ZS 2

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2. Direct methods for solving LSE's

Error: $\underbrace{\frac{\|\Delta x\|}{\|x\|}}_{\text{output error}} \leq \underbrace{\|A^{-1}\| \cdot \|A\|}_{\text{cond}(A)} \cdot \underbrace{\frac{\|\Delta b\|}{\|b\|}}_{\text{input error}}$

Terminology: $N(A) = \{x : Ax = 0\}$ lower Triangular: $\begin{pmatrix} 3 & & 0 \\ 6 & 4 & \\ 7 & 8 & 5 \end{pmatrix}$

$R(A) = \{Ax, \ x \in K^n\}$

2.1. Conventional Methods

Gauss:
1. Forward substitution (for "Zeilenstufenform") $O(n^3)$
2. Backwards substitution $O(n^2)$

- Not numerically stable (devision by small numbers, cancellation)
- $O(n^3)$, $O(N \cdot n^3)$

LU: $L \cdot U = P \cdot A$

1. LU-Factorization $O(n^3)$
2. Forward-substitution $Lz = b$ $O(n^2)$
3. backward-substitution $Ux = z$ $O(n^2)$

- Not numerically stable

+ $O(n^3)$, $O(n^3 + N \cdot n^2)$

2.2. Exploiting structure of LSE

Block elimination: $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$x_1 = A_{11}^{-1}(b_1 - A_{12} \cdot x_2)$$

$$x_2 = \frac{b_2 - A_{21} A_{11}^{-1} b_1}{A_{22} - A_{21} A_{11}^{-1} A_{12}}$$

+ If arrow format: $O(n)$

Low-rank-Modification:

You already solved $Ax = b$ and want to now solve $\tilde{A}x = \tilde{b}$ where $A - \tilde{A}$ is a low rank matrix

Rank-1: $\tilde{A} = A + u \cdot v^T$

$$\begin{bmatrix} A & u \\ v^T & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ \xi_1 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$\Rightarrow x = A^{-1}b - A^{-1}u \cdot \frac{v^T A^{-1} b}{\quad}$: $Ay_1 = u$

$1 + v^T(A^{-1}u)$: $Ay_2 = b$

LU-Composition of A available: $O(n^2)$

LU-Composition of A not available: $O(n^3)$

Rank-k: (Sherman-Morrison-Woodbury) $\tilde{A} = A + UV^H$

$$(\tilde{A})^{-1} = (A + U \cdot V^H)^{-1} = A^{-1} - A^{-1}U(I + V^H A^{-1} U)^{-1} V^H A^{-1}$$

$$x = \tilde{A}^{-1} b = A^{-1} b - A^{-1} U (I + V^H A^{-1} U)^{-1} V^H A^{-1} b$$

LU-Composition of A available: $O(n^2)$

LU-Composition of A not available: $O(n^3)$

Computing Factorization UV^H: $O(nk^2)$

2.3. Sparse LSE

Format:
1.) Nonzero entries (CRS → row major)
2.) Column index of the el. from 1.
3.) rowpointer: $rp[0]=0$, $rp[i] = rp[i-1] + nnz((i-1)\text{-th row})$

Complexity: $[O(nnz^{3/2}), O(nnz^{5/2})]$