1}\rangle^{\times}$, and $\overline{T} \subseteq R\langle S^{-1}\rangle^{\times}$. By Proposition 2.1, we have commutative diagrams of R-homomorphisms,

By Proposition 2.1, $fg = \mathrm{id}_{R(T^{-1})}$ and $gt = \mathrm{id}_{R(S^{-1})}$, and so f is an R-isomorphism. \square

If $S \subseteq T \subseteq R$ then $\operatorname{ass}_R(S) \subseteq \operatorname{ass}_R(T)$ and, by Proposition 2.1, there is a unique R-homomorphism

$$\phi_{TS}: R\langle S^{-1}\rangle \to R\langle T^{-1}\rangle,$$
 (7)

i.e. $\phi_{TS}(ra) = r\phi_{TS}(a)$ for all elements $a \in R\langle S^{-1}\rangle$. Clearly, $\phi_{SS} = \mathrm{id}_{R\langle S^{-1}\rangle}$ and

$$\ker(\phi_{TS}) \supseteq R\langle S^{-1} \rangle (\operatorname{ass}_R(T)/\operatorname{ass}_R(S)) R\langle S^{-1} \rangle. \tag{8}$$

If $S \subseteq T \subseteq U \subseteq R$ then

$$\phi_{US} = \phi_{UT}\phi_{TS}.\tag{9}$$

Criterion for a set to be a localizable set. Proposition 2.4 is a criterion for a set to be a localizable set.

Proposition 2.4 Let S be a subset of R. The following statements are equivalent:

- 1. $S \in L(R)$.
- 2. $\operatorname{ass}_R(S) \neq R$.
- 3. There is an ring homomorphism $f: R \to A$ such that $f(S) \subseteq A^{\times}$ where A^{\times} is a group of units of the ring A.
- *Proof.* $(1 \Leftrightarrow 2)$ $S \in L(R)$ iff $R\langle S^{-1} \rangle \neq \{0\}$ iff the image of the ring R under the R-homomorphism $R \to R\langle S^{-1} \rangle$ is not equal to 0 iff $\operatorname{ass}_R(S) \neq R$.
- $(1 \Rightarrow 3)$ If $S \in L(R)$ then the image of the set S under the homomorphism $\sigma_S : R \to R\langle S^{-1} \rangle$ consists of units of the ring $R\langle S^{-1} \rangle$.
- $(3 \Rightarrow 2)$ Suppose that there is a ring homomorphism $f: R \to A$ such that $f(S) \subseteq A^{\times}$. By the universal property of localization, there is a unique R-homomorphism $R\langle S^{-1}\rangle \to A$. Hence, $R\langle S^{-1}\rangle \neq \{0\}$ (since $f(S) \subseteq A^{\times}$), and so $\operatorname{ass}_R(S) \neq R$. \square

Proposition 2.5 gives a sufficient condition for a subset of a ring to be a localizable set.

Proposition 2.5 Let I be an ideal of a ring R, $\pi: R \to R/I$, $r \mapsto r + I$, and S be a subset of R such that $\pi(S) \subseteq \mathcal{C}_{R/I}$ and $\operatorname{ass}_R(S) = \mathfrak{a}(S)$. Then $S \in L(R, \mathfrak{a}(S))$ and $\mathfrak{a}(S) \subseteq I$.

Proof. By Corollary 2.11.(1), $\mathfrak{a}(S) \subseteq I$ since by induction $\mathfrak{a}_{\lambda} \subseteq I$ for all $\lambda \in \mathbb{N}$. Since $\mathfrak{a}(S) = \operatorname{ass}_R(S)$, the set S is a localizable set of R since $\operatorname{ass}_R(S) \neq R$ (Proposition 2.4) as $\operatorname{ass}_R(S) \subseteq I$. \square

Let S be a subset of the ring R and S_{mon} be the multiplicative submonoid of (R, \cdot) generated by the set S. Lemma 2.6 shows that the localization of the ring R at S is the same as the localization of the ring R at S_{mon} .

Lemma 2.6 Let S be a subset of the ring R. Then

- 1. $S \in L(R, \mathfrak{a})$ iff $S_{mon} \in L(R, \mathfrak{a})$.
- 2. If $S \in L(R, \mathfrak{a})$ then $R(S^{-1}) \simeq R(S^{-1}_{mon})$, an R-isomorphism
- *Proof.* 1. If $S \in L(R)$ then the image of the monoid S_{mon} under the R-homomorphism $\sigma_S: R \to R\langle S^{-1} \rangle$ consists of units, and so $S_{mon} \in L(R)$, by Proposition 2.4. Similarly, if $S_{mon} \in L(R)$ then the image of the set S under the R-homomorphism $\sigma_{S_{mon}}: R \to R\langle S_{mon}^{-1} \rangle$ consists of units, and so $S \in L(R)$, by Proposition 2.4.
- 2. Since $S \in L(R)$, $S_{mon} \in L(R)$, by statement 1. By the universal property of localization, there is a unique R-homomorphism $R\langle S^{-1} \rangle \to R\langle S_{mon}^{-1} \rangle$. By the universal property of localization, there is a unique R-homomorphism $R\langle S_{mon}^{-1} \rangle \to R\langle S^{-1} \rangle$. So, the rings $R\langle S^{-1} \rangle$ and $R\langle S_{mon}^{-1} \rangle$ are an R-isomorphic, and statement 2 follows. \square

The set $\mathcal{M}(S,R)$ of ideals of R and its minimal element. For a set $S \in L(R)$, let $\mathcal{M}(S,R)$ be the set of ideal \mathfrak{a} of the ring R such that there is a ring homomorphism $\varphi: R/\mathfrak{a} \to A$ for some ring A such that $\varphi(S+\mathfrak{a}) \subseteq A^{\times}$. Proposition 2.7 shows that $\operatorname{ass}_{R}(S)$ is the least element of the set $\mathcal{M}(S,R)$ w.r.t. inclusion.

Proposition 2.7 For every set $S \in L(R)$, $\min \mathcal{M}(S, R) = \{ ass_R(S) \}$.

Proof. Let $\mathfrak{a} \in \mathcal{M}(S,R)$ and

$$f: R \to R/\mathfrak{a} \stackrel{\varphi}{\to} A, \quad r \mapsto r + \mathfrak{a} \mapsto \varphi(r + \mathfrak{a})$$

for some ring A such that $\varphi(S + \mathfrak{a}) \subseteq A^{\times}$. Clearly, $\ker(f) = \mathfrak{a}$. By Proposition 2.1, there is an R-homomorphism $f' : R\langle S^{-1} \rangle \to A$ such that $f = f'\sigma_S$. Therefore,

$$\operatorname{ass}_R(S) = \ker(\sigma_S) \subseteq \ker(f'\sigma_S) = \ker(f) = \mathfrak{a},$$

and the proposition follows. \square

General results on localizations. Some useful general results on localizations are collected below that are used in the proofs of this paper.

Proposition 2.8 1. If $S \subseteq R^{\times}$ then $\operatorname{ass}_{R}(S) = 0$ and $R\langle S^{-1} \rangle = R$.

2. If $T \in L(R)$ and $S \subseteq T$. Then $S \in L(R)$, $ass_R(S) \subseteq ass_R(T)$, $\sigma_S(T) \in L(R\langle S^{-1} \rangle)$,

$$R\langle T^{-1}\rangle \simeq R\langle S^{-1}\rangle \langle \sigma_S(T)^{-1}\rangle,$$

an R-isomorphism, and $\operatorname{ass}_{R(S^{-1})}(\sigma_S(T)) \supseteq R(S^{-1})(\operatorname{ass}_R(T)/\operatorname{ass}_R(S))R(S^{-1})$.

3. If $S, T \in L(R)$ such that $S \cup T \in L(R)$ then $\sigma_S(T) \in L(R\langle S^{-1} \rangle)$ and $R\langle (S \cup T)^{-1} \rangle \simeq R\langle S^{-1} \rangle \langle \sigma_S(T)^{-1} \rangle$, an R-isomorphism.

Proof. 1. Both rings $R\langle S^{-1}\rangle$ and R satisfy the universal property of localization, Proposition 2.1. Therefore, $R\langle S^{-1}\rangle=R$. Hence, $\operatorname{ass}_R(S)=0$.

- 2. Since $T \in L(R)$ and $S \subseteq T$, there a unique R-homomorphism $R\langle S^{-1} \rangle \to R\langle T^{-1} \rangle$. Therefore, $S \in L(R)$. Both rings $R\langle T^{-1} \rangle$ and $R\langle S^{-1} \rangle \langle \sigma_S(T)^{-1} \rangle$ satisfy the universal property of localization for the set T (Proposition 2.1), and so they are R-isomorphic. It follows that $\sigma_S(T) \in L(R\langle S^{-1} \rangle)$, by statement 2 (since $T \subseteq S \cup T$).
 - 3. Statement 3 is a particular case of statement 2 (since $S \subseteq S \cup T$ and $S \cup T \in L(R)$. \square

Lemma 2.9 Suppose that $S \in L(R, \mathfrak{a}), T \subseteq R, \text{ and } \sigma_S : R \to R\langle S^{-1} \rangle$.

- 1. If $\sigma_S(T) \in L(R\langle S^{-1}\rangle)$ then $S \cup T \in L(R)$ and $R\langle (S \cup T)^{-1}\rangle \simeq R\langle S^{-1}\rangle \langle \sigma_S(T)^{-1}\rangle$, an R-isomorphism.
- 2. If $\sigma_S(T) \subseteq R\langle S^{-1} \rangle^{\times}$ then $S \cup T \in L(R,\mathfrak{a})$ and $R\langle (S \cup T)^{-1} \rangle \simeq R\langle S^{-1} \rangle$, an R-isomorphism.
- 3. Let $\mathcal{T} = \sigma^{-1}(R\langle S^{-1}\rangle^{\times})$. Then $\mathcal{T} \in L(R,\mathfrak{a})$ and $R\langle \mathcal{T}^{-1}\rangle \simeq R\langle S^{-1}\rangle$, an R-isomorphism.

Proof. 1. The image of the set $S \cup T$ in $R\langle S^{-1}\rangle \langle \sigma_S(T)^{-1}\rangle$ consists of units, hence $S \cup T \in L(R)$, by Proposition 2.4. By Proposition 2.8.(3), the rings $R\langle (S \cup T)^{-1}\rangle$ and $R\langle S^{-1}\rangle \langle \sigma_S(T)^{-1}\rangle$ are R-isomorphic.

2. Since $\sigma_S(T) \subseteq R\langle S^{-1}\rangle^{\times}$, $\sigma_S(T) \in L(R\langle S^{-1}\rangle)$. Now, by statement 1, $S \cup T \in L(R)$ and

$$R\langle (S \cup T)^{-1} \rangle \simeq R\langle S^{-1} \rangle \langle \sigma_S(T)^{-1} \rangle \simeq R\langle S^{-1} \rangle,$$

R-isomorphisms where the second R-isomorphism is due to Proposition 2.8.(1). Hence, $\operatorname{ass}_R(S \cup T) = \mathfrak{a}$.

3. Statement 3 follows from statement 2 where $T = \mathcal{T}$ since $S \cup \mathcal{T} = \mathcal{T}$. \square

The ideals $\mathfrak{a}(S)$, $'\mathfrak{a}(S)$ and $\mathfrak{a}'(S)$. The ideals $\mathfrak{a}(S)$, $'\mathfrak{a}(S)$ and $\mathfrak{a}'(S)$ are contained in the ideal $\mathrm{ass}_R(S)$. They are defined iteratively and it is difficult to compute them in general. They reveal a complex structure of the ideal $\mathrm{ass}_R(S)$.

For each element $r \in R$, let $r \cdot : R \to R$, $x \mapsto rx$ and $r : R \to R$, $x \mapsto xr$. The sets

$${}'\mathcal{C}_R := \{ r \in R \, | \, \ker(r) = 0 \} \text{ and } \mathcal{C}'_R := \{ r \in R \, | \, \ker(r) = 0 \}$$

are called the sets of left and right regular elements of R, respectively. Their intersection

$$C_R = {}'C_R \cap C'_R$$

is the set of regular elements of R. The rings

$$Q_{l,cl}(R) := \mathcal{C}_R^{-1} R$$
 and $Q_{r,cl}(R) := R \mathcal{C}_R^{-1}$

are called the *classical left and right quotient rings* of R, respectively. If both rings exist then they are isomorphic and the ring

$$Q_{cl}(R) := Q_{l,cl}(R) \simeq Q_{r,cl}(R)$$

is called the classical quotient ring of R. Goldie's Theorem states that the ring $Q_{l,cl}(R)$ is a semisimple Artinian ring iff the ring R is a semiprime left Goldie ring (the ring R is called a left Goldie ring if I udimI if udimI if udimI if udimI if udimI if udimI is a semiprime left Goldie ring if udimI if udimI is a semiprime left Goldie ring is defined. A left and right Goldie ring is called a Goldie ring.

Proposition 2.10 ([7, Proposition 1.1]) Let R be a ring and S be a non-empty subset of R.

- 1. There is the least ideal, $\mathfrak{a} = \mathfrak{a}(S)$, such that $(S + \mathfrak{a})/\mathfrak{a} \subseteq \mathcal{C}_{R/\mathfrak{b}}$.
- 2. There is the least ideal, $\mathfrak{a} = \mathfrak{a}(S)$, such that $(S + \mathfrak{a})/\mathfrak{a} \subseteq \mathcal{C}_{R/\mathfrak{b}}$.
- 3. There is the least ideal, $\mathfrak{a}' = \mathfrak{a}'(S)$, such that $(S + \mathfrak{a}')/\mathfrak{a}' \subseteq \mathcal{C}'_{R/h}$.

Notice that ${}'\mathfrak{a}(S) \subseteq \mathfrak{a}(S)$ and $\mathfrak{a}'(S) \subseteq \mathfrak{a}(S)$.

Recall that S_{mon} is the multiplicative submonoid of (R,\cdot) which is generated by the set S. If $0 \in S_{mon}$ then $\mathfrak{a}(S) = {}'\mathfrak{a}(S) = \mathfrak{a}'(S) = R$ and $R\langle S^{-1}\rangle = 0$. So, we can assume that $0 \notin S_{mon}$, i.e. the set S is a multiplicative set. Notice that (Lemma 2.6)

$$\mathfrak{a}(S) = \mathfrak{a}(S_{mon}), \quad \operatorname{ass}(S) = \operatorname{ass}(S_{mon}), \quad \operatorname{and} \quad R\langle S^{-1} \rangle = R\langle S_{mon}^{-1} \rangle.$$
 (10)

The proof of Proposition 2.10 ([7, Proposition 1.1]) contains an explicit description of the ideals $\mathfrak{a} = \mathfrak{a}(S)$, $'\mathfrak{a} = '\mathfrak{a}(S)$, and $\mathfrak{a}' = \mathfrak{a}'(S)$. In more detail, let Γ be the set of ordinals. The ideal \mathfrak{a} (resp., $'\mathfrak{a}$, \mathfrak{a}') is the union

$$\mathfrak{a} = \bigcup_{\lambda \in \Gamma} \mathfrak{a}_{\lambda} \quad (\text{resp.}, \ '\mathfrak{a} = \bigcup_{\lambda \in \Gamma} '\mathfrak{a}_{\lambda}, \ \mathfrak{a}' = \bigcup_{\lambda \in \Gamma} \mathfrak{a}'_{\lambda}) \tag{11}$$

of ascending chain of ideals $\{\mathfrak{a}_{\lambda}\}_{{\lambda}\in\Gamma}$ (resp., $\{{}'\mathfrak{a}_{\lambda}\}_{{\lambda}\in\Gamma}$, $\{\mathfrak{a}'_{\lambda}\}_{{\lambda}\in\Gamma}$), where ${\lambda}\leq \mu$ in ${\Gamma}$ implies $\mathfrak{a}_{\lambda}\subseteq \mathfrak{a}_{\mu}$ (resp., ${}'\mathfrak{a}_{\lambda}\subseteq {}'\mathfrak{a}_{\mu}$). The ideals \mathfrak{a}_{λ} (resp., ${}'\mathfrak{a}_{\lambda}$, \mathfrak{a}'_{λ}) are defined inductively as follows: the ideal \mathfrak{a}_{0} (resp., ${}'\mathfrak{a}_{0}$, \mathfrak{a}'_{0}) is generated by the set $\{r\in R\mid sr=0 \text{ or } rt=0 \text{ for some elements } s,t\in S\}$ (resp., $\{r\in R\mid rt=0 \text{ for some element } t\in S\}$, $\{r\in R\mid sr=0 \text{ for some element } s\in S\}$), and for ${\lambda}\in {\Gamma}$ such that ${\lambda}>0$ (where below $\{\{\ldots\}\}$) means the ideal of R generated by the set $\{\ldots\}$),

$$\mathfrak{a}_{\lambda} = \begin{cases} \bigcup_{\mu < \lambda \in \Gamma} \mathfrak{a}_{\mu} & \text{if } \lambda \text{ is a limit ordinal,} \\ \left(\left\{ r \in R \, | \, sr \in \mathfrak{a}_{\lambda - 1} \text{ or } rt \in \mathfrak{a}_{\lambda - 1} \text{ for some } s, t \in S \right\} \right) & \text{if if } \lambda \text{ is not a limit ordinal.} \end{cases}$$
 (12)

(resp.,

$${}'\mathfrak{a}_{\lambda} = \begin{cases} \bigcup_{\mu < \lambda \in \Gamma} {}'\mathfrak{a}_{\mu} & \text{if } \lambda \text{ is a limit ordinal,} \\ \left(\left\{ r \in R \, \middle| \, rt \in {}'\mathfrak{a}_{\lambda-1} \text{ for some } t \in S \right\} \right) & \text{if if } \lambda \text{ is not a limit ordinal.} \end{cases}$$
(13)

$$\mathfrak{a}'_{\lambda} = \begin{cases} \bigcup_{\mu < \lambda \in \Gamma} \mathfrak{a}'_{\mu} & \text{if } \lambda \text{ is a limit ordinal,} \\ \left(\{ r \in R \mid sr \in \mathfrak{a}'_{\lambda-1} \text{ for some } s \in S \} \right) & \text{if if } \lambda \text{ is not a limit ordinal).} \end{cases}$$
(14)

Let us define an ideal \mathfrak{b} of R which is the union $\mathfrak{b} = \bigcup_{\lambda \in \Gamma} \mathfrak{b}_{\lambda}$ of an ascending chain of ideals \mathfrak{b}_{λ} that are defined inductively

$$\mathfrak{b}_{\lambda} = \begin{cases} \bigcup_{\mu < \lambda \in \Gamma} \mathfrak{b}_{\mu} & \text{if } \lambda \text{ is a limit ordinal,} \\ \left(\left\{ r \in R \, | \, sr \in \mathfrak{b}_{\lambda - 1} \text{ or } rt \in \mathfrak{b}_{\lambda - 1} \text{ or } srt \in \mathfrak{b}_{\lambda - 1} \text{ for some } s, t \in S \right\} \right) & \text{if } if \lambda \text{ is not a limit ordinal.} \end{cases}$$

$$\tag{15}$$

where the ideal \mathfrak{b}_0 is generated by the set $\{r \in R \mid sr = 0 \text{ or } rt = 0 \text{ or } srt = 0 \text{ for some elements } s, t \in S\}$.

Corollary 2.11 Let R be a ring and S be a non-empty subset of R. Then

1.
$$\mathfrak{a}(S) = \bigcup_{\lambda \in \mathbb{N}} \mathfrak{a}_{\lambda} = \bigcup_{\lambda \in \mathbb{N}} \mathfrak{b}_{\lambda}$$
.

2.
$$'\mathfrak{a}(S) = \bigcup_{\lambda \in \mathbb{N}} '\mathfrak{a}_{\lambda}.$$

3.
$$\mathfrak{a}'(S) = \bigcup_{\lambda \in \mathbb{N}} \mathfrak{a}'_{\lambda}$$
.

Proof. The corollary follows from the definitions of the ideals $\mathfrak{a}_{\mathbb{N}}$, $\mathfrak{b}_{\mathbb{N}}$, $'\mathfrak{a}_{\mathbb{N}}$, and $\mathfrak{a}'_{\mathbb{N}}$. In more detail, let us show that $\mathfrak{a}(S) = \mathfrak{b}_{\mathbb{N}} := \bigcup_{\lambda \in \mathbb{N}} \mathfrak{b}_{\lambda}$. The other cases can be treated in a similar way. The inclusion $\mathfrak{a}(S) \supseteq \mathfrak{b}_{\mathbb{N}}$ is obvious. To prove that the reverse inclusion holds it suffices to show that the maps

$$s \cdot : R/\mathfrak{b}_{\mathbb{N}} \to R/\mathfrak{b}_{\mathbb{N}}, \ r \mapsto sr + \mathfrak{b}_{\mathbb{N}} \ \text{and} \ s : R/\mathfrak{b}_{\mathbb{N}} \to R/\mathfrak{b}_{\mathbb{N}}, \ r \mapsto rs + \mathfrak{b}_{\mathbb{N}}$$

are injections for all $s \in S$. Suppose that $r + \mathfrak{b}_{\mathbb{N}} \in \ker(s \cdot)$ or $r + \mathfrak{b}_{\mathbb{N}} \in \ker(s \cdot)$, that is $sr \in \mathfrak{b}_{\mathbb{N}}$ or $rs \in \mathfrak{b}_{\mathbb{N}}$. Then $sr \in \mathfrak{b}_{\lambda}$ or $rs \in \mathfrak{b}_{\lambda}$ for some $\lambda \in \mathbb{N}$, and so $r \in \mathfrak{b}_{\lambda+1}$ in both cases. This means that $r \in \mathfrak{b}_{\mathbb{N}}$, as required. \square

Definition. The first natural number such that the ascending chain of ideals in (15) (resp., (11)) stabilizes is called the Γ-length of the ideal denoted $l_{\Gamma}(\mathfrak{a})$ (resp., $l_{\Gamma}('\mathfrak{a})$, $l_{\Gamma}(\mathfrak{a}')$). If there is no such a number then $l_{\Gamma}(\mathfrak{a}) := \infty$ (resp., $l_{\Gamma}('\mathfrak{a}) := \infty$).

Lemma 2.12 shows that the Γ-length $l_{\Gamma}(\mathfrak{a})$ can be any natural number or ∞ .

Let K be a field. Consider the following rings

- 1. $R_0 = K\langle x, y_0 | xy_0 = 0 \rangle$,
- 2. $R_n = K\langle x, y_i, u_i, v_i | xy_0 = 0, xy_i = u_i y_{i-1} v_i, i = 0, \dots, n, j = 1, \dots, n \rangle$
- 3. $R_{\infty} = K\langle x, y_i, u_j, v_j | xy_0 = 0, xy_j = u_j y_{j-1} v_j, i \ge 0, j \ge 1 \rangle$.

Notice that $R_0/(y_0) \simeq K[x]$ and $R_n/(y_0, \ldots, y_n) \simeq K\langle x, u_j, v_j | j = 1, \ldots, n \rangle$ is a free algebra for all $n \geq 1$ and $n = \infty$.

Lemma 2.12 Let R_n , $n \in \mathbb{N} \cup \{\infty\}$, be the rings above, $S = \{x^i \mid i \geq 0\}$, and $\mathfrak{a}(S, R_n) := \mathfrak{a}(S)$ where $S \subseteq R_n$. Then $\mathfrak{a}(S, R_n) = \operatorname{ass}_{R_n}(S) = (y_0, \dots, y_n), \ l_{\Gamma}(\mathfrak{a}(S, R_n)) = n$, and

$$R_n \langle S^{-1} \rangle \simeq \begin{cases} K[x, x^{-1}] & \text{if } n = 0, \\ K\langle x^{\pm 1}, u_j, v_j \mid j = 1, \dots, n \rangle & \text{if } n = 1, \dots, \infty. \end{cases}$$

Proof. Suppose that n=0. Then $y_0 \in \mathfrak{b}_0 \subseteq \mathfrak{b} \subseteq \mathfrak{a}(S,R_0) \subseteq \operatorname{ass}_{R_0}(S)$, $R_0/(y_0) = K[x]$, and so $R_0\langle S^{-1}\rangle \simeq K[x,x^{-1}]$ and $\mathfrak{a}(S,R_0) = \operatorname{ass}_{R_0}(S) = (y_0) = \mathfrak{b}_0$. Hence, $l_{\Gamma}(\mathfrak{a}(S,R_0)) = 0$. Suppose that $n \geq 1$. Then

$$R_n = \bigoplus_{w \in W_n} Kw$$

where W_n is the set of all words in the alphabet $\{x, y_i, u_j, v_j \mid i = 0, ..., n, j = 1, ..., n\}$ that do not contained the subwords xy_i , where i = 0, ..., n. Then $\mathfrak{b}_0 = (y_0)$ since $\ker_{R_n}(x \cdot) = y_0 R_n$ and $\ker_{R_n}(x \cdot) = 0$. Using the same argument in the case of the factor ring

$$R_n/(\mathfrak{b}_0) = K\langle x, y_i, u_j, v_j | xy_1 = 0, xy_k = u_k y_{k-1} v_k, k = 2, \dots, n \rangle$$
 where $i = 1, \dots, n, j = 1, \dots, n$,

we conclude that $\mathfrak{b}_1 = (y_0, y_1)$ and

$$R_n/(\mathfrak{b}_1) = K\langle x, y_i, u_j, v_j | xy_2 = 0, xy_k = u_k y_{k-1} v_k, k = 3, \dots, n \rangle$$
 where $i = 2, \dots, n, j = 1, \dots, n$.

Repeating the same argument again and again (or use induction), we prove that for all m = 1, ..., n, $\mathfrak{b}_m = (y_0, ..., y_m)$ and

$$R_n/(\mathfrak{b}_m) = K\langle x, y_i, u_j, v_j | xy_{m+1} = 0, xy_k = u_k y_{k-1} v_k, k = m+2, \dots, n \rangle, i = m+1, \dots, n, j = 1, \dots, n.$$

Hence,

$$R_n \langle S^{-1} \rangle \simeq (R_n/\mathfrak{b}_n) \langle S^{-1} \rangle \simeq K \langle x^{\pm 1}, u_i, v_i | j = 1, \dots, n \rangle$$

and $\mathfrak{a}(S,R_n) = \operatorname{ass}_{R_n}(S) = \mathfrak{b}_n$. It follows that $l_{\Gamma}(\mathfrak{a}(S,R_n)) = n$. \square

Perfect localization set and perfect localizations. By [7, Lemma 1.2],

$$\mathfrak{a}(S) \subseteq \operatorname{ass}_R(S).$$
 (16)

Definition, [7]. If the equality holds, $\mathfrak{a}(S) = \operatorname{ass}_R(S)$, then set S is called a perfect localization set and the localization $R\langle S^{-1}\rangle$ is called a perfect localization.

Theorem 2.13 shows that there are plenty of perfect localizations (especially in applications).

Theorem 2.13 Let R be a K-algebra over a field K, $\mathfrak{a} := \mathfrak{a}(S)$, $\pi_{\mathfrak{a}} : R \to \overline{R} := R/\mathfrak{a}$, $r \mapsto \overline{r} := r + \mathfrak{a}$, S be a multiplicative set in R. Suppose that \overline{R} is a domain and for each pair of elements $s \in S$ and $r \in R$, the subalgebra $K\langle \overline{s}, \overline{r} \rangle$ of \overline{R} is not a free algebra. Then

- 1. $\operatorname{ass}_R(S) = \mathfrak{a}$.
- $2. \ \overline{S}:=\pi_{\mathfrak{a}}(S)\in \mathrm{Den}(\overline{R},0) \ and \ R\langle S^{-1}\rangle\simeq \overline{S}^{-1}\overline{R}\simeq \overline{RS}^{-1}.$

Proof. By the assumption, for each pair of elements $s \in S$ and $r \in R$, the subalgebra $K\langle \overline{s}, \overline{r} \rangle$ of \overline{R} is not a free algebra. Then $\overline{S} := \pi_{\mathfrak{a}}(S) \in \text{Den}(\overline{R}, 0)$ (by Jategaonkar's Lemma, see [11, Lemma 9.21]). Therefore, $\text{ass}_R(S) = \mathfrak{a}$ and $R\langle S^{-1} \rangle \simeq \overline{S}^{-1}\overline{R} \simeq \overline{RS}^{-1}$. \square

The localizable sets of a ring and the posets $(L(R), \subseteq)$ and $(Loc(R), \rightarrow)$. For an ideal \mathfrak{a} of R, let $L(R, \mathfrak{a}) := \{S \in L(R) \mid \operatorname{ass}_R(S) = \mathfrak{a}\}$. Then

$$L(R) = \coprod_{\mathfrak{a} \in \operatorname{ass} L(R)} L(R, \mathfrak{a}) \tag{17}$$

is a disjoint union of non-empty sets. A set with binary operation, (I, \leq) , is a partially ordered set (a poset, for short) if $i \leq i$ for all $i \in I$, and if $i \leq j$ and $j \leq k$ for some elements $i, j, k \in I$ then $i \leq k$. The set $(L(R), \subseteq)$ is a poset (w.r.t. inclusion \subseteq), and $(L(R, \mathfrak{a}), \subseteq)$ is a sub-poset of $(L(R), \subseteq)$ for every $\mathfrak{a} \in \operatorname{ass} L(R)$.

Let $\operatorname{Loc}(R)$ be the set of R-isomorphism classes of all the localizations of the ring R, i.e. R-isomorphism classes in the set $\{R\langle S^{-1}\rangle \mid S\in \operatorname{L}(R)\}$. If the rings $R\langle S^{-1}\rangle$ and $R\langle T^{-1}\rangle$ are R-isomorphic for some $S,T\in\operatorname{L}(R)$ then $\operatorname{ass}_R(S)=\operatorname{ass}_R(T)$. Therefore,

$$Loc(R) = \coprod_{\mathfrak{a} \in ass L(R)} Loc(R, \mathfrak{a})$$
(18)

is a disjoint union of non-empty sets where $Loc(R, \mathfrak{a})$ is the set of R-isomorphism classes in the set $\{R\langle S^{-1}\rangle \mid S \in L(R, \mathfrak{a})\}.$

The set $(\text{Loc}(R, \mathfrak{a}), \to)$ is a poset where $A \to B$ if there is a necessarily unique R-homomorphism from A to B which is denoted by

$$\phi_{BA}:A\to B.$$

In particular, the unique R-homomorphism $\phi_{TS}: R\langle S^{-1}\rangle \to R\langle T^{-1}\rangle$ is also denoted by $\phi_{R\langle S^{-1}\rangle R\langle T^{-1}\rangle}$. If $A\to B$ for some rings $A\in \operatorname{Loc}(R,\mathfrak{a})$ and $B\in \operatorname{Loc}(R,\mathfrak{b})$ then

$$\mathfrak{a} \subset \mathfrak{b}.$$
 (19)

If, in addition, $A = R\langle S^{-1}\rangle$ and $B = R\langle T^{-1}\rangle$ for some localizable sets $S, T \in L(R)$ then if necessary we can assume that $S \subseteq T$ (for example, by replacing T by $\sigma_T^{-1}(B^{\times})$ where $\sigma_T : R \to B$, see Proposition 3.18.(1)).

The map

$$R\langle (\cdot)^{-1} \rangle : L(R) \to Loc(R), \quad S \mapsto R\langle S^{-1} \rangle,$$
 (20)

is an epimorphism of posets. For each ideal $\mathfrak{a} \in \operatorname{ass} L(R)$, it induces the epimorphism of posets,

$$R\langle (\cdot)^{-1} \rangle : L(R, \mathfrak{a}) \to Loc(R, \mathfrak{a}), \quad S \mapsto R\langle S^{-1} \rangle.$$
 (21)

For each ideal $\mathfrak{a} \in \operatorname{ass} L(R)$,

$$L(R, \mathfrak{a}) = \bigsqcup_{A \in Loc(R, \mathfrak{a})} L(R, \mathfrak{a}, A)$$
(22)

where $L(R, \mathfrak{a}, A) := \{ S \in L(R, \mathfrak{a}) \mid R \langle S^{-1} \rangle \simeq A, \text{ an } R\text{-isomorphism (necessarily unique)} \}.$

The poset of localizable multiplicative sets $(L(R)_m, \subseteq)$. An Ore set or a denominator set of a ring is a multiplicative set. This fact suggests the following definitions. Let $L(R)_m$ be the set of all localizable multiplicative sets of R. By Lemma 2.6, the localization of a ring at a localizable set is the same as the localization of the ring at the localizable multiplicative set that the set generates. For an ideal \mathfrak{a} of R, let $L(R,\mathfrak{a})_m := \{S \in L(R)_m \mid \operatorname{ass}_R(S) = \mathfrak{a}\}$. By Lemma 2.6, ass $L(R) = \operatorname{ass} L(R)_m := \{\operatorname{ass}_R(S) \mid S \in L(R)_m\}$, and so

$$L(R)_m = \coprod_{\mathfrak{a} \in \operatorname{ass} L(R)} L(R, \mathfrak{a})_m \tag{23}$$

is a disjoint union of non-empty sets. The map

$$L(R) \to L(R)_m, S \mapsto S_{mon}$$
 (24)

is an epimorphism of posets with section

$$L(R)_m \to L(R), S \mapsto S.$$
 (25)

Similarly, for each $\mathfrak{a} \in \operatorname{ass} L(R)$, the map

$$L(R, \mathfrak{a}) \to L(R, \mathfrak{a})_m, \quad S \mapsto S_{mon}$$
 (26)

is an epimorphism of posets with section

$$L(R, \mathfrak{a})_m \to L(R, \mathfrak{a}), S \mapsto S.$$
 (27)

The map

$$R\langle (\cdot)^{-1} \rangle : L(R)_m \to Loc(R), \quad S \mapsto R\langle S^{-1} \rangle,$$
 (28)

is an epimorphism of posets. For each ideal $\mathfrak{a} \in \operatorname{ass} L(R)$, it induces the epimorphism of posets,

$$R\langle (\cdot)^{-1} \rangle : L(R, \mathfrak{a})_m \to Loc(R, \mathfrak{a}), \quad S \mapsto R\langle S^{-1} \rangle.$$
 (29)

For each ideal $\mathfrak{a} \in \operatorname{ass} L(R)$,

$$L(R, \mathfrak{a})_m = \bigsqcup_{A \in Loc(R, \mathfrak{a})} L(R, \mathfrak{a}, A)_m$$
(30)

where $L(R, \mathfrak{a}, A)_m := \{ S \in L(R, \mathfrak{a})_m \mid R(S^{-1}) \simeq A, \text{ an } R\text{-isomorphism (necessarily unique)} \}.$

Direct limits of rings. Let (I, \leq) be a poset, $\{R_i\}_{i\in I}$ be rings, and for each pair of elements i and j of I such that $i \leq j$ there is a ring homomorphism $f_{ji}: R_i \to R_j$ such that f_{ii} is the identity map for all $i \in I$, and

$$f_{ki} = f_{kj} f_{ji}$$

for all elements $i, j, k \in I$ such that $i \leq j \leq k$. The system (R_i, f_{ij}) is called a direct system of rings on I. A ring $\varinjlim_{i \in I} R_i$ together with ring homomorphisms $g_i : R_i \to \varinjlim_{i \in I} R_i$ such that $g_i = g_j f_{ji}$ for all $i \leq j$ is called the direct limit of the direct system (R_i, f_{ji}) if it satisfies the following universal property: For each system of ring homomorphisms $g_i' : R_i \to A$ such that $g_i' = g_j' f_{ji}$ for all $i \leq j$, there is a unique homomorphism

$$\alpha: \varinjlim_{i\in I} R_i \to A$$

such that $g'_i = \alpha g_i$ for all $i \in I$. It follows from the universal property that the direct limit is unique (up to isomorphism).

Suppose that the poset (I, \leq) is a directed set, that is for each pair of elements $i, j \in I$ there is an element $k \in I$ such that $i \leq k$ and $j \leq k$. On the disjoint union $\coprod_{i \in I} R_i$, we say that elements $a_i \in R_i$ and $a_j \in R_j$ are equivalent, $a_i \sim a_k$, if

$$f_{ki}(a_i) = f_{kj}(a_j)$$

for some element $k \in I$ such that $i, j \leq k$. Then the set of equivalence classes, $\coprod_{i \in I} R_i / \sim$, is isomorphic to $\varinjlim_{i \in I} R_i$ where for each $i \in I$, the map

$$g_i: R_i \to \varinjlim_{i \in I} R_i, \ r \mapsto \widetilde{r}$$

is a ring homomorphism and \tilde{r} is the equivalence class of the element r, such that for all $i \leq j$, $g_i = g_i f_{ii}$.

Let (I, \leq) be a poset, (R_i, f_{ij}) be a direct system of rings on I. Let \mathcal{I} be an additive subgroup of the direct sum of K-modules $\bigoplus_{i \in I} R_i$ generated by all the elements

$$\{a_i - a_j \mid f_{ki}(a_i) = f_{kj}(a_j), a_i \in R_i, a_j \in R_j, i \le k, j \le k\}.$$
(31)

By the definition, \mathcal{I} is a K-submodule of $\bigoplus_{i \in I} R_i$. Let $*_{i \in I} R_i$ be the free product of K-algebras R_i . For each $i \in I$, there is a natural ring homomorphism $\theta_i : R_i \to *_{i \in I} R_i$, and the ring $*_{i \in I} R_i$ is generated by the images of the rings R_i . Let

$$\theta: \bigoplus_{i \in I} R_i \to *_{i \in I} R_i, \ r_i \mapsto \theta_i(r_i) \text{ for } r_i \in R_i.$$
 (32)

An element $i \in I$ is the largest element of the poset I if for all elements $j \in I$, $j \leq i$.

Theorem 2.14 Let (I, \leq) be a poset and (R_i, f_{ij}) be a direct system of rings on I. Then

1. $\varinjlim_{i \in I} R_i \simeq *_{i \in I} R_i/(\theta(\mathcal{I}))$ where $(\theta(\mathcal{I}))$ is the ideal of the free product of K-algebras $*_{i \in I} R_i$ generated by the set $\theta(\mathcal{I})$.

2. Suppose that for each element $i \in I$ there is an element $j \in \max(I)$ such that $i \leq j$. Then

$$\lim_{i \in I} R_i \simeq *_{\mu \in \max(I)} R_{\mu} / \mathfrak{f}$$

where $*_{\mu \in \max(I)} R_{\mu}$ is a free product of the set of K-algebras $\{R_{\mu} \mid \mu \in \max(I)\}$ and \mathfrak{f} is an ideal of $*_{\mu \in \max(I)} R_{\mu}$ which is generated by the set $\theta(\bigoplus_{\mu \in \max(I)} R_{\mu}) \cap \mathcal{I}$.

3. Suppose that an element $i \in I$ is the largest element of the poset I. Then $\lim_{j \in I} R_j \simeq R_i$.

Proof. 1. To prove statement 1 it suffices to show that the ring $*_{i\in I}R_i/(\theta(\mathcal{I}))$ satisfies the universal property of the ring $\varinjlim_{i\in I}R_i$. Let $g_i':R_i\to A,\ i\in I$, be a set of ring homomorphisms such that $g_i'=g_j'f_{ji}$ for all $i\leq j$. Then there is a unique ring homomorphism $\rho:*_{i\in I}R_i\to A$ such that $g_i'=\rho\theta_i$ for all $i\in I$. Then

$$\rho\theta(\mathcal{I}) = 0$$

since for all elements $a_i \in R_i$ and $a_j \in R_j$ such that $f_{ki}(a_i) = f_{kj}(a_j)$ for some $k \in I$ such that $i, j \leq k$,

$$\rho(\theta_i(a_i) - \theta_j(a_j)) = g_i'(a_i) - g_j'(a_j) = g_k' f_{ki}(a_i) - g_k' f_{kj}(a_j) = g_k' (f_{ki}(a_i) - f_{kj}(a_j)) = g_k'(0) = 0.$$

- 2. Let $\varinjlim_{i \in I}^K R_i$ be the direct limit of the K-modules R_i .
- (i) $\varinjlim_{i\in I}^K R_i = \left(\bigoplus_{i\in I} R_i\right)/\mathcal{I}$: By the definition, the K-module $\left(\bigoplus_{i\in I} R_i\right)/\mathcal{I}$ satisfies the universal property of the direct limit of K-modules R_i and the statement (i) follows from the uniqueness of the direct limit of K-modules.
- (ii) $\left(\bigoplus_{i\in I} R_i\right)/\mathcal{I} \simeq \left(\bigoplus_{\mu\in\max(I)} R_\mu\right)/\left(\bigoplus_{\mu\in\max(I)} R_\mu\right)\cap \mathcal{I}$: By the assumption, for each element $i\in I$ there is an element $j\in\max(I)$ such that $i\leq j$. Then it follows from the inclusion $\bigoplus_{\mu\in\max(I)} R_\mu\subseteq\bigoplus_{i\in I} R_i$ that

$$\Big(\bigoplus_{i\in I} R_i\Big)/\mathcal{I} = \Big(\bigoplus_{\mu\in\max(I)} R_i + \mathcal{I}\Big)/\mathcal{I} \simeq \Big(\bigoplus_{\mu\in\max(I)} R_\mu\Big)/\Big(\bigoplus_{\mu\in\max(I)} R_\mu\Big)\bigcap \mathcal{I}.$$

- (iii) $\varinjlim_{i \in I}^K R_i = \left(\bigoplus_{i \in I} R_i\right) / \mathcal{I} \simeq \left(\bigoplus_{\mu \in \max(I)} R_\mu\right) / \left(\bigoplus_{\mu \in \max(I)} R_\mu\right) \cap \mathcal{I}$: The statement (iii) follows from the statements (i) and (ii).
- (iv) The $ring *_{\mu \in \max(I)} R_{\mu}/\mathfrak{f}$ satisfies the universal property of the $ring \varinjlim_{i \in I} R_i$: Let $g_i' : R_i \to A$, $i \in I$, be a set of ring homomorphisms such that $g_i' = g_j' f_{ji}$ for all $i \leq j$. By statement 1 and the statement (iii), the homomorphisms g_i' induce a unique ring homomorphism, $\xi : *_{\mu \in \max(I)} R_{\mu}/\mathfrak{f} \to A$ such that $g_{\mu}' = \xi \theta_{\mu}$ for all $\mu \in \max(I)$. Now, statement 2 follows from the statement (iv).
 - 3. Statement 3 follows from statement 2. \square

Let $T \in L(R)$. Then the power set of T, i.e. the set of all subsets of the set T, $(\mathcal{P}(T), \subseteq)$, is a poset. The system $\{R\langle S^{-1}\rangle, \phi_{SS'}\}$ is a direct system where $S, S' \in \mathcal{P}(T)$. Corollary 2.15.(1) describe the direct limit $\varinjlim_{\{S \subseteq T\}} R\langle S^{-1}\rangle$.

Corollary 2.15 Let R be a ring and $T \in L(R)$. Then

- 1. $R\langle T^{-1}\rangle \simeq \varinjlim_{S\subset T_s} R\langle S^{-1}\rangle$.
- 2. $\operatorname{ass}_R(T) = \bigcup_{S \subseteq T} \operatorname{ass}_R(S)$.

Proof. 1. Statement 1 follows from Proposition 2.8.(2).

2. Statement 2 follows from statement 1. \square

We collect two results on direct limits, Lemma 2.16 and Lemma 5.8, that are used later in the paper and are interesting on their own.

Lemma 2.16 Let (I, \leq) be a directed set and (R_i, f_{ij}) be a direct system of rings on I. Suppose that all rings R_i , where $i \in I$, are isomorphic to a ring R and all the maps f_{ij} are isomorphisms. Then $\lim_{i \in I} R_i \simeq R$.

Proof. Fix an element $i \in I$ and an isomorphism $g_i : R_i \to R$. For each element $j \in I$ such that $i \leq j$, let $g_j := g_i f_{ij} : R_j \to R$ where $f_{ij} := f_{ji}^{-1}$. For each element $k \in I$ such that $i \leq k$, fix an element $j \in I$ such that $i \leq j$ and $k \leq j$. Let

$$g_k := g_i f_{ij} f_{jk} : R_k \to R$$
 where $f_{jk} := f_{kj}$.

Then it is easy to verify that the map g_k does not depend on the choice of the element j and $(R, g_i)_{i \in I}$ satisfies the universal property of the direct limit for the direct system (R_i, f_{ij}) , and so $R = \varinjlim_{i \in I} R_i$. \square

Lemma 5.8 gives an example of a direct system of rings with zero direct limit.

Lemma 2.17 Let $J = \{1\} \coprod I$ be a disjoint union of two sets where the set I consists at least of two elements. The set J is a poset where 1 < i for all $i \in I$. Let $\{R_i \mid i \in I\}$ be a set of rings and $A_1 = \prod_{i \in I} R_i$. For each $i \in I$, let the map $f_{1i} : R_1 \to R_i$ be the projection onto R_i , $f_{11} = \operatorname{id}_{R_1}$ and $f_{ii} = \operatorname{id}_{R_i}$ (the identity maps). Then $\varinjlim_{i \in I} R_i \simeq \{0\}$.

Proof. For each $i \in I$, let 1_i be the identity element of the ring R_i , and $1'_i$ be an element of R_1 where on the *i*'th place is 1_i and zeros are elsewhere. Clearly, $\max(J) = I$. Now, by Theorem 2.14.(2), $\varinjlim_{i \in I} R_j \simeq *_{i \in I} R_i/\mathfrak{f}$. Since for all $i \in I$,

$$1_i = 1_i - 0 = (1_i - 1'_i) + (1'_i - p_j(1'_i)) \in \mathfrak{f}$$

where $j \neq i$, $\varinjlim_{i \in J} R_i \simeq \{0\}$ (see Theorem 2.14.(2) for the definition of the ideal \mathfrak{f}). \square

The absolute quotient ring $Q_a(R)$ of a ring R. The set $(L(R), \subseteq)$ is a poset and the system $\{R\langle S^{-1}\rangle, \phi_{ST}\}$ is a direct system of rings where $S, T \in L(R)$.

Definition. The direct limit $Q_a(R) := \varinjlim_{S \in \mathcal{L}(R)} R \langle S^{-1} \rangle$ is called the absolute quotient ring of the ring R. Let $\sigma_a : R \to Q_a(R)$ be a unique R-homomorphism, $\mathfrak{a}_a(R) := \ker(\sigma_a), R_a := R/\mathfrak{a}_a(R), \pi_a : R \to R_a, r \mapsto r+\mathfrak{a}_a$, and $\overline{\sigma_a} : R_a \to Q_a(R)$ is a unique R-homomorphism. Clearly, $\sigma_a = \overline{\sigma_a}\pi_a$.

$$\textbf{Theorem 2.18} \ \ Q_a(R) = \begin{cases} R \langle S^{-1} \rangle & \textit{if } \max \mathsf{L}(R) = \{S\}, \\ \{0\} & \textit{otherwise}, \end{cases} \ \textit{and} \ \mathfrak{a}_a(R) = \begin{cases} \mathsf{ass}_R(S) & \textit{if } \max \mathsf{L}(R) = \{S\}, \\ R & \textit{otherwise}. \end{cases}$$

Proof. Suppose that $\max L(R) = \{S\}$. Then, by Theorem 3.1.(2), the element S is the largest element of the poset L(R), and the statement follows from Theorem 2.14.(3).

Suppose that the set $\max L(R)$ contains at least two elements, say S and T. The set $S := \sigma_a^{-1}(Q_a(R)^{\times})$ is a localizable set in R since $\sigma_a(S) \subseteq Q_a(R)^{\times}$. The diagram below is commutative

$$\begin{array}{ccc} R & \stackrel{\sigma_S}{\rightarrow} & R\langle S^{-1} \rangle \\ & \stackrel{\sigma_a}{\searrow} & \downarrow^{\phi_S} \\ & & Q_a(R) \end{array}$$

where ϕ_S is a unique R-homomorphism. Then it follows from the inclusion $\phi_S(R\langle S^{-1}\rangle^{\times})\subseteq Q_a(R)^{\times}$ that

$$\mathcal{S} = \sigma_a^{-1}(Q_a(R)^{\times}) = (\phi_S \sigma_S)^{-1}(Q_a(R)^{\times}) = \sigma_S^{-1} \phi_S^{-1}(Q_a(R)^{\times}) \supseteq \sigma_S^{-1}(R \langle S^{-1} \rangle^{\times}) = S,$$

by Corollary 3.7.(1). Similarly, $S \supseteq T$. Therefore, $S \supseteq S \cup T$, and so the set $S \cup T$ is a localizable set of R as a subset of the localizable set S, a contradiction (since $S, T \in \max L(R)$ and $S \neq T$).

The proof of the first equality of the theorem is complete. The second equality of the theorem follows from the first. \Box

Example. If R is a commutative domain then the set $C_R = R \setminus \{0\}$ is the largest element of L(R). By Theorem 2.18, $Q_a(R) = Q_{cl}(R)$ is the classical quotient ring of R, i.e. the field of fractions of R, and $\mathfrak{a}_a(R) = 0$.

Lemma 2.19 If R is a simple Artinian rings then $\max L(R) = \{R^{\times}\}, Q_a(R) = R \text{ and } \mathfrak{a}_a(R) = 0.$

Proof. Every non-unit of the simple Artinian R ring is a zero divisor. Since the ring R is simple every non-unit $r \in R$ is a non-localizable element since $\operatorname{ass}_R(r) = R$. Therefore, $\mathcal{L}L(R) = R^{\times}$ and $\mathcal{NL}L(R) = R \setminus R^{\times}$. Hence, $\max L(R) = \{R^{\times}\}$. Now, by Theorem 2.18, $Q_a(R) = R \setminus (R^{\times})^{-1} = R$ and $\mathfrak{a}_a(R) = 0$. \square

We need the lemma below in order to prove Proposition 2.21 which gives a practical sufficient condition for $Q_a(R) = \{0\}$ and $|\max L(R)| \ge 2$.

- **Lemma 2.20** 1. $Q_a(R) \simeq (*_{S \in \max L(R)} R\langle S^{-1} \rangle)/\mathfrak{f}_a$ where \mathfrak{f}_a is an ideal of the K-algebra $*_{S \in \max L(R)} R\langle S^{-1} \rangle$ which is generated by the set $\theta(\bigoplus_{S \in \max L(R)} R\langle S^{-1} \rangle \cap \mathcal{I})$, see Theorem 2.14.(2).
 - 2. $\mathfrak{a}_a(R) \supseteq \sum_{S \in \max L(R)} \operatorname{ass}_R(S)$.
 - 3. $Q_a(R) = \{0\} \text{ iff } \mathfrak{a}_a(R) = R. \text{ If } \sum_{S \in \max L(R)} \operatorname{ass}_R(S) = R \text{ then } Q_a(R) = \{0\}.$

Proof. 1. Statement 1 is a particular case of Theorem 2.14.(2).

2. Let $S \in \max L(R)$. Then for each element $r \in \operatorname{ass}_R(S)$ and each $T \in \max L(R)$,

$$\sigma_T(r) = (\sigma_T(r) - r) - (0 - r) = (\sigma_T(r) - r) - (\sigma_S(r) - r) \in \theta \Big(\bigoplus_{S \in \max L(R)} R\langle S^{-1} \rangle \cap \mathcal{I} \Big),$$

and so $\operatorname{ass}_R(S) \subseteq \mathfrak{a}_a(R)$, and statement 2 follows.

3. $Q_a(R) = \{0\}$ iff 1 = 0 in $Q_a(R)$ iff $\mathfrak{a}_a(R) = R$. If $\sum_{S \in \max L(R)} \operatorname{ass}_R(S) = R$ then, by statement 2, $\mathfrak{a}_a(R) = R$, and so $Q_a(R) = \{0\}$. \square

Proposition 2.21 Given localizable sets $\{S_i | i \in I\}$ of a ring R such that $\sum_{i \in I} \operatorname{ass}_R(S_i) = R$. Then $Q_a(R) = \{0\}$ and $|\max L(R)| \ge 2$.

Proof. By Theorem 3.1.(2), every localizable set S_i is contained in a maximal localizable set S_i' . Since $\operatorname{ass}_R(S_I) \subseteq \operatorname{ass}_R(S_I')$ and $\sum_{i \in I} \operatorname{ass}_R(S_i) = R$, we have that $\sum_{i \in I} \operatorname{ass}_R(S_i') = R$, and so $\sum_{S \in \max L(R)} \operatorname{ass}_R(S) = R$. By Lemma 2.20.(2), $\mathfrak{a}_a(R) = R$, and so $Q_a(R) = 0$, by Lemma 2.20.(3). By Theorem 2.18, $|\max L(R)| \ge 2$. \square

Corollary 2.22 Let $A = \prod_{i \in I} A_i$ be a product of rings where I is an arbitrary set that contains at least two elements. Then $Q_a(A) = 0$ and $|\max L(R)| \ge 2$.

Proof. By the assumption, the set I contains at least two elements. For each $i \in I$, the set $S_i = A_i^{\times} \times \prod_{j \neq i} A_j$ is a localizable set of A with $\operatorname{ass}_R(S_i) = \{0\} \times \prod_{j \neq i} A_j$ since $A\langle S_i^{-1} \rangle \simeq A_i$. Let i and j be distinct elements of the set I. Since $A = \operatorname{ass}_A(S_i) + \operatorname{ass}_A(S_j)$, we have that $Q_a(A) = \{0\}$ and $|\max L(R)| \geq 2$, by Proposition 2.21. \square

Every Ore set is a localizable set. Let S be an Ore set of the ring R. Theorem 2.23 states that every Ore set is localizable, gives an explicit description of this ideal and the ring $R\langle S^{-1}\rangle$, and $\mathfrak{a}(S) = \mathrm{ass}_R(S)$. So, the localization an any Ore set is an example of the perfect localization. Furthermore, Theorem 2.23 also states that the ring $R\langle S^{-1}\rangle$ is R-isomorphic to the localization $\overline{S}^{-1}\overline{R}$ of the ring \overline{R} at the denominator set \overline{S} of \overline{R} .

Theorem 2.23 Let R be a ring and $S \in Ore(R)$.

- 1. [1, Theorem 4.15] Every Ore set is a localizable set.
- 2. [7, Theorem 1.6.(1)] $\mathfrak{a} := \{r \in R \mid srt = 0 \text{ for some elements } s, t \in S\}$ is an ideal of R such that $\mathfrak{a} \neq R$.
- 3. [7, Theorem 1.6.(2)] Let $\pi: R \to \overline{R} := R/\mathfrak{a}$, $r \mapsto \overline{r} = r + \mathfrak{a}$. Then $\overline{S} := \pi(S) \in \text{Den}(\overline{R}, 0)$, $\mathfrak{a} = \mathfrak{a}(S) = \text{ass}_R(S)$, $S \in L(R, \mathfrak{a})$, and $S^{-1}R \simeq \overline{S}^{-1}\overline{R}$, an R-isomorphism. In particular, every Ore set is localizable.

Criterion for a left Ore set to be a left localizable set. For a ring R and its ideal \mathfrak{a} , let

$$'\mathrm{Den}_{l}(R,\mathfrak{a}) := \{ S \in \mathrm{Den}_{l}(R) \mid \mathrm{ass}_{l}(S) = \mathfrak{a}, S \subseteq '\mathcal{C}_{R} \},$$

$$\mathrm{Den}'_{r}(R,\mathfrak{a}) := \{ S \in \mathrm{Den}_{r}(R) \mid \mathrm{ass}_{r}(S) = \mathfrak{a}, S \subseteq \mathcal{C}'_{R} \}.$$

Theorem 2.24.(1) is a criterion for a left Ore set to be a left localizable set and Theorem 2.24.(2) describes the structure of the localization of a ring at a localizable left Ore set.

Theorem 2.24 ([7, Theorem 1.5]) Let R be a ring, $S \in \text{Ore}_l(R)$, and $\mathfrak{a} = \text{ass}_R(S)$. Then

- 1. $S \in L(R)$ iff ' $\mathfrak{a} \neq R$ where the ideal ' $\mathfrak{a} = '\mathfrak{a}(S)$ of R is as in Proposition 2.10.(2) and (13).
- 2. Suppose that $\mathfrak{a} \neq R$. Let $\mathfrak{a} = R$ $\mathfrak{a} = R$
 - (a) $'S \in '\mathrm{Den}_l('R)$.
 - (b) $\mathfrak{a} = '\pi^{-1}(\text{ass}_{l}('S)).$
 - (c) $S^{-1}R \simeq 'S^{-1}'R$, an R-isomorphism.

[7, Theorem 2.11] is a criterion for a right Ore set to be a localizable set.

Definition 2.25 ([7]) A multiplicative set S of a ring R is called a left localizable set of R if

$$R\langle S^{-1}\rangle = \{\overline{s}^{-1}\overline{r} \mid \overline{s} \in \overline{S}, \overline{r} \in \overline{R}\} \neq \{0\}$$

where $\overline{R} = R/\mathfrak{a}$, $\mathfrak{a} = \operatorname{ass}_R(S)$ and $\overline{S} = (S + \mathfrak{a})/\mathfrak{a}$, i.e., every element of the ring $R\langle S^{-1} \rangle$ is a left fraction $\overline{s}^{-1}\overline{r}$ for some elements $\overline{s} \in \overline{S}$ and $\overline{r} \in \overline{R}$. Similarly, a multiplicative set S of a ring R is called a right localizable set of R if

$$R\langle S^{-1}\rangle = \{\overline{rs}^{-1} \mid \overline{s} \in \overline{S}, \overline{r} \in \overline{R}\} \neq \{0\},$$

i.e., every element of the ring $R\langle S^{-1}\rangle$ is a right fraction \overline{rs}^{-1} for some elements $\overline{s} \in \overline{S}$ and $\overline{r} \in \overline{R}$. A right and left localizable set of R is called a localizable set of R.

3 Maximal localizable sets, the localization radical, the sets of localizable and non-localizable elements

For a ring R, the following concepts are introduced: the maximal localizable sets, the localization radical $\operatorname{Lrad}(R)$, the set of localizable elements $\mathcal{L}L(R)$, the set of non-localizable elements $\mathcal{N}\mathcal{L}L(R)$, the set of completely localizable elements $\mathcal{C}L(R)$. All these objects are characteristic subsets of R, i.e. they are invariant under the action of the automorphism group of R. Proposition 3.10 shows that there are tight connections between the sets $\mathcal{L}L(R)$, $\mathcal{N}\mathcal{L}L(R)$, and $\operatorname{Lrad}(R)$. For an arbitrary ring, Theorem 3.1 states that the set of maximal localizable set is a non-empty set and every localizable set is a subset of a maximal localizable set.

Proposition 3.5.(1), is an explicit description of the largest element $S(R, \mathfrak{a}, A)$ of the poset $(L(R, \mathfrak{a}, A), \subseteq)$. Corollary 3.6 is a criterion for a set $S \in L(R, \mathfrak{a}, A)$ to be the largest element of the poset $L(R, \mathfrak{a}, A)$. Theorem 3.8 describes the sets $\max L(R)$ and $\max Loc(R)$, presents a bijection between them, and shows that $\max Loc(R) \neq \emptyset$. Similarly, for each ideal $\mathfrak{a} \in \operatorname{ass} L(R)$, Theorem 3.9

describes the sets $\max L(R, \mathfrak{a})$ and $\max Loc(R, \mathfrak{a})$, presents a bijection between them, and shows that they are nonempty sets.

The \mathfrak{a} -absolute quotient ring $Q_a(R,\mathfrak{a})$ is introduced for each ideal $\mathfrak{a} \in \operatorname{ass} L(R)$. Theorem 3.15 gives an explicit description of the ring $Q_a(R,\mathfrak{a})$, a criterion for $Q_a(R,\mathfrak{a}) \neq \{0\}$, and a criterion for $\mathfrak{a}_{a,\mathfrak{a}} = \mathfrak{a}$.

Theorem 3.17.(1) is an R-isomorphism criterion for localizations of the ring R and Theorem 3.17.(2) is a criterion for existing of an R-homomorphism between localizations of R. Proposition 3.14 contains some properties of CL(R) and \mathfrak{c}_R . Theorem 3.20 is a criterion for the ring $R\langle S^{-1}\rangle$ to be an R-isomorphic to a localization of R at a denominator set and it also describes all such denominator sets.

For a denominators set $T \in \operatorname{Den}_*(R,\mathfrak{a})$, Proposition 3.21 describes all localizable sets S of the ring R such that the ring $R\langle S^{-1}\rangle$ is an R-isomorphic to the localization of R at T. For a localizable multiplicative set S of the ring R, Proposition 3.22 is a criterion for the set S to be a left denominator set of R.

The set $\max L(R)$ of maximal elements of L(R) is a non-empty set. For a ring R, the set $\max. \mathrm{Den}_l(R)$ of maximal left denominator sets (w.r.t. \subseteq) is a non-empty set, [3, Lemma 3.7.(2)] and the sets of maximal localizable left or right or two-sided Ore sets are also non-empty sets [7, Theorem 1.9]. Let $\max L(R)$ be the set of maximal (w.r.t. \subseteq) elements of the set L(R). An element of the set $\max L(R)$ is called a maximal localizable set of the ring R.

Theorem 3.1 Let R be a ring. Then

- 1. $\max L(R) \neq \emptyset$.
- 2. Each set $S \in L(R)$ is contained in a set $S \in \max L(R)$.

Proof. 1. The theorem follows from Zorn's Lemma. Let $\{S_i\}_{i\in\Gamma}\subseteq \mathrm{L}(R)$ be an ordered by inclusion set of localizable sets in R indexed by an ordinal Γ , i.e. for each pair $i\leq j$ in Γ , $S_i\subseteq S_j$ and if $i\in\Gamma$ is a limit ordinal then $S_i=\bigcup_{j< i}S_j$. In view of Zorn's Lemma, it suffices to show that $\bigcup_{i\in\Gamma}S_i\in\mathrm{L}(R)$. By (7) and (9), we have the direct system of ring homomorphisms, $(R\langle S_i^{-1}\rangle,\phi_{S_jS_i})$, where for each pair $i\leq j$, the map $\phi_{S_jS_i}:R\langle S_i^{-1}\rangle\to R\langle S_j^{-1}\rangle$ is defined in (7). Since the rings $R\langle S_i^{-1}\rangle$ are unital and $\phi_{S_jS_i}(1)=1$ for all $i\leq j$, the direct limit $\varinjlim R\langle S_i^{-1}\rangle$ is a nonzero ring and and is isomorphic to $R\langle \left(\bigcup_{i\in\Gamma}S_i\right)^{-1}\rangle$. Therefore, $\bigcup_{i\in\Gamma}S_i\in\mathrm{L}(R)$.

2. Repeat the proof of statement 1 for the set of all localizable sets that contain the set S. \square

Corollary 3.2 Let R be a ring and $T \in \max L(R)$. Then

1.
$$R\langle T^{-1}\rangle \simeq \varinjlim_{\{S\subset T\}} R\langle S^{-1}\rangle$$
.

2.
$$\operatorname{ass}_R(T) = \bigcup_{S \subseteq T} \operatorname{ass}_R(S)$$
.

Proof. The corollary is a particular case of Corollary 2.15. \square

Let $\max \operatorname{ass} L(R)$ be the set of maximal elements of the set $\operatorname{ass} L(R) = \{\operatorname{ass}_R(S) \mid S \in L(R)\}$ and $\operatorname{ass} \max L(R) := \{\operatorname{ass}_R(S) \mid S \in \max L(R)\}.$

Corollary 3.3 $\max \operatorname{ass} L(R) \subseteq \operatorname{ass} \max L(R)$.

Proof. The corollary follows from Theorem 3.1. \square

A set of subsets is called an *antichain* if the sets are *incomparable*, that is none of the sets is a subset of another one.

Question. Is $\max \operatorname{ass} L(R) = \operatorname{ass} \max L(R)$? I.e. the set $\operatorname{ass} \max L(R)$ is an antichain.

The largest element $S(R, \mathfrak{a}, A)$ in $(L(R, \mathfrak{a}, A) \subseteq)$ and its characterizations where $A \in Loc(R, \mathfrak{a})$.

Lemma 3.4 Given sets $S, S \in L(R, \mathfrak{a}, A)$ such that $S \subseteq S$. Then $T \in L(R, \mathfrak{a}, A)$ for all subsets T of R such $S \subseteq T \subseteq S$.

Proof. The inclusions $S \subseteq T \subseteq \mathcal{S}$, imply that $\mathfrak{a} = \operatorname{ass}_R(S) \subseteq \operatorname{ass}_R(T) \subseteq \operatorname{ass}_R(\mathcal{S}) = \mathfrak{a}$, and so $\operatorname{ass}_R(T) = \mathfrak{a}$.

It suffices to show that the rings $R\langle T^{-1}\rangle$ and A are R-isomorphic. Since $S\subseteq \mathcal{S}$ and $S,\mathcal{S}\in L(R,\mathfrak{a},A)$, the unique R-homomorphism $A\simeq R\langle S^{-1}\rangle\to R\langle \mathcal{S}^{-1}\rangle\simeq A$ is the identity map id_A . In particular, $\sigma_S(T)\subseteq\sigma_S(\mathcal{S})\subseteq A^\times$ since $T\subseteq \mathcal{S}$ where $\sigma_S:R\to R\langle S^{-1}\rangle\simeq A$. Now,

$$R\langle T^{-1}\rangle \simeq R\langle S^{-1}\rangle \langle \sigma_S(T)^{-1}\rangle \simeq A\langle \sigma_S(T)^{-1}\rangle = A$$

since $\sigma_S(T) \subseteq A^{\times}$, as required. \square

Proposition 3.5.(1), is an explicit description of the largest element $S(R, \mathfrak{a}, A)$ of the poset $(L(R, \mathfrak{a}, A), \subset)$.

Proposition 3.5 Let $A \in \text{Loc}(R, \mathfrak{a})$, A^{\times} be the group of units of the ring A, and $\sigma : R \to A$, $r \mapsto r + \mathfrak{a}$. Then

1. The set

$$\mathcal{S}(R,\mathfrak{a},A) := \sigma^{-1}(A^{\times}) = \bigcup_{S \in \mathcal{L}(R,\mathfrak{a},A)} S$$

is the largest element of the poset $(L(R, \mathfrak{a}, A), \subseteq)$. In particular, the poset $(L(R, \mathfrak{a}, A), \subseteq)$ is a directed set such that if $S \in L(R, \mathfrak{a}, A)$ then $T \in L(R, \mathfrak{a}, A)$ for all subsets T of R such that $S \subseteq T \subseteq S(R, \mathfrak{a}, A)$.

- 2. The set $S(R, \mathfrak{a}, A)$ is a multiplicative set in R such that $S(R, \mathfrak{a}, A) + \mathfrak{a} = S(R, \mathfrak{a}, A)$.
- 3. $A \simeq R\langle \mathcal{S}(R,\mathfrak{a},A)^{-1}\rangle \simeq \varinjlim_{S \in L(R,\mathfrak{a},A)} R\langle S^{-1}\rangle$, R-isomorphisms.

Proof. 1. Let $S := \sigma^{-1}(A^{\times})$.

- (i) $S \in L(R, \mathfrak{a}, A)$: Fix an element $S \in L(R, \mathfrak{a}, A)$. Then the statement (i) follows from Lemma 2.9.(3).
- (ii) $S = \bigcup_{S \in L(R,\mathfrak{a},A)} S$: By the statement (i), $S \in L(R,\mathfrak{a},A)$. For all $S \in L(R,\mathfrak{a},A)$, $\sigma(S) \subseteq A^{\times}$, hence $S \subseteq S$, and the statement (ii) follows.

By the statement (ii), the set S is the largest element of the poset ($L(R, \mathfrak{a}, A), \subseteq$), and so the poset ($L(R, \mathfrak{a}, A), \subseteq$) is a directed set. By Lemma 3.4, if $S \in L(R, \mathfrak{a}, A)$ then $T \in L(R, \mathfrak{a}, A)$ for all subsets T of R such that $S \subseteq T \subseteq S(R, \mathfrak{a}, A)$.

- 2. By statement 1, $S(R, \mathfrak{a}, A) = \sigma^{-1}(A^{\times})$, and so the set $S(R, \mathfrak{a}, A)$ is a multiplicative set in R such that $S(R, \mathfrak{a}, A) + \mathfrak{a} = S(R, \mathfrak{a}, A)$.
- 3. Since $S \in L(R, \mathfrak{a}, A)$, there is an R-isomorphism $A \simeq R\langle S(R, \mathfrak{a}, A)^{-1} \rangle$. By statement 1, the set $L(R, \mathfrak{a}, A)$ is a directed set w.r.t. inclusion such that for all sets $S, T \in L(R, \mathfrak{a}, A)$ such that $S \subseteq T$, $R\langle S^{-1} \rangle \simeq R\langle T^{-1} \rangle \simeq A$, R-isomorphisms. By Lemma 2.16, $\varinjlim_{S \in L(R, \mathfrak{a}, A)} R\langle S^{-1} \rangle \simeq A$, an R-isomorphism. \square

Corollary 3.6 is a criterion for a set $S \in L(R, \mathfrak{a}, A)$ to be the largest element of the poset $L(R, \mathfrak{a}, A)$.

Corollary 3.6 Let $S \in L(R, \mathfrak{a}, A)$. Then $S = \mathcal{S}(R, \mathfrak{a}, A)$ iff for every element $r \in R \setminus S$ the rings A and $R(S \cup \{r\})^{-1}$ are not R-isomorphic.

Proof. (i) For all elements $x \in S \setminus S$, $S \cup \{x\} \in L(R, \mathfrak{a}, A)$: Since $S \in L(R, \mathfrak{a}, A)$, we must have that $S \subseteq S := S(R, \mathfrak{a}, A)$, by the maximality of the element S. Let $x \in S \setminus S$ and $T = S \cup \{x\}$. Then $S \subset T \subseteq S$, and so $\mathfrak{a} = \mathrm{ass}_R(S) \subseteq \mathrm{ass}_R(T) \subseteq \mathrm{ass}_R(S) = \mathfrak{a}$, i.e. $\mathrm{ass}_R(T) = \mathfrak{a}$ and there are unique R-homomorphisms

$$A \simeq R\langle S^{-1} \rangle \to R\langle T^{-1} \rangle \to R\langle S^{-1} \rangle \simeq A.$$

Hence, $T \in L(R, \mathfrak{a}, A)$.

(ii) For all elements $x \in R \setminus S$, $S \cup \{x\} \notin L(R, \mathfrak{a}, A)$: The statement (ii) follows from the maximality of the element S.

Now, the corollary follows from the statements (i) and (ii). \Box

By Proposition 3.5, for rings $A_1, A_2 \in Loc(R)$,

$$A_1 \to A_2$$
 iff $S_l(R, \mathfrak{a}, A_1) \subseteq S_l(R, \mathfrak{a}, A_2)$.

Corollary 3.7 is a description of the largest element of the directed set $(L(R, \mathfrak{a}, A), \subseteq)$.

Corollary 3.7 Let $S \in \max L(R)$, $\mathfrak{a} = \operatorname{ass}_R(S)$, $A = R(S^{-1})$, and $\sigma = \sigma_S : R \to A$. Then

1. The set

$$\mathcal{S}(A) := \mathcal{S}(R, \mathfrak{a}, A) = \sigma^{-1}(A^{\times}) = \bigcup_{S \in \mathcal{L}(R, \mathfrak{a}, A)}$$

is the largest element of the directed set $(L(R, \mathfrak{a}, A), \subseteq)$, and if $S \in L(R, \mathfrak{a}, A)$ then $T \in L(R, \mathfrak{a}, A)$ for all subsets T of R such that $S \subseteq T \subseteq S$.

- 2. The set S(A) is a multiplicative set in R such that $S(A) + \mathfrak{a} = S(A)$.
- 3. $A \simeq R\langle \mathcal{S}(A)^{-1} \rangle \simeq \varinjlim_{S \in L(R,\mathfrak{a},A)} R\langle S^{-1} \rangle$, R-isomorphisms.

Proof. The corollary is a particular case of Proposition 3.5. \square

By (18), for each ring $A \in \text{Loc}(R)$ there is a unique ideal $\mathfrak{a} \in \text{ass L}(R)$ such that $A \in \text{Loc}(R,\mathfrak{a})$. By Proposition 3.5.(1), for each ring $A \in \text{Loc}(R,\mathfrak{a})$, there is the largest element $\mathcal{S}(R,\mathfrak{a},A)$ of the set $L(R,\mathfrak{a},A)$. By Proposition 3.5.(1,2), the set

$$S(R, \mathfrak{a}, A)) = \sigma^{-1}(A^{\times})$$

is a multiplicative submonoid of R where $\sigma: R \to A, r \mapsto r + \mathfrak{a}$. Hence, the map

$$Loc(R) = \coprod_{\mathfrak{a} \in ass L(R)} Loc(R, \mathfrak{a}) \to L(R), \ Loc(R, \mathfrak{a}) \ni A \mapsto \mathcal{S}(R, \mathfrak{a}, A), \tag{33}$$

is a section of the epimorphism of the posets (20). The set

$$\mathcal{M}(R) = \{ \mathcal{S}(R, \mathfrak{a}, A) \mid \mathfrak{a} \in \operatorname{ass} L(R), A \in \operatorname{Loc}(R, \mathfrak{a}) \}$$
(34)

is the image of the section above. In particular, the map

$$Loc(R) = \coprod_{\mathfrak{a} \in ass L(R)} Loc(R, \mathfrak{a}) \to \mathcal{M}(R), \ Loc(R, \mathfrak{a}) \ni A \mapsto \mathcal{S}(R, \mathfrak{a}, A), \tag{35}$$

is an isomorphism of the posets $(Loc(R), \rightarrow)$ and $(\mathcal{M}(R), \subseteq)$ with inverse $\mathcal{S} \mapsto R(\mathcal{S}^{-1})$.

Theorem 3.8 describes the sets $\max L(R)$ and $\max Loc(R)$, presents a bijection between them, and shows that $\max Loc(R) \neq \emptyset$.

Theorem 3.8 1. The map $\max L(R) \to \max Loc(R)$, $S \mapsto R\langle S^{-1} \rangle$ is a bijection with inverse $A \mapsto \sigma_A^{-1}(A^{\times})$ where $\sigma_A : R \to A$.

- 2. $\max L(R) = {\sigma_A^{-1}(A^{\times}) \mid A \in \max Loc(R)}.$
- 3. $\max \operatorname{Loc}(R) = \{R\langle S^{-1}\rangle \mid S \in \max \operatorname{L}(R)\} \neq \emptyset$.

Proof. 1. Statement 1 follows from the isomorphisms of posets given in (35) and Corollary 3.7. 2 and 3. Statements 2 and 3 follow from statement 1 and Corollary 3.7 (max Loc(R) $\neq \emptyset$ since

2 and 3. Statements 2 and 3 follow from statement 1 and Corollary 3.7 (max Loc(R) \neq max L(R) \neq \emptyset , by Theorem 3.1).(1)). \square

Similarly, for each ideal $\mathfrak{a} \in \operatorname{assL}(R)$, Theorem 3.9 describes the sets $\max L(R,\mathfrak{a})$ and $\max \operatorname{Loc}(R,\mathfrak{a})$, presents a bijection between them, and shows that they are nonempty sets.

Theorem 3.9 Let $\mathfrak{a} \in \operatorname{ass} L(R)$.

- 1. The map $\max L(R, \mathfrak{a}) \to \max Loc(R, \mathfrak{a})$, $S \mapsto R\langle S^{-1} \rangle$ is a bijection with inverse $A \mapsto \sigma_A^{-1}(A^{\times})$ where $\sigma_A : R \to A$.
- 2. $\max L(R, \mathfrak{a}) = \{\sigma_A^{-1}(A^{\times}) \mid A \in \max Loc(R, \mathfrak{a})\} \neq \emptyset$.
- 3. $\max \operatorname{Loc}(R, \mathfrak{a}) = \{R\langle S^{-1}\rangle \mid S \in \max \operatorname{L}(R, \mathfrak{a})\} \neq \emptyset$.

Proof. 1. Statement 1 follows from the isomorphisms of posets given in (35) and Corollary 3.7. 2. (i) $\max L(R,\mathfrak{a}) \neq \emptyset$: The statement (i) follows from Zorn's Lemma. In more detail, let $\{S_i\}_{i\in\Gamma}\subseteq L(R,\mathfrak{a})$ be an ordered by inclusion set of localizable sets in R indexed by an ordinal Γ , i.e. for each pair $i\leq j$ in Γ , $S_i\subseteq S_j$ and if $i\in\Gamma$ is a limit ordinal then $S_i=\bigcup_{j< i}S_j$. In view of Zorn's Lemma, it suffices to show that $\bigcup_{i\in\Gamma}S_i\in L(R,\mathfrak{a})$. By (7) and (9), we have the direct system of ring homomorphisms, $(R\langle S_i^{-1}\rangle,\phi_{S_jS_i})$, where for each pair $i\leq j$, the map $\phi_{S_jS_i}:R\langle S_i^{-1}\rangle\to R\langle S_j^{-1}\rangle$ is defined in (7). Since the rings $R\langle S_i^{-1}\rangle$ are unital and $\phi_{S_jS_i}(1)=1$ for all $i\leq j$, the direct limit $\varinjlim R\langle S_i^{-1}\rangle$ is a nonzero ring and and is isomorphic to $R\langle \left(\bigcup_{i\in\Gamma}S_i\right)^{-1}\rangle$. Therefore, $\bigcup_{i\in\Gamma}S_i\in L(R)$. Finally,

$$\operatorname{ann}_R(\bigcup_{i\in\Gamma} S_i) = \bigcup_{i\in\Gamma} \operatorname{ann}_R(S_i) = \bigcup_{i\in\Gamma} \mathfrak{a} = \mathfrak{a},$$

and the statement (ii) follows.

- (ii) $\max L(R, \mathfrak{a}) = \{\sigma_A^{-1}(A^{\times}) \mid A \in \max Loc(R, \mathfrak{a})\}$: The statement (ii) follows from statement 1 and Corollary 3.7.
- 3. Statement 3 follows from statement 1 and Corollary 3.7 (by statements 1 and 2, $\max \operatorname{Loc}(R, \mathfrak{a}) \neq \emptyset$). \square

The L-radical Lrad(R) of the ring R, the sets of localizable, non-localizable and completely localizable elements of R.

Definition. An element S of the set $\max L(R)$ is called a maximal localizable set, and the rings $R\langle S^{-1}\rangle$ is called the maximal localization of R. The set of maximal localizations of the ring R is denoted my $\max Loc(R)$. The intersection and the sum,

$$\mathfrak{l}(R) := \operatorname{Lrad}(R) := \bigcap_{S \in \max L(R)} \operatorname{ass}_R(S) \text{ and } \mathfrak{cl}(R) := \operatorname{co-Lrad}(R) := \sum_{S \in \max L(R)} \operatorname{ass}_R(S), \quad (36)$$

are called the *localization radical* and the *localization coradical*, respectively, or the L-radical and the L-coradical of R, for short. By Theorem 3.1.(2),

$$\mathfrak{cl}(R) = \text{co-Lrad}(R) = \sum_{S \in \mathcal{L}(R)} \text{ass}_R(S).$$

For the ring R, there is the canonical exact sequence,

$$0 \to \operatorname{Lrad}(R) \to R \xrightarrow{\sigma} \prod_{S \in \max L(R)} R\langle S^{-1} \rangle, \quad \sigma := \prod_{S \in \max L(R)} \sigma_S, \tag{37}$$

where $\sigma_S: R \to R\langle S^{-1} \rangle$.

Definition. The sets $\mathcal{L}L(R) := \bigcup_{S \in L(R)} S = \bigcup_{S \in \max L(R)} S$ and $\mathcal{N}\mathcal{L}L(R) := R \setminus \mathcal{L}L(R)$ are called the set of localizable and non-localizable elements of R, resp., and the intersection

$$CL(R) := \bigcap_{S \in \max L(R)} S$$

is called the set of completely localizable elements of the ring R. The ring $Q_c(R) := R \langle \mathcal{C}L(R)^{-1} \rangle$ is called the complete localization or the \mathcal{C} -localization of R. Let $\mathfrak{c}_R := \operatorname{ass}_R(\mathcal{C}L(R)), R_c := R/\mathfrak{c}_R$, $\pi_{\mathfrak{c}_R} : R \to R_c, r \mapsto r + \mathfrak{c}_R$, and $\mathcal{C}_c := \mathcal{C}L(R)_c := \pi_{\mathfrak{c}_R}(\mathcal{C}L(R))$.

By Corollary 2.2.(2), $R_c \subseteq Q_c$, $C_c \in L(R_c)$, and

$$Q_c(R) \simeq R_c \langle \mathcal{C}_c^{-1} \rangle.$$
 (38)

By the definition, the sets $\mathcal{L}L(R)$, $\mathcal{N}\mathcal{L}L(R)$, and $\mathcal{C}L(R)$ are invariant under the action of the automorphism group of the ring R, i.e., they are *characteristic sets*.

Proposition 3.10 shows that there are tight connections between the sets $\mathcal{L}L(R)$, $\mathcal{N}\mathcal{L}L(R)$, and Lrad(R).

Proposition 3.10 1. $\mathcal{L}L(R) \cap Lrad(R) = \emptyset$.

- 2. $Lrad(R) \subseteq \mathcal{NL}(R)$.
- 3. $\mathcal{L}L(R) + Lrad(R) = \mathcal{L}L(R)$.
- 4. $\mathcal{NLL}(R) + \operatorname{Lrad}(R) = \mathcal{NLL}(R)$.

Proof. 1. For each element $s \in \mathcal{L}L(R)$, there is a maximal localizable set, say $S \in \max L(R)$, such that $s \in S$. Then $s \notin \operatorname{ass}_R(S)$ since the element s is a unit in the ring $R\langle S^{-1} \rangle$. Hence, $s \notin \operatorname{Lrad}(R)$, and so $\mathcal{L}L(R) \cap \operatorname{Lrad}(R) = \emptyset$.

- 2. Statement 2 follows from statement 1: $Lrad(R) \subseteq R \setminus \mathcal{L}L(R) = \mathcal{NL}L(R)$.
- 3. Since $0 \in \operatorname{Lrad}(R)$, to prove statement 3 it suffices to show that $\mathcal{L}L(R) + \operatorname{Lrad}(R) \subseteq \mathcal{L}L(R)$. For each element $s \in \mathcal{L}L(R)$, there is a maximal localizable set, say $S \in \max L(R)$, that contains the element s. Then the image of the elements $s + \operatorname{Lrad}(R)$ in the ring $R\langle S^{-1}\rangle$ are units (since $\operatorname{Lrad}(R) \subseteq \operatorname{ass}_R(S)$). Therefore, $s + \operatorname{Lrad}(R) \subseteq \mathcal{L}L(R)$, and statement 3 follows.
- 4. Recall that $R = \mathcal{L}L(R) \coprod \mathcal{N}\mathcal{L}L(R)$, a disjoint union. Suppose that statement 4 does not hold, i.e. there are elements $n \in \mathcal{N}\mathcal{L}L(R)$ and $r \in Lrad(R)$ such $n + r \notin \mathcal{N}\mathcal{L}L(R)$. Then $n + r \in \mathcal{L}L(R)$, and so

$$\mathcal{L}L(R) \not\ni n \in -r + \mathcal{L}L(R) \subseteq Lrad(R) + \mathcal{L}L(R) \stackrel{\text{st.3}}{=} \mathcal{L}L(R),$$

a contradiction. \Box

Corollary 3.11 provides a useful sufficient condition for the localization radical to be zero. Corollary 3.11 is used in the description of the localization radical of an arbitrary direct product of division rings (Theorem 5.11.(3)).

Corollary 3.11 If $\mathcal{L}L(R) = R \setminus \{0\}$ then Lrad(R) = 0.

Proof. By Proposition 3.10.(2), $\operatorname{Lrad}(R) \subseteq \mathcal{NL}(R) = R \setminus \mathcal{L}(R) = \{0\}$ since $\mathcal{L}(R) = R \setminus \{0\}$. Hence $\operatorname{Lrad}(R) = 0$. \square

Lemma 3.12 is a criterion for an element of R to be a localizable element.

Lemma 3.12 Let $r \in R$. The following statements are equivalent:

- 1. $r \in \mathcal{L}L(R)$.
- $2. \{r\} \in L(R).$
- 3. $\operatorname{ass}_R(r) \neq R$.
- 4. There is an R-homomorphism of rings $f: R \to A$ such that $f(r) \in A^{\times}$.

Proof. $(1 \Rightarrow 2)$ If $r \in S$ for some $S \in L(R)$, then $\{r\} \in L(R)$, by Proposition 2.8.(2). $(2 \Rightarrow 1)$ The implication follows from the inclusions $r \in \{r\}$ and $\{r\} \in L(R)$.

 $(2 \Leftrightarrow 3 \Leftrightarrow 4)$ Proposition 2.4. \square

Corollary 3.13 1. Every localizable set consists of localizable elements.

- 2. Every subset of a ring that consists of localizable elements is a localizable set iff there is the largest localizable set.
- *Proof.* 1. Statement 1 follows from Proposition 2.8.(2) or Lemma 3.12.(1,2).
- 2. If S is the largest localizable set of a ring R then $\mathcal{L}L(R) = S$, and so every subset of R that consists of localizable elements is a subset of S, hence localizable, by Proposition 2.8.(2).

Suppose that a ring R has two maximal localizable sets, \mathcal{S} and \mathcal{T} . Then their union consists of localizable elements but it is not a localizable set otherwise, by Theorem 3.1.(2), it would have contained in a maximal localizable set, which is not possible (since \mathcal{S} and \mathcal{T} are maximal localizable sets). \square

The set CL(R) of completely localizable elements of R. Let $\mathfrak{c}_R := \operatorname{ass}_R(CL(R))$. Consider the maps

$$0 \to \operatorname{Lrad}(R) \to R \overset{\pi_{\mathfrak{c}_R}}{\to} R/\mathfrak{c}_R \overset{\overline{\kappa}}{\to} Q_c(R) \to \prod_{S \in \max L(R)} R\langle S^{-1} \rangle \text{ and } \kappa := \overline{\kappa} \pi_{\mathfrak{c}_R} : R \to Q_c(R) \quad (39)$$

is the unique R-homomorphism from R to its localization $Q_c(R)$, $\pi_{\mathfrak{c}_R}(r) := r + \mathfrak{c}_R$, and $\overline{\kappa}$ is the unique R-homomorphism from R/\mathfrak{c}_R to $Q_c(R)$ (induced by κ).

Proposition 3.14 contains some properties of CL(R) and \mathfrak{c}_R .

Proposition 3.14 Let C := CL(R).

- 1. $C = \sigma^{-1} \left(\prod_{S \in \max L(R)} R \langle S^{-1} \rangle^{\times} \right) \text{ where } \sigma : R \to \prod_{S \in \max L(R)} R \langle S^{-1} \rangle.$
- 2. CL(R) + Lrad(R) = CL(R).
- 3. $\mathfrak{c}_R \subseteq \operatorname{Lrad}(R)$.
- 4. If the map $\overline{\kappa}: R/\mathfrak{c}_R \to Q_c(R)$ is an injection then $\mathfrak{c}_R = \operatorname{Lrad}(R)$.
- 5. $C + \mathfrak{c}_R = C$.

Proof. 1. Statement 1 follows from the definition of the set \mathcal{C} and Corollary 3.7.(1):

$$C = \bigcap_{S \in \max L(R)} S = \sigma^{-1} \Big(\prod_{S \in \max L(R)} R \langle S^{-1} \rangle^{\times} \Big).$$

- 2. By Corollary 3.7.(3), $S + \operatorname{ass}_R(S) = S$ for all $S \in \max L(R)$. Now, $S + \operatorname{Lrad}(R) = S$ for all $S \in \max L(R)$ since $\operatorname{Lrad}(R) = \bigcap_{S \in \max L(R)} \operatorname{ass}_R(S)$. It follows that $\operatorname{CL}(R) + \operatorname{Lrad}(R) = \operatorname{CL}(R)$ since $\operatorname{CL}(R) = \bigcap_{S \in \max L(R)} S$.
 - 3. The inclusion $\mathfrak{c}_R \subseteq \operatorname{Lrad}(R)$ follows from (39).
- 4. If the map $\overline{\kappa}$ is an injection then $\mathfrak{c}_R \supseteq \operatorname{Lrad}(R)$ (by (39)), and statement 4 follows from statement 3.
 - 5. Statement 5 follows from statements 2 and 3. \square

The \mathfrak{a} -absolute quotient ring $Q_a(R,\mathfrak{a})$) of a ring R where $\mathfrak{a} \in \operatorname{ass} L(R)$. Let $\mathfrak{a} \in \operatorname{ass} L(R)$ and

$$U(R,\mathfrak{a}) := \bigcup_{S \in \max L(R,\mathfrak{a})} S. \tag{40}$$

The set $(L(R, \mathfrak{a}), \subseteq)$ is a poset and the system $\{R\langle S^{-1}\rangle, \phi_{ST}\}$ is a direct system of rings where $S, T \in L(R, \mathfrak{a})$.

Definition. The direct limit $Q_a(R, \mathfrak{a}) := \varinjlim_{S \in L(R, \mathfrak{a})} R\langle S^{-1} \rangle$ is called the \mathfrak{a} -absolute quotient ring of the ring R. Let $\sigma_{a,\mathfrak{a}} : R \to Q_a(R, \mathfrak{a})$ be a unique R-homomorphism and $\mathfrak{a}_{a,\mathfrak{a}}(R) := \ker(\sigma_{a,\mathfrak{a}})$,

 $R_{a,\mathfrak{a}} := R/\mathfrak{a}_{a,\mathfrak{a}}(R), \ \pi_{a,\mathfrak{a}} : R \to R_{a,\mathfrak{a}}, \ r \mapsto r + \mathfrak{a}_{a,\mathfrak{a}}, \ \text{and} \ \overline{\sigma_{a,\mathfrak{a}}} : R_{a,\mathfrak{a}} \to Q_a(R,\mathfrak{a}) \text{ is a unique } R$ -homomorphism. Clearly, $\sigma_{a,\mathfrak{a}} = \overline{\sigma_{a,\mathfrak{a}}} \pi_{a,\mathfrak{a}} \text{ and } \mathfrak{a}_{a,\mathfrak{a}}(R) \supseteq \mathfrak{a}.$

Theorem 3.15 gives an explicit description of the ring $Q_a(R, \mathfrak{a})$, a criterion for $Q_a(R, \mathfrak{a}) \neq \{0\}$, and a criterion for $\mathfrak{a}_{a,\mathfrak{a}} = \mathfrak{a}$.

Theorem 3.15 Let $\mathfrak{a} \in \operatorname{ass} L(R)$ and $U = U(R, \mathfrak{a})$.

- 1. $Q_a(R, \mathfrak{a}) \neq \{0\}$ iff $U \in L(R)$.
- 2. Suppose that $U \in L(R, \mathfrak{b})$. Then
 - (a) The rings $Q_a(R,\mathfrak{a})$ and $R\langle U^{-1}\rangle$ are R-isomorphic and $\mathfrak{a}_{a,\mathfrak{a}}=\mathfrak{b}$.
 - (b) $\mathfrak{a}_{a,\mathfrak{a}} = \mathfrak{a}$ iff $\max L(R,\mathfrak{a}) = \{S\}$ iff the rings $Q_a(R,\mathfrak{a})$ and $R\langle S^{-1}\rangle$ are R-isomorphic for some $S \in \max L(R,\mathfrak{a})$.

Proof. 1. (\Rightarrow) Suppose that $Q_a(R, \mathfrak{a}) \neq \{0\}$. Then the set $S := \sigma_{a,\mathfrak{a}}^{-1}(Q_a(R,\mathfrak{a})^{\times})$ is a localizable set in R since $\sigma_{a,\mathfrak{a}}(S) \subseteq Q_a(R,\mathfrak{a})^{\times}$. Then for each $S \in \max L(R,\mathfrak{a})$, there is a commutative diagram

$$\begin{array}{ccc} R & \stackrel{\sigma_S}{\rightarrow} & R\langle S^{-1} \rangle \\ & \stackrel{\sigma_{a,\mathfrak{a}}}{\searrow} & \downarrow^{\phi_S} \\ & & Q_a(R,\mathfrak{a}) \end{array}$$

where ϕ_S is a unique R-homomorphism. Then it follows from the inclusion $\phi_S(R\langle S^{-1}\rangle^{\times}) \subseteq Q_a(R,\mathfrak{q})^{\times}$ that

$$\mathcal{S} = \sigma_{a,\mathfrak{a}}^{-1}(Q_a(R,\mathfrak{a})^{\times}) = (\phi_S \sigma_S)^{-1}(Q_a(R,\mathfrak{a})^{\times}) = \sigma_S^{-1} \phi_S^{-1}(Q_a(R,\mathfrak{a})^{\times}) \supseteq \sigma_S^{-1}(R\langle S^{-1}\rangle^{\times}) = S,$$

by Corollary 3.7.(1). Therefore, $S \supseteq \bigcup_{S \in \max L(R,\mathfrak{a})} S = U$, and so $U \in L(R)$ since $S \in L(R)$.

(\Leftarrow) Suppose that $U \in L(R)$. Let us show that the rings $Q_a(R,\mathfrak{a})$ and $R\langle U^{-1}\rangle$ are R-isomorphic. By the definition, the ring $Q_a(R,\mathfrak{a})$ is the direct limit of the direct system $\mathcal{D} := \{R\langle S^{-1}\rangle, \phi_{ST}\}$ where $S, T \in L(R,\mathfrak{a})$. Since for all $S \in L(R,\mathfrak{a})$, $S \subseteq U$, we can extend this direct system to a direct system \mathcal{D}' by adding the R-homomorphisms $R\langle S^{-1}\rangle \to R\langle U^{-1}\rangle$ for all $S \in L(R,\mathfrak{a})$. The element U is the largest element of the poset $L' := L(R,\mathfrak{a}) \cup \{U\}$. By Theorem 2.14.(3),

$$\varinjlim_{S\in \mathcal{L}'} R\langle S^{-1}\rangle \simeq R\langle U^{-1}\rangle,$$

an R-isomorphism. By the definitions of the two direct systems, \mathcal{D} and \mathcal{D}' , there is necessarily a unique R-homomorphism $\alpha: Q_a(R,\mathfrak{a}) \to R\langle U^{-1} \rangle$. Since

$$\sigma_{a,\mathfrak{a}}(U) \subseteq Q_a(R,\mathfrak{a})^{\times},$$

where $\sigma_{a,\mathfrak{a}}: R \to Q_a(R,\mathfrak{a})$, there is necessarily a unique R-homomorphism

$$\beta: R\langle U^{-1}\rangle \to Q_a(R,\mathfrak{a}).$$

Hence, the rings $Q_a(R, \mathfrak{a})$ and $R\langle U^{-1}\rangle$ are R-isomorphic, by the universal properties of localization and the direct limit. In particular, $\mathfrak{a}_{a,\mathfrak{a}} = \mathfrak{b}$. This finishes the proof of statements 1 and 2(a).

2(b). (i) $\mathfrak{a}_{a,\mathfrak{a}} = \mathfrak{a} \Rightarrow \max L(R,\mathfrak{a}) = \{S\}$: Suppose that the set $\max L(R,\mathfrak{a})$ contains at least two elements, say S and T. Then

$$S \subset S \cup T \subseteq U$$
 and $S, U \in L(R, \mathfrak{a})$.

This contradicts to the fact that $S \in \max L(R, \mathfrak{a})$, and the statement (i) follows.

- (ii) $\max L(R, \mathfrak{a}) = \{S\} \Rightarrow the \ rings \ Q_a(R, \mathfrak{a}) \ and \ R\langle S^{-1}\rangle \ are \ R$ -isomorphic: The implication follows from Theorem 2.14.(3).
- (iii) The rings $Q_a(R,\mathfrak{a})$ and $R\langle S^{-1}\rangle$ are R-isomorphic for some $S\in\max L(R,\mathfrak{a})$ then $\mathfrak{a}_{a,\mathfrak{a}}=\mathfrak{a}$: The implication is obvious. \square

Example. If R is a commutative domain then the set $C_R = R \setminus \{0\}$ is the largest element of L(R,0). By Theorem 3.15, $Q_a(R,0) = Q_{cl}(R)$ is the classical quotient ring of R, i.e. the field of fractions of R, and $\mathfrak{a}_{a,0}(R) = 0$.

Lemma 3.16 If R is a simple Artinian rings then $\max L(R,0) = \{R^{\times}\}, Q_a(R,0) = R$ and $a_{a,0}(R) = 0.$

Proof. Lemma follows from Lemma 2.19. \square

Isomorphism criterion for localizations of a ring. Theorem 3.17.(1) is an R-isomorphism criterion for localizations of the ring R and Theorem 3.17.(2) is a criterion for existing of an Rhomomorphism between localizations of R.

Theorem 3.17 Let $S, T \in L(R)$.

1. The rings $R\langle S^{-1}\rangle$ and $R\langle T^{-1}\rangle$ are R-isomorphic iff

$$\sigma_S^{-1}(R\langle S^{-1}\rangle^{\times}) = \sigma_T^{-1}(R\langle T^{-1}\rangle^{\times})$$

where $\sigma_S: R \to R\langle S^{-1} \rangle$ and $\sigma_T: R \to R\langle T^{-1} \rangle$.

2. There is an R-homomorphism $R\langle S^{-1}\rangle \to R\langle T^{-1}\rangle$ iff $\sigma_S^{-1}(R\langle S^{-1}\rangle^{\times})\subseteq \sigma_T^{-1}(R\langle T^{-1}\rangle^{\times})$.

Proof. 1. (\Rightarrow) The implication is obvious.

- (\Leftarrow) The implication follows from the fact that the rings $R(S^{-1})$ and $R(\sigma_S^{-1}(R(S^{-1})^{\times}))$ are isomorphic for all $S \in L(R)$, by Proposition 3.5.(1).
 - 2. (\Rightarrow) The implication is obvious.
- (\Leftarrow) The implication follows from the fact that the rings $R\langle S^{-1}\rangle$ and $R\langle \sigma_S^{-1}(R\langle S^{-1}\rangle^{\times})\rangle$ are isomorphic for all $S \in L(R)$ (Proposition 3.5.(1)) and the fact that if $S_1, S_2 \in L(R)$ and $S_1 \subseteq S_2$ then there is a unique R-homomorphism $R\langle S_1^{-1} \rangle \to R\langle S_2^{-1} \rangle$. \square

Proposition 3.18 is a practical criterion for a subset \bar{S} of a ring R to be an element of the set $L(R, \mathfrak{a}).$

Proposition 3.18 Let S be a subset of a ring R.

- 1. Suppose that there exists an ideal \mathfrak{a} of R such that $\mathfrak{a} \subseteq \operatorname{ass}_R(S)$ and $\overline{S} := \pi(S) \in L(\overline{R},0)$ where $\pi: R \to \overline{R} := R/\mathfrak{a}, r \mapsto r + \mathfrak{a}$. Then $S \in L(R, \mathfrak{a})$.
- 2. $S \in L(R, \mathfrak{b})$ iff there is an ideal \mathfrak{a} of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and $\overline{S} := \pi(S) \in L(\overline{R}, 0)$ where $\pi: R \to \overline{R} := R/\mathfrak{a}, r \mapsto r + \mathfrak{a}.$
- *Proof.* 1. Since the elements of the set \overline{S} are invertible in the ring $\overline{R}\langle \overline{S}^{-1} \rangle$, ass_R(S) $\subseteq \mathfrak{a}$, by Proposition 2.7. By the assumption, $\operatorname{ass}_R(S) \supseteq \mathfrak{a}$. Therefore, $\operatorname{ass}_R(S) = \mathfrak{a}$ and $S \in L(R, \mathfrak{a})$.
 - 2. (\Rightarrow) If $S \in L(R, \mathfrak{b})$ then it suffices to take $\mathfrak{a} = \mathfrak{b}$, by Corollary 2.2.(2).
 - (\Leftarrow) This implication follows from statement 1. \square

Proposition 3.19 Suppose that $S \in L(R, \mathfrak{a}), T \in L(R\langle S^{-1} \rangle, 0)$ and $\overline{S} \subseteq T$ where $\overline{S} = \sigma_S(S)$ and $\sigma_S: R \to R\langle S^{-1} \rangle$. Then

1. $\overline{T} := T \cap \overline{R} \in L(\overline{R}, 0), \overline{T} \in L(R\langle S^{-1} \rangle, 0),$

$$\overline{R} \langle \overline{S}^{-1} \rangle \subseteq \overline{R} \langle \overline{T}^{-1} \rangle = R \langle S^{-1} \rangle \langle \overline{T}^{-1} \rangle \subseteq R \langle S^{-1} \rangle \langle T^{-1} \rangle,$$

and $T' := \sigma_S^{-1}(T) \in L(R, \mathfrak{a}, \overline{R}\langle \overline{T}^{-1} \rangle).$

2. Let $\sigma: R \to A := R\langle S^{-1} \rangle \langle T^{-1} \rangle$ and $\mathcal{T}:=\sigma^{-1}(A^{\times})$. Then $\mathcal{T} \in L(R,\mathfrak{a},A)$.

Proof. 1. Recall that the rings $R\langle S^{-1}\rangle$ and $\overline{R}\langle \overline{S}^{-1}\rangle$ are R-isomorphic. (i) $\overline{T}\in L(\overline{R},0)$ and $\overline{T}\in L(R\langle S^{-1}\rangle,0)$: The chain of inclusions below follows from the inclusions $\overline{R} \subset \overline{R} \langle \overline{S}^{-1} \rangle$ and $\overline{T} \subset T$,

$$\operatorname{ass}_{\overline{R}}(\overline{T})\subseteq\operatorname{ass}_{\overline{R}(\overline{S}^{-1})}(\overline{T})=\operatorname{ass}_{R(S^{-1})}(\overline{T})\subseteq\operatorname{ass}_{R(S^{-1})}(T)=0,$$

and the statement (i) follows.

(ii) $\overline{S} \subseteq \overline{T}$ and $\overline{R}\langle \overline{S}^{-1} \rangle \subseteq \overline{R}\langle \overline{T}^{-1} \rangle = R\langle S^{-1} \rangle \langle \overline{T}^{-1} \rangle \subseteq R\langle S^{-1} \rangle \langle T^{-1} \rangle$: Since $\overline{S} \subseteq T$ and $\overline{S} \subseteq \overline{R}$, we have that $\overline{S} \subseteq T \cap \overline{R} = \overline{T}$. Hence

$$\overline{R}\langle \overline{S}^{-1}\rangle \subseteq \overline{R}\langle \overline{T}^{-1}\rangle$$

since $\overline{T} \in L(R(S^{-1}), 0) = L(\overline{R}(\overline{S}^{-1}), 0)$ (the statement (i)). Now,

$$\overline{R}\langle \overline{T}^{-1}\rangle = \overline{R}\langle (\overline{S} \cup \overline{T})^{-1}\rangle = \overline{R}\langle \overline{S}^{-1}\rangle \langle \overline{T}^{-1}\rangle = R\langle S^{-1}\rangle \langle \overline{T}^{-1}\rangle \subseteq R\langle S^{-1}\rangle \langle T^{-1}\rangle$$

since $\overline{S} \subseteq \overline{T} \subseteq T$ and $\overline{T}, T \in L(R\langle S^{-1} \rangle, 0)$.

Recall that $\pi_S: R \to \overline{R}, r \mapsto r + \mathfrak{a}$. (iii) $T' = \pi_S^{-1}(\overline{T}) \supseteq S$ and $\pi_S(T') = \overline{T}$: The map $\sigma_S: R \to R\langle S^{-1} \rangle$ is the composition of two maps

$$\sigma_S: R \stackrel{\pi_S}{\to} \overline{R} \to R\langle S^{-1} \rangle$$

where the second map is a natural inclusion. Hence, $T' = \pi_S^{-1}(\overline{T})$. Clearly, $\pi_S^{-1}(\overline{T}) \supseteq S$ since $\overline{S} \subseteq T$ and $\pi_S(S) = \overline{S}$. The map $\pi_S : R \to \overline{R}$ is an epimorphism, and so $\pi_S(T') = \pi_S(\pi_S^{-1}(\overline{T})) = \overline{T}$.

(iv) $T' \in L(R, \mathfrak{a}, \overline{R}(\overline{T}^{-1}))$: It follows from the inclusion $S \subseteq T'$ that

$$\mathfrak{a} = \operatorname{ass}_R(S) \subseteq \operatorname{ass}_R(T').$$

By the statements (i) and (iii), $\pi_S(T') = \overline{T} \in L(\overline{R}, 0)$. Therefore, $ass_R(T') = \mathfrak{a}$, by Proposition 3.18.(1). Now,

$$R\langle T'^{-1}\rangle = R\langle (S \cup T)'^{-1}\rangle = R\langle S^{-1}\rangle \langle \pi_S(T')^{-1}\rangle = R\langle S^{-1}\rangle \langle \overline{T}^{-1}\rangle = \overline{R}\langle \overline{T}^{-1}\rangle,$$

by the statement (ii), and the statement (iv) follows.

2. Since $T \in L(R(S^{-1}), 0)$, $\ker(\sigma) = \mathfrak{a}$. Now, statement 2 follows from Proposition 3.5.(1). \square

Localizable sets and denominator sets. Theorem 3.20 is a criterion for the ring $R(S^{-1})$ to be an R-isomorphic to a localization of R at a denominator set and it also describes all such denominator sets.

Theorem 3.20 Let $S \in L(R, \mathfrak{a})$, $\sigma_S : R \to R\langle S^{-1} \rangle$, and $S := \sigma_S^{-1}(R\langle S^{-1} \rangle^{\times})$. Then

- 1. The ring $R(S^{-1})$ is R-isomorphic to a localization of the ring R at a *-denominator set where $* \in \{l, r, \emptyset\}$ iff $S \in \mathrm{Den}_*(R, \mathfrak{a})$.
- 2. The set of all *-denominator sets of R with localization R-isomorphic to the ring $R(S^{-1})$ is equal to $\{T \in \mathrm{Den}_*(R,\mathfrak{a}) \mid T \subseteq \mathcal{S}, \mathcal{S} + \mathfrak{a} \subseteq (T^{-1}R)^{\times} \}.$

Proof. 1. Let us prove statement 1 for left denominator sets. The other two case can be proven in a similar way (or reduced to this case using an opposite ring construction).

- (\Rightarrow) Suppose that the ring $R(S^{-1})$ is R-isomorphic to a localization of the ring R at a left denominator set T of R. Then $T \in \text{Den}_l(R,\mathfrak{a})$. By [3, Proposition 3.1.(1)], $S \in \text{Den}_l(R,\mathfrak{a})$ and the rings $S^{-1}R$ and $T^{-1}R$ are R-isomorphic.
- (\Leftarrow) Suppose that $S \in \text{Den}_l(R, \mathfrak{a})$. Then the rings $S^{-1}R$ and $R(S^{-1})$ are R-isomorphic. By Proposition 3.5.(1), the rings $R(S^{-1})$ and $R(S^{-1})$ are R-isomorphic, and the implication follows.
 - 2. Statement follows from Corollary 2.3. \square

For a denominators set $T \in \mathrm{Den}_*(R,\mathfrak{a})$, Proposition 3.21 describes all localizable sets S of the ring R such that the ring $R(S^{-1})$ is an R-isomorphic to the localization of R at T.

Proposition 3.21 Let $T \in \text{Den}_*(R, \mathfrak{a})$ where $* \in \{l, r, \emptyset\}$, A be the localization of R at T, and $\pi_T: R \to \overline{R} = R/\mathfrak{a}, \ r \mapsto r + \mathfrak{a}.$ Then the set of all localizable sets S of R such that the rings $R\langle S^{-1}\rangle$ and A are R-isomorphic is equal to $\{S\in L(R,\mathfrak{a})\mid \pi_S(S)\subset A^{\times},\pi_S(T)\subset R\langle S^{-1}\rangle^{\times}\}.$

Proof. The statement follows from Corollary 2.3. \square

For a localizable multiplicative set S of the ring R, Proposition 3.22 is a criterion for the set S to be a left denominator set of R.

Proposition 3.22 A localizable multiplicative set $S \in L(R, \mathfrak{a})$ is a left denominator set of R iff

1.
$$\mathfrak{a} = \sum_{s \in S} \ker_R(s \cdot) \supseteq \sum_{s \in S} \ker_R(\cdot s)$$
, and

2. the left R-module $R\langle S^{-1}\rangle/\overline{R}$ is S-torsion where $\overline{R}=R\mathfrak{a}$.

Proof. (\Rightarrow) The implication is obvious.

 (\Leftarrow) Recall that $\operatorname{ass}_l(S) = \sum_{s \in S} \ker_R(s \cdot)$. By statement 1,

$$\mathfrak{a} = \operatorname{ass}_l(S).$$

Let $a \in R\langle S^{-1}\rangle$. By statement 2, $sa = \overline{r} \in \overline{R}$ for some elements $s \in S$ and $r \in \overline{R}$, and so the element $a = s^{-1}\overline{r}$ is a left fraction. Therefore, the set S is a left Ore set since $\mathfrak{a} = \operatorname{ass}_l(S)$. In fact, $S \in \operatorname{Den}_l(R,\mathfrak{a})$ since $\operatorname{ass}_l(S) \supseteq \sum_{s \in S} \ker_R(s)$, see statement 1. \square

Corollary 3.23 (resp., Corollary 3.24) is a criterion for the set S to be a right (resp., left and right) denominator set of R.

Corollary 3.23 A localizable multiplicative set $S \in L(R, \mathfrak{q})$ is a right denominator set of R iff

1.
$$\mathfrak{a} = \sum_{s \in S} \ker_R(\cdot s) \supseteq \sum_{s \in S} \ker_R(s \cdot)$$
, and

2. the right R-module $R\langle S^{-1}\rangle/\overline{R}$ is S-torsion where $\overline{R}=R\mathfrak{a}$.

Proof. Repeat the proof of Proposition 3.22 replacing 'left' to 'right' everywhere.

Corollary 3.24 A localizable multiplicative set $S \in L(R, \mathfrak{a})$ is a denominator set of R iff

1.
$$\mathfrak{a} = \sum_{s \in S} \ker_R(\cdot s) = \sum_{s \in S} \ker_R(s \cdot)$$
, and

2. $R\langle S^{-1}\rangle/\overline{R}$ is an S-torsion left and right R-module where $\overline{R}=R\mathfrak{a}$.

Proof. The corollary follows from Proposition 3.21 and Corollary 3.23. \square

4 Localizations of commutative rings

In this section, let R be a *commutative* ring and $\mathfrak{n} = \mathfrak{n}_R$ be its prime radical.

For an arbitrary commutative ring R, descriptions of the following sets are obtained: the sets of localizable and non-localizable elements (Lemma 4.1), the maximal localizable sets, maximal localization rings, the set of completely localizable elements, and the ideal \mathfrak{c}_R (Theorem 4.2). Theorem 4.2 also describes relations between the ideals $\operatorname{Lrad}(R)$, \mathfrak{n}_R , and \mathfrak{c}_R .

For a commutative ring R such that $|\min(R)| < \infty$ and \mathfrak{n}_R is a nilpotent ideal (eg, R is a commutative Noetherian ring), Proposition 4.3 describes $\operatorname{Spec}(Q_c(R))$ and the rings $Q_c(R)$ and $Q_c(R)/\mathfrak{n}_{Q_c(R)}$.

For a commutative ring R such that $|\min(R)| < \infty$ and $\mathfrak{n}_R = 0$ (eg, R is a commutative semiprime Noetherian ring), Proposition 4.3 shows that $\mathcal{C}L(R) = \mathcal{C}_R$, $Q_c(R) = Q_{cl}(R)$, and $\operatorname{Lrad}(R) = \mathfrak{c}_R = \mathfrak{n}_{Q_c(R)} = 0$. A characterization of commutative semiprime Goldie rings is given (Corollary 4.5).

Notice that the prime radical \mathfrak{n}_R of a commutative ring R contains precisely all the nilpotent elements of the ring R. Lemma 4.1 describes the sets of localizable and non-localizable elements of an arbitrary commutative ring.

Lemma 4.1 Let R be a commutative ring.

- 1. $\mathcal{L}L(R) = R \setminus \mathfrak{n}_R$.
- 2. $\mathcal{NL}(R) = \mathfrak{n}_R$.
- 3. Let $S \subseteq R$. Then $S \in L(R)$ iff S_{mon} is a multiplicative subset of R. If $S \in L(R)$ then $R\langle S^{-1} \rangle = S_{mon}^{-1}R$ and $\operatorname{ass}_R(S) = \operatorname{ass}_R(S_{mon}) = \{r \in R \mid tr = 0 \text{ for some } t \in S_{mon}\}.$

Proof. 1 and 2. The prime radical \mathfrak{n}_R contains precisely all the nilpotent elements of the ring R. Hence, $\mathfrak{n}_R \subseteq \mathcal{NL}(R)$. The ring R is a commutative. So, for each element $t \in R \setminus \mathfrak{n}_R$, the set $S_t := \{t^i \mid i \in \mathbb{N}\}$ is a multiplicative set of R, and so $s \in \mathcal{L}L(R)$, i.e. $R \setminus \mathfrak{n}_R \subseteq \mathcal{L}L(R)$. Since,

$$R = \mathcal{L}L(R) \prod \mathcal{N}\mathcal{L}L(R) = R \backslash \mathfrak{n}_R \prod \mathfrak{n}_R,$$

we must have that $\mathcal{L}L(R) = R \setminus \mathfrak{n}_R$ and $\mathcal{N}\mathcal{L}L(R) = \mathfrak{n}_R$.

3. Statement 3 follows from statement 1. \square

For an arbitrary commutative ring R, Theorem 4.2 describes its maximal localizable sets, maximal localization rings, the set of completely localizable elements, and the ideal \mathfrak{c}_R . It also describes relations between the ideals $\operatorname{Lrad}(R)$, \mathfrak{n}_R , and \mathfrak{c}_R .

Theorem 4.2 Let R be a commutative ring and C := CL(R). Then

- 1. $\max L(R) = \{S_{\mathfrak{p}} \mid \mathfrak{p} \in \min(R)\}\$ where $S_{\mathfrak{p}} = R \backslash \mathfrak{p}$ is a multiplicative subset of R.
- 2. $R\langle S_{\mathfrak{p}}^{-1}\rangle = S_{\mathfrak{p}}^{-1}R := R_{\mathfrak{p}}$ is the localization of the ring R at the prime ideal \mathfrak{p} .
- 3. $\operatorname{ass}_R(S_{\mathfrak{p}}) = \{ r \in R \mid sr = 0 \text{ for some } s \in S_{\mathfrak{p}} \} \subseteq \mathfrak{p}.$
- 4. Lrad $(R) \subseteq \mathfrak{n}_R$.
- 5. $C = R \setminus \left(\bigcup_{\mathfrak{p} \in \min(R)} \mathfrak{p}\right)$ is a multiplicative subset of R and $Q_c(R) = \left(R \setminus \left(\bigcup_{\mathfrak{p} \in \min(R)} \mathfrak{p}\right)\right)^{-1} R$.
- 6. $\mathfrak{c}_R = \{r \in R \mid cr = 0 \text{ for some } c \in \mathcal{C}\} \subseteq \mathfrak{n}_R.$
- 7. $\mathfrak{n}_{Q_c(R)} = \mathcal{C}^{-1}\mathfrak{n}_R$.

Proof. 1. Let $S \in L(R)$. By Lemma 4.1.(3), the monoid S_{mon} is a multiplicative subset of R and $R\langle S^{-1}\rangle = S_{mon}^{-1}R$. Choose a prime ideal $P \in \operatorname{Spec}(S_{mon}^{-1}R)$. Then $\mathfrak{p}' := \sigma_S^{-1}(P) \in \operatorname{Spec}(R)$ where $\sigma_S : R \to S_{mon}^{-1}R$, $r \mapsto \frac{r}{1}$, and so

$$S \subseteq S_{\mathfrak{p}'} = R \backslash \mathfrak{p}' \subseteq S_{\mathfrak{p}}$$

for each $\mathfrak{p} \in \min(R)$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$, and statement 1 follows.

- 2. Statement 2 follows from statement 1.
- 3. Clearly, $\operatorname{ass}_R(S_{\mathfrak{p}}) = \{r \in R \mid sr = 0 \text{ for some } s \in S_{\mathfrak{p}}\}$. If $r \in \operatorname{ass}_R(S_{\mathfrak{p}})$ then sr = 0 for some element $s \in S_{\mathfrak{p}} = R \setminus \mathfrak{p}$, and so $r \in \mathfrak{p}$ (since $s \notin \mathfrak{p}$ and the ideal \mathfrak{p} is a prime ideal). Hence, $\operatorname{ass}_R(S_{\mathfrak{p}}) \subseteq \mathfrak{p}$.
 - 4. By statements 1 and 3, $\operatorname{Lrad}(R) = \bigcap_{\mathfrak{p} \in \min(R)} \operatorname{ass}_R(S_{\mathfrak{p}}) \subseteq \bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p} = \mathfrak{n}_R$.
 - 5. By statement 1 and Lemma 4.1.(1), the set

$$\mathcal{C} = \bigcap_{\mathfrak{p} \in \min(R)} S_{\mathfrak{p}} = R \setminus \big(\bigcup_{\mathfrak{p} \in \min(R)} \mathfrak{p}\big) \subseteq R \setminus \big(\bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p}\big) = R \setminus \mathfrak{n}_R = \mathcal{L}L(R)$$

is a localizable set. Hence, the set C is a multiplicative subset of R.

- 6. By statement 5, $\mathfrak{c}_R = \{r \in R \mid cr = 0 \text{ for some } c \in \mathcal{C}\}$. Since $c \notin \mathfrak{p}$ for all $\mathfrak{p} \in \min(R)$, we must have $r \in \bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p} = \mathfrak{n}_R$, and so $\mathfrak{c}_R \subseteq \mathfrak{n}_R$.
- 7. An element $c^{-1}r \in Q_c(R)$, where $c \in \mathcal{C}$ and $r \in R$, belongs to the prime radical $\mathfrak{n}_{Q_c(R)}$ iff it is a nilpotent element iff $\frac{r}{1}$ is a nilpotent element of $Q_c(R)$ iff $c'r^n = 0$ for some element $c' \in \mathcal{C}$ and a natural number $n \geq 1$ iff $(c'r)^n = 0$ iff $c'r \in \mathfrak{n}_R$ iff $c^{-1}r \in \mathfrak{n}_{Q_c(R)}$. \square

For a commutative ring R such that $|\min(R)| < \infty$ and \mathfrak{n}_R is a nilpotent ideal (eg, R is a commutative Noetherian ring), Proposition 4.3 describes $\operatorname{Spec}(Q_c(R))$ and the rings $Q_c(R)$ and $Q_c(R)/\mathfrak{n}_{Q_c(R)}$.

Proposition 4.3 Let R be a commutative ring and C := CL(R). Then

- 1. Suppose that $|\min(R)| < \infty$. Then $K\dim(Q_c(R)) = 0$ and $\operatorname{Spec}(Q_c(R)) = \{\mathcal{C}^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R)\} = \min(Q_c(R))$.
- 2. Suppose that $|\min(R)| < \infty$ and \mathfrak{n}_R is a nilpotent ideal, eg, R is a commutative Noetherian ring. Then
 - (a) $Q_c(R) \simeq \prod_{\mathfrak{p} \in \min(R)} R_{\mathfrak{p}}$ is a direct product of local rings $(R_{\mathfrak{p}}, \mathfrak{p}_{\mathfrak{p}})$ and the maximal ideal $\mathfrak{p}_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$ is a nilpotent ideal.
 - (b) The rings $C^{-1}(R/\mathfrak{p}) \simeq (R/\mathfrak{p})_{\mathfrak{p}}$ are fields for all $\mathfrak{p} \in \min(R)$.
 - (c) $Q_c(R)/\mathfrak{n}_{Q_c(R)} \simeq \prod_{\mathfrak{p}\min(R)} (R/\mathfrak{p})_{\mathfrak{p}}$ is a direct product of fields.

Proof. 1. By Theorem 4.2.(5), $\mathcal{C} = R \setminus (\bigcup_{\mathfrak{p} \in \min(R)} \mathfrak{p})$ is a multiplicative subset of R and

$$Q_c(R) = \mathcal{C}^{-1}R = \left(R \setminus \left(\bigcup_{\mathfrak{p} \in \min(R)} \mathfrak{p}\right)\right)^{-1}R.$$

Since $|\min(R)| < \infty$, $K\dim(Q_c(R)) = 0$ and $\operatorname{Spec}(Q_c(R)) = \{C^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R)\} = \min(Q_c(R))$.

2. By statement 1, the set $\operatorname{Spec}(Q_c(R)) = \{\mathcal{C}^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R)\}$ consists precisely of maximal ideals of the ring $Q_c(R)$. Hence, they are co-prime ideals, that is $\mathcal{C}^{-1}\mathfrak{p} + \mathcal{C}^{-1}\mathfrak{p}' = Q_c(R)$ for all $\mathfrak{p} \neq \mathfrak{p}'$. Hence,

$$Q_c(R)/\mathfrak{n}_{Q_c(R)} = Q_c(R)/\bigcap_{\mathfrak{p} \min(R)} \mathcal{C}^{-1}\mathfrak{p} \simeq \prod_{\mathfrak{p} \min(R)} Q_c(R)/\mathcal{C}^{-1}\mathfrak{p} = \prod_{\mathfrak{p} \min(R)} \mathcal{C}^{-1}R/\mathcal{C}^{-1}\mathfrak{p} = \prod_{\mathfrak{p} \min(R)} \mathcal{C}^{-1}(R/\mathfrak{p})$$

is a direct product of fields. Therefore, $C^{-1}(R/\mathfrak{p}) \simeq (R/\mathfrak{p})_{\mathfrak{p}}$ for all $\mathfrak{p} \in \min(R)$. Since \mathfrak{n}_R is a nilpotent ideal, the orthogonal primitive idempotents that correspond to the direct product decomposition of the factor ring $Q_c(R)/\mathfrak{n}_{Q_c(R)}$ above can be lifted, i.e. the ring $Q_c(R)$ is a product of local rings. Hence,

$$Q_c(R) \simeq \prod_{\mathfrak{p} \in \min(R)} R_{\mathfrak{p}}. \ \Box$$

For a commutative ring R such that $|\min(R)| < \infty$ and $\mathfrak{n}_R = 0$ (eg, R is a commutative semiprime Noetherian ring), Proposition 4.3 shows that $\mathcal{C}L(R) = \mathcal{C}_R$ and $Q_c(R) = Q_{cl}(R)$.

Theorem 4.4 Let R be a commutative ring with $|\min(R)| < \infty$ and $\mathfrak{n}_R = 0$, eg, R is a commutative semiprime Noetherian ring, and C := CL(R). Then

- 1. The ring R is a semiprime Goldie ring.
- 2. $Q_{cl}(R) = \prod_{\mathfrak{p} \in \min(R)} R_{\mathfrak{p}}$ is a product of fields.
- 3. For all $\mathfrak{p} \in \min(R)$, $\operatorname{ass}_R(S_{\mathfrak{p}}) = \mathfrak{p}$.
- 4. $C = C_R$ and $Q_c(R) = Q_{cl}(R)$.
- 5. $\mathfrak{n}_{Q_c(R)} = 0$.
- 6. $\operatorname{Lrad}(R) = 0$.
- 7. $\mathfrak{c}_R = 0$.

Proof. 1. By the assumption, the ring R is a commutative ring with $|\min(R)| < \infty$ and $\mathfrak{n}_R = 0$. By [2, Theorem 5.1], the ring R is a semiprime Goldie ring since for each minimal prime ideal $\mathfrak{p} \in \min(R)$, the field R/\mathfrak{p} is a Goldie ring (any field is a Goldie ring).

2 and 3. Statements 2 and follow from statement 1 and [2, Theorem 4.1].

4. By statements 2 and 3, the union $\bigcup_{\mathfrak{p}\min(R)}\mathfrak{p}$ is the set of all zero divisors of the ring R, and so

$$C_R = R \setminus \bigcup_{\mathfrak{p} \min(R)} \mathfrak{p} = C,$$

by Theorem 4.2.(5). Hence, $Q_c(R) = Q_{cl}(R)$.

- 5. By Theorem 4.2.(7) (or by statements 2 and 4), $\mathfrak{n}_{Q_c(R)} = \mathcal{C}^{-1}\mathfrak{n}_R = 0$.
- 6. By Theorem 4.2.(4), Lrad(R) = 0.
- 7. Since $C = C_R$ (statement 4), $\mathfrak{c}_R = 0$. \square

Corollary 4.5 gives a characterization of commutative semiprime Goldie rings.

Corollary 4.5 Let R be a commutative semiprime ring. Then the ring R is a Goldie ring iff $|\min(R)| < \infty$.

Proof. Suppose that the ring R is a Goldie ring. Then the ring $Q_{cl}(R)$ is a commutative semisimple Artinian ring, i.e. a finite product of fields. Hence, $|\min(R)| < \infty$.

Suppose that the ring R has only finitely many minimal primes. Then, by Theorem 4.4.(1), the ring R is a Goldie ring. \square

5 Localizations of semiprime left Goldie rings and direct products of division rings

Proposition 5.1.(4) describes the maximal localizable sets of finite direct product of rings and their localizations. Proposition 5.1.(1) describes localizations of the direct product of rings via localizations of its components. For a direct product of simple rings $A = \prod_{i=1}^{s} A_i$ such that $\mathcal{L}L(A_i) = A_i^{\times}$ for i = 1, ..., s, Theorem 5.3 describes the sets $\max L(A)$, $\operatorname{ass}_A(S)$ for all $S \in \max L(A)$, the sets of localizable, non-localizable, and completely localizable elements of A. It also shows that $\operatorname{Lrad}(A) = 0$. Every semisimple Artinian ring satisfies the assumptions of Theorem 5.3, see Corollary 5.4 for detail. For a semiprime left Goldie ring, Lemma 5.8 describes all the maximal localizable sets that contain the set of regular elements of the ring.

Let $D = \prod_{i \in I} D_i$ be a direct product of division rings D_i where I is an arbitrary set. Proposition 5.9 describes all the localizable sets in the ring D and all the localizations of D. Theorem 5.11 describes the following sets: $\max L(D)$, $\max Loc(D)$, $\operatorname{Lrad}(D)$, $\operatorname{CL}(D)$, $\operatorname{Qc}(D)$, $\operatorname{LL}(D)$, and $\operatorname{NL}(D)$. Theorem 5.11.(2) shows that for every $S \in \max L(D)$, the ring $D(S^{-1})$ is a division ring.

The maximal localizable sets of finite direct product of rings. Let $A = \prod_{i=1}^{s} A_i$ be a finite direct product of rings A_i . For each i = 1, ..., n, let $p_i : A \to A_i$, $(a_1, ..., a_n) \mapsto a_i$ be the canonical projection and $\nu_i : A_i \to A$, $a_i \mapsto (0, ..., 0, a_i, 0, ..., 0)$ is the canonical injection.

Proposition 5.1.(4) describes the maximal localizable sets of finite direct product of rings and their localizations.

Proposition 5.1 Let $A = \prod_{i=1}^{s} A_i$ be a direct product of rings, $S \subseteq A$, and $S_i := p_i(S)$ for i = 1, ..., n. Then

- 1. $A\langle S^{-1}\rangle \simeq \prod_{i=1}^s A_i \langle S_i^{-1}\rangle$, an A-isomorphism.
- 2. $ass_A(S) = \prod_{i=1}^s ass_{A_i}(S_i)$.
- 3. $S \in L(A)$ iff $S_i \in L(A_i)$ for some i.
- 4. $\max L(A) = \coprod_{i=1}^{s} \max L(A_i)$, where the set $\max L(A_i)$ is identified with its image in $\max L(A)$ via the injection $\phi_i : \max L(A_i) \to \max L(A)$, $S_i \mapsto A_1 \times \cdots \times A_{i-1} \times S_i \times A_{i+1} \times \cdots \times A_s$, and $A(S_i^{-1}) \simeq A_i(S_i^{-1})$.

Proof. 1. The ring $\prod_{i=1}^{s} A_i \langle S_i^{-1} \rangle$ satisfies the universal property of localization for the set S, and statement 1 follows.

2 and 3. Statements 2 and 3 follow from statement 1.

4. By statement 1, $A\langle S_i^{-1}\rangle \simeq A_i\langle S_i^{-1}\rangle$ for all $S_i \in \max L(A_i)$. Therefore, the map ϕ_i is an injection for every $i = 1, \ldots, s$. By the definition of the maps ϕ_i ,

$$\max L(A) \supseteq \coprod_{i=1}^{s} \max L(A_i).$$

Let $S \in \max L(A)$. By statement 1, $A\langle S^{-1} \rangle \simeq \prod_{i=1}^s A_i \langle S_i^{-1} \rangle$, where $S_i := p_i(S)$ for $i = 1, \ldots, s$. There is an index i such that $A_i \langle S_i^{-1} \rangle \neq \{0\}$. Therefore,

$$S \subseteq T_i := A_1 \times \cdots \times A_{i-1} \times S_i \times A_{i+1} \times \cdots \times A_s$$

and $A\langle T_i^{-1}\rangle \simeq A_i\langle S_i^{-1}\rangle$. Hence,

$$\max L(A) \subseteq \coprod_{i=1}^{s} \max L(A_i),$$

and statement 4 follows. \square

Lemma 5.2 Let A be a ring such that $\mathcal{L}L(A) = A^{\times}$. Then

- 1. $\max L(A) = \{A^{\times}\}\ and\ A\langle (A^{\times})^{-1}\rangle = A.$
- 2. $\operatorname{Lrad}(A) = \operatorname{ass}_A(A^{\times}) = 0$.
- 3. $CL(A) = \mathcal{L}L(A) = A^{\times}$, and $Q_c(A) = A$, $\mathfrak{c}_A = 0$.

Proof. 1. By Proposition 2.8.(1), $A^{\times} \in L(A)$ and $A(A^{\times^{-1}}) = A$, and so $ass_A(A^{\times}) = 0$. Hence, $A^{\times} \in L(A)$. If $S \in L(A)$ then $S \subseteq \mathcal{L}L(A) = A^{\times}$, by Corollary 3.13.(1). Therefore, $max L(A) = \{A^{\times}\}$ and statement 1 follows.

- 2. By statement 1, $\max L(A) = \{A^{\times}\}\$, and so $\operatorname{Lrad}(A) = \operatorname{ass}_A(A^{\times}) = 0$.
- 3. Statement 3 follows from statement 1. \square

Example. Let (A, \mathfrak{m}) be a local ring where \mathfrak{m} is a maximal ideal of A which is assumed to be a *nil ideal* (that is every element of \mathfrak{m} is a nilpotent element). Then

$$\mathcal{L}L(A) = A^{\times} \text{ and } \mathcal{N}\mathcal{L}L(A) = \mathfrak{m}.$$

In particular, the ring A satisfies the conditions of Lemma 5.2, and so $\max L(A) = \{A^{\times}\}, A((A^{\times})^{-1}) = A$, and $\operatorname{Lrad}(A) = \operatorname{ass}_A(A^{\times}) = 0$.

Let $A = \prod_{i=1}^s A_i$ be a direct product of simple rings A_i . Then $\min(A) = \{\mathfrak{p}_i \mid i = 1, \ldots, s\}$ where $\mathfrak{p}_i := A_1 \times \cdots \times A_{i-1} \times \{0\} \times A_{i+1} \times \cdots \times A_s$, and $A/\mathfrak{p}_i \simeq A_i$.

For the ring $A = \prod_{i=1}^{s} A_i$ such that $\mathcal{L}L(A_i) = A_i^{\times}$ for i = 1, ..., n, Theorem 5.3 describes in an explicit way the following sets: $\max L(A)$, $\operatorname{Lrad}(A)$, $\operatorname{CL}(A)$, $\operatorname{Qc}(A)$, $\operatorname{LL}(A)$, and $\operatorname{NL}(A)$.

Theorem 5.3 Let $A = \prod_{i=1}^{s} A_i$ be a direct product of rings A_i such that $\mathcal{L}L(A_i) = A_i^{\times}$ for $i = 1, \ldots, n$. Then

- 1. $\max L(A) = \{S_i \mid i = 1, ..., s\}$ where $S_i := p_i^{-1}(A_i^{\times}) = \{(a_1, ..., a_s) \in A \mid a_i \in A_i^{\times}, a_j \in A_j \}$ for all $j \neq i\}$.
- 2. $\operatorname{ass}_A(S_i) = \mathfrak{p}_i \text{ and } A\langle S_i^{-1} \rangle \simeq A/\mathfrak{p}_i = A_i.$
- 3. $\operatorname{Lrad}(A) = 0$.
- 4. $CL(A) = A^{\times}$ and $Q_c(A) = A$.

5.
$$\mathcal{L}L(A) = \{(a_1, \ldots, a_s)\} \in A \mid a_i \in A_i^{\times} \text{ for some } i\} \text{ and } \mathcal{N}\mathcal{L}L(A) = \prod_{i=1}^s A_i \setminus A_i^{\times}.$$

Proof. 1. Statement 1 follows from Proposition 5.1.(4) and Lemma 5.2.(1).

- 2. Statement 2 follows from statement 1.
- 3. By statement 2, $Lrad(A) = \bigcap_{i=1}^{s} \mathfrak{p}_i = 0$.
- 4. By statement 1, $\mathcal{C}L(A) = \bigcap_{i=1}^s S_i = A^{\times}$, and so $Q_c(A) = A\langle \mathcal{C}L(A)^{-1} \rangle = A$.
- 5. Statement 5 follows from statement 1. \square

Example. $A = \prod_{i=1}^{s} A_i$ be a direct product of local rings (A_i, \mathfrak{m}_i) where the maximal ideals \mathfrak{m}_i are nil ideals. Then the ring A satisfies the assumption of Theorem 5.3 (see the Example above).

Let A be a semisimple Artinian ring. Then the set of minimal primes $\min(A)$ of A is a finite set, $A(\mathfrak{p}) := A/\mathfrak{p}$ is a simple Artinian ring for every $\mathfrak{p} \in \min(A)$, and

$$A \simeq \prod_{\mathfrak{p} \in \min(A)} A(\mathfrak{p}),$$

and vice versa. We identify the rings A and $\prod_{\mathfrak{p}\in\min(A)}A(\mathfrak{p})$. Then each element $a\in A$ is identified with $(a_{\mathfrak{p}})_{\mathfrak{p}\in\min(A)}\in\prod_{\mathfrak{p}\in\min(A)}A(\mathfrak{p})$, where $a_{\mathfrak{p}}\in A(\mathfrak{p})$, and for each $\mathfrak{p}\in\min(A)$,

$$\mathfrak{p} = \prod_{\mathfrak{q} \in \min(A) \setminus \{\mathfrak{p}\}} A(\mathfrak{q}), \ A = A(\mathfrak{p}) \oplus \mathfrak{p}, \ \pi_{\mathfrak{p}} : A \to A(\mathfrak{p}), \ a \mapsto a_{\mathfrak{p}} := a + \mathfrak{p}.$$

For the semisimple Artinian ring A, Corollary 5.4 describes the following sets: $\max L(A)$, $\operatorname{Lrad}(A)$, $\operatorname{CL}(A)$, $\operatorname{Q}_c(A)$, $\operatorname{LL}(A)$, and $\operatorname{NL}(A)$.

Corollary 5.4 Let A be a semisimple Artinian ring. We keep the notation as above. Then

- 1. $\max L(A) = \{ \mathcal{S}_{\mathfrak{p}} \mid \mathfrak{p} \in \min(A) \}$ where $\mathcal{S}_{\mathfrak{p}} := \pi_{\mathfrak{p}}^{-1}(A(\mathfrak{p})^{\times}) = \{ (a_{\mathfrak{q}})_{\mathfrak{q} \in \min(A)} \in A \mid a(\mathfrak{p}) \in A(\mathfrak{p})^{\times} \}$.
- 2. $\operatorname{ass}_A(\mathcal{S}_{\mathfrak{p}}) = \mathfrak{p} \ and \ A\langle \mathcal{S}_{\mathfrak{p}}^{-1} \rangle \simeq A/\mathfrak{p} = A(\mathfrak{p}).$
- 3. $\operatorname{Lrad}(A) = 0$.
- 4. $CL(A) = A^{\times} = C_A$ and $Q_c(A) = A$.
- 5. $\mathcal{L}L(A) = \{(a_{\mathfrak{p}})_{\mathfrak{p} \in \min(A)} \in \prod_{\mathfrak{p} \in \min(A)} A(\mathfrak{p}) \mid a_{\mathfrak{p}} \in A(\mathfrak{p})^{\times} \text{ for some } \mathfrak{p} \in \min(A) \} \text{ and } \mathcal{N}\mathcal{L}L(A) = \prod_{\mathfrak{p} \in \min(A)} A(\mathfrak{p}) \setminus A(\mathfrak{p})^{\times}.$

Proof. For each simple Artininian ring \mathcal{A} , $\mathcal{C}_{\mathcal{A}} = {}'\mathcal{C}_{\mathcal{A}} = \mathcal{C}'_{\mathcal{A}} = \mathcal{A}^{\times}$. Hence, $\operatorname{ass}_{\mathcal{A}}(a) = \mathcal{A}$ for all $a \in \mathcal{A} \backslash \mathcal{A}^{\times}$. Therefore, $\mathcal{L}L(\mathcal{A}) = \mathcal{A}^{\times}$, and so the condition of Theorem 5.3 hold, and the corollary follows from Theorem 5.3. \square

Classification of maximal left localizable sets of a semiprime left Goldie ring that contain the set of regular elements of the ring. It was proved that the set of maximal left denominators sets of R, max.Den $_l(R)$, is a finite set if the classical left quotient ring $Q_{l,cl}(R) := C_R^{-1}R$ of R is a semisimple Artinian ring, [2], or a left Artinian ring, [4], or a left Noetherian ring, [5]. In each of the three cases an explicit description of the set max.Den $_l(R)$ is given.

Theorem 5.5 is a classification of maximal left localizable sets of a semiprime left Goldie ring that contain the set of regular elements of the ring.

Theorem 5.5 Let R be a semiprime left Goldie ring. Then $\{S \in \max L(R) \mid C_R \subseteq S\} = \max Den_l(R) = \{C(\mathfrak{p}) \mid \mathfrak{p} \in \min(R)\}$ where $C(\mathfrak{p}) := \{c \in R \mid c + \mathfrak{p} \in C_{R/\mathfrak{p}}\}.$

Proof. (i) $\max \operatorname{Den}_l(R) = \{\mathcal{C}(\mathfrak{p}) \mid \mathfrak{p} \in \min(R)\}$: See [2, Theorem 3.1.(1-4)]. (ii) $\{\mathcal{C}(\mathfrak{p}) \mid \mathfrak{p} \in \min(R)\} \subseteq \max L(R)$: The inclusion follows from Corollary 5.4.(1).

Suppose that $S \in L(R)$ and $C(\mathfrak{p}) \subseteq S$. We have to show that $S \subseteq C(\mathfrak{p})$. By [2, Theorem 3.1.(3)], $\mathcal{A} := C(\mathfrak{p})^{-1}R$ is a simple Artinian ring. In particular, $\mathcal{L}L(\mathcal{A}) = \mathcal{A}^{\times}$. Let $\tau : R \mapsto \mathcal{A}$, $r \mapsto \frac{\tau}{1}$. By Proposition 2.8.(2),

$$R\langle S^{-1}\rangle \simeq R\langle \mathcal{C}(\mathfrak{p})^{-1}\rangle \langle \tau(S)^{-1}\rangle = \left(\mathcal{C}(\mathfrak{p})^{-1}R\right)\langle \tau(S)^{-1}\rangle = \mathcal{A}\langle \tau(S)^{-1}\rangle = \mathcal{A},$$

since $\tau(S) \subseteq \mathcal{L}L(\mathcal{A}) = \mathcal{A}^{\times}$. It follows from the definition of the set $\mathcal{C}(\mathfrak{p})$ that

$$S \subseteq \tau^{-1}(\tau(S)) \subseteq \tau^{-1}(\mathcal{A}^{\times}) = \mathcal{C}(\mathfrak{p}).$$

(iii) $\{S \in \max L(R) \mid C_R \subseteq S\} \subseteq \{C(\mathfrak{p}) \mid \mathfrak{p} \in \min(R)\}$: Let $S \in \max L(R)$ be such that $C_R \subseteq S$. By Proposition 2.8.(2),

$$R\langle \mathcal{S}^{-1}\rangle \simeq R\langle \mathcal{C}_R^{-1}\rangle \langle \sigma(\mathcal{S})^{-1}\rangle = Q_{l,cl}(R)\langle \sigma(\mathcal{S})^{-1}\rangle = \Big(\prod_{\mathfrak{p}\in \min(R)}Q(\mathfrak{p})\Big)\langle \sigma(\mathcal{S})^{-1}\rangle$$

where $\sigma: R \to Q_{l,cl}(R) = \prod_{\mathfrak{p} \in \min(R)} Q(\mathfrak{p})$ is a direct product of simple Artinian rings $Q(\mathfrak{p})$. By Corollary 5.4.(1), $\sigma(\mathcal{S}) \subseteq \mathcal{S}_{\mathfrak{p}}$ for some $\mathfrak{p} \in \min(R)$. By Corollary 5.4.(1),

$$\mathcal{S} \subseteq \sigma^{-1}(\mathcal{S}_{\mathfrak{p}}) = \mathcal{C}(\mathfrak{p}),$$

and so we must have $S = C(\mathfrak{p})$, by the statement (ii).

Now, the theorem follows from the satements (i)–(iii). \Box

So, every maximal left localizable set of a semiprime left Goldie ring that contains the set of regular elements of the ring is a maximal left denominator set, and vice versa.

Corollary 5.6 Let R be a semiprime left Goldie ring which is not a prime ring. Then $Q_a(R) = \{0\}$.

Proof. By Theorem 5.5, $|\max L(R)| \ge 2$, and the corollary follows from Theorem 2.18. \square

Classification of maximal Ore and denominator sets of a semiprime Goldie ring. The sets of localizable left, right or two-sided Ore sets of R are denoted by $\mathbb{L}_l(R)$, $\mathbb{L}_r(R)$ and $\mathbb{L}(R)$, respectively. Clearly, $\mathbb{L}(R) = \mathbb{L}_l(R) \cap \mathbb{L}_r(R)$. In order to work with these three sets simultaneously we use the following notation $\mathbb{L}_*(R)$ where $* \in \{l, r, \emptyset\}$ and \emptyset is the empty set $(\mathbb{L}(R) = \mathbb{L}_{\emptyset}(R))$. Then $\mathbb{L}_*(R) = \mathbb{L}(R) \cap \operatorname{Ore}_*(R)$ for $* \in \{l, r, \emptyset\}$. Ideals of a ring are called *incomparable* if none of them is contained in the other. For a ring R, $\min(R)$ is the set of its minimal prime ideals.

The next theorem is an explicit description of maximal Ore sets of a semiprime Goldie ring.

Theorem 5.7 ([7, Theorem 1.11]) Let R be a semiprime Goldie ring and $\mathcal{N}_* := \{S \in \max \mathbb{L}_*(R) \mid \mathcal{C}_R \subseteq S\}$ where $* \in \{l, r, \emptyset\}$. Then

- 1. $\max \operatorname{Ore}(R) = \max \operatorname{Den}(R) = \{\mathcal{C}(\mathfrak{p}) \mid \mathfrak{p} \in \min(R)\} = \mathcal{N}_* \text{ for all } * \in \{l, r, \emptyset\} \text{ where } \mathcal{C}(\mathfrak{p}) := \{c \in R \mid c + \mathfrak{p} \in \mathcal{C}_{R/\mathfrak{p}}\}.$ So, every maximal Ore set of R is a maximal denominator set, and vice versa.
- 2. For all $S \in \max \operatorname{Ore}(R)$ the ring $S^{-1}R$ is a simple Artinian ring.
- 3. $Q_{cl}(R) \simeq \prod_{S \in \max Ore(R)} S^{-1}R$.
- 4. $\max \operatorname{ass} \operatorname{Ore}(R) = \operatorname{ass} \max \operatorname{Ore}(R) = \min(R)$ where $\operatorname{ass} \max \operatorname{Ore}(R) := \{\operatorname{ass}_R(S) \mid S \in \max \operatorname{Ore}(R)\}$. In particular, the ideals in the set $\operatorname{ass} \max \operatorname{Ore}(R)$ are incomparable.

Corollary 5.8 Let R be a semiprime Goldie ring. Then $\{S \in \max L(R) \mid C_R \subseteq S\} = \max \operatorname{Ore}(R) = \max \operatorname{Den}(R) = \{C(\mathfrak{p}) \mid \mathfrak{p} \in \min(R)\} = \mathcal{N}_* \text{ for all } * \in \{l, r, \emptyset\}.$

Proof. The corollary follows from Theorem 5.5 and Theorem 5.7.(1). \square

Maximal localizable sets of a direct product of division rings. Let I be a set and $\mathcal{P}(I)$ be its power set of I, i.e. the set of all subsets of I (an element of $\mathcal{P}(I)$ is a subset of I). An element $F \in \mathcal{P}(I)$ is called a filter on I if the following conditions hold: $\emptyset \notin F$; if $\mathfrak{a} \in F$ then $\mathfrak{b} \in F$ for all $\mathfrak{b} \subseteq I$ such that $\mathfrak{a} \subseteq \mathfrak{b}$; and for all elements $\mathfrak{a}, \mathfrak{b} \in F$, $\mathfrak{a} \cap \mathfrak{b} \in F$. The set of all filters on I is denoted by $\mathcal{F}(I)$. A maximal element of $\mathcal{F}(I)$ is called an ultrafilter on I. The set of all ultrafilters on I is denoted by $\mathcal{U}(I)$. For each element $i \in I$, the set

$$F_i := \{ \mathfrak{a} \subseteq I \,|\, i \in \mathfrak{a} \} \in \mathcal{U}(I)$$

is called a *principal ultrafilter*. An ultrafilter is principal iff it contains a finite set. If the set I is a finite set then all ultrafilters are principal. If I is an infinite set then an ultrafilter is non-principal iff it contains the *Frechet filter* of co-finite subsets (a subset \mathfrak{a} of I is called *co-finite* if its *complement* $C\mathfrak{a} := I \setminus \mathfrak{a}$ is a finite set). A filter $F \in \mathcal{F}(I)$ is an ultrafilter iff for each subset \mathfrak{a} of I either $\mathfrak{a} \in F$ or $C\mathfrak{a} \in F$. The set $(\mathcal{F}(I), \subseteq)$ is a poset w.r.t. inclusion where $\mathcal{F} \subseteq \mathcal{F}'$ if for all elements $\mathfrak{f} \in \mathcal{F}$, $\mathfrak{f} \in \mathcal{F}'$.

Let $D = \prod_{i \in I} D_i$ be a direct product of division rings D_i . For each subset J of I, let $D_J := \prod_{j \in J} D_j$. We identify the set D_J with the ideal $D_J \times \prod_{k \in CJ} \{0\}$ of D where $CJ := I \setminus J$ is the complement of J in I. If $J \subseteq J' \subseteq I$ then $D_J \subseteq D_{J'}$, and $D_J = D_{J''}$ for some subset J'' of I iff J = J''. For each element $d = (d_i)_{i \in I} \in D$, where $d_i \in D_i$, the set $\mathrm{supp}(d) := \{i \in I \mid d_i \neq 0\}$ is called the *support* of d. For all elements $d, e \in D$,

$$\operatorname{supp}(de) = \operatorname{supp}(d) \cap \operatorname{supp}(e)$$
 and $\operatorname{supp}(d \pm e) \subseteq \operatorname{supp}(d) \cup \operatorname{supp}(e)$.

Each element of the ring D is either a unit or a zero divisor. An element $d \in D$ is a unit (resp., a zero divisor) iff $\operatorname{supp}(d) = I$ (resp., $\operatorname{supp}(d) \neq I$).

Proposition 5.9 describes all the localizable sets in the ring D and all the localizations of D.

Proposition 5.9 Let $D = \prod_{i \in I} D_i$ be a direct product of division rings D_i .

- 1. Suppose that S is a multiplicative submonoid of D. Then the following statements are equivalent:
 - (a) $S \in L(D)$.
 - (b) S is a multiplicative subset of D.
 - (c) $S \in \text{Den}(D)$.
- 2. Suppose that S is a multiplicative subset of D. Then $D\langle S^{-1}\rangle \simeq S^{-1}D \simeq DS^{-1} \simeq D/\mathrm{ass}_D(S)$ where $\mathrm{ass}_D(S) = \bigcup_{s \in S} D_{C\mathrm{supp}(s)} = \sum_{s \in S} D_{C\mathrm{supp}(s)}$.

Proof. 1. Straightforward.

2. By statement 1, $S \in \text{Den}(D)$, and so $D(S^{-1}) \simeq S^{-1}D \simeq DS^{-1}$ and

$$\operatorname{ass}_D(S) = \bigcup_{s \in S} D_{C\operatorname{supp}(s)} = \sum_{s \in S} D_{C\operatorname{supp}(s)}.$$

Since $D/\operatorname{ass}_D(S) \subseteq S^{-1}D$ and for every element $s \in S$, the element $s + \operatorname{ass}_D(S)$ is a unit of the factor ring $D/\operatorname{ass}_D(S)$, we must have $D/\operatorname{ass}_D(S) = S^{-1}D$. \square

By Proposition 5.9.(1), every localizable set of the ring D is a denominator set, and vice versa. By Proposition 5.9.(2), for all nonzero elements $d \in D$, $R(d^{-1}) \simeq R/D_{Csupp(s)}$.

Let S be a multiplicative subset of D, then the set

$$S_{sat} := \bigcup_{s \in S} \left(\prod_{i \in \text{supp}(s)} D_i^{\times} \times \prod_{j \in C \text{supp}(s)} D_j \right)$$

$$\tag{41}$$

is called the saturation of S. Clearly, $(S_{sat})_{sat} = S_{sat}$. For each subset S of D, let $\text{supp}(S) := \{\sup(s) \mid s \in S\} \in \mathcal{P}(I)$. For each filter $\mathcal{F} \in \mathcal{F}(I)$, the set

$$S(\mathcal{F}) := \{ s \in D \mid \text{supp}(s) \in \mathcal{F} \}$$
(42)

is a multiplicative subset of D. For all filters $\mathcal{F}, \mathcal{G} \in \mathcal{F}(I)$ such that $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{S}(\mathcal{F}) \subseteq \mathcal{S}(\mathcal{G})$. A subset S of D is called a *saturated subset* if $S = S_{sat}$. Let $\mathcal{P}(D)_{sat}$ be the set of all saturated subsets of D. The sets $(\mathcal{F}(I), \subseteq)$ and $(\mathcal{P}(D)_{sat}, \subseteq)$ are posets.

Proposition 5.10.(4) shows that the posets $\mathcal{F}(I)$ and $\mathrm{Sub}(D)_{sat}$ are isomorphic.

Proposition 5.10 Let $D = \prod_{i \in I} D_i$ be a direct product of division rings D_i , S be a multiplicative subset of D, and $\mathcal{F} \in \mathcal{F}(I)$. Then

- 1. S_{sat} is a multiplicative subset of D, $S^{-1}D \simeq S_{sat}^{-1}R$ (an R-isomorphism), and $\operatorname{ass}_R(S) = \operatorname{ass}_R(S_{sat})$.
- 2. $\operatorname{supp}(S_{sat}) \in \mathcal{F}(I) \text{ and } S_{sat} = \mathcal{S}(\operatorname{supp}(S_{sat})).$
- 3. $S(\mathcal{F})_{sat} = S(\mathcal{F})$ and $supp(S(\mathcal{F})) = \mathcal{F}$.
- 4. The map $\mathcal{F}(I) \to \mathcal{P}(D)_{sat}$, $\mathcal{F} \mapsto \mathcal{S}(\mathcal{F})$ is an isomorphism of posets with inverse $S \mapsto \sup_{i \in \mathcal{F}} S(i)$.

Proof. 1. By the definition of the set S_{sat} , it is a multiplicative subset of D such that $ass_R(S) = ass_R(S_{sat})$. Now, by Proposition 5.11.(2),

$$S^{-1}D \simeq D/\mathrm{ass}_R(S) = D/\mathrm{ass}_R(S_{sat}) \simeq S_{sat}^{-1}D.$$

- 2. Since the set S_{sat} is a saturated subset of D, $\operatorname{supp}(S_{sat}) \in \mathcal{F}(I)$. The equality $S_{sat} = \mathcal{S}(\operatorname{supp}(S_{sat}))$ follows from the definitions of the sets S_{sat} and $\mathcal{S}(\operatorname{supp}(S_{sat}))$.
- 3. The equality $S(\mathcal{F})_{sat} = S(\mathcal{F})$ is obvious. The equality $\operatorname{supp}(S(\mathcal{F})) = \mathcal{F}$ follows from the definitions of the sets $S(\mathcal{F})$ and $\operatorname{supp}(S(\mathcal{F}))$.
 - 4. Statement 4 follows from statements 2 and 3. \square

Let $\mathcal{F} \in \mathcal{F}(I)$. Then (\mathcal{F}, \supseteq) is a directed set since for all $\mathfrak{f}, \mathfrak{q} \in \mathcal{F}$, $f \supseteq \mathfrak{f} \cap \mathfrak{q}$ and $\mathfrak{q} \supseteq \mathfrak{f} \cap \mathfrak{q}$. For each $\mathfrak{f} \in \mathcal{F}$, let $D_{\mathfrak{f}} = \prod_{i \in \mathfrak{f}} D_i$, a direct product of division rings D_i . For each pair of elements $\mathfrak{f}, \mathfrak{q} \in \mathcal{F}$ such that $\mathfrak{f} \supseteq \mathfrak{q}$, let $p_{\mathfrak{q}\mathfrak{f}} : D_{\mathfrak{f}} \to D_{\mathfrak{q}}$ be the natural projection map. Then $(D_{\mathfrak{f}}, p_{\mathfrak{q}\mathfrak{f}})$ is a directed system.

Let $\mathcal{P}(D)_{mult}$ be the set of all multiplicative subsets of D. For the ring D, Theorem 5.11 describes the following sets: $\max L(D)$, $\max Loc(D)$, $\operatorname{Lrad}(D)$, $\operatorname{CL}(D)$, $\operatorname{Qc}(D)$, $\operatorname{LL}(D)$, and $\operatorname{NL}(D)$.

Theorem 5.11 Let $D = \prod_{i \in I} D_i$ be a direct product of division rings D_i . Then

- 1. $\max L(D) = \max Den_*(D) = \max Ore_*(D) = \max \mathcal{P}(D)_{mult} = \{S(\mathcal{F}) \mid \mathcal{F} \in \mathcal{U}(I)\}$ where $* \in \{l, r, \emptyset\}$.
- 2. For each $S = S(F) \in \max L(D)$,

$$D\langle \mathcal{S}^{-1}\rangle \simeq \varinjlim_{\mathfrak{f}\in\mathcal{F}} D_{\mathfrak{f}} \simeq \mathcal{S}^{-1}D \simeq D\mathcal{S}^{-1} \simeq D/\mathrm{ass}_D(\mathcal{S})$$

is a division ring and $ass_D(S) = \bigcup_{\mathfrak{f} \in \mathcal{F}} D_{C\mathfrak{f}} = \sum_{\mathfrak{f} \in \mathcal{F}} D_{C\mathfrak{f}}$ is a maximal ideal of the ring D.

- 3. $\operatorname{Lrad}(D) = 0$.
- 4. $CL(D) = C_D = D^{\times}$ and $Q_c(D) = Q_{cl}(D) = D$.
- 5. $\mathcal{L}L(D) = D \setminus \{0\}$ and $\mathcal{N}\mathcal{L}L(D) = \{0\}$.

Proof. 1. By Proposition 5.10.(4), $\max L(D) = \{ \mathcal{S}(\mathcal{F}) \mid \mathcal{F} \in \mathcal{U}(I) \}$. By Proposition 5.9.(1), $\max L(D) = \max \operatorname{Den}_*(D) = \max \operatorname{Ore}_*(D) = \max \mathcal{P}(D)_{mult}$ where $* \in \{l, r, \emptyset\}$.

2. By Proposition 5.9.(2),

$$D\langle \mathcal{S}^{-1} \rangle \simeq \mathcal{S}^{-1}D \simeq D\mathcal{S}^{-1} \simeq D/\mathrm{ass}_D(\mathcal{S}) \text{ and } \mathrm{ass}_D(\mathcal{S}) = \bigcup_{\mathfrak{f} \in \mathcal{F}} D_{C\mathfrak{f}} = \sum_{\mathfrak{f} \in \mathcal{F}} D_{C\mathfrak{f}}.$$

Let $s + \operatorname{ass}_D(S)$ be a nonzero element of the ring $D/\operatorname{ass}_R(S)$. Then $\operatorname{supp}(s) \in \mathcal{F}$ and the element $s + D_{C\operatorname{supp}(s)} \in D/D_{C\operatorname{supp}(s)}$ is a unit. The ring $D/\operatorname{ass}_D(S)$ is an epimorphic image of the ring $D/D_{C\operatorname{supp}(s)}$ since $D_{C\operatorname{supp}(s)} \subseteq \operatorname{ass}_D(S)$. Therefore the element

$$s + \operatorname{ass}_D(S) \in D/\operatorname{ass}_D(S)$$

is a unit (as an image of the unit $s + D_{Csupp(s)}$). Therefore, the ring $D/ass_D(S)$ is a division ring and the ideal $ass_D(S)$ is a maximal ideal of D.

For each element $\mathfrak{f} \in \mathcal{F}$, let

$$\phi_{\mathbf{f}}: D_{\mathbf{f}} \simeq D/D_{C\mathbf{f}} \to D/\mathrm{ass}_R(\mathcal{F})$$

be the natural epimorphism that is determined by the inclusion $D_{C\mathfrak{f}} \subseteq \operatorname{ass}_R(S)$. For each pair of elements $\mathfrak{f}, \mathfrak{q} \in \mathcal{F}$ such that $\mathfrak{f} \supseteq \mathfrak{q}, \phi_{\mathfrak{f}} = \phi_{\mathfrak{q}} p_{\mathfrak{q}\mathfrak{f}}$. Hence, there is a ring homomorphism

$$\varinjlim_{\mathfrak{f}\in\mathcal{F}} D_{\mathfrak{f}} \to D/\mathrm{ass}_D(S).$$

The ideal $\operatorname{ass}_D(S) = \bigcup_{\mathfrak{f} \in \mathcal{F}} D_{C\mathfrak{f}}$ belongs to the kernel of the homomorphism $\phi_I : D = D_I \to \varinjlim_{\mathfrak{f} \in \mathcal{F}} D_{\mathfrak{f}}$. Hence, there is a homomorphism

$$D/\mathrm{ass}_D(S) \to \varinjlim_{\mathfrak{f} \in \mathcal{F}} D_{\mathfrak{f}}.$$

Hence, $\varinjlim_{\mathfrak{f}\in\mathcal{F}} D_{\mathfrak{f}} \simeq D/\mathrm{ass}_D(S)$.

3. For each $i \in I$, F_i is a principal ultrafilter on I, and $\mathcal{S}(F_i) = D_i^{\times} \times \prod_{j \in I \setminus \{i\}} D_j$. Hence, $ass_D(\mathcal{S}(F_i)) = \{0\} \times \prod_{j \in I \setminus \{i\}} D_j$. Now, by statement 1,

$$\operatorname{Lrad}(D) = \bigcap_{\mathcal{F} \in \mathcal{U}(I)} \operatorname{ass}_D(\mathcal{S}(\mathcal{F})) \subseteq \bigcap_{i \in I} \operatorname{ass}_D(\mathcal{S}(F_i)) = 0,$$

and so Lrad(D) = 0.

4. Let C = CL(D). Recall that for each $i \in I$, F_i is a principal ultrafilter on I, and $S(F_i) = D_i^{\times} \times \prod_{i \in I \setminus \{i\}} D_j$. Hence,

$$D^{\times} \subseteq \mathcal{C} = \bigcap_{\mathcal{F} \in \mathcal{U}(I)} \mathcal{S}(\mathcal{F}) \subseteq \bigcap_{i \in I} \mathcal{S}(F_i) = D^{\times},$$

and so $C = D^{\times} = C_D$. Since $C = D^{\times}$, $Q_c(D) = D\langle C^{-1}\rangle = D\langle (D^{\times})^{-1}\rangle = D\langle D^{\times}\rangle = D = Q_{cl}(D)$. 5. Statement 5 is obvious. \square

6 Localization of a module at a localizable set

The aim of the section is to introduce the concept of localization of a module at a localizable set and to consider its basic properties. Proposition 6.1.(2) is a universal property of localization of a module. Theorem 6.2 is a criterion for the localization functor $M \to S^{-1}M$ to be an exact functor.

Definition. Let R be a ring, $S \in L(R, \mathfrak{a})$, and M be a left R-module. Then the left $R(S^{-1})$ -module

$$S^{-1}M := R\langle S^{-1}\rangle \otimes_R M$$

is called the *(left) localization* of M at S. If $S \in L(R, \mathfrak{a})$ and M be a right R-module. Then the right $R(S^{-1})$ -module

$$MS^{-1} := M \otimes_R R\langle S^{-1} \rangle$$

is called the (right) localization of M at S.

By the definition, $S^{-1}M$ is a left $R\langle S^{-1}\rangle$ -module. By applying $-\otimes_R M$ to the R-homomorphism $R\to R\langle S^{-1}\rangle$ we obtain the R-homomorphism $i_M:M\to S^{-1}M$. Proposition 6.1.(2) is the universal property of the $R\langle S^{-1}\rangle$ -module $S^{-1}M$.

Proposition 6.1 Let R be a ring, $S \in L(R, \mathfrak{a})$, M be an R-module, and $i_M : M \to S^{-1}M$. Then

- 1. $S^{-1}M \simeq \overline{S}^{-1}(M/\mathfrak{a}M)$ where $\overline{S} := (S + \mathfrak{a})/\mathfrak{a} \in L(\overline{R}, 0)$ and $\overline{R} = R/\mathfrak{a}$ (Corollary 2.2.(2)).
- 2. Let \mathcal{M} be an $R\langle S^{-1}\rangle$ -module and $f:M\to\mathcal{M}$ be an R-homomorphism. Then there is a unique $R\langle S^{-1}\rangle$ -homomorphism $S^{-1}f:S^{-1}M\to\mathcal{M}$ such that $f=S^{-1}f\circ i_M$,

$$\begin{array}{ccc} M & \stackrel{i_M}{\rightarrow} & S^{-1}M \\ & \stackrel{f}{\searrow} & \downarrow^{\exists!\,S^{-1}f} \\ & & \mathcal{M}. \end{array}$$

This property is uniquely characterized the $R(S^{-1})$ -module $S^{-1}M$ up to isomorphism.

Proof. 1. Let $\overline{M} = M/\mathfrak{a}M$. Then $S^{-1}(\mathfrak{a}M) = R\langle S^{-1}\rangle \otimes_R \mathfrak{a}M = R\langle S^{-1}\rangle \mathfrak{a} \otimes_R M = 0$, and so

$$S^{-1}M = S^{-1}\overline{M} = R\langle S^{-1}\rangle \otimes_R \overline{M} \simeq \overline{R}\langle \overline{S}^{-1}\rangle \otimes_{\overline{D}} \overline{M} = \overline{S}^{-1}\overline{M}.$$

2. The R-homomorphism $f: M \to \mathcal{M}$ determines a ring homomorphism

$$R \to \operatorname{End}_{\mathbb{Z}}(\mathcal{M}), \ r \mapsto (m \mapsto rm).$$

The images of the elements of the set S in $\operatorname{End}_{\mathbb{Z}}(\mathcal{M})$ are units. Now, by Proposition 2.1, there is a unique $R\langle S^{-1}\rangle$ -homomorphism $S^{-1}f: S^{-1}M \to \mathcal{M}$ such that $f = S^{-1}f \circ i_M$. By the uniqueness of the map $S^{-1}f$, this property is uniquely characterized the $R\langle S^{-1}\rangle$ -module $S^{-1}M$ up to isomorphism. \square

The R-homomorphism $i_M: M \to S^{-1}M$ is equal to the compositions of R-homomorphisms

$$i_M: M \to R/\mathfrak{a} \otimes_R M = M/\mathfrak{a}M \to S^{-1}M = \overline{S}^{-1}(M/\mathfrak{a}M).$$

Therefore, $\mathfrak{t}_S(M) := \ker(i_M) \supseteq \mathfrak{a}M$. We have the descending chain of R-modules

$$M \supseteq \mathfrak{t}_S(M) \supseteq \mathfrak{t}_S^2(M) \supseteq \cdots \supseteq \mathfrak{t}_S^n(M) \supseteq \cdots$$

where $\mathfrak{t}_S^n(M) = \mathfrak{t}_S \mathfrak{t}_S \cdots \mathfrak{t}_S(M)$, n times. An R-module M is called S-torsion (resp., S-torsionfree) if $S^{-1}M = 0$ (resp., $\mathfrak{t}_S(M) = 0$, i.e., the map $i_M : M \to S^{-1}M$, $m \mapsto 1 \otimes m$ is an R-module monomorphism). Let $\mathfrak{f}_S(M) = \operatorname{im}(i_M)$, the image of the map i_M , and we have a short exact sequence of R-modules

$$0 \to \mathfrak{t}_S(M) \to M \to \mathfrak{f}_S(M) \to 0. \tag{43}$$

We have the descending chain of R-module epimorphisms

$$M \to \mathfrak{f}_S(M) \to \mathfrak{f}_S^2(M) \to \cdots \to \mathfrak{f}_S^n(M) \to \cdots$$

where $\mathfrak{f}_S^n(M) = \mathfrak{f}_S \mathfrak{f}_S \cdots \mathfrak{f}_S(M)$, n times.

For a ring R, let R-mod be the category of left R-modules. By Proposition 6.1.(1), the localization at S functor,

$$S^{-1}: R - \text{mod} \to R\langle S^{-1} \rangle - \text{mod}, \ M \mapsto S^{-1}M,$$

is the composition of two functors

$$S^{-1} = \overline{S}^{-1} \circ (R/\mathfrak{a} \otimes_R -) \tag{44}$$

and the second one is a right exact functor.

Applying the right exact functor $-\otimes_R M$ to the short exact sequence $0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$ of right R-modules yields the short exact sequence $\mathfrak{a} \otimes_R M \to M \to R/\mathfrak{a} \otimes_R M \to 0$, and so

$$R/\mathfrak{a} \otimes_R M \simeq M/\mathfrak{a}M,$$

an isomorphism of left R-modules. Now, applying the right exact functor $R/\mathfrak{a} \otimes_R -$ to the short exact sequence of left R-modules, $0 \to M_1 \to M_2 \to M_3 \to 0$, yields the short exact sequence of left R-modules

$$0 \to M_1 \cap \mathfrak{a} M_2/\mathfrak{a} M_1 \to M_1/\mathfrak{a} M_1 \to M_2 \mathfrak{a} M_2 \to M_3 \mathfrak{a} M_3 \to 0.$$

Suppose that the functor S^{-1} is also a right exact functor for some $S \in L(R, \mathfrak{a})$. Then the sequence of left $R(S^{-1})$ -modules

$$0 \to S^{-1}(M_1 \cap \mathfrak{a}M_2/\mathfrak{a}M_1) \simeq \overline{S}^{-1}(M_1 \cap \mathfrak{a}M_2/\mathfrak{a}M_1) \to S^{-1}M_1 \to S^{-1}M_2 \to S^{-1}M_3 \to 0 \quad (45)$$

is exact. Notice that

$$M_1 \cap \mathfrak{a} M_2/\mathfrak{a} M_1 \simeq \overline{M}_1 \cap (\mathfrak{a} M_2/\mathfrak{a} M_1).$$

Theorem 6.2 is a criterion for the functor $S^{-1}: M \to S^{-1}M$ to be exact.

Theorem 6.2 Let R be a ring, $S \in L(R, \mathfrak{a})$, $\overline{R} = R/\mathfrak{a}$ and $\overline{S} := (S + \mathfrak{a})/\mathfrak{a}$. The functor S^{-1} is exact iff for all R-modules M_1 and M_2 such that $M_1 \subseteq M_2$, the \overline{R} -modules $M_1 \cap \mathfrak{a}M_2/\mathfrak{a}M_1$ is \overline{S} -torsion.

Proof. The theorem follows from the exact sequence (45). \square

Corollary 6.3 Let R be a ring, $S \in L(R, \mathfrak{a})$, $\overline{R} = R/\mathfrak{a}$ and $\overline{S} := (S + \mathfrak{a})/\mathfrak{a}$. If the functor S^{-1} is exact then the \overline{R} -module $\mathfrak{a}/\mathfrak{a}^2$ is \overline{S} -torsion or, equivalently, \overline{S} -torsion.

Proof. Applying Theorem 6.2 to the pair of R-modules $M_1 = \mathfrak{a} \subseteq M_2 = R$, we conclude that the R-module (resp., \overline{R} -module) $(M_1 \cap \mathfrak{a}M_2)/\mathfrak{a}M_1 = \mathfrak{a}/\mathfrak{a}^2$ is S-torsion (resp., \overline{S} -torsion). \square

Since $S^{-1}\mathfrak{a}M = R\langle S^{-1}\rangle \otimes_R \mathfrak{a}M = R\langle S^{-1}\rangle \mathfrak{a} \otimes_R M = 0$, $\mathfrak{a}M \subseteq \mathfrak{t}_{\underline{S}}(M)$. By taking the short exact sequence (43) modulo $\mathfrak{a}M$, we obtain a short exact sequence of \overline{R} -modules

$$0 \to \mathfrak{t}_S(M)/\mathfrak{a}M \to M/\mathfrak{a}M \to \mathfrak{f}_S(M) \to 0. \tag{46}$$

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