Probability Exercises, 2025-12-01

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Contents

1 Exercise (expected value, method of indicators): hat (or jacket check) problem

n visitors to the Komische Oper ('funny opera') in Berlin all check in black, Jack Wolfskin jackets before a performance of Mozart's Don Giovanni. When they come to pick up their jackets after the performance, with uniform probability, the coat check clerk gives them a random black, Jack Wolfskin jacket.

Let X be the random variable defined as the number of visitors who get the correct jacket back, so $X \in \{0, \ldots, n\}$.

On average, how many visitors do we expect to get back their own jacket?

Hint: Use indicator functions / variables, and expected value.

Cultural side notes:

1. I may have seen Ralf Fiennes, of Harry Potter Voldemort and English Gardner fame, waiting for a performance of Don Giovanni to start in Berlin back in 2008.

2. A large proportion of Germans wear black, Jack Wolfskin jackets. After a work party once when I went to pick up my jacket, I jokingly tried to help the coat clerk by telling her that mine was the black, Jack Wolfskin jacket.

A solution

Let J_k be the event that the k^{th} visitor gets their jacket back. By assumption of uniformity, $P(J_k) = 1/n$.

Define the indicator function I_k as

$$I_k = 1 \text{ if } A_k \text{ occurs; else } 0$$

In other words, I_k is the indicator function for the k^{th} person getting their correct jacket back.

Since $X = I_1 + \ldots I_n$, we can use linearity of expectation to calculate

$$E(X) = E\left(\sum_{j=1^n} I_j\right) = \sum_{j=1^n} E(I_j)$$

but $E(I_j) = 1 \cdot 1/n + 0 \cdot (n-1)/n = 1/n$ for each j, so

$$E(X) = n \cdot 1/n = 1$$

Hence we expect one person to get their correct jacket back.

Mathematical aside: Returning to the question about the police-officer and hooligan configurations, in this exercise, the trials in question (picking up your jacket) are *distinguishable*, as the setup requires that the similar looking jackets be identifiable as belonging to one owner.

2 Exercise (expected value, method of indicators): A variant on the blue and red pills

(Adapted from Stirzaker, Probability and Random Variables, Chapter 5, ex. 11)

In the movie the Matrix, the character Morpheus offers another character Neo two pills. If Neo chooses the blue pill, he will return to his old life. If he chooses the red one, his eyes will be opened to some truth, and he will join Morpheus' band.

In this variant (also of Pólyeva's urn, see MR, vaja 4), Neo has to choose blindly from a bucket initially containing one red and one blue pill. If he picks red, he must swallow it and the game is over. If he picks blue, then he returns the blue pill and Morpheus adds an extra red pill.

Let X be the number of draws. Calculate

1. The distribution $P(X > k)$
2. $\mathrm{E}(X)$ (hint: use the method of indicators)

A solution for 1, $P(X = k)$

This is a geometric like process (a number of successive failures followed by a single success), except that the probability of success or failure changes each time.

Now suppose $k \in \{1, \dots, 4\}$. By Morphius' rules, $X > k$ means that the first k draws were blue

$$P(X > k) = \prod_{j=1}^{k} (j+1) = \frac{1}{(k+1)!}$$

A solution with indicator functions for 2, $\mathbf{E}(X)$

Let I_j be the event that Neo draws again at step j. By construction, $X = \sum_{j=1}^{\infty} I_j$, and by linearity of expectation,

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{j=1}^{\infty} E(I_j) \\
&= \sum_{j=1}^{\infty} P(X \geq j) \\
&= \sum_{j=1}^{\infty} P(X > j-1) = \sum_{j=1}^{\infty} \frac{1}{j!}
\end{aligned}
$$

But $\sum_{j=1}^{\infty} \frac{1}{j!} = e - 1 \sim 1.718$, so $\mathrm{E}(X) = e - 1$.

3 Exercise (integral, transformation of continuous random variable)

Let the continuous random variable X be defined as having distribution

$$
f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi} x^{3/2}} \exp\left(\frac{-1}{2x}\right) & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}
$$

Show that, if $Z \sim N(0,1)$, for $x > 0$ we have

$$
F_X(x) = 2P_Z\left(Z \geq \frac{1}{\sqrt{x}}\right).
$$

A solution

Note: as mentioned during tutorials, I have a tendency to be verbose. This level of explanatory detail is likely not required for full marks on an exam answer.

This problem can be solved by finding an appropriate change of variable, and then using this change of variable correctly to convert the integral corresponding to (the definition of) $F_X(x)$ as $F_X(x) = \int_{-\infty}^{x} f_X(s)ds$ into an integral that involves the probability distribution function of the standard normal random variable. If you are already comfortable with the ins and outs of integrating combined with change of variables, then feel free to skip head to Section 3.0.2.

3.0.1 Solution background

In general, if $\psi : U \subseteq \mathbb{R} \to V = \psi(V) \subseteq \mathbb{R}$ is a continuous, 1x differentiable function with a well-defined inverse ψ^{-1}, and then the following integral I can be written under the change of variable ψ as

$$I = \int_c^d f(t)dt = \int_{\psi^{-1}(c)}^{\psi^{-1}(d)} f(\psi(s)) \frac{d\psi}{ds} ds \tag{1}$$

In even greater generality, the above only need hold off a set of Lebesgue measure 0 in $[c, d]$.

Note: Equation (1) assumes we have a **signed integral**. For the original integral we have chosen $c < d$. For the transformed version under ψ, it need not be the case that $\psi(c') \leq \psi(d')$ for all $c \leq c' < d \leq d'$. For calculations, we need to break the interval $[\psi^{-1}(c), \psi^{-1}(d)]$ (or $[\psi^{-1}(d), \psi^{-1}(c)]$ if $\psi(c) > \psi(d)$) into subintervals $[c', d']$ where ψ is either strictly non-increasing and non-decreasing, and then adjust the sign accordingly with a multiple of -1 if $\psi(c') > \psi(d')$.

3.0.2 A solutions sans background

Now we return to our specific example, with

$$\begin{aligned} I = I(x) &= \int_{-\infty}^{x} f_X(t)dt \\ &= \int_0^x \frac{1}{\sqrt{2}t^{3/2}} \exp\left(\frac{-1}{2t}\right) dt, \qquad \text{as } f_X(t) = 0 \text{ for } t \leq 0, \end{aligned}$$

Since the probability distribution of Z is $f_Z = \frac{\exp(-z^2)}{\sqrt{2\pi}}$, we want a transformation satisfying, for $s > 0$, $\frac{-1}{2\psi(s)} = -\frac{s^2}{2}$. Let's consider then

$$(t =)\, \psi(s) = \frac{1}{s^2}$$

with first derivative

$$\frac{d}{ds}\psi(s) = -\frac{2}{s^3}$$

This function is continuous (and differentiable) on $(0, \infty)$. It is strictly monotonically decreasing there as well, so it is invertible, and has inverse

$$(s =)\, \psi^{-1}(t) = \frac{1}{\sqrt{t}}$$

which is also at least once-differentiable on $(0, \infty)$.

Since $\lim_{t \to 0^+} \psi^{-1}(t) = +\infty$, we write the integral using the change of variable ψ now as

$$\begin{aligned}
I(x) &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\frac{1}{\sqrt{x}}} \frac{1}{\frac{1}{s^3}} \exp\left(\frac{-s^2}{2}\right)\left(-\frac{2}{s^3}\right) ds \\
&= -2\frac{1}{\sqrt{2\pi}} \int_{\infty}^{\frac{1}{\sqrt{x}}} \exp\left(\frac{-s^2}{2}\right) ds \\
&= 2\int_{\frac{1}{\sqrt{x}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-s^2}{2}\right) ds && \text{by switching integration order} \\
&= 2\int_{\frac{1}{\sqrt{x}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-s^2}{2}\right) ds \\
&= 2P\left(Z \geq \frac{1}{\sqrt{x}}\right)
\end{aligned}$$

where the final equality holds due to the definition of F_Z as

$$F_Z(z) = P(Z \leq z) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp^{-1/2s^2} ds,$$

and using that $P(Z > z) = 1 - F_Z(z)$ together with $P(Z = z) = 0$ (set of measure zero).

Notation aside: I first could reliably remember and felt comfortable with the above integration-under-change-of-variable in the context of pull-backs in differential geometry (see e.g. J.M. Lee, Introduction to Smooth Manifolds, Chapter 6, especially Proposition 6.13). Even without getting into differential geometry and cotangent bundles, the types of diagrams used in that context might be helpful.

$$\begin{tikzcd}
(a,b) \subseteq \mathbb{R} \arrow[r, "\psi"] \arrow[dr, "f \circ \psi"] & (c,d) \subseteq \mathbb{R} \arrow[d, "f"] \\
& \mathbb{R}_{\geq 0}
\end{tikzcd}$$

In our current exercise context, the original integral on the right with f turns out to be easier to calculate if we pull-it-back to $(a, b) = (\psi^{-1}(c), \psi^{-1}(d))$ and make sure we correctly handle the volume element dt, which is an example in differential geometry one-form.

3.1 Exercise (Transformation of random variable): uniform

Adapted from Stirzaker, Probability and Random Variables, Example 5.5.3

Let $X \sim U(a, b)$, $0 < a < b < 1 \subseteq \mathbb{R}$, and let $Y = -\frac{1}{\lambda} \log X$, where $\lambda > 0$ and log is the natural logarithm.

Determine $F_Y(y)$.

A solution

By definition of Y,

$$\begin{aligned}
F_Y(y) &= P_Y(Y \leq y) \\
&= P_X(-\frac{1}{\lambda} \log X \leq y) \\
&= P_X(\log X \geq -\lambda y), \text{ since } \lambda > 0 \\
&= P_X(X \geq \exp(-\lambda y)), \text{ since exp preserves ordering on } \mathbb{R}
\end{aligned}$$

The probability density function of $U(a, b)$ is $\frac{1}{b-a} \cdot I_{s \in [a,b]}$, where here we mean I as the indicator function ($= 1$ for $s \in [a, b]$, otherwise 0). Thus we have

$$
\begin{aligned}
F_Y(y) &= \int_{\exp(-\lambda y)}^{\infty} \frac{1}{b-a} \cdot I_{s\in[a,b]} ds \\
&= \frac{1}{b-a} \int_{\exp(-\lambda y)}^{\infty} I_{s\in[a,b]} ds
\end{aligned}
$$

We consider now cases on y values, first informally in words, then in formulae.

- If the bottom integration limit is bigger than b, then we are integrating over an identically 0 function, and the integral is 0.
- If the bottom integration limit is between a and b, then we have in integral whose value depends on the bottom integration limit.
- Finally, if the bottom integration limit is less than a then we are integrating the indicator over its support (=where it's non-zero). Since this indicator function is normalized to 1 when integrating over its support, the integral evaluates to 1 in this case.

If $\exp(-\lambda y) > b$, we have $F_Y(y) = 0$. This condition is equivalent to

$$
\begin{aligned}
&-\lambda y > \log(b), \text{ since log preserves ordering on } \mathbb{R}, \text{ and} \\
&y < -\frac{1}{\lambda}\log(b), \text{ since } \lambda > 0.
\end{aligned}
$$

Next, if $-\frac{1}{\lambda}\log(b) \leq y \leq -\frac{1}{\lambda}\log(a)$, we have $F_Y(y) = \frac{b-\exp(-\lambda y)}{b-a}$.

Finally, if $y > -\frac{1}{\lambda}\log(a)$, then the integral evaluates to $b-a$, and $F_Y(y) = 1$ for such y values.

Putting the pieces together,

$$
F_Y(y) = \begin{cases} 0 & \text{if } y < -\frac{1}{\lambda}\log(b) \\ \frac{b-\exp(-\lambda y)}{b-a} & \text{if } -\frac{1}{\lambda}\log(b) \leq y \leq -\frac{1}{\lambda}\log(a) \\ 1 & \text{if } y > -\frac{1}{\lambda}\log(a). \end{cases} \tag{2}
$$

The resulting Y is called a truncated exponential distribution.

3.2 Multivariable distribution, marginals, independence

Inspired by Ronald Meester, A Natural Introduction to Probability Theory, Example 2.4.10

Let (X, Y) have join distribution function

$$P(X = k, Y = \ell) = \begin{cases} \frac{1}{2\log 2}\frac{2^{-k}}{\ell} & \text{if } k \in \mathbb{N} \text{ and } \ell \in [1, k] \\ 0 & \text{otherwise.} \end{cases}$$

Calculate

1. The marginal distributions $P(X = k)$ and $P(Y = \ell)$
2. Are X and Y independent?

Hint: You may use that for $\ell \in \mathbb{N}$, $\sum_{k=\ell}^{\infty} \frac{1}{2^k} = \frac{1}{2^{\ell-1}}$

A solution

First, we calculate $P(X = k)$:

$$\begin{aligned} P(X = k) &= \sum_{\ell=1}^{\infty} P(X = k, Y = \ell) \\ &= \sum_{\ell=1}^{k} P(X = k, Y = \ell) \\ &= \frac{1}{2\log 2}\frac{1}{2^k}\sum_{\ell=1}^{k}\frac{1}{\ell}. \end{aligned}$$

The sum $\sum_{\ell=1}^{k} \frac{1}{\ell}$ is called the k^{th} harmonic number.

Next, let's calculate $P(Y = \ell)$:

$$
\begin{aligned}
P(Y=\ell) &= \sum_{k=1}^{\infty} P(X=k, Y=\ell) \\
&= \sum_{k=\ell}^{\infty} P(X=k, Y=\ell) \\
&= \frac{1}{2\log 2}\frac{1}{\ell}\sum_{k=\ell}^{\infty}\frac{1}{2^k} \\
&= \frac{1}{2\log 2}\frac{1}{\ell}\frac{1}{2^{\ell-1}} \text{ by the hint}
\end{aligned}
$$

Are X and Y independent?

Take $k=1, \ell=2$, then $P(X=1, Y=2)=0$, but $P(X=1)>0$ and $P(Y=2)>0$, so they are dependent.