ЛЕКЦ 4: Дурын үетэй функцийн Фурьегийн цуваа

Тодорхойлолт 0.1. *$f(x)$ функц нь $2l$ $(l \neq \pi)$ үетэй функц байг. $x \doteq \dfrac{l}{\pi}t$ орлуулга хийвэл $f\left(\dfrac{l}{\pi}t\right)$ функц t аргументын хувьд 2π үетэй болно. Тэгвэл Фурьегийн цуваа дараах хэлбэртэй болно.*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l} \right)$$

Үүнд:

$$\begin{aligned}
a_0 &= \frac{1}{l}\int_{-l}^{l} f(x)dx \\
a_k &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi x}{l}dx \\
b_k &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi x}{l}dx
\end{aligned}$$

Жишээ 0.1. $f(x) = x^2, \ \ -1 \leq x \leq 1$ *функцийг Фурьегийн цуваанд задал.*

Бодолт: *Энэ функц тэгш учраас* $b_k = 0, \ \ l = 1$

$$\begin{aligned}
a_0 &= \frac{2}{l}\int_{0}^{l} f(x)dx = 2\int_{0}^{1} x^2 dx = \frac{2}{3} \\
a_k &= \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{k\pi x}{l}dx = 2\int_{0}^{1} x^2\cos k\pi x dx = \frac{2}{k\pi}x^2\sin k\pi x\Big|_0^1 - \frac{4}{k\pi}\int_0^1 x\sin k\pi x dx \\
&= -\frac{4}{k\pi}\int_0^1 x\sin k\pi x dx = -\frac{4}{k\pi}\left(-\frac{1}{k\pi}x\cos k\pi x\Big|_0^1 + \frac{1}{k\pi}\int_0^1 \cos k\pi x dx\right) \\
&= -\frac{4}{k\pi}\left(-\frac{1}{k\pi}x\cos k\pi x + \frac{1}{(k\pi)^2}\sin k\pi x\right)\Big|_0^1 = -\frac{4}{k\pi}\left(-\frac{1}{k\pi}\cos k\pi + \frac{1}{(k\pi)^2}\sin k\pi + 0 - 0\right) \\
&= \frac{4}{(k\pi)^2}(-1)^k
\end{aligned}$$

Иймд

$$f(x) = x^2 = \frac{1}{3} - \frac{4}{\pi^2}\left(\cos\pi x - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - ...\right)$$

Жишээ 0.2. $f(x) = \begin{cases} 0, & -2 \leq x \leq 0 \\ x, & 0 \leq x \leq 2 \end{cases}$ *функцийг Фурьегийн цуваанд задал.*

Бодолт: *Энэ функц тэгш ч биш, сондгой ч биш функц ба* $l = 2$

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\left(\int_{-2}^{0} 0dx + \int_{0}^{2} xdx\right) = \frac{1}{2}\cdot\frac{x^2}{2}\Big|_0^2 = 1 \\
a_k &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{k\pi x}{2}dx = \frac{1}{2}\left(\int_{-2}^{0} 0\cdot\cos\frac{k\pi x}{2}dx + \int_{0}^{2} x\cos\frac{k\pi x}{2}dx\right) \\
&= \frac{1}{2}\int_{0}^{2} x\cos\frac{k\pi x}{2}dx = \frac{1}{2}\left(\frac{2}{k\pi}x\sin\frac{k\pi x}{2}\Big|_0^2 - \frac{2}{k\pi}\int_{0}^{2}\sin\frac{k\pi x}{2}dx\right) = \frac{1}{2}\frac{4}{(k\pi)^2}\cos\frac{k\pi x}{2}\Big|_0^2 \\
&= \frac{1}{2}\frac{4}{(k\pi)^2}\left(\cos k\pi - \cos 0\right) = \frac{2}{(k\pi)^2}\left((-1)^k - 1\right) \\
b_k &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{k\pi x}{2}dx = \frac{1}{2}\left(\int_{-2}^{0} 0\cdot\sin\frac{k\pi x}{2}dx + \int_{0}^{2} x\cdot\sin\frac{k\pi x}{2}dx\right) \\
&= \frac{1}{2}\left(-\frac{2}{k\pi}x\cos\frac{k\pi x}{2}\Big|_0^2 + \frac{2}{k\pi}\int_{0}^{2}\cos\frac{k\pi x}{2}dx\right) \\
&= \frac{1}{2}\left(-\frac{4}{k\pi}\cos k\pi + 0 + \frac{4}{(k\pi)^2}\sin\frac{k\pi x}{2}\Big|_0^2\right) = -\frac{2}{k\pi}\cdot(-1)^k = \frac{2}{k\pi}(-1)^{k+1}
\end{aligned}$$

Иймд

$$f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k^2}\cos\frac{k\pi x}{2}\right) + \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{(-1)^{k+1}}{k}\sin\frac{k\pi x}{2}\right)$$

Жишээ 0.3. $f(x) = x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$ *функцийг Фурьегийн цуваанд задал.*

Бодолт: *Энэ функц сондгой функц учраас* $a_0 = 0, \quad a_k = 0$ *байна.* $l = \frac{\pi}{2}$

$$\begin{aligned}
b_k &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\sin 2kxdx = \frac{4}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin 2kxdx = \frac{4}{\pi}\left(-x\frac{1}{2k}\cos 2kx\Big|_0^{\frac{\pi}{2}} + \frac{1}{2k}\int_{0}^{\frac{\pi}{2}}\cos 2kxdx\right) \\
&= \frac{4}{\pi}\left(-\frac{\pi}{4k}\cos k\pi + \frac{1}{4k^2}\sin 2kx\Big|_0^{\frac{\pi}{2}}\right) = -\frac{1}{k}(-1)^k = (-1)^{k+1}\frac{1}{k}
\end{aligned}$$

Иймд

$$f(x) = \sum_{k=1}^{\infty}\left((-1)^{k+1}\cdot\frac{1}{k}\cdot\sin 2kx\right) = \sin 2x - \frac{1}{2}\sin 4x + \frac{1}{3}\sin 6x - ...$$

Жишээ 0.4. $f(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < x \le \pi \end{cases}$ *функцийг* $(0;\pi)$ *завсарт* *a.* *зөвхөн косинусуудаар*

b. *зөвхөн синусүүдээр Фурьегийн цуваанд задал.*

Бодолт:

a. *Өгөгдсөн функцийг* $[-\pi;0]$ *завсарт үргэлжлүүлэхэд үүсэх* $f_1(x)$ *функц нь тэгш функц байна гэвэл* $f_1(x)=\begin{cases}\frac{\pi}{2}, & -\pi\le x<-\frac{\pi}{2}\\ -x, & -\frac{\pi}{2}\le x<0\\ x, & 0\le x\le\frac{\pi}{2}\\ \frac{\pi}{2}, & \frac{\pi}{2}<x\le\pi\end{cases}$

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int\limits_0^{\pi} f_1(x)dx=\frac{2}{\pi}\left(\int\limits_0^{\frac{\pi}{2}} xdx+\int\limits_{\frac{\pi}{2}}^{\pi}\frac{\pi}{2}dx\right)=\frac{2}{\pi}\left(\frac{x^2}{2}\Big|_0^{\frac{\pi}{2}}+\frac{\pi}{2}x\Big|_{\frac{\pi}{2}}^{\pi}\right)\\
&= \frac{2}{\pi}\left(\frac{\pi^2}{8}+\frac{\pi^2}{2}-\frac{\pi^2}{4}\right)=\frac{2}{\pi}\cdot\frac{3\pi^2}{8}=\frac{3\pi}{4}\\
a_k &= \frac{2}{\pi}\int\limits_0^{\pi} f_1(x)\cos kxdx=\frac{2}{\pi}\left(\int\limits_0^{\frac{\pi}{2}} x\cos kxdx+\int\limits_{\frac{\pi}{2}}^{\pi}\frac{\pi}{2}\cos kxdx\right)\\
&= \frac{2x\sin kx}{k\pi}\Big|_0^{\frac{\pi}{2}}-\frac{2}{k\pi}\int\limits_0^{\frac{\pi}{2}}\sin kxdx+\frac{\sin kx}{k}\Big|_{\frac{\pi}{2}}^{\pi}=\frac{1}{k}\sin\frac{k\pi}{2}+\frac{2}{\pi k^2}\cos kx\Big|_0^{\frac{\pi}{2}}-\frac{1}{k}\sin\frac{k\pi}{2}\\
&= \frac{2}{\pi k^2}\left(\cos\frac{k\pi}{2}-1\right)
\end{aligned}$$

Иймд

$$f(x)=f_1(x)=S(x)=\frac{3\pi}{8}+\sum_{k=1}^{\infty}\left(\frac{2}{\pi k^2}\left(\cos\frac{k\pi}{2}-1\right)\cos kx\right)$$

b. *Өгөдсөн функцийг* $[-\pi;0]$ *завсарт үргэлжлүүлэхэд үүсэх* $f_2(x)$ *функц нь сондгой функц байна гэвэл* $f_2(x)=\begin{cases}-\frac{\pi}{2}, & -\pi\le x<-\frac{\pi}{2}\\ x, & -\frac{\pi}{2}\le x\le\frac{\pi}{2}\\ \frac{\pi}{2}, & \frac{\pi}{2}<x\le\pi\end{cases}$

$$\begin{aligned}
b_k &= \frac{2}{\pi}\int\limits_0^{\pi} f_2(x)\sin kxdx=\frac{2}{\pi}\left(\int\limits_0^{\frac{\pi}{2}} x\sin kxdx+\int\limits_{\frac{\pi}{2}}^{\pi}\frac{\pi}{2}\sin kxdx\right)\\
&= -\frac{2x\cos kx}{k\pi}\Big|_0^{\frac{\pi}{2}}+\frac{2}{k\pi}\int\limits_0^{\frac{\pi}{2}}\cos kxdx-\frac{\cos kx}{k}\Big|_{\frac{\pi}{2}}^{\pi}\\
&= -\frac{1}{k}\cos\frac{k\pi}{2}+\frac{2}{\pi k^2}\sin kx\Big|_0^{\frac{\pi}{2}}-\frac{1}{k}\cos k\pi+\frac{1}{k}\cos\frac{k\pi}{2}=\frac{2}{\pi k^2}\sin\frac{k\pi}{2}-\frac{(-1)^k}{k}
\end{aligned}$$

Иймд

$$f(x) = f_2(x) = S(x) = \sum_{k=1}^{\infty} \left(\frac{2}{\pi k^2} \sin \frac{k\pi}{2} - \frac{(-1)^k}{k} \right) \sin kx$$

Жишээ 0.5. $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$ *функцийг* $(0;2)$ *завсарт*

a. *зөвхөн косинусуудаар*

b. *зөвхөн синусүүдээр Фурьегийн цуваанд задал.*

Бодолт:

a. *Өгөгдсөн функцийг* $[-2;0]$ *завсарт үргэлжлүүлэхэд үүсэх* $f_1(x)$ *функц нь тэгш функц байна гэвэл* $f_1(x) = \begin{cases} 0, & -2 \le x < -1 \\ 1, & -1 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$ *болно.*

$$\begin{aligned} a_0 &= \frac{2}{l} \int_0^l f_1(x)dx = \int_0^2 f_1(x)dx = \int_0^1 1dx + \int_1^2 0 \cdot dx = x|_0^1 = 1 \\ a_k &= \int_0^2 f_1(x) \cos \frac{k\pi x}{2} dx = \int_0^1 \cos \frac{k\pi x}{2} dx = \frac{2}{k\pi} \sin \frac{k\pi x}{2} \bigg|_0^1 = \frac{2}{k\pi} \sin \frac{k\pi}{2} \end{aligned}$$

$$f(x) = f_1(x) = S(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{2}{k\pi} \sin \frac{k\pi}{2} \cos \frac{k\pi x}{2} \right)$$

b. *Өгөгдсөн функцийг* $[-2;0]$ *завсарт үргэлжлүүлэхэд үүсэх* $f_2(x)$ *функц нь сондгой функц байна гэвэл* $f_2(x) = \begin{cases} 0, & -2 \le x < -1 \\ -1, & -1 \le x < 0 \\ 1, & 0 < x \le 1 \\ 0, & 1 < x \le 2 \end{cases}$ *болно.*

$$b_k = \int_0^2 f_2(x) \sin \frac{k\pi x}{2} dx = \int_0^1 \sin \frac{k\pi x}{2} dx = -\frac{2}{k\pi} \cos \frac{k\pi x}{2} \bigg|_0^1 = \frac{2}{k\pi} \left(1 - \cos \frac{k\pi}{2} \right)$$

Иймд

$$f(x) = f_2(x) = S(x) = \sum_{k=1}^{\infty} \left(\frac{2}{k\pi} \left(1 - \cos \frac{k\pi}{2} \right) \sin \frac{k\pi x}{2} \right)$$