Optimal Risk Policies and Periodic Dividend Strategies for an Insurance Company

Mark Kelbert[1,2]
Harold A. Moreno-Franco[1,3]
Nikolai P. Pogorelov[1]

National Research University Higher School of Economics
Department of Statistics and Data Analysis
Laboratory of Stochastic Analysis and its Applications
Moscow, Russian Federation

Abstract

We study the problem of optimal risk policies and dividend strategies for an insurance company operating under the constraint that the timing of shareholder payouts is governed by the arrival times of a Poisson process. Concurrently, risk control is continuously managed through proportional reinsurance. Our analysis confirms the optimality of a periodic-classical barrier strategy for maximizing the expected net present value until the first instance of bankruptcy across all admissible periodic-classical strategies.

Keywords: Proportional reinsurance, optimal periodic dividend, cheap and non-cheap reinsurance.

2000 MR Subject Classification 60J60; 91B70; 91G50; 93E20.

1 Introduction

In the field of actuarial risk theory, significant research has focused on optimal risk policies and/or dividend strategies, especially within the context of a diffusion approximation of the classical risk model (e.g., see [3, 6, 12]). This approach has been widely adopted when both the dividend stream and the risk exposure are continuously modelled over time (e.g., see [5, 4, 9, 13]).

This is exemplified in the works of Højgaard and Taksar [5], as well as Taksar [13], where the authors delve into the control of risk through proportional reinsurance and the management of cumulative dividend payments via singular control. In both papers, it is demonstrated that a two-barrier strategy is optimal for maximizing expected net discounted dividend payouts until the first time of bankruptcy. One barrier signifies the threshold for maximum risk-taking by the insurance company, while the other designates the level at which the excess between the surplus process and the threshold is paid out to the shareholders (after discounts) if the surplus process exceeds this threshold. The authors explored this problem in the context of cheap and non-cheap reinsurance, respectively.

In light of the recent wide-ranging exploration of periodic observation models in insurance literature (e.g., [1, 2, 10]), this paper extends the aforementioned problem with a periodic dividend constraint. This modification dictates that the decision times for paying out to the shareholders are governed by the arrivals times of a Poisson process with intensity $\gamma > 0$, while the control of risk is continuously managed by proportional reinsurance as in [5, 13]. The primary objective of this paper is to establish that the periodic-classical barrier strategy is optimal for the problem of maximizing the expected net present value (NPV) until the first time of bankruptcy across all the admissible periodic-classical strategies.

A notable contrast with the results obtained by Højgaard and Taksar [5], and Taksar [13] is that the level of maximum risk-taking by the company, as determined by the optimal barrier, does

[1]M. Kelbert and H. A. Moreno-Franco made equal contributions to this paper, while N. P. Pogorelov worked only on Subsection 3.2 of this paper as part of his master's thesis under the supervision of H. A. Moreno-Franco.
[2]M. Kelbert was supported by the Russian Science Foundation Grant number 23-21-00052.
[3]Corresponding author: hmoreno@hse.ru.

not consistently remain lower than the level at which payouts should be made, as these levels are dependent on γ. Moreover, for certain values of γ, the strategy of making the payout equal to the entire amount of assets that the company holds at the first arrival of the Poisson process is considered optimal.

The remainder of this document is organized as follows: in Section 2, we consistently formulate the problem of optimal risk policies and periodic dividend strategies. This section introduce the HJB equation associated with the value function as defined in (2.4); see (2.5). Subsequently, in Subsection 2.1, the classical-barrier strategy is well-defined. Assuming that the solution v to (2.5) satisfies certain properties, we verify that v is also a solution to the non-linear partial differential system (NLPDS) given by (2.8)–(2.10); see Proposition 3.9. Following this, a verification theorem is presented; see Theorem 2.3. Moving to Section 3, by solving the NLPDS (2.8)–(2.10), we obtain different explicit solutions to the HJB equation (2.5). These explicit solutions are dependent on the various scenarios, covering cases of cheap reinsurance and non-cheap reinsurance; see Propositions 3.4, 3.8 and 3.13. Additionally, explicit forms of the levels associated with the optimal periodic-classical barrier strategy proposed in Subsection 2.1 are provided. Subsequently, in Section 4, we discuss the behaviour of our solutions in relation to the parameter γ. Specifically, as γ tends to infinity, we recover the value functions for the case of singular dividend strategies obtained by Hôgaard and Taksar [5], and Taksar [13], in the cases of cheap and non-cheap reinsurance, respectively. In conclusion, Section 5 presents numerical examples for further illustration.

2 Formulation of the problem

In order to mitigate the risk, the cedent (or primary insurance) enters into a contract with a reinsurance firm. According to this contract, at time t, the insurer assumes the responsibility to pay a fraction u^π_t of each claim, while the reinsurance firm takes care of the remaining payment $1 - u^\pi_t$ for each claim. The starting point is the classical Cramér-Lundberg ruin problem with reinsurance and dividend payments, i.e. the capital of the primary insurer is governed by

$$X^\pi_t = x_0 + \lambda a[1+\eta]t - \lambda a[1-u^\pi_t][1+\mu]t - u^\pi_t \sum_{j=1}^{N^\lambda_t} Y_j - L^\pi_t, \quad \text{for } t \geq 0, \tag{2.1}$$

where $X_0 = x_0 > 0$ is the initial surplus of the insurance firm. Here $\{N^\lambda_t, t \geq 0\}$ stands for a Poisson process of rate λ , and $\{Y_j\}_{j\geq 1}$ represents i.i.d. non-negative random variables corresponding to the sizes of individual claims, which are also independent of N^λ. The premiums are paid continuously at constant rates $\lambda a[1+\eta]$ and $\lambda a[1-u^\pi_t][1+\mu]$ based on the expected value principle, where $u^\pi_t \in [0,1]$, $a = \mathbb{E}[Y_1]$, $\bar{\sigma}^2 = \mathrm{Var}[Y_1]$, and safety loadings $\mu \geq \eta > 0$ are involved. The payment rate η represents the input payment rate of the insurance firm, while μ is the insurer's payment rate to the reinsurance firm as per the contract mentioned earlier. Additionally, L^π_t is a non-decreasing random function that quantifies the total amount of dividends paid by the primary insurer to the shareholders and will be formally defined later. When $\mu > \eta$, we call it non-cheap reinsurance, and in the case of $\mu = \eta$, it is usually referred as a cheap reinsurance.

Notice that (2.1) can be rewritten in a different form as follows

$$X^\pi_t = x_0 + \lambda a\{\eta - [1-u^\pi_t]\mu\}t + u^\pi_t Z_t - L^\pi_t, \quad \text{for } t \geq 0,$$

with $Z_t := \lambda a t - \sum_{j=1}^{N^\lambda_t} Y_j$. Then $\mathbb{E}[Z_t] = 0$, $\mathrm{Var}[Z_t] = \bar{\sigma}^2\lambda t$ and for$\lambda \gg 1$, $a \ll 1$ and $\lambda a \approx 1$, the process Z_t is well approximated by the Brownian motion σW_t, with $\sigma := \bar{\sigma}\lambda^{1/2}$; for more details, see for example [7]. This approximation is valid under the condition that the random variables $\{Y_j\}_{j\geq 1}$ are stochastically small, and it serves as the intuitive background for the following model

$$\mathrm{d}X^\pi_t = \{\eta - [1-u^\pi_t]\mu\}\mathrm{d}t + \sigma u^\pi_t \mathrm{d}W_t - \mathrm{d}L^\pi_t, \quad \text{for } t \geq 0, \tag{2.2}$$

which is the main object of our study. Additionally, let us consider that the insurance firm can pay out to its shareholders at the arrival times $\mathcal{T}^\gamma := \{T_i : i \geq 1\}$ of a Poisson process $N^\gamma := \{N_t^\gamma : t \geq 0\}$ with intensity $\gamma > 0$. Then, in order that (2.2) is well-defined, consider that W, N^γ are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{H}, \mathbb{P})$, where $\mathbb{H} = \{\mathcal{H}_t\}_{t\geq 0}$ is the right-continuous complete filtration generated by (W, N^γ), and the processes W and N^γ are independent. Then, a dividend strategy L^π has the following form

$$L_t^\pi = \int_{[0,t]} \nu_s^\pi \mathrm{d}N_s^\gamma, \quad \text{for } t \geq 0, \tag{2.3}$$

for some non-negative càglàd process $\nu^\pi := \{\nu_t^\pi : t \geq 0\}$ adapted to the filtration $\mathbb{H}$, where $\nu_{T_i}^\pi$ represents the payout made at time T_i, with $T_i \in \mathcal{T}^\gamma$.

Therefore, under the strategy $\pi = (u^\pi, L^\pi)$, the surplus process $X^\pi := \{X_t^\pi : t \geq 0\}$ evolves as in (2.2), with $X_0^\pi = x_0$. Let $\mathcal{A}$ be the collection of strategies $\pi = (u^\pi, L^\pi)$ such that u^π, ν^π are $\mathbb{H}$-adapted such that $u^\pi \in [0,1]$ and ν^π satisfies (2.3) and $0 \leq \nu_t^\pi \leq X_{t-}^\pi$. We say that a strategy π is admissible if $\pi \in \mathcal{A}$.

For each $\pi \in \mathcal{A}$, the expected NPV is given by

$$V^\pi(x) := \mathbb{E}_x \left[\int_0^{\tau^\pi} \mathrm{e}^{-\delta t} \, \mathrm{d}L_t^\pi \right], \quad x \geq 0.$$

Here $\mathbb{E}_x$ represents the conditional expectation given an initial surplus x, $\delta > 0$ denotes the discount rate, and $\tau^\pi := \inf\{t > 0 : X_t^\pi < 0\}$ is the first time of bankruptcy. Then, our goal is to find the value function V for the problem, which is

$$V(x) := \sup_{\pi \in \mathcal{A}} V^\pi(x), \quad x \geq 0. \tag{2.4}$$

Furthermore, we aim to find an optimal strategy $\pi^* \in \mathcal{A}$ for which the expected NPV V^{π^*} coincides with V, if such a strategy exists.

Using standard dynamic programming arguments, we can associate the value function V with the Hamilton Jacobi Bellman (HJB) equation given by

$$\begin{aligned} \max_{0\leq u\leq 1,\, 0\leq \xi\leq x} \left\{ \mathcal{L}^u v + \gamma[\xi + v(x-\xi) - v(x)] \right\} = 0 \text{ for } x > 0, \\ \text{s.t. } v(0) = 0. \end{aligned} \tag{2.5}$$

with

$$\mathcal{L}^u v(x) := \frac{1}{2}[\sigma u]^2 v''(x) + a\lambda[\eta - [1-u]\mu] v'(x) - \delta v(x). \tag{2.6}$$

Remark 2.1. *It is noteworthy that optimal risk policies and periodic dividend strategies with debt liabilities can be reformulated as a non-cheap reinsurance problem. This is achieved by a simply reparametrization in* (2.2) *and* (2.6)*. Essentially, this implies that the results obtained in this paper are applicable to this scenario as well, provided that we replace η with $\mu - \hat{\delta}$, where $\hat{\delta}$ represents the debt repayment.*

2.1 Choosing an optimal strategy and verification theorem

To find an optimal solution to the problem (2.4), we focus our study on the type of periodic-classical barrier strategies. We define a periodic-classical barrier strategy $\pi^b = (u^b, L^b) \in \mathcal{A}$ as having dividend payments given by

$$\nu_{T_i}^b = \max\{X_{T_i}^b - b, 0\}, \quad \text{for } T_i \in \mathcal{T}^\gamma,$$

where $L_t^b = \int_{[0,t]} \nu_s^b \mathrm{d}N_s^\gamma$ and $X^b := X^{\pi^b}$ is the solution to the SDE

$$\begin{aligned} \mathrm{d}X_t^b &= \left\{\eta - \left[1 - u_t^b\right]\mu\right\}\mathrm{d}t + \sigma u_t^b \mathrm{d}W_t - \mathrm{d}L_t^b, \quad \text{for } t > 0, \\ X_0^b &= x. \end{aligned}$$

To choose our candidate periodic-classical barrier strategy $\pi^{b_\gamma} = (u^{b_\gamma}, L^{b_\gamma})$ to be optimal, we need to first look at solving equation (2.5). This is the same as finding a solution to the non-linear partial differential system (NLPDS) (2.8)–(2.10), under certain assumptions on v.

Proposition 2.2. *Let v be a C^2-continuous, concave and increasing function on $(0,\infty)$ such that v'/v'' is decreasing on $(0,\infty)$ and*

$$\begin{aligned} x_\gamma &:= \inf\left\{x > 0 : -\frac{\mu v'(x)}{\sigma^2 v''(x)} \geq 1\right\} < \infty, \\ b_\gamma &:= \inf\{x > 0 : v'(x) < 1\} < \infty. \end{aligned} \tag{2.7}$$

Then, v is a solution to (2.5) if and only if v is a solution to

$$-\frac{\mu^2[v'(x)]^2}{2\sigma^2 v''(x)} + [\eta - \mu]v'(x) - \delta v(x) + \gamma\{x - b_\gamma + v(b_\gamma) - v(x)\}\mathbb{1}_{\{x - b_\gamma > 0\}} = 0 \quad \textit{for } x \in (0, x_\gamma), \tag{2.8}$$

$$\frac{1}{2}\sigma^2 v''(x) + \eta v'(x) - \delta v(x) + \gamma\{x - b_\gamma + v(b_\gamma) - v(x)\}\mathbb{1}_{\{x - b_\gamma > 0\}} = 0 \quad \textit{for } x \in (x_\gamma, \infty), \tag{2.9}$$

$$\textit{s.t. } v(0) = 0. \tag{2.10}$$

Proof. Since v is C^2-continuous, increasing and concave on $(0,\infty)$, which satisfies $b_\gamma \in [0,\infty)$, it is easy to check that

$$\max_{0 \leq \xi \leq x}\{\xi + v(x - \xi) - v(x)\} = \begin{cases} 0 & \text{if } x \leq b_\gamma, \\ x - b_\gamma + v(b_\gamma) - v(x) & \text{if } x > b_\gamma. \end{cases} \tag{2.11}$$

Substituting (2.11) into (2.5), it yields

$$\begin{aligned} \max_{0 \leq u \leq 1}\left\{\mathcal{L}^u v(x)\right\} + \gamma\{x - b_\gamma + v(b_\gamma) - v(x)\}\mathbb{1}_{\{x - b_\gamma > 0\}} &= 0 \text{ for } x > 0, \\ \text{s.t. } v(0) &= 0. \end{aligned} \tag{2.12}$$

Maximizing w.r.t. u in (2.12) we see that the maximizer operator u^* is given by

$$u^*(x) := \begin{cases} -\frac{\mu v'(x)}{\sigma^2 v''(x)}, & \text{if } x \in (0, x_\gamma), \\ 1 & \text{if } x \in (x_\gamma, \infty), \end{cases} \tag{2.13}$$

which is non-negative and increasing on $(0, x_\gamma)$ since v'/v'' is decreasing. Therefore, applying (2.13) in (2.12), we get that the double implication presented in the proposition above holds. ∎

2.1.1 Verification Theorem

Consider v as a solution to the HJB equation (2.5), satisfying the hypotheses in Proposition 2.2. Then, the SDE

$$\begin{aligned} \mathrm{d}X_t^{b_\gamma} &= \left\{\eta - \left[1 - u_t^{b_\gamma}\right]\mu\right\}\mathrm{d}t + \sigma u_t^{b_\gamma} \mathrm{d}W_t - \mathrm{d}L_t^{b_\gamma}, \quad \text{for } t > 0. \\ X_0^{b_\gamma} &= x, \end{aligned} \tag{2.14}$$

with

$$
\begin{aligned}
u_t^{b_\gamma} &:= \begin{cases} -\frac{\mu v'(X_t^{b_\gamma})}{\sigma^2 v''(X_t^{b_\gamma})} & \text{if } X_t^{b_\gamma} \in (0, x_\gamma), \\ 1 & \text{if } X_t^{b_\gamma} \in (x_\gamma, \infty), \end{cases} \\
L_t^{b_\gamma} &= \int_{[0,t]} \nu_s^{b_\gamma} \mathrm{d}N_s^\gamma \text{ where } \nu_t^{b_\gamma} = \max\{X_t^{b_\gamma} - b_\gamma, 0\},
\end{aligned}
\tag{2.15}
$$

admits a unique solution $X^{b_\gamma} = \{X_t^{b_\gamma} : t \geq 0\}$; see [8]. Thus, the candidate strategy to be optimal is given by the barrier strategy $\pi^{b_\gamma} = (u^{b_\gamma}, L^{b_\gamma})$ where u^{b_γ} and L^{b_γ} are as in (2.15). We will verify below that its expected NPV

$$
V^{b_\gamma}(x) := V^{\pi^{b_\gamma}}(x) = \mathbb{E}_x \left[\int_0^{\tau^{b_\gamma}} \mathrm{e}^{-\delta t} \, \mathrm{d}L_t^{b_\gamma} \right], \quad \text{for } t \geq 0,
\tag{2.16}
$$

with $\tau^{b_\gamma} := \inf\{t > 0 : X_t^{b_\gamma} < 0\}$, satisfies the HJB equation (2.5), proving that $V^{b_\gamma} = V$. From here, it follows that the strategy π^{b_γ} is indeed an optimal strategy.

Theorem 2.3 (Verification theorem)**.** *Let v be a solution to the HJB equation* (2.5)*, satisfying the hypotheses in Proposition 2.2. Then, v agrees with the value function V defined in* (2.4)*. Furthermore, the strategy $\pi^{b_\gamma} = (u^{b_\gamma}, L^{b_\gamma})$ given by* (2.15) *is an optimal strategy and $V^{b_\gamma} = v = V$.*

Proof. Let $\pi = (u^\pi, L^\pi)$ be in $\mathcal{A}$, where X^π evolves as in (2.2), with $X_0^\pi = x$. For each $n \in \mathbb{N}$ such that $v'(n) < 1$, take τ_n in the following manner $\tau_n := \inf\{t > 0 : X_t^\pi > n \text{ or } X_t^\pi < 1/n\}$. Notice that there is an $\bar{n} \in \mathbb{N}$ such that $v'(n) < 1$ for each $n > \bar{n}$, because of (2.7). Using integration by parts and Itô's formula in $\mathrm{e}^{-\delta[t\wedge\tau_n]} v(X_{t\wedge\tau_n}^\pi)$ (since X^π is a semi-martingale and v is a C^2-continuous function); see [11, Ch. II, Theorem 33], and taking into account (2.2), it gives

$$
\begin{aligned}
\mathrm{e}^{-\delta[t\wedge\tau_n]} v(X_{t\wedge\tau_n}^\pi) - v(x) &= \int_0^{t\wedge\tau_n} \mathrm{e}^{-\delta s} \mathcal{L}^{u_s^\pi} v(X_s^\pi) \mathrm{d}s + M_{t\wedge\tau_n} \\
&\quad + \int_{[0,t\wedge\tau_n]} \mathrm{e}^{-\delta s} [v(X_{s-}^\pi - \nu_s^\pi) - v(X_{s-}^\pi)] \mathrm{d}N_s^\gamma,
\end{aligned}
\tag{2.17}
$$

where $M_{t\wedge\tau_n} := \sigma \int_0^{t\wedge\tau_n} \mathrm{e}^{-\delta s} u_s^\pi v'(X_s^\pi) \mathrm{d}W_s$. Notice that

$$
\begin{aligned}
&\int_{[0,t\wedge\tau_n]} \mathrm{e}^{-\delta s} [v(X_{s-}^\pi - \nu_s^\pi) - v(X_{s-}^\pi)] \mathrm{d}N_s^\gamma \\
&\quad = - \int_{[0,t\wedge\tau_n]} \mathrm{e}^{-\delta s} \nu_s^\pi \mathrm{d}N_s^\gamma + \int_0^{t\wedge\tau_n} \mathrm{e}^{-\delta s} \gamma [\nu_s^\pi + v(X_{s-}^\pi - \nu_s^\pi) - v(X_{s-}^\pi)] \mathrm{d}s + H_{t\wedge\tau_n},
\end{aligned}
\tag{2.18}
$$

with $H_{t\wedge\tau_n} := \int_{[0,t\wedge\tau_n]} \mathrm{e}^{-\delta s} [\nu_s^\pi + v(X_{s-}^\pi - \nu_s^\pi) - v(X_{s-}^\pi)] \mathrm{d}\widetilde{N}_s^\gamma$ and $\widetilde{N} := N_s^\gamma - \gamma s$ is the compensated Poisson process. Applying (2.18) in (2.17), it gives

$$
\begin{aligned}
\mathrm{e}^{-\delta[t\wedge\tau_n]} v(X_{t\wedge\tau_n}^\pi) - v(x) &= \int_0^{t\wedge\tau_n} \mathrm{e}^{-\delta s} \Big\{ \mathcal{L}^{u_s^\pi} v(X_s^\pi) + \gamma [\nu_s^\pi + v(X_{s-}^\pi - \nu_s^\pi) - v(X_{s-}^\pi)] \Big\} \mathrm{d}s \\
&\quad - \int_{[0,t\wedge\tau_n]} \mathrm{e}^{-\delta s} \nu_s^\pi \mathrm{d}N_s^\gamma + M_{t\wedge\tau_n} + H_{t\wedge\tau_n}.
\end{aligned}
\tag{2.19}
$$

Considering that for each $x > 0$, $\tilde{u} \in [0,1]$ and $\tilde{\xi} \in [0,x]$,

$$\mathcal{L}^{\tilde{u}} v(x) + \gamma[\tilde{\xi} + v(x-\tilde{\xi}) - v(x)] \leq \max_{0 \leq u \leq 1,\, 0 \leq \xi \leq x} \left\{ \mathcal{L}^{u} v(x) + \gamma[\xi + v(x-\xi) - v(x)] \right\} = 0,$$

it implies that

$$\mathrm{e}^{-\delta[t\wedge\tau_n]}\, v(X^{\pi}_{t\wedge\tau_n}) - v(x) \leq -\int_{[0,t\wedge\tau_n]} \mathrm{e}^{-\delta s}\, \nu^{\pi}_{s} \mathrm{d}N^{\gamma}_{s} + M_{t\wedge\tau_n} + H_{t\wedge\tau_n}. \tag{2.20}$$

Since $v' \in (0, v'(0+))$ is decreasing on $(0,\infty)$, observe that $\mathrm{e}^{-\delta s}\, u^{\pi}_{s} v'(X^{\pi}_{s}) \leq v'(1/n) < \infty$ for $s \in (0, t\wedge\tau_n)$. Meanwhile, since v is concave, it follows that $\mathrm{e}^{-\delta s}[\nu^{\pi}_{s} + v(X^{\pi}_{s-} - \nu^{\pi}_{s}) - v(X^{\pi}_{s-})] \leq X^{\pi}_{s-}[1-v'(n)] < n[1-v'(n)] < \infty$ for $s \in [0, t\wedge\tau_n]$. Thus, it follows that the processes $\{M_{t\wedge\tau_n} : t \geq 0\}$ and $\{H_{t\wedge\tau_n} : t \geq 0\}$ are zero-mean $\mathbb{P}_x$-martingales. Taking expectation in (2.20), it gives

$$v(x) \geq \mathbb{E}_x\Big[\mathrm{e}^{-\delta[t\wedge\tau_n]}\, v(X^{\pi}_{t\wedge\tau_n})\Big] + \mathbb{E}_x\Big[\int_{[0,t\wedge\tau_n]} \mathrm{e}^{-\delta s}\, \nu^{\pi}_{s} \mathrm{d}N^{\gamma}_{s}\Big]. \tag{2.21}$$

Taking into account again the concavity property of v, it is known that $v(X^{\pi}_{t\wedge\tau_n}) \leq K[1 + X^{\pi}_{t\wedge\tau_n}]$, for some positive constant K independent of n. Moreover, notice that

$$\begin{aligned}
\mathbb{E}_x[X^{\pi}_{t\wedge\tau_n}] &= \mathbb{E}_x\left[x + \int_0^{t\wedge\tau_n} \{\eta - [1-u^{\pi}_{s}]\mu\}\mathrm{d}s + \sigma\int_0^{t\wedge\tau_n} u^{\pi}_{t}\mathrm{d}W_t - \int_{[0,t]} \nu^{\pi}_{s}\mathrm{d}N^{\gamma}_{s}\right] \\
&= \mathbb{E}_x\left[x + \int_0^{t\wedge\tau_n} \{\eta - [1-u^{\pi}_{s}]\mu\}\mathrm{d}s - \int_{[0,t]} \nu^{\pi}_{s}\mathrm{d}N^{\gamma}_{s}\right] \leq x + \eta\mathbb{E}_x[t\wedge\tau_n].
\end{aligned}$$

Then, it follows that $\lim_{t,n\to\infty} \mathbb{E}_x\Big[\mathrm{e}^{-\delta[t\wedge\tau_n]}\, v(X^{\pi}_{t\wedge\tau_n})\Big] \leq \lim_{t,n\to\infty} K\mathbb{E}_x[\mathrm{e}^{-\delta[t\wedge\tau_n]}\{1+x+\eta[t\wedge\tau_n]\}] = 0$. Letting $t,n\to\infty$ in (2.21), considering the previous identity and using dominated convergence theorem, we get that

$$v(x) \geq \mathbb{E}_x\left[\int_0^{\tau^{\pi}} \mathrm{e}^{-\delta s}\, \nu^{\pi}_{s}\mathrm{d}N^{\gamma}_{s}\right] = V^{\pi}(x), \tag{2.22}$$

due to $\tau_n \xrightarrow[n\to\infty]{} \tau^{\pi}$ $\mathbb{P}_x$-a.s.. Therefore, $v(x) \geq V(x)$ for $x \geq 0$. Now, consider the strategy $\pi^{b_\gamma} = (u^{b_\gamma}, L^{b_\gamma})$ given by (2.15) where the controlled process X^{b_γ} associated to this strategy evolves as (2.14). Defining $\bar{\tau}_n = \inf\{t > 0 : X^{b_\gamma}_t > n \text{ or } X^{b_\gamma}_t < 1/n\}$, it can be observed that (2.19) holds, but with the substitution of X^{π}, u^{π} and τ^{π}_n with X^{b_γ}, u^{b_γ} and $\tau^{b_\gamma}_n$, because of $\pi^{b_\gamma} \in \mathcal{A}$. Considering that v satisfies (2.5) and (2.11)–(2.13), we have that

$$\mathrm{e}^{-\delta[t\wedge\bar{\tau}_n]}\, v(X^{b_\gamma}_{t\wedge\tau_n}) - v(x) = -\int_{[0,t\wedge\bar{\tau}_n]} \mathrm{e}^{-\delta s}\, \nu^{\pi}_{s}\mathrm{d}N^{\gamma}_{s} + M_{t\wedge\bar{\tau}_n} + H_{t\wedge\bar{\tau}_n},$$

with $M_{t\wedge\bar{\tau}_n} = \sigma\int_0^{t\wedge\bar{\tau}_n} \mathrm{e}^{-\delta s}\, u^{b_\gamma}_{s} v'(X^{b_\gamma}_{s})\mathrm{d}W_s$ and $H_{t\wedge\bar{\tau}_n} = \int_{[0,t\wedge\bar{\tau}_n]} \mathrm{e}^{-\delta s}[\nu^{b_\gamma}_{s} + v(X^{b_\gamma}_{s-} - \nu^{b_\gamma}_{s}) - v(X^{b_\gamma}_{s-})]\mathrm{d}\widetilde{N}^{\gamma}_{s}$. From here and by similar arguments seen in (2.21)–(2.22), it follows that

$$v(x) = \mathbb{E}_x\left[\int_0^{\tau^{b_\gamma}} \mathrm{e}^{-\delta s}\, \nu^{b_\gamma}_{s}\mathrm{d}N^{\gamma}_{s}\right] = V^{b_\gamma}(x) \leq V(x) \qquad \text{for } t \geq 0.$$

Therefore, we conclude that $V^{b_\gamma} = v = V$, and that π^{b_γ} is an optimal strategy. ■

3 Constructing a solution to the HJB equation and describing the optimal barrier strategy

In this section, we will solve the NLPDS (2.8)–(2.10), providing an explicit solution to the HJB equation (2.5). We will consider the cases of non-cheap ($\mu > \eta$) and cheap ($\mu = \eta$) reinsurance separately. The proofs of the results presented in this section can be found in the Appendix.

3.1 Non-cheap reinsurance

3.1.1 The case $\mu \geq 2\eta$

Let us consider the scenario where the insurance company takes the maximum risk at any time first, which implies that $u^* \equiv 1$ on the interval $(0, \infty)$, and

$$\max_{0\leq u\leq 1} \left\{\mathcal{L}^u v(x)\right\} = \frac{1}{2}\sigma^2 v''(x) + \eta v'(x) - \delta v(x), \quad \text{for } x > 0.$$

To ensure that this scenario holds, we must have $\mu \geq 2\eta$ as we will show in Proposition 3.4. Recall that $\mu > \eta$.

Now, let us proceed by finding a solution $v_{\gamma,b}$ to (3.1), which depends on the parameter $b > 0$.

Proposition 3.1. *Let $b > 0$ be fixed. Then, a solution of the NLPDS*

$$\begin{aligned} \frac{1}{2}\sigma^2 v''(x) + \eta v'(x) - \delta v(x) + \gamma\{(x-b) + v(b) - v(x)\}\mathbb{1}_{\{x-b\}>0\}} = 0, \quad \textit{for } x > 0,\\ \textit{s.t. } v(0) = 0, \end{aligned} \tag{3.1}$$

is a C^2*-continuous function, whose form is given by*

$$v_{\gamma,b}(x) = \begin{cases} c_{1,1}(b)h_1(x) & \textit{for } x \in (0,b),\\ c_{1,2}(b)\,\mathrm{e}^{\lambda_\gamma[x-b]} + \frac{\gamma}{\gamma+\delta}\left[x - b + v_{\gamma,b}(b) + \frac{\eta}{\gamma+\delta}\right] & \textit{for } x \in [b,\infty), \end{cases} \tag{3.2}$$

where λ_γ is the negative root of

$$\frac{\sigma^2}{2}r^2 + \eta r - (\delta+\gamma) = 0, \tag{3.3}$$

h_1 is defined as

$$h_1(x) = \mathrm{e}^{\theta_+ x} - \mathrm{e}^{\theta_- x} \quad \textit{for } x \in (0,\infty), \tag{3.4}$$

with θ_+, θ_- as the positive and negative root, respectively, of

$$\frac{\sigma^2}{2}r^2 + \eta r - \delta = 0, \tag{3.5}$$

and

$$c_{1,1}(b) = \frac{\frac{\gamma}{\gamma+\delta}\left[1 - \frac{\eta\lambda_\gamma}{\gamma+\delta}\right]}{h_1'(b) - \frac{\delta\lambda_\gamma}{\gamma+\delta}h_1(b)} \quad \textit{and} \quad c_{1,2}(b) = \frac{1}{\gamma+\delta}\left[\delta c_{1,1}(b)h_1(b) - \frac{\gamma\eta}{\gamma+\delta}\right]. \tag{3.6}$$

Supposing that b_γ as in (2.7) belongs in $(0,\infty)$, and v_{γ,b_γ} given by (3.2) (when $b = b_\gamma$) is both concave and increasing on $(0,\infty)$, it can be immediately observed that v'_{γ,b_γ} is decreasing on $(0,\infty)$. Then, in order to determine when

$$v'_{\gamma,b_\gamma}(b_\gamma-) = 1 \tag{3.7}$$

holds, calculating the first derivative of (3.2) on $(0, b_\gamma)$, we obtain that (3.7) is equivalent to verifying whether

$$g_1(b_\gamma) = \frac{1}{\delta\lambda_\gamma}\left[\delta + \frac{\gamma\eta\lambda_\gamma}{\gamma+\delta}\right] \tag{3.8}$$

is true, where

$$g_1(b) := \frac{h_1(b)}{h_1'(b)} \quad \text{for } b > 0. \tag{3.9}$$

To further analyse (3.8), let us first establish some properties of $g_1(b)$, with $b > 0$.

Lemma 3.2. *Let g_1 be as in (3.9). Then, g_1 is strictly increasing on $(0, \infty)$ satisfying*

$$\lim_{b\downarrow 0} g_1(b) = 0 \quad \textit{and} \quad \lim_{b\uparrow\infty} g_1(b) = \frac{1}{\theta_+}.$$

So, by (3.8) and Lemma 3.2, we deduce that there exists a $b_\gamma > 0$ such that (3.7) holds if and only if

$$0 < \frac{1}{\delta\lambda_\gamma}\left[\delta + \frac{\gamma\eta\lambda_\gamma}{\gamma+\delta}\right] < \frac{1}{\theta_+} \tag{3.10}$$

is true. Taking f_1 as follows

$$f_1(\gamma) := \frac{1}{\lambda_\gamma\delta}\left[\delta + \frac{\lambda_\gamma\eta\gamma}{\delta+\gamma}\right] = \frac{\eta\gamma}{\delta[\delta+\gamma]} - \frac{\sigma^2}{\eta + \sqrt{\eta^2 + 2\sigma^2(\delta+\gamma)}} \quad \text{for } \gamma > 0, \tag{3.11}$$

the next lemma, provides the necessary conditions on γ so that (3.10) is satisfied.

Lemma 3.3. *Let f_1 be as in (3.11). Then f_1 is strictly increasing on $(0, \infty)$ satisfying*

$$\lim_{\gamma\downarrow 0} f_1(\gamma) = \frac{1}{\theta_-} \quad \textit{and} \quad \lim_{\gamma\uparrow\infty} f_1(\gamma) = \frac{\eta}{\delta}. \tag{3.12}$$

Since $\frac{1}{\theta_-} < \frac{\eta}{\delta} < \frac{1}{\theta_+}$ and by Lemma 3.3, we conclude that there exists a unique $\gamma_0 > 0$ such that $f_1(\gamma_0) = 0$. Thus, for any $\gamma \in (\gamma_0, \infty)$ fixed, (3.10) is satisfied, and there exists a unique $b_\gamma \in (0, \infty)$ where (3.7) holds. On the other hand, calculating the first and second derivatives of v_{γ,b_γ} on the interval $(0, b_\gamma)$, notice that

$$\lim_{x\downarrow 0} \frac{\mu v'_{\gamma,b_\gamma}(x)}{\sigma^2 v''_{\gamma,b_\gamma}(x)} = \lim_{x\downarrow 0} \frac{\mu h_1'(x)}{\sigma^2 h_1''(x)} = -\frac{\mu}{2\eta}. \tag{3.13}$$

Then, assuming $v'_{\gamma,b_\gamma}/v''_{\gamma,b_\gamma}$ is decreasing on $(0, \infty)$, by (3.13), it follows immediately that if $\mu \geq 2\eta$, $u^* \equiv 1$, with u^* as in (2.13).

In the case that $\gamma \in (0, \gamma_0]$, we take $b_\gamma = 0$, and the solution to (2.5), when $b = 0$, is given by

$$v_{\gamma,0}(x) = -\frac{\gamma\eta}{[\gamma+\delta]^2} \mathrm{e}^{\lambda_\gamma x} + \frac{\gamma}{\gamma+\delta}\left\{x + \frac{\eta}{\gamma+\delta}\right\} \quad \text{for } x > 0. \tag{3.14}$$

Then, calculating the first and second derivatives of (3.14), it gives that $\lim_{x\downarrow 0} \frac{\mu v'_{\gamma,0}(x)}{\sigma^2 v''_{\gamma,0}(x)} = \frac{\mu}{\sigma^2}\Big\{\frac{1}{\lambda_{\gamma_0}} - \frac{[\gamma_0+\delta]}{\eta[\lambda_{\gamma_0}]^2}\Big\} = -\frac{\mu}{2\eta}$. We have again that $u^* \equiv 1$ if $\mu \geq 2\eta$ and $v'_{\gamma,0}/v''_{\gamma,0}$ is decreasing on $(0, \infty)$.

Proposition 3.4. *Let $\mu \geq 2\eta$ and $\gamma_0 = f_1^{-1}(0)$.*

(i) If $\gamma > \gamma_0$, then (3.2) *is a* C^2*-continuous, increasing and concave solution to* (2.5)*, when*

$$b = b_\gamma = g_1^{-1}(f_1(\gamma)). \tag{3.15}$$

Furthermore, $v'_{\gamma,b_\gamma}/v''_{\gamma,b_\gamma}$ is decreasing on $(0,\infty)$ and $x_\gamma = 0$.

(ii) If $\gamma < \gamma_0$, then (3.14) *is a* C^2*-continuous, increasing and concave solution to* (2.5)*, with linear growth, when $b = b_\gamma = 0$. Furthermore, $v'_{\gamma,0}/v''_{\gamma,0}$ is decreasing on $(0,\infty)$ and $x_\gamma = 0$.*

Optimal strategy. By Theorem 2.3 and Proposition 3.4, we conclude that if $\mu \geq 2\eta$ and $\gamma > \gamma_0$, the optimal strategy is given by (2.15), where $x_\gamma = 0$ and b_γ is determined by (3.15). The controlled process X^{b_γ} is given as the solution to the SDE

$$\begin{aligned} \mathrm{d}X_t^{b_\gamma} &= \eta \mathrm{d}t + \sigma \mathrm{d}W_t - \mathrm{d}L_t^{b_\gamma}, \quad \text{for } t > 0. \\ X_0^{b_\gamma} &= x. \end{aligned}$$

Moreover, the value function V and the expected NPV V^{b_γ} (defined in (2.4) and (2.16) respectively) are given by (3.2). In this context, the optimal strategy for the insurance company is to take the maximum risk while the payout $\max\{X_{T_i}^{b_\gamma} - b_\gamma, 0\}$ is made at time $T_i < \tau^{b_\gamma}$.

In the case that $\mu \geq 2\eta$ and $\gamma \in (0, \gamma_0]$, by Theorem 2.3 and Proposition 3.4, we get that $x_\gamma = b_\gamma = 0$. Furthermore, the optimal strategy for the company in this scenario is to take the maximum risk while making the payout equal to the entire amount of the company's assets at the first arrival of the Poisson process N^γ. Additionally, V^{b_γ} and V are given by (3.14).

3.1.2 The case $\mu < 2\eta$

Now, assuming $\mu < 2\eta$, we will examine the scenarios where $x_\gamma \leq b_\gamma$ and $x_\gamma > b_\gamma$. Recall that x_γ indicates the level at which maximum retention is reached, whereas b_γ represents the comparison level at which the insurer determines if payouts to shareholders are appropriate at the arrival times $\mathcal{T}^\gamma$.

Given a $b > 0$ fixed, we will determine an explicit solution $v_{\gamma,b}$ to the NLPDS (3.20)–(3.22) and (3.39)–(3.41) first. To derive this solution, we will consider that $v_{\gamma,b}$ must have linear growth, and there exists an $x_b \in (0,\infty)$ such that

$$u_b^*(x) = \begin{cases} -\frac{\mu v'_{\gamma,b}(x)}{\sigma^2 v''_{\gamma,b}(x)}, & \text{if } x \in (0, x_b), \\ 1 & \text{if } x \in [x_b, \infty). \end{cases} \tag{3.16}$$

Subsequently, we will derive sufficient conditions regarding the parameter γ to ensure the existence of b_γ, which is defined in (2.7). After, it will be shown that $v'_{\gamma,b_\gamma}/v''_{\gamma,b_\gamma}$ is decreasing on $(0,\infty)$ satisfying

$$\lim_{x\downarrow 0} \frac{\mu v'_{\gamma,b_\gamma}(x)}{\sigma^2 v''_{\gamma,b_\gamma}(x)} > -1, \tag{3.17}$$

which is enough to obtain the existence of x_γ given in (2.7). The conditions will be presented in Propositions 3.8 and 3.13.

For the case where $x_\gamma \leq b_\gamma$, we take

$$\bar{x} := \frac{c_{2,1}}{1+\delta\bar{\eta}} \ln[c_{2,2}] + \frac{\bar{\eta}[2\eta - \mu]}{2[1+\delta\bar{\eta}]}, \tag{3.18}$$

where

$$\bar{\eta} := \frac{2\sigma^2}{\mu^2}, \ c_{2,1} := \frac{\bar{\eta}[\mu - \eta]}{1+\delta\bar{\eta}}, \text{ and } c_{2,2} := \frac{\delta\bar{\eta}\mu + 2\eta - \mu}{2\delta\bar{\eta}[\mu - \eta]}. \tag{3.19}$$

Proposition 3.5. *Let $\mu < 2\eta$. If $b > \bar{x}$ is fixed, a solution to the NLPDS*

$$-\frac{\mu^2[v'(x)]^2}{2\sigma^2 v''(x)} + [\eta-\mu]v'(x) - \delta v(x) = 0 \quad \text{for } x \in (0, x_b), \tag{3.20}$$

$$\frac{1}{2}\sigma^2 v''(x) + \eta v'(x) - \delta v(x) + \gamma\{x - b + v(b) - v(x)\}\mathbb{1}_{\{x-b>0\}} = 0 \quad \text{for } x \in (x_b, \infty), \tag{3.21}$$

$$\text{s.t. } v(0) = 0, \tag{3.22}$$

is a C^2*-continuous function on* $(0,\infty)$*, whose form is given by*

$$v_{\gamma,b}(x) = \begin{cases} c_{2,1}\,\mathrm{e}^{-\chi_1^{-1}(x)}\left[\mathrm{e}^{[1+\delta\bar{\eta}][M(b)+\chi_1^{-1}(x)]} - 1\right] & \text{if } x \in (0, x_b), \\ \mathrm{e}^{M(b)}\, c_{2,2}^{-1/[1+\delta\bar{\eta}]} h_2(x - x_b) & \text{if } x \in (x_b, b), \\ c_{2,3}(b)\,\mathrm{e}^{\lambda_\gamma[x-b]} + \frac{\gamma}{\gamma+\delta}\left[x - b + v_{\gamma,b}(b) + \frac{\eta}{\delta+\gamma}\right] & \text{if } x \in (b, \infty), \end{cases} \tag{3.23}$$

where χ_1^{-1} is the inverse function of

$$\chi_1(z) = k_{2,1}\,\mathrm{e}^{[1+\delta\bar{\eta}][z+M(b)]} + c_{2,1}z + k_{2,2}(b), \text{ for } z \in [-M(b), \bar{z}(b)], \tag{3.24}$$

with

$$\begin{aligned} k_{2,1} &:= \frac{\delta\bar{\eta}c_{2,1}}{1+\delta\bar{\eta}}, \quad k_{2,2}(b) := c_{2,1}\left[M(b) - \frac{\delta\bar{\eta}}{1+\delta\bar{\eta}}\right], \\ \bar{z}(b) &:= \frac{1}{1+\delta\bar{\eta}}\ln[c_{2,2}] - M(b), \end{aligned} \tag{3.25}$$

the function h_2 is taken as

$$h_2(x) := \frac{1}{\theta_+ - \theta_-}\left[a_1\,\mathrm{e}^{\theta_+ x} - a_2\,\mathrm{e}^{\theta_- x}\right], \tag{3.26}$$

with θ_+, θ_-, λ_γ as in Proposition 3.1, and

$$\begin{aligned} x_b &= \bar{x}, \quad a_1 := 1 - \theta_-\frac{[2\eta-\mu]}{2\delta}, \quad a_2 := 1 - \theta_+\frac{[2\eta-\mu]}{2\delta}, \\ M(b) &= \ln\left[\frac{\frac{\gamma}{\gamma+\delta} - \frac{\lambda_\gamma\eta\gamma}{[\delta+\gamma]^2}}{c_{2,2}^{-1/[1+\delta\bar{\eta}]}\left[h_2'(b-\bar{x}) - \frac{\lambda_\gamma\delta}{\delta+\gamma}h_2(b-\bar{x})\right]}\right], \\ c_{2,3}(b) &:= \frac{\mathrm{e}^{M(b)}\,\delta c_{2,2}^{-1/[1+\delta\bar{\eta}]}}{\delta+\gamma}h_2(b-\bar{x}) - \frac{\eta\gamma}{[\delta+\gamma]^2}. \end{aligned} \tag{3.27}$$

Remark 3.6. *Observe that χ_1, given in* (3.24)*, is a positive and increasing function on* $(0,\bar{x})$*. Then, by the construction of $v_{\gamma,b}$, which is given by* (3.23)*, it obtains immediately that $v_{\gamma,b}$ is concave and increasing on* $(0,\bar{x})$*, since $v_{\gamma,b}$ satisfies*

$$v_{\gamma,b}'(x) = \mathrm{e}^{-\chi_1^{-1}(x)} \quad \text{and} \quad v_{\gamma,b}''(x) = -\frac{\mathrm{e}^{-\chi_1^{-1}(x)}}{\chi_1'(\chi_1^{-1}(x))}, \quad \text{for } x \in (0,\bar{x}). \tag{3.28}$$

It implies that $u_b^*(x) < 1$ for $x \in (0,\bar{x})$, and $u_b^*(x) = 1$ at $x = \bar{x}$. For a more detail exposition, see Subsection A.2.

Taking the first derivative in (3.23) on $(\bar{x}, b)$, we see that (3.7) is achieved if and only if

$$g_2(b_\gamma - \bar{x}) = \frac{1}{\lambda_\gamma \delta}\left[\delta + \frac{\lambda_\gamma \eta \gamma}{\delta + \gamma}\right] \tag{3.29}$$

is true, where

$$g_2(b) = \frac{h_2(b)}{h_2'(b)} \quad \text{for } b \in (0, \infty). \tag{3.30}$$

In order to guarantee the veracity of (3.29), let us see some properties of the function g_2 first.

Lemma 3.7. *Let g_2 be as in* (3.30)*. Then, g_2 is increasing on $(0, \infty)$, satisfying*

$$\lim_{b \downarrow 0} g_2(b) = \frac{2\eta - \mu}{2\delta} \quad \textit{and} \quad \lim_{b \uparrow \infty} g_2(b) = \frac{1}{\theta_+}.$$

By Lemma 3.7, we get that $b \mapsto g_2(b - \bar{x})$ is increasing on $(\bar{x}, \infty)$ satisfying $\lim_{b \downarrow \bar{x}} g_2(b - \bar{x}) = \frac{2\eta - \mu}{2\delta}$ and $\lim_{b \uparrow \infty} g_2(b - \bar{x}) = \frac{1}{\theta_+}$. Then, we have that (3.29) is equivalent to verify that $\frac{2\eta-\mu}{2\delta} \leq \frac{1}{\lambda_\gamma \delta}\big[\delta + \frac{\lambda_\gamma \eta \gamma}{\delta + \gamma}\big] < \frac{1}{\theta_+}$, which is true only for $\gamma > \gamma_1 := f_1^{-1}\big(\frac{2\eta - \mu}{2\delta}\big)$, where f_1^{-1} is the inverse of f_1 given in (3.11), because of $\frac{1}{\theta_-} < \frac{2\eta - \mu}{2\delta} < \frac{\eta}{\delta} < \frac{1}{\theta_+}$ and Lemma 3.3. Thus, for $\gamma > \gamma_1$ fixed, there exists a unique $b_\gamma \in (0, \infty)$ that satisfies (3.29). By the seen above, we get the following proposition.

Proposition 3.8. *If $\mu < 2\eta$ and $\gamma > \gamma_1$, then* (3.23) *is a C^2-continuous, increasing and concave solution to* (2.5)*, with*

$$b = b_\gamma = \bar{x} + g_2^{-1}(f_1(\gamma)). \tag{3.31}$$

Furthermore, $v'_{\gamma, b_\gamma}/v''_{\gamma, b_\gamma}$ is decreasing on $(0, \infty)$ satisfying (3.17)*. Then, $x_\gamma = \bar{x}$.*

For the case that $b_\gamma < x_\gamma$, we will analyse the behaviour of the solution v_{γ, b_γ} to the NLPDS (3.39)–(3.41) for $\gamma \in (0, \gamma_1)$, as shown in Proposition 3.13. For $\beta \in \mathbb{R}$ fixed, take H_β and $\bar{f}_\beta$ as

$$H_\beta(z) := \int_{\mathrm{e}^{-\beta}}^{z} \frac{1}{y^2 g(y)} \mathrm{d}y, \quad \text{for } z > \mathrm{e}^{-\beta}, \tag{3.32}$$

$$\bar{f}_\beta(z) := c_{3,1}(\beta)[G(z) - G(\mathrm{e}^{-\beta})] - \bar{\eta}[\mu - \eta]\left[G(z) H_\beta(z) - \int_{\mathrm{e}^{-\beta}}^{z} \frac{G(y)}{y^2 g(y)} \mathrm{d}y\right], \quad \text{for } z > \mathrm{e}^{-\beta}, \tag{3.33}$$

with

$$c_{3,1}(\beta) := \frac{\sigma^2}{\mu \alpha_\gamma g(\alpha_\gamma)} + \bar{\eta}[\mu - \eta] H_\beta(\alpha_\gamma) \quad \text{and} \quad \alpha_\gamma := \frac{\gamma + \delta}{\gamma}\left[1 + \frac{\mu}{\sigma^2 \lambda_\gamma}\right]. \tag{3.34}$$

Here, G is a gamma accumulative distribution function with parameters $(\bar{\eta}[\delta + \gamma] + 1, [\gamma \bar{\eta}]^{-1}])$, and its density function is represented by g which has the following form

$$g(x) = \frac{[\bar{\eta}\gamma]^{\bar{\eta}[\delta + \gamma] + 1}}{\Gamma(\bar{\eta}[\delta + \gamma] + 1)} x^{\bar{\eta}[\delta + \gamma]} \mathrm{e}^{-\gamma \bar{\eta} x} \quad \text{for } x > 0, \tag{3.35}$$

where $\Gamma(\cdot)$ is the gamma function. Observe that α_γ is a positive constant due to $2\eta - \mu > 0$, and $\bar{f}_\beta$ is an invertible function for $\beta \in (0, \infty)$. For more details, refer to Subsection A.3. Additionally, for each $\gamma \in (0, \gamma_1)$ fixed, it is worth noticing that $\beta \mapsto H_\beta(\alpha_\gamma)$ is well-defined on $(0, \infty)$, because of $\alpha_\gamma > 1$; as stated in Lemma 3.11.(i).

Proposition 3.13 presents a solution $v_{\gamma, b}$ to the NPDS (3.39)–(3.41), which depends on $b > 0$ satisfying

$$[1 + \delta \bar{\eta}]\left[1 + \frac{b}{c_{2,1}}\right] \in ([1 + \delta \bar{\eta}], \bar{g}(0+)], \tag{3.36}$$

with

$$\bar{g}(\beta) := \frac{\mathrm{e}^{-\beta}\, g(\mathrm{e}^{-\beta})}{c_{2,1}} c_{3,1}(\beta) + \ln\left[\frac{1}{\delta\bar{\eta}}\left\{\frac{\mathrm{e}^{-\beta}\, g(\mathrm{e}^{-\beta})}{c_{2,1}} c_{3,1}(\beta) - 1\right\}\right], \quad \text{for } \beta \in (0, \underline{b}), \tag{3.37}$$

and $\underline{b} := \inf\{\beta > 0 : \bar{g}(\beta) < 0\} < \infty$. The function $\bar{g}$ is a decreasing function on $(0, \underline{b})$, as is verified in Subsection A.3. Also $\bar{g}(0+) > 1 + \delta\bar{\eta}$ holds if and only if $\frac{1}{\delta\bar{\eta}}\left\{\frac{g(1)}{c_{2,1}} c_{3,1}(0) - 1\right\} > 1$, which, using (3.19) and (3.32)–(3.34), is rewritten as follows

$$\frac{\sigma^2\, \mathrm{e}^{\bar{\eta}\gamma\alpha_\gamma}}{\mu[\alpha_\gamma]^{\bar{\eta}[\delta+\gamma]+1}} + \bar{\eta}[\mu - \eta] \int_1^{\alpha_\gamma} \frac{\mathrm{e}^{\bar{\eta}\gamma y}}{y^{\bar{\eta}[\delta+\gamma]+2}} \mathrm{d}y > \bar{\eta}[\mu - \eta]\, \mathrm{e}^{\bar{\eta}\gamma}. \tag{3.38}$$

Proposition 3.9. *Let $\mu < 2\eta$ and $\gamma \in (0, \gamma_1)$ satisfying* (3.38)*. If $b > 0$ is such that* (3.36) *holds, a solution to the NLPDS*

$$-\frac{\mu^2[v'(x)]^2}{2\sigma^2 v''(x)} + [\eta - \mu]v'(x) - \delta v(x) + \gamma\{x - b + v(b) - v(x)\}\mathbb{1}_{\{x-b>0\}} = 0 \quad \text{for } x \in (0, x_b), \tag{3.39}$$

$$\frac{1}{2}\sigma^2 v''(x) + \eta v'(x) - [\delta + \gamma]v(x) + \gamma\{x - b + v(b)\} = 0 \quad \text{for } x \in (x_b, \infty), \tag{3.40}$$

$$\text{s.t. } v(0) = 0, \tag{3.41}$$

is a C^2-continuous, whose form is given by

$$v_{\gamma,b}(x) = \begin{cases} c_{2,1}\, \mathrm{e}^{-\chi_1^{-1}(x)} \left[\mathrm{e}^{[1+\delta\bar{\eta}][M_\gamma + \chi_1^{-1}(x)]} - 1\right] & \text{if } x \in (0, b), \\ \frac{\mathrm{e}^{-\chi_2^{-1}(x)}}{\delta+\gamma}\Big\{\frac{g(\mathrm{e}^{\chi_2^{-1}(x)})\, \mathrm{e}^{\chi_2^{-1}(x)}}{\bar{\eta}} & \\ \times[c_{3,1}(M_2) - \bar{\eta}[\mu-\eta] H_{M_2}(\mathrm{e}^{\chi_2^{-1}(x)})] - [\mu - \eta]\Big\} + \frac{\gamma[v_{\gamma,b}(b)+x-b]}{\delta+\gamma} & \text{if } x \in (b, x_b), \\ c_{3,2}(\alpha_\gamma)\, \mathrm{e}^{\lambda_\gamma[x-x_b]} + \frac{\gamma}{\gamma+\delta}\left[x - b + v_{\gamma,b}(b) + \frac{\eta}{\delta+\gamma}\right] & \text{if } x \in (x_b, \infty), \end{cases} \tag{3.42}$$

where $\bar{\eta}$, $c_{2,1}$, $c_{2,2}$ are as in (3.19)*, χ_1^{-1} is the inverse function of $\chi_1(z) = k_{2,1}\, \mathrm{e}^{[1+\delta\bar{\eta}][z+M_\gamma]} + c_{2,1}z + k_{2,2}^{(\gamma)}$, for $z \in [-M_\gamma, -M_2]$, with $k_{2,1}$ is as in* (3.25)*, χ_2^{-1} is the inverse function of $\chi_2(z) = \bar{f}_{M_2}(\mathrm{e}^z) + b$ for $z \in [-M_2, \ln[\alpha_\gamma]]$, with $\bar{f}_{M_2}$ defined as in* (3.33) *when $\beta = M_2$, and*

$$\begin{aligned} x_b &= \bar{f}_{M_2}(\alpha_\gamma) + b, \quad c_{3,2}(\alpha_\gamma) := -\frac{\mu}{\sigma^2\alpha_\gamma\lambda_\gamma^2}, \quad k_{2,2}^{(\gamma)} := c_{2,1}\left[M_\gamma - \frac{\delta\bar{\eta}}{1+\delta\bar{\eta}}\right], \\ M_\gamma &= \frac{1}{1+\delta\bar{\eta}} \ln\left[\frac{1}{\delta\bar{\eta}}\left\{\frac{\mathrm{e}^{-M_2}\, g(\mathrm{e}^{-M_2})}{c_{2,1}} c_{3,1}(M_2) - 1\right\}\right] + M_2, \end{aligned} \tag{3.43}$$

where $M_2 \in (0, \underline{b})$ is the solution to

$$\bar{g}(M_2) = [1 + \delta\bar{\eta}]\left[1 + \frac{b}{c_{2,1}}\right]. \tag{3.44}$$

Remark 3.10. *Taking $\gamma \in (0, \gamma_1)$ such that* (3.38) *holds, we get that $\bar{g}$ is a positive and decreasing function on $(0, \underline{b})$. It implies that M_2 is the unique solution to* (3.44)*, for any $b > 0$ satisfying* (3.36)*. Additionally, by the construction of $v_{\gamma,b}$, which is given by* (3.42)*, it is known that $v_{\gamma,b}$ is concave and increasing on $(0, \infty)$, since $c_{3,2}(\alpha_\gamma) < 0$ and $v_{\gamma,b}$ satisfies* (3.28) *on $(0, b)$, and*

$$v'_{\gamma,b}(x) = \mathrm{e}^{-\chi_2^{-1}(x)} \quad \text{and} \quad v''_{\gamma,b}(x) = -\frac{\mathrm{e}^{-\chi_2^{-1}(x)}}{\chi_2'(\chi_2^{-1}(x))}, \quad \text{for } x \in (b, x_b). \tag{3.45}$$

It implies also that $u_b^*(x) < 1$ for $x \in (0, x_b)$, and $u_b^*(x) = 1$ at $x = x_b$. For a more detail exposition, see Subsection A.3.

By the seen in the remark above and considering that (3.7) must be true, it follows that $\mathrm{e}^{-\chi_1^{-1}(b_\gamma)} = \mathrm{e}^{-\chi_2^{-1}(b_\gamma)} = \mathrm{e}^{M_2} = 1$, which implies that $M_2 = 0$ and

$$\chi_1^{-1}(b_\gamma) = 0 \quad \iff \quad b_\gamma = c_{2,1}\left[\frac{\bar{\eta}\delta}{1+\delta\bar{\eta}}[\mathrm{e}^{M_\gamma[1+\delta\bar{\eta}]} - 1] + M_\gamma\right]. \tag{3.46}$$

We have also that

$$M_\gamma = \frac{1}{1+\delta\bar{\eta}} \ln\left[\frac{1}{\delta\bar{\eta}}\left\{\frac{g(1)}{c_{2,1}} c_{3,1}(0) - 1\right\}\right]. \tag{3.47}$$

To establish the values of $\gamma \in (0, \gamma_1)$ for which the condition $M_\gamma > 0$ holds, due to $v'_{\gamma, b_\gamma}(0+) > 1$, we must study when (3.38) is true. To achieve this, let us highlight some properties of the following functions

$$\begin{aligned}
f_2(\gamma) &:= \alpha_\gamma = \frac{\gamma+\delta}{\gamma}\left[1 - \frac{\mu}{\eta + \sqrt{\eta^2 + 2\sigma^2[\delta+\gamma]}}\right], \\
f_3(\gamma) &:= \frac{\sigma^2\, \mathrm{e}^{\bar{\eta}\gamma f_2(\gamma)}}{\mu[f_2(\gamma)]^{\bar{\eta}[\delta+\gamma]+1}} + \bar{\eta}[\mu-\eta]\int_1^{f_2(\gamma)} \frac{\mathrm{e}^{\bar{\eta}\gamma y}}{y^{\bar{\eta}[\delta+\gamma]+2}} \mathrm{d}y, \qquad \text{for } \gamma \in (0, \gamma_1). \\
f_4(\gamma) &:= \bar{\eta}[\mu-\eta]\, \mathrm{e}^{\bar{\eta}\gamma},
\end{aligned} \tag{3.48}$$

Lemma 3.11. *Let f_2, f_3, f_4 be as in (3.48). Then,*

(i) f_2 is decreasing on $(0, \gamma_1)$ satisfying

$$\lim_{\gamma\downarrow 0} f_2(\gamma) = \infty \quad \text{and} \quad \lim_{\gamma\uparrow\gamma_1} f_2(\gamma) = 1. \tag{3.49}$$

(ii) f_3 and f_4 are increasing on $(0, \gamma_1)$ satisfying

$$\begin{aligned}
&\lim_{\gamma\downarrow 0} f_3(\gamma) = \frac{\bar{\eta}[\mu-\eta]}{\bar{\eta}\delta + 1} \quad \text{and} \quad \lim_{\gamma\uparrow\gamma_1} f_3(\gamma) = \frac{\sigma^2\, \mathrm{e}^{\bar{\eta}\gamma_1}}{\mu}, \\
&\lim_{\gamma\downarrow 0} f_4(\gamma) = \bar{\eta}[\mu-\eta] \quad \text{and} \quad \lim_{\gamma\uparrow\gamma_1} f_4(\gamma) = \bar{\eta}[\mu-\eta]\, \mathrm{e}^{\bar{\eta}\gamma_1}.
\end{aligned} \tag{3.50}$$

Remark 3.12. *By (3.19), (3.49)–(3.50) and since $\mu > \eta$ and $2\eta > \mu$, it is easy to check that $f_3(0+) < f_4(0+)$ and $f_3(\gamma_1-) > f_4(\gamma_1-)$. It implies that there exists a unique $\gamma_2 \in (0, \gamma_1)$ such that*

$$f_3(\gamma_2) = f_4(\gamma_2), \tag{3.51}$$

due to the increasing property of f_3 and f_4. Therefore, for any $\gamma \in (\gamma_2, \gamma_1)$, (3.38) holds, demonstrating that $M > 0$. it is also observed that $\bar{g}(0+) > 1 + \delta\bar{\eta}$ for $\gamma \in (\gamma_2, \gamma_1)$.

In case that $\gamma \in (0, \gamma_2)$, we take $b = 0$ and $x_0 = \bar{f}_{-M_\gamma}(\alpha_\gamma)$, where

$$\begin{aligned}
\bar{f}_\beta(z) &= c_{3,3}(\beta)[G(z) - G(\mathrm{e}^{-\beta})] - \bar{\eta}[\mu-\eta]\left[G(z)H_\beta(z) - \int_{\mathrm{e}^{-\beta}}^{z} \frac{G(y)}{y^2 g(y)} \mathrm{d}y\right], \\
c_{3,3}(\beta) &:= \frac{\bar{\eta}[\mu-\eta]}{\mathrm{e}^{-\beta}\, g(\mathrm{e}^{-\beta})},
\end{aligned}$$

and M_γ is the unique positive solution to

$$\frac{\mu-\eta}{\mathrm{e}^{M_\gamma}\,g(\mathrm{e}^{M_\gamma})}=\frac{\mu}{2\alpha_\gamma g(\alpha_\gamma)}+[\mu-\eta]H_{-M_\gamma}(\alpha_\gamma). \tag{3.52}$$

Then, a solution to (3.39)–(3.41) is given by

$$v_{\gamma,0}(x)=\begin{cases}\frac{\mathrm{e}^{-\chi_2^{-1}(x)}}{\delta+\gamma}\Big\{\frac{g(\mathrm{e}^{\chi_2^{-1}(x)})\,\mathrm{e}^{\chi_2^{-1}(x)}}{\bar{\eta}} & \\ \times\big[c_{3,3}(M_\gamma)-\bar{\eta}[\mu-\eta]H_{-M_\gamma}(\mathrm{e}^{\chi_2^{-1}(x)})\big]-[\mu-\eta]\Big\}+\frac{\gamma x}{\delta+\gamma} & \text{if } x\in(0,x_0),\\ c_{3,2}(\alpha_\gamma)\,\mathrm{e}^{\lambda_\gamma[x-x_0]}+\frac{\gamma}{\gamma+\delta}\Big[x+\frac{\eta}{\delta+\gamma}\Big] & \text{if } x\in(x_0,\infty),\end{cases} \tag{3.53}$$

where $c_{3,2}(\alpha_\gamma)$ is as in (3.43) and χ_2^{-1} is the inverse function of $\chi_2:[M_\gamma,\bar{z}]\longrightarrow[0,x_\gamma]$, with $\bar{z}=\ln[\alpha_\gamma]$, which has the following form $\chi_2(z)=\bar{f}_{-M_\gamma}(\mathrm{e}^z)$. For more details about the construction of (3.53), see the proof of Proposition 3.13 in Section A.3.

Proposition 3.13. *Let $\mu<2\eta$ and $\gamma_2\in(0,\gamma_1)$ satisfying* (3.51)*.*

(i) If $\gamma\in(\gamma_2,\gamma_1)$, then (3.42) *is a C^2-continuous, increasing and concave solution to* (2.5)*, with $M_2=0$, $b=b_\gamma$ and M_γ as in* (3.46) *and* (3.47)*. Furthermore, $v'_{\gamma,b_\gamma}/v''_{\gamma,b_\gamma}$ is decreasing on $(0,\infty)$. Then,*

$$x_\gamma=\bar{f}_0(\alpha_\gamma)+b_\gamma \tag{3.54}$$

where $\bar{f}_0$ is as in (3.33) *when $\beta=0$.*

(ii) If $\gamma\in(0,\gamma_2)$, then $v_{\gamma,0}$ as in (3.53) *is a solution to the equation* (2.5)*, with $b_\gamma=0$. Furthermore, $v'_{\gamma,0}/v''_{\gamma,0}$ is decreasing on $(0,\infty)$. Then,*

$$x_\gamma=\bar{f}_{-M}(\alpha_\gamma). \tag{3.55}$$

Optimal strategy. In light of Theorem 2.3 and Proposition 3.8, we deduce under the conditions $\mu<2\eta$ and $\gamma>\gamma_1$, the optimal strategy is determined by (2.15), where x_γ and b_γ are given by (3.18) and (3.31), respectively. The controlled process X^{b_γ} is given as the solution to the SDE described in (2.14). Moreover, the value function V and the expected NPV V^{b_γ} are given by (3.23) with $b=b_\gamma$ and $x_b=\bar{x}$,. In this context, the optimal strategy for the insurance company is to assume the maximum risk exclusively when $X^{b_\gamma}_t\geq x_\gamma$, while the payout $\max\{X^{b_\gamma}_{T_i}-b_\gamma,0\}$ is made at time $T_i<\tau^{b_\gamma}$. It is noteworthy that in the strategy π^{b_γ}, shareholder payments are made precisely when the insurance company is exposed to the maximum risk, as a consequence of $b_\gamma>x_\gamma$.

On the other hand, if $\gamma\in(\gamma_2,\gamma_1)$, by Theorem 2.3 and Proposition 3.13.(i), we have that the optimal strategy is determined by (3.46) and (3.54), respectively. The value function V and the expected NPV V^{b_γ} are given by (3.42), with $b=b_\gamma$ and $x_\gamma=\bar{f}_0(\alpha_\gamma)+b_\gamma$. It is remarkable that in the strategy π^{b_γ}, shareholder payments are made precisely when the insurance company does not wish to be exposed to the maximum risk, as a consequence of $b_\gamma<x_\gamma$.

In the scenario where $\mu<2\eta$ and $\gamma\in(0,\gamma_2]$, according to Theorem 2.3 and Proposition 3.13.(ii), we observe that $b_\gamma=0$ and $x_\gamma=\bar{f}_0(\alpha_\gamma)$. Furthermore, the optimal strategy for the company in this scenario is to accept the maximum risk when $X^0_t>\bar{f}_0(\alpha_\gamma)$, while making the payout equal to the entire amount that the company has at the first arrival of the Poisson process N^γ. Additionally, V^0 and V are given by (3.53).

3.2 Cheap reinsurance

In this subsection, we analyse the case where $\eta = \mu$. Similar to our previous analysis, we examine the scenarios $x_\gamma \leq b_\gamma$ and $x_\gamma > b_\gamma$. However, since $\mu < 2\mu$, we can leverage the arguments presented in Sub-subsection 3.1.2 to verify the validity of this subsection. Consequently, we will omit the proofs of the main results of this part.

For the first case, let us consider $b > \hat{x}$, with

$$\hat{x} := \frac{\sigma^2}{\mu[1+\delta\bar{\eta}]}. \tag{3.56}$$

Proposition 3.14. *Let $\eta = \mu$. If $b > \hat{x}$ is fixed, a solution to the NLPDS*

$$-\frac{\mu^2[v'(x)]^2}{2\sigma^2 v''(x)} - \delta v(x) = 0 \quad \text{for } x \in (0, x_b), \tag{3.57}$$

$$\frac{1}{2}\sigma^2 v''(x) + \mu v'(x) - \delta v(x) + \gamma\{x - b + v(b) - v(x)\}\mathbb{1}_{\{x-b>0\}} = 0 \quad \text{for } x \in (x_b, \infty), \tag{3.58}$$

$$\text{s.t. } v(0) = 0, \tag{3.59}$$

is a C^2-continuous function on $(0,\infty)$, whose form is given by

$$v_{\gamma,b}(x) = \begin{cases} c_{4,1}(b) x^{\delta\bar{\eta}/[1+\delta\bar{\eta}]} & \text{if } x \in (0, x_b), \\ c_{4,1}(b) h_3(x - x_b) & \text{if } x \in (x_b, b), \\ c_{4,2}(b)\, \mathrm{e}^{\lambda_\gamma[x-b]} + \frac{\gamma}{\gamma+\delta}\left[x - b + v_{\gamma,b}(b) + \frac{\mu}{\delta+\gamma}\right] & \text{if } x \in (b, \infty), \end{cases} \tag{3.60}$$

where the function h_3 is taken as $h_3(x) := \frac{1}{\theta_+ - \theta_-}\left[\bar{a}_1\, \mathrm{e}^{\theta_+ x} - \bar{a}_2\, \mathrm{e}^{\theta_- x}\right]$, with θ_+, θ_-, λ_γ as in Proposition 3.1, and

$$x_b = \hat{x}, \quad \bar{a}_1 := \left[\frac{\delta\bar{\eta}}{1+\delta\bar{\eta}}\right] x_b^{-1/[1+\delta\bar{\eta}]} - \theta_- x_b^{\delta\bar{\eta}/[1+\delta\bar{\eta}]},$$

$$\bar{a}_2 := \left[\frac{\delta\bar{\eta}}{1+\delta\bar{\eta}}\right] x_b^{-1/[1+\delta\bar{\eta}]} - \theta_+ x_b^{\delta\bar{\eta}/[1+\delta\bar{\eta}]},$$

$$c_{4,1}(b) = \frac{\frac{\gamma}{\gamma+\delta} - \frac{\mu\gamma\lambda_\gamma}{[\delta+\gamma]^2}}{h_3'(b - x_b) - \frac{\lambda_\gamma\delta}{\gamma+\delta} h_3(b - x_b)},$$

$$c_{4,2}(b) := \frac{\delta}{\delta+\gamma} c_{4,1}(b) h_3(b - x_b) - \frac{\mu\gamma}{[\delta+\gamma]^2}.$$

Remark 3.15. *By the construction of $v_{\gamma,b}$, which is given by (3.60), for any $b > \hat{x}$, it is easy to observe that $v_{\gamma,b}$ is concave and increasing on $(0, \hat{x})$, due to $c_{4,1}(b) > 0$. Moreover, $u_b^*(x) = \frac{\mu}{\sigma^2}[1 + \delta\bar{\eta}]x$, which increases on $(0, \hat{x})$ and satisfies $u_b^*(\hat{x}) = 1$. For a more detail exposition, see Subsection A.3.*

Taking the first derivative in (3.60) on $(\hat{x}, b_\gamma)$, we see that (3.7) is achieved if and only if

$$g_4(b_\gamma - \hat{x}) = \frac{1}{\lambda_\gamma \delta}\left\{\delta + \frac{\mu\gamma\lambda_\gamma}{\delta+\gamma}\right\} \tag{3.61}$$

is true, where

$$g_4(b) := \frac{h_3(b)}{h_3'(b)} \quad \text{for } b \in (0, \infty). \tag{3.62}$$

In order to guarantee the veracity of (3.61), let us mention some properties of the function g_4 first.

Lemma 3.16. *Let g_4 be as in (3.62). Then, g_4 is increasing on $(0,\infty)$, satisfying*

$$\lim_{b\downarrow 0} g_4(b) = \frac{\mu}{2\delta} \quad \text{and} \quad \lim_{b\uparrow\infty} g_4(b) = \frac{1}{\theta_+}.$$

By Lemma 3.16, we have that (3.61) is equivalent to verify that $\frac{\mu}{2\delta} \le \frac{1}{\lambda_\gamma\delta}\big[\delta + \frac{\lambda_\gamma\mu\gamma}{\delta+\gamma}\big] < \frac{1}{\theta_+}$, which is true only for $\gamma > \bar{\gamma}_1 := f_1^{-1}\big(\frac{\mu}{2\delta}\big)$, where f_1^{-1} is the inverse of f_1 given in (3.11) when $\eta = \mu$, because of $\frac{1}{\theta_-} < \frac{\mu}{2\delta} < \frac{\mu}{\delta} < \frac{1}{\theta_+}$ and Lemma 3.3. Thus, for $\gamma > \bar{\gamma}_1$ fixed, there exists a unique $b_\gamma \in (0,\infty)$ that satisfies (3.61). By the seen above, we get the following proposition.

Proposition 3.17. *If $\eta = \mu$ and $\gamma > \bar{\gamma}_1$, then (3.60) is a C^2-continuous, increasing and concave solution to (3.57)–(3.59), with*

$$b = b_\gamma = \hat{x} + g_4^{-1}\left(\frac{1}{\lambda_\gamma\delta}\left[\delta + \frac{\lambda_\gamma\mu\gamma}{\delta+\gamma}\right]\right). \tag{3.63}$$

Furthermore, $v'_{\gamma,b_\gamma}/v''_{\gamma,b_\gamma}$ is decreasing on $(0,\infty)$ satisfying (3.17). Then, $x_\gamma = \hat{x}$.

The following proposition presents a solution $v_{\gamma,b}$ to the NPDS (3.65)–(3.67), which depends on $b \in (0,\bar{b})$ where

$$\bar{b}_1 := \frac{c_{5,1}(\alpha_\gamma)g(1)}{1+\delta\bar{\eta}}, \quad \text{and} \quad c_{5,1}(\alpha_\gamma) := \frac{\sigma^2}{\mu\alpha_\gamma g(\alpha_\gamma)}, \tag{3.64}$$

with α_γ as in (3.34) when $\eta = \mu$. Recall that G is the gamma accumulative distribution whose density function g is given by (3.35), and its inverse function is represented by G^{-1}.

Proposition 3.18. *Let $\eta = \mu$. If $b \in (0,\bar{b}_1)$ is fixed, with $\bar{b}_1$ as in (3.64), a solution to the NLPDS*

$$-\frac{\mu^2[v'(x)]^2}{2\sigma^2 v''(x)} - \delta v(x) + \gamma\{x - b + v(b) - v(x)\}\mathbb{1}_{\{x-b>0\}} = 0 \quad \text{for } x \in (0,x_b), \tag{3.65}$$

$$\frac{1}{2}\sigma^2 v''(x) + \mu v'(x) - [\delta+\gamma]v(x) + \gamma\{x - b + v(b)\} = 0 \quad \text{for } x \in (x_b,\infty), \tag{3.66}$$

$$\text{s.t. } v(0) = 0, \tag{3.67}$$

is a C^2-continuous, whose form is given by

$$v_{\gamma,b}(x) = \begin{cases} \frac{[1+\delta\bar{\eta}]\,\mathrm{e}^{M_2}\,b^{1/[1+\delta\bar{\eta}]}}{\delta\bar{\eta}} x^{\delta\bar{\eta}/[1+\delta\bar{\eta}]} & \text{if } x \in (0,b), \\ \frac{1}{\delta+\gamma}\big\{\frac{c_{5,1}(\alpha_\gamma)}{\bar{\eta}} g\big(G^{-1}\big(\frac{x-b}{c_{5,1}(\alpha_\gamma)} + G(\mathrm{e}^{-M_2})\big)\big) + \gamma[v_{\gamma,b}(b) + x - b]\big\} & \text{if } x \in (b,x_b), \\ c_{3,2}(\alpha_\gamma)\,\mathrm{e}^{\lambda_\gamma[x-x_b]} + \frac{\gamma}{\gamma+\delta}\big[x - b + v_{\gamma,b}(b) + \frac{\mu}{\delta+\gamma}\big] & \text{if } x \in (x_b,\infty), \end{cases} \tag{3.68}$$

where $c_{3,2}(\alpha_\gamma)$, $c_{5,1}(\alpha_\gamma)$ are given in (3.43) and (3.64), respectively, and $x_b = c_{5,1}(\alpha_\gamma)[G(\alpha_\gamma) - G(\mathrm{e}^{-M_2})] + b$. The parameter $M_2 > 0$ is the unique solution to $\mathrm{e}^{-M_2}\,g(\mathrm{e}^{-M_2}) = \frac{b[1+\delta\bar{\eta}]}{c_{5,1}(\alpha_\gamma)}$.

Remark 3.19. *Since $\beta \mapsto \mathrm{e}^{-\beta}\,g(\mathrm{e}^{-\beta})$ is a positive and decreasing function on $(0,\infty)$, we have a unique solution M_2 to (3.44), for any $b \in (0,\bar{b}_1)$. Additionally, $v_{\gamma,b}$, given in (3.68), is concave and increasing on $(0,x_b)$, due to its construction. Additionally, we have also that $u_b^*(x) < 1$ for $x \in (0,\bar{x})$, and $u_b^*(x) = 1$ at $x = x_b$. For a more detail exposition, see Subsection A.3.*

Since (3.7) should hold, it follows that $e^{M_2} = 1$, which implies that $M_2 = 0$ and

$$b_\gamma = \frac{c_{5,1}(\alpha_\gamma) g(1)}{1 + \delta\bar{\eta}}. \tag{3.69}$$

Notice that b_γ as before is a positive real value because $\gamma \mapsto \alpha_\gamma$ is a positive and decreasing function on $(0, \bar{\gamma}_1)$, due to Lemma 3.11.(i) when $\eta \downarrow \mu$.

Proposition 3.20. *If $\eta = \mu$ and $\gamma \in (0, \bar{\gamma}_1)$, then* (3.68) *is a* C^2*-continuous, increasing and concave solution to* (3.65)–(3.67)*, with $b = b_\gamma = \frac{c_{5,1}(\alpha_\gamma) g(1)}{1+\delta\bar{\eta}}$. Furthermore, $v'_{\gamma,b_\gamma}/v''_{\gamma,b_\gamma}$ is decreasing on $(0, \infty)$ satisfying* (3.17)*. Then,*

$$x_\gamma = c_{5,1}(\alpha_\gamma)[G(\alpha_\gamma) - G(1)] + b_\gamma. \tag{3.70}$$

Optimal strategy. When $\gamma \in (\bar{\gamma}_1, \infty)$, according to Theorem 2.3 and Proposition 3.17, the optimal strategy π^{b_γ} is given by equations (3.56) and (3.63). In this scenario, the expressions for V and V^{b_γ} are represented by (3.60) when $b = b_\gamma$ as in (3.63). It is worth noting that in the strategy π^{b_γ}, shareholder payments occur precisely when the insurance company is exposed to the maximum risk, as a consequence of $b_\gamma > x_\gamma$.

On the other hand, when $\gamma \in (0, \bar{\gamma}_1)$, based on Theorem 2.3 and Proposition 3.20, the optimal strategy π^{b_γ} is given by equations (3.69)–(3.70). At the same time, the representations for V and V^{b_γ} are given by (3.68), when $b = b_\gamma$ as in (3.69). It is remarkable that in the strategy π^{b_γ}, dividend payments to shareholders occur precisely when the insurance company does not wish to be exposed to the maximum risk, as a consequence of $b_\gamma < x_\gamma$.

4 Analysing the limit behaviour for γ

4.1 Non-cheap reinsurance

4.1.1 The case $\mu \geq 2\eta$

Notice that the trajectory $\gamma \mapsto b_\gamma = g_1^{-1}(f_1(\gamma))$ is continuous and strictly increasing on (γ_0, ∞), due to Lemmas 3.2–3.3, which satisfies $b_{\gamma_0} = 0$ and $b_\infty := \lim_{\gamma \uparrow \infty} b_\gamma = g_1^{-1}(\frac{\eta}{\delta}) = \frac{1}{\theta_+ - \theta_-} \ln\left[\frac{\delta - \eta\theta_-}{\delta - \eta\theta_+}\right]$. Furthermore, By (3.12) and taking into account that

$$\frac{\delta\lambda_\gamma}{\gamma + \delta} \xrightarrow[\gamma\uparrow\infty]{} 0 \quad \text{and} \quad \frac{\gamma}{\gamma + \delta}\left[1 - \frac{\eta\lambda_\gamma}{\gamma + \delta}\right] \xrightarrow[\gamma\uparrow\infty]{} 1, \tag{4.1}$$

we have that

$$\lim_{\gamma\uparrow\infty} c_{1,1}(b_\gamma) = \frac{1}{h'_1(b_\infty)} \text{ and } \lim_{\gamma\uparrow\infty} c_{1,2}(b_\gamma) = 0. \tag{4.2}$$

Then, taking v_{γ,b_γ} as in (3.2), with $b = b_\gamma$, and using (4.2), it follows that

$$v_{b_\infty}(x) := \lim_{\gamma\uparrow\infty} v_{\gamma,b_\gamma}(x) = \begin{cases} \frac{h_1(x)}{h'_1(b_\infty)} & \text{for } x \in (0, b_\infty), \\ x - b_\infty + v_{b_\infty}(b_\infty) & \text{for } x \in [b_\infty, \infty). \end{cases} \tag{4.3}$$

Here $x_\infty = 0$. These asymptotic limits allow us to recover the value function for the case of singular dividend strategies for the optimal problem obtained by Taksar [13] in Theorem 5.2.

4.1.2 The case $\mu < 2\eta$

Notice that the trajectory $\gamma \mapsto b_\gamma = \bar{x} + g_2^{-1}(f_1(\gamma))$ is continuous and strictly increasing on (γ_1, ∞), due to Lemmas 3.2 and 3.7, which satisfies $b_{\gamma_1} = \bar{x} + g_2^{-1}\big(\frac{2\eta-\mu}{2\delta}\big) = \bar{x}$ and $b_\infty := \lim_{\gamma\uparrow\infty} b_\gamma = \bar{x} + g_2^{-1}\big(\frac{\eta}{\delta}\big) = \bar{x} + \frac{1}{\theta_+-\theta_-} \ln\Big[\frac{a_2[\delta-\eta\theta_-]}{a_1[\delta-\eta\theta_+]}\Big]$. On the other hand, considering (4.1), we get that $\lim_{\gamma\uparrow\infty} c_{2,3}(b_\gamma) = 0$ and $M(b_\infty) := \lim_{\gamma\uparrow\infty} M(b_\gamma) = \ln\Big[\frac{c_{2,2}^{1/[1+\delta\bar{\eta}]}}{h_2'(b_\infty-\bar{x})}\Big]$. Then, taking v_{γ,b_γ} as in (3.42), with $b = b_\gamma$, it follows that

$$v_{b_\infty}(x) = \lim_{\gamma\uparrow\infty} v_{\gamma,b}(x) = \begin{cases} c_{2,1}\,\mathrm{e}^{-\varkappa_1^{-1}(x)}\Big[\mathrm{e}^{[1+\delta\bar{\eta}][M(b_\infty)+\varkappa_1^{-1}(x)]} - 1\Big] & \text{if } x \in (0,\bar{x}),\\ \frac{h_2(x-\bar{x})}{h_2'(b_\infty-\bar{x})} & \text{if } x \in (\bar{x}, b_\infty),\\ x - b_\infty + v_{b_\infty}(b_\infty) & \text{if } x \in (b_\infty,\infty), \end{cases} \tag{4.4}$$

where $\varkappa_1^{-1}$ is the inverse function of

$$\varkappa_1(z) = k_{2,1}\,\mathrm{e}^{[1+\delta\bar{\eta}][z+M(b_\infty)]} + c_{2,1}z + k_{2,2}(b_\infty) \quad \text{for } z \in [-M(b_\infty), \bar{z}(b_\infty)],$$

with $k_{2,2}(b_\infty) = c_{2,1}\big[M(b_\infty) - \frac{\delta\bar{\eta}}{1+\delta\bar{\eta}}\big]$ and $\bar{z}(b_\infty) = \frac{1}{1+\delta\bar{\eta}}\ln[c_{2,2}] - M(b_\infty)$. Here $x_\infty = \bar{x}$. These asymptotic limits allow us to recover the value function for the case of singular dividend strategies for the optimal problem obtained by Taksar [13] in Theorem 5.3.

Additionally, a solution to the HJB equation (2.5), when $\gamma \downarrow \gamma_1$, is given by

$$v_{\gamma,\bar{x}}(x) = \begin{cases} c_{2,1}\,\mathrm{e}^{-\varkappa_1^{-1}(x)}\Big[\mathrm{e}^{[1+\delta\bar{\eta}][M(\bar{x})+\varkappa_1^{-1}(x)]} - 1\Big] & \text{if } x \in (0,\bar{x}),\\ c_{2,3}(\bar{x})\,\mathrm{e}^{\lambda_\gamma[x-\bar{x}]} + \frac{\gamma}{\gamma+\delta}\Big[x - \bar{x} + v_{\gamma,\bar{x}}(\bar{x}) + \frac{\eta}{\delta+\gamma}\Big] & \text{if } x \in (\bar{x},\infty), \end{cases} \tag{4.5}$$

with

$$M(\bar{x}) = \ln\left[\frac{\frac{\gamma_1}{\gamma_1+\delta} - \frac{\lambda_{\gamma_1}\eta\gamma_1}{[\delta+\gamma_1]^2}}{c_{2,2}^{-1/[1+\delta\bar{\eta}]}\Big[1 - \frac{\lambda_{\gamma_1}[2\eta-\mu]}{2[\delta+\gamma_1]}\Big]}\right] = \frac{1}{1+\delta\bar{\eta}}\ln[c_{2,2}],$$

because of

$$f_1(\gamma_1) = \frac{2\eta-\mu}{2\delta} \quad\Longleftrightarrow\quad \frac{1}{\lambda_{\gamma_1}} = \frac{2\eta-\mu}{2\delta} - \frac{\eta\gamma_1}{\delta[\delta+\gamma_1]}. \tag{4.6}$$

It implies that $c_{2,3}(\bar{x}) = \frac{2\eta-\mu}{2[\delta+\gamma_1]} - \frac{\eta\gamma_1}{[\delta+\gamma_1]^2} = -\frac{\mu}{\lambda_{\gamma_1}^2\sigma^2} = c_{3,2}(1)$, due to

$$f_2(\gamma_1) = 1 \quad\Longleftrightarrow\quad \frac{1}{\lambda_{\gamma_1}} = -\frac{\delta\sigma^2}{[\gamma_1+\delta]\mu}. \tag{4.7}$$

Meanwhile, using (3.48), we see that b_γ and M as in (3.46) and (3.47), respectively, can be written in the following way

$$\begin{aligned} M_\gamma &= \frac{1}{1+\delta\bar{\eta}}\ln\left[\frac{\mathrm{e}^{-\gamma\bar{\eta}}[f_3(\gamma)-f_4(\gamma)]}{\delta\bar{\eta}c_{2,1}} + 1\right],\\ b_\gamma &= \frac{\mathrm{e}^{-\gamma\bar{\eta}}[f_3(\gamma)-f_4(\gamma)]}{c_{2,1}^{-1}\bar{\eta}[\mu-\eta]} + \frac{c_{2,1}}{1+\delta\bar{\eta}}\ln\left[\frac{\mathrm{e}^{-\gamma\bar{\eta}}[f_3(\gamma)-f_4(\gamma)]}{\delta\bar{\eta}c_{2,1}} + 1\right]. \end{aligned} \tag{4.8}$$

Then, the trajectory $\gamma \mapsto b_\gamma$ is continuous on (γ_2, γ_1). Moreover, by Remark 3.13, it follows that $b_{\gamma_2} = 0$. Additionally, using (3.50), it gives that $f_3(\gamma_1) - f_4(\gamma_1) = \left[\frac{\sigma^2}{\mu} - \bar{\eta}[\mu - \eta]\right] \mathrm{e}^{\bar{\eta}\gamma_1}$. From here and (4.8), it follows that $b_{\gamma_1} = \bar{x}$, with $\bar{x}$ as in (3.18), and $M_\gamma = M(\bar{x})$. On the other hand, notice that $f_0(\alpha_{\gamma_1}) = 0$, which implies that $x_{\gamma_1} = f_0(\alpha_{\gamma_1}) + \bar{x} = \bar{x}$. Therefore, for each $x \in (0, \infty)$ fixed, $v_{\gamma, b_\gamma}(x) \xrightarrow[\gamma\uparrow\gamma_1]{} v_{\gamma,\bar{x}}(x)$ where $v_{\gamma,\bar{x}}$ is as in (4.5).

4.2 Cheap reinsurance

Observe that the trajectory $\gamma \mapsto b_\gamma = \hat{x} + g_4^{-1}(f_1(\gamma))$ is continuous and strictly increasing on (γ_1, ∞), due to Lemmas 3.2 and 3.16, which satisfies $b_{\gamma_1} = \hat{x} + g_4^{-1}\left(\frac{\mu}{2\delta}\right) = \hat{x}$ and $b_\infty := \lim_{\gamma\uparrow\infty} b_\gamma = \hat{x} + g_4^{-1}\left(\frac{\eta}{\delta}\right) = \hat{x} + \frac{1}{\theta_+ - \theta_-} \ln\left[\frac{\bar{a}_2[\delta - \eta\theta_-]}{\bar{a}_1[\delta - \eta\theta_+]}\right]$. Additionally, since

$$\lim_{\gamma\uparrow\infty} c_{4,1}(b_\gamma) = \frac{1}{h_3'(b_\infty - \hat{x})} \quad \text{and} \quad \lim_{\gamma\uparrow\infty} c_{4,2}(b_\gamma) = 0,$$

it gives that

$$v_{b_\infty}(x) = \lim_{\gamma\uparrow\infty} v_{\gamma,b_\gamma}(x) = \begin{cases} \frac{x^{\delta\bar{\eta}/[1+\delta\bar{\eta}]}}{h_3'(b_\infty - \hat{x})} & \text{if } x \in (0, \hat{x}), \\ \frac{h_3(x-\hat{x})}{h_3'(b_\infty - \hat{x})} & \text{if } x \in (\hat{x}, b_\infty), \\ x - b_\infty + v_{b_\infty}(b_\infty) & \text{if } x \in (b_\infty, \infty). \end{cases} \tag{4.9}$$

These asymptotic limits allow us to recover the value function for the case of singular dividend strategies for the optimal problem obtained by Højgaard and Taksar [5] in Theorem 3.1. Furthermore, a solution to the HJB equation (2.5), when $\gamma \downarrow \bar{\gamma}_1$, is given by

$$v_{\gamma,\hat{x}}(x) = \begin{cases} c_{4,1}(\hat{x}) x^{\delta\bar{\eta}/[1+\delta\bar{\eta}]} & \text{if } x \in (0, \hat{x}), \\ c_{4,2}(\hat{x})\, \mathrm{e}^{\lambda_{\bar{\gamma}_1}[x-\hat{x}]} + \frac{\bar{\gamma}_1}{\bar{\gamma}_1+\delta}\left[x - b + v_{\gamma,\hat{x}}(\hat{x}) + \frac{\mu}{\delta+\bar{\gamma}_1}\right] & \text{if } x \in (\hat{x}, \infty), \end{cases}$$

with $c_{4,1}(\hat{x}) = \frac{\hat{x}^{1/[1+\delta\bar{\eta}]}\left[\frac{\bar{\gamma}_1}{\bar{\gamma}_1+\delta} - \frac{\mu\bar{\gamma}_1\lambda_{\bar{\gamma}_1}}{[\delta+\bar{\gamma}_1]^2}\right]}{\frac{\delta\bar{\eta}}{1+\delta\bar{\eta}}\left[1 - \frac{\mu\lambda_{\bar{\gamma}_1}}{2[\bar{\gamma}_1+\delta]}\right]} = \frac{[1+\delta\bar{\eta}]}{\delta\bar{\eta}}\hat{x}^{1/[1+\delta\bar{\eta}]}$ because of $\frac{1}{\lambda_{\bar{\gamma}_1}} = \frac{\mu}{2\delta} - \frac{\mu\bar{\gamma}_1}{\delta[\delta+\bar{\gamma}_1]}$. It implies that $c_{4,2}(\hat{x}) = \frac{\sigma^2}{\bar{\eta}\mu[\delta+\bar{\gamma}_1]} - \frac{\mu\bar{\gamma}_1}{[\delta+\bar{\gamma}_1]^2} = c_{3,2}(1)$, due to $\frac{1}{\lambda_{\gamma_1}} = -\frac{\delta\sigma^2}{\mu[\delta+\bar{\gamma}_1]}$. Taking b_γ and $\hat{x}$ as in (3.69) and (3.70), respectively, it is easy to check that $\lim_{\gamma\uparrow\gamma_1} b_\gamma = \lim_{\gamma\uparrow\gamma_1} x_\gamma = \hat{x}$. Moreover, considering v_{γ,b_γ} as in (3.68), we have that $v_{\gamma,b_\gamma}(x) \xrightarrow[\gamma\uparrow\gamma_1]{} v_{\gamma,\hat{x}}(x)$.

5 Numerical results

This section presents the numerical results for the various scenarios discussed in Section 3, taking into account the parameters outlined in Table 1. Each row of Table 1 represents a distinct scenario, as mentioned earlier. In Figures 1–3, we illustrate the behaviours of different $v_{\gamma,b}$, considering (3.2), (3.14), (3.23), (3.42), and (3.53).

Taking into account Propositions 3.1 and 3.4, in the upper-left corner of Figure 1 depicts the behaviours of the optimal solution v_{γ,b_γ} concerning the parameter $\gamma = 2^N$ as $N \in [-4, 50]$ with a growth rate of 0.2. It is evident that as N increases, v_{γ,b_γ} (blue line) closely approaches v_∞ (red line). Here, v_∞, given by (4.3), represents the value function for the singular dividend strategies for the optimal problem derived by Taksar [13] in Theorem 5.2.

In addition, recalling that b_γ is given by (3.15) if $\gamma > \gamma_0 \approx 0.2812$, otherwise $b_\gamma = 0$, we can appreciate the behaviour of the trajectory $\gamma \mapsto (b_\gamma, v_{\gamma,b_\gamma}(b_\gamma))$ (triangular points) in this graphic. It

δ	σ	μ	η
0.5	0.3	1.2	0.2
1.5	0.3	0.8	0.5
1.5	0.5	0.7	0.7

Table 1: Parameter values of the HJB equation (2.5) where each row of the table above represents the different scenarios presented in Section 3.

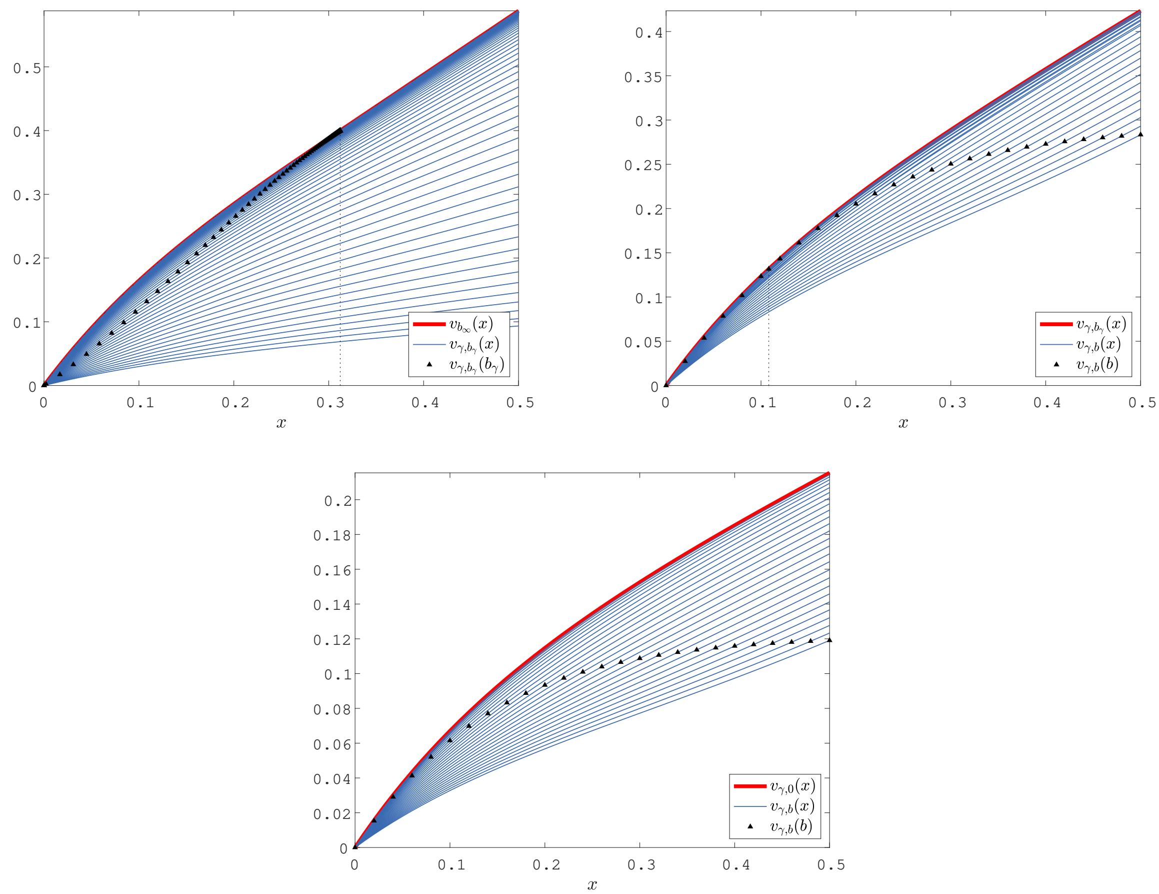

Figure 1: Plots of $v_{\gamma,b}$ considering (3.2) and (3.14) with respect the parameters outlined in the first row of Table 1. *Upper- left corner*: plots of the optimal solution v_{γ,b_γ} and v_∞ (given by (4.3)) with the points $(b_\gamma, v_{\gamma,b_\gamma}(b_\gamma))$ indicated by the triangles for $\gamma = 2^N$ as $N \in [-4, 50]$ grows at a rate of 0.2. *Upper-right corner and center below*: plots of v_{γ,b_γ} when γ is equal to $2^{-0.2}$ and $2^{-2.4}$, respectively, are being compared with $v_{\gamma,b}$ as $b \in [0, 0.5]$ grows at a rate of 0.02.

exhibits an increasing trend and is converging to the point $(b_\infty, v_\infty(b_\infty)) \approx (0.3121, 0.4)$ as $N \uparrow \infty$, where b_∞ is as in Sub-subsection 4.1.1. Moreover, according to Proposition 3.4, in the case where $\mu \geq 2\eta$ (with $\mu > \eta$), the optimal strategy is to take the maximum risk, resulting in $x_\gamma = 0$ for any value of γ. Conversely, in the upper-right corner and in the center below of Figure 1, the plots of v_{γ,b_γ} (red line) are compared when γ takes values of $2^{-0.2}$ and $2^{-2.4}$, respectively, with the plots of $v_{\gamma,b}$ (blue line) as $b \in [0, 0.5]$ grows at a rate of 0.02. For the former, $b_{2^{-0.2}} \approx 0.1082$, and for the latter, $b_{2^{-2.4}} = 0$.

Utilizing Propositions 3.5, 3.8, 3.14 and 3.17 and considering the second and the third row of Table 1, in the upper-left corner of Figures 2 and 3 depict the behaviours of the optimal solution

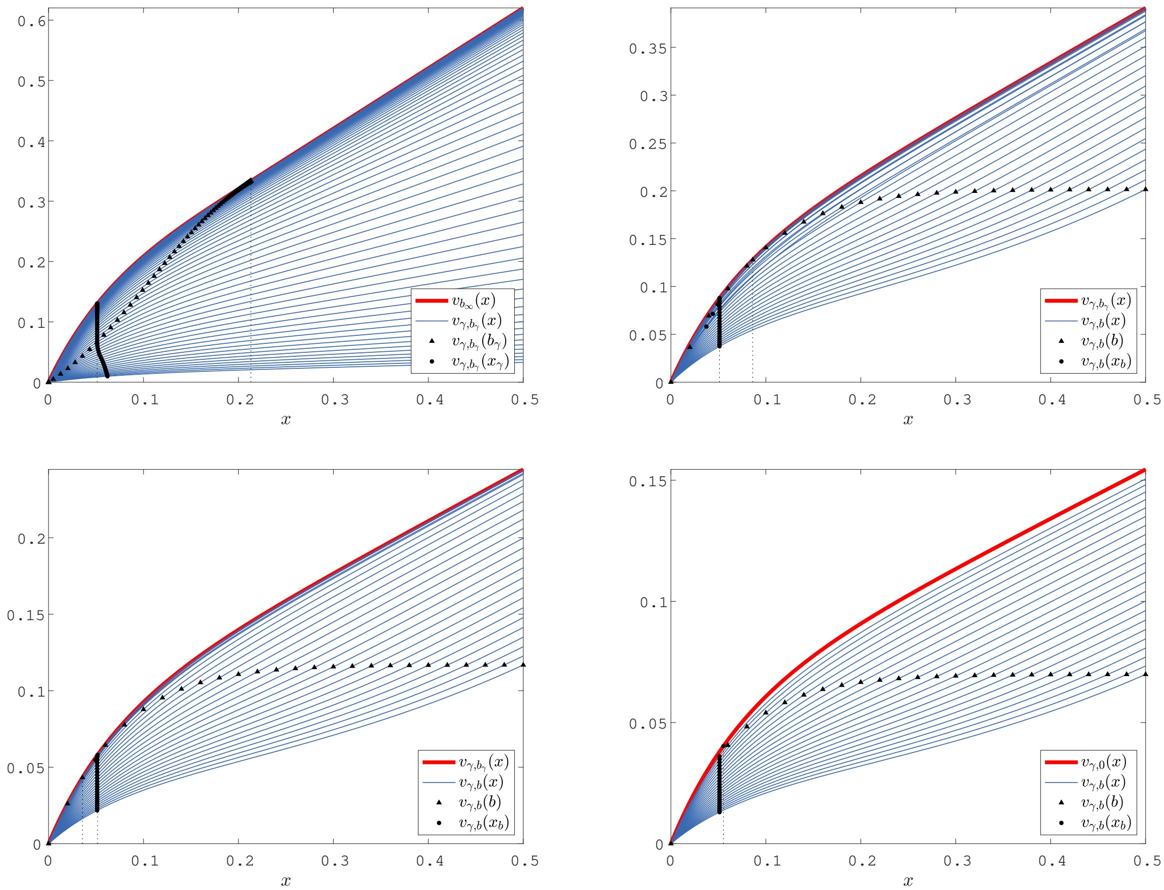

Figure 2: Plots of $v_{\gamma,b}$ considering (3.23), (3.42) and (3.53) with respect the parameters outlined in the second row of Table 1. *Upper- left corner*: plots of the optimal solution v_{γ,b_γ} and v_∞ (given by (4.4)) with the points $(x_\gamma, v_{\gamma,b_\gamma}(x_\gamma))$ and $(b_\gamma, v_{\gamma,b_\gamma}(b_\gamma))$ indicated by the circles and triangles, respectively, for $\gamma = 2^N$ as $N \in [-4, 50]$ grows at a rate of 0.2. *Upper-right, left-below and right-below corners*: plots of v_{γ,b_γ} when γ is equal to 2, $2^{-0.4}$ and $2^{-1.4}$, respectively, are compared with $v_{\gamma,b}$ as $b \in [0, 0.5]$ grows at a rate of 0.02.

v_{γ,b_γ} with respect to $\gamma = 2^N$ with N as previously defined. We observe that v_{γ,b_γ} (blue line) goes to v_∞ (red line) when $N \uparrow \infty$. Here, v_∞, as described in (4.4) and (4.9) respectively, stands the value function for the singular dividend strategies for the optimal problem obtained by Taksar [13] in Theorem 5.3 and Højgaard and Taksar [5] in Theorem 3.1, respectively.

In the case that $\mu < 2\eta$ (with $\mu > \eta$), as indicated in Propositions 3.8 and 3.13, the values of b_γ is determined by (3.31) if $\gamma > \gamma_1 \approx 1.0078$, and by (3.46) if $\gamma \in (\gamma_2, \gamma_1)$, with $\gamma_2 \approx 0.3979$; otherwise, it is zero. In the upper-left corner of Figure 2, we observe the trajectory $\gamma \mapsto (b_\gamma, v_{\gamma,b_\gamma}(b_\gamma))$ (triangular points), which exhibits an increasing trend and converges to the point $(b_\infty, v_\infty(b_\infty)) \approx (0.2131, 0.3333)$ as $N \uparrow \infty$, where b_∞ is as in Sub-subsection 4.1.2. Additionally, we can also observe the behaviour of the trajectory $\gamma \mapsto (x_\gamma, v_{\gamma,b_\gamma}(x_\gamma))$ (circular point) plotted with x_γ determined by (3.18) if $\gamma > \gamma_1$, by (3.54) if $\gamma \in (\gamma_2, \gamma_1)$ and by (3.55) if $\gamma \in (0, \gamma_2)$. It is noteworthy that $(x_\gamma, v_{\gamma,b_\gamma}(x_\gamma))$ converges to the point $(x_\infty, v_{b_\gamma}(x_\infty)) \approx (0.0512, 0.1299)$ as $N \uparrow \infty$. On the other hand, in the upper-right, left-below and right-below corners of Figure 2, the plots of v_{γ,b_γ} (red line) when γ is equal to 2, $2^{-0.4}$, and $2^{-1.4}$, respectively, are compared with $v_{\gamma,b}$ (blue line) as $b \in [0, 0.5]$ increases at a rate of 0.02. In the first case $x_2 \approx 0.0512$ and $b_2 \approx 0.0862$; in the second case $x_{2^{-0.4}} \approx 0.0516$ and $b_{2^{-0.4}} = 0.0357$; and in the last case $x_{2^{-1.4}} \approx 0.0553$ and $b_{2^{-1.4}} = 0$.

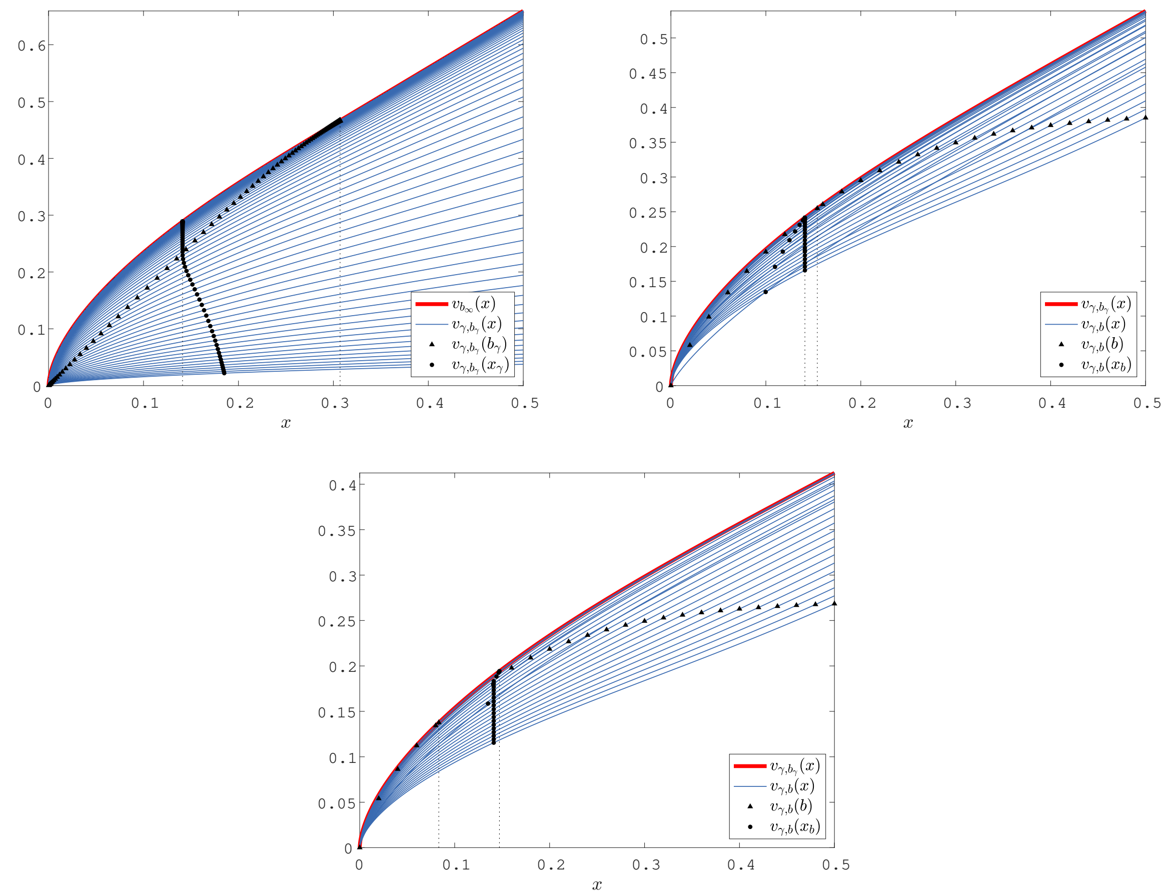

Figure 3: Plots of $v_{\gamma,b}$ considering (3.60) and (3.68) with respect the parameters outlined in the first row of Table 1. *Upper-left corner*: plots of the optimal solution v_{γ,b_γ} and v_∞ (given by (4.9)) with the points $(x_\gamma, v_{\gamma,b_\gamma}(x_\gamma))$ and $(b_\gamma, v_{\gamma,b_\gamma}(b_\gamma))$ indicated by the circles and triangles, respectively, for $\gamma = 2^N$ as $N \in [-4, 50]$ grows with at a rate of 0.2. *Upper-right corner and center below*: plots of v_{γ,b_γ} when γ is equal to $2^{2.2}$ and $2^{0.8}$, respectively, are compared with $v_{\gamma,b}$ as $b \in [0, 0.5]$ grows at a rate of 0.02.

In the case where $\mu = \eta$, similar results to those described in the paragraph above are obtained. However, it should be noted that the strategy involving $b = 0$ is not considered optimal for any value of γ, as depicted in Figure 3. This scenario is characterized by $\bar{\gamma}_1 \approx 3.7959$, along with the values $(x_\infty, v_\infty(x_\infty)) \approx (0.1411, 0.2888)$ and $(b_\infty, v_\infty(b_\infty)) \approx (0.3071, 0.4667)$. Furthermore, in the upper-right corner and in the center below of Figure 3, the plots of v_{γ,b_γ} (red line) when γ is equal to $2^{2.2}$ and $2^{0.8}$, respectively, are compared with $v_{\gamma,b}$ (blue line) when b is as before. In the former case $x_{2^{2.2}} \approx 0.1411$ and $b_{2^{2.2}} \approx 0.1542$, and in the latter $x_{2^{0.8}} \approx 0.1472$ and $b_{2^{0.8}} = 0.0833$.

A Proofs of some technical results

In the upcoming subsections, we will present the proofs for the results discussed in Section 3. To establish $v_{\gamma,b}$ as in (3.2), (3.23) and (3.42), we will utilize the following results. Given the assumption that $v_{\gamma,b}$ is concave on some open set $\mathcal{O}$, there exists a function χ satisfying $-\ln[v'_{\gamma,b}(\chi(z))] = z$. This transformation has been applied in numerous optimal control problems; see e.g. [13, 5] and references therein. Additionally, it is worth noting that considering $\lambda_\gamma, \lambda_+$ as the negative and positive roots of (3.3), respectively, the function

$$v(x) = c_3\,\mathrm{e}^{\lambda_\gamma[x-\max\{x_b,b\}]} + c_4\,\mathrm{e}^{\lambda_+[x-\max\{x_b,b\}]} + \frac{\gamma}{\gamma+\delta}\left[x - b + v(b) + \frac{\eta}{\gamma+\delta}\right] \tag{A.1}$$

satisfies the equation $\frac{1}{2}\sigma^2 v''(x) + \eta v'(x) - \delta v(x) + \gamma\{x - b_\gamma + v(b_\gamma) - v(x)\} = 0$, for $x > \max\{x_b, b\}$. Here, c_3 and c_4 are parameters to be found. However, since our solution must have a linear growth, we conclude that $c_4 = 0$.

A.1 Proofs of Propositions 3.1, 3.4, and Lemmas 3.2, 3.3

Proof of Proposition 3.1. Since $v_{\gamma,b}(0) = 0$, a solution of (3.1) has the form $v_{\gamma,b}(x) = k_1 h_1(x)$ for $x \in (0, b)$, otherwise, i.e. for $x > b$, $v_{\gamma,b}(x)$ is given by (A.1), with $\max\{x_b, b\} = b$ and $c_4 = 0$. Here k_1, c_3 are parameters to be found, and h_1 is as in (3.4). To determine the values of the parameters, applying smooth fit at $b > 0$, we have that

$$k_1 h_1(b) = c_3 + \frac{\gamma}{\gamma+\delta}\left\{k_1 h_1(b) + \frac{\eta}{\gamma+\delta}\right\} \quad \text{and} \quad k_1 h_1'(b) = c_3\lambda_\gamma + \frac{\gamma}{\gamma+\delta}. \tag{A.2}$$

Solving the system of equations above w.r.t. k_1, c_3, it can be easily checked that $c_{1,1}(b)$ and $c_{1,2}(b)$ as in (3.6) satisfy (A.2). From here, we have immediately that $v_{\gamma,b}$ given as in (3.2) is C^2-continuous on $(0,\infty) \setminus \{b\}$. By (3.1), we have for $x > b$, $v''_{\gamma,b}(x) = -\frac{2}{\sigma^2}\{\eta v'_{\gamma,b}(x) - \delta v_{\gamma,b}(x) + \gamma[x - b + v_{\gamma,b}(b) - v_{\gamma,b}(x)]\}$. Then, letting $x \downarrow b$ in the equation above, it follows that $v''_{\gamma,b}(b+) = -\frac{2}{\sigma^2}\{\eta v'_{\gamma,b}(b) - [\delta+\gamma]v_{\gamma,b}(b) = v''_{\gamma,b}(b-)$. Therefore, v is C^2-continuous at b. ■

Proof of Lemma 3.2. Let g_1 be as in (3.9). Calculating the first derivative of g_1 and considering h_1 as in (3.4) and its first and second derivative, we get that $g_1'(b) = \frac{[h_1'(b)]^2 - h_1(b)h_1''(b)}{[h_1'(b)]^2} = \frac{[\theta_+-\theta_-]^2\,\mathrm{e}^{[\theta_+-\theta_-]b}}{[h_1'(b)]^2}$ which is positive. Then, by the seen before, we have that $g_1' > 0$ on $(0,\infty)$, which is enough to conclude that g is strictly increasing on $(0,\infty)$. By definition of h_1, it is easy to verify that $\lim_{b\downarrow 0} g_1 = 0$ and $\lim_{b\uparrow\infty} g_1(x) = \lim_{b\to\infty} \frac{1-\mathrm{e}^{[\theta_- - \theta_+]b}}{\theta_+ - \theta_-\,\mathrm{e}^{[\theta_- - \theta_+]b}} = \frac{1}{\theta_+}$. ■

Proof of Lemma 3.3. Let f_1 be as in (3.11). Since the functions $\gamma \mapsto \frac{\eta\gamma}{\delta[\delta+\gamma]}$ and $\gamma \mapsto \frac{\sigma^2}{\eta + \sqrt{\eta^2 + 2\sigma^2(\delta+\gamma)}}$ are strictly increasing and strictly decreasing on $(0,\infty)$, respectively, it immediately follows that f_1 is strictly increasing on $(0,\infty)$. The reader can quickly verify that the limits given in Lemma 3.3 hold. ■

Proof of Proposition 3.4. Assuming that $\mu \geq 2\eta$ and $\gamma > \gamma_0$, it is known that there exists $b_\gamma > 0$ satisfying (3.8). Take v_{γ,b_γ} as in (3.2) when $b = b_\gamma$. Observe that $c_{1,1}(b_\gamma)$, defined in (3.6), is positive because of $\lambda_\gamma < 0$ and $h_1'(b_\gamma) - \frac{\delta\lambda_\gamma}{\gamma+\delta}h_1(b_\gamma) = \frac{h_1'(b_\gamma)}{\gamma+\delta}\big[\gamma - \frac{\gamma\eta\lambda_\gamma}{\gamma+\delta}\big] > 0$. It is worth noting that v_{γ,b_γ} inherits the increasing property of h_1 on $(0, b_\gamma)$, due to $h_1' > 0$ on $(0, b_\gamma)$. Since h_1' attains its unique minimum value at $x_1 = \frac{2}{\theta_+ - \theta_-}\ln\big[-\frac{\theta_-}{\theta_+}\big]$, it is sufficient to verify that $h_1''(b_\gamma) < h_1''(x_1) = 0$ in order to establish that $b_\gamma < x_1$, and consequently conclude that v_{γ,b_γ} is concave on $(0, b_\gamma)$. It is known

$$\frac{\sigma^2}{2}h_1'' + \eta h_1' - \delta h_1 = 0 \quad \text{on } (0,\infty). \tag{A.3}$$

From here and using (3.8), it gives that $h_1'' = \frac{2h_1'}{\sigma^2}[\delta g_1 - \eta]$ on $(0,\infty)$. Then $h_1''(b_\gamma) = \frac{2h_1'(b_\gamma)}{\sigma^2}\big[\frac{1}{\lambda_\gamma}\big[\delta + \frac{\gamma\eta\lambda_\gamma}{\gamma+\delta}\big] - \eta\big] = \frac{2\delta h_1'(b_\gamma)}{\sigma^2}\big[\frac{1}{\lambda_\gamma} - \frac{\eta}{\gamma+\delta}\big] < 0$, because of $h_1'(b_\gamma) > 0$ and $\lambda_\gamma < 0$. Now, let us see that v_{γ,b_γ} is increasing and concave on (b_γ, ∞). For this purpose, it is sufficient to check that

$$c_{1,2}(b_\gamma) < 0, \tag{A.4}$$

because if (A.4) holds, it follows immediately that $v'_{\gamma,b_\gamma} > 0$ and $v''_{\gamma,b_\gamma} < 0$ on (b_γ,∞). Using (3.6), (3.9) and (3.15), we see that (A.4) is equivalent to $0 < \delta[\delta+\gamma-\delta\eta\lambda_\gamma]$, which is true due to $\lambda_\gamma < 0 < \delta$. By the seen before, it is known that v'_{γ,b_γ} is strictly decreasing on $(0,\infty)$. Moreover, considering that b_γ is the unique solution to (3.8), it follows that $v'_{\gamma,b_\gamma}(b_\gamma-) = \frac{\frac{\gamma}{\gamma+\delta}\left[1-\frac{\eta\lambda_\gamma}{\gamma+\delta}\right]}{1-\frac{\delta\lambda_\gamma}{\gamma+\delta}g_1(b_\gamma)} = 1$. It implies that $b_\gamma = \inf\{x > 0 : v'_{\gamma,b_\gamma}(x) < 1\} < \infty$. Let us prove now that $\hat{f}_1 := v'_{\gamma,b_\gamma}/v''_{\gamma,b_\gamma}$ is decreasing on $(0,\infty)$. Calculating the first and second derivatives of v_{γ,b_γ}, and taking into account (A.3), it gives

$$\hat{f}_1(x) = \begin{cases} \frac{\sigma^2}{2[\delta g_1(x)-\eta]} & \text{if } x \in (0,b_\gamma),\\ \frac{\gamma\,\mathrm{e}^{-\lambda_\gamma[x-b_\gamma]}}{c_{1,2}(b_\gamma)\lambda_\gamma^2[\gamma+\delta]} + \frac{1}{\lambda_\gamma} & \text{if } x \in (b_\gamma,\infty). \end{cases}$$

From here we see that $\hat{f}_1$ is decreasing on $(0,\infty)$ since $\delta g_1 - \eta$ is increasing in $(0,b_\gamma)$, and $\hat{f}_1'(x) = -\frac{\gamma\,\mathrm{e}^{-\lambda_\gamma[x-b_\gamma]}}{c_{1,2}(b_\gamma)\lambda_\gamma[\gamma+\delta]} < 0$ for $x \in (b_\gamma,\infty)$. By the seen before, we get that $-\frac{\mu}{\sigma^2}\hat{f}_1$ is increasing on $(0,\infty)$ and $-\frac{\mu}{\sigma^2}\hat{f}_1(0+) = \frac{\mu}{2\eta} \geq 1$. Therefore $x_\gamma = 0$. Since v_{γ,b_γ} satisfies the hypotheses of Proposition 2.2, we conclude that v_{γ,b_γ} is a solution to the HJB equation (2.5).

Now, let us check the veracity of Item (ii). For each $\gamma \in (0,\gamma_0)$ fixed, let us take $v_{\gamma,0}$ as in (3.14). By similar arguments seen before, it is easy to check that $v_{\gamma,0}$ is an increasing and concave solution to (3.1). Notice that $v'_{\gamma,0}(0+) = -\frac{\gamma\eta\lambda_\gamma}{[\gamma+\delta]^2} + \frac{\gamma}{\gamma+\delta}$, which is less or equal than one if and only if

$$-\lambda_\gamma < \frac{\delta[\gamma+\delta]}{\eta\gamma}. \tag{A.5}$$

Since

$$\lambda_\gamma = \frac{-\eta - \sqrt{\eta^2 + 2\sigma^2[\delta+\gamma]}}{\sigma^2}, \tag{A.6}$$

we see that (A.5) is equivalent to verify that $\gamma < \frac{1}{2}\left[\frac{\sigma\delta}{\eta}\right]^2$. Meanwhile, observe that $f_1\left(\frac{1}{2}\left[\frac{\sigma\delta}{\eta}\right]^2\right) = 0$, with f_1 as in (3.11). It implies that $\gamma_0 = \frac{1}{2}\left[\frac{\sigma\delta}{\eta}\right]^2$, since f_1 is strictly decreasing on $(0,\infty)$ and γ_0 is the unique point where $f_1(\gamma_0) = 0$. From here (A.5) is true. Concluding that $v'_{\gamma,0}(0+) \leq 1$ and $b_\gamma = \inf\{x > 0 : v'_{\gamma,0}(x) \leq 1\} = 0$. On the other hand, $\frac{v'_{\gamma,0}(x)}{v''_{\gamma,0}(x)} = \frac{1}{\lambda_\gamma} - \frac{[\gamma+\delta]\,\mathrm{e}^{-\lambda_\gamma x}}{\eta\lambda_\gamma^2}$ is decreasing on $(0,\infty)$. Thus, $-\frac{\mu v'_{\gamma,0}}{\sigma^2 v''_{\gamma,0}}$ is increasing on the same domain mentioned before, and

$$-\frac{\mu v'_{\gamma,0}(0+)}{\sigma^2 v''_{\gamma,0}(0+)} = -\frac{\mu}{\sigma^2\lambda_\gamma}\left[1 - \frac{\gamma+\delta}{\lambda_\gamma\eta}\right] = \frac{\mu}{2\eta}. \tag{A.7}$$

The last equality in (A.7) is obtained because of (A.6). By the seen before and invoking Proposition 2.2, we conclude that $v_{\gamma,0}$, given in (3.14) is a solution to the HJB equation (2.5).

To finish, let us see that $v_{\gamma,0}$ as in (3.14), when $\gamma = \gamma_0$, satisfies also (2.5). Consider v_{γ,b_γ} as in (3.2) when $b = b_\gamma$. Then, by the seen in Subsection 4.1, we get that for each $x \in (0,\infty)$, $v_{\gamma,b_\gamma}(x) \xrightarrow[\gamma\downarrow\gamma_0]{} v_{\gamma_0,0}(x)$, with $v_{\gamma_0,0}$ as in (3.14). Furthermore, $v_{\gamma,b_\gamma}(b_\gamma) \xrightarrow[\gamma\downarrow\gamma_0]{} v_{\gamma_0,0}(0) = 0$ and $v'_{\gamma,b_\gamma}(b_\gamma) \xrightarrow[\gamma\downarrow\gamma_0]{} v'_{\gamma_0,0}(0) = 1$. By the same arguments seen in the case $\gamma < \gamma_0$, it follows that $v_{\gamma,0}$ as in (3.14), when $\gamma = \gamma_0$, satisfies also (2.5). ∎

A.2 Proofs of Propositions 3.5, 3.8, and Lemma 3.7

Proof of Proposition 3.5. Consider $b > \bar{x}$, with $\bar{x}$ as in (3.18). Let us suppose that the solution $v_{\gamma,b}$ to (3.20)–(3.22) is concave on $(0,x_b)$ which will be proven later on. Here, $x_b \leq b$ is a parameter to be determined. Then, by this assumption, it is known that there exists $\chi_1 : (-\infty, \bar{z}) \longrightarrow [0, \infty)$ satisfying $-\ln(v'_{\gamma,b}(\chi_1(z))) = z$, for some $\bar{z} \in \mathbb{R}$. It implies (3.28). Applying (3.28) in (3.20), it follows that

$$\frac{\mu^2 \chi_1'(z)\,\mathrm{e}^{-z}}{2\sigma^2} - [\mu - \eta]\,\mathrm{e}^{-z} - \delta v_{\gamma,b}(\chi_1(z)) = 0. \tag{A.8}$$

Taking first derivatives w.r.t. z in (A.8),

$$\chi_1''(z) - [1 + \delta\bar{\eta}]\chi_1'(z) + \bar{\eta}[\mu - \eta] = 0, \tag{A.9}$$

where $\bar{\eta}$ is defined in (3.19). A solution to (A.9) is given by

$$\chi_1(z) = k_1\,\mathrm{e}^{[1+\delta\bar{\eta}]z} + \frac{\bar{\eta}[\mu - \eta]}{1 + \delta\bar{\eta}} z + k_2. \tag{A.10}$$

where k_1 and k_2 are positive constants that shall be given after. Taking $M > 0$ such that $\chi_1(-M+) = 0$ and $v'_{\gamma,b}(\chi_1(-M+)) = \mathrm{e}^{M}$, the function χ_1 is from $[-M, \bar{z})$ to $[0, x_b)$, where $\bar{z} > -M$ such that $\chi_1(\bar{z}) = x_b$. It implies that $k_1 = -\,\mathrm{e}^{M[1+\delta\bar{\eta}]}\left[k_2 - \frac{\bar{\eta}M[\mu-\eta]}{1+\delta\bar{\eta}}\right]$. By the assumption of concavity of $v_{\gamma,b}$ on $(0, x_b)$, it implies that χ_1^{-1} is an increasing function on $[0, x_b)$. Notice that

$$\chi_1'(z) = k_1[1 + \delta\bar{\eta}]\,\mathrm{e}^{[1+\delta\bar{\eta}]z} + \frac{\bar{\eta}[\mu - \eta]}{1 + \delta\bar{\eta}}. \tag{A.11}$$

Then, from here and (A.8), we have that a solution to (3.20) is represented by

$$\begin{aligned}
v_{\gamma,b}(x) &= \frac{\mathrm{e}^{-\chi_1^{-1}(x)}}{\bar{\eta}\delta}[\chi_1'(\chi_1^{-1}(x)) + \bar{\eta}[\eta - \mu]] \\
&= \frac{\mathrm{e}^{-\chi_1^{-1}(x)}}{\bar{\eta}\delta}\left[k_1[1 + \delta\bar{\eta}]\,\mathrm{e}^{[1+\delta\bar{\eta}]\chi_1^{-1}(x)} - \frac{\bar{\eta}[\eta - \mu]}{1 + \delta\bar{\eta}} + \bar{\eta}[\eta - \mu]\right] \\
&= \frac{\mathrm{e}^{-\chi_1^{-1}(x)}[\eta - \mu]}{\delta}\left[-\left[k_2 + \frac{\bar{\eta}M[\eta - \mu]}{1 + \delta\bar{\eta}}\right]\left[\frac{1 + \delta\bar{\eta}}{\bar{\eta}[\eta - \mu]}\right]\mathrm{e}^{[1+\delta\bar{\eta}][M+\chi_1^{-1}(x)]} + \frac{\delta\bar{\eta}}{1 + \delta\bar{\eta}}\right].
\end{aligned}$$

Since $v_{\gamma,b}(0) = 0$, we have that

$$k_2 = c_{2,1}\left[M - \frac{\delta\bar{\eta}}{1 + \delta\bar{\eta}}\right]. \tag{A.12}$$

From here notice that

$$k_1 = \frac{\delta\bar{\eta}c_{2,1}}{1 + \delta\bar{\eta}}\,\mathrm{e}^{M[1+\delta\bar{\eta}]}. \tag{A.13}$$

Then,

$$\begin{aligned}
v_{\gamma,b}(x) &= \frac{\mathrm{e}^{-\chi_1^{-1}(x)}[\eta - \mu]}{\delta}\left[-\frac{\delta\bar{\eta}}{1 + \delta\bar{\eta}}\,\mathrm{e}^{[1+\delta\bar{\eta}][M+\chi_1^{-1}(x)]} + \frac{\delta\bar{\eta}}{1 + \delta\bar{\eta}}\right] \\
&= c_{2,1}\,\mathrm{e}^{-\chi_1^{-1}(x)}\left[\mathrm{e}^{[1+\delta\bar{\eta}][M+\chi_1^{-1}(x)]} - 1\right] \quad \text{for } x \in (0, x_b),
\end{aligned} \tag{A.14}$$

with $c_{2,1}$ as in (3.19). Observe that $v_{\gamma,b}$ is concave on $(0, x_b)$, since $v''_{\gamma,b}$ is non-positive on $(0, x_b)$, because of (3.28) and $\chi_1' \geq 0$ on $(-M, \bar{z})$. From (3.16), (A.11) and (A.14), it yields that u_b^* has the following form for each $x \in (0, x_b)$, $u_b^*(x) = \frac{\mu}{\sigma^2}\chi_1'(\chi_1^{-1}(x)) = \frac{2[\mu-\eta]}{\mu[1+\delta\bar{\eta}]}\{\delta\bar{\eta}\,\mathrm{e}^{[1+\delta\bar{\eta}][M+\chi_1^{-1}(x)]}+1\}$. We see that u_b^* is increasing on $(0, x_b)$ and

$$\lim_{x\downarrow 0} u_b^*(x) = \frac{2[\mu-\eta]}{\mu} < 1, \tag{A.15}$$

because of $\mu < 2\eta$. Then, by (3.16), we have that x_b is determined by the condition $u_b^*(x_b-) = 1$, which is equivalent to $\frac{2[\mu-\eta]}{\mu[1+\delta\bar{\eta}]}\{\delta\bar{\eta}\,\mathrm{e}^{[1+\delta\bar{\eta}][M+\chi_1^{-1}(x_b)]}+1\} = 1$. From here, using (A.10) and considering that χ_1^{-1} is the inverse function of χ_1, it gives that $\bar{z}$ is as in (3.25) and

$$\begin{aligned}
x_b &= \chi_1\left(\frac{1}{1+\delta\bar{\eta}}\ln[c_{2,2}] - M\right)\\
&= k_1 c_{2,2}\,\mathrm{e}^{-[1+\delta\bar{\eta}]M} + \frac{\bar{\eta}[\mu-\eta]}{[1+\delta\bar{\eta}]^2}\ln[c_{2,2}] - \frac{\bar{\eta}[\mu-\eta]}{1+\delta\bar{\eta}}\frac{\delta\bar{\eta}}{1+\delta\bar{\eta}}\\
&= \frac{c_{2,1}}{1+\delta\bar{\eta}}\ln[c_{2,2}] + \frac{\bar{\eta}[2\eta-\mu]}{2[1+\delta\bar{\eta}]} = \bar{x}.
\end{aligned}$$

Recall that $c_{2,2}$ and k_1 are as in (3.19) and (A.13). On the other hand, The NLPD equation (3.21) admits the following solution

$$v_{\gamma,b}(x) = \begin{cases} c_1\,\mathrm{e}^{\theta_-[x-x_b]} + c_2\,\mathrm{e}^{\theta_+[x-x_b]} & \text{for } x \in (x_b, b),\\ c_3\,\mathrm{e}^{\lambda_\gamma[x-b]} + \frac{\gamma}{\gamma+\delta}\left[x - b + v_{\gamma,b}(b) + \frac{\eta}{\delta+\gamma}\right] & \text{for } x \in (b, \infty),\end{cases}$$

where c_1, c_2, c_3 are free constants. Notice that

$$\begin{aligned}
v_{\gamma,b}(x_b-) &= \frac{\mathrm{e}^{-\chi_1^{-1}(x_b)}\,\bar{\eta}[\mu-\eta]}{1+\delta\bar{\eta}}\left[\mathrm{e}^{[1+\delta\bar{\eta}][M+\chi_1^{-1}(x_b)]} - 1\right] = \mathrm{e}^{M}\,\frac{c_{2,2}^{-1/[1+\delta\bar{\eta}]}[2\eta-\mu]}{2\delta},\\
v'_{\gamma,b}(x_b-) &= \mathrm{e}^{-\chi_1^{-1}(x_b)} = \mathrm{e}^{M}\,c_{2,2}^{-1/[1+\delta\bar{\eta}]},
\end{aligned}$$

because of

$$\begin{aligned}
\chi_1^{-1}(x_b) &= \chi_1^{-1}\left(\chi_1\left(\frac{1}{1+\delta\bar{\eta}}\ln[c_{2,2}] - M\right)\right) = \frac{1}{1+\delta\bar{\eta}}\ln[c_{2,2}] - M,\\
\mathrm{e}^{-\chi_1^{-1}(x_b)} &= \exp\left[-\frac{1}{1+\delta\bar{\eta}}\ln[c_{2,2}] + M\right] = c_{2,2}^{-1/[1+\delta\bar{\eta}]}\,\mathrm{e}^{M},\\
\mathrm{e}^{[1+\delta\bar{\eta}]\chi_1^{-1}(x_b)} &= \exp\left[\ln[c_{2,2}] - [1+\delta\bar{\eta}]M\right] = c_{2,2}\,\mathrm{e}^{-[1+\delta\bar{\eta}]M}.
\end{aligned}$$

Meanwhile, $v_{\gamma,b}(x_b+) = c_1 + c_2$ and $v'_{\gamma,b}(x_b+) = c_1\theta_- + c_2\theta_+$. In order that $v_{\gamma,b}$ is C^1-continuous at x_b, by smooth fit, we have the following system of equations

$$c_1 + c_2 = \mathrm{e}^{M}\,\frac{c_{2,2}^{-1/[1+\delta\bar{\eta}]}[2\eta-\mu]}{2\delta} \quad \text{and} \quad c_1\theta_- + c_2\theta_+ = \mathrm{e}^{M}\,c_{2,2}^{-1/[1+\delta\bar{\eta}]},$$

whose solution is given by $c_1 = -\frac{\mathrm{e}^{M}\,c_{2,2}^{-1/[1+\delta\bar{\eta}]}}{\theta_+-\theta_-}\left[1 - \theta_+\frac{[2\eta-\mu]}{2\delta}\right]$ and $c_2 = \frac{\mathrm{e}^{M}\,c_{2,2}^{-1/[1+\delta\bar{\eta}]}}{\theta_+-\theta_-}\left[1 - \theta_-\frac{[2\eta-\mu]}{2\delta}\right]$. From here, we have that for $x \in (x_b, b)$, $v_{\gamma,b}(x) = \mathrm{e}^{M}\,c_{2,2}^{-1/[1+\delta\bar{\eta}]}h_2(x - x_b)$, where h_2 is defined in (3.26). Then,

$$v_{\gamma,b}(b-) = \mathrm{e}^{M}\,c_{2,2}^{-1/[1+\delta\bar{\eta}]}h(b - x_b), \quad v'_{\gamma,b}(b-) = \mathrm{e}^{M}\,c_{2,2}^{-1/[1+\delta\bar{\eta}]}h'(b - x_b),$$

and

$$\begin{aligned} v_{\gamma,b}(b+) &= c_3 + \frac{\gamma}{\gamma+\delta}\left[\mathrm{e}^{M}\, c_{2,2}^{-1/[1+\delta\bar{\eta}]} h(b-x_b) + \frac{\eta}{\delta+\gamma}\right], \\ v'_{\gamma,b}(b+) &= c_3\lambda_\gamma + \frac{\gamma}{\gamma+\delta}. \end{aligned} \tag{A.16}$$

Since $v_{\gamma,b}(b-) = v_{\gamma,b}(b+)$, we have that

$$c_3 = \frac{\mathrm{e}^{M}\, \delta c_{2,2}^{-1/[1+\delta\bar{\eta}]}}{\delta+\gamma} h(b-x_b) - \frac{\eta\gamma}{[\delta+\gamma]^2}. \tag{A.17}$$

Substituting (A.17) into (A.16), and taking into account that $v'_{\gamma,b}(b-) = v'(b+)$ must be occurred, it gives that

$$\mathrm{e}^{M}\, c_{2,2}^{-1/[1+\delta\bar{\eta}]} h'(b-x_b) = \lambda_\gamma \left\{ \frac{\mathrm{e}^{M}\, \delta c_{2,2}^{-1/[1+\delta\bar{\eta}]}}{[\delta+\gamma]} h(b-x_b) - \frac{\eta\gamma}{[\delta+\gamma]^2}\right\} + \frac{\gamma}{\gamma+\delta}.$$

Solving the previous equation w.r.t. M, it yields that $M = M(b)$, where $M(b)$ is given by (3.27), which depends on b. Therefore, (3.23) is a solution to the NLPDS (3.20)–(3.22). By the construction seen above, we know that $v_{\gamma,b}$ is is $\mathrm{C}^1((0,\infty)) \cap \mathrm{C}^2((0,\infty)\setminus\{x_b, b\})$. However, since $v_{\gamma,b}$ is C^1-continuous on $(0,\infty)$ and satisfies (3.20)–(3.21) in the neighbourhoods of x_b and b, it follows $v''_{\gamma,b}$ is continuous at x_b and at b. ∎

Proof of Lemma 3.7. Let g_2 be as in (3.30). In order to demonstrate that g_2 is an increasing function on $(0,\infty)$, it suffices to show that $g'_2 > 0$ on $(0,\infty)$. Notice that $g'_2(b) = \frac{[h'_2(b)]^2 - h_2(b)h''_2(b)}{[h'_2(b)]^2} = \frac{a_1a_2\,\mathrm{e}^{[\theta_+ + \theta_-]b}}{[h'_2(b)]^2}$ for $b \in (0,\infty)$, where θ_- and θ_+ are the negative and positive root of (3.5), and a_1 and a_2 are defined in (3.27). Considering that $-\mu[2\eta-\mu] < 2\delta\sigma^2$, it can be verified easily that $\theta_+ < \frac{2\delta}{2\eta-\mu}$. From here it gives that $a_1a_2 > 0$. Thus, $g'_2 > 0$ on $(0,\infty)$. By definition of h_2, a_1 and a_2, we get that

$$\begin{aligned} \lim_{b\downarrow 0} \frac{h_2(b)}{h'_2(b)} &= \frac{a_1 - a_2}{a_1\theta_+ - a_2\theta_-} = \frac{2\eta-\mu}{2\delta}, \\ \lim_{b\uparrow\infty} \frac{h_2(b)}{h'_2(b)} &= \lim_{b\uparrow\infty} \frac{a_1 - a_2\,\mathrm{e}^{[\theta_- - \theta_+]x}}{a_1\theta_+ - a_2\theta_-\,\mathrm{e}^{[\theta_- - \theta_+]x}} = \frac{1}{\theta_+}. \end{aligned}$$

∎

Proof of Proposition 3.8. We know that b_γ is given by (3.31), due to $\mu < 2\eta$ and $\gamma > \gamma_1$. Consider v_{γ,b_γ} as in (3.23) (when $b = b_\gamma$). By (3.28) we see that (3.28) holds. Therefore v_{γ,b_γ} is concave and increasing on $(0,\bar{x})$, with $\bar{x}$ as in (3.18). notice that $h'_2 > 0$ on $(0,\infty)$. It implies that v_{γ,b_γ} is increasing on $(\bar{x}, b_\gamma)$. To verify that v_{γ,b_γ} is concave on $(b_\gamma,\bar{x})$, by similar arguments seen in the proof of Proposition 3.4, we only need to verify that $h''_2(b_\gamma - \bar{x}) < h''_2(x_2 - \bar{x}) = 0$, where $x_2 := \bar{x} + \frac{1}{\theta_+ - \theta_-}\ln\left[\frac{a_2\theta_-^2}{a_1\theta_+^2}\right]$ is the point where $h'_2(x-\bar{x})$ attains its global minimum. Since h_2 satisfies (A.3), it follows that $h''_2(b_\gamma - \bar{x}) = \frac{2h'_2(b_\gamma-\bar{x})}{\sigma^2}\big[\frac{1}{\lambda_\gamma}\big[\delta + \frac{\gamma\eta\lambda_\gamma}{\gamma+\delta}\big] - \eta\big] = \frac{2\delta h'_2(b_\gamma-\bar{x})}{\sigma^2}\big[\frac{1}{\lambda_\gamma} - \frac{\eta}{\gamma+\delta}\big] < 0$. To verify that v_{γ,b_γ} is increasing and concave on (b,∞), we only need to check that $c_{2,3}(b_\gamma) < 0$, which is true because of $0 < \delta[\delta+\gamma-\delta\eta\lambda_\gamma]$. By the seen before, v'_{γ,b_γ} is decreasing on $(0,\infty)$. Furthermore, since b_γ as in (3.31) satisfies (3.29), it immediately follows that $v'_{\gamma,b_\gamma}(b_\gamma-) = \frac{\frac{\gamma}{\gamma+\delta}\big[1 - \frac{\eta\lambda_\gamma}{\gamma+\delta}\big]}{1 - \frac{\delta\lambda_\gamma}{\gamma+\delta}g_2(b_\gamma - \bar{x})} = 1$. Thus, $b_\gamma = \inf\{x > 0 : v'_{\gamma,b_\gamma}(x) < 1\} < \infty$. Let us prove now that $\hat{f}_1 = v'_{\gamma,b_\gamma}/v''_{\gamma,b_\gamma}$ is decreasing on $(0,\infty)$. Calculating the first and second derivatives of v_{γ,b_γ}, and taking into account (3.28) and (A.3), it gives

$$\hat{f}_1(x) = \begin{cases} -\chi_1'(\chi_1^{-1}(x)) & \text{if } x \in (0,\bar{x}), \\ \frac{\sigma^2}{2[\delta g_2(x)-\eta]} & \text{if } x \in (\bar{x}, b_\gamma), \\ \frac{\gamma\, \mathrm{e}^{-\lambda_\gamma[x-b_\gamma]}}{c_{2,3}(b_\gamma)\lambda_\gamma^2[\gamma+\delta]} + \frac{1}{\lambda_\gamma} & \text{if } x \in (b_\gamma,\infty). \end{cases}$$

Using similar arguments seen in the Proof of Proposition 3.4, it can be seen that $\hat{f}_1$ is decreasing on $(0,\infty)$. It implies that $-\frac{\mu}{\sigma^2}\hat{f}_1$ is increasing on $(0,\infty)$ and $-\frac{\mu}{\sigma^2}\hat{f}_1(0+) = \frac{2[\mu-\eta]}{\mu} < 1$. Therefore $x_\gamma = \bar{x}$. Since v_{γ,b_γ} satisfies the hypotheses of Proposition 2.2, we conclude that v_{γ,b_γ} is a solution to the HJB equation (2.5). ■

A.3 Proofs of Proposition 3.9, 3.13 and Lemma 3.11

Proof of Proposition 3.9. Let $b \in (0,\underline{b})$, with $\underline{b}$ as in (3.36). Let us assume that $b < x_b$, which will be proven later on. Here, x_b is a parameter to be determined. By the seen in the proof of Proposition 3.5, we know that $v_{\gamma,b}(x) = c_{2,1}\,\mathrm{e}^{-\chi_1^{-1}(x)}\left[\mathrm{e}^{[1+\delta\bar{\eta}][M+\chi_1^{-1}(x)]} - 1\right]$ for $x \in (0,b)$, is a solution to (3.39) on $(0,b)$ where χ_1^{-1} is the inverse function of $\chi_1 : [-M,-M_2] \longrightarrow [0,b]$ which is given by (A.10) with k_2 and k_1 as in (A.12) and (A.13), respectively. Here, M and M_2 (with $M_2 \le M$) are unknown parameters where $\chi_1(-M) = 0$ and $\chi_1(-M_2) = b$. So, we shall construct the solution to (3.39) on (b, x_b). Denoting $\bar{v}$ as a solution to

$$-\frac{\mu^2[v'(x)]^2}{2\sigma^2 v''(x)} - [\mu-\eta]\bar{v}'(x) - [\delta+\gamma]v(x) + \gamma x = 0 \quad \text{for } x \in (b, x_b), \tag{A.18}$$

we see easily that

$$v_{\gamma,b}(x) = \bar{v}(x) + \frac{\gamma[v_{\gamma,b}(b) - b]}{\delta+\gamma} \tag{A.19}$$

is a solution to (3.39) for $x \in (b, x_b)$. Assuming concavity of $\bar{v}$, which will be proven later on, it is known that there exists $\chi_2 : \mathbb{R} \longrightarrow [0,\infty)$ satisfying $-\ln(\bar{v}'(\chi_2(z))) = z$. Then, (3.45) holds. Substituting (3.45) in (A.18), it follows that

$$\frac{\mu^2 \chi_2'(z)\,\mathrm{e}^{-z}}{2\sigma^2} - [\mu-\eta]\,\mathrm{e}^{-z} - [\delta+\gamma]\bar{v}(\chi_2(z)) + \gamma\chi_2(z) = 0. \tag{A.20}$$

Taking first derivatives w.r.t. z in (A.20), it yields

$$\chi_2''(z) - \{1 + \bar{\eta}[\delta+\gamma] - \gamma\bar{\eta}\,\mathrm{e}^z\}\chi_2'(z) + \bar{\eta}[\mu-\eta] = 0 \tag{A.21}$$

where $\bar{\eta}$ is as in (3.19). Taking $M_2 \ge 0$ such that $\chi_2(-M_2+) = b$ and $v'(\chi_2(-M_2+)) = \mathrm{e}^{M_2}$, the function χ_2 is from $[-M_2, \bar{z}_1)$ to $[b, x_b)$, with $\bar{z}_1 > -M_2$ such that $\chi_2(\bar{z}_1) = x_b$. It implies that χ_2' has the following solution

$$\begin{aligned} \chi_2'(z) &= k_3 \exp[\{1+\bar{\eta}[\delta+\gamma]\}z - \gamma\bar{\eta}\,\mathrm{e}^z] \\ &\quad - \bar{\eta}[\mu-\eta]\exp[\{1+\bar{\eta}[\delta+\gamma]\}z - \gamma\bar{\eta}\,\mathrm{e}^z]\int_{-M_2}^{z} \exp[-\{1+\bar{\eta}[\delta+\gamma]\}y + \gamma\bar{\eta}\,\mathrm{e}^y]\mathrm{d}y \\ &= \mathrm{e}^z\, g(\mathrm{e}^z)\left[k_3 - \bar{\eta}[\mu-\eta]\int_{-M_2}^{z} \frac{1}{\mathrm{e}^y\, g(\mathrm{e}^y)}\mathrm{d}y\right], \end{aligned} \tag{A.22}$$

where k_3 is a parameter to be determined, and g is the density function of a gamma distribution given by (3.35). From here, it follows that

$$\varkappa_2(z) = \bar{f}_{M_2}(\mathrm{e}^z) + b \tag{A.23}$$

is a solution to (A.21), with H_{M_2} as in (3.32) when $\beta = M_2$, and

$$\begin{aligned} \bar{f}_{M_2}(z) &:= \int_{\mathrm{e}^{-M_2}}^{z} \big\{ -\bar{\eta}[\mu-\eta]g(y)H_{M_2}(y) + k_3 g(y)\big\}\mathrm{d}y \\ &= k_3[G(z) - G(\mathrm{e}^{-M_2})] - \bar{\eta}[\mu-\eta]\bigg[G(z)H_{M_2}(z) - \int_{\mathrm{e}^{-M_2}}^{z} \frac{G(y)}{y^2 g(y)}\mathrm{d}y\bigg]. \end{aligned} \tag{A.24}$$

Recall that G represents the gamma accumulative distribution function of g. By the assumption of concavity of $\bar{v}$, it implies that $\varkappa_2^{-1}$ is an increasing function on $[0,\infty)$. From here and since e^z is also increasing, wee have that $\bar{f}_{M_2}^{-1}$ is an increasing function. Using (A.20) and (A.22), we have that for $x \in (b, x_b)$

$$\begin{aligned} \bar{v}(x) &= \frac{\mathrm{e}^{-\varkappa_2^{-1}(x)}}{\delta+\gamma}\bigg\{\frac{\mu^2 \varkappa_2'(\varkappa_2^{-1}(x))}{2\sigma^2} - [\mu-\eta]\bigg\} + \frac{\gamma x}{\delta+\gamma} \\ &= \frac{\mathrm{e}^{-\varkappa_2^{-1}(x)}}{\delta+\gamma}\bigg\{\frac{\mu^2 g(\mathrm{e}^{\varkappa_2^{-1}(x)})\,\mathrm{e}^{\varkappa_2^{-1}(x)}}{2\sigma^2}\big[k_3 - \bar{\eta}[\mu-\eta]H_{M_2}(\mathrm{e}^{\varkappa_2^{-1}(x)})\big] - [\mu-\eta]\bigg\} + \frac{\gamma x}{\delta+\gamma} \end{aligned} \tag{A.25}$$

is a solution to (A.18). Applying (A.25) in (A.19), it follows that for $x \in (b, x_b)$,

$$v_{\gamma,b}(x) = \frac{\mathrm{e}^{-\varkappa_2^{-1}(x)}}{\delta+\gamma}\bigg\{\frac{\mu^2 g(\mathrm{e}^{\varkappa_2^{-1}(x)})\,\mathrm{e}^{\varkappa_2^{-1}(x)}}{2\sigma^2}[k_3 - \bar{\eta}[\mu-\eta]H_{M_2}(\mathrm{e}^{\varkappa_2^{-1}(x)})] - [\mu-\eta]\bigg\} + \frac{\gamma[v_{\gamma,b}(b)+x-b]}{\delta+\gamma}$$

is a solution to (3.39). Since $v'_{\gamma,b}$ must be continuous at b, by smooth fit, (3.45), (3.28), and (A.23)–(A.24), we get that $\mathrm{e}^{-\varkappa_1^{-1}(b)} = \mathrm{e}^{-\varkappa_2^{-1}(b)} = \mathrm{e}^{M_2}$, where $\varkappa_2^{-1}$ is the inverse function of (A.22). Thus,

$$\varkappa_1(-M_2) = b \quad \Longleftrightarrow \quad \bigg[\frac{\delta\bar{\eta}}{1+\delta\bar{\eta}}\bigg]\,\mathrm{e}^{[1+\delta\bar{\eta}][M-M_2]} + M - M_2 = \frac{b}{c_{2,1}} + \frac{\delta\bar{\eta}}{1+\delta\bar{\eta}}, \tag{A.26}$$

where $c_{2,1}$ is as in (3.19). On the other hand, in order that $v_{\gamma,b}$ is C^2-continuous at b, by smooth fit again and using (3.28) and (A.13), we have that $v''_{\gamma,b}(b-) = -\frac{\mathrm{e}^{M_2}}{\varkappa_1'(-M_2)} = -\frac{\mathrm{e}^{M_2}}{\varkappa_2'(-M_2)} = v''_{\gamma,b}(b+)$. From here, and by (A.11) and (A.22), it follows that

$$k_3 = \frac{c_{2,1}}{\mathrm{e}^{-M_2}\, g(\mathrm{e}^{-M_2})}\big[\delta\bar{\eta}\,\mathrm{e}^{[M-M_2][1+\delta\bar{\eta}]} + 1\big]. \tag{A.27}$$

A solution to the equation (3.40), is given by (A.1), with $\max\{x_b, b\} = x_b$ and $c_4 = 0$, where c_3 is a free constant. By smooth fit, it gives the following identities

$$v'_{\gamma,b}(x_b-) = c_3\lambda_\gamma + \frac{\gamma}{\gamma+\delta} \quad \text{and} \quad v''_{\gamma,b}(x_b-) = c_3\lambda_\gamma^2.$$

Since $v'_{\gamma,b}(x_b-) = \mathrm{e}^{-\varkappa_2^{-1}(x_b)} = \frac{1}{\bar{f}_{M_2}^{-1}(x_b-b)}$ (because of $\varkappa_2^{-1}(x) = \ln[\bar{f}_{M_2}^{-1}(x-b)]$), and considering that x_b is a point where $-\frac{\mu v'_{\gamma,b}(x_b)}{\sigma^2 v''_{\gamma,b}(x_b)} = 1$, we have the following system of equations

$$\frac{1}{\hat{\alpha}} = c_3\lambda_\gamma + \frac{\gamma}{\gamma+\delta} \quad \text{and} \quad -\frac{\mu}{\sigma^2}\frac{1}{\hat{\alpha}} = c_3\lambda_\gamma^2. \tag{A.28}$$

with $\hat{\alpha} = \bar{f}_{M_2}^{-1}(x_b - b)$. Then, solving the system (A.28), we see that $\hat{\alpha} = \alpha_\gamma$ and $c_3 = c_{3,2}(\alpha_\gamma)$ are as in (3.34) and (3.43), respectively. It implies that $x_b = \bar{f}_{M_2}(\alpha_\gamma) + b$. On the other hand, since $u_b^*(x_b) = \frac{\mu}{\sigma^2}\chi_2'(\chi_2^{-1}(x_b)) = 1$ must be occurred, it gives that

$$\frac{\mu}{\sigma^2}\,\mathrm{e}^{\chi_2^{-1}(x_b)}\,g(\mathrm{e}^{\chi_2^{-1}(x_b)})\{k_3 - \bar{\eta}[\mu-\eta]H_{M_2}(\mathrm{e}^{\chi_2^{-1}(x_b)})\} = 1 \quad \Longleftrightarrow \quad k_3 = \frac{\sigma^2}{\mu\alpha_\gamma g(\alpha_\gamma)} + \bar{\eta}[\mu-\eta]H_{M_2}(\alpha_\gamma). \tag{A.29}$$

Notice that $\bar{z}_1 = \ln[\alpha_\gamma]$. Substituting (A.27) in (A.29), we obtain that

$$\begin{aligned} \mathrm{e}^{[M-M_2][1+\delta\bar{\eta}]} &= \frac{1}{\delta\bar{\eta}}\left\{\frac{\mathrm{e}^{-M_2}\,g(\mathrm{e}^{-M_2})}{c_{2,1}}\left[\frac{\sigma^2}{\mu\alpha_\gamma g(\alpha_\gamma)} + \bar{\eta}[\mu-\eta]H_{M_2}(\alpha_\gamma)\right] - 1\right\}, \\ [M-M_2] &= \frac{1}{1+\delta\bar{\eta}}\ln\left[\frac{1}{\delta\bar{\eta}}\left\{\frac{\mathrm{e}^{-M_2}\,g(\mathrm{e}^{-M_2})}{c_{2,1}}\left[\frac{\sigma^2}{\mu\alpha_\gamma g(\alpha_\gamma)} + \bar{\eta}[\mu-\eta]H_{M_2}(\alpha_\gamma)\right] - 1\right\}\right]. \end{aligned} \tag{A.30}$$

Applying (A.30) in (A.26), (3.44) holds. If there is a unique M_2 solving (3.44), by the seen before, we conclude the results announced in Proposition 3.9.

Concavity property of $v_{\gamma,b}$. Let us now verify that $v_{\gamma,b}$ is concave on $(0,\infty)$. For this, it is sufficient to check that $v''_{\gamma,b}$ is non-positive on $(0,\infty)$, which is true on $(0,b)$ because of (3.28) and $\chi_1' \geq 0$ on $(-M,-M_2)$. To verify that $v''_{\gamma,b} \leq 0$ on (b,x_b), by (3.45), it is sufficient to show that

$$\chi_2'(\chi_2^{-1}(x)) \geq 0 \quad \text{on } (b,x_b). \tag{A.31}$$

Taking k_3 as in (A.29) and substituting this in (A.22), it gives that

$$\chi_2'(\chi_2^{-1}(x)) = \mathrm{e}^{\chi_2^{-1}(x)}\,g(\mathrm{e}^{\chi_2^{-1}(x)})\left[\frac{\sigma^2}{\mu\alpha_\gamma g(\alpha_\gamma)} + \bar{\eta}[\mu-\eta]\{H_{M_2}(\alpha_\gamma) - H_{M_2}(\mathrm{e}^{\chi_2^{-1}(x)})\}\right].$$

From here it is clear that (A.31) is true, due to $H_{M_2}(\mathrm{e}^{\chi_2^{-1}(x)}) < H_{M_2}(\alpha_\gamma)$ for $x \in (b,x_b)$. Since $c_{3,2}(\alpha_\gamma)$ given in (3.43) is negative, we conclude that $v'' < 0$ on (x_b,∞).

Increasing property of u_b^* on $(0,x_b)$. By the seen in the proof of Proposition 3.8, it is clear that $u_b^* = -\frac{\mu v'_{\gamma,b}}{\sigma^2 v''_{\gamma,b}}$ is increasing on $(0,b)$ satisfying (A.15). Notice that

$$u_b^*(x) = \frac{\mu}{\sigma^2}\chi_2'(\chi_2^{-1}(x)) = \frac{\mu}{\sigma^2}\,\mathrm{e}^{\chi_2^{-1}(x)}\,\bar{f}'_{M_2}(\mathrm{e}^{\chi_2^{-1}(x)}) = \frac{\mu}{\sigma}\bar{f}_{M_2}^{-1}(x-b)\bar{f}'_{M_2}(\bar{f}_{M_2}^{-1}(x-b)). \tag{A.32}$$

Then, to verify that u_b^* is increasing on (b,x_b), it is enough to check that $f_5(x) := \frac{\mu}{\sigma^2}x\bar{f}'_{M_2}(x)$ is increasing on $(\mathrm{e}^{-M_2},\alpha_\gamma)$, since $u_b^*(x) = f_5(\bar{f}_{M_2}^{-1}(x-b))$, and $\bar{f}_{M_2}^{-1}(x-b)$ is increasing on (b,x_b). Observe that $f_5 \geq 0$ on $(\mathrm{e}^{-M_2},\alpha_\gamma)$ and $f_5(\alpha_\gamma-) = 1$. Furthermore,

$$f_5'(x) = \frac{1}{x}\left\{[\bar{\eta}[\delta+\gamma] - \bar{\eta}\gamma x + 1]f_5(x) - \frac{\mu\bar{\eta}[\mu-\eta]}{\sigma^2}\right\}, \tag{A.33}$$

because of

$$xg'(x) + g(x) = g(x)\{\bar{\eta}[\delta+\gamma] - \bar{\eta}\gamma x + 1\}. \tag{A.34}$$

From here $f_5''(x) = \frac{1}{x}\big\{[\bar{\eta}[\delta+\gamma]-\bar{\eta}\gamma x]f_5'(x) - \gamma f_5(x)\big\}$. Assuming there exists a point $x^* \in (\mathrm{e}^{-M_2},\alpha_\gamma)$ such that $f_5'(x^*) = 0$, it gives $f_5''(x^*) = -\frac{\gamma f_5(x^*)}{x^*} < 0$, which implies that there are no local minimums on $(\mathrm{e}^{-M_2},\alpha_\gamma)$. Hence, f_5 is concave on $(\mathrm{e}^{-M_2},\alpha_\gamma)$. Moreover, letting $x \uparrow \alpha_\gamma$ in (A.33), we see that

$$f_5'(\alpha_\gamma-) = \frac{1}{\alpha_\gamma}\left\{\bar{\eta}[[\delta+\gamma] - \gamma\alpha_\gamma] + 1 - \frac{\mu\bar{\eta}[\mu-\eta]}{\sigma^2}\right\} > 0, \tag{A.35}$$

due to $1-\frac{\mu\bar{\eta}[\mu-\eta]}{\sigma^2}>0$ and $\bar{\eta}[[\delta+\gamma]-\gamma\alpha_\gamma]=-\frac{2}{\mu\lambda_\gamma}>0$. Therefore, f_5 is increasing on $(\mathrm{e}^{-M_2},\alpha_\gamma)$, concluding that u_b^* is increasing on $(0,x_b)$, and $u_b^*<1$ on $(0,x_b)$.

Decreasing property of $\bar{g}$ *on* $(0,\underline{b})$. To finish, let us determine the well solvability of (3.44). For this, it is sufficient to check that $\bar{g}$ defined in (3.37) is positive and decreasing on $(0,\underline{b})$, concluding that for each $b>0$ satisfying (3.36), there exists a unique $M_2>0$ such that (3.44) is true. For this purpose, let us first check the following function $f_6(\beta):=\frac{\mathrm{e}^{-\beta}\,g(\mathrm{e}^{-\beta})}{c_{2,1}}c_{3,1}(\beta)$, for $\beta>0$, is positive and decreasing. Calculating the first and second derivatives and taking into account (A.34), it gives that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}f_6(\beta)=f_7(\beta)\left[\frac{\bar{\eta}[\mu-\eta]}{c_{2,1}f_7(\beta)}-f_6(\beta)\right]\quad\text{and}\quad\frac{\mathrm{d}^2}{\mathrm{d}\beta^2}f_6(\beta)=-\bar{\eta}\gamma\,\mathrm{e}^{-\beta}\,f_6(\beta)-f_7(\beta)\frac{\mathrm{d}}{\mathrm{d}\beta}f_6(\beta),$$

with $f_7(\beta):=\bar{\eta}[\delta+\gamma]+1-\bar{\eta}\gamma\,\mathrm{e}^{-\beta}$. Notice that $\beta\mapsto\frac{\bar{\eta}[\mu-\eta]}{c_{2,1}f_7(\beta)}$ is decreasing and positive on $(0,\infty)$, satisfying $\lim_{\beta\downarrow0}\frac{\bar{\eta}[\mu-\eta]}{c_{2,1}f_7(\beta)}=1$ and $\lim_{\beta\uparrow\infty}\frac{\bar{\eta}[\mu-\eta]}{c_{2,1}f_7(\beta)}=\frac{\bar{\eta}[\mu-\eta]}{c_{2,1}[\bar{\eta}[\delta+\gamma]+1]}$. Meanwhile, observe that $f_6(0+)>1+\delta\bar{\eta}>\lim_{\beta\downarrow0}\frac{\bar{\eta}[\mu-\eta]}{c_{2,1}f_7(\beta)}$ due to (3.38), and by L'Hôpital's rule $\lim_{\beta\downarrow0}f_6(\beta)=\frac{\bar{\eta}[\mu-\eta]}{c_{2,1}[\bar{\eta}[\delta+\gamma]+1]}$. It implies that f_6 is decreasing on $(0,\bar{\beta})$, for some $\bar{\beta}>0$. On the other hand, assuming there exists a point $\beta^*\in(\bar{\beta},\infty)$ such that $f_6'(\beta^*)=0$, it gives $f_6''(\beta^*)=-\bar{\eta}\gamma\,\mathrm{e}^{-\beta^*}\,f_6(\beta^*)<0$, which implies that there are no local minimums on (β^*,∞). Thus, f_6 is decreasing on $(0,\infty)$. Since there exists a unique point $\bar{\beta}$ such that $f_6(\beta)>1$, for $\beta\in(0,\bar{\beta})$, and $f_6(\bar{\beta})=0$, it gives that $\lim_{\beta\uparrow\bar{\beta}}\bar{g}(\beta)=-\infty$, and thus $\bar{b}<\infty$. Calculating the first derivatives and considering that $\frac{\mathrm{d}}{\mathrm{d}\beta}f_6(\beta)<0$ for $\beta\in(0,\infty)$, we have $\frac{\mathrm{d}}{\mathrm{d}\beta}\bar{g}(\beta)=\frac{\mathrm{d}}{\mathrm{d}\beta}f_6(\beta)\big(1+\frac{1}{f_6(\beta)-1}\big)<0,\quad$ for $\beta\in(0,\bar{\beta})$. Therefore $\bar{g}$ is decreasing and positive on $(0,\bar{\beta})$. ■

Proof of Lemma 3.11. (i) Taking f_2 as in (3.48), it is easy to corroborate that $f_2(\gamma)\xrightarrow[\gamma\downarrow0]{}\infty$. In order that $f_2(\gamma_1)=1$ holds, we see that γ_1 must satisfy (4.7). Moreover, since $f_1(\gamma_1)=\frac{2\eta-\mu}{2\delta}$, with f_1 as in (3.11), it follows that (4.6) holds. Applying (4.7) in (4.6), it follows that $\gamma_1=\frac{\delta}{\mu}\big[\frac{2\delta\sigma^2}{\mu}+2\eta-\mu\big]$. With this representation of γ_1 we obtain easily that $f_2(\gamma)\xrightarrow[\gamma\uparrow\gamma_1]{}1$.

(ii) Let f_3 be as in (3.48). Notice that $\gamma f_2(\gamma)\xrightarrow[\gamma\downarrow0]{}\delta\Big[1-\frac{\mu}{\eta+\sqrt{\eta^2+2\sigma^2\delta}}\Big]$. It implies that $\lim_{\gamma\downarrow0}f_3(\gamma)=\bar{\eta}[\mu-\eta]\int_1^\infty\frac{1}{y^{\bar{\eta}\delta+2}}\mathrm{d}y=\frac{\bar{\eta}[\mu-\eta]}{\bar{\eta}\delta+1}$. On the other hand, considering that $f_2(\gamma_1)=1$, it is easy to verify that $f_3(\gamma)\xrightarrow[\gamma\uparrow\gamma_1]{}\frac{\sigma^2\,\mathrm{e}^{\bar{\eta}\gamma_1}}{\mu}$. Take $\bar{g}_1(\gamma):=\sqrt{\eta^2+2\sigma^2[\delta+\gamma]}$, $\bar{g}_2(\gamma):=\frac{\mu\sigma^2}{\bar{g}(\gamma)[\bar{g}_1(\gamma)+\eta][\bar{g}_1(\gamma)+\eta-\mu]}$ and $\bar{g}_3(\gamma):=\frac{\delta}{\gamma[\gamma+\delta]}$. Then

$$f_2'(\gamma)=\frac{f_2(\gamma)}{\gamma\bar{g}_1(\gamma)}[\gamma\bar{g}_1(\gamma)\bar{g}_2(\gamma)-\gamma\bar{g}_1(\gamma)\bar{g}_3(\gamma)].\tag{A.36}$$

Observe that $\gamma\bar{g}_1(\gamma)\bar{g}_2(\gamma)$ and $\gamma\bar{g}_1(\gamma)\bar{g}_3(\gamma)$ are increasing and decreasing on $(0,\infty)$, respectively, due to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\gamma}[\gamma\bar{g}_1(\gamma)\bar{g}_2(\gamma)]&=\frac{\mu\sigma^2[2\eta-\mu]\{\eta[\eta+\bar{g}_1(\gamma)]+\sigma^2[\gamma+2\delta]\}+2\mu\sigma^4\delta\bar{g}_1(\gamma)}{\bar{g}_1(\gamma)\{[\bar{g}_1(\gamma)+\eta][\bar{g}_1(\gamma)+\eta-\mu]\}^2}>0,\\ \frac{\mathrm{d}}{\mathrm{d}\gamma}[\gamma\bar{g}_1(\gamma)\bar{g}_3(\gamma)]&=-\frac{\eta^2+\delta^2[x+\delta]}{[x+\delta]^2\bar{g}_1(\gamma)}<0,\end{aligned}$$

for $\gamma>0$. From here, it implies that there exists $\bar{\gamma}>0$ such that $f_2'(\gamma)<0$ for $\gamma\in(0,\bar{\gamma})$. To finish the proof, let us verify that $\gamma_1\leq\bar{\gamma}$, i.e.

$$\gamma\bar{g}_1(\gamma_1)\bar{g}_2(\gamma_1)=\frac{\mu\sigma^2\gamma_1}{[\bar{g}_1(\gamma_1)+\eta][\bar{g}_1(\gamma_1)+\eta-\mu]}<\frac{\delta\bar{g}_1(\gamma_1)}{\gamma_1+\delta}=\gamma_1\bar{g}_1(\gamma)\bar{g}_3(\gamma_1).\tag{A.37}$$

Considering that $\bar{g}_1(\gamma_1) = \frac{\mu[\gamma_1+\delta]}{\delta} - \eta$, because of $f_2(\gamma_1) = 1$, we see that (A.37) is equivalent to show that

$$\frac{[\sigma\delta]^2}{\mu} < \frac{\mu\gamma_1 + \delta[\mu - \eta]}{\delta}. \tag{A.38}$$

Since $\gamma_1 = \frac{\delta}{\mu}\big[\frac{2\delta\sigma^2}{\mu} + 2\eta - \mu\big]$, it follows that (A.38) is true, due to $\eta < 2\eta$. Therefore, by the seen before, we conclude that f_2 is decreasing on $(0, \gamma_1)$.

(ii) Taking first derivatives in f_3 and using (A.36), it gives that

$$\begin{aligned} f_3'(\gamma) &= \bar{\eta}^2[\mu - \eta] \int_1^{f_2(\gamma)} \frac{\mathrm{e}^{\bar{\eta}\gamma y}}{y^{\bar{\eta}[\delta+\gamma]+2}}[y - \ln[y]]\mathrm{d}y + \frac{\mathrm{e}^{\bar{\eta}\gamma f_2(\gamma)}}{[f_2(\gamma)]^{\bar{\eta}[\delta+\gamma]+1}} \\ &\quad \times \left\{ \frac{\sigma^2}{\mu}\bar{\eta}[f_2(\gamma) + \gamma f_2'(\gamma)] + \frac{\sigma^2}{\mu}\left[-\frac{f_2'(\gamma)}{f_2(\gamma)}\bar{g}_4(\gamma) - \bar{\eta}\ln[f_2(\gamma)]\right]\right\} \\ &= \bar{\eta}^2[\mu - \eta] \int_1^{f_2(\gamma)} \frac{\mathrm{e}^{\bar{\eta}\gamma y}}{y^{\bar{\eta}[\delta+\gamma]+2}}[y - \ln[y]]\mathrm{d}y + \frac{\mathrm{e}^{\bar{\eta}\gamma f_2(\gamma)}}{[f_2(\gamma)]^{\bar{\eta}[\delta+\gamma]+1}} \\ &\quad \times \left\{ \frac{\sigma^2}{\mu}\bar{\eta}[f_2(\gamma) + \gamma f_2'(\gamma)] + \frac{\sigma^2}{\mu}\left[\bar{\eta}\left[\frac{\delta}{\gamma} - \ln[f_2(\gamma)]\right] + \frac{\delta[2\eta - \mu]}{\mu\gamma[\gamma + \delta]} - \bar{g}_2(\gamma)\bar{g}_4(\gamma)\right]\right\} \end{aligned} \tag{A.39}$$

with $\bar{g}_4(\gamma) := \bar{\eta}[\delta + \gamma] + \frac{1}{\mu}[2\eta - \mu]$. Observe that the terms $\int_1^{f_2(\gamma)} \frac{\mathrm{e}^{\bar{\eta}\gamma y}}{y^{\bar{\eta}[\delta+\gamma]+2}}\{y - \ln[y]\}\mathrm{d}y$, $\frac{\mathrm{e}^{\bar{\eta}\gamma f_2(\gamma)}}{[f_2(\gamma)]^{\bar{\eta}[\delta+\gamma]+1}}$ and $f_2(\gamma) + \gamma f_2'(\gamma)$ are positive for $\gamma \in (0, \gamma_1)$. So, we only need to verify the positiveness of the last term in (A.39) to show that f_3 is increasing on $(0, \gamma_1)$. Notice that $\gamma \mapsto \frac{\delta}{\gamma} - \ln[f_2(\gamma)]$ is positive and decreasing on $(0, \gamma_1)$, since $\frac{\mathrm{d}}{\mathrm{d}\gamma}\big[\frac{\delta}{\gamma}\big] < \frac{\mathrm{d}}{\mathrm{d}\gamma}[\ln[f_2(\gamma)]] < 0$ and $\lim_{\gamma\uparrow\gamma_1} \ln[f_2(\gamma)] = 0 < \frac{\delta}{\gamma_1}$. On the other hand, we have that $\bar{g}_2(\gamma)\bar{g}_4(\gamma)$ is also positive and decreasing on $(0, \gamma_1)$, since for each $\gamma \in (0, \gamma_1)$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\gamma}[\bar{g}_2(\gamma)\bar{g}_4(\gamma)] \\ &= -\frac{\sigma^4\{4\sigma^4[\gamma^2 + \delta[\delta + 2\gamma]] + [2\eta - \mu][[3\mu - \eta][\delta + \gamma] - \eta^3 - [\eta - \mu]^2[2\bar{g}_1(\gamma) + \eta]]\}}{\mu[\bar{g}_2(\gamma)]^3[\bar{g}_2(\gamma) + \eta]^2[\bar{g}_2(\gamma) + \eta - \mu]^2} < 0. \end{aligned} \tag{A.40}$$

The last inequality in (A.40) is true because $\gamma \mapsto [3\mu - \eta][\delta + \gamma] - \eta^3 - [\eta - \mu]^2[2\bar{g}_1(\gamma) + \eta]$ is positive and increasing on $(0, \gamma_1)$. Additionally, it is easy to check that $\lim_{\gamma\downarrow 0} \bar{g}_2(\gamma)\bar{g}_4(\gamma) < \lim_{\gamma\downarrow 0}\Big\{\bar{\eta}\Big[\frac{\delta}{\gamma} - \ln[f_2(\gamma)]\Big] + \frac{\delta[2\eta-\mu]}{\mu\gamma[\gamma+\delta]}\Big\} = \infty$. Taking into account that $\bar{g}_1(\gamma_1) = \frac{\mu[\gamma_1+\delta]}{\delta} - \eta$, $\gamma_1 = \frac{\delta}{\mu}\big[\frac{2\delta\sigma^2}{\mu} + 2\eta - \mu\big]$ and $\ln[f_2(\gamma_1)] = 0$, it gives that $\bar{\eta}\frac{\delta}{\gamma_1} + \frac{\delta[2\eta-\mu]}{\mu\gamma_1[\gamma_1+\delta]} - \bar{g}_2(\gamma_1)\bar{g}_4(\gamma_1) = \frac{4\delta\sigma^2[\delta\sigma^2+\eta\mu]+\mu^3[2\eta-\mu]}{2\delta(2\delta\sigma^2+\eta\mu)(\mu(2\eta-\mu)+2\delta\sigma^2)} > 0$.

By the seen before we conclude that f_3 is increasing on $(0, \gamma_1)$. Taking f_4 as in (3.48), it is easy to verify its properties. So, it does not require of a formal proof. ■

Proof of Proposition 3.13. Considering $\gamma \in (\gamma_2, \gamma_1)$, it is known (3.38) is true due to Remark 3.12. Then, taking b_γ and M_γ as in (3.46) and (3.47) respectively, by the proof of Proposition 3.9, we get that v_{γ,b_γ} given as in (3.42), when $b = b_\gamma$, is increasing and concave on $(0, \infty)$, which is a solution to (2.8)–(2.10). Furthermore, $v'_{\gamma,b_\gamma}(0+) > 1$, because of $M_\gamma > 0$. Therefore $v'_{\gamma,b_\gamma}(b_\gamma) = 1$. Defining $\hat{f}_2 = v'_{\gamma,b_\gamma}/v''_{\gamma,b_\gamma}$ and calculating the first and second derivatives of v_{γ,b_γ}, and taking into account (3.28) and (3.45), it gives

$$\hat{f}_2(x) = \begin{cases} -\chi_1'(\chi_1^{-1}(x)) & \text{if } x \in (0, b_\gamma), \\ -\chi_2'(\chi_2^{-1}(x)) & \text{if } x \in (b_\gamma, \bar{x}), \\ \frac{\gamma\,\mathrm{e}^{-\lambda_\gamma[x-b_\gamma]}}{c_{3,2}(\alpha_\gamma)\lambda_\gamma^2[\gamma+\delta]} + \frac{1}{\lambda_\gamma} & \text{if } x \in (\bar{x}, \infty). \end{cases}$$

By the seen in the proofs of Propositions 3.8 and 3.9, it is easy to check that $\hat{f}_2$ is decreasing on $(0,\infty)$. It implies that $-\frac{\mu}{\sigma^2}\hat{f}_2$ is increasing on $(0,\infty)$ and $-\frac{\mu}{\sigma^2}\hat{f}_2(0+) < 1$. Therefore $x_\gamma = x_{b_\gamma}$, with x_{b_γ} as in (3.43). Since v_{γ,b_γ} satisfies the hypotheses of Proposition 2.2, we conclude that v_{γ,b_γ} is a solution to the HJB equation (2.5).

Now, let us check the validity of Item (ii). For each $\gamma \in (0,\gamma_2)$ fixed, let us take $b = 0 < x_0$, where x_0 is unknown parameter. Through analogous reasoning as seen in (A.20)–(A.25), we derive that $v_{\gamma,0}(x) = \bar{v}(x)$, with $\bar{v}$ as in (A.25), serves as a solution to (3.39) when $b = 0$. Here $M_2 = -M$, and $\chi_2 : [M,\bar{z}] \longrightarrow [0,x_0]$ is given by $\chi_2(z) = \bar{f}_{-M}(\mathrm{e}^z)$, with $\bar{f}_{-M}(z) = k_3[G(z) - G(\mathrm{e}^M)] - \bar{\eta}[\mu - \eta]\Big[G(z)H_{-M}(z) - \int_{\mathrm{e}^M}^{z}\frac{G(y)}{y^2 g(y)}\mathrm{d}y\Big]$, which is a solution to (A.21). The constants M, $\bar{z}$ and k_3 are unknown positive parameters. Notice that (3.45) is satisfied. Since $v_{\gamma,0}(0) = 0$, it gives that

$$k_3 = \frac{\bar{\eta}[\mu-\eta]}{\mathrm{e}^M\, g(\mathrm{e}^M)}. \tag{A.41}$$

On the other hand, observing that $v_{\gamma,0}$ must satisfy (3.40), we have that $v_{0,\gamma}$ is as in (A.1), with $\max\{x_b, b\} = x_0$, $b = 0$, $v(b) = 0$ and $c_4 = 0$. Then, since $v'_{\gamma,0}(x_0-) = \mathrm{e}^{-\chi_2^{-1}(x_0)} = \frac{1}{\bar{f}_{-M}^{-1}(x_0)}$ (because of $\chi_2^{-1}(x) = \ln[\bar{f}_{-M}^{-1}(x)]$), and considering that x_0 is a point where $-\frac{\mu v'_{\gamma,0}(x_0)}{\sigma^2 v''_{\gamma,0}(x_0)} = 1$, we have the system of equations (A.28), with $\hat{\alpha} = \bar{f}_{-M}^{-1}(x_0)$. Solving the aforementioned system, it yields that $x_0 = \bar{f}_{-M}(\alpha_\gamma)$, $c_3 = c_{3,2}(\alpha_\gamma)$ and $\bar{z} = \ln[\alpha_\gamma]$. Furthermore, as $u_0^*(x_0) = \frac{\mu}{\sigma^2}\chi_2'(\chi_2^{-1}(x_0)) = 1$ must be occurred, we have that

$$k_3 = \frac{\sigma^2}{\mu\alpha_\gamma g(\alpha_\gamma)} + \bar{\eta}[\mu-\eta]H_{-M}(\alpha_\gamma). \tag{A.42}$$

By applying (A.42) in (A.41), we deduce that M must be a solution to (3.52). To ensure this, notice that $\beta \mapsto \frac{\mu-\eta}{\mathrm{e}^\beta g(\mathrm{e}^\beta)}$ is decreasing for $\beta \in (0,\ln[\alpha_\gamma])$, because of $\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\frac{1}{\mathrm{e}^\beta g(\mathrm{e}^\beta)}\right] = -\frac{[g(\mathrm{e}^\beta)+\mathrm{e}^\beta g'(\mathrm{e}^\beta)]}{\mathrm{e}^\beta g(\mathrm{e}^\beta)^2} = -\frac{\{\bar{\eta}[\delta+\gamma]-\bar{\eta}\gamma\,\mathrm{e}^\beta+1\}}{\mathrm{e}^\beta g(\mathrm{e}^\beta)}$ and $\bar{\eta}[\delta+\gamma] - \bar{\eta}\gamma\,\mathrm{e}^\beta + 1 > 0$ for $\beta \in (0,\ln[\alpha_\gamma])$. It can be also verified easily that $\beta \mapsto H_{-\beta}(\alpha_\gamma)$ is decreasing on $(0,\beta)$. Moreover, by Remark 3.12 and considering that $\mu - \eta < \mu/2$, respectively, it yields that $\frac{\mu-\eta}{g(1)} > \frac{\mu}{2\alpha_\gamma g(\alpha_\gamma)} + [\mu-\eta]H_0(\alpha_\gamma)$ and $\frac{\mu-\eta}{\alpha_\gamma g(\alpha_\gamma)} < \frac{\mu}{2\alpha_\gamma g(\alpha_\gamma)}$. Therefore, there exists a unique $M_\gamma > 0$ such that (3.52) holds.

By employing arguments akin to those seen in Item (i), it is easy to check that $v_{\gamma,0}$ is an increasing and concave solution to (3.39)–(3.41), when $b = 0$. Notice that $v'_{\gamma,0}(0+) = \mathrm{e}^{-M_\gamma} \leq 1$, because of $M_\gamma > 0$, leading to $b_\gamma = \inf\{x > 0 : v'_{\gamma,0}(x) \leq 1\} = 0$. On the other hand, following the reasoning in (A.32)–(A.35), it can be demonstrated that $-\frac{\mu v'_{\gamma,0}}{\sigma^2 v''_{\gamma,0}}$ is increasing on $(0,\infty)$, satisfying $-\frac{\mu v'_{\gamma,0}(0+)}{\sigma^2 v''_{\gamma,0}(0+)} < 1$. By invoking Proposition 2.2 and the aforementioned arguments, we conclude that the solution $v_{\gamma,0}$ given in (3.53) satisfies the HJB equation (2.5).

To finish, let us see that $v_{\gamma,0}$ as in (3.53), when $\gamma = \gamma_2$, satisfies also (2.5). Considering v_{γ,b_γ} as in (3.42) when $b = b_\gamma$ and by the observations made in Subsection 4.1.2, it is evident that for each $x \in (0,\infty)$, $v_{\gamma,b_\gamma}(x) \xrightarrow[\gamma\downarrow\gamma_2]{} v_{\gamma_2,0}(x)$, where $v_{\gamma_2,0}$ is as in (3.53), with $M_{\gamma_2} = 0$ solving (3.52) when $\gamma = \gamma_2$, due to (3.51). Furthermore, $\lim_{\gamma\downarrow\gamma_2} v_{\gamma,b_\gamma}(b_\gamma) = v_{\gamma,0}(0) = 0$ and $v'_{\gamma,b_\gamma}(b_\gamma) \xrightarrow[\gamma\downarrow\gamma_2]{} v'_{\gamma,0}(0) = 1$. By the same arguments seen in the case $\gamma < \gamma_2$, we get that $v_{\gamma_2,0}$ satisfies also (2.5). ■

References

[1] Albrecher, H., Bäuerle, N., Thonhauser, S. Optimal dividend-payout in random discrete time. *Statistics & Risk Modeling with Applications in Finance and Insurance* **28**(3), 251–276, (2011).

[2] Albrecher, H., Cheung, E. C., Thonhauser, S. Randomized observation periods for the compound Poisson risk model: dividends. *ASTIN Bulletin* **41**(2), 645–672, (2011).

[3] Asmussen, S. and Taksar, M., Controlled Diffusion Models for Optimal Dividend Payout *Insurance: Math. Econ.* **20**, 1–15, (1997).

[4] Cadenillas, A., Choulli, T., Taksar, M. and Zhang, L. Classical and impulse stochastic control for the optimization of the dividend and risk policies of an insurance firm. *Math. Finance* **16**(1),181–202, (2006).

[5] Højgaard, B. and Taksar, M. Controlling risk exposure and dividends payout schemes: insurance company example. *Math. Finance.*, **9**(2), 153–182, (1999).

[6] Hubalek, F. and Schachermayer, W. Optimizing expected utility of dividend payments for a Brownian risk process and a peculiar nonlinear ODE. *Insurance Math. Econom.*, **34**(2), 193–225, (2004).

[7] Iglehart, D. L. Diffusion approximations in collective risk theory *J. Appl. Probability*, **6**, 285–292, (1969).

[8] Kulinich, G. and Kushnirenko, S. Strong uniqueness of solutions of stochastic differential equations with jumps and non-Lipschitz random coefficients. *Mod. Stoch. Theory Appl.*, **1**, 65–72, (2014).

[9] Luo, X., Zhu, Q., Zhang, Y. and Chen, P. Optimal impulse dividend and capital injection model with proportional and fixed transaction costs. *Math. Methods Appl. Sci.*, **46**(5), 4942–4964, (2023).

[10] Noba, K., Pérez, J. L., Yamazaki, K. and Yano, K. On optimal periodic dividend strategies for Lévy risk processes. *Insurance Math. Econom.*, **80**, 29–44, (2018).

[11] Protter, P.E. Stochastic integration and differential equations. *Second edition. Version 2.1. Corrected third printing. Stochastic Modelling and Applied Probability, 21. Springer-Verlag, Berlin*, (2005).

[12] Radner, R. and Shepp, L. Risk vs. Profit Potential: A Model for Corporate Strategy, *J. Econ. Dynam. Control* **20**, 1373–1393, (1996).

[13] Taksar, M. Optimal risk and dividend distribution control models for an insurance company. *Math. Methods Oper. Res.*, **51**(1), 1–42, (2000).