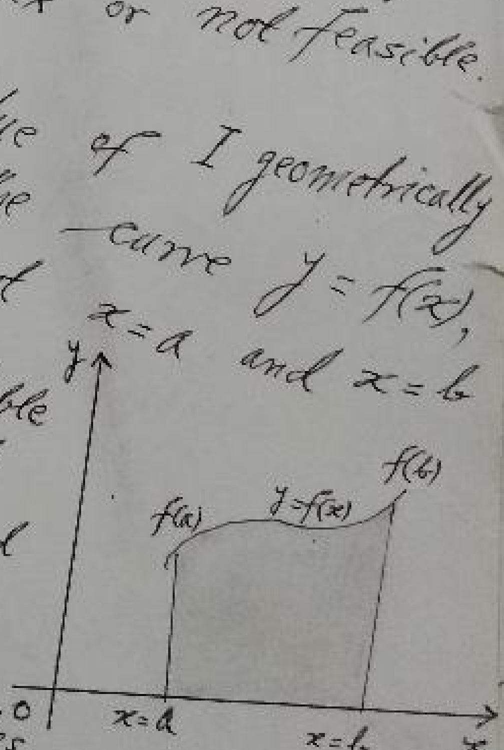
Ch. Numerical Integration or numerical quadrature

is the process or technique of integrating a function between two specified limits. The function to be integrated may be given explicitly or as a set of numerical values.

Note: (1) It is necessary to point out that when the function is known explicitly, the analytical integration may either be complex or not feasible.

(2) If $I = \int_a^b f(x)\,dx$, the value of I geometrically represents the area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$



There are several methods available for numerical integration, but the most commonly used methods be classified in two groups.

(i) Newton - Cotes Quadrature Rules.
(ii) Gaussian " " .

(1) Newton - Cotes Quadrature Rules

(A general quadrature formulae for equidistant arguments)

These formulae are used for evaluating definite integrals $I = \int_a^b f(x)\,dx$, where $f(x)$ is known explicitly or as a set of points

Let the function be known through its values as shown in the table, where $x_i - x_{i-1} = h$ for $i = 1(1)n$.

x :	$a = x_0$	x_1	x_2	$\dots$	$b = x_n$
y :	f_0	f_1	f_2	$\dots$	f_n

Let $u = \dfrac{x - x_0}{h} \Rightarrow x = x_0 + hu$ —— (1)

$\Rightarrow dx = h\,du$

We notice that when $x = a$, $u = 0$ and when $x = b$, $u = \dfrac{x_n - x_0}{h} = \dfrac{nh}{h} = n$

So, we have

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \int_0^n f(x_0+hu)\,h\,du$$

$$= h\int_0^n f(x_0+hu)\,du = h\int_0^n E^u f(x_0)\,du$$

$$= h\int_0^n (1+\Delta)^u f(x_0)\,du, \qquad [\because E = 1+\Delta]$$

$$= h\int_0^n \left[1 + u\Delta + \frac{u(u-1)}{2!}\Delta^2 + \frac{u(u-1)(u-2)}{3!}\Delta^3 + \dots\right] f(x_0)\,du$$

$$= h\int_0^n \left[f_0 + u\Delta f_0 + \frac{1}{2!}u(u-1)\Delta^2 f_0 + \frac{1}{3!}u(u-1)(u-2)\Delta^3 f_0 + \dots\right] du \quad \text{—— (2)}$$

$$= hu \left[nf_0 + \frac{n^2}{2}\Delta f_0 + \frac{1}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 f_0 + \frac{1}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 f_0 + \frac{1}{4!}\left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right)\Delta^4 f_0 + \frac{1}{5!}\left(\frac{n^6}{6} - \frac{2n^5}{1} + \frac{35n^4}{4} - \frac{50n^3}{3} + 12n^2\right)\Delta^5 f_0 + \dots \right] \quad \text{——③}$$

By substituting $n = 1, 2, 3, \dots$ several quadrature formulae can be obtained.

(ii) <u>Trapezoidal Rule</u>

(A particular case for $n = 1$)

As $n = 1$, so the interval of integration will be from x_0 to $x_0 + h$ and there are only two functional values y_0 and y_1 in the interval so that there will not be any difference higher than the first.

Thus, taking $n = 1$ in ② and neglecting all the differences of higher order than 1, we have

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx$$

$$= h\int_0^1 (f_0 + u\,\Delta f_0)\,du$$

$$= h\left[f_0 u + \Delta f_0 \cdot \frac{u^2}{2}\right]_0^1 = h\left[f_0 + \frac{1}{2}\Delta f_0\right]$$

$$= h\left[f_0 + \frac{1}{2}(f_1 - f_0)\right] = h\left[\frac{f_0}{2} + \frac{f_1}{2}\right]$$

$$= \frac{1}{2}h\left[f_0 + f_1\right]$$

This result can be extended by taking n intervals each of length h such that

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+3h} f(x)\,dx + \cdots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{1}{2}h[f_0 + f_1] + \frac{1}{2}h[f_1 + f_2] + \frac{1}{2}h[f_2 + f_3] + \cdots + \frac{h}{2}[f_{n-1} + f_n]$$

$$= \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + 2f_3 + \cdots + 2f_{n-1} + f_n\right]$$

$$= h\left[\frac{1}{2}(f_0 + f_n) + (f_1 + f_2 + f_3 + \cdots + f_{n-1})\right]$$

$$= h\left[m + M\right],$$

where $m = \frac{1}{2}(f_0 + f_1)$, $M = (f_1 + f_2 + f_3 + \cdots + f_{n-1})$.

Examples:

Ex 1. Use trapezoidal rule, to evaluate the definite integral $\int_4^7 f(x)\,dx$, given that

x:	1	2	3	4	5	6	7
y:	2.105	2.808	3.614	4.604	5.857	7.451	9.40

Sol: Here the table is

x:	1	2	3	4	5	6	7
y:	2.105	2.808	3.614	4.604	5.857	7.451	9.467,

n = number of equispaced intervals = 6

By trapezoidal rule, we have

$$I = \int_{1}^{7} f(x)\,dx = \frac{1}{2}h\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

Using the values,

$$I = \frac{1}{2}\left[(2.105 + 9.467) + 2(2.808 + 3.614 + 4.604 + 5.857 + 7.451)\right]$$

$$= \frac{1}{2}\left[11.572 + 48.668\right]$$

So, $\int_{1}^{7} f(x)\,dx = 30.12$.

Hence the area bounded by the curve is 30.12 sq. unit

Ex. 2. Use trapezoidal rule to evaluate the area bounded by the curve and the x-axis from 7.47 to 7.52 from the following table

x:	7.47	7.48	7.49	7.50	7.51	7.52
y:	1.93	1.95	1.98	2.01	2.03	2.06

Sol: By trapezoidal rule

$$\int_{7.47}^{7.52} f(x)\,dx = \frac{0.01}{2}\left[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)\right]$$

$$= 0.005\left[3.99 + 15.94\right] = 0.09965$$

Ex. 3: Evaluate the integral $\int_1^2 \frac{1}{x^2}\,dx$ using trapezoidal rule for (i) three points, (ii) five points. Also, calculate the exact value and comment on your answer.

Sol: Here $I = \int_1^2 \frac{1}{x^2}\,dx$, $f(x) = \frac{1}{x^2}$,

$h = \frac{b-a}{n}$, where $a = 1$, $b = 2$

(i) For three points: $n = 2$ $\therefore h = \frac{2-1}{2} = 0.5$

So, the table is

x :	1	1.5	2
y :	1	0.44	0.25

By trapezoidal rule, we have $\int_1^2 f(x)\,dx$

$$= \int_1^2 \frac{1}{x^2}\,dx = \frac{1}{2}h\left[(f_0 + f_2) + 2f_1\right] = \frac{0.5}{2}\left[(1+0.25) + 2(0.44)\right]$$

$$= 0.5325$$

(ii) For five points: $n = 4$ $\therefore h = \frac{2-1}{4} = 0.25$

So, the table is

x :	1	1.25	1.50	1.75	2.00
y :	1	0.64	0.44	0.33	0.25

So, by trapezoidal rule,

$$\int_1^2 f(x)\,dx = \int_1^2 \frac{1}{x^2}\,dx = \frac{1}{2}h\left[(f_0 + f_4) + 2(f_1 + f_2 + f_3)\right]$$

$$= \frac{0.25}{2}\left[(1+0.25) + 2(0.64+0.44+0.33)\right]$$

$$= 0.5088$$

Exact value:

$$I = \int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^2 = 1 - \frac{1}{2} = \frac{1}{2} = 0.50$$

Comments

By decreasing the step-size, the error will be less and by increasing the number of intervals, we can get more accurate values.

(iii) Simpson's 1/3rd - Rule.

(A particular case for $n = 2$)

In this case interval of integration is from x_0 to $x_0 + 2h$, i.e., there are three functional values, y_0, y_1, y_2, in the interval and consequently there can be no difference higher than 2.

Now taking $n = 2$ in (2), neglecting all the differences higher than 2,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = h\int_0^2 \left\{ f_0 + u\Delta f_0 + \frac{1}{2!}u(u-1)\Delta^2 f_0 \right\} du$$

$$= h\left[f_0 u + \frac{u^2}{2}\Delta f_0 + \frac{1}{2!}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 f_0 \right]_0^2$$

$$= h\left[2f_0 + 2\Delta f_0 + \frac{1}{2}\left(\frac{8}{3} - 2\right)\Delta^2 f_0 \right]$$

$$= h\left[2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0 \right]$$

$$= h\left[2f_0 + 2(f_1 - f_0) + \frac{1}{3}(f_2 - 2f_1 + f_0) \right]$$

$$= h\left[\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2 \right]$$

$$= \frac{1}{3}h\left[f_0 + f_2 + 4f_1 \right]$$

$$= \frac{1}{3}h\left[f_0 + 4f_1 + f_2 \right],$$

due to this $\frac{1}{3}$, this rule is called Simpson's $\frac{1}{3}$rd-rule.

This result can be extended by taking n (multiple of 2)-intervals each of length 'h'.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+4h} f(x)\,dx + \dots + \int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{1}{3}h[f_0 + 4f_1 + f_2] + \frac{1}{3}h[f_2 + 4f_3 + f_4] + \dots + \frac{1}{3}h[f_{n-2} + 4f_{n-1} + f_n]$$

$$= \frac{h}{3}\left[f_0 + 4(f_1 + f_3 + \dots + f_{n-1}) + 2(f_2 + f_4 + \dots + f_{n-2}) + f_n \right].$$

I_{Simp} (Soft) $= 0.2828$
Boole $= 0.2853$

Examples:

Ex. 1. Apply Simpson's $\frac{1}{3}$rd rule to find the approximate value of π from $\int_0^1 \frac{1}{1+x^2}\,dx$ taking 4-intervals.

Sol: We have $$I = \int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^1$$
$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

i.e., $I = \pi/4 \Rightarrow \pi = 4I$.

Now we find the approximate value of I using Simpson's $\frac{1}{3}$rd-Rule.

Here $a = 0$, $b = 1$ and $n = 4$

$$\Rightarrow h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25.$$

So, the table is

x :	0	0.25	0.50	0.75	1.0
$y = \frac{1}{1+x^2} = f(x)$:	1	0.9412	0.8000	0.6400	0.50

Now by Simpson $\frac{1}{3}$-rule, we have

$$\int_a^b f(x)\,dx = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{3}h\left[f_0 + f_4 + 4(f_1 + f_3) + 2f_2\right]$$

Using the values,

$$I = \frac{1}{3}(0.25)\left[1+0.5+4(0.9412+0.6401)+2(0.80)\right]$$
$$= 0.7854$$

Hence, the approximate value of π is given by

$$\pi = 4I = 4(0.7854) = 3.1416.$$

Ex. 2. Evaluate the integral $\int_2^{10} \frac{dx}{1+x}$ by dividing the range into 8 equal parts.

Sol: Here $f(x) = \frac{1}{1+x}$, $a=2$, $b=10$, $n=8$, so

$h = \frac{b-a}{n} = \frac{10-2}{8} = \frac{8}{8} = 1$.

$\therefore$ The table is

x :	2	3	4	5	6	7	8	9	10
$f(x)$:	0.333	0.25	0.20	0.1667	0.1429	0.1250	0.1111	0.10	0.0909

$\therefore$ By Simpson's $\frac{1}{3}$rd-rule,

$$\int_2^{10} \frac{dx}{1+x} = \frac{1}{3}h\left[f_0+f_8+4(f_1+f_3+f_5+f_7)+2(f_2+f_4+f_6)\right]$$

Using the values, we have

$$\int_2^{10} \frac{dx}{1+x} = \frac{1}{3}(1)[0.333 + 0.0909 + 2.5668 + 0.908]$$

$$= 1.2996$$

Ex. 3 Apply Simpson's $\frac{1}{3}$-rule to evaluate $\int_0^1 \frac{1}{1+x}dx$ correct to 3 dp taking $h = 0.25$.

Sol: Here $f(x) = \frac{1}{1+x}$, $h = 0.25$.

So, the table is,

x:	0	0.25	0.50	0.75	2.0
$f(x)$:	1	0.8	0.667	0.571	0.5

$\therefore$ By Simpson's $\frac{1}{3}$-rule,

$$I = \int_0^1 \frac{1}{1+x}dx = \frac{1}{3}h[f_0 + f_4 + 4(f_1 + f_3) + 2f_2]$$

Using the values, $I = \frac{1}{3}(0.25)[1 + 0.5 + 4(0.8 + 0.571) + 2(0.667)]$

$$= \frac{0.25}{3}[1.5 + 5.484 + 1.334]$$

$$= 0.693$$

Ex. 4. Evaluate the integral $\int_0^1 \frac{1}{1+x^2}dx$, using (i) Trapezoidal rule, (ii) Simpson's $\frac{1}{3}$-rule, for $n=4$. Obtain the true solution and justify which method is superior.

Sol: (i) Trapezoidal rule: $f(x) = \frac{1}{1+x^2}$, $a=0$, $b=1$, $n=4$

$\therefore h = \frac{1-0}{4} = 0.25$. So, the table is,

x:	0	0.25	0.500	0.75	1
$f(x) = y$:	1	0.94	0.8	0.64	0.5

$\therefore$ By trapezoidal rule,

$$I = \int_0^1 \frac{dx}{1+x^2} = \frac{1}{2} h \left[f_0 + f_4 + 2(f_1 + f_2 + f_3) \right]$$

$$= \frac{1}{2}(0.25)\left[1 + 0.5 + 2(0.94 + 0.8 + 0.64) \right]$$

$$= 0.7825$$

(ii) <u>Simpson's $\frac{1}{3}$-rule</u>: We have already calculated it in Ex. 1 as $\int_0^1 \frac{1}{1+x^2} dx = 0.7854$

Also, true solution is $\int_0^1 \frac{1}{1+x^2} dx = \tan^{-1} x \Big|_0^1 = \tan^{-1} 1 = \pi/4$.

$$= 0.7854$$

<u>Justification</u>: From above, we conclude that Simpson's $\frac{1}{3}$-rule is closer to the exact value than that obtained by Trapezoidal rule. Thus, Simpson's $\frac{1}{3}$-rule is better than the trapezoidal rule.

<u>Ex. 5</u> Evaluate the integral $\int_0^{10} y\,dx$ using (i) 5-point Trapezoidal rule, (ii) 5-point Simpson's rule, where the values of y are given in the following table.

x :	0	2.5	5.0	7.5	10.0
$y = f(x)$:	0	18.75	75.0	168.75	300.0

If $y = 3x^2$, compare your answer with the exact value and comment on the superiority of the method applied.

Sol: (i) Trapezoidal rule:

$$I = \int_0^{10} y\,dx = \frac{1}{2}h\left[f_0 + 2(f_1 + f_2 + f_3) + f_4\right]$$

Using the values, we have

$$I = \int_0^{10} 3x^2\,dx = \frac{1}{2}(2.5)\left[0 + 2(18.75 + 75.0 + 168.75) + 300\right]$$

$$= 1.25\left[300 + 2(262.5)\right.$$

$$= 1.25\left[300 + 525\right]$$

$$= 1031.25$$

(ii) Simpson's rule:

$$I = \int_0^{10} f(x)\,dx = \frac{1}{3}h\left[f_0 + 4(f_1 + f_3) + 2f_2 + f_4\right]$$

Using the values, we get

$$I = \int_0^{10} 3x^2\,dx = \frac{1}{3}(2.5)\left[0 + 4(18.75 + 168.75) + 2(75.0) + 300\right]$$

$$= \frac{2.5}{3}\left[750 + 150 + 300\right]$$

$$= 1000$$

(iii) Exact Value: $I = \int_0^{10} f(x)\,dx = \int_0^{10} 3x^2\,dx = x^3\Big|_0^{10} = 1000$

Comments: Simpson's five points rule gives the value which coincides with the exact value. Clearly, Simpson's method is better than Trapezoidal rule.

Ex. 6 A rocket is launched from the ground and its acceleration during the first 80 seconds is given in the following table

t (time in secs) :	0	10	20	30	40	50	60	70	80
$f(t) = a$ (accel. in m/sec^2) :	31.0	31.63	33.44	35.47	37.75	40.33	43.29	46.69	50.67

Use Simpson's rule to calculate the velocity.

Sol: Here

$$I = \int_0^{80} f(t)\,dt = \frac{1}{3}h\left[f_0 + f_8 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6)\right]$$

$$= \frac{1}{3}(10)\left[30 + 50.67 + 4(31.63 + 35.47 + 40.33 + 46.67) + 2(33.44 + 37.75 + 43.29)\right]$$

$$= \frac{10}{3}\left[80.67 + 4(154.12) + 2(114.48)\right]$$

$$= \frac{10}{3}\left[80.67 + 616.48 + 228.96\right]$$

$$= 3087 \text{ (approx.)}$$

Hence the velocity of the rocket,

$$v(t) \approx 3087 \; m/sec.$$

Simpson's $\frac{3}{8}$th - Rule (Particular case for $n=3$)

In this case, interval of integration is from x_0 to x_0+3h, i.e., there are four functional values, viz, y_0, y_1, y_2, y_3 in the interval and consequently there can be no difference higher than 3.

Now taking $n=3$ in the general result (2), neglecting all the differences higher than 3, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = h\int_0^3 \left[f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 f_0\right] du$$

$$= h\left[uf_0 + \frac{u^2}{2}\Delta f_0 + \frac{1}{2!}\left(\frac{u^3}{3}-\frac{u^2}{2}\right)\Delta^2 f_0 + \frac{1}{3!}\left(\frac{u^4}{4}-\frac{3u^3}{3}+\frac{2u^2}{2}\right)\Delta^3 f_0\right]_0^3$$

$$= h\left[3f_0 + \frac{9}{2}\Delta f_0 + \frac{1}{2}\left(9-\frac{9}{2}\right)\Delta^2 f_0 + \frac{1}{6}\left(\frac{81}{4}-27+9\right)\Delta^3 f_0\right]$$

$$= h\left[3f_0 + \frac{9}{2}(f_1-f_0) + \frac{9}{4}(f_2-2f_1+f_0) + \frac{3}{8}(f_3-3f_2+3f_1-f_0)\right]$$

$$= h\left[\frac{3}{8}f_0 + \frac{9}{8}f_1 + \frac{9}{8}f_2 + \frac{3}{8}f_3\right]$$

$$= \frac{3}{8}h\left[f_0 + 3(f_1+f_2) + f_3\right]$$

The result can be extended by taking n (multiple of 3)-intervals each of length h.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{3}{8}h\left[f_0 + 3(f_1+f_2) + f_3\right] + \frac{3}{8}h\left[f_3 + 3(f_4+f_5) + f_6\right] + \cdots + \frac{3h}{8}\left[f_{n-3} + 3(f_{n-2}+f_{n-1}) + f_n\right].$$

$$= \frac{3}{8}h\left[f_0 + f_n + 3(f_1 + f_2 + f_4 + f_5 + \cdots + f_{n-2} + f_{n-1}) + 2(f_3 + f_6 + f_9 + \cdots + f_{n-3})\right]$$

Examples:

Ex. 1 Compute $\int_0^{7.5} y\,dx$, where y is given in the table below

x:	0	2.5	5.0	7.5
y:	0	18.75	75.0	168.75

using (i) Simpson's $\frac{3}{8}$th rule, (ii) Trapezoidal rule.

If $y = 3x^2$, comment on the superiority of the method.

Sol: Here $a = 0$, $b = 7.5$, $n = 3$, so

$$h = \frac{b-a}{n} = \frac{7.5-0}{3} = 2.5$$

(i) Simpson's $\frac{3}{8}$th rule: We have

$$\int_0^{7.5} y\,dx = \frac{3}{8}h\left[f_0 + f_3 + 3(f_1 + f_2)\right]$$

$$= \frac{3}{8}(2.5)\left[0 + 168.75 + 3(18.75 + 75)\right]$$

$$= 0.9375\left[168.75 + 281.25\right]$$

$$= 0.9375(450)$$

$$= 421.875$$

(ii) Trapezoidal rule: We have

$$\int_0^{7.5} y\,dx = \frac{1}{2}h\left[f_0 + f_3 + 2(f_1 + f_2)\right]$$

$$= \frac{1}{2}(2\cdot5)\left[0 + 168\cdot75 + 2(18\cdot75 + 75)\right]$$

$$= \frac{2\cdot5}{2}\left[168\cdot75 + 187\cdot5\right]$$

$$= 1\cdot25\,[356\cdot25]$$

$$= 445\cdot3125$$

(iii) Exact value: $I = \int_0^{7\cdot5} y\,dx = \int_0^{7\cdot5} 3x^2\,dx = x^3\Big|_0^{7\cdot5}$

$$= (7\cdot5)^3 = 421\cdot875$$

Comments: Here we notice that the value of the integral obtained from Simpson's $\frac{3}{8}$th-rule coincides with the exact value. So, Simpson's $\frac{3}{8}$th rule is far better than Trapezoidal rule. Hence, on the basis of our observation, Simpson's $\frac{3}{8}$th-rule is superior than Trapezoidal rule.

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v.v.imp.

Ex. 2. Evaluate the integral $\int_{\pi/8}^{\pi/2} y\,dx$, where y is given as under:

x:	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
y:	$0\cdot382$	$0\cdot707$	$0\cdot924$	$1\cdot0$

using (i) Trapezoidal rule, (ii) Simpson's $\frac{3}{8}$th rule.

Sol: Here $a = \pi/8$, $b = \pi/2$, $n = 3$

So, $h = \dfrac{b-a}{n} = \dfrac{\pi/2 - \pi/8}{3} = \dfrac{3\pi/8}{3} = \pi/8$

(i) Trapezoidal rule:

$$\int_{\pi/8}^{\pi/2} y\,dx = \frac{1}{2}h\left[f_0 + f_3 + 2(f_1 + f_2)\right]$$

$$= \frac{1}{2}(\pi/8)\left[0.382 + 1.0 + 2(0.707 + 0.924)\right]$$

$$= \frac{\pi}{16}\left[1.382 + 3.262\right]$$

$$= 0.9118$$

(ii) Simpson's $\frac{3}{8}$th - rule:

$$\int_{\pi/8}^{\pi/2} y\,dx = \frac{3}{8}h\left[f_0 + f_3 + 3(f_1 + f_2)\right]$$

$$= \frac{3}{8}(\pi/8)\left[0.382 + 1.0 + 3(0.707 + 0.924)\right]$$

$$= \frac{3\pi}{64}\left[1.382 + 4.893\right]$$

$$= 0.9241$$

v.v. Imp.

Ex. 3 Evaluate $\int_2^3 \frac{1}{x^2}\,dx$, using (i) Trapezoidal rule, (ii) Simpson's rule. Compare your results with the exact value and compute the error incurred in these methods. [Ans: (i) 0.1667; 0.00[illegible] (ii) 0.1652; 0.0015]

Ex. 4 Apply Simpson's $\frac{3}{8}$th rule to evaluate the following:

(i) $\displaystyle\int_0^{\pi/2} e^{\sin x}\,dx$, (ii) $\displaystyle\int_0^{6} \frac{1}{1+x^2}\,dx$.

Sol: (i) Here $a=0$, $b=\pi/2$, $n=3$,

$\therefore h = \frac{1}{n}(b-a) = \frac{1}{3}(\pi/2 - 0) = \pi/6$

So, the table of values is:

x:	0	$\pi/6$	$\pi/3$	$\pi/2$
y:	1	1.6487	2.3774	2.7183

$\therefore$ By Simpson's $3/8$th-rule, we have

$$\int_0^{\pi/2} e^{\sin x}\,dx = \frac{3}{8}h\left[f_0 + f_3 + 3(f_1+f_2)\right]$$

$$= \frac{3}{8}(\pi/6)\left[1 + 2.7183 + 3(1.6487 + 2.3774)\right]$$

$$= \frac{\pi}{16}\left[3.7183 + 3(4.026)\right] = \cancel{1.5206}\ 3.1016$$

(ii) Here $a=0$, $b=6$ and let $n=6$ $\therefore h = \frac{1}{n}(b-a) = \frac{6}{6} = 1$.

So, the table of values is

x:	0	1	2	3	4	5	6
y:	1	0.5	0.2	0.10	0.588	0.0385	0.0270

$\therefore$ By Simpson's $\frac{3}{8}$th-rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3}{8}h\left[f_0 + f_6 + 3(f_1+f_2+f_4+f_5) + 2f_3\right]$$

$$= \frac{3}{8}(1)\left[1 + 0.0270 + 3(0.5+0.2+0.0588+0.0385) + 2(0.1)\right]$$

$$= \frac{3}{8}\left[1 + .0270 + 2.3919 + 0.2\right]$$

$$= 1.3571$$

Boole's Rule: (Particular case for $n=4$

Formula:

$$\int_{x_0}^{x_0+4h} f(x)\,dx = \frac{2}{45}h\left[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4\right]$$

Extension: (multiple of 4)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2}{45}h\Big[7(f_0+f_n) + 14(f_4+f_8+\dots+f_{n-4}) + 32(f_1+f_3+\dots+f_{n-1}) + 12(f_2+f_6+\dots+f_{n-2})\Big]$$

Weddle's Rule: (Particular case for $n=6$)

Formula:

$$\int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3}{10}h\left[f_0 + 5f_1 + f_2 + 6f_3 + f_4 + 5f_5 + f_6\right]$$

Extension: (multiple of 6)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3}{10}h\Big[(f_0+f_2+f_4+f_8+\dots+f_n) + 2(f_6+f_{12}+\dots+f_{n-6}) + 5(f_1+f_5+f_7+f_{11}+\dots+f_{n-5}+f_{n-1}) + 6(f_3+f_9+f_{15}+\dots+f_{n-3})\Big]$$

Examples:

Ex.1. Calculate the integral $\int_0^1 \frac{1}{1+x^2}\,dx$ using (i) Simpson's $\frac{1}{3}$-rule, (ii) Boole's rule, and calculate its exact value. Compare your answer and discuss the superiority of the method.

[Ans: (i) 0.7854 (ii) 0.78554; exact value = 0.785398; Simpson's $\frac{1}{3}$ > Boole]

Ex.2. Evaluate $\int_0^{\pi/2} \sin x\,dx$ by using (i) Simpson's rule, (ii) Boole's rule.

[Ans: (i) 1.00015, (ii) 1.0000].

Ex. 3. Evaluate $\int_2^3 \frac{1}{1+x^4}\,dx$ by using (i) Simpson's rule for 5-points, (ii) Boole's rule

[Ans: (i) 0.02832, (ii) 0.02831].

Ex 4. Using (i) Trapezoidal Rule, (ii) Weddle's rule, evaluate the integral $\int_{1.8}^{3} f(x)\,dx$ when

x:	1.8	2	2.2	2.4	2.6	2.8	3
y:	6.050	7.389	9.025	11.023	13.464	16.445	20.086

Calculate the exact value of the integral if $f(x) = e^x$ and compare the results with the exact value. Which method is superior and why?

[Ans: (i) 14.0828, (ii) 14.03598; Exact value = 14.036, Here Weddle's method > Trapezoidal rule].

Ex. 5. Consider the function given by the table

x:	0	0.1	0.2	0.3	0.4	0.5	0.6
y:	0	0.0998	0.1981	0.2955	0.3894	0.4794	0.5646.

Compute $\int_0^{0.6} f(x)\,dx$ by using (i) Simpson's $\frac{1}{3}$rd rule, (ii) Trapezoidal rule, (iii) Simpson's $\frac{3}{8}$th-rule, (iv) Weddle's rule.

[Ans: (i) 0.17461, (ii) 0.17445, (iii) 0.1746, (iv) 0.17463]

Ex. 6. Evaluate $\int_{0.5}^{0.7} x^{\frac{1}{2}} e^{-x}\,dx$ approximately by using a suitable formula.

[Ans: (i) Simpson's $\frac{1}{3}$rd rule gives 0.08483].

v.v.Imp.

Ex. 7 The prime number theorem states that the number of primes in the interval $a < x < b$ is approximately $\int_a^b \frac{1}{\ln x}\,dx$. If $a = 100$, $b = 200$, using Simpson's $\frac{1}{3}$rd rule for $n = 4$, calculate the above integral and compare it with its exact value.

[Ans: 20.066; exact value = 21.]

Ex. 8. Evaluate $\int_{0.1}^{0.7} \frac{x^4 e^x}{(e^x - 1)^2}\,dx$ by using Trapezoidal rule when $h = 0.1$.

[Ans: 0.1122]