Weekly Assignment 1

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Algorithm Design and Analysis

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Solution 2:

To prove an undirected graph $G = \langle V, E \rangle$ for vertices $n > 2$, with $|E| > \frac{(n-1)(n-2)}{2}$, we will follow the 'proof by contradiction method'.
Assume for contradiction, G is not connected and $|E|$ has more than $\frac{(n-1)(n-2)}{2}$ edges. Let's define C of size k is the smallest connected component of G where $k \geq 1$. Other vertices forms the set $V \backslash C$, hence, $|C| = k$ and $|V \backslash C| = n - k$. Suppose, $k = min|C|, |V \backslash C|$ such that, $k \epsilon [1, [n/2]]$.

- **Case - 1:** For $k = 1$, C is just an isolated vertex. The maximum number of edges in this case is within the $V \backslash C$ component, which is $|E| = \frac{(n-1)(n-2)}{2}$. But if G has more edges than $\frac{(n-1)(n-2)}{2}$, then there cannot be an isolated vertex since that violates our assumption, hence $k > 1$.

- **Case - 2:** For $k > 1$, G has two partitions, one with k vertices and another with $n - k$ vertices. The maximum no. of edges for $E_C = \frac{k(k-1)}{2}$ and for $E_{V \backslash C} = \frac{(n-k)(n-k-1)}{2}$. Since, it is assumed that G is disconnected, the total no. edges would be $|E| = |E_C| + |E_{V \backslash C}|$. Simplifying the inequality that G cannot have more edges than $\frac{(n-1)(n-2)}{2}$ we get,
$$k(k-1) + (n-k)(n-k-1) \leq (n-1)(n-2)$$
Further simplifying,
$$2k^2 - 2nk + n^2 - n \leq n^2 - 3n + 2$$
Finally,
$$2k^2 - 2nk + 2n \leq 0 \tag{1}$$

- **Case - 3:** Now, for a value of $k \leq \frac{n}{2}$, the inequality becomes, $k(n-k) \leq \frac{n^2}{4}$, so eqn.(1) holds as k being small enough keeps the maximum number of edges below $\frac{(n-1)(n-2)}{2}$.

Thus by proof of contradiction and using the case analysis, we can say that $G = \langle V, E \rangle$ can have at most $|E| = \frac{(n-1)(n-2)}{2}$. Therefore, if G has more than that many edges, then it becomes connected.