Formulas

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[redacted]

(4) $$\int_{x_0}^{x_1} f(x)\,dx \approx \frac{h}{2}(f_0 + f_1)$$ (trapezoidal rule),

(5) $$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2)$$ (Simpson's rule),

(6) $$\int_{x_0}^{x_3} f(x)\,dx \approx \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3)$$ (Simpson's $\frac{3}{8}$ rule),

(7) $$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4)$$ (Boole's rule).

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Example 7.2. Consider the integration of the function $f(x) = 1 + e^{-x}\sin(4x)$ over the fixed interval $[a, b] = [0, 1]$. Apply the various formulas (4) through (7).

For the trapezoidal rule, $h = 1$ and

$$\int_0^1 f(x)\,dx \approx \frac{1}{2}(f(0) + f(1))$$
$$= \frac{1}{2}(1.00000 + 0.72159) = 0.86079.$$

For Simpson's rule, $h = 1/2$, and we get

$$\int_0^1 f(x)\,dx \approx \frac{1/2}{3}(f(0) + 4f(\tfrac{1}{2}) + f(1))$$
$$= \frac{1}{6}(1.00000 + 4(1.55152) + 0.72159) = 1.32128.$$

For Simpson's $\frac{3}{8}$ rule, $h = 1/3$, and we obtain

$$\int_0^1 f(x)\,dx \approx \frac{3(1/3)}{8}(f(0) + 3f(\tfrac{1}{3}) + 3f(\tfrac{2}{3}) + f(1))$$
$$= \frac{1}{8}(1.00000 + 3(1.69642) + 3(1.23447) + 0.72159) = 1.31440.$$

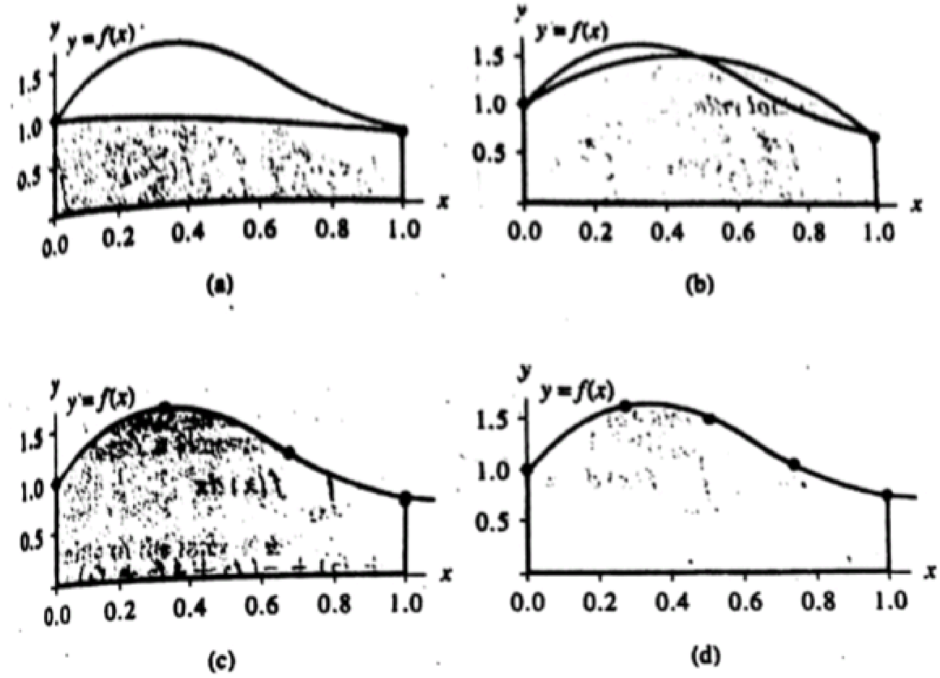

Figure 7.3 (a) The trapezoidal rule used over [0, 1] yields the approximation 0.86079. (b) Simpson's rule used over [0, 1] yields the approximation 1.32128. (c) Simpson's $\frac{3}{8}$ rule used over [0, 1] yields the approximation 1.31440. (d) Boole's rule used over [0, 1] yields the approximation 1.30859.

For Boole's rule, $h = 1/4$, and the result is

$$\int_0^1 f(x)\,dx \approx \frac{2(1/4)}{45}(7f(0) + 32f(\tfrac{1}{4}) + 12f(\tfrac{1}{2}) + 32f(\tfrac{3}{4}) + 7f(1))$$
$$= \frac{1}{90}(7(1.00000) + 32(1.65534) + 12(1.55152)$$
$$+ 32(1.06666) + 7(0.72159)) = 1.30859.$$

The true value of the definite integral is

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...tively.

To make a fair comparison of quadrature methods, we must use the same number of function evaluations in each method. Our final example is concerned with comparing

integration over a fixed interval $[a, b]$ using exactly five function evaluations $f_k = f(x_k)$, for $k = 0, 1, \ldots, 4$ for each method. When the trapezoidal rule is applied on the four subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, and $[x_3, x_4]$; it is called a *composite trapezoidal rule*:

$$
\begin{aligned}
\int_{x_0}^{x_4} f(x)\,dx &= \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \int_{x_3}^{x_4} f(x)\,dx \\
&\approx \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \frac{h}{2}(f_2 + f_3) + \frac{h}{2}(f_3 + f_4) \qquad (17) \\
&= \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + f_4).
\end{aligned}
$$

Simpson's rule can also be used in this manner. When Simpson's rule is applied on the two subintervals $[x_0, x_2]$ and $[x_2, x_4]$, it is called a ***composite Simpson's rule***:

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$$\begin{aligned}\int_{x_0}^{x_4} f(x)\,dx &= \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx \\ &\approx \frac{h}{3}(f_0 + 4f_1 + f_2) + \frac{h}{3}(f_2 + 4f_3 + f_4) \\ &= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4).\end{aligned} \tag{18}$$

The next example compares the values obtained with (17), (18), and (7).

Example 7.3. Consider the integration of the function $f(x) = 1 + e^{-x}\sin(4x)$ over $[a, b] = [0, 1]$. Use exactly five function evaluations and compare the results from the composite trapezoidal rule, composite Simpson rule, and Boole's rule.

The uniform step size is $h = 1/4$. The composite trapezoidal rule (17) produces

$$\int_0^1 f(x)\,dx \approx \frac{1/4}{2}\left(f(0) + 2f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 2f(\tfrac{3}{4}) + f(1)\right)$$

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$$= \frac{1}{8}(1.00000 + 2(1.65534) + 2(1.55152) + 2(1.06666) + 0.72159)$$
$$= 1.28358.$$

Using the composite Simpson's rule (18), we get

$$\int_0^1 f(x)\,dx \approx \frac{1/4}{3}\left(f(0) + 4f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 4f(\tfrac{3}{4}) + f(1)\right)$$
$$= \frac{1}{12}(1.00000 + 4(1.65534) + 2(1.55152) + 4(1.06666) + 0.72159)$$
$$= 1.30938.$$

We have already seen the result of Boole's rule in Example 7.2:

$$\int_0^1 f(x)\,dx \approx \frac{2(1/4)}{45}\left(7f(0) + 32f(\tfrac{1}{4}) + 12f(\tfrac{1}{2}) + 32f(\tfrac{3}{4}) + 7f(1)\right)$$
$$= 1.30859.$$

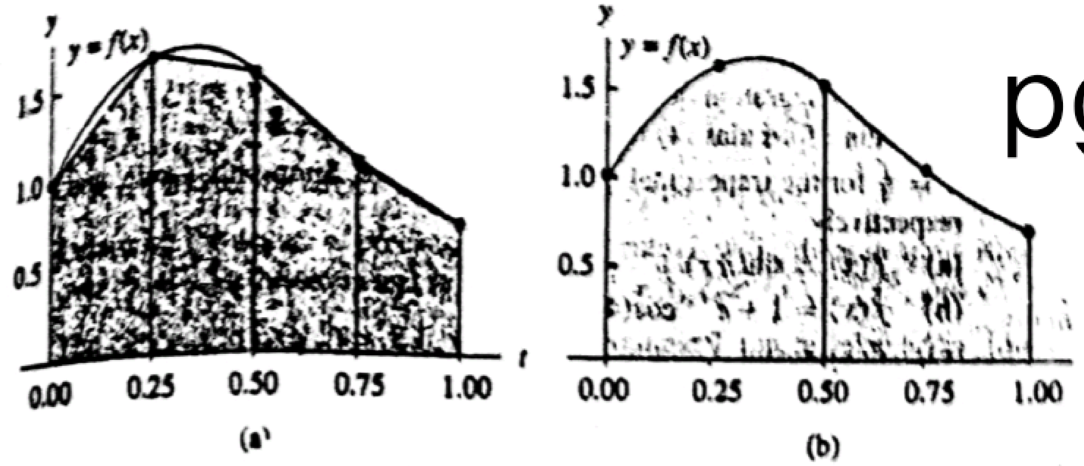

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Figure 7.4 (a) The composite trapezoidal rule yields the approximation 1.28358. (b) The composite Simpson rule yields the approximation 1.30938.

The true value of the integral is

$$\int_0^1 f(x)\,dx = \frac{21e - 4\cos(4) - \sin(4)}{17e} = 1.3082506046426\ldots,$$

and the approximation 1.30938 from Simpson's rule is much better than the value 1.28358 obtained from the trapezoidal rule. Again, the approximation 1.30859 from Boole's rule is closest. Graphs for the areas under the trapezoids and parabolas are shown in Figure 7.4(a) and (b), respectively. ∎

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Exercises for Introduction to Quadrature

1. Consider integration of $f(x)$ over the fixed interval $[a, b] = [0, 1]$. Apply the various quadrature formulas (4) through (7). The step sizes are $h = 1$, $h = \frac{1}{2}$, $h = \frac{1}{3}$, and $h = \frac{1}{4}$ for the trapezoidal rule, Simpson's rule, Simpson's $\frac{3}{8}$ rule, and Boole's rule, respectively.
   (a) $f(x) = \sin(\pi x)$
   (b) $f(x) = 1 + e^{-x}\cos(4x)$
   (c) $f(x) = \sin(\sqrt{x})$
   Remark. The true values of the definite integrals are (a) $2/\pi = 0.636619772367\ldots$, (b) $(18e - \cos(4) + 4\sin(4))/(17e) = 1.007459631397\ldots$, and (c) $2(\sin(1) - \cos(1)) = 0.602337357879\ldots$. Graphs of the functions are shown in Figure 7.5(a) through (c), respectively.
2. Consider integration of $f(x)$ over the fixed interval $[a, b] = [0, 1]$. Apply the various quadrature formulas: the composite trapezoidal rule (17), the composite Simpson rule (18), and Boole's rule (7). Use five function evaluations at equally spaced nodes. The uniform step size is $h = \frac{1}{4}$.
   (a) $f(x) = \sin(\pi x)$
   (b) $f(x) = 1 + e^{-x}\cos(4x)$
   (c) $f(x) = \sin(\sqrt{x})$.
3. Consider a general interval $[a, b]$. Show that Simpson's rule produces exact results