Lecture 3

Normal Distribution: Inverse use of Area Table

(Area → Z → Get X as Answer)

Review of Previous Lecture

- **Direct use of Area Table: Probability under Normal Distribution**
- **(X → Z → Get Area as Answer)**

In this lecture, the general idea is

- **Inverse use of Area Table: Find one point (Percentile) and two points containing specific area between them.**
- **(Area → Z → Get X as Answer)**

Exercise Questions: 9.19 to 9.37 (page 390 to 392)

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Inverse Use of Area Table (Find μ or σ)

Example 9.11 (page. 370): In a normal distribution $\mu = 40$ and $P(25 \le X \le 55) = 0.8662$. Find $P(20 \le X \le 60)$.

Solution: As given $X \sim N(40, \sigma)$ so $\mu = 40,\ \sigma = ?$

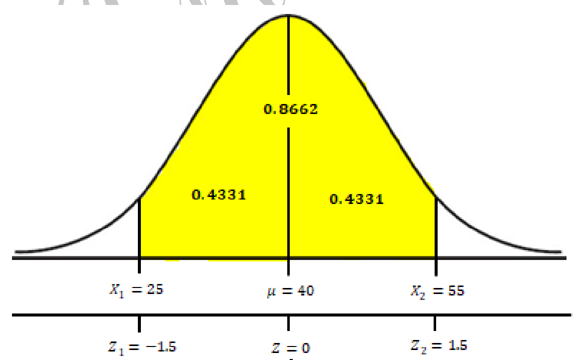
$$P(25 \le X \le 55) = 0.8662.$$

i. $\boldsymbol{P(25 \le X \le 55) = 0.8662,\ \ \sigma = ?}$

$Z_1 = \frac{X_1 - \mu}{\sigma}$, at $X_1 = 25,\ \mu = 40,$

We have $Z_1 = \frac{X_1 - \mu}{\sigma}$

$$-1.5 = \frac{25-40}{\sigma} = \frac{-15}{\sigma} \quad \Longrightarrow \quad \sigma = 10$$



ii. $\boldsymbol{P(20 \le X \le 60) = ?}$

At $X_1 = 20$, we have $Z_1 = \frac{X_1 - \mu}{\sigma} = \frac{20-40}{10} = -2$

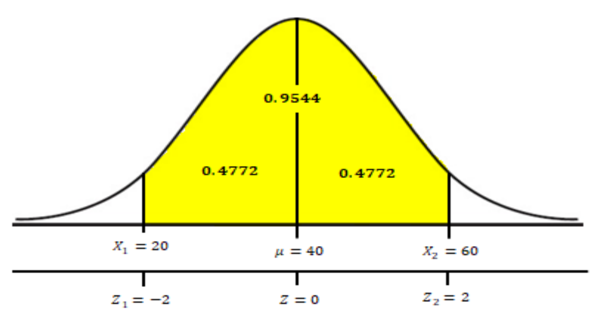
At $X_2 = 40$, we have $Z_2 = \frac{X_2 - \mu}{\sigma} = \frac{60-40}{10} = 2$

$P(20 \le X \le 60) = P(-2 \le Z \le 2)$

$= P(-2 \le Z \le 0) + P(0 \le Z \le 2)$

$= 0.4772 + 0.4772$

$P(20 \le X \le 60) = 0.9544$



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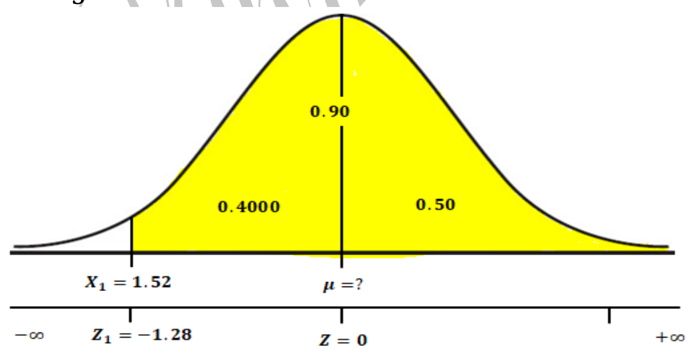
Example 9.14 (page. 371): An athlete find that in a high jump he can clear height of 1.68m once in five attempts and a height of 1.52m nine times out of ten attempts. Assuming the heights he can clear in various jumps from a normal distribution, estimate the mean and standard deviation of the distribution.

Solution: As given $X \sim N(\mu, \sigma)$ so $\mu = ?,\ \sigma = ?$

$P(X \geq 1.52) = \frac{9}{10} = 0.90$ and $P(X \geq 1.68) = \frac{1}{5} = 0.20$

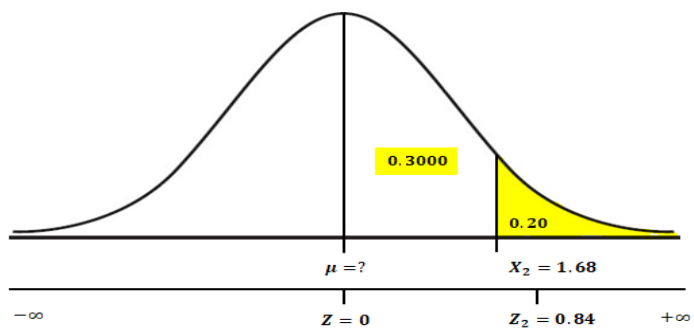


$\boldsymbol{P(X \geq 1.52) = 0.90}$

$$Z_1 = \frac{X_1 - \mu}{\sigma} \quad \Longrightarrow -1.28 = \frac{1.52 - \mu}{\sigma}$$

$$\mu - 1.28\sigma = 1.52 \qquad (1)$$

$\boldsymbol{P(X \geq 1.68) = 0.20}$



$$Z_2 = \frac{X_2 - \mu}{\sigma} \quad \Longrightarrow 0.84 = \frac{1.68 - \mu}{\sigma}$$

$$\mu + 0.84\sigma = 1.68 \qquad (2)$$

Subtract equation (1) from equation (2)

$$(\mu + 0.84\sigma) - (\mu - 1.28\sigma) = 1.68 - 1.52$$

$$\mu + 0.84\sigma - \mu + 1.28\sigma = 0.16$$

$$2.12\sigma = 0.16 \qquad \Longrightarrow \qquad \sigma = 0.075$$

Put equation $\sigma = 0.075$ in (1)

$$\mu - 1.28(0.075) = 1.52 \qquad \Longrightarrow \qquad \mu = 1.617$$

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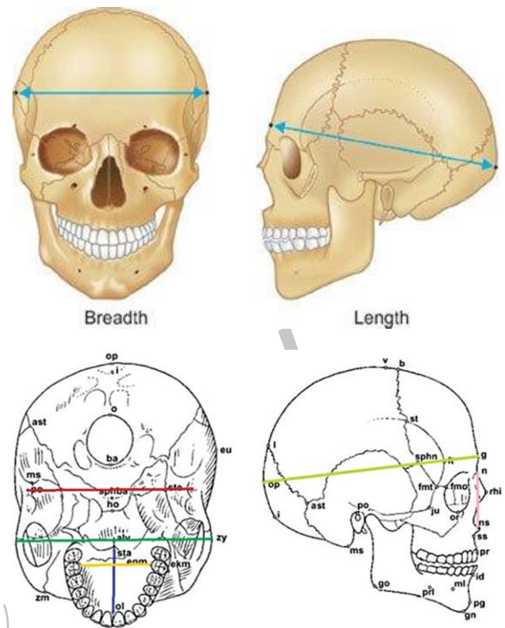
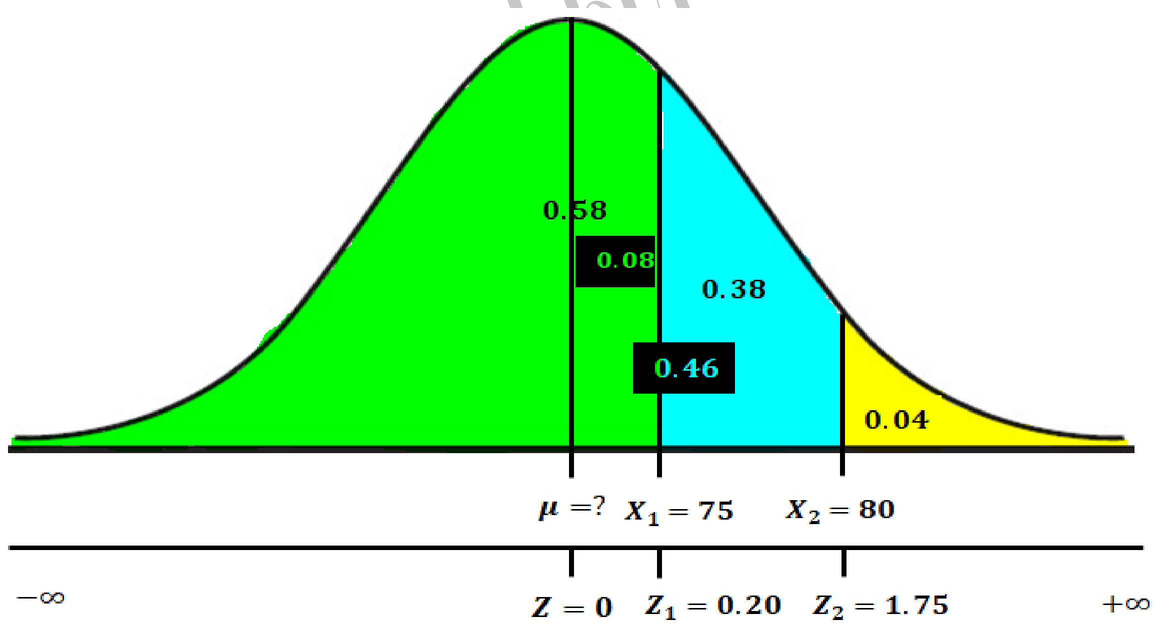
Example 9.15 (page. 372): A collection of human skulls divide into three classes **A**, **B** and **C** according to the value of a "*length-breadth index*" X. Skulls with $X < 75$ are classified as **A** (*long-headed*), those with $75 < X < 80$ as **B** (*medium-headed*) and those with $X > 80$ as **C** (*short-headed*). The percentages in the three classes in the collection are 58, 38 and 4 respectively. Find approximately the mean and the standard deviation of X, on the assumption that X in normally distributed.



Solution: As given $X \sim N(\mu, \sigma)$ so $\mu = ?,\ \sigma = ?$

A: $P(X < 75) = 0.58$, **B**: $P(75 < X < 80) = 0.38$, **C**: $P(X > 80) = 0.04$



$$Z_1 = \frac{X_1 - \mu}{\sigma} \implies 0.20 = \frac{75 - \mu}{\sigma} \qquad Z_2 = \frac{X_2 - \mu}{\sigma} \implies 1.75 = \frac{80 - \mu}{\sigma}$$

$$\mu + 0.20\sigma = 75 \qquad (1) \qquad \mu + 1.75\sigma = 80 \qquad (2)$$

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Subtract equation (1) from equation (2)

$$(\mu + 1.75\sigma) - (\mu + 0.20\sigma) = 80 - 75$$

$$\mu + 1.75\sigma - \mu - 0.20\sigma = 5$$

$$1.55\sigma = 5 \qquad \Longrightarrow \qquad \sigma = 3.23$$

Put equation $\sigma = 3.23$ in (1)

$$\mu + 0.20(3.23) = 75 \qquad \Longrightarrow \qquad \mu = 74.4$$

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Inverse Use of Area Table (Find X)

Example 9.12 (page. 370): The time required by a nurse to inject a shot of penicillin has been observed to be normally distributed, with a mean of $\mu = 30$ seconds and a standard deviation of $\sigma = 10$ seconds. Find the following:

- **i.** 10^{th} percentile (The time that has 10% area below it)
- **ii.** 90^{th} percentile (The time that has 90% area below it)
- **iii.** The time (seconds) that has 35% area above it.
- **iv.** The time (seconds) that has 77% area above it.
- **v.** Two points such that a single observation has 97% area between them.

Solution: As given $X \sim N(30, 10)$ so $\mu = 30,\ \sigma = 10$

a) $P_{10} = ?$ **The value of X (seconds) that has 10% area blow it**

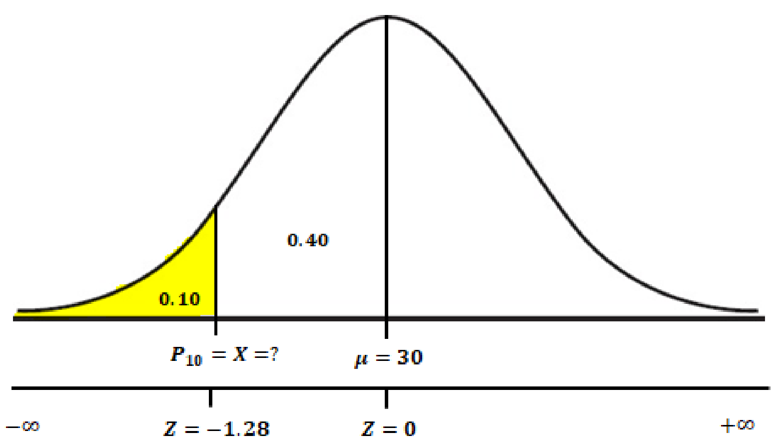
As $Z = \frac{X-\mu}{\sigma}$

$$-1.28 = \frac{P_{10}-30}{10}$$

$$-1.28(10) = P_{10} - 30$$

$$-12.8 + 30 = P_{10}$$

$$P_{10} = 17.2 \text{ Seconds}$$



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b) P_{90}=? **The value of X (seconds) that has 90% area blow it**

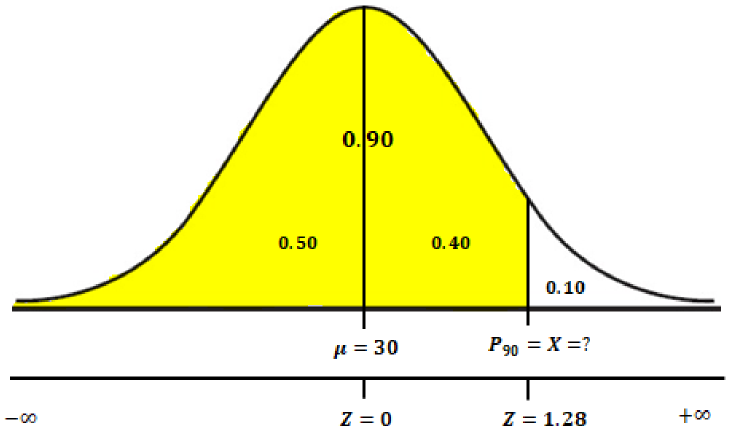
As $Z = \frac{X-\mu}{\sigma}$

$1.28 = \frac{P_{90}-30}{10}$

$1.28(10) = P_{90} - 30$

$12.8 + 30 = P_{90}$

$P_{10} = 42.8$ Seconds



c) The time X (seconds) that has 35% area above it.

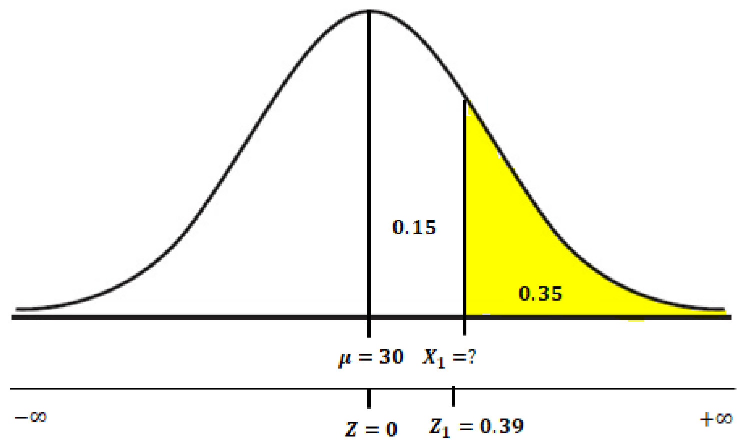
As $Z = \frac{X-\mu}{\sigma}$

$0.39 = \frac{X_1-30}{10}$

$0.39(10) = X_1 - 30$

$3.9 + 30 = X_1$

$X_1 = 33.9$ Seconds



d) The time X (seconds) that has 77% area above it.

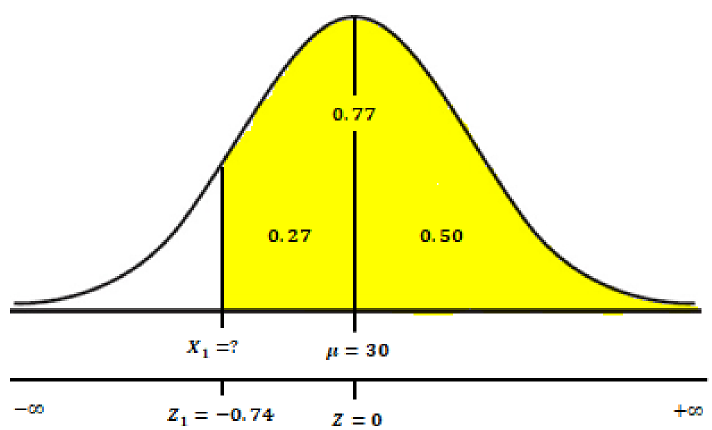
As $Z = \frac{X-\mu}{\sigma}$

$-0.74 = \frac{X_1-30}{10}$

$-0.74(10) = X_1 - 30$

$-7.4 + 30 = X_1$

$X_1 = 22.6$ Seconds



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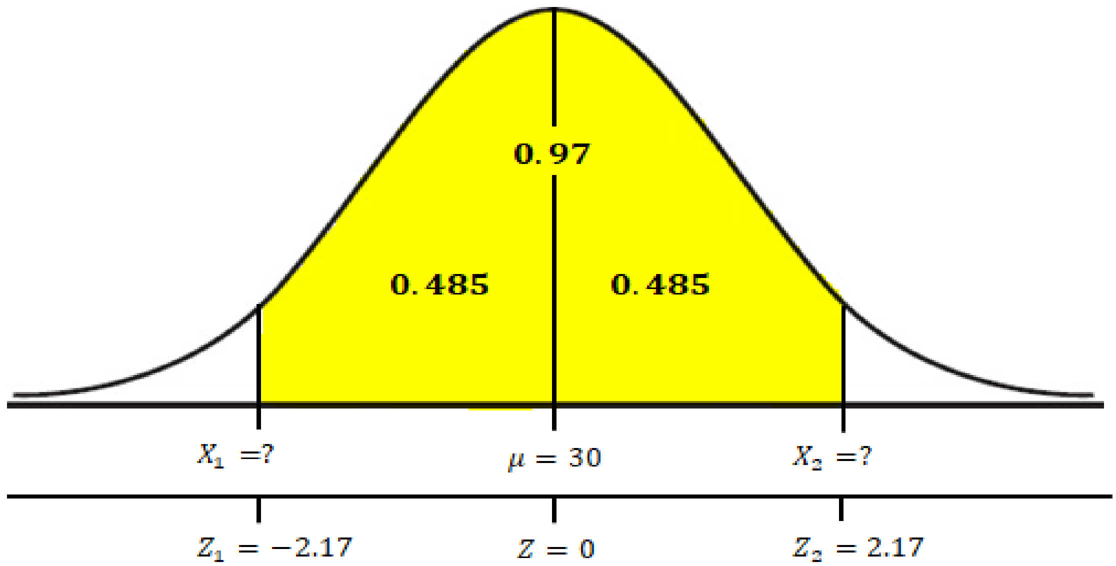
e) Two points such that a single observation has 97% area between them



As $Z_1 = \frac{X_1-\mu}{\sigma}$

$-2.17 = \frac{X_1-30}{10}$

$-2.17(10) = X_1 - 30$

$-21.7 + 30 = X_1$

$X_1 = 8.3$ Seconds

As $Z_{21} = \frac{X_2-\mu}{\sigma}$

$2.17 = \frac{X_2-30}{10}$

$2.17(10) = X_2 - 30$

$21.7 + 30 = X_2$

$X_2 = 51.7$ Seconds

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Example 9.16 (page. 372): A lawyer commutes daily from his suburban home to his midtown office. On average, the trip one way takes 24 minutes, with a standard deviation of 3.8 minutes. Assume distribution of trip times to be normally distributed.

a) What is the probability that a trip will take at least $\frac{1}{2}$ hour?

b) If the office opens at 9:00 AM and he leaves his house at 8:45 AM daily, what percentage of the time is he late for work?

c) If he leaves the house at 8:35 AM and coffee is served at the office from 8:50 AM until 9:00AM, what is the probability that he misses coffee?

d) Find the length of time above which we find the slowest 15% of the trips.

e) Find the probability that 2 of the next 3 trips will take at least $\frac{1}{2}$ hour.

Solution: Let X be the trip time (min), so $X \sim N(24,\ 3.8)$ so $\mu = 24$ and $\sigma = 3.8$.

a. $\boldsymbol{P(X \geq 30) = ?}$ **The trip will take at least $\frac{1}{2}$ hour**

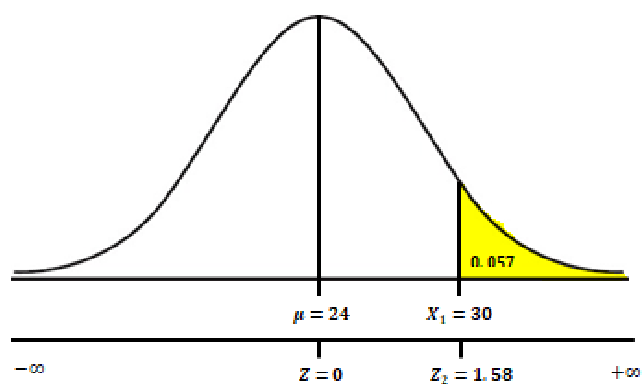
At $X = 30$, we have $Z = \frac{X-\mu}{\sigma} = \frac{30-24}{3.8} = 1.58$

$P(Z \geq 1.58)$

$= P(0 \leq Z \leq +\infty) - P(0 \leq Z \leq 1.58)$

$= 0.50 - 0.4430$

$P(X \geq 30) = 0.057$



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b. $\boldsymbol{P(X \geq 15) = ?}$ He leaves home at 8:45 AM and the office opens at 9.00 AM implies that he has 15 minutes to reach the office. He will be late for work if he takes more than 15 minutes, so we have to find $P(X \geq 15)$.

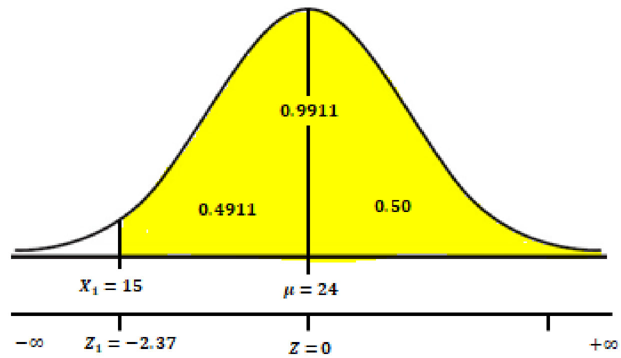
At $X = 15$, we have $Z = \frac{X-\mu}{\sigma} = \frac{15-24}{3.8} = -2.37$

$P(Z \geq -2.37)$

$= P(-2.37 \leq Z \leq 0) + P(0 \leq Z \leq \infty)$

$= 0.4911 + 0.5$

$P(X \geq 15) = 0.9911$



c. $\boldsymbol{P(X \geq 25) = ?}$ He leaves home at 8:35 AM and coffee is served from 8:50 AM until 9:00 AM. He will miss coffee if he reaches office after 9:00 AM, i.e. if he takes 25 minutes or more time. Thus we need $P(X \geq 25)$.

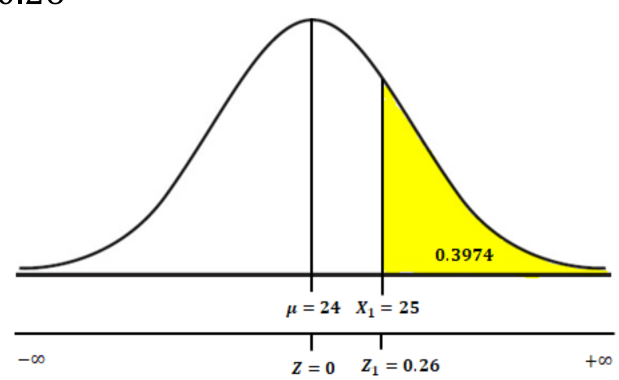
At $X = 25$, we have $Z = \frac{X-\mu}{\sigma} = \frac{25-24}{3.8} = 0.26$

$P(Z \geq 0.26)$

$= P(0 \leq Z \leq \infty) - P(0 \leq Z \leq 0.26)$

$= 0.5 - 0.1026$

$P(X \geq 25) = 0.3974$



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d. The length of time above which we find the slowest 15% of the trips

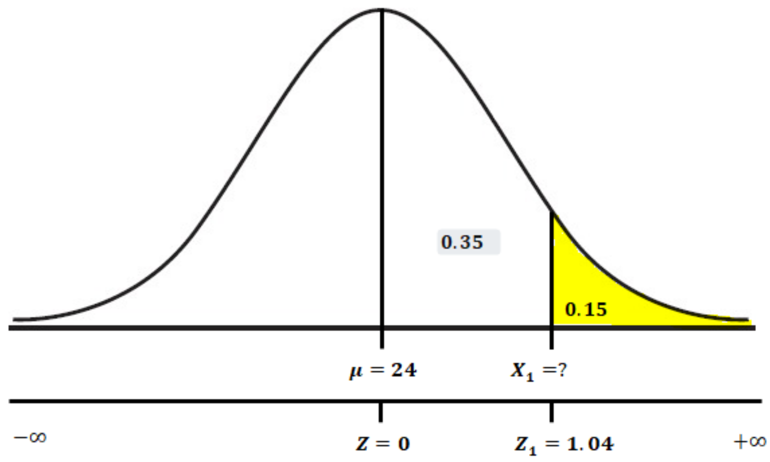
As $Z = \frac{X-\mu}{\sigma}$

$1.04 = \frac{X_1-24}{3.8}$

$1.04(3.8) = X_1 - 30$

$3.952 + 24 = X_1$

$X_1 = 27.952$ Seconds



e. The probability that 2 of the next 3 trips will take at least $\frac{1}{2}$ hour

$P(X \geq 30) = 0.057 = p$ From part (a)

$n = 3$ and $q = 1 - p = 1 - 0.057 = 0.943$

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

$$P(X = 2) = \binom{3}{2}(0.057)^2(0.943)^1$$

$$P(X = 2) = 0.0092$$

In next Lecture

- **Sampling, Probability Sampling, Types and Application**

THE END ☺

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