CHAPTER 1

SET THEORY

[Part 1: Set & Subset]

Sets

- The concept of set is basic to all of mathematics and mathematical applications.
- A set is a well-defined collection of distinct objects.
- These objects are called members or elements of the set.

A = {x, y, z}

Example

- A is a set of all positive integers less than 10,

  $A=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- B is a set of first 5 positive odd integers,

  $B=\{1, 3, 5, 7, 9\}$

- C is a set of vowels,

  $C=\{a, e, i, o, u\}$

Sets

- A set is determined by its elements and not by any particular order in which the element might be listed.
- **Example**, $A=\{1, 2, 3, 4\}$, A might just as well be specified as $\{2, 3, 4, 1\}$ or $\{4, 1, 3, 2\}$

- The elements making up a set are assumed to be distinct, we may have duplicates in our list, only one occurrence of each element is in the set.
- **Example**,

  $\{a, b, c, a, c\} \rightarrow \{a, b, c\}$

  $\{1, 3, 3, 5, 1\} \rightarrow \{1, 3, 5\}$

A = {x,x,y,y,z}
same as
A = {x ,y, z}

Sets

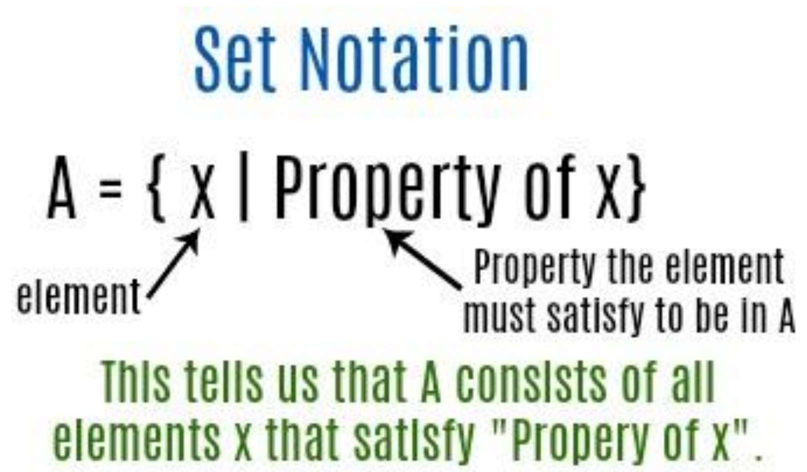
- Use uppercase letters *A, B, C …* to denote sets, lowercase denote the elements of set.

- The symbol $\in$ stands for ‘belongs to’

- The symbol $\notin$ stands for ‘does not belong to’

Example

- $X=\{a, b, c, d, e\}$,
  $b \in X$ and $m \notin X$
- $A=\{\{1\}, \{2\}, 3, 4\}$,
  $\{2\} \in A$ and $1 \notin A$



Sets

- If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for memberships

- Let S be a set, the notation,

  $A = \{x \mid x \in S, P(x)\}$ or $A = \{x \in S \mid P(x)\}$ means that A is the set of all elements x of S such that x satisfies the property P.

Example

- $A=\{1, 2, 3, 4, 5, 6\}$

  $A=\{x \mid x \in Z, 0< x <7\}$

  if Z denotes the set of integers.

- $B=\{1, 2, 3, 4, \ldots\}$

  $B=\{x \mid x \in Z, x>0\}$,

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Set notation

$\mathbf{N}$	$=$	the set of all natural numbers $= \{0, 1, 2, 3, \ldots\}$.
$\mathbf{Z}$	$=$	the set of all integers $= \{0, -1, 1, -2, 2, \ldots\}$.
$\mathbf{Z}^+$	$=$	the set of all positive integers.
$\mathbf{Z}^-$	$=$	the set of all negative integers.
$\mathbf{R}$	$=$	the set of all real numbers.
$\mathbf{R}^+$	$=$	the set of all positive real numbers.
$\mathbf{R}^-$	$=$	the set of all negative real numbers.
$\mathbf{R}^2$	$=$	the set of all points in the plane.
$\mathbf{Q}$	$=$	the set of all rational numbers.
$\mathbf{Q}^+$	$=$	the set of all positive rational numbers.
$\mathbf{Q}^-$	$=$	the set of all negative rational numbers.
$\emptyset$	$=$	the empty set $=$ the set containing no elements.

Set notation

"$\forall$" stands for "for every"

"$\bigcup$" stands for "union"

"$\subseteq$" stands for "is a subset of"

"$\not\subset$" stands for "is a not a (proper) subset of"

"$\in$" stands for "is an element of"

"$\times$" stands for "cartesian cross product"

"$\exists$" stands for "there exists"

"$\bigcap$" stands for "intersection"

"$\subset$" stands for "is a (proper) subset of"

"$\varnothing$" stands for the "empty set"

"$\notin$" stands for "is not an element of"

"$=$" stands for "is equal to"

Subset

- If every element of A is an element of B, we say that A is a subset of B and write $A \subseteq B$.

  $A=B$,

  if $A \subseteq B$ and $B \subseteq A$

- The empty set ($\varnothing$) is a subset of every set.

Example

$A=\{1, 2, 3\}$

Subset of A,

$\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$,
$\{1, 2\}$, $\{1, 3\}$,
$\{2, 3\}$ $\{1, 2, 3\}$

Note:

A is a subset of A

Proper subset

- If A is a subset of B and A does not equal B, we say that A is a proper subset of B ($A \subseteq B$ and $A \neq B$ ($B \not\subseteq A$))

- A proper subset of a set A is a subset of A that is not equal to A ($\{1,2,3\} \not\subset A$)

Example

- $A=\{1, 2, 3\}$

Proper subset of A,

$\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$

- $B=\{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3\}$.

Thus, Proper subset of A.

- $A=\{a,b,c,d,e,f,g,h\}$, $B=\{b,d,e\}$
  $C=\{a,b,c,d,e\}$, $D=\{r,s,d,e\}$

Thus, B *and* C are proper subset of A

Empty set

The **empty set** $\varnothing$ or {} but not $\{\varnothing\}$ is the set without elements.

- Empty set has no elements
- Empty set is a subset of any set
- There is exactly one empty set
- Properties of empty set:
  $A \cup \varnothing = A$, $A \cap \varnothing = \varnothing$
  $A \cap A' = \varnothing$, $A \cup A' = U$
  $U' = \varnothing$, $\varnothing' = U$

Example

$\varnothing = \{x \mid x$ is a real number and $x^2 = -3\}$

$\varnothing = \{x \mid x$ is positive integer and $x^3 < 0\}$

Equal set

The sets A and B are **equal** (A=B) if and only if each element of A is an element of B and vice versa.

Formally: A=B means $\forall x\ [x \in A \leftrightarrow x \in B]$.

Example

$A=\{a, b, c\}$,

$B=\{b, c, a\}$, $A=B$

$C=\{1, 2, 3, 4\}$,

$D=\{x \mid x$ is a positive integer and $2x < 10\}$, $C=D$

Equivalent set

Two sets, A and B, are equivalent if there exists a one-to-one correspondence between them.

When we say sets "have the same size", we mean that they are equivalent.

Example

A: {A, B, C, D, E}

B: {1, 2, 3, 4, 5}, A and B is equivalent.

- **Note :** An **equivalent set** is simply a **set** with an **equal** number of elements. The **sets** do not have to have the same exact elements, just the same number of elements.

Finite sets

A set A is finite

if it is empty

or

if there is a natural number n such that set A is equivalent to $\{1, 2, 3, \ldots n\}$.

Example

$A = \{1, 2, 3, 4\}$

$B = \{x \mid x \text{ is an integer}, 1 \leq x \leq 4\}$

Note : There exists a nonnegative integer n such that *A* has *n* elements (*A* is called a finite set with *n* elements)

Infinite sets

A set A is infinite

if there is NOT a natural number n such that set A is equivalent to $\{1, 2, 3, \ldots n\}$.

Infinite sets are uncountable.

Are all infinite sets equivalent?

A set is infinite if it is equivalent to a proper subset of itself!

Example

- $C = \{5, 6, 7, 8, 9, 10\}$ **(finite set)**
- $B = \{x \mid x \text{ is an integer}, 10 < x < 20\}$ **(finite set)**
- $D = \{x \mid x \text{ is an integer}, x > 0\}$ **(infinite set)**

Universal set

Typically we consider a set A
a part of a **universal set** $\mathcal{U}$,
which consists of all possible elements.

Example

- The sets A={1,2,3}, B={2,4,6,8} and C={5,7}
- Universal set, U={1,2,3,4,5,6,7,8}

Cardinality of Set

- Let S be a finite set with n distinct elements, where $n \geq 0$.
- Then we write $|S|=n$ and say that the cardinality (or the number of elements) of S is n.
- **Example**

  $A = \{1, 2, 3\}$, $|A|=3$

  $B = \{a,b,c,d,e,f,g\}$, $|B|=7$

Power Set

- The set of all subsets of a set A, denoted $P(A)$, is called the power set of A., $P(A)= \{X \mid X \subseteq A\}$
- If $|A|=n$, then $|P(A)| = 2^n$

Example

- $A=\{1,2,3\}$
- The power set of A,

  $P(A)= \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$

  Notice that $|A| = 3$, and $|P(A)| = 2^3 = 8$

Summary

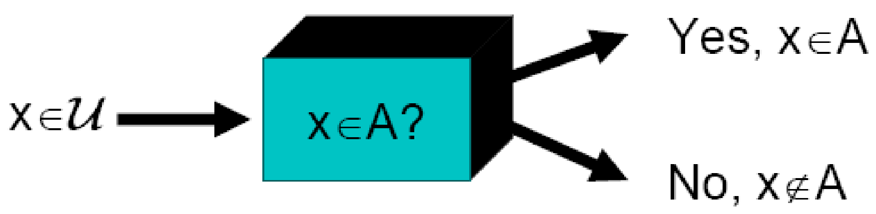
How to Think of Sets

The elements of a set do not have an ordering, hence $\{a,b,c\} = \{b,c,a\}$

The elements of a set do not have multitudes, hence $\{a,a,a\} = \{a,a\} = \{a\}$

All that matters is: "Is x an element of A or not?"

The size of A is thus the number of *different* elements



CHAPTER 1

SET THEORY

[Part 2: Operation on Set]

Union

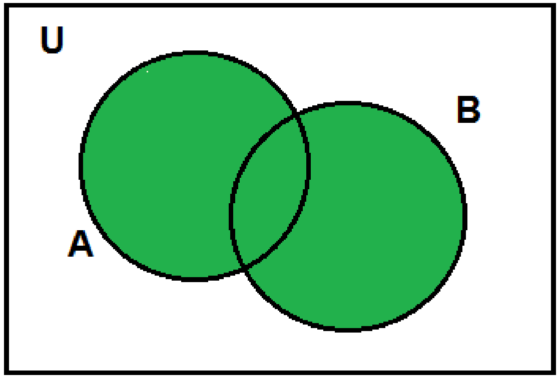
- The union of two sets A and B, denoted by $A \cup B$, is defined to be the set

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B\}$$

- The union consists of all elements belonging to either A or B (or both)

Union

- Venn diagram of $A \cup B$



If A and B are finite sets, the cardinality of $A \cup B$,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example

$A=\{1, 2, 3, 4, 5\}$, $B=\{2, 4, 6\}$ and $C=\{8, 9\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

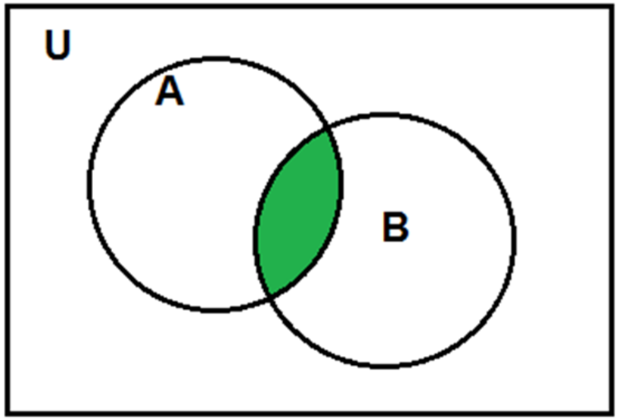
$A \cup C = \{1, 2, 3, 4, 5, 8, 9\}$

$B \cup C = \{2, 4, 6, 8, 9\}$

$A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 8, 9\}$

Intersection

- Venn diagram of $A \cap B$



- The intersection of two sets A and B, denoted by $A \cap B$, is defined to be the set

  $A \cap B = \{ x \mid x \in A \text{ and } x \in B\}$
- The intersection consists of all elements belonging to both A and B.

Example

$A=\{1, 2, 3, 4, 5, 6\}$, $B=\{2, 4, 6, 8, 10\}$ and $C=\{ 1, 2, 8, 10 \}$

$A \cap B = \{2, 4, 6\}$

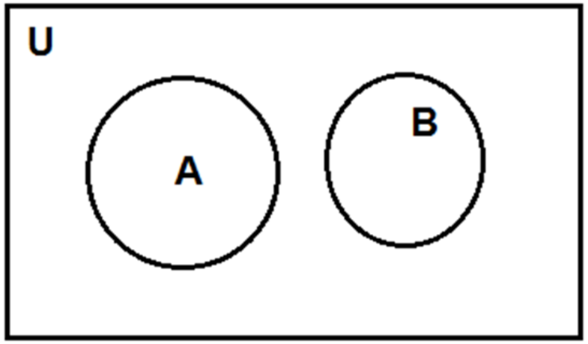
$A \cap C = \{1, 2\}$

$C \cap B = \{2, 8, 10\}$

$A \cap B \cap C = \{2\}$

Disjoint

- Venn diagram, $A \cap B = \varnothing$



- Two sets A and B are said to be disjoint if,

  $A \cap B = \varnothing$

Example

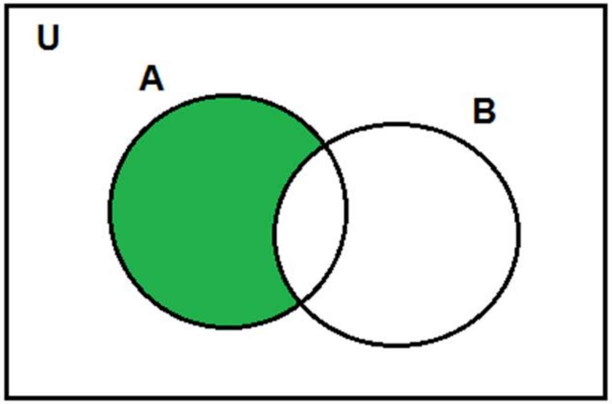
$A = \{1, 3, 5, 7, 9, 11\}$

$B = \{2, 4, 6, 8, 10\}$

$A \cap B = \varnothing$

Difference

- Venn diagram of $A - B$



Example

$A = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$

$B = \{ 2, 4, 6, 8 \}$

$A - B = \{ 1, 3, 5, 7 \}$

- The set

  $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

  is called the difference.

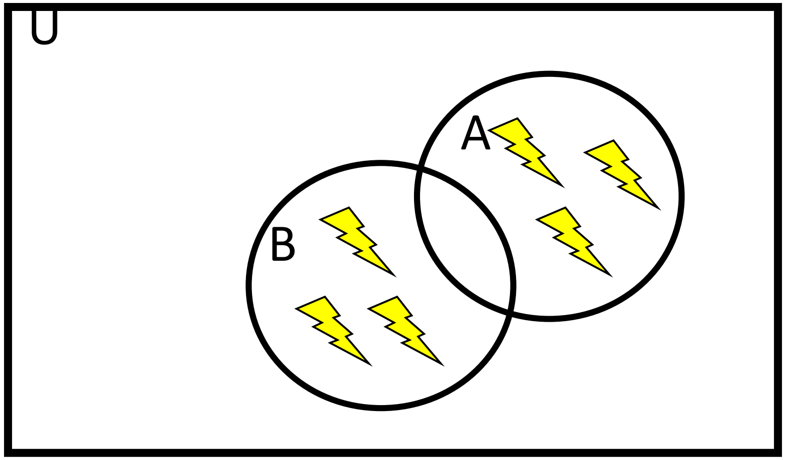
- The difference $A - B$ consists of all elements in A that are not in B.

Symmetric Difference

- The *symmetric difference*,

$$A \oplus B = \{ x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$

$$= (A - B) \cup (B - A)$$

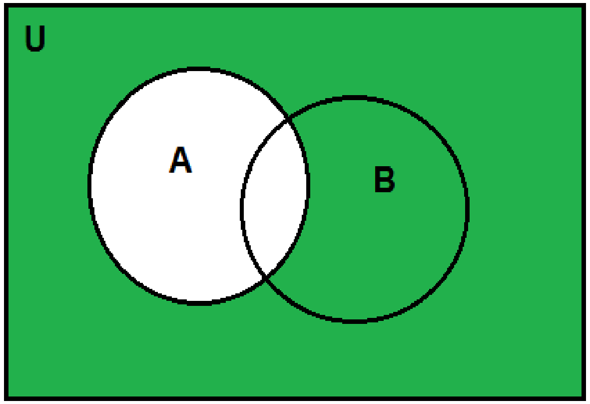


Complement

- The complement of a set A with respect to a universal set U, denoted by A' is defined to be

$$A' = \{x \in U \mid x \notin A\}$$

$$A' = U - A$$



Example

Let U be a universal set,

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$A = \{ 2, 4, 6 \}$

$A' = U - A = \{ 1, 3, 5, 7 \}$

Properties of Sets

- Commutative laws

  $A \cap B = B \cap A$ $\quad$ $A \cup B = B \cup A$

- Associative laws

  $A \cap (B \cap C) = (A \cap B) \cap C$ $\quad$ $A \cup (B \cup C) = (A \cup B) \cup C$

- Distributive laws

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\quad$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- Absorption laws

  $A \cup (A \cap B) = A$ $\quad$ $A \cap (A \cup B) = A$

- Idempotent laws

  $A \cap A = A$ $\quad$ $A \cup A = A$

Properties of Sets

- Complement laws

  $(A')' = A \qquad A \cap A' = \varnothing \qquad A \cup A' = \mathrm{U} \qquad \varnothing' = U \qquad U' = \varnothing$

- De Morgan's laws

  $(A \cap B)' = A' \cup B' \qquad (A \cup B)' = A' \cap B'$

- Properties of universal set

  $A \cup U = U \qquad A \cap U = A$

- Properties of empty set

  $A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing$

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Example

- Let A, B and C denote the subsets of a set S and let C' denote a complement of C in S.
- If $A \cap C = B \cap C$ and $A \cap C' = B \cap C'$, then prove that $A = B$

Solution

$$
\begin{aligned}
A &= A \cap S \\
&= A \cap (C \cup C') \\
&= (A \cap C) \cup (A \cap C') && \text{Distributive laws} \\
&= (B \cap C) \cup (B \cap C') && \text{by the given conditions} \\
&= B \cap (C \cup C') && \text{Distributive laws} \\
&= B \cap S \\
&= B
\end{aligned}
$$

Example

Simplify the set

$(((A \cup B) \cap C)' \cup B')' =$

$= ((A \cup B) \cap C)'' \cap B''$	[DeMorgan]
$= ((A \cup B) \cap C) \cap B$	[Double Complement]
$= (A \cup B) \cap (C \cap B)$	[Associativity of $\cap$]
$= (A \cup B) \cap (B \cap C)$	[Commutativity of $\cap$]
$= ((A \cup B) \cap B) \cap C$	[Associativity of $\cap$]
$= B \cap C$	[Absorption]

Generalized Union/Intersection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \qquad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Example

$$A_i = \{1,2,3,\ldots,i\} \quad i = 1,2,3,\ldots$$

$$\bigcup_{i=1}^{\infty} A_i = ? \qquad \bigcup_{i=1}^{\infty} A_i = Z^+$$

$$\bigcap_{i=1}^{\infty} A_i = ? \qquad \bigcap_{i=1}^{\infty} A_i = \{1\}$$

Cartesian Product

- Let A and B be sets, an ordered pair of elements $a \in A$ dan $b \in B$ written (a, b) is a listing of the elements a and b in a specific order.
- The ordered pair (a, b) specifies that a is the first element and b is the second element. An ordered pair (a, b) is considered distinct from ordered pair (b, a), unless $a=b$. , example $(1, 2) \neq (2, 1)$
- The Cartesian product of two sets A and B, written $A \times B$ is the set, $A \times B = \{(a,b) \mid a \in A, b \in B\}$. For any set A, $A \times \varnothing = \varnothing \times A = \varnothing$. If $A \neq B$, then $A \times B \neq B \times A$. if $|A| = m$ and $|B| = n$, then $|A \times B| = mn$.

Example

$A = \{a, b\}$, $B = \{1, 2\}$.

$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

$B \times A = \{(1, a), (1, b), (2, a), (2, b)\}$

Example

$A = \{1, 3\}$, $B = \{2, 4, 6\}$.

$A \times B = \{(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6)\}$

$B \times A = \{(2, 1), (2, 3), (4, 1), (4, 3), (6, 1), (6, 3)\}$

$A \neq B$, $A \times B \neq B \times A$

$|A| = 2$, $|B| = 3$,

$|A \times B| = 2.3 = 6$.

Cartesian Product

- The Cartesian product of sets $A_1, A_2, \ldots., A_n$ is defined to be the set of all n-tuples

  $(a_1, a_2, \ldots a_n)$ where $a_i \in A_i$ for $i=1,\ldots,n$;

- It is denoted $A_1 \times A_2 \times \ldots. \times A_n$

  $|A_1 \times A_2 \times \ldots. \times A_n| = |A_1|.|A_2| \ldots. |A_n|$

Example

$A = \{a, b\}$, $B=\{1, 2\}$, $C=\{x, y\}$

$A\times B\times C = \{(a,1,x),(a,1,y), (a,2,x), (a,2,y), (b,1,x), (b,1,y), (b,2,x), (b,2,y)\}$

$|A\times B\times C| = 2.\ 2.\ 2 = 8$

CHAPTER 1

Part 3:
Fundamental and Elements of Logic

Why Are We Studying Logic?

Some of the reasons:

- Logic is the foundation for computer operation
- Logical conditions are common in programs and programs can be proven correct.
- All manner of structures in computing have properties that need to be proven (and proofs that need to be understood), example Trees, Graphs, Recursive Algorithms, . . .
- Computational linguistics must represent and reason about human language, and language represents thought (and thus also logic).

PROPOSITION

A **statement** or a **proposition**, is a declarative sentence that is **either TRUE or FALSE, but not both.**

Example:

- 4 is less than 3.
- 7 is an even integer.
- Washington, DC, is the capital of United State.

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Example

i) Why do we study mathematics?
ii) Study logic.
iii) What is your name?
iv) Quiet, please.

Not propositions. Why ?

(i) & (iii) : is question, not a statement.
(ii) & (iv) : is a command.

i) The temperature on the surface of the planet Venus is 800 F.
ii) The sun will come out tomorrow.

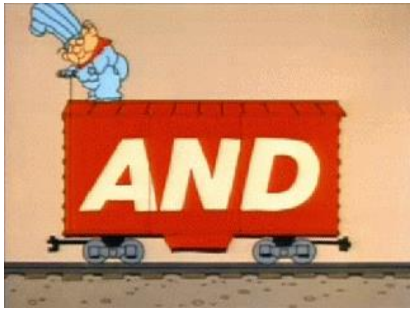
Propositions? Why?

- Is a statement since it is either true or false, but not both.
- However, we do not know at this time to determine whether it is true or false.

CONJUNCTIONS

Conjunctions are:

- Compound propositions formed in English with the word "**and**",
- Formed in logic with the caret symbol ("$\wedge$"), and
- True only when both participating propositions are true.



CONJUNCTIONS (cont.)

TRUTH TABLE: This tables aid in the evaluation of **compound propositions**.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

True (T)
False (F)

Example

p : 2 is an even integer
q : 3 is an odd number

propositions

$p \wedge q$

symbols

2 is an even integer and 3 is an odd number

statements

Example

p : today is Monday
q : it is hot

$p \wedge q$: today is Monday and it is hot

Example

Proposition

p : 2 divides 4

q : 2 divides 6

Symbol: Statement

$p \wedge q$: 2 divides 4 and 2 divides 6.

or,

$p \wedge q$: 2 divides both 4 and 6.

Proposition

p : 5 is an integer

q : 5 is not an odd integer

Symbol: Statement

$p \wedge q$: 5 is an integer and 5 is not an odd integer.

or,

$p \wedge q$: 5 is an integer but 5 is not an odd integer.

DISJUNCTION

- Compound propositions formed in English with the word “**or**”,
- Formed in logic with the caret symbol (“ $\vee$ ”), and,
- True when one or both participating propositions are true.

DISJUNCTION (cont.)

- Let ***p*** and ***q*** be propositions.
- The **disjunction** of ***p*** and ***q***, written ***p*** V ***q*** is the statement formed by putting statements ***p*** and ***q*** together using the word "**or**".
- The symbol V is called "or"

DISJUNCTION (cont.)

The truth table for $p \vee q$:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example

i) ***p***: 2 is an integer ; ***q***: 3 is greater than 5

$p \vee q$ — 2 is an integer or 3 is greater than 5

ii) ***p*** : 1+1=3 ; ***q*** *:* A decade is 10 years

$p \vee q$ — 1+1=3 or a decade is 10 years

Example

iii) ***p*** : 3 is an even integer ; ***q*** : 3 is an odd integer

p $\vee$ **q** 3 is an even integer or 3 is an odd integer

or

3 is an even integer or an odd integer

NEGATION

Negating a proposition simply flips its value. Symbols representing negation include: $\boxed{\neg x\,,\ \bar{x}\,,\ \sim x,\ x'\ \ \text{(NOT)}}$

Let p be a proposition.
The negation of p, written $\neg\, p$
is the statement obtained by negating statement p.

NEGATION (cont.)

The truth table of $\neg p$:

p	$\neg p$
T	F
F	T

p : 2 is positive

$\neg p$

2 is not positive

Exercise

Suppose x is a particular real number. Let ***p***, ***q*** and ***r*** symbolize "$0 < x$", "$x < 3$" and "$x = 3$", respectively. Write the following inequalities symbolically:

a) $x \leq 3$

b) $0 < x < 3$

c) $0 < x \leq 3$

CONDITIONAL PROPOSITIONS

Let ***p*** *and* ***q*** be propositions.

“if *p*, then *q*”

is a statement called a **conditional proposition**, written as

$$p \rightarrow q$$

CONDITIONAL PROPOSITIONS(cont.)

The **truth table** of $p \rightarrow q$
(Cause and effect relationship)

FALSE if *p* = True and *q* =false

p	q	p→ q
T	T	T
T	F	F
F	T	T
F	F	T

TRUE if both true OR *p*=false for any value of *q*

Example

p : today is Sunday ; ***q*** : I will go for a walk

p → q : If today is Sunday, then I will go for a walk.

p : I get a bonus ; ***q*** : I will buy a new car

p → ***q***: If I get a bonus, then I will buy a new car

Example

p : x/2 is an integer.

q : x is an even integer.

p → q : if x/2 is an integer, then x is an even integer.

BICONDITIONAL

Let ***p*** *and* ***q*** *be propositions.*

“*p* if and only if *q*”

is a statement called a **biconditional proposition**, written as

$$p \leftrightarrow q$$

BICONDITIONAL (cont.)

The **truth table** of $p \leftrightarrow q$:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example

p *:* my program will compile
q *:* it has no syntax error.

p ↔ ***q*** :

My program will compile if and only if it has no syntax error.

p : x is divisible by 3
q : x is divisible by 9

p ↔ ***q:***

x is divisible by 3 if and only if x is divisible by 9.

Neither ..nor..

Neither ***p*** **nor** ***q*** **[*~p and ~ q*]** is a TRUE statement if neither *p* nor *q* is true.

p	q	$\sim p \wedge \sim q$
T	T	F
T	F	F
F	T	F
F	F	T

Example

p : It is hot.

q : It is sunny.

$\sim p \wedge \sim q$: It is neither hot nor sunny, or
It is not hot and it is not sunny.

LOGICAL EQUIVALENCE

- The compound propositions ***Q*** and ***R*** are made up of the propositions $p_1, \ldots, p_n$.

- ***Q*** and ***R*** are logically equivalent and write,

$$Q \equiv R$$

provided that given any truth values of $p_1, \ldots, p_n$, either ***Q*** and ***R*** are **both true** or ***Q*** and ***R*** are **both false**.

Example

$\boldsymbol{Q} = p \rightarrow q$ $\boldsymbol{R} = \neg q \rightarrow \neg p$

Show that, $\boldsymbol{Q} \equiv \boldsymbol{R}$

The truth table shows that, $\boldsymbol{Q} \equiv \boldsymbol{R}$

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Example

Show that, $\neg (p \rightarrow q) \equiv p \wedge \neg q$

The truth table shows that, $\neg (p \rightarrow q) \equiv p \wedge \neg q$

p	q	$\neg(p \rightarrow q)$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

PRECEDENCE OF LOGICAL CONNECTIVES

	Precedence of logical connectives is as follows:		
not	¬	↑	Highest
and	∧		
or	∨		
If…then	→		
If and only if	↔		Lowest

Example

Construct the truth table for,

A $= \neg(p \lor q) \to (q \land p)$

Solution:

p	q	$(p \lor q)$	$\neg(p \lor q)$	$(q \land p)$	A
T	T	T	F	T	T
T	F	T	F	F	T
F	T	T	F	F	T
F	F	F	T	F	F

LOGIC & SET THEORY

Logic and set theory go very well togather. The previous definitions can be made very succinct:

$x \notin A$ if and only if $\neg(x \in A)$

$A \subseteq B$ if and only if $(x \in A \rightarrow x \in B)$ is True

$x \in (A \cap B)$ if and only if $(x \in A \wedge x \in B)$

$x \in (A \cup B)$ if and only if $(x \in A \vee x \in B)$

$x \in A - B$ if and only if $(x \in A \wedge x \notin B)$

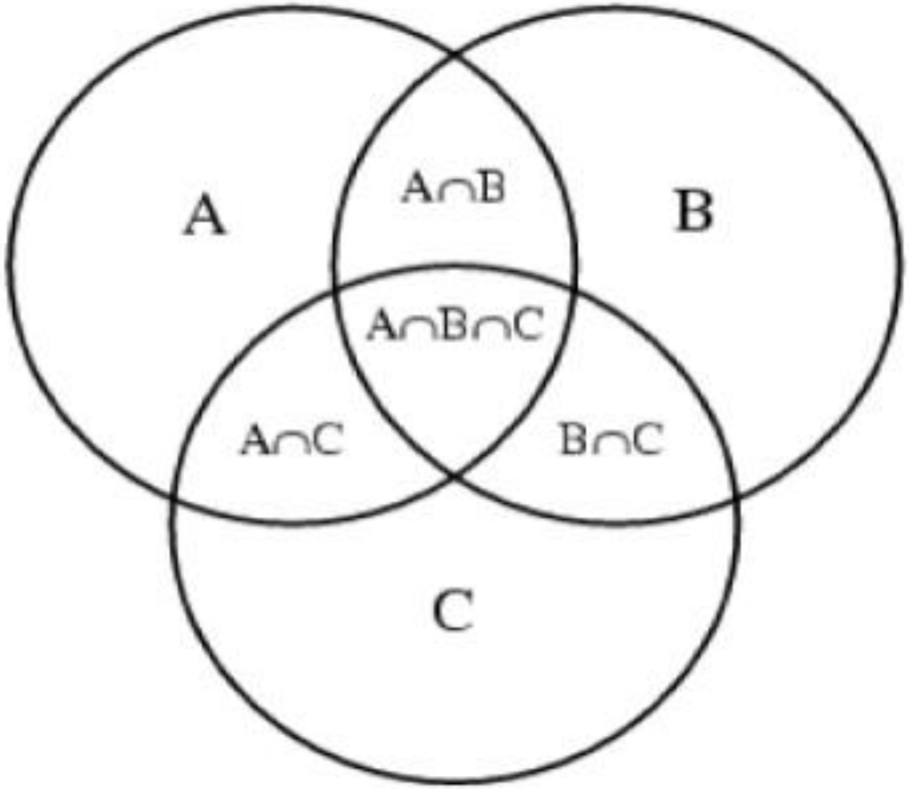
$x \in A \Delta B$ if and only if $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$

$x \in A'$ if and only if $\neg(x \in A)$

$X \in P(A)$ if and only if $X \subseteq A$

Venn Diagrams

Venn Diagrams are used to depict the various unions, subsets, complements, intersections etc. of sets.



Logic and Sets
are closely related

Tautology	Set Operation Identity
$p \vee q \leftrightarrow q \vee p$	$A \cup B = B \cup A$
$p \wedge q \leftrightarrow q \wedge p$	$A \cap B = B \cap A$
$p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$	$A \cup (B \cup C) = (A \cup B) \cup C$
$p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$	$A \cap (B \cap C) = (A \cap B) \cap C$
$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$p \wedge \neg q \leftrightarrow p \wedge \neg (p \wedge q)$	$A - B = A - (A \cap B)$
$p \wedge \neg (q \vee r) \leftrightarrow (p \wedge \neg q) \wedge (p \wedge \neg r)$	$A - (B \cap C) = (A - B) \cup (A - C)$
$p \wedge \neg (q \wedge r) \leftrightarrow (p \wedge \neg q) \vee (p \wedge \neg r)$	$A - (B \cup C) = (A - B) \cap (A - C)$
$p \wedge (q \wedge \neg r) \leftrightarrow (p \wedge q) \wedge \neg (p \wedge \neg r)$	$A \cap (B - C) = (A \cap B) - (A \cap C)$
$p \vee (q \wedge \neg r) \leftrightarrow (p \vee q) \wedge \neg (r \wedge \neg p)$	$A \cup (B - C) = (A \cup B) - (C - A)$
$p \wedge \neg \vee (q \wedge \neg r) \leftrightarrow (p \wedge \neg q) \vee (p \wedge r)$	$A - (B - C) = (A - B) \cup (A \cap C)$

The above identities serve as the basis for an "algebra of sets".

Logic and Sets

are closely related

Tautology

$p \wedge p \leftrightarrow p$

$p \vee p \leftrightarrow p$

$p \wedge \neg(q \wedge \neg q) \leftrightarrow p$

$p \vee \neg(q \wedge \neg q) \leftrightarrow p$

Set Operation Identity

$A \cap A = A$

$A \cup A = A$

$A - \varnothing = A$

$A \cup \varnothing = A$

Contradiction

$p \wedge \neg p$

$p \wedge (q \wedge \neg q)$

$p \wedge \neg p$

Set Operation Identity

$A - A = \varnothing$

$A \cap \varnothing = \varnothing$

$A - A = \varnothing$

The above identities serve as the basis for an "algebra of sets".

Theorem for Logic

Let ***p***, ***q*** and ***r*** be propositions.

Idempotent laws:

$$p \land p \equiv p$$

$$p \lor p \equiv p$$

Truth table:

p	$p \land p$	$p \lor p$
T	T	T
F	F	F

Theorem for Logic (cont.)

Double negation law:

$$\neg \neg p \equiv p$$

Commutative laws:

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

Theorem for Logic (cont.)

Associative laws:

$$(p \land q) \land r \equiv p \land (q \land r)$$
$$(p \lor q) \lor r \equiv p \lor (q \lor r)$$

Distributive laws:

$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$
$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$

PROVE

Prove: Distributive Laws

p	q	r	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Theorem for Logic (cont.)

Absorption laws:

$$p \wedge (p \vee q) \equiv p$$
$$p \vee (p \wedge q) \equiv p$$

PROVE

Prove: Absorption Laws

p	q	$p \wedge (p \vee q)$	$p \vee (p \wedge q)$
T	T	T	T
T	F	T	T
F	T	F	F
F	F	F	F

Theorem for Logic (cont.)

De Morgan's laws:

$\neg(p \land q) \equiv (\neg\, p) \lor (\neg\, q)$
$\neg(p \lor q) \equiv (\neg\, p) \land (\neg\, q)$

The truth table for $\neg(p \lor q) \equiv (\neg\, p) \land (\neg\, q)$

p	q	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Exercise

Propositional functions ***p***, ***q*** and ***r*** are defined as follows:

p is "$n = 7$"

q is "$a > 5$"

r is "$x = 0$"

Write the following expressions in terms of ***p***, ***q*** and ***r***, and show that each pair of expressions is **logically equivalent**. State carefully which of the above laws are used at each stage.

(a) $((n = 7)$ or $(a > 5))$ *and* $(x = 0)$

$((n = 7)$ *and* $(x = 0))$ or $((a > 5)$ *and* $(x = 0))$

(b) $\neg((n = 7)$ *and* $(a \leq 5))$

$(n \neq 7)$ or $(a > 5)$

(c) $(n = 7)$ or $(\neg((a \leq 5)$ *and* $(x = 0)))$

$((n = 7)$ or $(a > 5))$ or $(x \neq 0)$

Exercise

Propositions ***p***, ***q***, ***r*** and ***s*** are defined as follows:
p is "I shall finish my Coursework Assignment"
q is "I shall work for forty hours this week"
r is "I shall pass Maths"
s is "I like Maths"

Write each sentence in symbols:
(a) I shall not finish my Coursework Assignment.
(b) I don't like Maths, but I shall finish my Coursework Assignment.
(c) If I finish my Coursework Assignment, I shall pass Maths.
(d) I shall pass Maths only if I work for forty hours this week and finish my Coursework Assignment.

Write each expression as a sensible (if untrue!) English sentence:
(e) $q \vee p$
(f) $\neg p \rightarrow \neg r$

CHAPTER 1

[Part 4 : Quantifiers & Proof Technique]

QUANTIFIERS

- Most of the statements in mathematics and computer science are not described properly by the propositions.
- Since most of the statements in mathematics and computer science use **variables**, the system of logic must be extended to include **statements with the variables**.

QUANTIFIERS (cont.)

- Let *P(x)* is a statement with variable x and *A* is a set.
- *P* is a **propositional function** or also known as **predicate** if for each x in *A*, *P(x)* is a proposition.
- Set *A* is the **domain of discourse** of *P*.
- Domain of discourse -> the particular domain of the variable in a propositional function.

QUANTIFIERS (cont.)

- A **predicate** is a statement that contains variables.

- **Example:**

  $P(x) : x > 3$

  $Q(x,y) : x = y + 3$

  $R(x,y,z) : x + y = z$

Example

- $x^2 + 4x$ is an odd integer

  (domain of discourse is set of **positive numbers**).
- $x^2 - x - 6 = 0$

  (domain of discourse is set of **real numbers**).
- UTM is rated as Research University in Malaysia

  (domain of discourse is set of **research university** in Malaysia).

QUANTIFIERS (cont.)

- A predicate becomes a proposition if the variable(s) contained is(are)
  - **Assigned specific value(s)**
  - **Quantified**

Example

- $P(x) : x > 3$.

  What are the truth values of $P(4)$ and $P(2)$?
- $Q(x,y) : x = y + 3$.

  What are the truth values of $Q(1,2)$ and $Q(3,0)$?

QUANTIFIERS (cont.)

- Two types of quantifiers:
  - **Universal**
  - **Existential**

QUANTIFIERS (cont.)

- Let A be a propositional function with domain of discourse B. The statement

  for every x, $A(x)$

  is **universally quantified statement**

- Symbol ∀ called a **universal quantifier** is used "**for every**".
- Can be read as "**for all**", "**for any**".

QUANTIFIERS (cont.)

- The statement can be written as

$$\forall x\, A(x)$$

- Above statement is true if $A(x)$ is true for every x in B (false if $A(x)$ is false for at least one x in B).
- A value x in the domain of discourse that makes the statement $A(x)$ *false* is called a **counterexample** to the statement.

Example

- Let the universally quantified statement is

$$\forall x\,(x^2 \geq 0)$$

- Domain of discourse is the set of real numbers.

- **This statement is true** because for every real number x, it is true that the square of x is positive or zero.

Example

- Let the universally quantified statement is

$$\forall x\ (x^2 \leq 9)$$

- Domain of discourse is a set B = {1, 2, 3, 4}
- When x = 4, the statement produce false value.
- Thus, **the above statement is false** and the counterexample is 4.

QUANTIFIERS (cont.)

- Easy to prove a universally quantified statement is true or false if the domain of discourse is not too large.
- What happen if the domain of discourse contains a large number of elements?
- For example, a set of integer from 1 to 100, the set of positive integers, the set of real numbers or a set of students in Faculty of Computing. It will be hard to show that every element in the set is *true*.

Use existential quantifier!!

QUANTIFIERS (cont.)

- Let A be a propositional function with domain of discourse B. The statement

  There exist x, $A(x)$

  is **existentially quantified statement**
- Symbol ∃ called an **existential quantifier** is used "**there exist**".
- Can be read as "**for some**", "**for at least one**".

QUANTIFIERS (cont.)

- The statement can be written as

$$\exists x\, A(x)$$

- Above statement is true if $A(x)$ is true for at least one x in B (false if every x in B makes the statement $A(x)$ false).

- **Just find one x that makes $A(x)$ true!**

Example

- Let the existentially quantified statement is

$$\exists x\left(\frac{x}{x^2+1}=\frac{2}{5}\right)$$

- Domain of discourse is the set of real numbers.
- **Statement is true** because it is possible to find at least one real number x to make the proposition true.
- For example, if $x = 2$, we obtain the true proposition as below

$$\left(\frac{x}{x^2+1}=\frac{2}{5}\right)=\left(\frac{2}{2^2+1}=\frac{2}{5}\right)$$

Negation of Quantifiers

- Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg (\forall x\, P(x)) \;;\; \exists x \neg P(x)$$

$$\neg (\exists x\, P(x)) \;;\; \forall x \neg P(x)$$

Example

- Let $P(x)$ = x is taking Discrete Structure course with the domain of discourse is the set of all students.

    - $\forall x\, P(x)$: All students are taking Discrete Structure course.
    - $\exists x\, P(x)$: There is some students who are taking Discrete Structure course.

$\neg (\exists x\, P(x))$; $\forall x \neg P(x)$

$\neg\exists x\, P(x)$: None of the students are taking Discrete Structure course.

$\forall x \neg P(x)$: All students are not taking Discrete Structure course.

$\neg (\forall x\, P(x))$; $\exists x \neg P(x)$

$\neg\forall x\, P(x)$: Not all students are taking Discrete Structure course.

$\exists x \neg P(x)$: There is some students who are not taking Discrete Structure course

Proofs of Mathematical Statements

- A **proof** is a valid argument that establishes the truth of a statement.
- In math, CS, and other disciplines, informal proofs which are generally shorter, are generally used.
- Proofs have many practical applications:
  - verification that computer programs are correct
  - establishing that operating systems are secure
  - enabling programs to make inferences in artificial intelligence
  - showing that system specifications are consistent

Forms of Theorems

- Often the universal quantifier (needed for a precise statement of a theorem) is omitted by standard mathematical convention.

  For example, the statement:

  “If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$ ”

  really means

  “For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$.”

Proving Theorems

- Many theorems have the form: $\forall x(P(x) \rightarrow Q(x))$
- To prove them, we show that where *c* is an arbitrary element of the domain, $P(c) \rightarrow Q(c)$
- By universal generalization the truth of the original formula follows.
- So, we must prove something of the form: $p \rightarrow q$

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Even and Odd Integers

- **Definition**: The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k, such that $n = 2k + 1$. Note that every integer is either even or odd and no integer is both even and odd.
- We will need this basic fact about the integers in some of the example proofs to follow.

Proving Conditional Statements: $p \rightarrow q$

Direct Proof: Assume that p is true. Use rules of inference, axioms, and logical equivalences to show that q must also be true.

Example: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Assume that n is odd. Then $n = 2k + 1$ for an integer k. Squaring both sides of the equation, we get:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2r + 1,$$

where $r = 2k^2 + 2k$, an integer.

We have proved that if n is an odd integer, then n^2 is an odd integer.

Proving Conditional Statements: $p \to q$

Indirect Proof : Assume $\neg q$ and show $\neg p$ is true also. If we give a direct proof of $\neg q \to \neg p$ then we have a proof of $p \to q$.

Example: Prove that for an integer *n,* if n^2 is odd, then *n* is odd.

Solution: Use proof by contraposition. Assume *n* is even (i.e., not odd). Therefore, there exists an integer *k* such that $n = 2k$. Hence,

$$n^2 = 4k^2 = 2\,(2k^2)$$

and n^2 is even(i.e., not odd).

We have shown that if *n* is an even integer, then n^2 is even. Therefore by indirect proof, for an integer *n,* if n^2 is odd, then *n* is odd.

Proving Conditional Statements: $p \rightarrow q$

- ***Proof by Contradiction.*** To prove p, assume $\neg p$ and derive a contradiction such as $p \wedge \neg p$. (an indirect form of proof). To prove p, assume $\neg p$ and derive a contradiction such as $p \wedge \neg p$. (an indirect form of proof). Since we have shown that $\neg p \rightarrow \mathbf{F}$ is true , it follows that the contrapositive $\mathbf{T} \rightarrow p$ also holds.

- **Example**: Use a proof by contradiction to give a proof that $\sqrt{2}$ is irrational.

Solution: Suppose $\sqrt{2}$ is rational. Then there exists integers a and b with $\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors.
Then

$$2 = \frac{a^2}{b^2} \qquad 2b^2 = a^2$$

Therefore a^2 must be even. If a^2 is even then a must be even (an exercise). Since a is even, $a = 2c$ for some integer c. Thus,

$$2b^2 = 4c^2 \quad b^2 = 2c^2$$

Therefore b^2 is even. Again then b must be even as well.

But then 2 must divide both a and b. This contradicts our assumption that a and b have no common factors. We have proved by contradiction that our initial assumption must be false and therefore $\sqrt{2}$ is irrational .