Two Main Mathematical Actors

- **Eigenvector**
- **Eigenvalue**

We know the Arabic numerals 0, 1, 2, . . . ; and if we begin to arrange these numbers in the form of rows and columns and then add them like we add numerals and multiply them as follows:

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 8 & 1 \\ 9 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 12 & 10 \end{bmatrix} \qquad (1)$$

Similarly, we can multiply them as follows:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (1 \times 1) + (2 \times 1) & (1 \times 1) + (2 \times 2) \\ (3 \times 1) + (4 \times 1) & (3 \times 1) + (4 \times 2) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix} \qquad (2)$$

What do we realize from equations (1) and (2)? The objects on the right-hand side are new matrices after adding or multiplying two matrices the same way we do for numbers.

Before we proceed, we can define one more multiplication of a matrix with a scalar as follows:

$$\begin{bmatrix} 5 & 1 \\ 6 & 1 \end{bmatrix} \times 2 = \begin{bmatrix} 10 & 2 \\ 12 & 2 \end{bmatrix} \qquad (3)$$

However, things start getting interesting when we multiply a matrix by a vector. What does a vector mean?

Well, any matrix with one row or column can be called a vector. Graphically, one can say that a matrix formed by the components of a vector in mutually perpendicular directions can be said to be a vector.

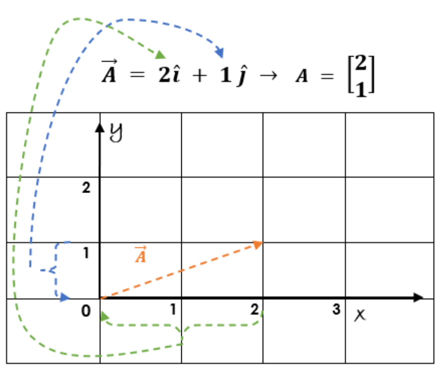

Figure 1: Illustration of a vector and its components.

Playing with Matrices and Vectors

Consider a matrix:

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$$

and a vector:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

.

Now, let's multiply both of them and see what happens:

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + (-2) \times 2 \\ -1 \times 1 + 4 \times 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

So, we put a vector as input and we get another vector as output but, that is scaled and rotated as compared to the original one. This is what it means that 'Matrix Does Sometimes Interesting to Vectors.'

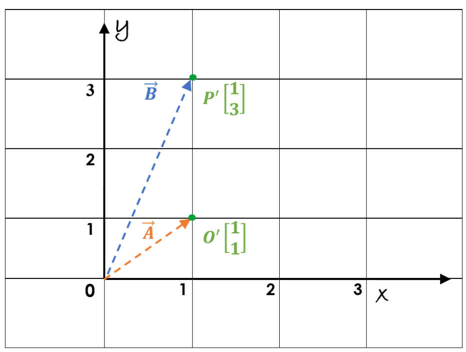

Figure 2: Illustration of matrix transformation on vectors.

Different matrices can scale and rotate vectors into completely different vectors. Let us take a few examples to understand this rotation dance:

- The following matrix, if multiplied with a vector, will rotate the given vector by 90°:

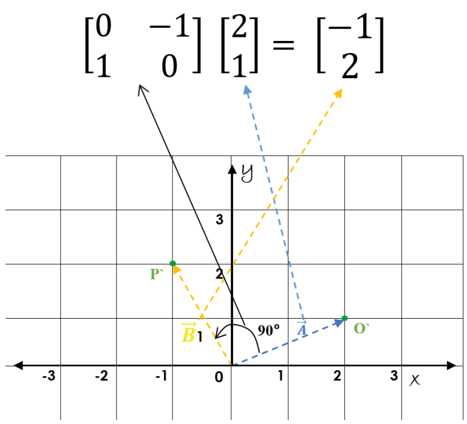

Figure 3: Illustration of matrix transformation on vectors.

Different matrices can also scale vectors. Consider the following example:

- The following matrix will scale the input vector by a factor of 2, since the diagonal elements contain 2:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

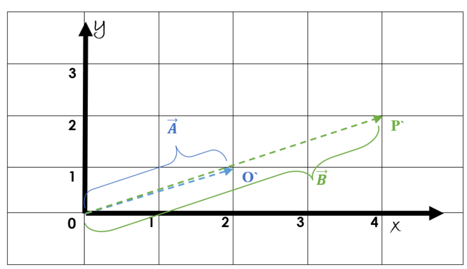

Figure 4: Scaling transformation on a vector.

So, different input matrices get scaled and rotated differently based on how we operate upon them. However, all the transformations are linear, meaning that any vector on the same line as one of those inputs will be mapped onto the same line as the corresponding outputs.

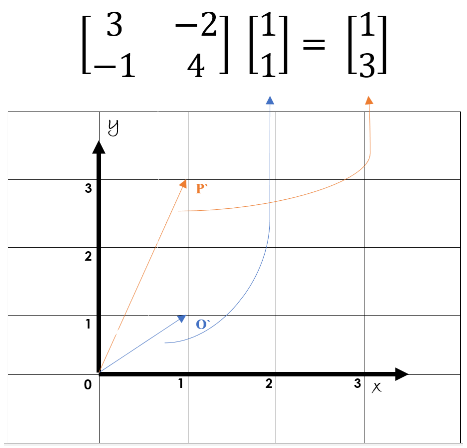

Figure 5: Another example of matrix transformation.

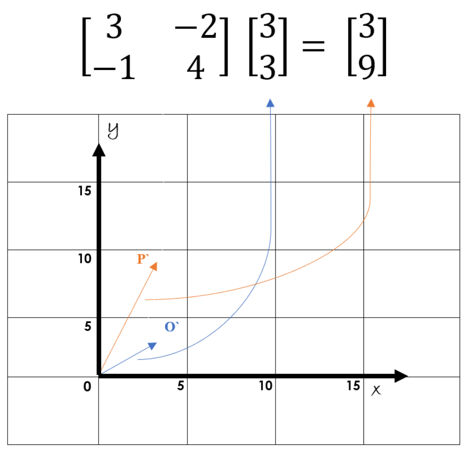

Figure 6: Matrix multiplication example.

Now consider a matrix:

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$$

and calculate its eigenvalues and eigenvectors.

The eigenvalues come out to be 2 and 5 respectively, and the corresponding eigenvectors are:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Now, let's see what happens when we multiply the matrix A with its eigenvectors:

$$\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Thus, we can see that:

$$A \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \times \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

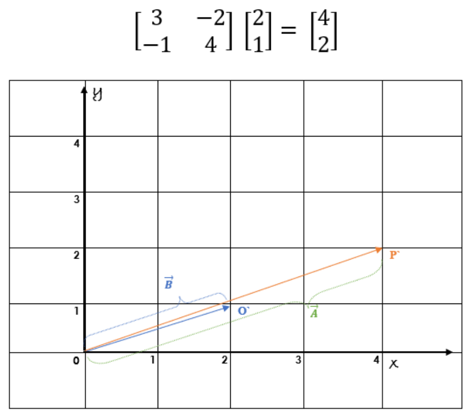

Figure 7: Eigenvector transformation.

So, what we notice is that we took a vector A. We call it an eigenvector of matrix A. Then when we operate this vector with matrix A, no change occurred in the direction, although the same matrix A was responsible for rotating and scaling other vectors earlier. So, any vector that is only scaled by a matrix is called an Eigen Vactor (Eigen in German means 'Special') of that matrix, and how much the vector is scaled by is called its 'Eigen Value'. So, eigen vectors and eigenvalues are the vectors and values that remain unchanged in direction and only get scaled when operated upon by a matrix. In summary:

- Eigenvectors are the vectors that remain unchanged in direction when operated upon by a matrix.
- Eigenvalues are the corresponding values that indicate how much the eigenvector is scaled.