Weekly Assignment 6

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Algorithm Design and Analysis

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Solution 1:

The claim: "In a directed graph G with no self-loops, if a vertex u has both incoming and outgoing edges, then there is at least one other node in the tree to which u belongs in the DFS forest after running DFS(G)." needs to be disproved.

Finding a case where a vertex u has both incoming and outgoing edges, but is alone in its DFS tree can disprove this claim.

Suppose a directed graph G with vertices A, B, C, have edges (A, B), (B, C), and (C, A), forming a cycle: $A \to B \to C \to A$ with no self-loops. Here, each vertex has both incoming and outgoing edges.

- Let's pick the first root to be A. DFS will visit B, then C, then try to visit A again but find it's already visited. The DFS tree will be: $A|B|C$
- If we pick B as the starting vertex, we get: $B|C|A$
- Similarly, starting from C gives: $C|A|B$.

In each case, the vertex we start from (let's call it u) ends up as the root of a tree containing all vertices. However, if we consider only the tree to which u belongs i.e., only the vertices that u can reach through DFS, u is alone in its tree.

This process disproves the claim in question.

Solution 2:

Corollary 24.3: Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \to R$. Then for each vertex $v \in V$, there is a path from s to v if and only if BELLMAN-FORD terminates with $d[v] < \infty$ when it is run on G.

Proof: Suppose there is a path from s to v.

- There must be a shortest such path of length $\delta(s, v)$
- This path must have finite length since it contains at most $|V| - 1$ edges and each edge has finite length.
- By Lemma 24.2 from the textbook, $d[v] = \delta(s, v) < \infty$ upon termination of BELLMAN-FORD algorithm.

Again, suppose $d[v] < \infty$ when BELLMAN-FORD terminates.

- $d[v]$ is monotonically decreasing throughout the algorithm.
- RELAX will update $d[v]$ only if $d[u] + w(u, v) < d[v]$ for some u adjacent to v.
- When $d[v]$ is updated, $\pi[v] = u$ is also updated, so v has an ancestor in the predecessor subgraph.

Since this is a tree rooted at s, there must be a path from s to v in this tree. Therefore, there is a path from s to v if and only if BELLMAN-FORD terminates with $d[v] < \infty$.

Solution 3:

Proof: Understanding the $Initialize - Single - Source(G, s)$ function:

- its sets $d[s] = 0$
- For all vertices $v \neq s$ it sets $d[v] = \infty$
- For all vertices v (including s), it sets $\pi[v] = NIL$

Now, suppose u is the vertex which first caused $\pi[s] \neq NIL$. This means that during $Relax(u, s, w) : 0 = d[s] > d[u] + w(u, s)$.

There must be some path from s to u in the graph for $d[u]$ to get its value. Let's call this path p. So we can form a cycle C by:

- Following path p from s to u
- Then taking the edge (u, s) back to s

The weight of this cycle C is $w(C) = w(p) + w(u, s) = d[u] + w(u, s) < 0$ because $d[u]$ is the weight of the current shortest path from s to u.

Therefore, we've found a cycle C with negative weight.