Lecture No.

Introduction To Statistics, Statistics And Probability

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Introduction To Probability

Basic Concepts

In this lecture

- Set Theory and Algebra of Sets
- Basics of Probability
- Random and Non-random Experiment
- Event and Sample Space
- Simple and Compound Events
- Mutually Exclusive Events
- Exhaustive Events, Equally Likely Events
- Rules of Counting

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Set Theory:

- A set is any well-defined collection or list of distinct objects, e.g. a group of students, the books in a library, the integers between 1 to 100, all human beings on earth, etc.
- The term *well-defined* here means that any object must be classified as either belonging or not belonging to the set under study.
- The objects that are in a set, are called *members* or *elements* of the set.
- Sets are usually denoted by capital letters such as: A, B, E, Y, etc.
- The elements of the sets are represented by small letters, such as: a, b, c, e, y, etc.
- Elements are enclosed by braces to represent a set, e.g. A is the set of vowels, i.e.,

$$A = \{a, e, i, o, u\}$$

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Algebra of Sets:

$S=\{1,2,3,4\}$

Since events and sample spaces are just sets, let's review the algebra of sets:

$S=\{\emptyset\}$

$C=\{1,2\}$

$D=\{3,4\}$

$C\cup D=\{1,2,3,4\}$

Ø is the "null set" (or "empty set")

C ∪ D = "union" = the elements in C or D or both

D' = Dc = "complement" = the elements not in D

A ∩ B = "intersection" = the elements in A and B. If A ∩ B = Ø, then A and B are called "mutually exclusive events" (or "disjoint events")

If E ∪ F ∪ G ∪ ... = S, then E, F, G, and so on are called "exhaustive events."

Basics of Probability

- A planned activity or process whose results yield a set of data (called outcomes), is called and *experiment*.
- An *outcome* is the result of a single trial of a probability experiment.
- An *event* consists any collection of results or outcomes of a procedure.
- A *simple event* is an outcome or an event that cannot be further broken down into simpler components.
- A *sample space (S)* is the set of all possible outcomes or simple events of a probability experiment.

$S = \{H, T\} \rightarrow$ a Coin

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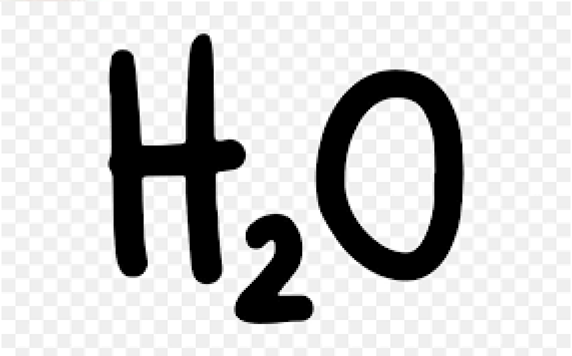
Random and Non-random Experiment:

- An experiment which produces different results even though it is repeated a large number of times under essentially similar conditions, is called a *random experiment*.
- For example: the tossing of a fair coin, throwing of balanced die, selecting a sample, etc.
- An experiment which produces fix results even though it is repeated a large number of times, is called a *non random experiment*.
- For example: mixture of two hydrogen and one oxygen element make water, Area of circle, etc.



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Event and Sample Space

Sample Space

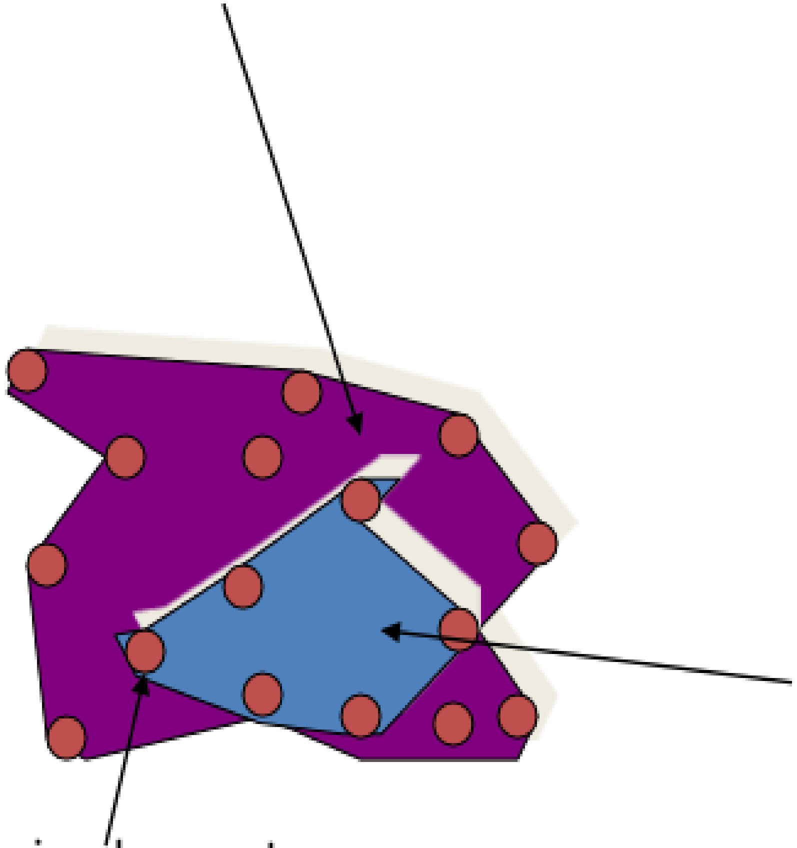
The sample space is the set of all possible outcomes.

a die

$S = \{1,2,3,4,5,6\}$

5 D

odd → ?



Event

An event is any collection of one or more simple events

Simple Events

The individual outcomes are called simple events.

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Sample Space Example 1:

Experiment → Toss a Coin

Possible outcomes

Head or Tail

Thus the sample space is:

$$S = \{Head, Tail\}$$

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Sample Space Example 2:

Experiment $\longrightarrow$ Roll a Dice

Possible outcomes

1, 2, 3, 4, 5, 6

Thus the sample space is:

$$S = \{1, 2, 3, 4, 5, 6\}$$

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Events

- An event is an individual outcome or any number of outcomes (sample points) of a random experiment or a trial

In set terminology, any subset of a sample space S of the experiment, is called an event.

- The events are classified as:
  - Simple Events
  - Compound Events
  - Mutually Exclusive Events
  - Exhaustive Events
  - Equally Likely Events

Simple and Compound Events:

- An event that contains exactly one sample point, is defined a *simple event*.
- A *compound event* contains more than one sample points and is produced by the union of simple events.

For example:

$$\overset{1}{6} \times \overset{2}{6} = 36 \longrightarrow 5$$

- The occurrence of six when a die is thrown, is a *simple event*,
- while the occurrence of a sum of 10 with a pair of dice, is a *compound event*, as it can be decomposed into three simple events $(4, 6), (5, 5), and\ (6, 4)$.

When two dice are rolled

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Example 3: Simple Events and Sample Spaces:

In the following table, we use *“b”* to denote a baby boy and *“g”* to denote a baby girl.

Procedure	Example of Event	Sample Space
Single birth	1 girl (simple event)	{b, g}
3 births	2 boys and 1 girl (bbg, bgb, and gbb are all simple events)	{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg} 2^3

1 birth: The result of 1 girl is a simple event and so is the result of 1 boy.

~~*3 births:* The result of 2 girls followed by a boy *(ggb)* is a simple event.~~

3 births: The event of “2 girls and 1 boy” is not a simple event because it can occur with these different simple events: *ggb, gbg, bgg*.

Mutually Exclusive Events

- Two events A & B of a single experiment are said to be *mutually exclusive* or *disjoint* if and only if they cannot both occur at same time. For example,
- When we toss a coin, we get either a head or a tail, but not both, the two events head and tail are therefore mutually exclusive.
- When a die is rolled, the outcome are mutually exclusive as we get one and only one of six possible outcomes 1, 2, 3, 4, 5, 6.

Exhaustive Events:

- Events are said to be *collectively exhaustive*, when the union of mutually exclusive events is the entire sample space S.
- Thus in our coin-tossing experiment, head and tail are collectively exhaustive set of events.

$S \to \{H, T\}$

Equally Likely Events:

- Two events A & B are said to be *equally likely*, when one event is as likely to occur as the other.
- In other words, each event should occur in equal number in repeated trials.
- For example, when a fair coin is tossed, the head is as likely to appear as tail, and the proportion of times each side is expected to appear is $\frac{1}{2}$.

- Similarly, when a die is rolled, each possible outcome i.e., 1, 2, 3, 4, 5 ,6 are ec to appear, and the proportion of times each side is expected to appear is $\frac{1}{6}$.

Counting Sample Points:

- The *sample space S*, is the set of all possible outcomes of a statistical experiment.
- Each outcome in a sample space is called a *sample point*. It is also called an *element* or a *member* of the sample space.

Fundamental Principle of Counting:

- When the number of sample points in a sample space S is very large, it becomes very difficult to list them all and to count the sample points.
- In such cases, a probability problem may be solved by counting sample points in S, without actually listing each element.

Counting Sample Points:

The basic rules or method which helps us to count the number of sample points without actually listing them all are:

1. Rule of Multiplication
2. Rule of Permutation
3. Rule of Combination

1. Rule of Multiplication:

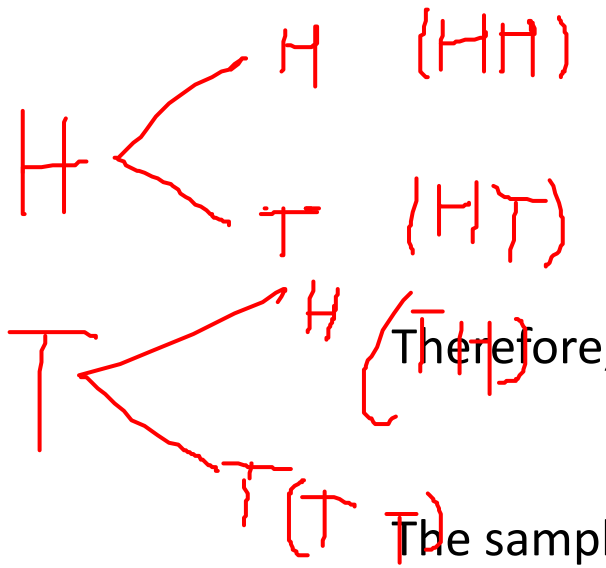
- If a compound experiment consists of two experiment such that, the first has exactly n_1 distinct outcomes and corresponding to each outcome of the first there can be n_2 distinct outcomes of the second experiment, then the compound experiment has exactly $n_1 \times n_2$ outcomes.
- In general if there are *k* experiments, then total sample points of compound experiment are:

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

Example 1:

Statement: How many sample points are there if two coins are tossed together?

Solution: Total no. of outcomes the first coin has= n_1= 2, and Total no. of outcomes the second coin has=$n_2 = 2$,



	H	T
H	(H, H)	(H, T)
T	(T, H)	(T, T)

Therefore, a pair of coins can lend in:

$$n_1 \times n_2 = 2 * 2 = 4\, points$$

The sample space will be as:

$$S = \{HH, HT, TH, TT\}$$

Example 2:

Statement: How many sample points are there if the compound experiment consist of tossing a coin and throwing a die together?

Solution: Total possible outcomes the first experiment (coin) has= $n_1 = 2$, and Total possible outcomes the second experiment (die) has=$n_2 = 6$,

	H	T
1	(1, H)	(1, T)
2	(2, H)	(2, T)
3	(3, H)	(3, T)
4	(4, H)	(4, T)
5	(5, H)	(5, T)
6	(6, H)	(6, T)

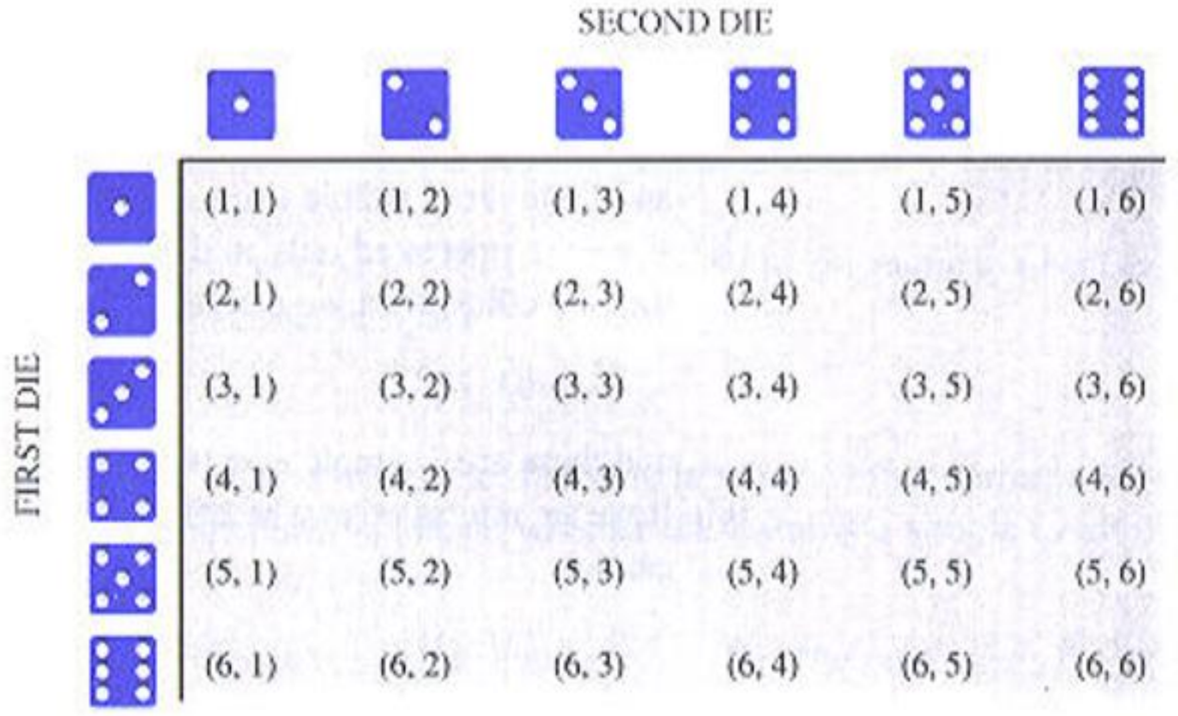
Total sample points: $n_1 \times n_2 = 2 \times 6 = 12$ points

$S = \{ \qquad \}$

Example 3:

Statement: How many sample points are in the sample space when a pair of dice are thrown together?

Solution: the first die can lend in $n_1 = 6$ ways and corresponding to each the second die can also lend in $n_2 = 6$ ways, therefore a pair of dice can lend in:



SECOND DIE

FIRST DIE

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Thus, Sample Space Points are:

$$n_1 \times n_2 = 6 \times 6 = 36 \text{ points}$$

Example 4:

Three Coins: How many sample points are there if three coins are tossed at once?

Solution: Each of three coins can lend in 2 possible ways, i.e. Head and Tail, Thus,

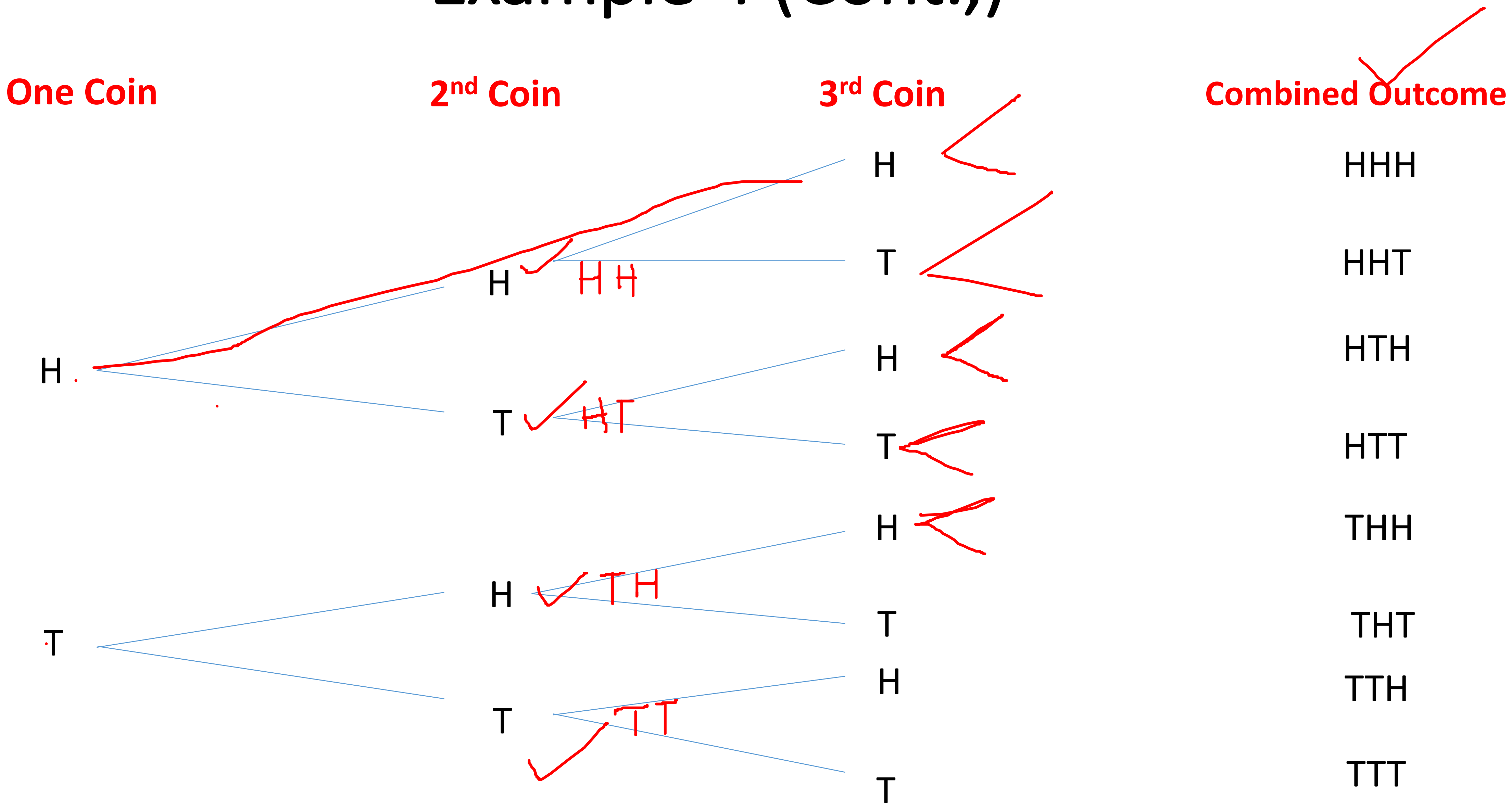
$$n_1 = n_2 = n_3 = 2$$

The possible situations are:

		One Coin	
		H	T
Two Coins	HH	HHH	HHT
	HT	HTH	HTT
	TH	THH	THT
	TT	TTH	TTT

Thus Sample Space Points: $n_1 \times n_2 \times n_3 = 2 \times 2 \times 2 = 8$ points

Example 4 (Cont.,)



Example 5:

Two Coins and One Die:

Solution: Each of two coins has exactly two possible outcomes, i.e. $n_1 = n_2 = 2$**,** and one die has exactly 6 possible outcomes, i.e. $n_3 = 6$**,** thus the possible situations are:

Two Coins

One Die		HH	HT	TH	TT
	1	(1, H, H)	(1, H, T)	(1, T, H)	(1, T, T)
	2	(2, H, H)	(2, H, T)	(2, T, H)	(2, T, T)
	3	(3, H, H)	(3, H, T)	(3, T, H)	(3, T, T)
	4	(4, H, H)	(4, H, T)	(4, T, H)	(4, T, T)
	5	(5, H, H)	(5, H, T)	(5, T, H)	(5, T, T)
	6	(6, H, H)	(6, H, T)	(6, T, H)	(6, T, T)

Thus possible Sample Points: $n_1 \times n_2 \times n_3 = 2 \times 2 \times 6 = 24 \text{ points}$

Practice Questions:

1. How many sample points are in the sample space, when a pair of dice and one coin are thrown together?
2. How many sample points are in the sample space when three dice are thrown together?
3. How many sample points are in the sample space when four coins are thrown together?
4. If an experiment consists of throwing a die and then drawing a letter at random from the English alphabet, how many points are there in the sample space?

2. Rule of Permutation:

- A *permutation* is any ordered subset of r objects from a set of n distinct objects.

OR

- A *permutation* is an arrangement of all or part of a set of objects.
- The number of permutations of n objects is $n!$ $(read\ n\ factorial)$.
- The number of permutations of n distinct objects taken r at a time is denoted by symbol $\boldsymbol{{}^nP_r}$**,** is defined as:

$$\boldsymbol{{}^nP_r = \frac{n!}{(n-r)!}}, \qquad r \le n$$

Where $n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1$, it is relevant to note that $1! = 1$ and that we define $0! = 1$.

3. Rule of Combination:

- A *combination* is any subset of r objects, selecting without regarding their order, from a set of n distinct objects.

- The number of combinations of n distinct objects taken r at a time is denoted by symbol $\boldsymbol{{}^{n}C_{r}}$ or $\boldsymbol{\binom{n}{r}}$ **(**read "$n\ C\ r$" or "$n\ above\ r$"**) ,** is defined as:

$$\boldsymbol{{}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}}$$

Where $r \leq n$.

Example 1:

In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Solution: Given $n = 25, r = 3,$

Since the awards are distinguishable, it is a permutation problem. The total number of sample points is:

$$^{n}P_{r} = \frac{n!}{(n-r)!} = \frac{25!}{(25-3)!} = \frac{22! \times 23 \times 24 \times 25}{22!} = (23)(24)(25) = 13{,}800$$

Example 2:

A young boy asks his mother to get 5 Game-Boy cartridges from his collection of 10 cartridges and 5 sports games. How many ways are there that his mother can get 3 cartridges and 2 sports games?

Solution: The number of ways of selecting 3 cartridges from 10 is:

$$^{n}\boldsymbol{C_r} = \frac{\boldsymbol{n}!}{\boldsymbol{r}!\,(\boldsymbol{n}-\boldsymbol{r})!}$$

$$\frac{\binom{10}{3}+\binom{5}{2}}{\binom{15}{5}}$$

$$^{10}C_3 = \frac{10!}{3!\,(10-3)!} = \frac{7!\times 8\times 9\times 10}{1\times 2\times 3\times 7!} = \frac{(8)(9)(10)}{(1)(2)(3)} = 120$$

Example 2 (cont.):

Solution: The number of ways of selecting 2 cartridges from 5 is:

$$^{n}\boldsymbol{C_r} = \frac{\boldsymbol{n}!}{\boldsymbol{r}!\,(\boldsymbol{n}-\boldsymbol{r})!}$$

$$^{5}C_2 = \frac{5!}{2!\,(5-2)!} = \frac{3! \times 4 \times 5}{1 \times 2 \times 3!} = 10$$

Example 3:

- From a student club consisting of 50 people. How many different choices of officers are possible if :

a) 3 members are chosen randomly.

b) president and a treasurer are to be chosen.

Solution: a) Given $n = 50, r = 3$ **,** since 3 members are to be chosen at random, it is combination problem. The total no. of sample points are:

$$^{50}C_3 = \frac{50!}{3!\,(50-3)!} = \frac{47! \times 48 \times 49 \times 50}{1 \times 2 \times 3 \times 47!} = \frac{(48)(49)(50)}{(1)(2)(3)} = 19{,}600$$

Example 3 (cont,):

b) president and a treasurer are to be chosen.

Solution: Given $n = 50, r = 2$, since president and treasurer are distinguishable, it is permutation problem. The total no. of sample points are:

$$^{50}P_2 = \frac{50!}{(50-2)!} = \frac{48! \times 49 \times 50}{(48)!} = (49)(50) = 2450$$

Practice Questions:

1. How many ways are there to select 3 candidates from 8 equally qualified recent graduates for openings in an accounting firm?
2. In how many different ways can a true-false test consisting of 9 questions be answered?
3. A drug for the relief of asthma can be purchased from 5 different manufacturers in liquid, tablet, or capsule form, all of which come in regular and extra strength. How many different ways can a doctor prescribe the drug for a patient suffering from asthma?

ANY QUESTION