Probability Exercises, 2025-12-01

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1 Exercise (expected value, method of indicators): hat (or jacket check) problem

n visitors to the Komische Opera ('funny opera') in Berlin all check in black, Jack Wolfskin jackets before a performance of Mozart's Don Giovanni. When they come to pick up their jackets after the performance, with uniform probability, the coat check clerk gives them a random black, Jack Wolfskin jacket.

Let X be the random variable defined as the number of visitors who get the correct jacket back, so $X \in \{0, \ldots, n\}$.

On average, how many visitors do we expect to get back their own jacket?

Hint: Use indicator functions / variables, and expected value.

Cultural side notes:

1. I may have seen Ralf Fiennes, of Harry Potter Voldemort and English Gardner fame, waiting for a performance of Don Giovanni to start in Berlin back in 2008.

2. A large proportion of Germans wear black, Jack Wolfskin jackets. After a work party once when I went to pick up my jacket, I jokingly tried to help the coat clerk by telling her that mine was the black, Jack Wolfskin jacket.

2 Exercise (expected value, method of indicators): A variant on the blue and red pills

(Adapted from Stirzaker, Probability and Random Variables, Chapter 5, ex. 11)

In the movie the Matrix, the character Morpheus offers another character Neo two pills. If Neo chooses the blue pill, he will return to his old life. If he chooses the red one, his eyes will be opened to some truth, and he will join Morpheus' band.

In this variant (also of Pólyeva's urn, see MR, vaja 4), Neo has to choose blindly from a bucket initially containing one red and one blue pill. If he picks red, he must swallow it and the game is over. If he picks blue, then he returns the blue pill and Morpheus adds an extra red pill.

Let X be the number of draws. Calculate

1. The distribution $P(X > k)$

2. E(X) (hint: use the method of indicators)

3 Exercise (integral, transformation of continuous random variable)

Let the continuous random variable X be defined as having distribution

$$f_X(x) = \begin{cases} \frac{c}{x^{3/2}} \exp\left(\frac{-1}{2x}\right) & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is some constant chosen so that $\int_{-\infty}^{\infty} f_X(x) = 1$.
Part 1: Show that, if $Z \sim N(0,1)$, for $x > 0$ we have

$$F_X(x) = 2P_Z\left(Z \geq \frac{1}{\sqrt{x}}\right).$$