# Classification of multiplication modules over multiplication rings with finitely many minimal primes

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#### Abstract

A classification of multiplication modules over multiplication rings with finitely many minimal primes is obtained. A characterisation of multiplication rings with finitely many minimal primes is given via faithful, Noetherian, distributive modules. It is proven that for a multiplication ring with finitely many minimal primes every faithful, Noetherian, distributive module is a faithful multiplication module, and vice versa.

Key Words: a multiplication module, a multiplication ring, a Dedekind domain, an Artinian local principal ideal ring.

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## 1 Introduction

In this paper, all rings are commutative with 1 and all modules are unital. A ring R is called a **multiplication ring** if I and J are ideals of R such that  $J \subseteq I$  then J = I'I for some ideal I' of R. An R-module M is called a **multiplication module** if each submodule of M is equal to IM for some ideal I of the ring R. The concept of multiplication ring was introduced by Krull in [5]. In [6], Mott proved that a multiplication ring has finitely many minimal prime ideals iff it is a Noetherian ring.

The next theorem is a description of multiplication rings with finitely many minimal primes.

**Theorem 1.1** ([1, Theorem 1.1]) Let R be a ring with finitely many minimal prime ideals. Then the ring R is a multiplication ring iff  $R \cong \prod_{i=1}^{n} D_i$  is a finite direct product of rings where  $D_i$  is either a Dedekind domain or an Artinian, local principal ideal ring.

Classification of multiplication modules over multiplication rings with finitely many minimal primes. Using Theorem 1.1, a criterion for a direct sum of modules to be a multiplication module (Theorem 2.4) and some other results, a classification of multiplication modules over a multiplication ring with finitely many minimal primes is given, Theorem 1.2.

**Theorem 1.2** Let R be a multiplication ring with finitely many minimal primes, i.e.,  $R \cong \prod_{i=1}^{n} D_i$ 

is a finite direct product of rings where  $D_i$  is either a Dedekind domain or an Artinian, local principal ideal ring and  $1 = e_1 + \cdots + e_n$  be the corresponding sum of central orthogonal idempotents of the ring R. Let M be an R-modules and  $M = \bigoplus_{i=1}^n M_i$  where  $M_i := e_i M$ . Then the R-module M is a multiplication R-module iff each  $D_i$ -module  $M_i$  is either isomorphic to  $D_i$  or to  $D_i/I_i$  where  $I_i$  is a nonzero ideal of  $D_i$  or to a nonzero ideal of the ring  $D_i$  in case when the ring  $D_i$  is a Dedekind domain.

Classification of faithful multiplication modules over a multiplication ring with finitely many minimal primes.

**Theorem 1.3** Let R be a multiplication ring with finitely many minimal primes. We keep the notation of Theorem 1.2 ( $R \cong \prod_{i=1}^{n} D_i$ ). Then an R-module  $M = \bigoplus_{i=1}^{n} M_i$  (where  $M_i = e_i M$ ) is a faithful multiplication R-module iff for each  $i = 1, \ldots, n$ , either  ${}_{R}M_i \cong D_i$  or  ${}_{R}M_i \cong I_i$  where  $I_i$  is a nonzero ideal of the ring  $D_i$  in case when  $D_i$  is a Dedekind domain.

*Proof.* The theorem follows at once from Theorem 1.2.  $\square$ 

Characterisation of multiplication rings with finitely many minimal primes via faithful, Noetherian, distributive modules. Let R be a ring and M be an R-module. A submodule N of M is called a *distributive submodule* if one of the following equivalent conditions holds:

$$(M_1 + M_2) \cap N = M_1 \cap N + M_2 \cap N,$$
  
 $M_1 \cap M_2 + N = (M_1 + N) \cap (M_2 + N).$ 

The R-module M is called a **distributive module** if all submodules of M are distributive submodules.

**Theorem 1.4** A commutative ring R is a multiplication ring with finitely many minimal primes iff there is a faithful, Noetherian, distributive R-module.

Classification of faithful, Noetherian, distributive modules over a multiplication ring with finitely many minimal primes.

**Theorem 1.5** Let R be a multiplication ring with finitely many minimal primes. Then every faithful, Noetherian, distributive R-module is a faithful multiplication R-module, and vice versa.

## 2 Proofs

In this section we prove the results from the Introduction.

**Definition 2.1** We say that the intersection condition holds for a direct sum  $M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}$  of nonzero R-modules  $M_{\lambda}$  if for all submodules N of M,  $N = \bigoplus_{\lambda \in \Lambda} (N \cap M_{\lambda})$ .

**Definition 2.2** Let  $M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}$  be a direct sum of nonzero R-modules with  $\operatorname{card}(\Lambda) \geq 2$ ,  $\mathfrak{a}_{\lambda} = \operatorname{ann}_{R}(M_{\lambda})$  and  $\mathfrak{a}'_{\lambda} = \bigcap_{\mu \neq \lambda} \mathfrak{a}_{\mu}$ . We say that the **orthogonality condition** holds for the direct sum  $M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}$  if  $\mathfrak{a}'_{\lambda} M_{\mu} = \delta_{\lambda \mu} M_{\mu}$  for all  $\lambda, \mu \in \Lambda$ . Clearly,  $\mathfrak{a}'_{\lambda} \neq 0$  for all  $\lambda \in \Lambda$  (since all  $M_{\lambda} \neq 0$ ). In particular,  $\mathfrak{a}_{\lambda} \neq 0$  for all  $\lambda \in \Lambda$ .

**Definition 2.3** Let  $M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}$  be a direct sum of nonzero R-modules with  $\operatorname{card}(\Lambda) \geq 2$ . We say that the **strong orthogonality condition** holds for M if for each set of R-modules  $\{N_{\lambda}\}_{{\lambda} \in \Lambda}$  such that  $N_{\lambda} \subseteq M_{\lambda}$ , there is a set of ideals  $\{I_{\lambda}\}_{{\lambda} \in \Lambda}$  of R such that  $I_{\lambda}M_{\mu} = \delta_{{\lambda}{\mu}}N_{{\lambda}}$  for all  ${\lambda}$ ,  ${\mu} \in {\Lambda}$  where  $\delta_{{\lambda}{\mu}}$  is the Kronecker delta. The set of ideals  $\{I_{\lambda}\}_{{\lambda} \in {\Lambda}}$  is called an **orthogonalizer** of  $\{N_{\lambda}\}_{{\lambda} \in {\Lambda}}$ .

Theorem 2.4 is one of the criteria for a direct sum of modules to be a multiplication module that are obtained in [1]. It is given via the intersection and strong orthogonality conditions.

**Theorem 2.4** [2] Let  $M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}$  be a direct sum of nonzero R-modules with  $\operatorname{card}(\Lambda) \geq 2$ . Then M is a multiplication module iff the intersection and strong orthogonality conditions hold for the direct sum  $M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}$ . An R-module is called a cyclic if it is 1-generated. For an R-module M, let  $\mathrm{Cyc}_R(M)$  be the set of its cyclic submodules. For an R-module M, we denote by  $\mathrm{ann}_R(M)$  its annihilator. An R-module M is called faithful if  $\mathrm{ann}_R(M)=0$ . For a submodule N of M, the set  $[N:M]:=\mathrm{ann}_R(M/N)=\{r\in R\,|\,rM\subseteq N\}$  is an ideal of the ring R that contains the  $annihilator\,\mathrm{ann}_R(M)=[0:M]$  of the module M. The set  $\theta(M):=\sum_{C\in\mathrm{Cyc}_R(M)}[C:M]$  is an ideal of R. Clearly,  $\mathrm{ann}_R(M)\subseteq\theta(M)$ . If M is an ideal of R then  $M\subseteq\theta(M)$ .

**Proof of Theorem 1.2.** ( $\Leftarrow$ ) All the  $D_i$ -modules  $M_i$  of the theorem are multiplication  $D_i$ -modules. Hence, the direct sum  $\bigoplus_{i=1}^n M_i$  is a multiplication module over the direct product rings  $R = \prod_{i=1}^n D_i$ .

- ( $\Rightarrow$ ) Suppose that the *R*-module  $M = \bigoplus_{i=1}^n M_i$  is a multiplication *R*-module where  $M_i = e_i M$  for  $i = 1, \ldots, n$ .
- (i) The  $D_i$ -module  $M_i$  is a multiplication  $D_i$ -module: The statement is obvious since  $R = \prod_{i=1}^n D_i$ .
- (ii) The  $D_i$ -module  $M_i$  is a finitely generated  $D_i$ -module: Since  $M_i$  is a multiplication  $D_i$ -module,

$$M_i = \sum_{C \in \operatorname{Cyc}_{D_i}(M_i)} C = \sum_{C \in \operatorname{Cyc}_{D_i}(M_i)} [C:M_i] M_i = (\sum_{C \in \operatorname{Cyc}_{D_i}(M_i)} [C:M_i]) M_i = \theta(M_i) M_i.$$

The ideal  $\theta(M_i) = \sum_{C \in \text{Cyc}_{D_i}(M_i)} [C:M_i]$  of the Noetherian ring  $D_i$  is a finitely generated  $D_i$ -module, i.e.,  $\theta(M_i) = \sum_{i=1}^{n_i} D_i \theta_i$  for some elements  $\theta_i \in \theta(M_i)$ . Then

$$M_i = \theta(M_i)M_i = \sum_{i=1}^{n_i} D_i \theta_i M_i \subseteq \sum_{i=1}^{n_i} C_i \subseteq M_i,$$

and so the  $D_i$ -module  $M_i = \sum_{i=1}^{n_i} C_i$  is finitely generated.

(iii) Suppose that the ring  $D_i$  is a Dedekind domain. Then the  $D_i$ -module  $M_i$  is isomorphic either to  $D_i$  or to  $D_i/I_i$  or to  $J_i$  where  $I_i$  and  $J_i$  are ideals of the ring  $D_i$ : It is well-known that a nonzero finitely generated module  $\mathcal{M}$  over a Dedekind domain D is a direct sum  $\mathcal{M} = \mathcal{F} \oplus \mathcal{T}$  of a torsionfree D-module  $\mathcal{F}$  and a torsion D-module  $\mathcal{T}$ ;  $\mathcal{F} = I \oplus D^m$  for some ideal I of D and  $m \geq 0$ ; and  $\mathcal{T} = \bigoplus_{i=1}^{t_i} D/\mathfrak{p}_i^{m_i}$  where  $\mathfrak{p}_i$  are maximal ideals of the ring D and  $m_i \in \mathbb{N}$ . Suppose that the D-module  $\mathcal{M}$  is a multiplication D-module. By Theorem 2.4, the direct sum of D-modules

$$\mathcal{M} = I \oplus D^m \oplus \bigoplus_{i=1}^{t_i} D/\mathfrak{p}_i^{m_i}$$

must satisfy the strong orthogonality conditions. Hence, either  $\mathcal{M} = I$  of  $\mathcal{M} = D$  or  $\mathcal{M} = \bigoplus_{i=1}^{t_i} D/\mathfrak{p}_i^{m_i}$  where  $\mathfrak{p}_1, \ldots, \mathfrak{p}_{t_i}$  are distinct maximal ideals of the ring D, and so  $\mathcal{M} = \bigoplus_{i=1}^{t_i} D/\mathfrak{p}_i^{m_i} \simeq D/\prod_{i=1}^{t_i} \mathfrak{p}_i^{m_i}$ .

(iv) Suppose that  $D_i$  is an Artinian, local, principal ideal ring. Then the  $D_i$ -module  $M_i$  is isomorphic either to  $D_i$  or to  $D_i/I_i$  where  $I_i$  is a nonzero ideal of  $D_i$ : Let  $D = D_i$  and  $\mathfrak{m}$  be the maximal ideal of the local ring  $D_i$  and  $\mathfrak{m}^{\nu} \neq 0$  and  $\mathfrak{m}^{\nu+1} = 0$  for some natural number  $\nu$ . Then

$$\{D, \mathfrak{m}, \mathfrak{m}^2, \dots, \mathfrak{m}^{\nu}, \mathfrak{m}^{\nu+1} = 0\}$$

is the set of all the ideals of the ring D. The D-module  $M_i$  is a nonzero finitely generated multiplication D-module. Hence,  $\{M_i, \mathfrak{m} M_i, \mathfrak{m}^2 M_i, \dots, \mathfrak{m}^{\mu} M_i, \mathfrak{m}^{\mu+1} M_i = 0\}$  is the set of all D-submodules of  $M_i$  for some natural number  $\mu$  such that  $\mu \leq \nu$ . In particular, the D-module  $M_i$  is

a uniserial D-module since

$$M_i \supset \mathfrak{m} M_i \supset \mathfrak{m}^2 M_i \supset \cdots \supset \mathfrak{m}^{\mu} M_i \supset \mathfrak{m}^{\mu+1} M_i = 0.$$

Therefore,

$$\dim_{k_{\mathfrak{m}}}(M_i/\mathfrak{m}M_i)=1$$

where  $k_{\mathfrak{m}} := D/\mathfrak{m}$ , and so  $M_i = Dm_i + \mathfrak{m}M_i$  for some element  $m_i \in M_i \setminus \mathfrak{m}M_i$ . By the Nakayama Lemma,  $M_i = D\mathfrak{m}_i$ , and the statement (iv) follows.  $\square$ 

Corollary 2.5 Let R be an Artinian multiplication ring. Then every multiplication R-module if an epimorphic image of the R-module R.

*Proof.* The corollary follows at once from Theorem 1.2.  $\square$ 

Corollary 2.6 Let R be a multiplication ring with finitely many minimal primes and M be a multiplication R-module. Then

- 1. The endomorphism ring  $\operatorname{End}_R(M)$  is also a multiplication ring.
- 2.  $\operatorname{End}_R(M) \simeq R/\operatorname{ann}_R(M)$ .
- 3. The  $\operatorname{End}_R(M)$ -module M is a faithful multiplication  $\operatorname{End}_R(M)$ -module.

*Proof.* The corollary follows at once from Theorem 1.2.  $\square$ 

In the proof of Theorem 1.4 we will use the following results.

### **Theorem 2.7** Let R be a commutative ring.

- 1. ([3, Corollary, p.177]) Let M be a Noetherian distributive R-module. Then every submodule of M which is locally nonzero at every maximal ideal of R, is of the form IM where I is a unique product of maximal ideals of R.
- 2. ([3, Lemma 2.(ii)]) A finitely generated R-module M is a multiplication module iff the  $R_{\mathfrak{p}}$ module  $M_{\mathfrak{p}}$  is a multiplication module for all prime/maximal ideals  $\mathfrak{p}$  of R.
- 3. ([4, Theorem 1.3.(ii)]) (CANCELLATION LAW) If M is a finitely generated, faithful multiplication R-module then for any two ideals A and B of R,  $MA \subseteq BM$  iff  $A \subseteq B$ .

**Proof of Theorem 1.4.** ( $\Rightarrow$ ) By Theorem 1.2, the *R*-module *R* is a faithful, Noetheriaan, distributive *R*-module.

- $(\Leftarrow)$  Let M be faithful, Noetheriaan, distributive R-module.
- (i) The ring R is a Noetherian ring: The R-module M is Noetherian, hence finitely generated,  $M = \sum_{i=1}^{n} Rm_{i}$  for some elements  $m_{1}, \ldots, m_{n} \in M$ . The R-module M is a faithful module. Hence, the map  $R \to \bigoplus_{i=1}^{n} Rm_{i}$ ,  $r \mapsto (rm_{1}, \ldots, rm_{n})$  is an R-monomorphism. The direct sum is a Noetherian R-module (as a finite direct sum of Noetherian modules), and the statement (i) follows.
- (ii) The ring R has only finitely many minimal primes: The statement (ii) follows from the statement (i).
- (ii) For all maximal ideals  $\mathfrak{m}$  of the ring R, the  $R_{\mathfrak{m}}$ -module  $M_{\mathfrak{m}}$  is faithful, Noetherian and distributive: The R-module M is finitely generated. Hence,  $\operatorname{ann}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \operatorname{ann}_{R}(M)_{\mathfrak{m}} = 0$  since  $\operatorname{ann}_{R}(M) = 0$ . Clearly, the  $R_{\mathfrak{m}}$ -module  $M_{\mathfrak{m}}$  is Noetherian and distributive (since the R-module M is so and localizations respect finite intersections).
- (iv) The  $R_{\mathfrak{m}}$ -module  $M_{\mathfrak{m}}$  is a multiplication  $R_{\mathfrak{m}}$ -module: The R-module M is finitely generated. By the statement (iv) and Theorem 2.7.(2), the R-module M is a multiplication R-module.

Let  $(\mathcal{I}(R), \subseteq)$  be the lattice of ideals of the ring R and  $(\operatorname{Sub}_R(M), \subseteq)$  be the lattice of R-submodules of the R-module M.

- (vi) The map  $\mathcal{I}(R) \to \operatorname{Sub}_R(M)$ ,  $I \mapsto IM$  is an isomorphism of latices: The R-module M is a finitely generated, faithful multiplication module (the statement (v)), and the statement (vi) follows from Theorem 2.7.(3).
- (vii) The ring R is a multiplication ring: The statement (vii) follows from the statements (v) and (vi).

Now, the theorem follows from the statement (ii) and (vii).  $\Box$ 

**Proof of Theorem 1.5.**  $(\Rightarrow)$  See the statement (vi) in the proof of Theorem 1.4.

 $(\Leftarrow)$  This implication follows at once from Theorem 1.3.  $\square$ 

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# References

- [1] T. Alsuraiheed and V. V. Bavula, Characterization of multiplication commutative rings with finitely many minimal prime ideals, *Comm. in Algebra*, **47** (2019), no. 11, 4533–4540.
- [2] T. Alsuraiheed and V. V. Bavula, Multiplication modules over noncommutative rings, J. Algebra, 584 (2021), 69–88.
- [3] A. Barnard, Multiplication modules, J. of Algebra, 71 (1981) 174–178.
- [4] Z. A. El-Bast and P. F. Smith, Multiplication modules, Comm. Algebra, 16 (1988) 755-779.
- [5] W. Krull, Ideal Theory, New York, 1948.
- [6] J. L. Mott, Multiplication rings containing only finitely many minimal prime ideals, J. Sci. Hiroshima. UNIV. SER., 16 (1969) 73–83.

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