Weekly Assignment 10

Munia Humaira
Student ID: 21116435
Algorithm Design and Analysis

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Solution 1:

Let's consider for the counterexample $n = 3$ and $A_1 = 10 \times 5$, $A_2 = 5 \times 20$, $A_3 = 20 \times 10$.

According to Alice's greedy choice, the parenthesization can be written as $((A_1 \cdot A_2) \cdot A_3)$.

The number of scalar multiplications for the greedy choice:

$A_1 \cdot A_2 = 10 \times 5 \times 20 = 1000$

$((A_1 \cdot A_2) \cdot A_3) = 10 \times 20 \times 10 = 2000$

Total operation $= 3000$

If we consider different parenthesization $(A_1 \cdot (A_2 \cdot A_3))$ the number of operations become:

$A_2 \cdot A_3 = 5 \times 20 \times 10 = 1000$

$(A_1 \cdot (A_2 \cdot A_3)) = 10 \times 5 \times 10 = 500$

Total operation $= 1500$

This gives the optimum number of operations. So we can say that the greedy choice does not necessarily lead to a global optimum.

Solution 2:

Modified recurrence relation,

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} \text{ if } d_{ij}^{(k-1)} < d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \\ \pi_{kj}^{(k-1)} \text{ if } d_{ij}^{(k-1)} \geq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \end{cases}$$

The example graph $G = \langle V, E, w \rangle$ considered such that $V = 4$ and $E = 6$.

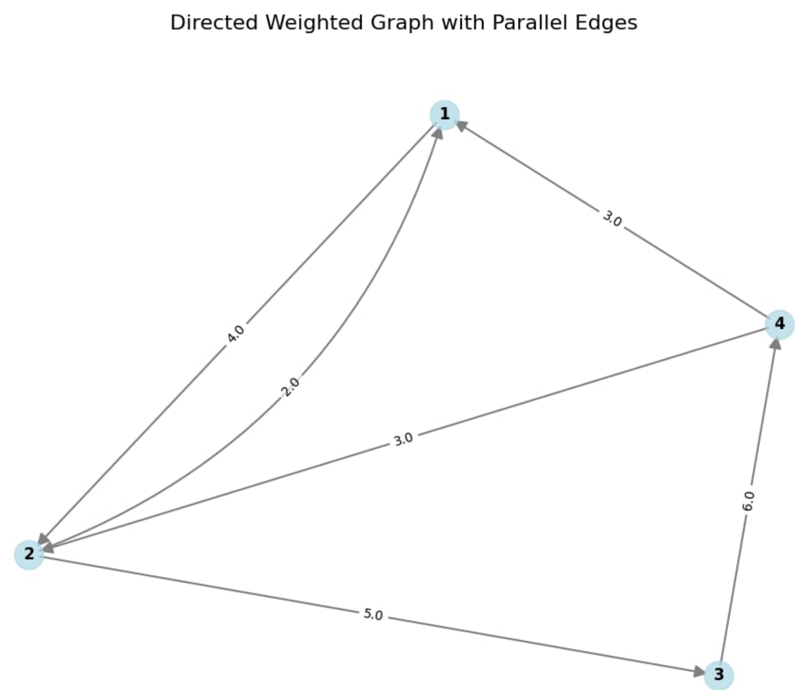

Figure 1: An example directed graph

Let's write the $D^{(0)}$ and $\Pi^{(0)}$ matrices for this graph:

$$D^{(0)} = \begin{pmatrix} 0 & 4 & \infty & \infty \\ 2 & 0 & 5 & \infty \\ \infty & \infty & 0 & 6 \\ 3 & 3 & \infty & 0 \end{pmatrix}$$

and,

$$\Pi^{(0)} = \begin{pmatrix} NILL & 1 & NILL & NILL \\ 2 & NILL & 2 & NILL \\ NILL & NILL & NILL & 3 \\ 4 & 4 & NILL & NILL \end{pmatrix}$$

Now for $k \geq 1$ let's calculate both of the matrices. The $1st$ row and $1st$ column will stay same. I am calculating both the conditions to determine the element of the predecessor matrix.

$$d_{23}^{(0)} < d_{21}^{(0)} + d_{13}^{(0)} = 5 < 2 + \infty$$
$$d_{23}^{(0)} \geq d_{21}^{(0)} + d_{13}^{(0)} = 5 \geq 2 + \infty$$

It can be seen that the $2nd$ condition is not possible as 5 is not greater than ∞. So the $1st$ condition gives us the value of the predecessor value $\pi_{23}^{(1)} = 1$, which is not correct. If we would have considered the recurrence given in the textbook, the value returned should have been $\pi_{23}^{(1)} = 2$, which would have been the correct value.

Similarly,

$$d_{24}^{(0)} < d_{21}^{(0)} + d_{14}^{(0)} = \infty < 2 + \infty$$
$$d_{24}^{(0)} \geq d_{21}^{(0)} + d_{14}^{(0)} = \infty \geq 2 + \infty \approx \infty = \infty$$

It also gives wrong value for the predecessor matrix $\pi_{24}^{(1)} = 1$ whereas it should have been $\pi_{24}^{(1)} = NILL$ if computed using the correct relation.

Since we have found 2 wrong values already for the predecessor matrix $\Pi^{(1)}$ using the relation given, we can safely say that given recurrence for $\pi_{ij}^{(k)}$ is no longer correct.

Solution 3:

A set of hash functions $\mathcal{H}$ is said to be universal if for any two distinct keys $k, l \in \mathcal{U}$ in the input domain, the probability of a collision is at most $1/m$, where m is the size of the codomain. This definition ensures that the hash functions distribute keys uniformly across the hash table and minimize collisions. To prove this let's consider the following example:

$\mathcal{U} = \{a, b, c\}$ and $m = 2$. This gives, $\mathcal{F} = 8$. Now, the set of all possible hash functions that will map elements from $\mathcal{U}$ to $\{0, 1\}$:

- $f_1(a) = 0, f_1(b) = 0, f_1(c) = 0$
- $f_2(a) = 0, f_2(b) = 0, f_2(c) = 1$
- $f_3(a) = 0, f_3(b) = 1, f_3(c) = 0$
- $f_4(a) = 0, f_4(b) = 1, f_4(c) = 1$
- $f_5(a) = 1, f_5(b) = 0, f_5(c) = 0$
- $f_6(a) = 1, f_6(b) = 0, f_6(c) = 1$
- $f_7(a) = 1, f_7(b) = 1, f_7(c) = 0$
- $f_8(a) = 1, f_8(b) = 1, f_8(c) = 1$

The colliding pairs are:

- For (a, b) : 4 functions (f_1, f_2, f_7, f_8)
- For (a, c) : 4 functions (f_1, f_3, f_6, f_8)
- For (b, c) : 4 functions (f_1, f_4, f_5, f_8)

The probability of collision per pair, $\mathcal{P} = \frac{4}{8} = \frac{1}{2} = \frac{1}{m}$.

This example satisfies the condition for the set of all functions with domain $\mathcal{U}$ and codomain $0, ..., m-1$ being universal.

Solution 4:

For a key x_i inserted as the $i-$th element into the table:

All elements $x_1, x_2, x_3, ..., x_{i-1}$ that hash to the same slot as x_i must be examined during a search for x_i. Let X_{ij} be the indicator random variable:

$$X_{ij} = \begin{cases} 1, \text{if } h(k_i = h(k_j) \\ 0, \text{otherwise} \end{cases}$$

The number of elements examined during the search for x_i is:

$$1 + \sum_{j=1}^{i-1} X_{ij}$$

Thus, the expected number of elements examined in a successful search is:

$$\mathbf{E}\left[\text{elements examined}\right] = 1 + \mathbf{E}\left[\sum_{j=1}^{i-1} X_{ij}\right]$$

Using linearity of expectation,

$$\mathbf{E}\left[\sum_{j=1}^{i-1} X_{ij}\right] = \sum_{j=1}^{i-1} \mathbf{E}\left[X_{ij}\right]$$

$$= \sum_{j=1}^{i-1} \frac{1}{m} = \frac{i-1}{m}$$

The expected number of elements examined for x_i is therefore:

$$\mathbf{E}\left[\text{elements examined}\right] = 1 + \frac{i-1}{m}$$

Averaging over all n elements,

$$\mathbf{E}\left[\text{total}\right] = \frac{1}{n}\left(n + \frac{1}{m}\sum_{i=1}^{n}(i-1)\right)$$

$$\mathbf{E}\left[\text{total}\right] = \frac{1}{n}\left(n + \frac{1}{m}\frac{n(n-1)}{2}\right) = 1 + \frac{n(n-1)}{2mn}$$

Let, $\alpha = \frac{n}{m}$. Then:

$$\mathbf{E}\left[\text{total}\right] = 1 + \frac{\alpha(n-1)}{2n}$$

For large enough n, this can be approximated as,

$$\mathbf{E}\left[\text{total}\right] \approx 1 + \frac{\alpha}{2}$$

So, the total time required for a successful search, $\theta(1 + \alpha)$.