Lecture No.8

Introduction To Statistics, Statistics And Probability

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Measures of Moments

Raw Moments and Pure Moments

Relationship Between Pure and Raw Moments

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In this lecture

- **What are Moments?**
- **Why Moments are Important?**
- **Types of Moments**
- **Moments about mean or central moments**
- **Raw moments**
- **Arbitrary origin moments**
- **Relationship among moments**

Moments

- A moment is a quantitative measure of the shape of a set of points.

OR

- Moments describe the distribution of data.

- Moments are used to describe the basic characteristics of the data from its frequency distribution like

  – **measure of the central tendency** is given by the **first raw moment**

  – **measure of dispersion** is given by the **2nd moment about mean**

  – **symmetry/ skewness of the curve** is given by **3rd moment about mean**

  – **The peakendness or flatness of the distribution** is given by **4th moment about mean**.

Types of Moments

There are three types of moments

1. Moments about mean or central moments.

2. Raw moments.

3. Arbitrary origin moments.

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1. Moments about mean or central moments

- Which obtained through Arithmetic mean $\bar{X}$.
- The general formula for finding moments about mean is:

For ungroup data:

$$m_r = \frac{\sum (X - \bar{X})^r}{n}$$

For group data:

$$m_r = \frac{\sum f (X - \bar{X})^r}{\sum f}$$

Where $r = 1, 2, 3, 4$

Steps

1. Calculate mean, i.e.,

$$\bar{X} = \frac{\sum fx}{\sum f}$$

2. Take deviation from mean, i.e.,

$$(X_i - \bar{X})$$

3. Power 'r', i.e.,

$$(X_i - \bar{X})^r$$

Where,

For first moment, $r = 1$

For 2nd moment $r = 2$

For 3rd moment, $r = 3$

And for 4th moment, $r = 4$

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Example 1. Find first four moments about mean 3,6,2,1,7,5.
Solution.

$$\bar{X} = \frac{\sum X}{n} = \frac{24}{6}$$

$$\bar{X} = 4$$

X	$(X-\bar{X})$	$(X-\bar{X})^2$	$(X-\bar{X})^3$	$(X-\bar{X})^4$
3	-1	1	-1	1
6	2	4	8	16
2	-2	4	-8	16
1	-3	9	-27	81
7	3	9	27	81
5	1	1	1	1
$\sum X = 24$	$\sum(X-\bar{X}) = 0$	$\sum(X-\bar{X})^2 = 28$	$\sum(X-\bar{X})^3 = 0$	$\sum(X-\bar{X})^4 = 196$

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Now to find first four moments about mean:

$$m_1 = \frac{\sum(X - \bar{X})}{n} = \frac{0}{6}$$

$$\boldsymbol{m_1 = 0}$$

$$m_2 = \frac{\sum(X - \bar{X})^2}{n} = \frac{28}{6}$$

$$\boldsymbol{m_2 = 4.67}$$

2nd moment about mean is its variance also.

$$m_3 = \frac{\sum(X - \bar{X})^3}{n} = \frac{0}{6}$$

$$\boldsymbol{m_3 = 0}$$

$$m_4 = \frac{\sum(X - \bar{X})^4}{n} = \frac{196}{6}$$

$$\boldsymbol{m_4 = 32.67}$$

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Moments about Mean (Grouped data)
Example:

Calculate the first four moments about mean from the following frequency distribution:

Classes	110-119	120-129	130-139	140-149	150-159	160-169	170-179	180-189	190-199	200-209	210-219
f	1	4	17	28	25	18	13	6	5	2	1

Solution:

$$\boldsymbol{m_r = \frac{\sum f(X-\bar{X})^r}{\sum f}}$$

Where $r = 1,2,3,4$

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Calculations

For mean, $i.e.,$

$$\bar{X} = \frac{\sum fx}{\sum f}$$

$$\bar{X} =$$

$$\frac{18740}{120} = 156.667$$

Classes	$\boldsymbol{x}$	$\boldsymbol{f}$	$\boldsymbol{fx}$
$110-119$	114.5	1	114.5
$120-129$	124.5	4	498
$130-139$	134.5	17	2286.5
$140-149$	144.5	28	4046
$150-159$	154.5	25	3862.5
$160-169$	164.5	18	2961
$170-179$	174.5	13	2268.5
$180-189$	184.5	6	1107
$190-199$	194.5	5	972.5
$200-209$	204.5	2	409
$210-219$	214.5	1	214.5
Total		**120**	**18740**

X	f	fx	$X-\overline{X}$	$f(X-\overline{X})$	$(X-\overline{X})^2$	$f(X-\overline{X})^2$	$(X-\overline{X})^3$	$f(X-\overline{X})^3$	$(X-\overline{X})^4$	$f(X-\overline{X})^4$
114.5	1	114.5	−41.66666667	−41.66666667	1736.111111	1736.111111	−72337.96296	−72337.96296	3014081.79	3014081.79
124.5	4	498	−31.66666667	−126.6666667	1002.777778	4011.111111	−31754.62963	−127018.5185	1005563.272	4022253.09
134.5	17	2286.5	−21.66666667	−368.3333333	469.4444444	7980.555556	−10171.2963	−172912.037	220378.0864	3746427.47
144.5	28	4046	−11.66666667	−326.6666667	136.1111111	3811.111111	−1587.962963	−44462.96296	18526.23457	518734.568
154.5	25	3862.5	−1.666666667	−41.66666667	2.777777778	69.44444444	−4.62962963	−115.7407407	7.716049383	192.901235
164.5	18	2961	8.333333333	150	69.44444444	1250	578.7037037	10416.66667	4822.530864	86805.5556
174.5	13	2268.5	18.33333333	238.3333333	336.1111111	4369.444444	6162.037037	80106.48148	112970.679	1468618.83
184.5	6	1107	28.33333333	170	802.7777778	4816.666667	22745.37037	136472.2222	644452.1605	3866712.96
194.5	5	972.5	38.33333333	191.6666667	1469.444444	7347.222222	56328.7037	281643.5185	2159266.975	10796334.9
204.5	2	409	48.33333333	96.66666667	2336.111111	4672.222222	112912.037	225824.0741	5457415.123	10914830.2
214.5	1	214.5	58.33333333	58.33333333	3402.777778	3402.777778	198495.3704	198495.3704	11578896.6	11578896.6
$\sum$	**120**	**18740**		**0**		**43466.6666**		**516111.111**		**50013888.9**

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Now the four moments about mean are:

First moment: put ***r=1***

$$m_1 = \frac{\sum f(X-\bar{X})^1}{\sum f}$$

$$m_1 = \frac{0}{120} = 0$$

2nd Moment: put ***r=2***

$$m_2 = \frac{\sum f(X-\bar{X})^2}{\sum f}$$

$$m_2 = \frac{43466.66667}{120} = 362.222$$

2nd Moment about mean is also its Variance.

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3rd Moment: put *r=3*

$$m_3 = \frac{\sum f(X - \bar{X})^3}{\sum f}$$

$$m_3 = \frac{516111.1111}{120}$$

$$m_3 = 4300.925926$$

4th Moment: put *r=4*

$$m_4 = \frac{\sum f(X - \bar{X})^4}{\sum f}$$

$$m_4 = \frac{50013888.9}{120}$$

$$m_4 = 416782.4$$

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Example 2. Find first four moments about mean

classes	1-3	3-5	5-7	7-9
f	40	30	20	10

Solution.

$$\bar{X} = \frac{\sum fX}{\sum f} = \frac{400}{100}$$

$$\bar{\boldsymbol{X}} = \mathbf{4}$$

classes	f	X	fX	$f(X-\bar{X})$	$f(X-\bar{X})^2$	$f(X-\bar{X})^3$	$f(X-\bar{X})^4$
1-3	40	2	80	-80	160	-320	640
3-5	30	4	120	0	0	0	0
5-7	20	6	120	40	80	160	320
7-9	10	8	80	40	160	640	2560
	$\sum f = 100$		$\sum fX = 400$	$\sum f(X-\bar{X}) = 0$	$\sum f(X-\bar{X})^2 = 400$	$\sum f(X-\bar{X})^3 = 480$	$\sum f(X-\bar{X})^4 = 3520$

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Now to find first four moments about mean is

$$m_1 = \frac{\sum f(X-\bar{X})}{\sum f} = \frac{0}{100}$$

$$\boldsymbol{m_1 = 0}$$

$$m_2 = \frac{\sum f(X-\bar{X})^2}{\sum f} = \frac{400}{100}$$

$$\boldsymbol{m_2 = 4}$$

$$m_3 = \frac{\sum f(X-\bar{X})^3}{\sum f} = \frac{480}{100}$$

$$\boldsymbol{m_3 = 4.8}$$

$$m_4 = \frac{\sum f(X-\bar{X})^4}{\sum f} = \frac{3520}{100}$$

$$\boldsymbol{m_4 = 35.2}$$

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2. Raw moments

- Raw moments are obtained from origin or zero.
- The general formula for finding raw moments is:

For ungroup data

$$m'_r = \frac{\sum (X)^r}{n}$$

For group data

$$m'_r = \frac{\sum f(X)^r}{\sum f}$$

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Example 1. **Find first four raw moments, and convert into moments about mean 3,6,2,1,7,5.**

Solution.

X	X^2	X^3	X^4
3	9	27	81
6	36	216	1296
2	4	8	16
1	1	1	1
7	49	343	2401
5	25	125	625
$\sum X$ =**24**	$\sum X^2$ =124	$\sum X^3$ =720	$\sum X^4$ =4420

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Calculations:

First Raw Moment:

$$m_1' = \frac{\sum X}{n} = \frac{24}{6}$$

$$\boldsymbol{m_1' = 4}$$

Second Raw Moment:

$$m_2' = \frac{\sum X^2}{n} = \frac{124}{6}$$

$$\boldsymbol{m_2' = 20.67}$$

Third Raw Moment:

$$m_3' = \frac{\sum X^3}{n} = \frac{720}{6}$$

$$\boldsymbol{m_3' = 120}$$

Fourth Raw Moment:

$$m_4' = \frac{\sum X^4}{n} = \frac{4420}{6}$$

$$\boldsymbol{m_4' = 736.67}$$

Convert raw moments into moments about mean

$$m_1 = m_1' - m_1'$$

$$m_1 = 4 - 4$$

$$\boldsymbol{m_1} = \mathit{0}$$

$$m_2 = m_2' - (m_1')^2$$

$$m_2 = 20.67 - 16$$

$$\boldsymbol{m_2} = \mathit{4.67}$$

$$m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3$$

$$m_3 = 120 - 3(20.67)(4) + 2(4)^3$$

$$\boldsymbol{m_3} = \mathit{0}$$

$$m_4 = m_4' - 4m_3'm_1' + 6m_2'(m_1')^2 - 3(m_1')^4$$

$$m_4 = 736.67 - 4(120)(4) + 6(20.67)(4)^2 - 3(4)^4$$

$$\boldsymbol{m_4} = 32.67$$

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Example 2. Find Raw moments and convert into moments about mean

X	*2*	*4*	*6*	*8*
f	*40*	*30*	*20*	*10*

Solution:

X	f	fX	fX^2	fX^3	fX^4
2	40	80	160	320	640
4	30	120	480	1920	7680
6	20	120	720	4320	25920
8	10	80	640	5120	40960
	$\sum f = 100$	$\sum fX$ =40 0	$\sum fX^2$ =200 0	$\sum fX^3$ =11680	$\sum fX^4$ =7520 0

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Calculations:

First Raw Moment:

$$m_1' = \frac{\sum fX}{\sum f} = \frac{400}{100}$$

$$\boldsymbol{m_1' = 4}$$

Second Raw Moment:

$$m_2' = \frac{\sum fX^2}{\sum f} = \frac{2000}{100}$$

$$\boldsymbol{m_2' = 20}$$

Third Raw Moment:

$$m_3' = \frac{\sum fX^3}{\sum f} = \frac{11680}{100}$$

$$\boldsymbol{m_3' = 116.8}$$

Fourth Raw Moment:

$$m_4' = \frac{\sum fX^4}{\sum f} = \frac{75200}{100}$$

$$\boldsymbol{m_4' = 752}$$

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Convert raw moments into moments about mean

$$m_1 = m_1' - m_1'$$

$$m_1 = 4 - 4$$

$$\boldsymbol{m_1} = 0$$

$$m_2 = m_2' - (m_1')^2$$

$$m_2 = 20 - 16$$

$$\boldsymbol{m_2} = 4$$

$$m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3$$

$$m_3 = 116.8 - 3(20)(4) + 2(4)^3$$

$$\boldsymbol{m_3} = 4.8$$

$$m_4 = m_4' - 4m_3'm_1' + 6m_2'(m_1')^2 - 3(m_1')^4$$

$$m_4 = 752 - 4(116.8)(4) + 6(20)(4)^2 - 3(4)^4$$

$$\boldsymbol{m_4} = 35.2$$

3. Arbitrary origin moments

- Arbitrary origin moments are obtained from an assume value A (A is constant).
- The general formula for finding arbitrary moments is:

for ungroup data

$$m'_r = \frac{\sum (X - A)^r}{n}$$

for group data

$$m'_r = \frac{\sum f(X - A)^r}{\sum f}$$

Example 1. **Find first four arbitrary moments where A=7, and convert into moments about mean 3,7,7,7,7,8,8,8,18.**

Solution:

X	$(X-A)$	$(X-A)^2$	$(X-A)^3$	$(X-A)^4$
3	-4	16	-64	256
7	0	0	0	0
7	0	0	0	0
7	0	0	0	0
7	0	0	0	0
8	1	1	1	1
8	1	1	1	1
8	1	1	1	1
18	11	121	1331	14641
	$\sum(X-A)=10$	$\sum(X-A)^2=140$	$\sum(X-A)^3=1270$	$\sum(X-A)^4=14900$

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Calculations:

First Arbitrary Moment:

$$m_1' = \frac{\sum(X - A)}{n} = \frac{10}{9}$$

$$\boldsymbol{m_1' = 1.11}$$

Second Arbitrary Moment:

$$m_2' = \frac{\sum(X - A)^2}{n} = \frac{140}{9}$$

$$\boldsymbol{m_2' = 15.56}$$

Third Arbitrary Moment:

$$m_3' = \frac{\sum(X - A)^3}{n} = \frac{1270}{9}$$

$$\boldsymbol{m_3' = 141.11}$$

Fourth Arbitrary Moment:

$$m_4' = \frac{\sum(X - A)^4}{n} = \frac{14900}{9}$$

$$\boldsymbol{m_4' = 1655.56}$$

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Convert arbitrary moments into moments about mean

$$m_1 = m_1' - m_1'$$

$$m_1 = 1.11 - 1.11$$

$$\boldsymbol{m_1 = 0}$$

$$m_2 = m_2' - (m_1')^2$$

$$m_2 = 15.56 - 1.2321$$

$$\boldsymbol{m_2 = 14.33}$$

$$m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3$$

$$m_3 = 141.11 - 3(15.56)(1.11) + 2(1.11)^3$$

$$\boldsymbol{m_3 = 92.03}$$

$$m_4 = m_4' - 4m_3'm_1' + 6m_2'(m_1')^2 - 3(m_1')^4$$

$$m_4 = 1655.56 - 4(141.11)(1.11) + 6(15.56)(1.11)^2 - 3(1.11)^4$$

$$\boldsymbol{m_4 = 1139.51}$$

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Example 2. Find first four arbitrary moments, and convert into moments about mean, A=3

X	*1*	*2*	*3*	*4*
f	*2*	*3*	*4*	*1*

Solution:

X	f	$(X-A)$	$f(X-A)^1$	$f(X-A)^2$	$f(X-A)^3$	$f(X-A)^4$
1	2	-2	-4	8	-16	32
2	3	-1	-3	3	-3	3
3	4	0	0	0	0	0
4	1	1	1	1	1	1
	$\sum f=$ 10		$\sum f(X-A)=$-6	$\sum f(X-A)^2=$12	$\sum f(X-A)^3=$-18	$\sum f(X-A)^4=$36

Calculations:

First Arbitrary Moment:

$$m_1' = \frac{\sum f(X-A)}{\sum f} = \frac{-6}{10}$$

$$\boldsymbol{m_1' = -0.6}$$

Second Arbitrary Moment:

$$m_2' = \frac{\sum f(X-A)^2}{\sum f} = \frac{12}{10}$$

$$\boldsymbol{m_2' = 1.2}$$

Third Arbitrary Moment:

$$m_3' = \frac{\sum f(X-A)^3}{\sum f} = \frac{-18}{10}$$

$$\boldsymbol{m_3' = -1.8}$$

Fourth Arbitrary Moment:

$$m_4' = \frac{\sum f(X-A)^4}{\sum f} = \frac{36}{10}$$

$$\boldsymbol{m_4' = 3.6}$$

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Convert arbitrary moments into moments about mean

$$m_1 = m_1' - m_1'$$

$$m_1 = -0.6 + 0.6$$

$$\boldsymbol{m_1} = \mathbf{0}$$

$$m_2 = m_2' - (m_1')^2$$

$$m_2 = 1.2 - 0.36$$

$$\boldsymbol{m_2} = \mathbf{0.84}$$

$$m_3 = m_3' - 3m_2'm_1' + 2(m_1')^3$$

$$m_3 = -1.8 - 3(1.2)(-0.6) + 2(-0.6)^3$$

$$\boldsymbol{m_3} = \mathbf{-0.072}$$

$$m_4 = m_4' - 4m_3'm_1' + 6m_2'(m_1')^2 - 3(m_1')^4$$

$$m_4 = 3.6 - 4(-1.8)(-0.6) + 6(1.2)(-0.6)^2 - 3(-0.6)^4$$

$$\boldsymbol{m_4} = \mathbf{1.48}$$

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ANY QUESTION