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SCSR1013 DIGITAL LOGIC
Module 4b

FACULTY OF COMPUTING

Simplify using Rules of Boolean Algebra

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1	$A + 0 = A$
2	$A + 1 = 1$
3	$A \bullet 0 = 0$
4	$A \bullet 1 = A$
5	$A + A = A$
6	$A + \overline{A} = 1$
7	$A \bullet A = A$
8	$A \bullet \overline{A} = 0$
9	$\overline{\overline{A}} = A$
10	$A + AB = A$
11	$A + \overline{A}B = A + B$
12	$(A + B)(A + C) = A + BC$

Simplify this expression

$$AB + A(B + C) + B(B + C)$$

$$AB + A(B + C) + B(B + C)$$

Step 1: Apply distributive law to the red terms

$$AB + AB + AC + BB + BC$$

Step 2: Apply rule 7 (BB = B) to the green term

$$AB + AB + AC + B + BC$$

Step 3: Apply rule 5 (AB + AB = AB) to the red terms

$$AB + AC + B + BC$$

Step 4: Apply rule 10 (B + BC = B) to the green terms

$$AB + AC + B$$

Step 5: Apply rule 10 (B + AB = B) to the red terms

$$AC + B$$

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8	$A \bullet \overline{A} = 0$
9	$\overline{\overline{A}} = A$
10	$A + AB = A$
11	$A + \overline{A}B = A + B$
12	$(A + B)(A + C) = A + BC$

Simplify this expression

$$\overline{A}BC + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + ABC$$

$$\overline{A}BC + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + ABC$$

Step 1: Factor BC for the red terms

$$BC(\overline{A} + A) + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C$$

Step 2: Apply rule 6 to the green term and factor the blue term

$$BC\,.\,1 + A\overline{B}(\overline{C} + C) + \overline{A}\,\overline{B}\,\overline{C}$$

Step 3: Apply rule 4 to the red term and rule 6 to the blue term

$$BC + A\overline{B}\,.\,1 + \overline{A}\,\overline{B}\,\overline{C}$$

Step 4: Apply rule 4 to the green term

$$BC + A\overline{B} + \overline{A}\,\overline{B}\,\overline{C}$$

Step 5: Factor the red terms

$$BC + \overline{B}(A+\overline{A}\,\overline{C})$$

Step 6: Apply rule 11 to the blue term

$$BC + \overline{B}(A + \overline{C})$$

Step 7: Use the distributive and commutative laws to get the following expression

$$BC + A\overline{B} + \overline{B}\,\overline{C}$$

Extra

Exercise 4b.2: According the the example, draw the logic circuit for the original expression and the last expression simplified.

Original expression:

$$\overline{A}BC + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + ABC$$

Simplified expression: $BC + A\overline{B} + \overline{B}\,\overline{C}$

Forms of Boolean expressions

- Boolean expression can be converted into one of 2 forms.

$\overline{A}$

Forms of Boolean expressions

SOP (Sum-of-Products)

POS (Product-of-Sum)

$$AB\overline{C} + \overline{A}B\overline{C}$$

$$(A + B + C)(A + \overline{B} + C)$$

Product term = a term with the product (Boolean multiplication) of literals

Sum term = a term with the sum (Boolean addition) of literals

- Standard form
  - SOP $ABC + \bar{A}B\bar{C} + \bar{A}\bar{B}C$
  - POS $(A+B+\bar{C})(\bar{A}+B+\bar{C})(A+\bar{B}+\bar{C})$
  - A standard SOP / POS form is when ALL the variables appear in each product term / sum term

- Non standard
  - SOP $AB + \bar{C} + \bar{A}\bar{B}C$
  - POS $(B+\bar{C})(\bar{A}+\bar{C})(A+\bar{B})$

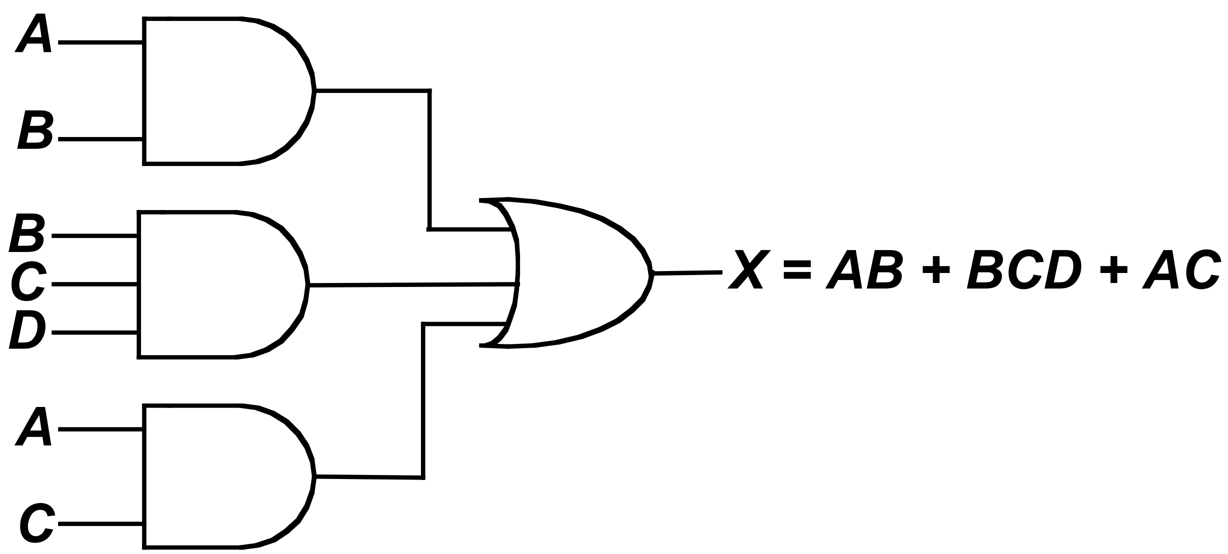
Sum-of-Product (SOP)

- The bit 0 is $\bar{A}$ and 1 is A
- Product term ➔ multiplication '.'
- The operator is summation +
- The output truth table / K-Map is **1**
- The symbol is $\sum_{ABC}$
- **SOP** = when 2 or more product terms are summed.
- e.g $AB_{P1} + ABC_{P2}$

  $ABC_{P1} + CDE_{P2} + BCD_{P3}$
- A good writing $\bar{A}\bar{B}\bar{C}$ ☑ $\overline{ABC}$ ☒

- Implementation of the SOP expression

  $AB + BCD + AC$



Exercise

Convert each of the following Boolean expressions to SOP form:

(i) $AB + B(CD + EF)$

(ii) $(A + B)(B + C + D)$

(iii) $\overline{(\overline{A + B}) + C}$

Exercise Solution

(i) $AB + B(CD + EF) \quad = AB + BCD + BEF$

(ii) $(A + B)(B + C + D)$

$$= (A + B)B + (A + B)C + (A + B)D$$

$$= AB + BB + AC + BC + AD + BD$$

(iii) $\overline{(\overline{A + B}) + C}$

$$= (\overline{\overline{A + B}})\overline{C}$$ (DeMorgan's Theorem II)

$$= (A + B)\overline{C}$$ (Apply rule 9)

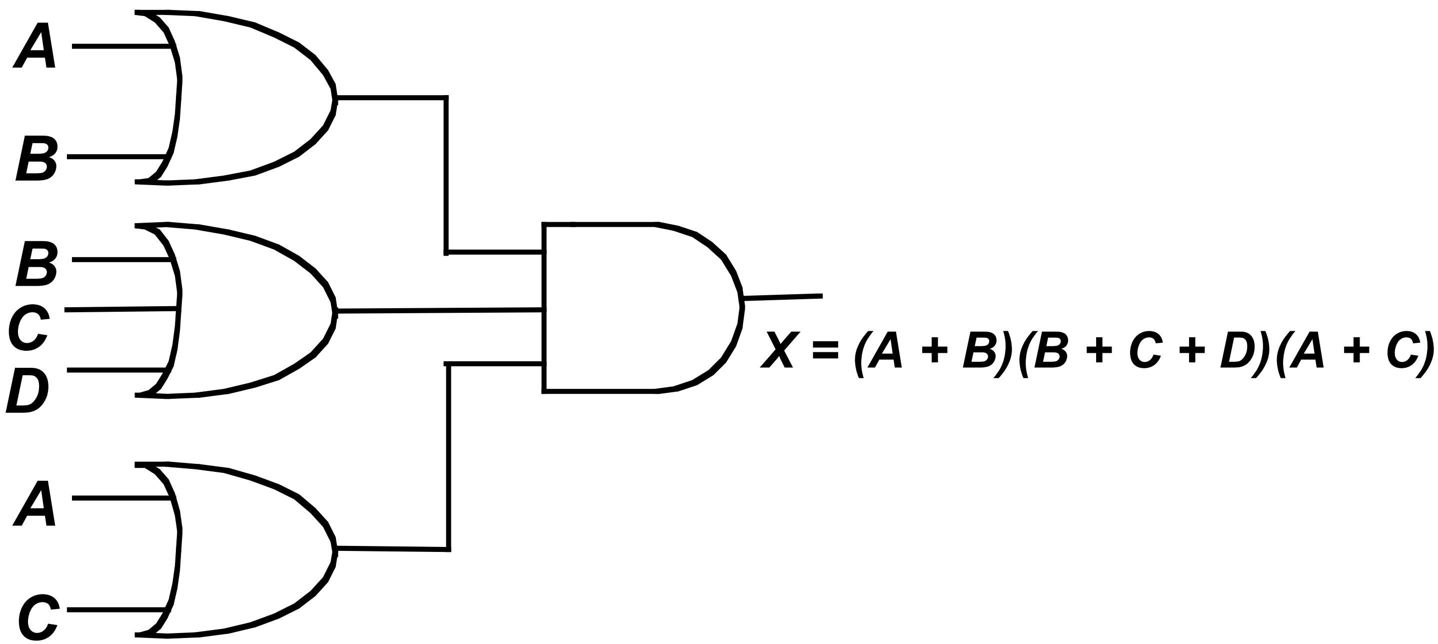
$$= A\overline{C} + B\overline{C}$$

Product-of-Sum (SOP)

- The bit 1 is $\overline{A}$ and 0 is A
- Sum term ➔ addition '+'
- The operator is multiplication .
- The output truth table / K-Map is **0**
- The symbol is $\prod_{ABC}$
- **POS** = when 2 or more sum terms are multiplied.
  - $(A + B)_{S1}(A + B + C)_{S2}$
  - $(A + B + C)_{S1}(C + D + E)_{S2}(B + C + D)_{S3}$
- A good writing $\overline{A} + \overline{B} + \overline{C}$ ☑ $\overline{A + B + C}$ ☒

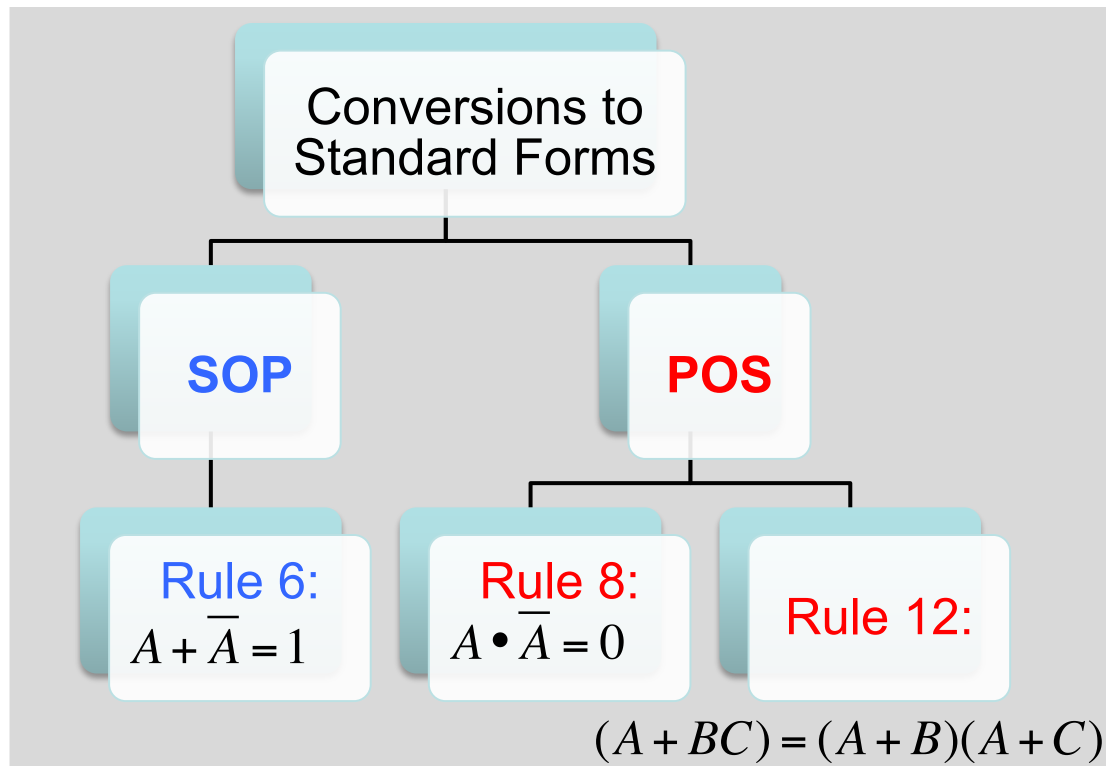
- Implementation of the POS expression

  (A + B)(B + C + D)(A + C)



Standard for Boolean Expressions (BE)

- Standardization makes the evaluation, simplification, and implementation of Boolean expressions more systematic and easier.
- A standard SOP / POS form is when ALL the variables appear in each product term / sum term of the expression.
- There are **two** method for each SOP/POS to solve the BE.



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(a) Sum-of-Product (SOP) Method 1

- A logic expression can be changed to SOP form using Boolean algebra techniques.
  - $A(B + CD) = AB + ACD$
  - $AB + B(CD + EF) = AB + BCD + BEF$

- Standard SOP form is when all the variables appear in each product term in the expression.

$$A\overline{B}CD + \overline{A}\overline{B}C\overline{D} + AB\overline{C}\overline{D}$$

- To convert product terms to standard SOP
  - Multiply each of the nonstandard term with the missing term using Boolean algebra Rule 6: $(A + \bar{A})=1$
  - Repeat until all variables appear in each product term.

(a) Sum-of-Product (SOP) Method 1

- Convert this Boolean expression to standard SOP form:

$$\underbrace{A\overline{B}C}_{\textit{Term 1}} + \underbrace{\overline{A}\,\overline{B}}_{\textit{Term 2}} + \underbrace{AB\overline{C}D}_{\textit{Term 3}}$$

Solution:

- Variables = A, B, C, D.
- What is missing?
  - Term 1: missing D
  - Term 2: missing C and D
- Complete these terms by applying Boolean rule 6

(a) Sum-of-Product (SOP) Method 1

$$\underbrace{A\overline{B}C}_{\text{Term 1}} + \underbrace{\overline{A}\,\overline{B}}_{\text{Term 2}} + AB\overline{C}D$$

Rule 6: (A + Ā)=1

$$\text{Term 1}: ABC = ABC(D + D) = ABCD + ABCD$$

$$\text{Term 2}: \overline{A}\,\overline{B} = AB(C + \overline{C}) = \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C}$$

$$= \overline{A}\,\overline{B}C(D + \overline{D}) + \overline{A}\,\overline{B}\,\overline{C}(D + \overline{D}) = \overline{A}\,\overline{B}CD + \overline{A}\,\overline{B}C\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$$

- Now we have the standard of SOP expression:

$$A\overline{B}C + \overline{A}\,\overline{B} + AB\overline{C}D$$

$$= A\overline{B}CD + A\overline{B}C\overline{D} + \overline{A}\,\overline{B}CD + \overline{A}\,\overline{B}C\overline{D} + \overline{A}\,\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D} + AB\overline{C}D$$

Exercise

Define the variables of SOP expression $A\overline{C}+B\overline{C}$ and convert the expression to standard SOP form.

Solution:

$$A\overline{C}+B\overline{C} = A\overline{C}(B+\overline{B})+B\overline{C}(A+\overline{A})$$ (Apply rule 6)

$$= AB\overline{C}+A\overline{B}\,\overline{C}+AB\overline{C}+\overline{A}B\overline{C}$$ (Apply rule 5)

$$= AB\overline{C}+A\overline{B}\,\overline{C}+\overline{A}B\overline{C}$$

(a) Sum-of-Product (SOP) Method 2 – Binary Representation

$$\underbrace{A\bar{B}C}_{\text{Term 1}} + \underbrace{\bar{A}\bar{B}}_{\text{Term 2}} + AB\bar{C}D$$

- The bit 0 is $\bar{A}$ and 1 is A
- Identify a variable that is missing in each term using binary

$A\bar{B}C$ Term 1 – missing D ➔ it can be **0** $A\bar{B}C\bar{D}$

1 $A\bar{B}CD$

$\bar{A}\bar{B}$ Term 2 – missing C, D ➔ **00** $\bar{A}\bar{B}\bar{C}\bar{D}$

01 $\bar{A}\bar{B}\bar{C}D$

10 $\bar{A}\bar{B}C\bar{D}$

11 $\bar{A}\bar{B}CD$

From the standard SOP it is easy to generate the truth table.

$$ABCD + A\bar{B}\bar{C}D + \bar{A}\bar{B}\bar{C}\bar{D}$$

$ABCD$
1 1 1 1

$A\bar{B}\bar{C}D$
1 0 0 1

$\bar{A}\bar{B}\bar{C}\bar{D}$
0 0 0 0

INPUT				OUTPUT
A	B	C	D	F
0	**0**	**0**	**0**	**1**
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	**0**	**0**	**1**	**1**
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	**1**	**1**	**1**	**1**

(b) Product-of-Sum (POS) Method 1

- Standard POS form = where all the variables appear in each sum term in the expression.
- To convert product terms to standard POS
  - Multiply each of the nonstandard term with the missing term using Boolean algebra Rule 8: $(A \cdot \bar{A}) = 0$
  - Apply Rule 12: $(A + BC) = (A + B)(A + C)$
  - Repeat until all variables appear in each sum term.

(b) Product-of-Sum (POS) Method 1

- Convert this Boolean expression to standard POS form

$$\underbrace{(A+\overline{B}+C)}_{\textit{Term 1}}\underbrace{(\overline{B}+C+\overline{D})}_{\textit{Term 2}}\underbrace{(A+\overline{B}+\overline{C}+D)}_{\textit{Term 3}}$$

Solution:

- Variables = A, B, C, D.
- What is missing?
  - Term 1: missing D
  - Term 2: missing A

$$(A+\overline{B}+C)(\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

$\underbrace{(A+\overline{B}+C)}_{\text{Term 1}}\underbrace{(\overline{B}+C+\overline{D})}_{\text{Term 2}}$

- Apply rules 8 and 12

Rule 8: (A . Ā) = 0

Rule 12: (A + BC) = (A + B)(A + C)

$$\text{Term 1}: A+\overline{B}+C = A+\overline{B}+C+D\overline{D} = (A+\overline{B}+C+D)(A+\overline{B}+C+\overline{D})$$

$$\text{Term 2}: \overline{B}+C+\overline{D} = \overline{B}+C+\overline{D}+A\overline{A}$$

$$= (A+\overline{B}+C+\overline{D})(\overline{A}+\overline{B}+C+\overline{D})$$

- Now we have the standard of POS expression:

$$=(A+\overline{B}+C+D)(A+\overline{B}+C+\overline{D})(A+\overline{B}+C+\overline{D})(\overline{A}+\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

Exercise

- Convert the following Boolean expressions to standard POS form: $(A+B)(\overline{B}+C)$

Rule 8: $(A \cdot \bar{A}) = 0$

Rule 12: $(A + BC) = (A + B)(A + C)$

Solution

$$(A+B)(\overline{B}+C)$$

$$= (A+B+C\cdot\overline{C})(A\cdot\overline{A}+\overline{B}+C)$$

$$= (A+B+C)(A+B+\overline{C})(A+\overline{B}+C)(\overline{A}+\overline{B}+C)$$

(b) Product-of-Sum (POS) Method 2 – Binary Representation

$(A+B)(\overline{B}+C)$

Term 1 *Term 2*

- The bit 1 is $\overline{A}$ and 0 is A
- Identify a variable that is missing in each term using binary

$(A+B+\boxed{})$ Term 1 – missing C ➔ it can be **0** $A+B+\boxed{C}$

1 $A+B+\boxed{\overline{C}}$

$(\boxed{}+B+C)$ Term 2 – missing A ➔

0 $\boxed{A}+B+C$

1 $\boxed{\overline{A}}+B+C$

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Exercise

Example: Determine the binary value for which following standard POS expression is equal to 0

$$(A+B+C+D)(A+\overline{B}+\overline{C}+D)(\overline{A}+\overline{B}+\overline{C}+\overline{D})$$

Solution:

$(A + B + C + D) = (0 + 0 + 0 + 0) = 0;$ A=0, B=0, C=0, D=0

$(A + \overline{B} + \overline{C} + D) = (0 + 1 + 1 + 0) = 0;$ A=0, B=1, C=1, D=0

$(\overline{A} + \overline{B} + \overline{C} + \overline{D}) = (1 + 1 + 1 + 1) = 0;$ A=1, B=1, C=1, D=1

The POS expression equal 0 when ALL of the terms are 0.

From the standard POS, $(A+B+C+D)(A+\overline{B}+\overline{C}+D)(\overline{A}+\overline{B}+\overline{C}+\overline{D})$ it is easy to generate the truth table.

$(A+B+C+D)$

0 0 0 0

$(A+\overline{B}+\overline{C}+D)$

0 1 1 0

$(\overline{A}+\overline{B}+\overline{C}+\overline{D})$

1 1 1 1

INPUT				OUTPUT
A	B	C	D	F
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

Exercise 4b.6:

Represent the following Boolean expression:

(i) $AB\overline{C} + A\overline{B}\,\overline{C} + \overline{A}B\overline{C}$ as a sigma notation

(ii) $(A+B+C)(A+B+\overline{C})(A+\overline{B}+C)(\overline{A}+B+C)$

as a PI notation.

Solution 4b.6(i):

Expression: $AB\overline{C} + A\overline{B}\,\overline{C} + \overline{A}B\overline{C}$

110 100 010

Sigma notation: $\sum_{ABC}(6,4,2)$

Solution 4b.6(ii):

Expression: $(A+B+C)(A+B+\overline{C})(A+\overline{B}+C)(\overline{A}+B+C)$

000 001 010 100

PI notation: $\prod_{ABC}(0,1,2,4)$

Exercise 4b.7:

A Boolean expression is written in sigma notation as X= $\sum_{ABC}(7,4,3)$. Determine the logic level (binary value) for each product term and write whole expression.

Solution 4b.7:

$\sum_{ABC}(7,4,3)$

Logic level: 111 100 011

Expression: (SOP) $ABC + A\overline{B}\overline{C} + \overline{A}BC$

Exercise 4b.8:

A Boolean expression is written in PI notation as $X = \prod_{ABC}(7,4,3)$. Determine the logic level (binary value) for each sum term and write whole expression.

Solution 4b.8:

$\prod_{ABC}(7,4,3)$

Logic level: 111 100 011

Expression: (POS)

$$(\overline{A}+\overline{B}+\overline{C})(\overline{A}+B+C)(A+\overline{B}+\overline{C})$$

Converting Standard (SOP → POS)

Example: Convert the following SOP expression to an equivalent POS expression.

$$\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC + A\bar{B}C + ABC$$

Solution:

- Variables = (A, B, C) = 3. So, 2^3 = 8 possible combinations.

$$\begin{array}{ccccc} \bar{A}\bar{B}\bar{C} & +\bar{A}B\bar{C} & +\bar{A}BC & +A\bar{B}C & +ABC \\ 000 & 010 & 011 & 101 & 111 \end{array}$$

- The SOP have 5 of 8, so POS have the other 3 (001, 100, 110)

  ➔ These 3 make

  sum term = 0

A	B	C	Output
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$\begin{array}{ccccccccc} (A + & B + & \bar{C})(& \bar{A} + & B + & C)(& \bar{A} + & \bar{B} + & C) \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}$$

Exercise 4b.9: Convert the following SOP expressions to an equivalent POS expression: $A\overline{C} + B\overline{C}$

Solution 4b.9:

- **Step 1**: Need to convert the expression into standard SOP (refer Exercise 4b.7)

$$AB\overline{C} + A\overline{B}\,\overline{C} + \overline{A}B\overline{C}$$

- **Step 2**: Binary number for each SOP term.

  Variables = 3 (A, B, C); 2^3 = 8 possible combinations.

$$\begin{array}{ccccc} AB\overline{C} & + & A\overline{B}\,\overline{C} & + & \overline{A}B\overline{C} \\ 1\ 1\ 0 & & 1\ 0\ 0 & & 0\ 1\ 0 \end{array} \qquad (\leftarrow 3 \text{ combinations})$$

Boolean Expressions and Truth Tables

Standard SOP → truth tables

Develop a truth table for the standard SOP expression

$$\bar{A}\bar{B}C + A\bar{B}\bar{C} + ABC$$

Solution

- Domain = A, B, C. combinations = 2^3 = 8
- What binary value makes the product term = 1?

$\bar{A}\bar{B}C$
0 0 1 = 1 1 1 = 1

$A\bar{B}\bar{C}$
1 0 0 = 1 1 1 = 1

ABC
1 1 1 = 1 1 1 = 1

- Fill the truth table

INPUTS			OUTPUT	PRODUCT TERM
A	B	C	X	
0	0	0	0	
0	0	1	1	$\bar{A}\bar{B}C$
0	1	0	0	
0	1	1	0	
1	0	0	1	$A\bar{B}\bar{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	ABC

Standard POS → truth tables

Develop a truth table for the standard POS expression

$$(A + B + C)(A + \overline{B} + C)(A + \overline{B} + \overline{C})(\overline{A} + B + \overline{C})(\overline{A} + \overline{B} + C)$$

- Domain = A, B, C. combinations = $2^3 = 8$
- What binary value makes the sum term = 0?

Solution

$(A + B + C)$	$(A + \overline{B} + C)$	$(A + \overline{B} + \overline{C})$	$(\overline{A} + B + \overline{C})$	$(\overline{A} + \overline{B} + C)$
0 0 0	0 1 0	0 1 1	1 0 1	1 1 0
$\underbrace{0 \quad 0 \quad 0}_{0}$	$\underbrace{0 \quad 0 \quad 0}_{0}$	$\underbrace{0 \quad 0 \quad 0}_{0}$	$\underbrace{0 \quad 0 \quad 0}_{0}$	$\underbrace{0 \quad 0 \quad 0}_{0}$

- Fill the truth table

INPUTS			OUTPUT	PRODUCT TERM
A	B	C	X	
0	0	0	0	$(A + B + C)$
0	0	1	1	
0	1	0	0	$(A + \overline{B} + C)$
0	1	1	0	$(A + \overline{B} + \overline{C})$
1	0	0	1	
1	0	1	0	$(\overline{A} + B + \overline{C})$
1	1	0	0	$(\overline{A} + \overline{B} + C)$
1	1	1	1	

Exercise

Produce the truth table for the following expression:

a) $A\overline{B}+BC+\overline{A}D$

b) $(A+\overline{B})(B+C)(\overline{A}+D)$