CHAPTER 2

RELATIONS & FUNCTIONS

PART 1

RELATIONS

Definition

- If R is a relation from set A into itself, we say that R is a relation on A.

  $a \in A$, $b \in A$ $(a,b) \in A \times A$ and $R \subseteq A \times A$

- Example

  Let A = (1,2,3,4,5) and R be defined by $a,b \in A$,

$$aRb \leftrightarrow b - a = 2$$

R = {(1,3),(2,4),(3,5)}

Example

Let $A = \{1, 2, 3, 4\}$ *and* $B = \{p, q, r\}$

$R = \{(1, q), (2, r), (3, q), (4, p)\}$

$R \subseteq A \times B$

R is the relation from A to B

$1Rq$ (*1* is related to *q*)

$3\not{R}p$ (*1* is not related to *p*)

Relations

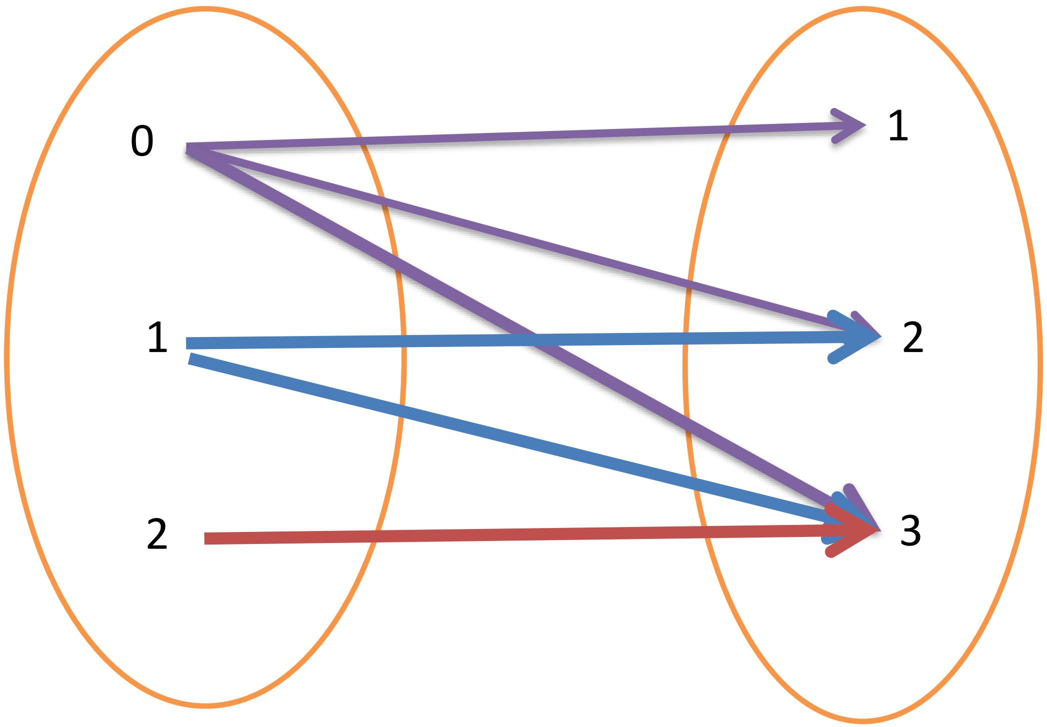
- Binary relations: xRy

  On sets $x \in X$ $y \in Y$

  $R \subseteq X \times Y$
- Example:

  "less than" relation from $A=\{0,1,2\}$ to $B=\{1,2,3\}$

Use Arrow Diagrams



$R=\{(0,1),(0,2),(0,3), (1,2),(1,3),(2,3)\}$

Use traditional notation

$0 < 1$, $0 < 2$, $0 < 3$, $1 < 2$, $1 < 3$, $2 < 3$

Use set notation

$A \times B=\{(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$

Example

A = {New Delhi, Ottawa, London, Paris, Washington}
B = {Canada, England, India, France, United States}

Let $x \in A$, $y \in B$.

Define the relation between x and y by "x is the capital of y"

R = {(New Delhi, India), (Ottawa, Canada), (London, England), (Paris, France), (Washington, United States)}

Domain and Range

Let R, a relation **from A to B.**

The set, **$\{ a \in A \mid (a,b) \in R$ for some $b \in B \}$**
is called the **domain** of R.

The set, **$\{ b \in B \mid (a,b) \in R$ for some $a \in A \}$**
is called **the range** of R.

Example

Let R be a relation on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ *if* $x \leq y$, *and* $x,y \in X$.

Then,
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

The **domain and range** of R are both equal to X.

Example

Let X = {2, 3, 4} and Y = {3, 4, 5, 6, 7}

If we **define a relation R from X to Y by,**

$(x,y) \in R$ if y/x (*with zero* remainder)

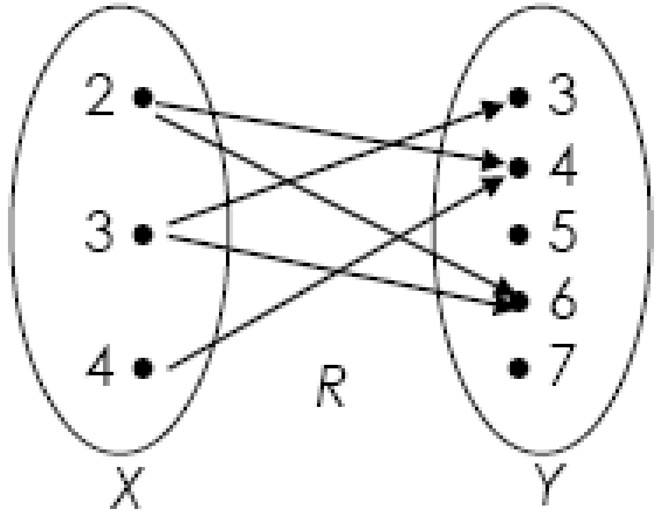
We obtain,

R = { (2,4), (2,6), (3,3), (3,6), (4,4) }

The domain of R is {2,3,4}

The **range** of R is {3,4,6}

R = { (2,4), (2,6), (3,3), (3,6), (4,4) }



Arrow diagram

Diagraph

An informative way to picture a relation on a set is to draw its digraph.

❖Let R be a relation on a finite set A.

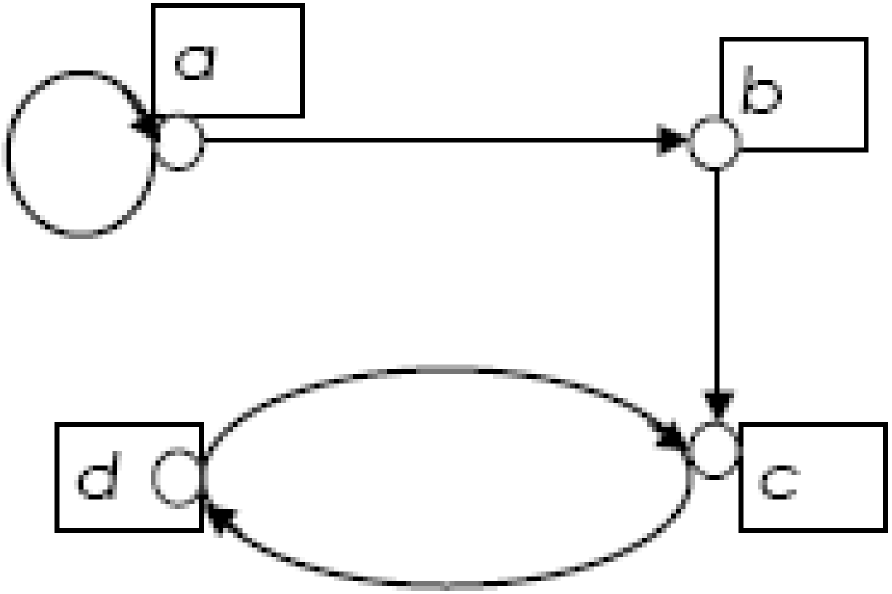
❖Draw dots (vertices) to represent the elements of A.

❖If the element $(a,b) \in R$, draw an arrow (called a directed edge) from a to b

Example

The relation R on $A = \{a, b, c, d\}$,

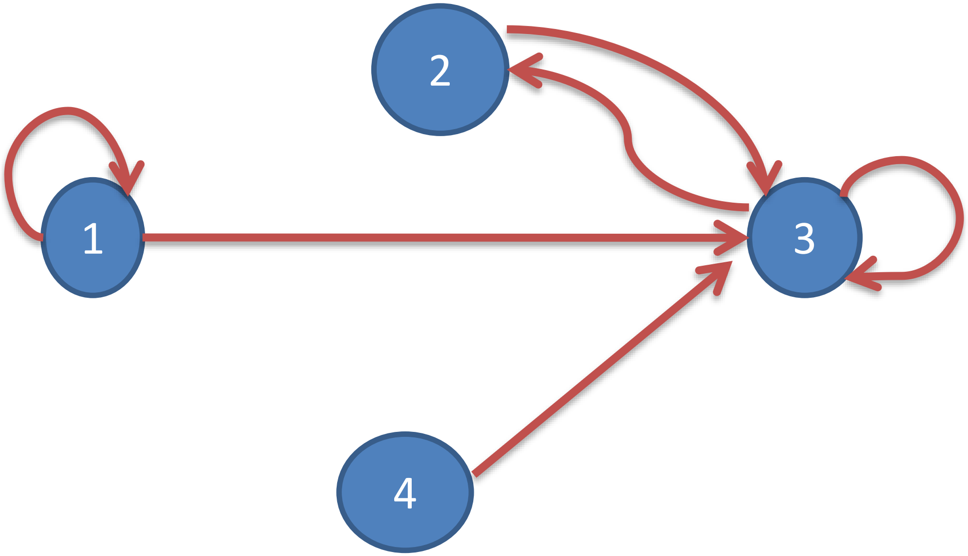
$R = \{(a, a), (a, b), (c, d), (d, c), (b, c)\}$



Exercise

1. Let $A = \{ 1,2,3,4\}$ and $R = \{ (1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4) , (4,1)\}$. Draw the digraph of R.

2. Find the relation determined by digraph below



Matrices of Relations

A matrix is a convenient way to represent a relation *R* from *A* to *B.*

- Label the rows with the elements of *A* (in some arbitrary order)
- Label the columns with the elements of *B* (in some arbitrary order)

Matrices of Relations

- Let $M_R = [m_{ij}]_{n\times p}$ be the Boolean n x p matrix

$$M_R = \begin{bmatrix} m_{11} & m_{12} & .. & .. & m_{1p} \\ m_{21} & m_{22} & .. & .. & m_{2p} \\ \vdots & \vdots & .. & .. & \vdots \\ \vdots & \vdots & .. & .. & \vdots \\ m_{n1} & m_{n2} & .. & .. & m_{np} \end{bmatrix}$$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

Example

- The relation,
  $R = \{ (1,b),(1,d),(2,c),(3,c),(3,b),(4,a) \}$
  from, $X = \{ 1, 2, 3, 4 \}$ to $Y = \{ a, b, c, d \}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \text{ or } M_R = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Example

The matrix of the relation *R* from { 2, 3, 4 } to { 5, 6, 7, 8 } defined by

x R y if x divides *y*

$$\begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example

Let A={ a, b, c, d }

Let R be a relation on A.

R = { (a,a),(b,b),(c,c),(d,d),(b,c),(c,b) }

$$M_R = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Exercise

An airline services the five cities c_1, c_2, c_3, c_4 and c_5. Table below gives the cost (in dollars) of going from c_i to c_j. Thus the cost of going from c_1 to c_3 is RM100, while the cost of going from c_4 to c_2 is RM200

To from	c_1	c_2	c_3	c_4	c_5
c_1		140	100	150	200
c_2	190		200	160	220
c_3	110	180		190	250
c_4	190	200	120		150
c_5	200	100	200	150	

If the relation R on the set of cities $A = \{c_1, c_2, c_3, c_4, c_5\} : c_i \, R \, c_j$ if and only if the cost of going from c_i to c_j is defined and less than or equal to RM180.

i) Find R.

ii) Matrices of relations for R

In degree and out degree

If R is a relation on a set A and $a \in A$, then the in-degree of a (relative to relation R) is the number of $b \in A$ such that $(b,a) \in R$.

The out degree of a is the number of $b \in A$ such that $(a,b) \in R$.

Meaning that, in terms of the digraph of R, is that the in-degree of a vertex is

"the number of edges terminating at the vertex"

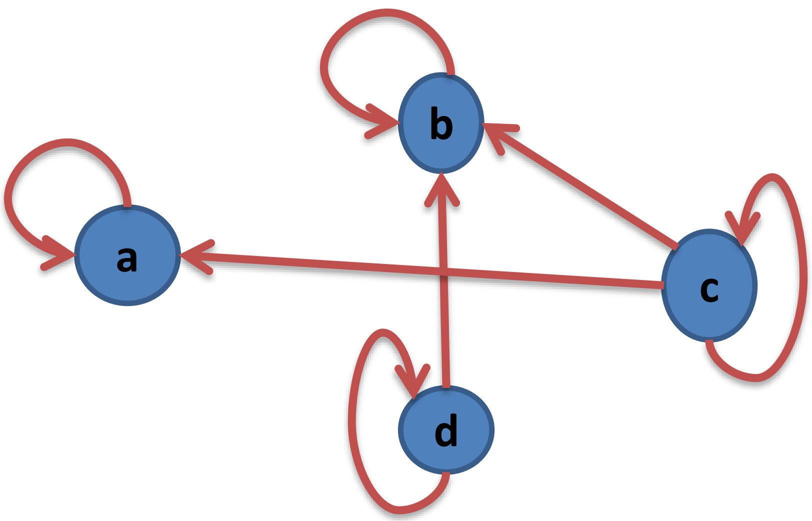
The out-degree of a vertex is

" the number of edges leaving the vertex"

Example

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

	a	b	c	d
In-degree	2	3	1	1
Out-degree	1	1	3	2



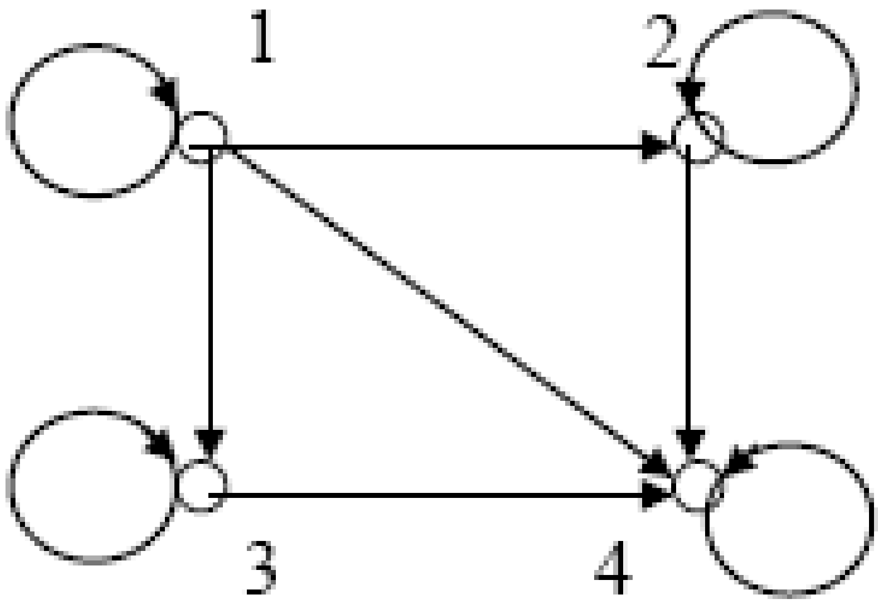
Reflexive Relations

- Reflexive
  - A Relation *R* on set *A* is called **reflexive** if every $a \in A$ is related to itself OR
  - A relation *R* on a set *X* is called **reflexive** if all pair $(x,x) \in R;\ \forall x : x \in X$
- Irreflexive
  - A relation *R* on a set *A* is **irreflexive** if $x \cancel{R} x$ or $(x,x) \notin R;\ \forall x : x \in X$
- Not Reflexive
  - A Relation *R* is **not reflexive** if at least one pair of $(x,x) \in R,\ \forall x : x \in X$

Reflexive Relations

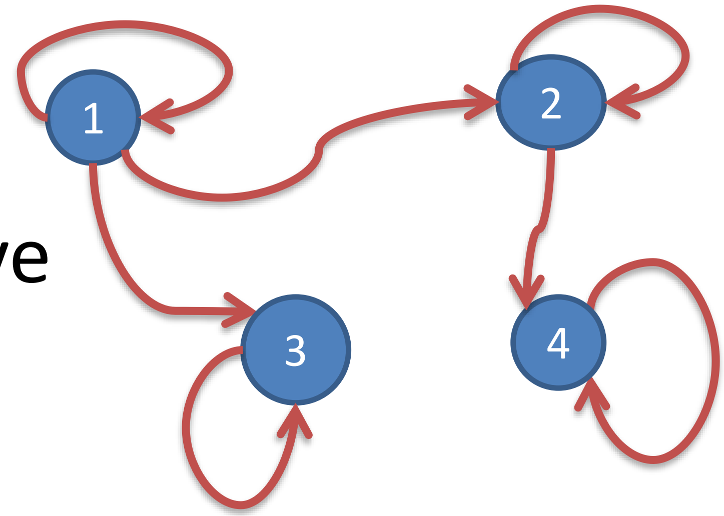
The digraph of a reflexive relation has a loop at every vertex.

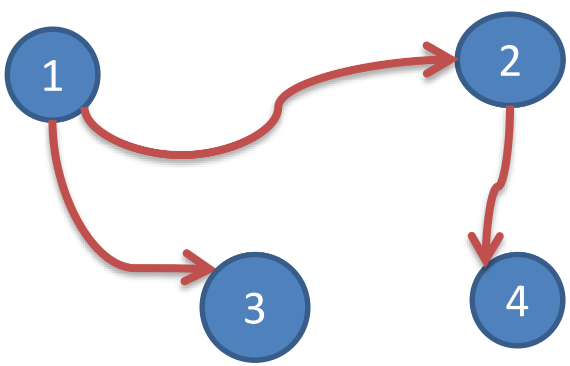
example



Reflexive Relations

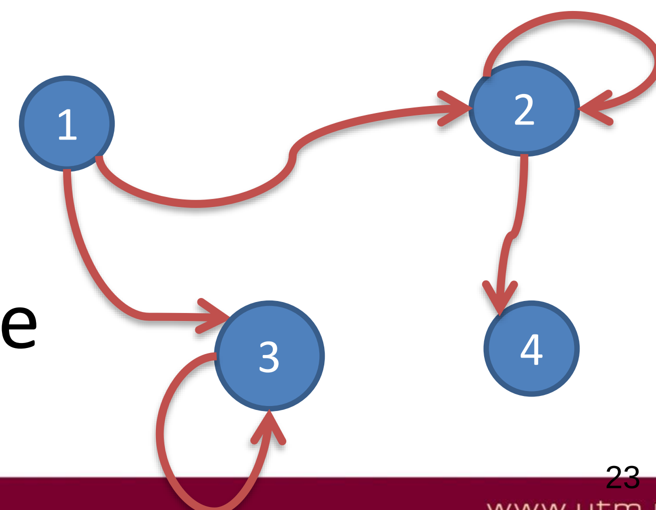
- Reflexive





- Irreflexive

- Not Reflexive



Reflexive Relations

The relation R is reflexive if and only if the matrix of relation has 1's on the main diagonal.

example

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

Reflexive Relations

The relation R is *irreflexive* if and only if the matrix relation have all 0's on its main diagonal

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Reflexive Relations

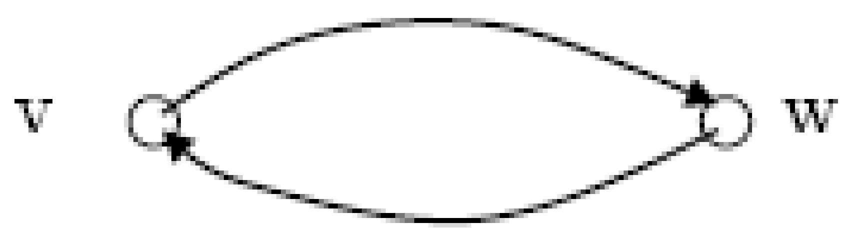
The relation R is not reflexive.

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\boxed{\begin{array}{l} b \in X \\ (b,b) \notin R \end{array}}$$

Symmetric Relations

The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w, there is also a directed edge from w to v.



Example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$\boxed{\begin{array}{l}(b,c) \in R \\ (c,b) \in R\end{array}}$$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

symmetric

Antisymmetric Relations

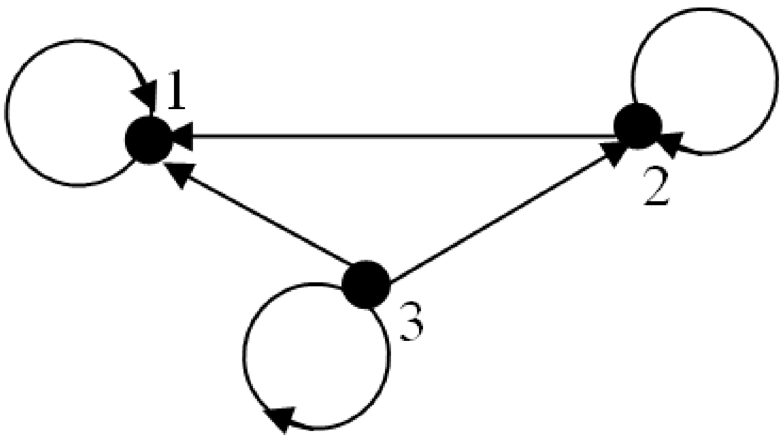
- Matrix $M_R = [M_{ij}]$ of an antisymmetric relation R satisfies the property that if $i \neq j$, then $m_{ij}=0$ or $m_{ji}=0$
- If R is antisymmetric relation, then for different vertices i and j there cannot be an edge from vertex i to vertex j and an edge from vertex j to vertex i
- At least one directed relation and one way

Example

- Let R be a relation on $A = \{1, 2, 3\}$ defined as $(a, b) \in R$ if $a \geq b$, $a, b \in A$ is an antisymmetric relation because for all $a, b \in A$, $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, for example

  $(3, 2) \in R$ but $(2, 3) \notin R$

  $(3, 3) \in R$ and $(3, 3) \in R$ implies $a = b$



Example

- The relation R on $X = \{1, 2, 3, 4\}$ defined by,

  $$(x,y) \in R \quad \text{if } x \le y,\ x,y \in X$$

$$\boxed{\begin{array}{l}(1,2) \in R \\ (2,1) \notin R\end{array}}$$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

antisymmetric

Example

The relation

$R = \{ (a,a), (b,b), (c,c) \}$

on $X = \{ a, b, c \}$

R has no members of the form (x,y) *with* $x \neq y$, then R is antisymmetric

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Asymmetric

- A relation is asymmetric if and only if it is both antisymmetric and irreflexive.
- The matrix $M_R = [m_{ij}]$ of an asymmetric relation R satisfies the property that

  If $m_{ij} = 1$ then $m_{ji} = 0$

  $m_{ii} = 0$ for all i (the main diagonal of matrix M_R consists entirely of 0's or otherwise)
- If R is asymmetric relation, then the digraph of R cannot simultaneously have an edge from vertex i to vertex j and an edge from vertex j to vertex i
- All edges are "one way street" and no loop at every vertex

Example

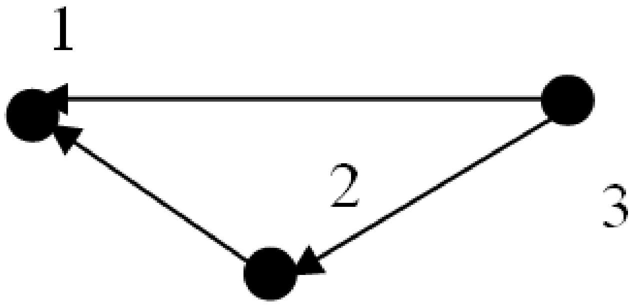
- Let R be the relation on A = {1, 2, 3} defined by $(a, b) \in R$ if $a > b$, $a,b \in A$ is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$

$$(3, 1) \in R \text{ but } (1, 3) \notin R$$

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(1, 1) \notin R, (2, 2) \notin R, (3, 3) \notin R$$



Not Symmetric

- Let R be a relation on a set A. Then R is called ***not symmetric***, if for all a, $b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$.

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

Not Symmetric AND not antisymmetric

- Let R be a relation on a set A. Then R is called ***not symmetric*** and ***not antisymmetric***, if for all $a, b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$ and if $(a, b) \in R$, there exist $(b, a) \in R$.

$$\exists a,b \in A, (a, b) \in R \to (b, a) \in R$$

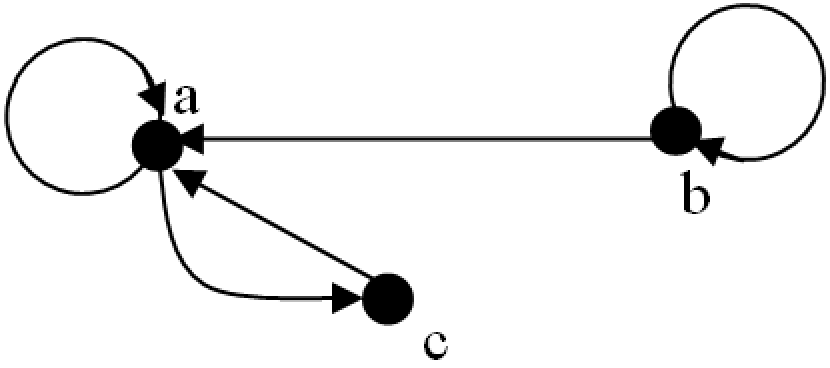
AND

$$\exists a,b \in A, (a, b) \in R \to (b, a) \notin R$$

Example

- Relation $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$ on $A = \{a, b, c\}$ is not symmetric and not antisymmetric relation because there is,

$$(a,c), (c,a) \in R \text{ and also } (b,a) \in R \text{ but } (a,b) \notin R$$



Example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$\boxed{\begin{array}{l}(b,c) \in R \\ (c,b) \in R\end{array}}$$

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Symmetric and

not antisymmetric

Exercise

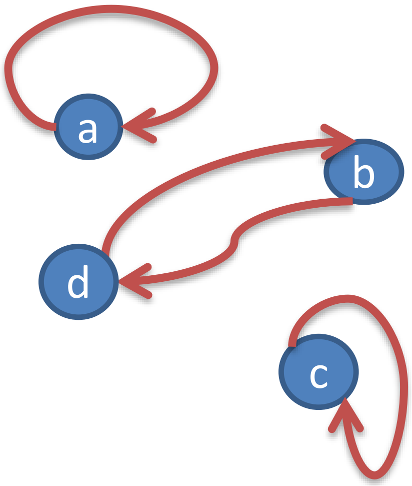
1. Let $A=Z$, the set of integers and let $R=\{(a,b) \in A \times A \mid a<b\}$. So that R is the relation "less than".

Is R symmetric, asymmetric or antisymmetric?

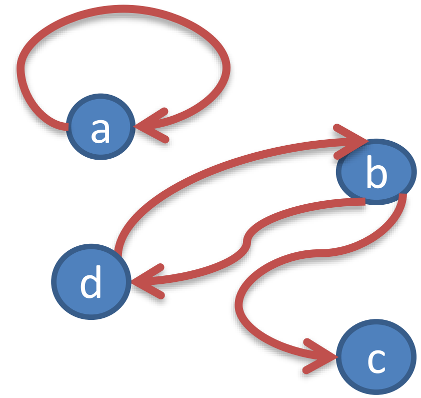
2. Let $A=\{1,2,3,4\}$ and let $R = \{(1,2), (2,2), (3,4), (4,1)\}$

Determine whether R symmetric, asymmetric or antisymmetric.

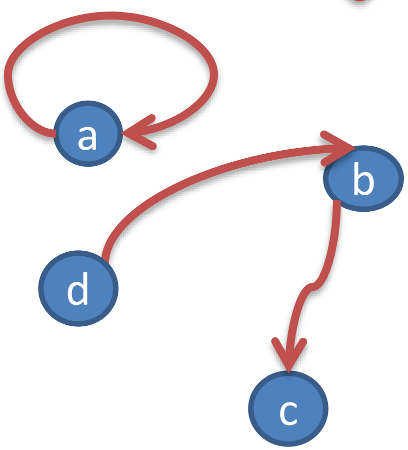
Summary on Symmetric



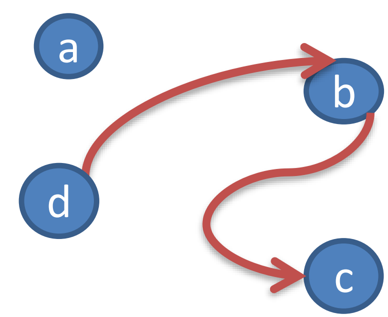
Symmetric



Not Symmetric



Antisymmetric



Asymmetric

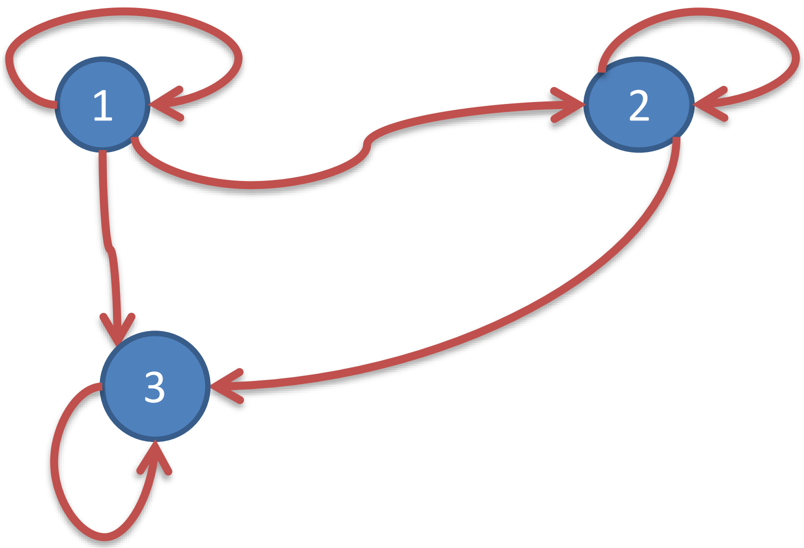
Transitive Relations

- A relation R on set A is transitive if for all $a,b \in A$, $(a,b) \in R$ and $(b,c) \in R$ implies that $(a,c) \in R$
- In the diagraph of R, R is a transitive relation if and only if there is a directed edge from one vertex a to another vertex b, an if there exits a directed edge from vertex b to vertex c, then there must exists a directed edge from a to c

Example

R={(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}

The diagraph:



Transitive Relations

The matrix of the relation M_R is transitive if

$$M_R \otimes M_R = M_R$$

$\otimes$ is the product of boolean

Matrix multiplication

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}}_{C}$$

A, B and C are square metrices of size N x N

a, b, c and d are submatrices of A, of size N/2 x N/2

e, f, g and h are submatrices of B, of size N/2 x N/2

Matrix multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} =$$

$$\begin{bmatrix} aj + bm + cp & ak + bn + cq & al + bo + cr \\ dj + em + fp & dk + en + fq & dl + eo + fr \\ gj + hm + ip & gk + hn + iq & gl + ho + ir \end{bmatrix}$$

Matrix multiplication

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} \otimes \begin{pmatrix} a & d \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \otimes \begin{pmatrix} b & e \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \otimes \begin{pmatrix} c & f \end{pmatrix}$$

$$= \begin{pmatrix} 1a & 1d \\ 4a & 4d \\ 7a & 7d \end{pmatrix} + \begin{pmatrix} 2b & 2e \\ 5b & 5e \\ 8b & 8e \end{pmatrix} + \begin{pmatrix} 3c & 3f \\ 6c & 6f \\ 9c & 9f \end{pmatrix}$$

$$= \begin{pmatrix} 1a + 2b + 3c & 1d + 2e + 3f \\ 4a + 5b + 6c & 4d + 5e + 6f \\ 7a + 8b + 9c & 7d + 8e + 9f \end{pmatrix}.$$

Example

The relation R on A={1,2,3} defined by $(a,b) \in R$ if $a \leq b$, $a,b \in A$, is a transitive. The matrix of relation M_{R},

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that,(1,2) and (2,3)$\in R$, (1,3)$\in R$

Example

The relation R on A={a,b,c,d}

r={(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)} is not transitive. The matrix of relation $M_{R,}$

$$M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

$M_R \otimes M_R \neq M_R$

The product of boolean,

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array} \otimes \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

Note that,(a,c) and (c,b)$\in R$, (a,b)$\notin R$

Example

Let *R* be a relation on *A*={1,2,3} is defined by *(a,b)* $\in$*R* if *a*$\leq$*b*, *a,b* $\in$*A*. Find *R*. Is *R* a transitive relation?

Solution:

R={(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

R is a transitive relation

Equivalence Relations

Relation *R* on set *A* is called an equivalence relation if it is a **reflexive, symmetric and transitive**.

Example

Let R={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matirx relation M_R,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1 and the matris is **reflexive.**

Example - cont

The transpose matrix M_R, M_R^T is equal to M_R, so R is **symmetric**

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \qquad M_R^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

The product of boolean show that the matrix is **transitive**

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

So R is an **equivalence relation.**

Partial Order Relations

Relation R on set A is called a partial order relation if it is a **reflexive, antisymmetric and transitive.**

Example:

Let R be a relation on a set $A=\{1,2,3\}$ defined by $(a,b) \in R$ if $a \leq b$, $a,b \in R$.

$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

R is **reflexive, antisymmetric and transitive.**

So R is a **partial order relation.**

PART 2

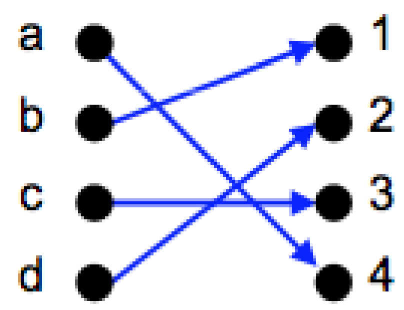
FUNCTIONS

FUNCTION

- Let **X** and **Y** are nonempty sets.
- A **function (*f*)** from **X** to **Y** is a relation from **X** to **Y** having a properties:
  - The domain of f is **X**
  - If $(x,y), (x,y') \in f$, then $y = y'$

Relations vs. Functions

- Not all relations are functions
- But consider the following function:



- All functions are relations!

Relations vs Functions

When to use which?

- A function is used when you need to obtain a SINGLE result for any element in the domain
  - Example: sin, cos, tan
- A relation is when there are multiple mappings between the domain and the co-domain
  - Example: students enrolled in multiple courses

Domain, Co-domain, Range

- A function from **X** to **Y** is denoted, f : **X** → **Y**
- The domain of f is the set **X**.
- The set **Y** is called the co-domain or target of f.
- The set $\{ y \mid (x,y) \in f \}$ is called the range.

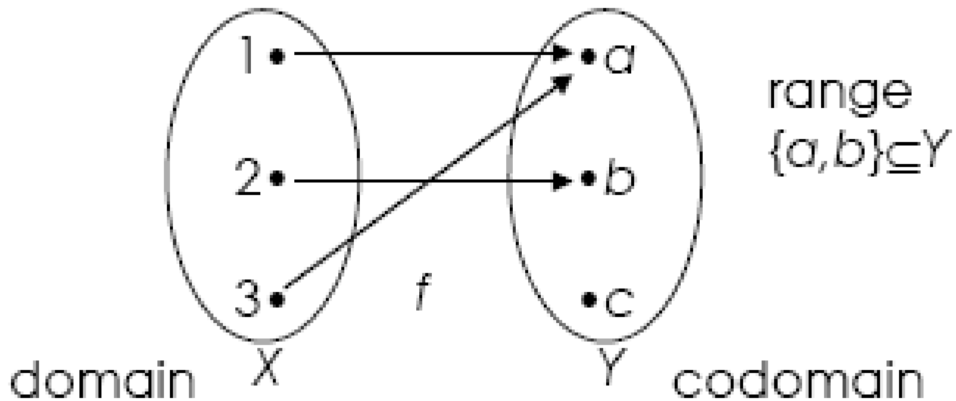
Example

Given the relation, $f = \{ (1,a), (2,b), (3,a) \}$ from **X** = { 1, 2, 3 } to **Y**= { a, b, c } is a function from **X** to **Y**. State the domain, co-domain and range.

Solution:

- ✓ The domain of f is **X**
- ✓ Co-domain of f is **Y**
- ✓ The range of f is $\{a, b\}$

- $f = \{ (1,a), (2,b), (3,a) \}$

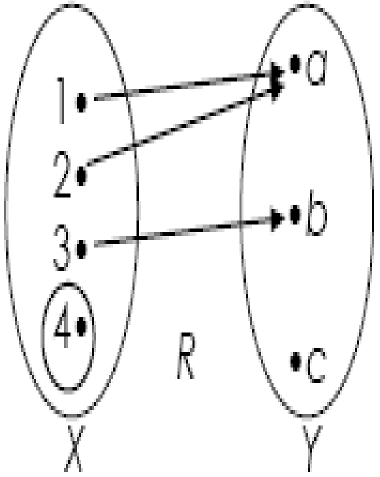


Example

- The relation, $R = \{(1,a), (2,a), (3,b)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$ is NOT a function from X to Y.

- The domain of R, $\{ 1,2,3 \}$ is not equal to X.

$R = \{(1,a), (2,a), (3,b)\}$



There is no arrow from 4

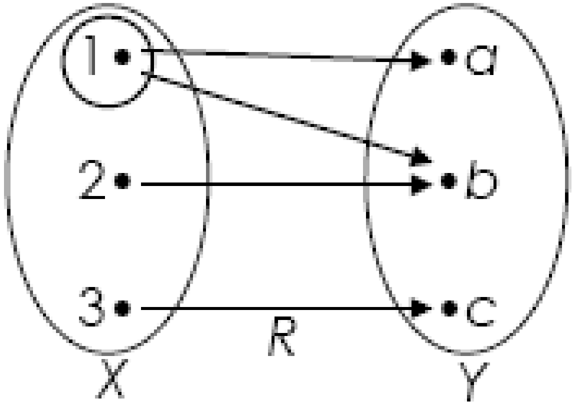
Example

- The relation, $R = \{(1,a), (2,b), (3,c), (1,b)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is NOT a funtion from X to Y

- (1,a) and (1,b) in R but a ≠ b.

$R = \{(1,a), (2,b), (3,c), (1,b)\}$

There are 2 arrows from 1



Notation of function: $f(x)$

- For the function, $f = \{(1,a), (2,b), (3,a)\}$
- We may write, $f(1)=a$, $f(2)=b$, $f(3)=a$
- Notation $f(x)$ is used to define a function.

Example :

- Defined: $f(x) = x^2$
- $f(2) = 4$, $f(-3.5) = 12.25$, $f(0) = 0$
- Notation : $\boldsymbol{f = \{(x, x^2)\mid x}$ **is a real number}**

One-to-One Function

- A function f from X to Y, is said one-to-one (or injective) if for each $y \in Y$, there is at most one $x \in X$, with $f(x)=y$.

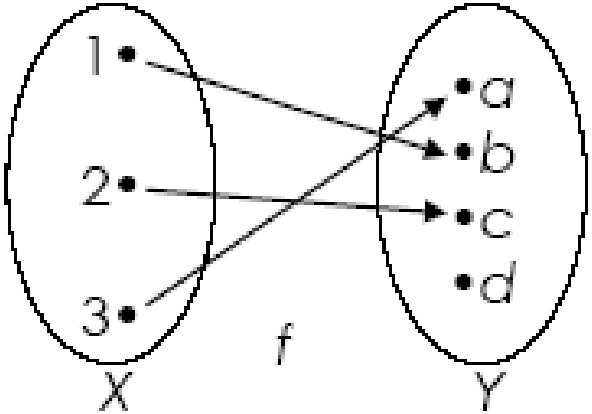
- For all x_1, x_2, if $f(x_1) = f(x_2)$, then $x_1=x_2$.

  $$\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1=x_2))$$

Example

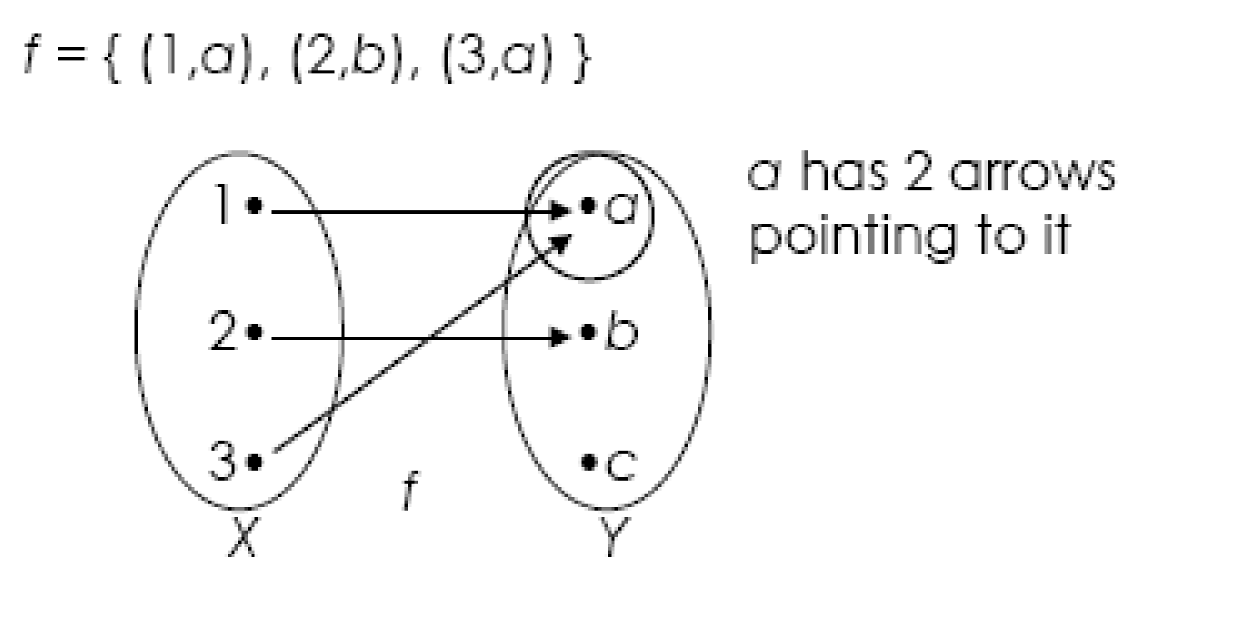
- The function, $f = \{ (1,b), (3,a), (2,c) \}$ from **X** = { 1, 2, 3 } to **Y** = { *a, b, c, d* } is one-to-one.

Each element in *Y* has at most one arrow pointing to it



Example

- The function, $f = \{ (1,a), (2,b), (3,a) \}$ from **X** = { 1, 2, 3 } to **Y** = { *a, b, c*} is NOT one-to-one.
- $f(1) = a = f(3)$



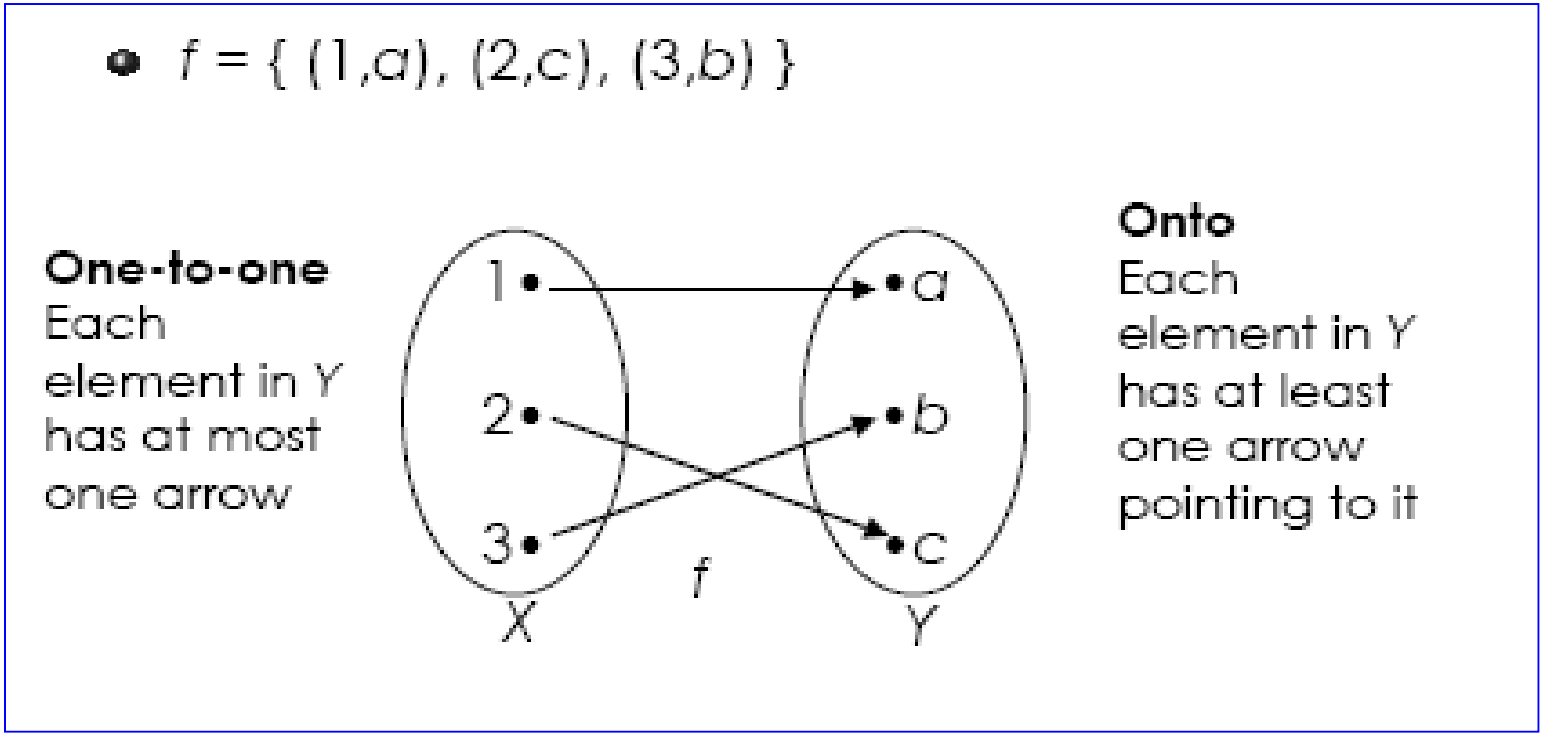
Onto Function

- If f is a function from **X** to **Y** and the range of f is **Y**, f is said to be onto **Y** (or an onto function or a surjective function)
- For every $y \in$ **Y**, there exists at least one $x \in$ **X** such that $f(x) = y$

$$\forall y \in \mathbf{Y}\ \exists x \in \mathbf{X}\ (f(x) = y)$$

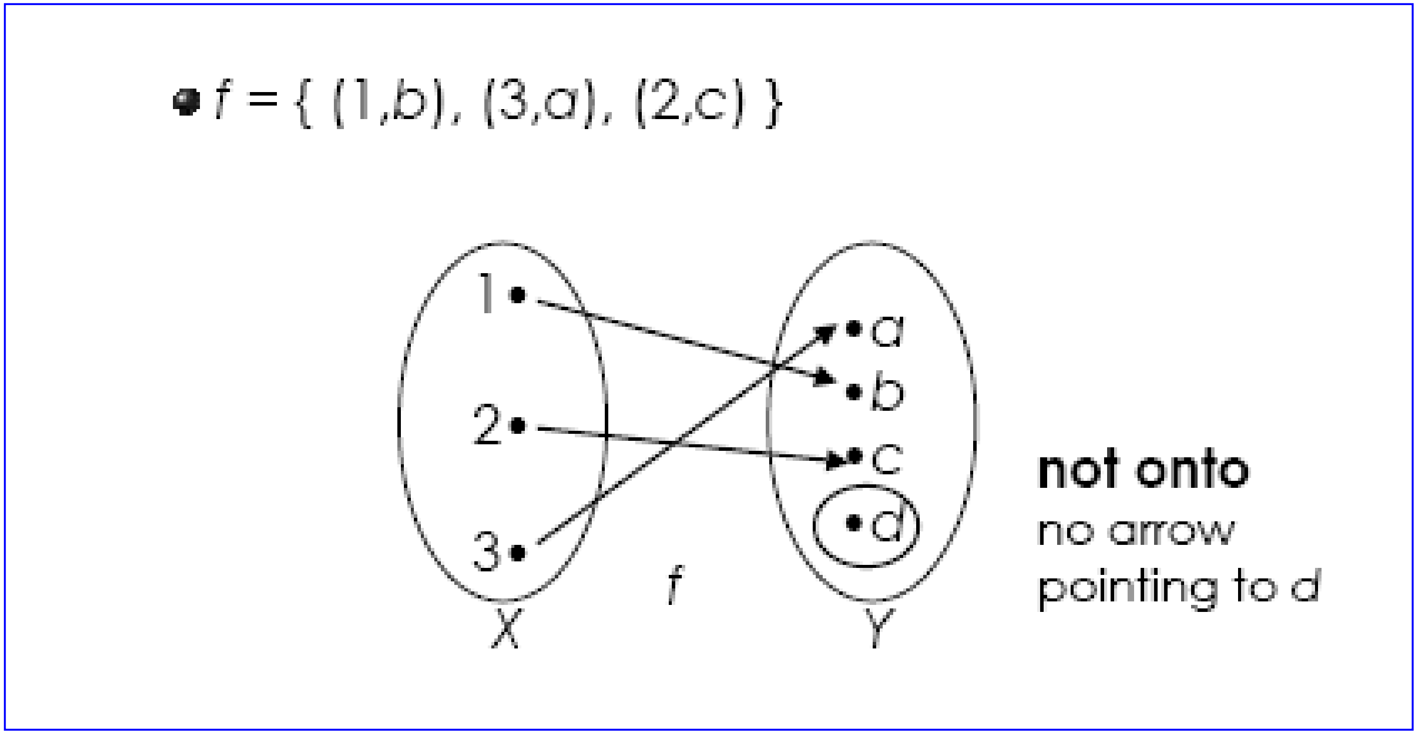
Example

- The function, $f = \{ (1,a), (2,c), (3,b) \}$ from **X** = { 1, 2, 3 } to **Y** = { *a, b, c* } is one-to-one and onto **Y**.



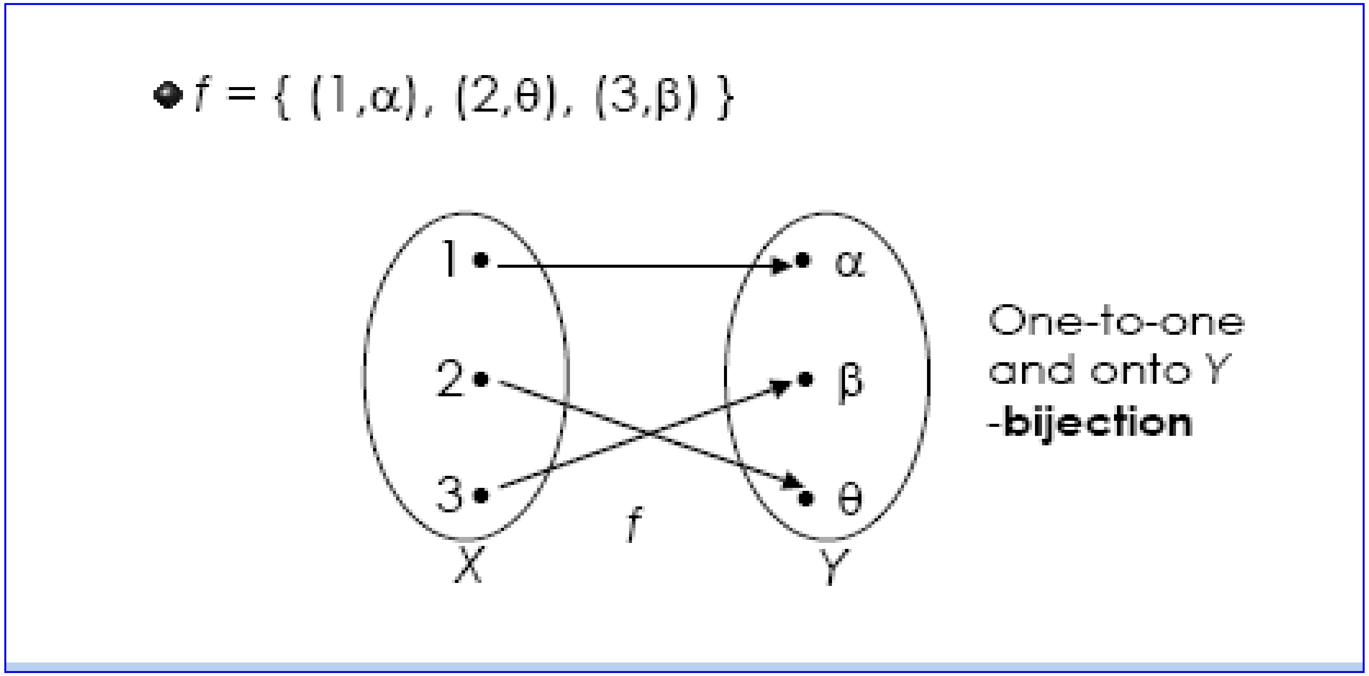
Example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ is not onto **Y** = $\{a, b, c, d\}$



Bijection Function

- A function, f is called one-to-one correspondence (or bijective/bijection) if f is both one-to-one and onto.
- Example



Exercise

Determine which of the relations f are functions from the set **X** to the set **Y**. In case any of these relations are functions, determine if they are one-to-one, onto **Y**, and/or bijection.

a) **X** = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
$f = \{ (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) \}$

b) **X** = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
$f = \{ (-2,-3), (1,4), (2,5)\}$

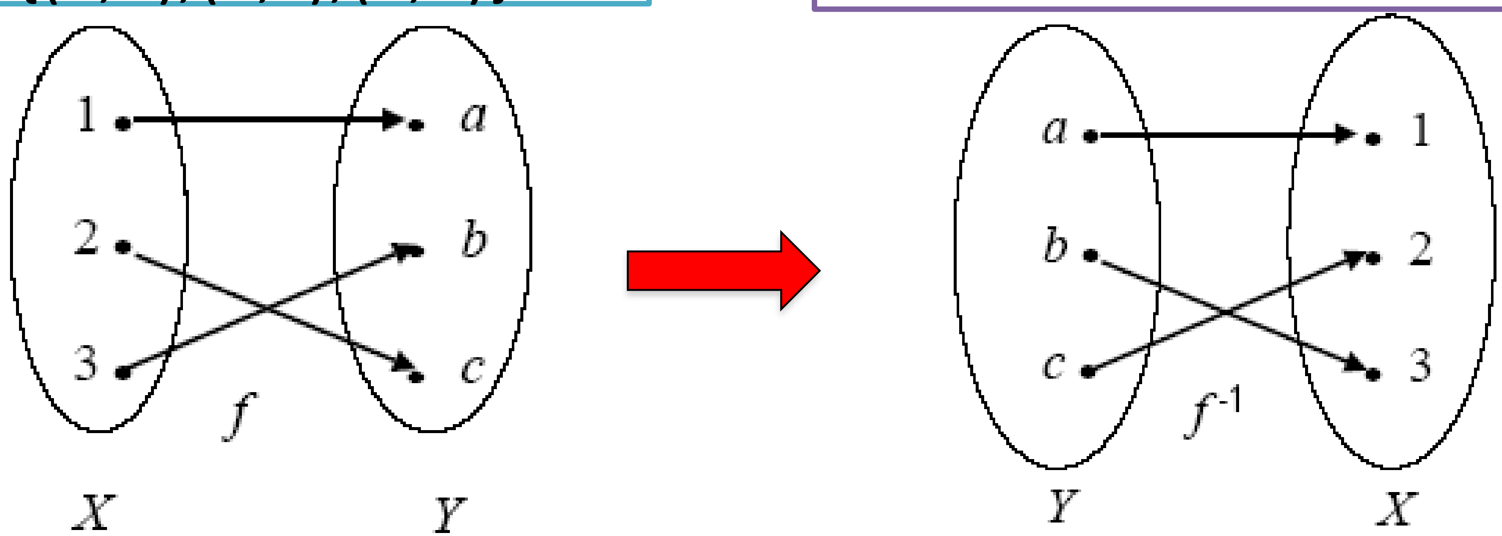
c) **X** = **Y**= { -3, -1, 0, 2 } and
$f = \{ (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) \}$

Inverse Function

- Let f: **X**→**Y** be a function.
- The inverse relation $f^{-1} \subseteq$ **Y** × **X** is a function from **Y** to **X**, if and only if f is both one-to-one and onto **Y**.
- **Example:**

$f = \{(1,a),(2,c),(3,b)\}$

$f^{-1} = \{(a,1),(c,2),(b,3)\}$



Composition

- Suppose that g is a function from **X** to **Y** and f is a function from **Y** to **Z**.

- The composition of f with g,

$$f \circ g$$

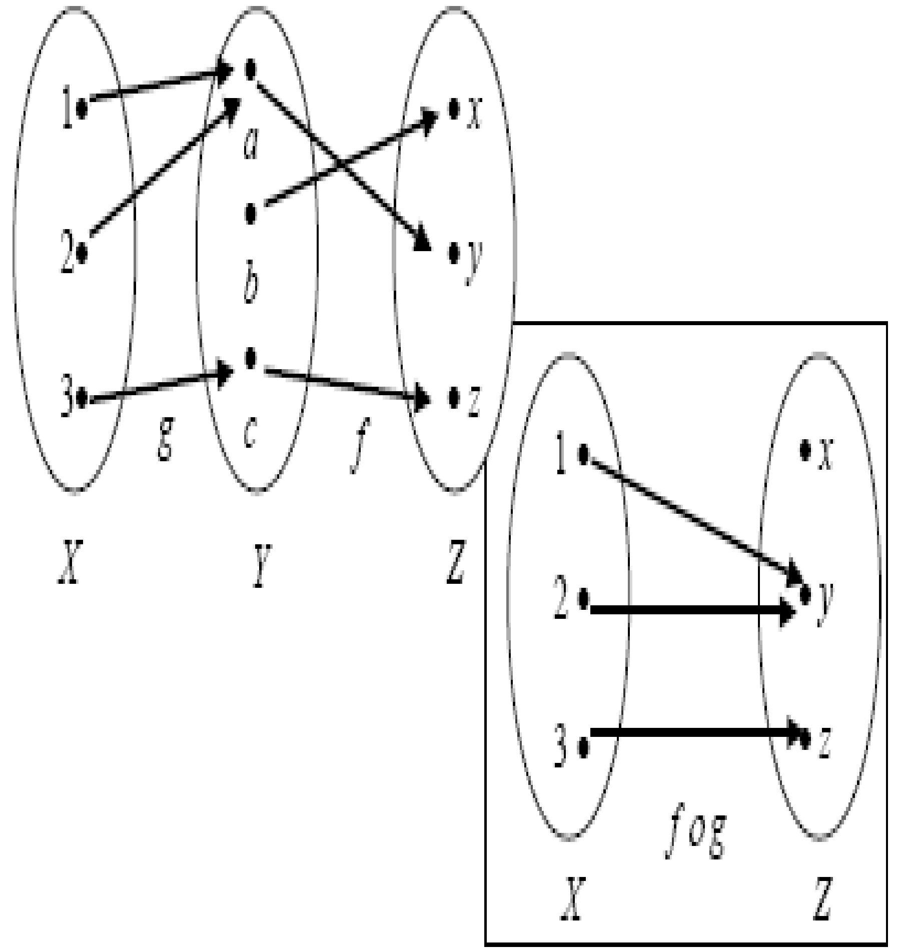
is a function

$$(f \circ g)(x) = f(g(x))$$

from **X** to **Z**.

Example

- Given, g = { (1,a), (2,a), (3,c) } a function from **X** = {1, 2, 3} to **Y** = {a, b, c} and, f = { (a,y), (b,x), (c,z) } a function from **Y** to **Z** = { x, y, z }.
- The composition function from **X** to **Z** is the function f o g = { (1,y), (2,y), (3,z) }



Example

$f(x) = \log_3 x$ and $g(x) = x^4$

- $f(g(x)) = \log_3 (x^4)$
- $g(f(x)) = (\log_3 x)^4$
- Note: $f \circ g \neq g \circ f$

Example

$$f(x) = \frac{1}{5}x \qquad\qquad g(x) = x^2 + 1$$

$$(g \circ f)(x) = g(f(x)) = g(\frac{x}{5})$$

$$= (\frac{x}{5})^2 + 1 = \frac{x^2}{25} + 1$$

PART 3

RECURRENCE RELATION

Recursion

- "A description of something that refers to itself is called a *recursive definition.*"
- In mathematics, certain recursive definitions are used all the time.
- The classic recursive example in mathematics is the definition of **factorial**.

Recursive

- A *recursive procedure* is a procedure that invokes itself
  - Example: given a positive integer n, **factorial of n** is defined as the product of n by all numbers less than n and greater than 0. Notation: $n! = n(n-1)(n-2)\ldots3.2.1$
- Observe that $n! = n(n-1)! = n(n-1)(n-2)!$, etc.
- A *recursive algorithm* is an algorithm that contains a recursive procedure

Recursive vs. Iteration

- Iteration can be used in place of recursion
  - An iterative algorithm uses a *looping construct*
  - A recursive algorithm uses a *branching structure*
- Recursive solutions are often less efficient, in terms of both *time* and *space*, than iterative solutions
- Recursion can simplify the solution of a problem, often resulting in shorter, more easily understood source code

Recursive vs. Iteration

Q: Does using recursion usually make your code **faster**?

A: No, it's usually slower (due to the overhead of maintaining the stack)

Q: Does using recursion usually use **less memory**?

A: No, it usually uses **more** memory (for the stack)

Q: Then **why** use recursion?

A: It sometimes makes your code much **simpler!**

Recursive vs. Iteration

Recursive version

```
int factorial (int n)
{
    if (n == 0)
        return 1;
    else
        return  n * factorial (n-1);
}
```

Recursive Call

Iterative version

```
int factorial (int n)
{
    int i, product=1;
    for (i=n; i>1; --i)
        product=product * i;

    return product;
}
```

Factorial

Factorial Notation

If n is a positive integer, the notation n! is the product of all positive integers from n down through 1.

$n! = n(n-1)(n-2)\ldots(3)(2)(1)$

0! , by definition is 1.

Factorial

Factorial Notation (n!)

- For n as a positive integer, n factorial is defined by
  $n! = n\times(n-1)\times(n-2)\times \dots \times 2\times 1$
- $0! = 1$
- Note that n! can also be written as
  $n! = n\times(n-1)! = n\times(n-1)\times(n-2)!$ and so on

Recursive Factorial

A simple problem - Factorial

- We have to calculate the factorial of a number 'N'.
- We know how to calculate the factorial.
- Logic 1:
  N! = N * N-1 * N-2 * …* N-(N-1) {N-(N-1)=1}
- Logic 2:
  N! = 1 * 2 * 3 * … * N
- Logic 3:
  N! = N * (N-1)! (Recursion)

Recursive Factorial algorithm

Example:

Give a recursive algorithm for computing $n!$, where n is a nonnegative integer.

Solution:

Use the recursive definition of the factorial function.

procedure *factorial*(n: nonnegative integer)
if $n = 0$
  then return 1
else
  return $n \cdot$ *factorial*$(n - 1)$
{output is $n!$}

Fibonacci

What is the Fibonacci Sequence of Numbers?

The Fibonacci numbers are a unique sequence of integers, starting with 1, where each element is the sum of the two previous numbers. For example:
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, etc.

Fibonacci

Fibonacci relationship

$F_1 = 1$

$F_2 = 1$

$F_3 = 1 + 1 = 2$

$F_4 = 2 + 1 = 3$

$F_5 = 3 + 2 = 5$

In general :

$$F_n = F_{n-1} + F_{n-2}$$

or

$$F_{n+1} = F_n + F_{n-1}$$

Recursive Fibonacci algorithm

procedure fib(n: nonnegative integer)

if n = 0 **then** fib(0) := 0

else if n = 1 **then** fib(1) := 1

else fib(n) := fib(n – 1) + fib(n – 2)

Arithmetic sequence

- Sequence in which each term after the first is obtained by adding a fixed number, called the difference, to the previous term.

+3 +3 +3 +3

5, 8, 11, 14, 17, . . .

Common difference is 3.

(d = 3)

-2 -2 -2 -2

16, 14, 12, 10, 8, . . .

Common difference is -2.

(d = -2)

Arithmetic sequence

An arithmetic sequence has a common difference.

The formula for the n^{th} term is

$$a_n = a + (n - 1)d$$

where a_n = n^{th} term of the sequence
a = first term of the sequence
d = common difference

Recursive Arithmetic sequence

The nth term is of an arithmetic sequence can be found by:

Explicit Formula:

$$a_n = dn + c$$

d = common difference

n = # of the term

$c = a_1 - d$

Recursive Formula:

$$a_{n+1} = a_n + d$$

a_{n+1} = next term

a_n = current term

d = common difference

Recursive Arithmetic sequence

Explicit Formula:

$a_n = 3n - 2$

$a_n = 3(1) - 2, 3(2) - 2, 3(3) - 2, 3(4) - 2, \ldots$

$a_n = 1, 4, 7, 10, \ldots$

Recursive Formula:

$a_1 = 1, a_{n+1} = a_n + 3$

$a_n = 1, 1+3, (1+3)+3, [(1+3)+3]+3\ldots$

$a_n = 1, 4, 7, 10, \ldots$

Recursive Arithmetic algorithm

Recursion*

- Recursive formula has a correspondence in programming language: recursive function calls:

$$\begin{cases} a_1 = 0 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$$

- Pseudo-code for function a(n)
  - int a(n)
  - {
    - If n==1, return 0;
    - If n==2, return 1
    - Return (a(n-1)+a(n-2));
  - }