Interpolation

Interpolation and Extrapolation:

Suppose we are given the following data:

x	0	3	6	9	12	15
$f(x)$	5	7	10	14	18	23

Then the process of finding $f(2), f(4), f(10)$, ... etc is known as <u>interpolation</u> and that of finding $f(-1), f(16), f(18)$, ... etc. is categorized as <u>extrapolation</u>.

Note:

In order to solve the problem of interpolation, we can assume a function $f(x)$ as a polynomial in x of degree $(n-1)$ if n values of function are given in the data.

Interpolation with Equal Intervals:

(i.e., when the values of the function are given at equidistant intervals).

(A) Newton - Gregory Formula for Forward Interpolation

Suppose that the table of values of $y = f(x)$ corresponding to different values of x are given below:

x	x_0	x_1	x_2	$\ldots$	x_n
y	y_0	y_1	y_2	$\ldots$	y_n

Here the values of the independent variable x, i.e., $x_0, x_1, x_2, \ldots, x_n$ are equidistant with h as the width of the intervals,

$$h = x_1 - x_0,\ x_2 - x_1,\ \ldots \text{ etc.}$$

Assuming $y = f(x)$ to be a polynomial of degree n in x, we may write

$$y = f(x) \approx A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1) + A_3(x - x_0)(x - x_1)(x - x_2) + \ldots + A_n(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1}) \quad \text{--- (1)}$$

where $A_0, A_1, A_2, \ldots, A_n$ are constants yet to be determined.

Putting values from the table we have:

At $x = x_0$, $y = y_0$. So, ① $\Longrightarrow y_0 = A_0$

At $x = x_1$, $y = y_1$. So, ① $\Rightarrow y_1 = A_0 + A_1(x_1 - x_0)$

$\Rightarrow y_1 = y_0 + A_1(x_1 - x_0) \Rightarrow A_1 = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{\Delta y_0}{h}$

At $x = x_2$, $y = y_2$. So, ① $\Longrightarrow$

$$y_2 = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_0)(x_2 - x_1)$$

$$\Rightarrow A_2(x_2 - x_0)(x_2 - x_1) = (y_2 - y_0) - \frac{\Delta y_0}{h}(x_2 - x_0)$$

$$\Rightarrow A_2 = \left\{(y_2 - y_0) - \frac{\Delta y_0}{h}(x_2 - x_0)\right\} \Big/ (x_2 - x_0)(x_2 - x_1)$$

$$= \left\{(y_2 - y_0) - \frac{(y_1 - y_0)}{(x_1 - x_0)}(x_2 - x_0)\right\} \Big/ (x_2 - x_0)(x_2 - x_1)$$

$$= \frac{(y_2 - y_0)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{(y_2 - y_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{(y_1 - y_0)}{(x_2 - x_1)(x_1 - x_0)}$$

$$= \frac{(y_2 - y_0)}{(2h)(h)} - \frac{(y_1 - y_0)}{(h)(h)}$$

$$= \frac{(y_2 - y_0) - 2(y_1 - y_0)}{2h^2}$$

$$= \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$$\Rightarrow \quad A_2 = \frac{1}{2!} \cdot \frac{\Delta^2 y_0}{h^2}$$

Continuing on these lines, we have

$$A_3 = \frac{1}{3!} \cdot \frac{\Delta^3 y_0}{h^3},$$

$$A_4 = \frac{1}{4!} \cdot \frac{\Delta^4 y_0}{h^4}$$

So, the function $f(x)$ can be written as

$$f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{1}{2!} \cdot \frac{\Delta^2 y_0}{h^2}(x - x_0)(x - x_1)$$
$$+ \dots + \frac{1}{n!} \cdot \frac{\Delta^n y_0}{h^n}(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

This is called <u>Newton - Gregory Formula for forward interpolation</u>.

<u>Note</u>:

Let $u = \dfrac{x - x_0}{h} \Rightarrow hu = x - x_0$

$$\Rightarrow \quad x = x_0 + hu.$$

$$\therefore \quad x - x_1 = x_0 + hu - x_1$$
$$= (x_0 - x_1) + hu = -h + hu$$
$$= h(u-1)$$

or $\dfrac{x - x_1}{h} = u - 1$

So, the above function becomes

$$f(x) = y_0 + u\Delta y_0 + \frac{1}{2!}u(u-1)\Delta^2 y_0 + \cdots + \frac{1}{n!}[u(u-1)(u-2)\cdots\{u-(n-1)\}]\Delta^n y_0$$

and if we write $u(u-1)(u-2)\cdots(u-n+1) = u^{(n)}$,
then the above formula takes the form

$$f(x) = y_0 + u^{(1)}\Delta y_0 + \frac{1}{2!}u^{(2)}\Delta^2 y_0 + \frac{1}{3!}u^{(3)}\Delta^3 y_0 + \cdots + \frac{1}{n!}u^{(n)}\Delta^n y_0$$

Examples:

Ex. 1. The following table gives the population of a town during last six censuses. Estimate using a suitable formula, the increase in the population during the period from 1946 to 1948.

Years	1911	1921	1931	1941	1951	1961
Population (in thousands)	12	15	20	27	39	52

Sol: The given table can be written as

Years x	Population (in thousands) y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1911	12					
		3				
1921	15		2			
		5		0		
1931	20		2		3	
		7		3		−10
1941	27		5		−7	
		12		−4		
1951	39		1			
		13				
1961	52					

(i) 1946

Take $u = \left(\frac{x-x_0}{h}\right) = \frac{1946-1911}{10} = \frac{35}{10} = 3.5$

The Newton-Gregory Formula is

$$f(x) = y_0 + u^{(1)}\Delta y_0 + \frac{1}{2!}u^{(2)}\Delta^2 y_0 + \frac{1}{3!}u^{(3)}\Delta^3 y_0 + \frac{1}{4!}u^{(4)}\Delta^4 y_0 + \frac{1}{5!}u^{(5)}\Delta^5 y_0$$

or

$$f(x) = y_0 + u\Delta y_0 + \frac{1}{2!}u(u-1)\Delta^2 y_0 + \frac{1}{3!}u(u-1)(u-2)\Delta^3 y_0 + \frac{1}{4!}u(u-1)(u-2)(u-3)\Delta^4 y_0 + \frac{1}{5!}u(u-1)(u-2)(u-3)(u-4)\Delta^5 y_0 \quad \ldots (A)$$

Using values from the table,

$$f(1946) = 12 + (3.5)(3) + \frac{1}{2}(3.5)(2.5)(2) + \frac{1}{6}(3.5)(2.5)(1.5)(0) + \frac{1}{24}(3.5)(2.5)(1.5)(0.5)(3) + \frac{1}{120}(3.5)(2.5)(1.5)(0.5)(-0.5)(-10)$$

$$= 12 + 10.5 + 8.75 + 0.8203 + 0.2734$$

$$\therefore f(1946) = 32.34$$

$$f(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{1}{2!}\frac{\Delta^2 y_0}{h^2}(x-x_0)(x-x_1)$$

$$+\frac{1}{3!}\frac{\Delta^3 y_0}{h^3}(x-x_0)(x-x_1)(x-x_2) + \frac{1}{4!}\frac{\Delta^4 y_0}{h^4}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$+\frac{1}{5!}\frac{\Delta^5 y_0}{h^5}(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

$\underline{h = 10}$

$$f(1946) = 12 + \frac{3}{10}(1946-1911) + \frac{1}{2!}\cdot\frac{2}{10^2}(1946-1911)(1946-1921)$$

$$+ \frac{1}{3!}\cdot 0 + \frac{1}{4!}\cdot\frac{3}{10^4}(1946-1911)(1946-1921)(1946-1931)(1946-1941)$$

$$+ \frac{1}{5!}\cdot\frac{(-10)}{10^5}(1946-1911)(1946-1921)(1946-1931)(1946-1941)(1946-1951)$$

$$= 12 + \frac{3}{10}(35) + \frac{1}{100}(35)(25) + 0 + \frac{1}{8}\cdot\frac{1}{10000}(35)(25)(15)(5)$$

$$+ \frac{1}{120}\left(-\frac{1}{10000}\right)(35)(25)(15)(5)(-5)$$

$$= 32.34375$$

Ex. 2. Given

$\sin 45°$	$\sin 50°$	$\sin 55°$	$\sin 60°$
0.7071	0.7660	0.8192	0.8660

Find $\sin 52°$ by using interpolation method.

Sol: The given data can be written

as :

Angles x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		-0.0057	
		0.0532		-0.0007
55°	0.8192		-0.0064	
		0.0468		
60°	0.8660			

To find $\sin 52^\circ$;

let $x = 52^\circ$, then

$$u = \frac{x - x_0}{h} = \frac{52^\circ - 45^\circ}{5} = \frac{7}{5} = 1.4$$

The Newton - Gregory Formula is

$$f(x) = y_0 + u^{(1)} \Delta y_0 + \frac{1}{2!} u^{(2)} \Delta^2 y_0 + \frac{1}{3!} u^{(3)} \Delta^3 y_0$$

$$\therefore f(52^\circ) = y_0 + u \Delta y_0 + \frac{1}{2} u(u-1) \Delta^2 y_0 + \frac{1}{6} u(u-1)(u-2) \Delta^3 y_0$$

$$\therefore \sin 52^\circ = 0.7071 + (1.4)(0.0589) + \frac{1}{2}(1.4)(0.4)(-0.0057) + \frac{1}{6}(1.4)(0.4)(-0.6)(-0.0007)$$

$$\therefore \sin 52^\circ = 0.7071 + 0.0825 - 0.0016 + 0.0000$$

$$\therefore \sin 52^\circ = 0.7880.$$

Hence, $\sin 52^\circ = 0.7880.$

$$f(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{1}{2!}\frac{\Delta^2 y_0}{h^2}(x-x_0)(x-x_1)$$

$$+ \frac{1}{3!}\cdot\frac{\Delta^3 y_0}{h^3}(x-x_0)(x-x_1)(x-x_2)$$

$\underline{h = 5}$

$$f(52) = 0.7071 + \frac{0.0589}{5}(52-45) + \frac{1}{2}\cdot\frac{(-0.0057)}{25}(52-45)(52-50)$$

$$+ \frac{1}{6}\frac{(-0.0007)}{125}(52-45)(52-50)(59-55)$$

$$= 0.7071 + 0.08246 - \overset{0.001596}{\cancel{0.003192}} + 0.00000392$$

$$= \cancel{0.78637}$$

$$= 0.78796792$$

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Ex. 5. Write $y = \sin \frac{\pi}{2} x$ as a polynomial whose values coincide with it at $x = 0, 1, 2, 3$.

Sol: The given function is $y = \sin \frac{\pi}{2} x$

$\therefore$ The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0			
		1		
1	1		−2	
		−1		2
2	0		0	
		−1		
3	−1			

Take $u = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$

The Newton-Gregory formula is

$$f(x) \approx y_0 + u \Delta y_0 + \frac{1}{2!} u(u-1) \Delta^2 y_0 + \frac{1}{3!} u(u-1)(u-2) \Delta^3 y_0$$

Putting the values,

$$f(x) \approx 0 + x(1) + \frac{1}{2} x(x-1)(-2) + \frac{1}{6} x(x-1)(x-2)(2)$$

$$= x - x(x-1) + \frac{1}{3} x(x-1)(x-2)$$

$$= x - x^2 + x + \frac{1}{3} x(x^2 - 3x + 2)$$

$$= \frac{1}{3} x^3 - 2x^2 + \frac{8}{3} x$$

At $x = 0$, $f(x) = 0$, At $x = 1$, $f(x) = 1$, At $x = 2$, $f(x) = 0$, At $x = 3$, $f(x) = -1$

$\therefore$ $f(x) = \frac{1}{3} x^3 - 2x^2 + \frac{8}{3} x$ is the required polynomial

$$f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{1}{2!}\frac{\Delta^2 y_0}{h^2}(x-x_0)(x-x_1)$$
$$+ \frac{1}{3!}\frac{\Delta^3 y_0}{h^3}(x-x_0)(x-x_1)(x-x_2)$$

$\underline{h=1}$

$$f(x) = 0 + \frac{1}{1}(x-0) + \frac{1}{2}\cdot\frac{(-2)}{1^2}(x-0)(x-1)$$
$$+ \frac{1}{6}\cdot\frac{2}{1^3}(x-0)(x-1)(x-2)$$

$$= x - (x^2 - x) + \frac{1}{3}x(x^2 - 3x + 2)$$
$$= \frac{1}{3}x^3 - 2x^2 + \frac{8}{3}x$$

Ex. 6. Write $y = x^4$ as a quadratic polynomial whose values coincide with it at $x = 0, 1, 2$.

Sol: The given function is $y = x^4$.

The difference table is

x	y	Δy	$\Delta^2 y$
0	0		
		1	
1	1		14
		15	
2	16		

Take $u = \dfrac{x - x_0}{h} = \dfrac{x - 0}{1} = x$

The Newton-Gregory formula is

$$f(x) \approx y_0 + u\,\Delta y_0 + \frac{1}{2!}\,u(u-1)\,\Delta^2 y_0$$

Putting the values,

$$f(x) \approx 0 + x(1) + \frac{1}{2!}\,x(x-1)(14)$$

$$= x + 7x^2 - 7x$$

$$f(x) = 7x^2 - 6x$$

At $x = 0$, $f(x) = 0$

At $x = 1$, $f(x) = 1$

At $x = 2$, $f(x) = 16$

Hence $f(x) \approx 7x^2 - 6x$ is the reqd. polynomial.

$\underline{h=1}$

$$f(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{1}{2!}\frac{\Delta^2 y_0}{h^2}(x-x_0)(x-x_1)$$

$$= 0 + \frac{1}{1}(x-0) + \frac{1}{2}\cdot\frac{14}{1}(x-0)(x-1)$$

$$= x + 7(x^2 - x)$$

$$= 7x^2 - 6x$$

(B) **Newton - Gregory Formula for Backward Interpolation.**

Suppose the table of values of $y = f(x)$ corresponding to different values of x is given below

x :	x_0	x_1	x_2	$\cdots$	x_n
$y = f(x)$:	y_0	y_1	y_2	$\cdots$	y_n

Here the values of the independent variable x i.e., $x_0, x_1, x_2, \cdots, x_n$ are equidistant with 'h' step size.

Assuming $f(x)$ to be the polynomial of degree in x, we can write

$$f(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) + \cdots$$
$$+ A_n(x - x_n)(x - x_{n-1}) \cdots (x - x_1), \quad (1)$$

where $A_0, A_1, \cdots, A_n$ are unknown constants yet to be determined

Putting the values from the table, we get at $x = x_n$, $y = y_n$ so that

$$(1) \Rightarrow \quad y_n = A_0$$

At $x = x_{n-1}$, $y = y_{n-1}$, so

$$(1) \Rightarrow \quad y_{n-1} = A_0 + A_1(x_{n-1} - x_n)$$

$$\Rightarrow \quad y_{n-1} - y_n = A_1(x_{n-1} - x_n)$$

$$\Rightarrow \quad A_1 = \frac{y_{n-1} - y_n}{x_{n-1} - x_n}$$

$$\Rightarrow \quad A_1 = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{\nabla y_n}{h}$$

At $x = x_{n-2}$, $y = y_{n-2}$, so (1) gives

$$y_{n-2} = A_0 + A_1(x_{n-2} - x_n) + A_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\Rightarrow \quad y_{n-2} - y_n - \frac{(y_n - y_{n-1})}{h}(x_{n-2} - x_n) = A_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\Rightarrow \quad A_2(-2h)(-h) = y_{n-2} - y_n - \frac{(y_n - y_{n-1})}{h}(-2h)$$

$$\Rightarrow \quad 2h^2 \cdot A_2 = y_{n-2} - y_n + 2y_n - 2y_{n-1}$$

$$\Rightarrow \quad A_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2!\, h^2}$$

Continuing in this way, we get

$$A_3 = \frac{\nabla^3 y_n}{3!\, h^3}, \quad A_4 = \frac{\nabla^4 y_n}{4!\, h^4} \text{ and so on.}$$

Finally, $A_n = \dfrac{1}{n!} \cdot \dfrac{\nabla^n y_n}{h^n}$.

Substituting these values in (1), we get

$$f(x) = y_n + \frac{\nabla y_n}{h}(x-x_n) + \frac{1}{2!}\cdot\frac{\nabla^2 y_n}{h^2}(x-x_n)(x-x_{n-1}) + \dots + \frac{1}{n!}\,\frac{\nabla^n y_n}{h^n}(x-x_n)(x-x_{n-1})\dots(x-x_1).$$

This is Newton - Gregory formula for backward interpolation.

Note: Let $\frac{(x-x_n)}{h} = k$ so that

$$\frac{x-x_{n-1}}{h} = \frac{x-(x_n-h)}{h} = \frac{(x-x_n)+h}{h} = \frac{(x-x_n)}{h}+1$$
$$= k+1.$$

Therefore, the above formula becomes

$$f(x) = y_n + k\nabla y_n + \frac{1}{2!}k(k+1)\nabla^2 y_n + \dots + \frac{1}{n!}k(k+1)(k+2)\dots(k+n-1)\nabla^n y_n$$

Examples:

Ex. 1. Using Newton - Gregory forward and backward formulae, find the polynomial which deficits the following data:

x:	1	2	3	4	5
y:	1	-1	1	-1	1

Ex. 3. A second degree polynomial possesses the points (0, 1), (1, 3), (2, 7), (3, 13). Find the polynomial.

Sol: The given difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

Take $k = \frac{x - x_n}{h} = \frac{x-3}{1} = (x-3)$

Now Newton-Gregory backward interpolation formula is

$$f(x) = y_n + k\nabla y_n + \frac{1}{2!}k(k+1)\nabla^2 y_n$$

$$= 13 + (x-3)(6) + \frac{1}{2}(x-3)(x-2)(2)$$

$$= 13 + 6x - 18 + x^2 - 5x + 6$$

$$= x^2 + x + 1,$$

which is the required polynomial.

$$\underline{h=1}$$

$$f(x) = y_3 + \frac{\nabla y_3}{h}(x-x_3) + \frac{1}{2!}\frac{\nabla^2 y_3}{h^2}(x-x_3)(x-x_2) + \frac{1}{3!}\frac{\nabla^3 y_3}{h^3}(x-x_3)(x-x_2)(x-x_1)$$

$$= 13 + \frac{6}{1}(x-3) + \frac{1}{2}\cdot\frac{2}{1^2}(x-3)(x-2) + 0$$

$$= 13 + 6(x-3) + (x^2-5x+6)$$

$$= x^2 + x + 1$$

Ex. 4. Find the polynomial which satisfies the values 7)

x:	3	4	5	6
y:	6	24	60	120.

Sol: **Backward Interpolation:** The given difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
3	6			
4	24	18		
5	60	36	18	
6	120	60	24	6

Take $k = \frac{x - x_n}{h} = \frac{x-6}{1} = (x-6)$

$\therefore$ By Newton-Gregory backward interpolation formula,

$$f(x) = y_n + k\nabla y_n + \frac{1}{2!}k(k+1)\nabla^2 y_n + \frac{1}{3!}k(k+1)(k+2)\nabla^3 y_n$$

Using the values;

$$f(x) = 120 + (x-6)(60) + \frac{1}{2}(x-6)(x-5)(24) + \frac{1}{6}(x-6)(x-5)(x-4)(6)$$

$$= 120 + 60x - 360 + 12(x^2 - 11x + 30) + (x^2 - 11x + 30)(x-$$

$$= 120 + 60x - 360 + 12x^2 - 132x + 360 + x^3 - 15x^2 + 74x - 12$$

$\therefore f(x) = x^3 - 3x^2 + 2x,$

which is the required polynomial.

$\underline{h=1}$

$$f(x) = y_3 + \frac{\nabla y_3}{h}(x-x_3) + \frac{1}{2!}\frac{\nabla^2 y_3}{h^2}(x-x_3)(x-x_2) + \frac{\nabla^3 y_3}{h^3}(x-x_3)(x-x_2)(x-x_1)$$

$$= 120 + \frac{60}{1}(x-6) + \frac{1}{2}\cdot\frac{24}{1^2}(x-6)(x-5) + \frac{1}{6}\cdot\frac{6}{1^3}(x-6)(x-5)(x-4)$$

$$= 120 + 60(x-6) + 12(x^2-11x+30) + (x^2-11x+30)(x-4)$$

$$= 120 + 60(x-6) + 12(x^2-11x+30) + (x^3-15x^2+74x-120)$$

$$= x^3 - 3x^2 + 2x$$