UNIVERSITY FOR DEVELOPMENT STUDIES

FACULTY OF MATHEMATICAL SCIENCES

LINEAR PROGRAMMING AND ITS APPLICATION IN RADIATION THERAPY PLANNING.

MOHAMMED MUNIR YUSIF

2019

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BY

MOHAMMED MUNIR YUSIF

(FMS/2334/15)

A Dissertation submitted by the Department of Mathematics, Faculty of Mathematical Sciences, University for Development Studies in partial fulfilment of the requirements for the award of Bachelor of Science degree in Mathematics with Economics.

July, 2019

DECLARATION

I hereby declare that this dissertation is the result of my own original work, except for references to the work of others which have been duly acknowledged; and that no part of the work has been presented for another degree in this university or elsewhere.

Candidate's Name: MOHAMMED MUNIR YUSIF

Candidate's Signature: ……………………………………......

Date: …………………………….

Supervisor's Name: PROF. STEPHEN BOAKYE TWUM.

Supervisor's Signature: ………………………………...

Date: …………………………….

Head of Department's Name: DR. KWARA NANTOMAH.

Head of Department's Signature: …………………………..

Date: …………………………………

ABSTRACT

The main objective of the study is to discuss linear programming and how it is applied in radiation therapy treatment planning. Radiation therapy treatment involves using an external beam (in most cases) to pass ionized radiation through a patient's body. Normally, several beams are precisely administered from different angles in a two dimensional plane. Due to the effects of attenuation and scatter, tissues at entry point of a beam may receive more than those at the exit point, and delivery of radiation to tissues outside the direct path of the beam is a possibility. The study aims at formulating a linear programming model that will provide optimal radiation treatment plan to prevent harmful effects on cells in nearby organs and tissues. The case study tackled is a brain tumor problem with four beam angles. A simple linear problem model formulated for the case study's objective is to minimize a linear combination of the n directional beam strengths. The mean dose intensity for both critical and normal tissues of the four beams is acquired after implementing the model. The mean dose strength from the left, right, auterior and posterior beam directions is 3768*rads*, 1307*rads*, 2645*rads* and 3503*rads* respectively. The entire anatomy in all received an integral dose of 53952.00082*rads*.

ACKNOWLEDGEMENT

I thank the Almighty Allah for his maximum protection and guidance throughout this project period.

I express much gratitude and appreciation to my supervisor Prof. Stephen Boakye Twum for his advice and good supervision throughout this research work.

Lastly, I thank my parents and Uncle Munir Kuta for their care, love and great support which made this project work possible.

DEDICATION

I dedicate this work to my late cousin Hakeem Munir Kuta who have been an inspiration to me.

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CHAPTER ONE

INTRODUCTION

Mathematics can be found in our everyday life. It involves the use of patterns to help formulate solutions to problems. We always have to find some sort of strategy or plan for every situation we come across. A plan is said to be a good one if most is made out of it with limited available resource or input. Maximum output is expected with the input of minimum effort and cost. This area relating to this is called optimization. Linear programming is a branch of optimization and can be simply described as a mathematical technique for maximizing or minimizing a linear function of several variables, subject to a set of linear constraints.

1.1 Background of the Study

Linear programming and its application stretches back to the early 1900s. It was used as a secret tool during the World War 2 until it was published. George B. Dantzig and John Von Newman can be seen as the pioneers of it as they published the simplex method and the theory of duality, respectively. Linear programming methods were adopted after the World War for its usefulness in planning optimization (Wikipedia). Today it is applied in almost all fields including medicine and healthcare. Radiation therapy treatment planning is an example of field in medicine where optimization s applied.

The World Health Organization (WHO) estimates that globally, more than 11 million people are diagnosed with cancer every year. Cancer causes more death every year compared to the combined total of HIV/AIDS, TB and malaria. (Ministry of Health, 2011). In Ghana, it is estimated that 16,600 cases of cancer occur annually. (Ministry of Health, 2011). The most common cancers are cancer of liver, cervix, breast, prostate and Non-Hodgkin's Lymphoma (NHL). Cancer treatment come in many forms. Treatment options are determined by the type and stage of the cancer. Cancer treatments include chemotherapy, surgery, radiation therapy, immunotherapy, etc. This dissertation focuses on optimization in radiation therapy treatment planning.

Radiation therapy treatment is a form of therapy that uses high doses of ionizing radiation to kill cancer cells and shrink them. It can be delivered from outside from outside the patient using special machines (teletherapy) or deposited from radioactive substances within the patient (brachytherapy). (Lim, 2002). Treatment works because the repair mechanism for cancer cells is less efficient than for normal cells. Radiation therapy or radiotherapy has improved over the years due to modern technology. The treatment however can be harmful to surrounding organs and cells. The application of mathematical optimization sets in here to help apportion correct doses to the tumor while reducing any harmful effects on surrounding organs and cells.

1.2 Statement of the Problem

Radiation therapy treatment has its fair share of problems. The treatment which is effective also is harmful to tissues surrounding the target area when beams are released. Mathematical models are formulated through the application of linear optimization to minimize doses to areas with normal tissues and maximize does to the target area.

The oncology unit of every hospital faces these same problems with its cancer patient during radiotherapy treatment. The problem faced during treatment is how to optimize the right doses to treat patients. This study intends to formulate and design the problem as a linear programming problem model that will provide an optimal radiation treatment plan to help prevent harmful effects on normal tissues whiles the right amount pf doses are delivered to cancerous cells.

1.3 Objectives of the Study

The main objective of this study is to

- Discuss linear programming and its application in radiation therapy treatment planning.
- Formulate the problem as a linear programming model that will provide optimal radiation treatment plan to prevent harmful effects on cells in nearby organs and tissues.
- Analyze the results and make recommendations on basis of the results for radiotherapy treatment planning.

1.4 Research Questions

- What is linear programming and how it may be applied?
- How can the LP model be developed and what are the practical implications?

1.5 Significance of The Study

The study is of great significance to the medical society. The study will help broaden the knowledge on linear programming and optimization in radiation therapy treatment planning and demonstrate results for the benefit of the mathematics community and the country.

1.6 Scope of the Study

This dissertation will cover radiation therapy and mathematical models that can be applied to this treatment.

1.7 Organization of the Study

The organization of this dissertation is structured in chapters to help breakdown the study.

The first chapter consist of the background of the study, statement of the problem, objectives, research questions and the significance of the study.

Chapter two dives into the literature review of the study. This chapter gives a more detailed information about previous studies and their outcome.

Chapter three discusses the various methods and approaches that will be employed.

Chapter Four is concerned with an orderly presentation and discussion of the results of the study.

Chapter Five consist of the conclusions reached while making relevant recommendations.

CHAPTER TWO

LITERATURE REVIEW

2.1 Brief History on Linear Programming

Linear programming was developed as a discipline in the 1940's, motivated initially by the need to solve complex planning problems in wartime operations. Its development accelerated rapidly in the postwar period as many industries found valuable uses for linear programming. The founders of the subject are generally regarded as George B. Dantzig, who devised the simplex method in 1947, and John Neuman who established the theory of duality the same year (Overton, 1997). The Nobel Prize winner in economics was awarded in 1975 to the mathematician Leonid Kantorovich (USSR) and the economist Tjalling Koopmans (USA) for their contributions to the theory of optimal allocation of resources, in which linear programming played a key role. (Overton, 1997).

The simplex method has been the standard technique for solving a linear program since the 1940's. In brief, the simplex method assesses from the vertex to vertex on the boundary of feasible polyhedron, repeatedly increasing the objective function until either an optimal solution is found, or it is established that no solution exists. In principle the time required might be an exponential function of the number of variables, and this can happen in some contrived cases. (Overton, 1997). In practice, however, the method is highly efficient, typically requiring a number of steps which is just a small multiple of the number of variables. Linear programs in thousands or even in millions of variables are routinely solved using the simplex method on modern computers.

Efficiently, highly sophisticated implementations are available in the form of computer software packages.

In 1979, Leonid Khaciyan presented the ellipsoid method, guaranteed to solve any linear program in a number of steps which is a polynomial function of the amount of data defining the linear program. Consequently, the ellipsoid method is faster than the simplex method in contrived cases where the simplex method performs poorly. (Overton, 1997). In 1984, Narenda Karmarkar introduced an interior-point method for linear programming, combining the desirable theoretical properties of the ellipsoid method and practical advantages of the simplex method. (Overton, 1997). Its success initiated an explosion in the development of interior-point methods. These do not pass vertex to vertex, but pass only through the interior of the feasible region. Though this property is easy to state, the analysis of the interior-point method is a subtle subject which is much less understood than the behavior of the simplex method. Interior-point methods are now generally considered competitive with the simplex method in most, though not all, applications, and sophisticated software packages. (Overton, 1997).

An essential component of both the simplex and interior-point methods is the solution of system of linear equations, which use techniques developed by C.F. Gauss and A. Cholesky in the 18th and 19th centuries (Overton, 1997).

2.2. Characteristics of Linear Programming.

All linear programming problems must have the following characteristics;

- There must be clearly defined objective which can be stated in quantitative way. In business problems the objective is generally profit maximization or cost minimization.
- All constraints regarding resources should be fully spelt out in mathematical form.
- The value of the variables must be zero or positive or not negative.
- The relationship between the variables must be linear.
- The number of inputs need to be finite. In the case of infinite factors, computation of feasible solution is not possible.

2.3 Application in Radiotherapy

Olafsson and Wright (2004) thesis on linear programming formulations and algorithms for radiotherapy treatment planning focused on three linear programming models. Formulation with explicit bounds on voxel doses, formulation with DV constraints and formulation with range constraints and penalties are models 1, 2 and 3 respectively. We will discuss the Intensity Modulated Radiation Therapy (IMRT) data set for the case study of nasopharyngeal tumor. The IMRT data is divided into 5 region-subclass (i.e. target-target with 884 voxels, target-regional with 4246 voxels, critical-spinal cord with 406 voxels, critical-parotids with 692 voxels and normal with 17772 voxels). There are 51 beam angles, with 39 beamlets from each angle, giving a total of 1,989 beamlets. In summary, the dose matrix A has 24,000 rows and 1,989 columns. For the IMRT data set, the normal voxel penalty vector is c_N to $e = (1, 1, .1)^T$ and the bounds on target voxel dose is $x^L=50e$ $x^U = 75e$. The nonbounded solution had maximum doses to the parotids of 56.15Gy (where 'Gy' denotes 'Grey', the unit of radiation) and to the spinal cord of 14.13Gy.We set the bounds as follows: 75Gy (target and normal tissue), 50Gy (parotids), and 10Gy (spinal cord). The results and findings of the three models was solved with the linear programming software CPLX (Simplex and Barrier) with gave the integral dose, iterations and time spent on the treatment.

Caron et al (1994) researched on application of linear programming in radiation therapy with the help of articles from University of Windsor's Leddy Library and University of Michigan's Taubman Medical Library. The thesis included among others a linear programming model developed for a whole bladder lesion and a 4-beam treatment plan for a brain tumor. With the model developed for the bladder, the general LP problem is to minimize the objective function, $J = J(w_1, \ldots\ldots w_{72}) = \sum^{72} w_i \ (s_i^{\ 1} + s_i^{\ 2} + s_i^{\ 3})$, where the least volume to the target region is 6*Gy* and the integral dose to the non-target area is 3Gy. With target entries of when treatment matrix when i=18 and listed a^i at an entry angle of 90 degrees we have;

s^1=0.0, s^2=0.421 and s^3=0.0

target = 0.699 and t_{max} = 0.700

For the example of the case study for a brain tumor, the 4-beam has 4 variables; w_1, w_2, w_3 and w_4. Eight points around the border of the tumor were picked with constraints placed on them. The objective is to minimize the integral dose which is the total radiation dose received by the entire anatomy. The objective function is given as; minimize J = J(w_1, w_2, w_3, w_4) with the dose volume being at least

6000Gy. There are 13 constraints including, the 4 non-negative constraints (where, w_1, w_2, w_3, $w_4 \geq 0$).The integral dose for each decision variable after the problem was solve is $w_1 = 3720$ Gy, $w_2 = 3200$ Gy, $w_3 = 670$ Gy and $w_4 = 4040$ Gy.

Craft et al (2014) provided a dataset for researchers to use developing and contrasting radiation treatment planning optimization algorithms in a published journal. The dataset consisted of four radiation therapy cases. They provide datasets for a prostate case, a liver case, a head and neck case, and a standard IMRT phantom with their respective CT (Computed Tomography) scans. The dose-influence matrix from a variety of beam/couch angle pairs for each dataset was provided. Each cae was model in to a simple single-objective linear programming problem. The optimization problem for all four cases was solved using MATLAB linear programming solver. They discussed the datasets as a benchmark cases for the development of fast and efficient solvers customized to Fluence Map Optimization (FMO) and its variants.

CHAPTER THREE

METHODOLOGY

3.1 Linear Programming

A linear programming problem (LP) is an optimization problem for which we attempt to maximize or minimize a linear function of the decision variables. The function that is to be maximized or minimized is called the objective function. The Graphical method and the Simplex method are the two most used method solution in solving LP problems. This thesis will apply the simplex method in solving of the radiation treatment planning problem.

3.1.1 General form of Linear Programming

We wish to optimize a linear function subject to a set of linear inequalities. Given a set of real numbers: a_1, a_2, a_3, a_n and a set of variables x_1, x_2, x_3, …. . , x_n.

Linear function *f* on those variables is defined by

$F\ (x_1, x_2, x_3 \ldots \ldots x_n) = a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n =$

$$\sum^{n}{}_{j=1} a_jx_j$$

If b is a real number and *f* is a linear function, then the equation *f*

$(x_1, x_2, x_3, \ldots x_n) \geq b$ are linear inequalities.

3.1.2 Converting a LP Problem to a Standard Form

An LP problem must be converted to a standard form before it can be solved. It must be converted into an equivalent problem where all constraints are equations and all variables are non-negative. Suppose we have converted an LP with m constraints into standard form. Assuming that the standard form contains n variables (labeled for convenience $x_1, x_2 ... x_n$), the standard form for such an LP is (Winston, 2003)

$$\text{Max } Z \text{ (or Min)} = c_1x_1 + c_2x_2 + \ldots\ldots.. + c_nx_n$$

$$\text{Subject to} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_n$$

$$x_i \geq 0 \ (i = 1, n.)$$

3.2 Assumptions of Linear Programming

- The **proportionality** assumption: the contribution of a decision variable to the objective function or constant is proportional to the value of the decision variable.
- The **additivity** assumption: the contribution of a decision variables to the objective function or constraints is the sum of the individual contributions of the variables.
- The **divisibility** assumption: requires that each decision variable be allowed to assume fractional values.

- The **certainty** assumption: is that each parameter (i.e. objective function coefficient, right- hand side, and technological coefficient) is known with certainty.

3.3 Feasible Region and Optimal Solution.

Two of the most basic concepts associated with a linear programming problem are feasible region and optimal solution. The **feasible region** for an LP is the set of all points that satisfies all the LP's constraints and sign restrictions (Winston, 2003). Any point that is not in an LP's feasible region is said to be an **infeasible point**. For a maximization problem, an **optimal solution** to an LP is a point in the feasible region with the largest objective function value. Similarly, for a minimization problem, an optimal solution is a point in the feasible region with the smallest objective function value (Winston, 2003).

3.4 The Graphical Method for LP Model.

This model is commonly used to solve a linear programming problem with two decision variables. It's an easy and effective method. We find the intercepts of the decision variables of each constraint set and plot it on a graph. The constraints within the area where the plotted lines intercept. Here is a simple example of a two decision variable LP problem. The example (Ferguson, n.d) below gives a simple illustration of the graphical method.

Example 3.1

Find numbers x_1 and x_2 that maximize the sum $x_1 + x_2$ subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, and

$$x_1 + 2x_2 \leq 4$$

$$4x_1 + 2x_2 \leq 12$$

$$-x_1 + \quad x_2 \leq 1$$

We seek the point (x_1,x_2), that achieves the maximum of $x_1 + x_2$ as (x_1,x_2) ranges over this constraint set. The function $x1 + x2$ is constant on lines with slope −1, for example the line $x_1 + x_2 = 1$, and as we move this line further from the origin up and to the right, the value of $x^1 + x^2$ increases. Therefore, we seek the line of slope −1 that is farthest from the origin and still touches the constraint set. This occurs at the intersection of the lines $x_1 + 2x_2 = 4$ and $4x_1 + 2x_2 = 12$, namely, $(x_1,x_2) = (8/3,2/3)$. The value of the objective function there is $(8/3) + (2/3) = 10/3$.

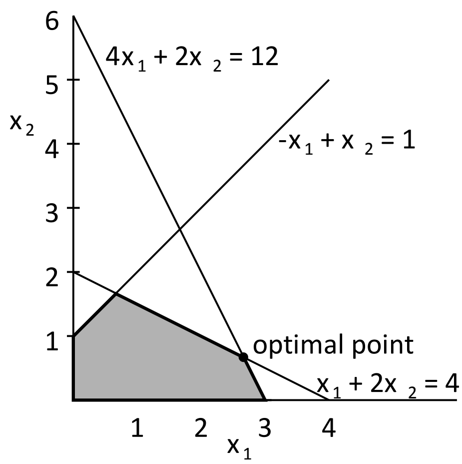


Figure 3.1 Graphical representation of the LP problem.

The shaded region is where all the inequalities are satisfied. This is the feasible region which is in this case is a five sided polygon with five vertices. The points in the feasible region are the feasible points and the coordinates of the feasible points are the feasible solution. The single point that yields the largest objective value provides the optimal solution. In this case it occurs at the intersection of the lines $x_1 + 2x_2 = 4$ and $4x_1 + 2x_2 = 12$, namely, $(x_1, x_2) = (8/3, 2/3)$. The value of the objective function there is $(8/3) + (2/3) = 10/3$.

3.5 The Simplex Algorithm.

The simplex method is the most common and effective LP algorithm used for solving LP problems with more than two decision variables. The model can also be used to solve a two decision variable problem. The simplex method uses elementary row operations to iterate from one basic feasible solution to another until the optimal solution is reached. Computer programs based on this method can routinely solve linear programming problems with thousands of decision variables and constraints.

3.5.1 Summary of the Simplex Algorithm for a Maximum Problem.

- Convert the LP to standard form.
- Find a basic feasible solution.
- If all non-basic variables have nonnegative coefficients in objective row, then the current BFS (basic feasible solution) is optimal. If any variables in objective row have negative coefficients, then choose the variable with the most negative coefficient in objective row to enter the basis.
- Use EROs (Elementary Row Operations) to make the entering variable the basic variable in any row that wins the ratio test. After the EROs

have been used to create a new canonical form, return to the previous step, using the current canonical form (Winston, 2003).

3.6 Integer Programming and Mixed Integer Programming.

Integer programming model is commonly used for improving radiotherapy treatment efficiency. In a situation where the fractional solutions of a decision variables are not realistic, the problem is referred to as the integer programming problem (Bradley et al, 1977). We consider the LP problem to be,

$$\text{Maximize } \sum_{j=1}^{n} cj\ x\ j$$

$$\text{Subject to} \qquad \sum_{j=1}^{n} ai\ j\ x\ j\ =\ b \qquad (i = 1, 2, \ldots m),$$

$$x\ j \geq 0 \qquad (j = 1, 2, \ldots, n),$$

$$x\ j \text{ integer (for some or all } j = 1, 2, \ldots, n).$$

Also, a linear programming problem is said to be a mixed integer program when some, but not all, variables are restricted to be integers (Bradley et al, 1977). The Beam Selection problem, the Fluence Map Optimization problem (in order to model dose volume constraints) and Segmentation problem are areas where (mixed) integer programming models appear in radiation oncology. (Erghott and Holder, 2014).

3.7 Treatment Planning Problem.

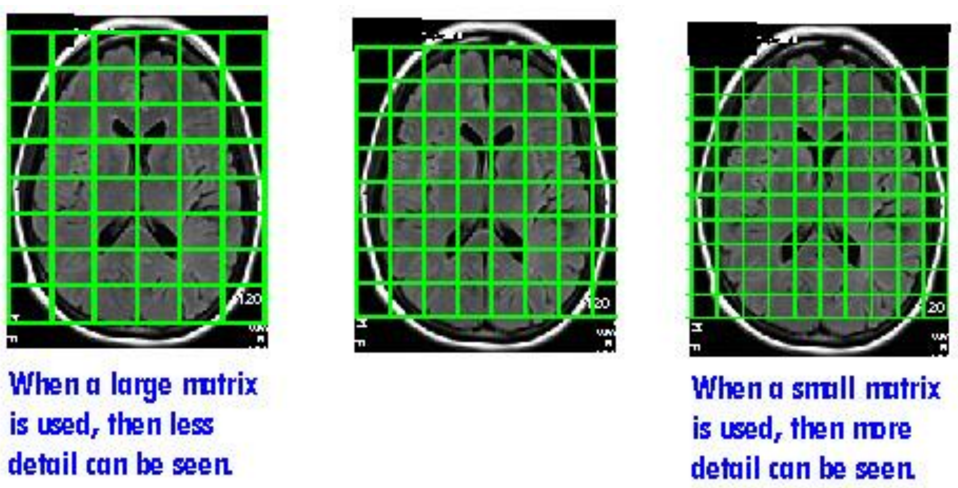
Epelman (2013) discussed the different types of treatment planning problems in radiation therapy and the models for these problems. There is a treatment planning problem each;

- For a given tumor and given critical areas.
- For a given set of possible “beamlet” origins and angles.
- To determine the intensity of each beamlet such that:
    - ➢ Dosage over the tumor area will be at least a target level.
    - ➢ Dosage over the critical area will be at most a target level

3.7.1 A Simple Model for a Treatment Planning Problem.

A given tumor treatment region is divided into a 2-dimensional (or 3-dimensional) grid of voxels.

Figure 3.2. Tumor region divided into voxels.



The figure shows an example of a tumor affected area divided various voxels (MriShark, 2017). We then create the beamlet data for each of p =1,. .., n possible beamlets. D^p is an example matrix of unit doses delivered by beam p.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0.8 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 1.0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$D^p{}_{ij}$ = dose delivered to voxel (i , j) by beamlet p at unit intensity

A model for the treatment planning problem is given as;

$$\text{Minimize}_{\text{w.D}} \sum_{(i,j)\in S} (D_{ij})$$

$$\text{Subject to} \qquad D_{ij} = \sum_{p=1}^{n} D_{ij}\, w_p \quad (i,j) \in S$$

$$w_p \geq 0 \qquad p = 1,. .., n$$

$$D_{ij} \geq \Gamma l \qquad (i\,,j) \in T$$

$$D_{ij} \leq \gamma U \qquad (i\,,j) \in C$$

3.8 Software for Solving LPs

The more the decision variables the more difficult it becomes to solve a linear programming problem. The average mathematician can only solve a linear programming problem up to a certain number. Software codes makes it easier to solve these type of problems as they can be used to solve LP problems with over thousand decision variables. Analytic Solver, MATLAB, CPLEX and AMPL are among software codes used to solve LP problems.

3.9 Summary

In this chapter, we talk about LP with detailed explanations in symbolic form. We discussed the models and algorithms used in solving LP problems and models for radiation treatment planning problems. The next chapter will focus on the application, results and discussions of the application of linear programming radiotherapy treatment planning. Case study will be based on a brain tumor problem.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1 Introduction

In the previous chapter, we discussed linear programming models and radiotherapy treatment planning model. In this chapter, we are going to formulate an LP model based on a case study which will be seen later in the chapter. Analysis and discussion of the results will be made.

4.2 Source of Data

The case study to be analyzed and discussed is a formulated word problem by my project supervisor based on a published journal by Caron et al (1994). Data for the project was difficult to acquire so we had to do with this.

4.3 General Formulation of the LP Model.

Considering an n beam treatment plan for a patient from all angles with beam intensity x_1, $x_2 \ldots x_n$. The number of tissue points (m) surrounding the tumor region with a required unit dose matrix (a_{ij}) $m \times n$ and the combined dose of n beams per tissue point should not exceed a certain limit k_i (where i = 1, 2, 3,….m). If the objective is to minimize a linear combination of the n directional beam strength, then the treatment planning LP model is given as;

$$\text{Minimize } f\,(x_1, x_2 \ldots x_n)$$

Subject to, $\sum_{j=1}^{n} a_{ij}\, x_j \quad \leq \quad k_i$ i= 1, 2,…,m.

$x_j \geq 0$

4.4 A Case Study Related to the Model.

Suppose a brain tumor treatment is to be undertaken with four beam from directions; left, right, auterior and posterior. If there are eight locations around the tumor, with required dose strength per beam direction given in table 4.1 and the center of tumor dose strength also given in the table. We can find the dose strengths that minimizes the linear combination of the four beams' strengths.

Table 4.1. Dose strength to anatomy. (*All doses in kilorads*).

Location	Left	Right	Auterior	Posterior	Limit
1	0.85	0.53	0.37	0.32	6000
2	0.82	0.54	0.33	0.38	6000
3	0.72	0.61	0.35	0.44	6000
4	0.67	0.67	0.40	0.44	6000
5	0.65	0.72	0.47	0.39	6000
6	0.69	0.66	0.53	0.32	6000
7	0.75	0.60	0.50	0.30	6000
8	0.83	0.55	0.43	0.29	6000
Center	0.75	0.62	0.42	0.36	6000

4.5 Formulation of LP Problem for Case Study.

In the previous chapter we discussed various treatment planning models applied in the field of radiotherapy. We are going to apply the model to the case study and for formulate an LP problem. The model is given as;

$$\text{Minimize } f\,(x_1, x_2\ldots\ldots x_n)$$

$$\text{Subject to, } \quad \sum_{j=1}^{n} a_{ij}\, x_j \quad \leq \quad k_i \quad i = 1, 2, \ldots, m.$$

$$x_j \geq 0$$

Applying the above model to the dataset of the case study we have;

Minimize $6.73x_1 + 5.50x_2 + 3.86x_3 + 3.24x_4$

Subject to,

$0.85x_1 + 0.53\,x_2 + 0.37x_3 + 0.32x_4 \leq 6000$

$0.82x_1 + 0.54\,x_2 + 0.33x_3 + 0.38x_4 \leq 6000$

$0.72x_1 + 0.61\,x_2 + 0.35x_3 + 0.44x_4 \leq 6000$

$0.67x_1 + 0.67\,x_2 + 0.40x_3 + 0.44x_4 \leq 6000$

$0.65x_1 + 0.72\,x_2 + 0.47x_3 + 0.39x_4 \leq 6000$

$0.69x_1 + 0.66\,x_2 + 0.53x_3 + 0.32x_4 \leq 6000$

$0.75x_1 + 0.60\,x_2 + 0.50x_3 + 0.30x_4 \leq 6000$

$0.83x_1 + 0.55\,x_2 + 0.49x_3 + 0.29x_4 \leq 6000$

$0.75x_1 + 0.62\,x_2 + 0.42x_3 + 0.36x_4 \leq 6000$

Where, $x_1, x_2, x_3, x_4 \geq 0$

4.6 Solution of the Model

Analytical solver in the Microsoft Excel was used to solve the problem. The solver was used to find the optimal radiation dose to the four beam angles (x_1, x_2, x_3,x_4). The objective value is 53952.00 and the values of the decision variables are shown table 4.2.

Table 4.2 Integral radiation generated by analytic solver.

Decision Variables	Values
X_1	3768.45
X_2	1307.21
X_3	2645.06
X_4	3502.92

4.7 Sensitivity Analysis of the Model.

Here, we check for the stability of the model by sensitivity analysis. Sensitivity analysis is a typical measure to quantify the impact of parameter uncertainty on overall simulation/prediction uncertainty, and a variety of sensitivity analysis techniques have been developed (Helton, 1993; Saltelli et al., 2008). It is a powerful tool for examining issues relating to uncertainties in model structure, or in input or parameter values (Landsberg & Sands, 2011). We are going to increase and decrease the dose strengths of all the beams by 1%. Analysis will also be made on the right hand side (RHS) which is the limit of the beam directional dose strength. Here, both a 1% increase and decrease and a 5% increase and decrease.

4.7.1 Formulation of LP Problem (One Percent Increase).

The dose strengths of the four beam directions and the locations after 1% increase is shown in the table 4.3.

Table 4.3. Dose strength increased by 1%. (*All doses in kilorads*)

Location	Left	Right	Auterior	Posterior	Limit
1	0.8585	0.5353	0.3737	0.3232	6000
2	0.8282	0.5454	0.3333	0.3838	6000
3	0.7272	0.6161	0.3535	0.4444	6000
4	0.6767	0.6767	0.4040	0.4444	6000
5	0.6565	0.7272	0.4747	0.3939	6000
6	0.6969	0.6666	0.5353	0.3232	6000
7	0.7575	0.6060	0.5050	0.3030	6000
8	0.8383	0.5555	0.4343	0.2929	6000
Center	0.7575	0.6262	0.4242	0.3636	6000

Applying the model in the previous section, we have;

Minimize $6.7973x_1 + 5.555x_2 + 3.838x_3 + 3.2724x_4$

Subject to,

$0.8585x_1 + 0.5353x_2 + 0.3737x_3 + 0.3232x_4 \leq 6000$

$0.8282x_1 + 0.5454x_2 + 0.3333x_3 + 0.3838x_4 \leq 6000$

$0.7272x_1 + 0.6161x_2 + 0.3535x_3 + 0.4444x_4 \leq 6000$

$0.6767x_1 + 0.6767x_2 + 0.4040x_3 + 0.4444x_4 \leq 6000$

$0.6565x_1 + 0.7272x_2 + 0.4747x_3 + 0.3939x_4 \leq 6000$

$0.6969x_1 + 0.6666x_2 + 0.5353x_3 + 0.3232x_4 \leq 6000$

$0.7575x_1 + 0.6060x_2 + 0.5050x_3 + 0.3030x_4 \leq 6000$

$0.8383x_1 + 0.5555x_2 + 0.4949x_3 + 0.2929x_4 \leq 6000$

$0.7575x_1 + 0.6262\, x_2 + 0.4242x_3 + 0.3636x_4 \leq 6000$

Where, $x_1, x_2, x_3, x_4 \geq 0$

4.7.2. Solution of the Model.

Here, the objective value after the problem has been solved by analytical solver is 53952.00. The value of the decision variables after a 1% increase in all parameters is shown in table 4.4.

Table 4.4 Integral radiation generated by analytic solver.

Decision Variables	Values
X_1	3731.14
X_2	1294.27
X_3	2618.87
X_4	3468.24

4.7.3 Formulation of LP Problem (One Percent Decrease).

We are going to decrease the dose intensity of all the beams by 1% to check the stability of the model applied. The new dose strengths after the 1% decrease is shown in table 4.5.

Table 4.5. Dose strength decreased by 1%. (*All doses in kilorads*).

Location	Left	Right	Auterior	Posterior	Limit
1	0.8415	0.5247	0.3663	0.3168	6000
2	0.8118	0.5346	0.3267	0.3762	6000
3	0.7128	0.6039	0.3465	0.4356	6000
4	0.6633	0.6633	0.396	0.4356	6000
5	0.6435	0.7128	0.4653	0.3861	6000
6	0.6831	0.6534	0.5247	0.3168	6000
7	0.7425	0.594	0.495	0.297	6000
8	0.8217	0.5445	0.4257	0.2871	6000
Center	0.7425	0.6138	0.4158	0.3564	6000

Applying the given model, we have an LP problem;

Minimize $6.6627x_1 + 5.445x_2 + 3.762x_3 + 3.2076x_4$

Subject to,

$0.8415x_1 + 0.5247x_2 + 0.3663x_3 + 0.3168x_4 \leq 6000$

$0.8118x_1 + 0.5346x_2 + 0.3267x_3 + 0.3762x_4 \leq 6000$

$0.7128x_1 + 0.6039x_2 + 0.3465x_3 + 0.4356x_4 \leq 6000$

$0.6633x_1 + 0.6633x_2 + 0.396x_3 + 0.4356x_4 \leq 6000$

$0.6435x_1 + 0.7128x_2 + 0.4653x_3 + 0.3861x_4 \leq 6000$

$0.6831x_1 + 0.6534x_2 + 0.5247x_3 + 0.3168x_4 \leq 6000$

$0.7425x_1 + 0.594x_2 + 0.495x_3 + 0.297x_4 \leq 6000$

$0.8217x_1 + 0.5445x_2 + 0.4257x_3 + 0.2871x_4 \leq 6000$

$0.7425x_1 + 0.6138x_2 + 0.4158x_3 + 0.3564x_4 \leq 6000$

Where, $x_1, x_2, x_3, x_4 \geq 0$

4.7.4. Solution of the Model

The solver produced an objective value of 53952.00. The values of the four decision variables can be seen in table 4.6.

Table 4.6 Integral radiation generated by analytic solver.

Decision Variables	Values
X_1	3806.51
X_2	1320.42
X_3	2671.78
X_4	3538.30

4.7.5 Formulation of LP Problem (1% Increase in Limit).

We are going to increase the limit of all the beams by 1% to check the stability of the model applied. The new limit after the 1% increase is shown in table 4.7.

Table 4.7. 1% increase in limit. (*All doses in kilorads*)

Location	Left	Right	Auterior	Posterior	Limit
1	0.85	0.53	0.37	0.32	6060
2	0.82	0.54	0.33	0.38	6060
3	0.72	0.61	0.35	0.44	6060
4	0.67	0.67	0.40	0.44	6060
5	0.65	0.72	0.47	0.39	6060
6	0.69	0.66	0.53	0.32	6060
7	0.75	0.60	0.50	0.30	6060
8	0.83	0.55	0.43	0.29	6060
Center	0.75	0.62	0.42	0.36	6060

Applying the given model we have;

Minimize $6.73x_1 + 5.50x_2 + 3.86x_3 + 3.24x_4$

Subject to,

$0.85x_1 + 0.53\ x_2 + 0.37x_3 + 0.32x_4 \leq 6060$

$0.82x_1 + 0.54\ x_2 + 0.33x_3 + 0.38x_4 \leq 6060$

$0.72x_1 + 0.61\ x_2 + 0.35x_3 + 0.44x_4 \leq 6060$

$0.67x_1 + 0.67\ x_2 + 0.40x_3 + 0.44x_4 \leq 6060$

$0.65x_1 + 0.72\ x_2 + 0.47x_3 + 0.39x_4 \leq 6060$

$0.69x_1 + 0.66\ x_2 + 0.53x_3 + 0.32x_4 \leq 6060$

$0.75x_1 + 0.60\ x_2 + 0.50x_3 + 0.30x_4 \leq 6060$

$0.83x_1 + 0.55\ x_2 + 0.49x_3 + 0.29x_4 \leq 6060$

$0.75x_1 + 0.62\ x_2 + 0.42x_3 + 0.36x_4 \leq 6060$

Where, $x_1, x_2, x_3, x_4 \geq 0$

4.6.4. Solution of the Model.

The solver produced an objective value of 54491.52. The values of the four decision variables can be seen in table 4.8.

Table 4.8 Integral radiation generated by analytic solver.

Decision Variables	Values
X_1	3806.13
X_2	1320.28
X_3	2671.51
X_4	3537.95

4.7.6. Formulation of LP Problem (1% Decrease in Limit).

We are going to decrease the limit of all the beams by 1% to check the stability of the model applied. The new limit after the 1% decrease is shown in table 4.9.

Table 4.9. 1% Decrease in limit. (*All doses in kilorads*)

Location	Left	Right	Auterior	Posterior	Limit
1	0.85	0.53	0.37	0.32	5940
2	0.82	0.54	0.33	0.38	5940
3	0.72	0.61	0.35	0.44	5940
4	0.67	0.67	0.40	0.44	5940
5	0.65	0.72	0.47	0.39	5940
6	0.69	0.66	0.53	0.32	5940
7	0.75	0.60	0.50	0.30	5940
8	0.83	0.55	0.43	0.29	5940
Center	0.75	0.62	0.42	0.36	5940

Applying the given model we have;

Minimize $6.73x_1 + 5.50x_2 + 3.86x_3 + 3.24x_4$

Subject to,

$0.85x_1 + 0.53\,x_2 + 0.37x_3 + 0.32x_4 \leq 5940$

$0.82x_1 + 0.54\,x_2 + 0.33x_3 + 0.38x_4 \leq 5940$

$0.72x_1 + 0.61\,x_2 + 0.35x_3 + 0.44x_4 \leq 5940$

$0.67x_1 + 0.67\,x_2 + 0.40x_3 + 0.44x_4 \leq 5940$

$0.65x_1 + 0.72\,x_2 + 0.47x_3 + 0.39x_4 \leq 5940$

$0.69x_1 + 0.66\ x_2 + 0.53x_3 + 0.32x_4 \leq 5940$

$0.75x_1 + 0.60\ x_2 + 0.50x_3 + 0.30x_4 \leq 5940$

$0.83x_1 + 0.55\ x_2 + 0.49x_3 + 0.29x_4 \leq 5940$

$0.75x_1 + 0.62\ x_2 + 0.42x_3 + 0.36x_4 \leq 5940$

Where, $x_1, x_2, x_3, x_4 \geq 0$

4.7.7. Solution of the Model.

The solver produced an objective value of 53412.48. The values of the four decision variables can be seen in table 4.10.

Table 4.10 Integral radiation generated by analytic solver.

Decision Variables	Values
X_1	3730.76
X_2	1294.14
X_3	2618.61
X_4	3467.89

4.7.8. Formulation of LP Problem (5% Decrease in Limit)

We are going to decrease the limit of all the beams by 5% to check the stability of the model applied. The new limit after the 5% decrease is shown in table 4.11.

Table 4.11. 5% decrease in limit. (*All doses in kilorads*)

Location	Left	Right	Auterior	Posterior	Limit
1	0.85	0.53	0.37	0.32	5700
2	0.82	0.54	0.33	0.38	5700
3	0.72	0.61	0.35	0.44	5700
4	0.67	0.67	0.40	0.44	5700
5	0.65	0.72	0.47	0.39	5700
6	0.69	0.66	0.53	0.32	5700
7	0.75	0.60	0.50	0.30	5700
8	0.83	0.55	0.43	0.29	5700
Center	0.75	0.62	0.42	0.36	5700

Applying the given model we have;

Minimize $6.73x_1 + 5.50x_2 + 3.86x_3 + 3.24x_4$

Subject to,

$0.85x_1 + 0.53x_2 + 0.37x_3 + 0.32x_4 \leq 5700$

$0.82x_1 + 0.54x_2 + 0.33x_3 + 0.38x_4 \leq 5700$

$0.72x_1 + 0.61x_2 + 0.35x_3 + 0.44x_4 \leq 5700$

$0.67x_1 + 0.67x_2 + 0.40x_3 + 0.44x_4 \leq 5700$

$0.65x_1 + 0.72x_2 + 0.47x_3 + 0.39x_4 \leq 5700$

$0.69x_1 + 0.66x_2 + 0.53x_3 + 0.32x_4 \leq 5700$

$0.75x_1 + 0.60x_2 + 0.50x_3 + 0.30x_4 \leq 5700$

$0.83x_1 + 0.55x_2 + 0.49x_3 + 0.29x_4 \leq 5700$

$0.75x_1 + 0.62x_2 + 0.42x_3 + 0.36x_4 \leq 5700$

Where, $x_1, x_2, x_3, x_4 \geq 0$

4.7.9. Solution of the Model

The solver produced an objective value of 51254.40. The values of the four decision variables can be seen in table 4.12.

Table 4.12 Integral radiation generated by analytic solver.

Decision Variables	Values
X_1	3580.02
X_2	1241.85
X_3	2512.81
X_4	3327.77

4.7.10 Formulation of LP Problem (5% Increase in Limit).

We are going to increase the limit of all the beams by 5% to check the stability of the model applied. The new limit after the 5% increase is shown in table 4.12.

Table 4.13. 5% Increase in limit. (*All doses in kilorads*)

Location	Left	Right	Auterior	Posterior	Limit
1	0.85	0.53	0.37	0.32	6300
2	0.82	0.54	0.33	0.38	6300
3	0.72	0.61	0.35	0.44	6300
4	0.67	0.67	0.40	0.44	6300
5	0.65	0.72	0.47	0.39	6300
6	0.69	0.66	0.53	0.32	6300
7	0.75	0.60	0.50	0.30	6300
8	0.83	0.55	0.43	0.29	6300
Center	0.75	0.62	0.42	0.36	6300

Applying the given model we have;

Minimize $6.73x_1 + 5.50x_2 + 3.86x_3 + 3.24x_4$

Subject to,

$0.85x_1 + 0.53\,x_2 + 0.37x_3 + 0.32x_4 \leq 6300$

$0.82x_1 + 0.54\,x_2 + 0.33x_3 + 0.38x_4 \leq 6300$

$0.72x_1 + 0.61\,x_2 + 0.35x_3 + 0.44x_4 \leq 6300$

$0.67x_1 + 0.67\,x_2 + 0.40x_3 + 0.44x_4 \leq 6300$

$0.65x_1 + 0.72\,x_2 + 0.47x_3 + 0.39x_4 \leq 6300$

$0.69x_1 + 0.66\,x_2 + 0.53x_3 + 0.32x_4 \leq 6300$

$0.75x_1 + 0.60\,x_2 + 0.50x_3 + 0.30x_4 \leq 6300$

$0.83x_1 + 0.55\,x_2 + 0.49x_3 + 0.29x_4 \leq 6300$

$0.75x_1 + 0.62\,x_2 + 0.42x_3 + 0.36x_4 \leq 6300$

Where, $x_1, x_2, x_3, x_4 \geq 0$

4.6.4. Solution of the Model.

The solver produced an objective value of 56649.60. The values of the four decision variables can be seen in table 4.11.

Table 4.14 Integral radiation generated by analytic solver.

Decision Variables	Values
X_1	3956.87
X_2	1372.57
X_3	2777.31
X_4	3678.07

4.8 Discussion and Interpretation of Results.

From Table 4.2, the decision variables (x_1, x_2, x_3 and x_4) are the dose strengths of the beams direction (left, right, auterior and posterior). The mean dose strength form each beam left, right, auterior and posterior is 3768*rads*, 1307*rads*, 2645*rads* and 3503*rads* respectively. The mean dose strength distributed to all the locations ranges from 5996*rads* to 6000*rads*. Locations 1, 3, 6 and 7 each received a mean dose strength of 5996*rads*, 5978*rads*, 5986*rads* and 5984*rads* respectively. The remaining areas (locations 2, 4, 5, 8 and the center) each received a dose of 6000*rads*. The entire anatomy in all received an integral dose of 53952.00*rads*.

A 1% increase in the parameters led to a new set of dose strengths as seen in Table 4.4 where the mean dose from left, right, auterior and posterior are 3731.14*rads*, 1294.27*rads*, 2618.87*rads* and 3468.24*rads* respectively. Also, Table 4.6 shows the new mean dose released from all four directions after a 1% decrease in all the parameters. 3806.51*rads* was released for the left whiles the right, auterior and posterior directions released 1320.42*rads*, 2671.78*rads* and 3538.30*rads* respectively.

Comparison of the new set of mean dose after the percentage changes can be seen in Table 4.15. The percentage increase in parameters reduced the dose release from the four directions whiles the percentage decrease in the parameters increased the dose released.

Table 4.15 Analysis after 1% increment and decrement in parameters.

Beam Direction	Integral Dose	After 1% increase in dose strength	Positive Difference	After 1% decrease in dose strength	Negative Difference
Left	3768.45	3731.14	37.31	3806.51	38.06
Right	1307.21	1294.27	12.94	1320.42	13.21
Auterior	2645.06	2618.87	26.19	2671.78	26.72
Posterior	3502.92	3468.24	34.68	3538.30	35.38

The mean dose strength released from all four directions are supposed to be at most 6000*rads*. Here the limits was increased and decreased by 1% and 5%. A percentage increase in the limit results in the rise in the mean dose whiles a percentage decrease also results in reduction in the mean dose strength released. Table 4.16 displays the new set of dose strength after 1% increase and decrease in the limits. The mean dose after 1% addition increased from 3768.45*rads*, 1307.21*rads*, 2645.06*rads* and 3502.92*rads* to 3806.13*rads*, 1320.28*rads*, 2671.51*rads* and 3537.95*rads* respectively. Also, new dose after 1% decrease in the limit for left, right, auterior and posterior is 3730.76*rads*, 1294.14*rads*, 2618.61*rads* and 3467.89*rads* respectively.

Table 4.16 Analysis after 1% increment and decrement in limits.

Beam Direction	Integral Dose	After 1% increase in limit	Negative Difference	After 1% decrease in limit	Positive Difference
Left	3768.45	3806.13	37.68	3730.76	37.69
Right	1307.21	1320.28	13.07	1294.14	13.07
Auterior	2645.06	2671.51	26.45	2618.61	26.45
Posterior	3502.92	3537.95	35.03	3467.89	35.03

However, as seen in table 4.17, the 5% percent increment and decrement in the limit results in a larger change in the integral dose released form the all four directions. The dose strength from the left, right, auterior and posterior directional beams increased from 3768.45*rads*, 1307.21*rads*, 2645.06*rads* and 3502.92*rads* to 3956.87*rads*, 1372.57*rads*, 2777.31*rads* and 3678.07*rads* respectively after a 5% increment in the limit. On the other hand, the dose strength after 5% decrement decreased from 3768.45*rads*, 1307.21*rads*, 2645.06*rads* and 3502.92*rads* to 3580.02*rads*, 1241.85*rads*, 2512.81*rads* and 3327.77*rads* respectively. The difference after both the 5% increment and decrement as seen in table 4.17 are in hundreds for the left, auterior and posterior directional beams whiles that of the right directional beam is in the tens.

Table 4.17 Analysis after 5% increment and decrement in limits.

Beam Direction	Integral Dose	After 5% increase in limit	Negative Difference	After 5% decrease in limit	Positive Difference
Left	3768.45	3956.87	188.42	3580.02	188.43
Right	1307.21	1372.57	65.36	1241.85	65.36
Auterior	2645.06	2777.31	132.25	2512.81	132.25
Posterior	3502.92	3678.07	175.15	3327.77	175.15

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 Summary of Main Findings

The main objective if the study was to be able to design a radiation planning problem as a linear programming model that will provide optimal radiation treatment plan to prevent harmful effects on cells in nearby organs and tissues. The treatment planning model used was a single-objective model. The goal was to use the model to minimize radiation dose to the entire anatomy.

The secondary data acquired was on three cases of radiation therapy problem (IMRT Phantom case, prostrate case and liver case). The model was applied to the dataset which resulted in mean dose radiations for each case. The model was solved with the Analytic Solver via Microsoft excel.

5.2. Conclusion

The model used in this study was able to provide the mean dose needed for both critical and normal tissues. The mean dose strength from the left, right, auterior and posterior beam directions is 3768*rads*, 1307*rads*, 2645*rads* and 3503*rads* respectively. The entire anatomy in all received an integral dose of 53952.00082*rads*. The problem faced in oncology units in hospitals during the treatment period of patients can be prevented with the application of this mathematical model. This application can be used in the stead of the try-and-error method applied by some practitioners.

5.3. Recommendations

This recommended that the use of linear programming models in the healthcare especially in oncology r should incorporate linear programming in their planning. Mathematical programs can be used to the advantage and make the work of practitioners easier.

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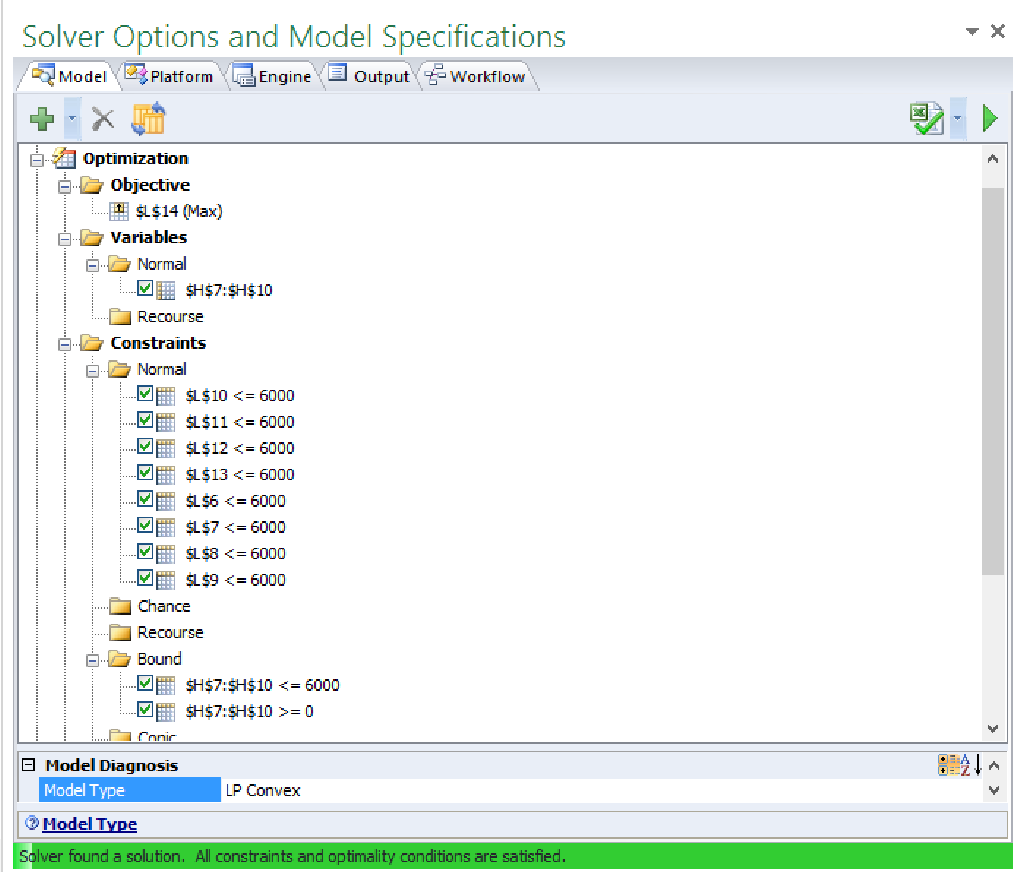
APPENDIX I

Optimization of Case Study with Analytic Solver.



Solver found a solution. All constraints and optimality conditions are satisfied.

Solve time: 5.13 Seconds.

---- Start Solve ----

No uncertain input cells.

Using: Full Reparse.

Parsing started...

Diagnosis started...

Convexity testing started...

Model diagnosed as "LP Convex".

Automatic engine selection: Gurobi Solver V8.1.0.0

Model: [Brain Tumor.xlsx]Sheet1

Using: Psi Interpreter

Parse time: 0.81 Seconds.

Engine: Gurobi Solver V8.1.0.0

Setup time: 0.01 Seconds.

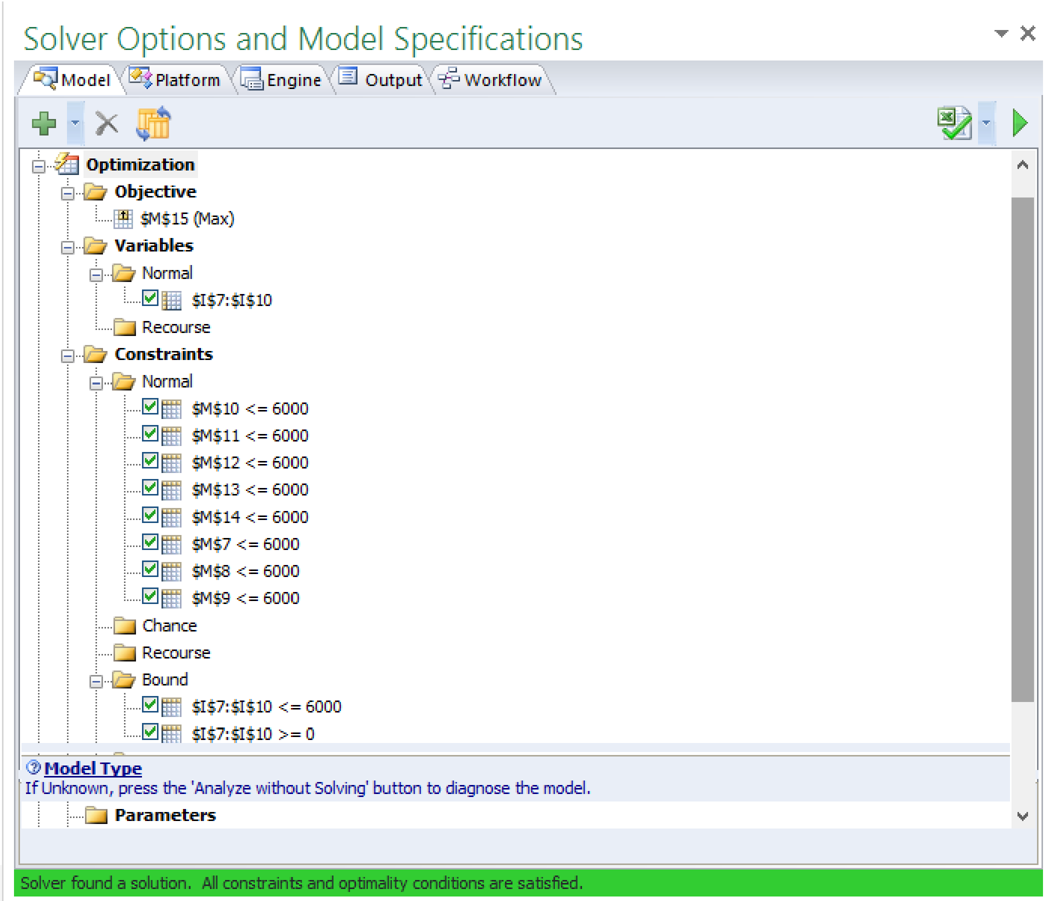
Engine Solve time: 0.03 Seconds.

Solver found a solution. All constraints and optimality conditions are satisfied.

Solve time: 4.30 Seconds.

APPENDIX II

Optimization of Case Study with Analytical Solver (1% increase in all dose strengths).




```
---- Start Solve ----

No uncertain input cells.

Using: Full Reparse.

Parsing started...

Diagnosis started...

Convexity testing started...

Model diagnosed as "LP Convex".

Automatic engine selection: Gurobi Solver V8.1.0.0

Model: [Sensitivity Analysis 1.xlsx]Sheet1

Using: Psi Interpreter

Parse time: 1.22 Seconds.
```

Engine: Gurobi Solver V8.1.0.0

Setup time: 0.03 Seconds.

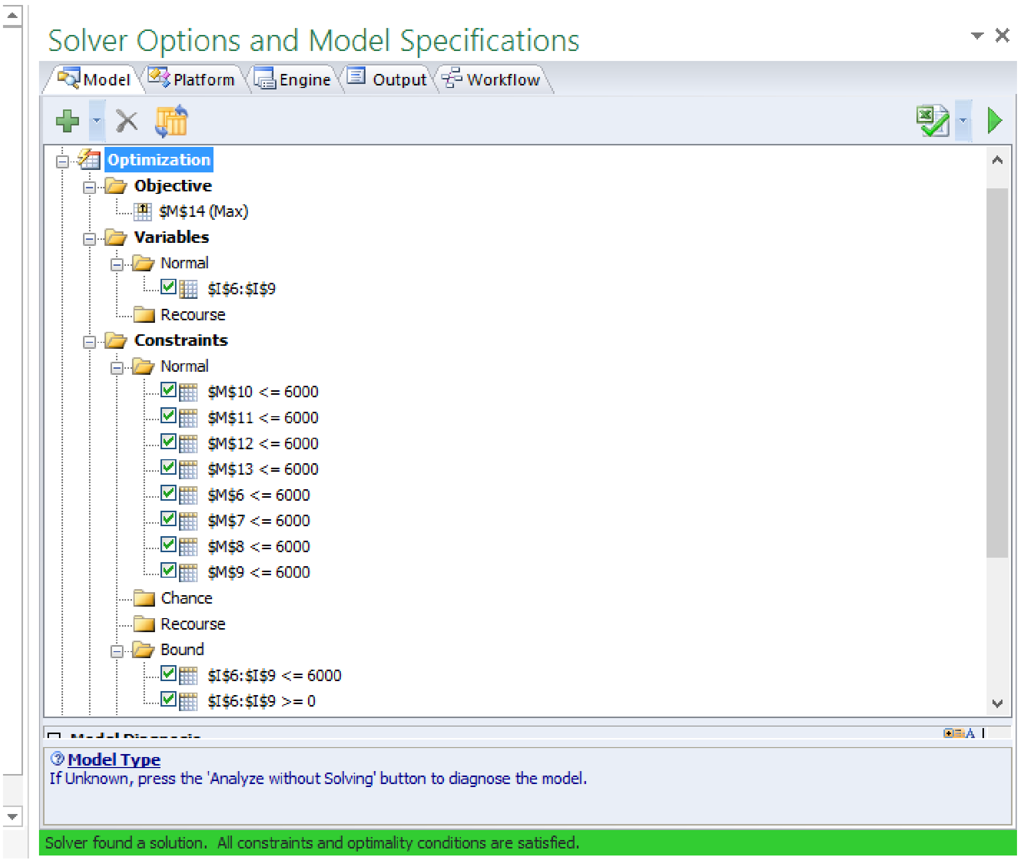
Engine Solve time: 0.05 Seconds.

Solver found a solution. All constraints and optimality conditions are satisfied.

Solve time: 10.94 Seconds.

APPENDIX III

Optimization of Case Study with Analytical Solver (1% decrease in all dose strengths).




```
---- Start Solve ----

No uncertain input cells.

Using: Full Reparse.

Parsing started...

Diagnosis started...

Convexity testing started...

Model diagnosed as "LP Convex".

Automatic engine selection: Gurobi Solver V8.1.0.0

Model: [Sensitivity Analysis 2.xlsx]Sheet1

Using: Psi Interpreter

Parse time: 4.67 Seconds.
```

Engine: Gurobi Solver V8.1.0.0

Setup time: 0.73 Seconds.

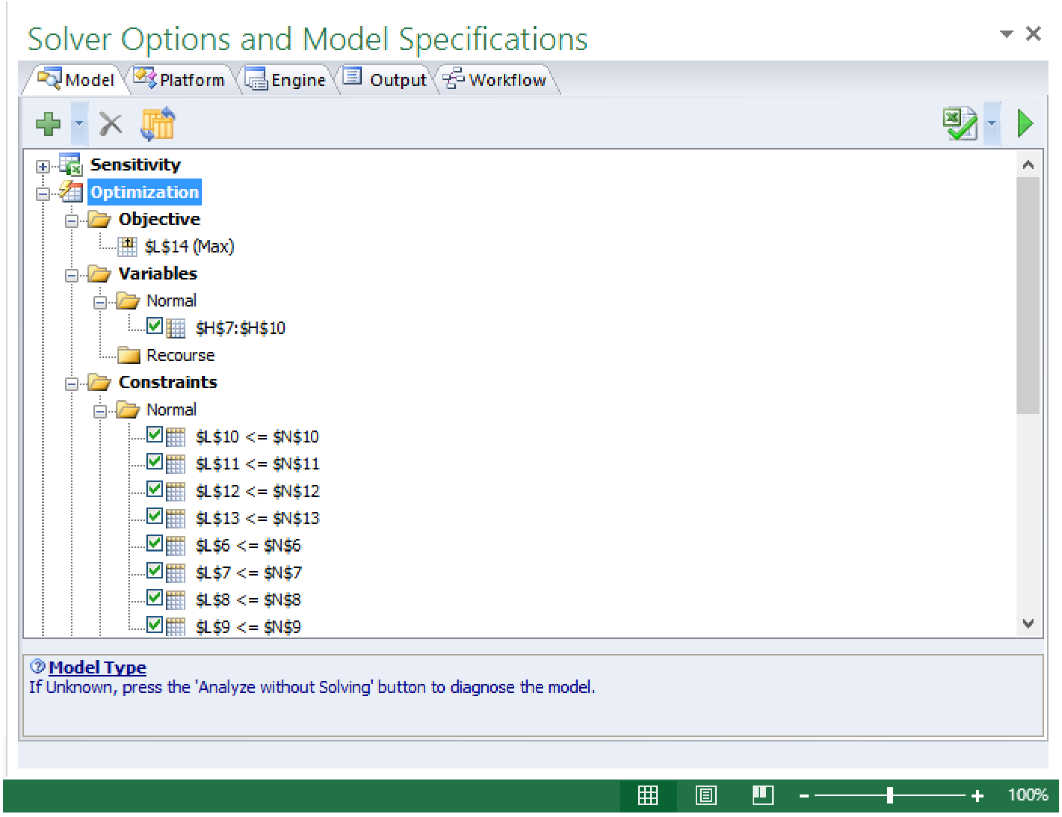
Engine Solve time: 0.03 Seconds.

Solver found a solution. All constraints and optimality conditions are satisfied.

Solve time: 9.58 Seconds.

APPENDIX IV

Optimization of Case Study with Analytical Solver (1% increase in limit).




```
---- Start Solve ----

No uncertain input cells.

Using: Full Reparse.

Parsing started...

Diagnosis started...

Convexity testing started...

Model diagnosed as "LP Convex".

Automatic engine selection: Gurobi Solver V8.1.0.0

Model: [Brain Tumor rhs 1.xlsx]Sheet1

Using: Psi Interpreter

Parse time: 2.94 Seconds.


Engine: Gurobi Solver V8.1.0.0

Setup time: 0.05 Seconds.
```

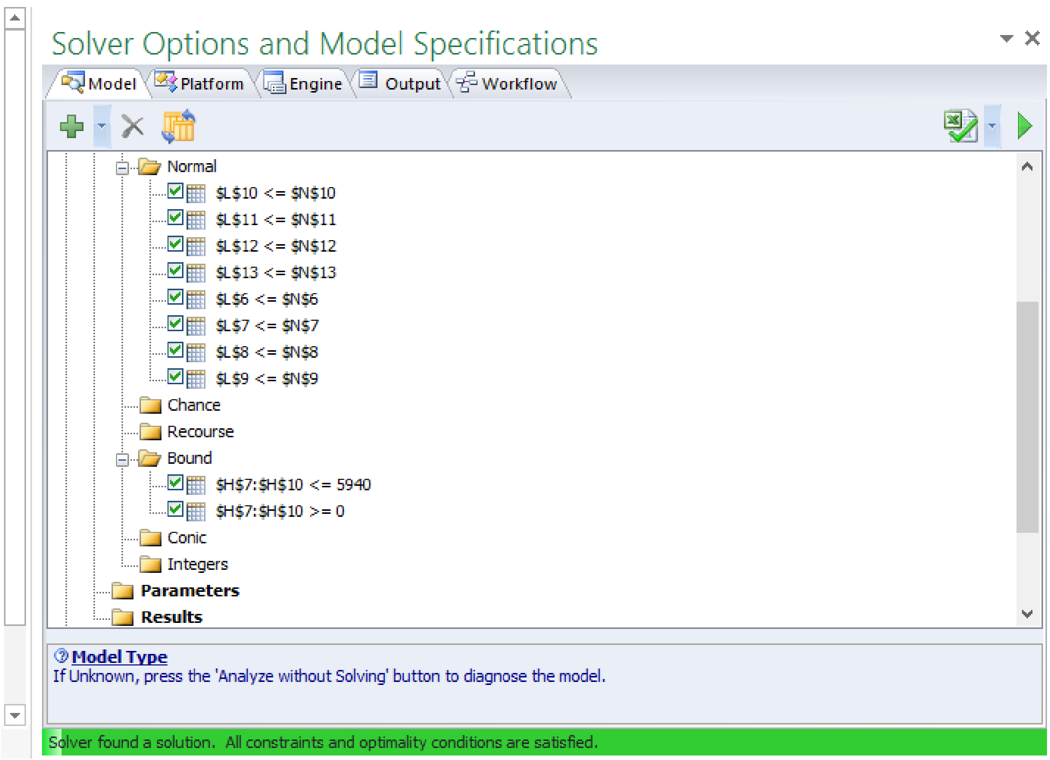
Engine Solve time: 0.03 Seconds.

Solver found a solution. All constraints and optimality conditions are satisfied.

Solve time: 6.59 Seconds.

APPENDIX V

Optimization of Case Study with Analytical Solver (1% decrease in limit).




```
---- Start Solve ----

No uncertain input cells.

Using: Full Reparse.

Parsing started...

Diagnosis started...

Convexity testing started...

Model diagnosed as "LP Convex".

Automatic engine selection: Gurobi Solver V8.1.0.0

Model: [Brain Tumor rhs 2.xlsx]Sheet1

Using: Psi Interpreter

Parse time: 1.30 Seconds.


Engine: Gurobi Solver V8.1.0.0

Setup time: 0.33 Seconds.
```

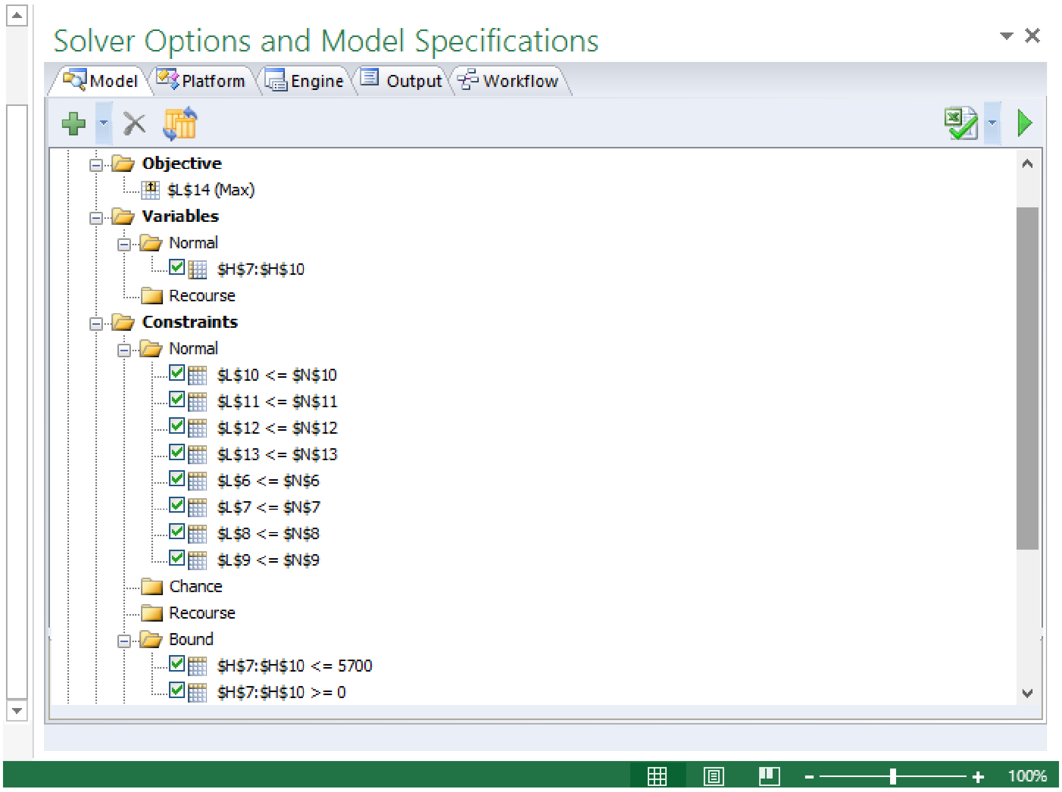
Engine Solve time: 0.00 Seconds.

Solver found a solution. All constraints and optimality conditions are satisfied.

Solve time: 4.03 Seconds.

APPENDIX VI

Optimization of Case Study with Analytical Solver (5% increase in limit).




```
---- Start Solve ----

No uncertain input cells.

Using: Full Reparse.

Parsing started...

Diagnosis started...

Convexity testing started...

Model diagnosed as "LP Convex".

Automatic engine selection: Gurobi Solver V8.1.0.0

Model: [Brain Tumor rhs 3.xlsx]Sheet1

Using: Psi Interpreter

Parse time: 1.33 Seconds.


Engine: Gurobi Solver V8.1.0.0

Setup time: 0.03 Seconds.
```

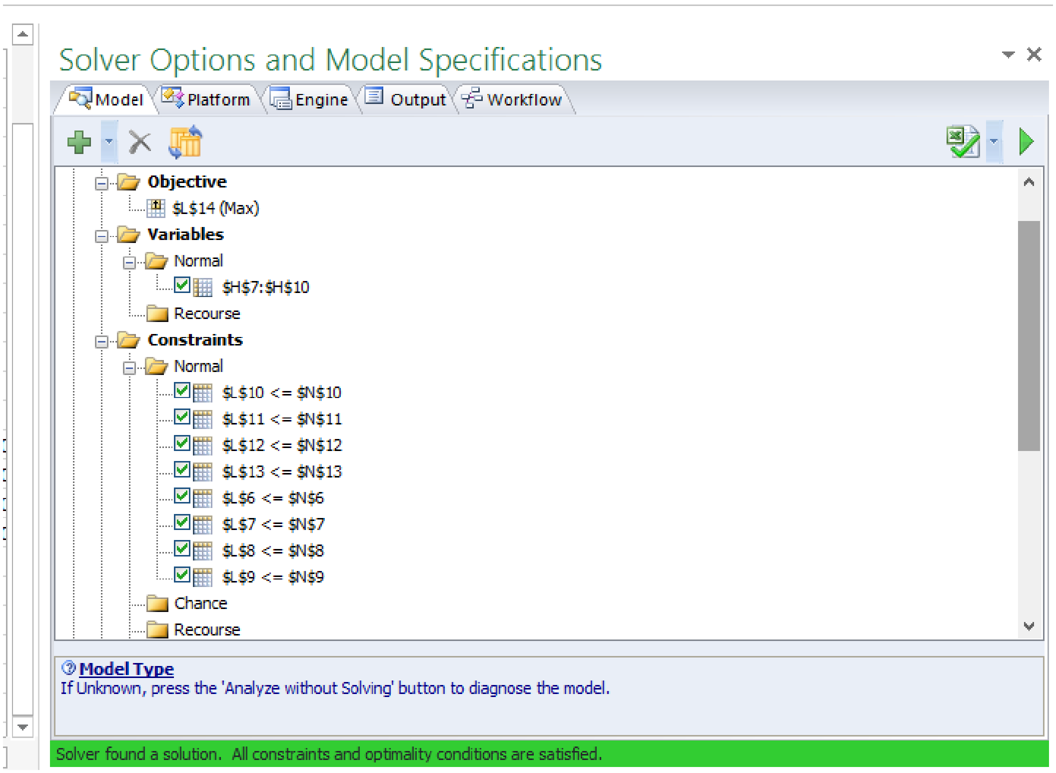
Engine Solve time: 0.02 Seconds.

Solver found a solution. All constraints and optimality conditions are satisfied.

Solve time: 5.02 Seconds.

APPENDIX VII

Optimization of Case Study with Analytical Solver (5% decrease in limit).




```
---- Start Solve ----

No uncertain input cells.

Using: Full Reparse.

Parsing started...

Diagnosis started...

Convexity testing started...

Model diagnosed as "LP Convex".

Automatic engine selection: Gurobi Solver V8.1.0.0

Model: [Brain Tumor rhs 4.xlsx]Sheet1

Using: Psi Interpreter

Parse time: 1.64 Seconds.


Engine: Gurobi Solver V8.1.0.0

Setup time: 0.02 Seconds.
```

Engine Solve time: 0.01 Seconds.

Solver found a solution. All constraints and optimality conditions are satisfied.

Solve time: 4.03 Seconds.