7.1 Conformal Mapping

In Section 2.3 we saw that a nonconstant linear mapping acts by rotating, magnifying, and translating points in the complex plane. As a result, the angle between any two intersecting arcs in the z-plane is equal to the angle between the images of the arcs in the w-plane under a linear mapping. Complex mappings that have this angle-preserving property are called **conformal mappings**. In this section we will formally define and discuss conformal mappings. We show that any analytic complex function is conformal at points where the derivative is nonzero. Consequently, all of the elementary functions studied in Chapter 4 are conformal in some domain D. Later in this chapter we will see that conformal mappings have important applications to boundary-value problems involving Laplace's equation.

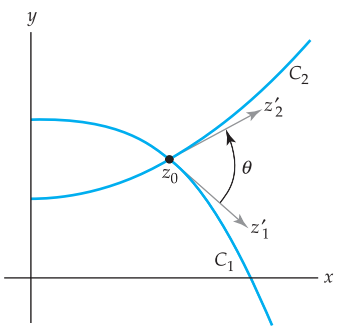
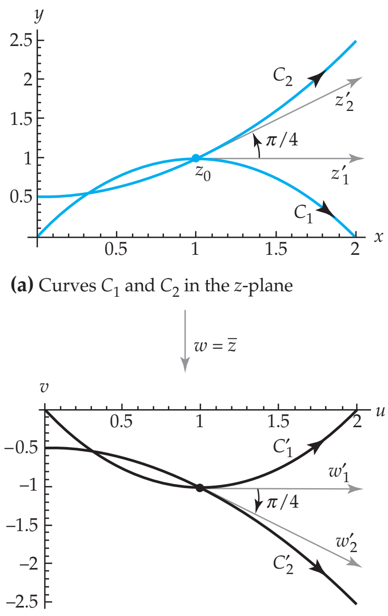


Figure 7.1 The angle θ between C_1 and C_2



(a) Curves C_1 and C_2 in the z-plane

(b) Images of the curves in (a) under $w = \bar{z}$

Figure 7.2 Figure for Example 1

Conformal Mapping Suppose that $w = f(z)$ is a complex mapping defined in a domain D. The mapping is said to be conformal at a point z_0 in D if it "preserves the angle" between any two curves intersecting at z_0. To make this concept precise, assume that C_1 and C_2 are smooth curves in D that intersect at z_0 and have a fixed orientation as described in Section 5.1. Let $z_1(t)$ and $z_2(t)$ be parametrizations of C_1 and C_2 such that $z_1(t_0) = z_2(t_0) = z_0$, and such that the orientations on C_1 and C_2 correspond to the increasing values of the parameter t. Because C_1 and C_2 are smooth, the tangent vectors $z_1' = z_1'(t_0)$ and $z_2' = z_2'(t_0)$ are both nonzero. We define the **angle** between C_1 and C_2 to be the angle θ in the interval $[0, \pi]$ between the tangent vectors z_1' and z_2'. See Figure 7.1. Now suppose that under the complex mapping $w = f(z)$ the curves C_1 and C_2 in the z-plane are mapped onto the curves C_1' and C_2' in the w-plane, respectively. Because C_1 and C_2 intersect at z_0, we must have that C_1' and C_2' intersect at $f(z_0)$. If C_1' and C_2' are smooth, then the angle between C_1' and C_2' at $f(z_0)$ is similarly defined to be the angle ϕ in the interval $[0, \pi]$ between similarly defined tangent vectors w_1' and w_2'. We say that the angles θ and ϕ are **equal in magnitude** if $\theta = \phi$.

In the z-plane, the vector z_1', whose initial point is z_0, can be rotated through the angle θ onto the vector z_2'. This rotation in the z-plane can be either in the counterclockwise or the clockwise direction. Similarly, in the w-plane, the vector w_1', whose initial point is $f(z_0)$, can be rotated in either the counterclockwise or clockwise direction through an angle of ϕ onto the vector w_2'. If the rotation in the z-plane is the same direction as the rotation in the w-plane, we say that the angles θ and ϕ are **equal in sense**. The following example illustrates these concepts.

EXAMPLE 1 Magnitude and Sense of Angles

The smooth curves C_1 and C_2 shown in Figure 7.2(a) are given by $z_1(t) = t + \left(2t - t^2\right) i$ and $z_2(t) = t + \frac{1}{2}\left(t^2 + 1\right) i$, $0 \le t \le 2$, respectively. These curves intersect at the point $z_0 = z_1(1) = z_2(1) = 1 + i$. The tangent vectors at z_0 are $z_1' = z_1'(1) = 1$ and $z_2' = z_2'(1) = 1 + i$. Furthermore, from Figure 7.2(a) we see that the angle between C_1 and C_2 at z_0 is $\theta = \pi/4$. Under the complex mapping $w = \bar{z}$, the images of C_1 and C_2 are the curves C_1' and C_2', respectively, shown in Figure 7.2(b). The image curves are parametrized by $w_1(t) = t - \left(2t - t^2\right) i$ and $w_2(t) = t - \frac{1}{2}\left(t^2 + 1\right) i$, $0 \leq t \leq 2$, and intersect at the point $w_0 = f(z_0) = 1 - i$. In addition, at w_0 we have the tangent vectors $w_1' = w_1'(1) = 1$ and $w_2' = w_2'(1) = 1 - i$ to C_1' and C_2', respectively. Inspection of Figure 7.2(b) indicates that the angle between C_1' and C_2' at w_0 is $\phi = \pi/4$. Therefore, the angles θ and ϕ are equal in magnitude. However, because the rotation through $\pi/4$ of the vector z_1' onto z_2' must be *counterclockwise*, whereas the rotation through $\pi/4$ of w_1' onto w_2' must be *clockwise*, we conclude that θ and ϕ are not equal in sense.

With the terminology regarding the magnitude and sense of an angle established, we are now in a position to give the following precise definition of a conformal mapping.

Definition 7.1 Conformal Mapping

Let $w = f(z)$ be a complex mapping defined in a domain D and let z_0 be a point in D. Then we say that $w = f(z)$ is **conformal** at z_0 if for every pair of smooth oriented curves C_1 and C_2 in D intersecting at z_0 the angle between C_1 and C_2 at z_0 is equal to the angle between the image curves C_1' and C_2' at $f(z_0)$ in both magnitude and sense.

We will also use the term **conformal mapping** to refer to a complex mapping $w = f(z)$ that is conformal at z_0. In addition, if $w = f(z)$ maps a domain D onto a domain D' and if $w = f(z)$ is conformal at every point in D, then we call $w = f(z)$ a conformal mapping of D onto D'. From Section 2.3 it should be intuitively clear that if $f(z) = az + b$ is a linear function with $a \neq 0$, then $w = f(z)$ is conformal at every point in the complex plane. In Example 1 we have shown that the $w = \bar{z}$ is not a conformal mapping at the point $z_0 = 1 + i$ because the angles θ and ϕ are equal in magnitude but not in sense.

Angles between Curves

Definition 7.1 is seldom used directly to show that a complex mapping is conformal. Rather, we will prove in Theorem 7.1 that an analytic function f is a conformal mapping at z whenever $f'(z) \neq 0$. In order to prove this result we need a procedure to determine the angle (in both magnitude and sense) between two smooth curves in the complex plane. For our purposes, the most efficient way to do this is to use the argument of a complex number.

Let us again adopt the notation of Figure 7.1, where C_1 and C_2 are smooth curves parametrized by $z_1(t)$ and $z_2(t)$, respectively, which intersect at $z_1(t_0) = z_2(t_0) = z_0$. The requirement that C_1 is smooth ensures that the tangent vector to C_1 at z_0, given by $z_1' = z_1'(t_0)$, is nonzero, and so $\arg\left(z_1'\right)$ is defined and represents an angle between the position vector z_1' and the positive x-axis. Similarly, the tangent vector to C_2 at z_0, given by $z_2' = z_2'(t_0)$, is nonzero, and $\arg\left(z_2'\right)$ represents an angle between the position

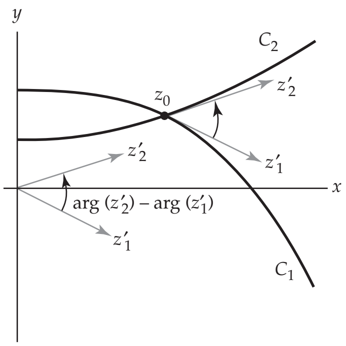


Figure 7.3 The angle between C_1 and C_2

vector z'_2 and the positive x-axis. Inspection of Figure 7.3 shows that the angle θ between C_1 and C_2 at z_0 is the value of

$$\arg(z'_2) - \arg(z'_1) \tag{1}$$

in the interval $[0, \pi]$, provided that we can rotate z'_1 counterclockwise about 0 through the angle θ onto z'_2. In the case that a clockwise rotation is needed, then $-\theta$ is the value of (1) in the interval $(-\pi, 0)$. In either case, we see that (1) gives both the magnitude and sense of the angle between C_1 and C_2 at z_0. As an example of this discussion, consider the curves C_1, C_2, and their images under the complex mapping $w = \bar{z}$ in Example 1. Notice that the unique value of

$$\arg(z'_2) - \arg(z'_1) = \arg(1+i) - \arg(1) = \frac{\pi}{4} + 2n\pi,$$

$n = 0, \pm1, \pm2, \ldots$, that lies in the interval $[0, \pi]$ is $\pi/4$. Therefore, the angle between C_1 and C_2 is $\theta = \pi/4$, and the rotation of z'_1 onto z'_2 is counterclockwise. On the other hand,

$$\arg(w'_2) - \arg(w'_2) = \arg(1-i) - \arg(1) = -\frac{\pi}{4} + 2n\pi,$$

$n = 0, \pm1, \pm2, \ldots$, has no value in $[0, \pi]$, but has the unique value $-\pi/4$ in the interval $(-\pi, 0)$. Thus, the angle between C'_1 and C'_2 is $\phi = \pi/4$, and the rotation of w'_1 onto w'_2 is clockwise.

Analytic Functions

We will now use (1) to prove the following theorem.

Theorem 7.1 Conformal Mapping

If f is an analytic function in a domain D containing z_0, and if $f'(z_0) \neq 0$, then $w = f(z)$ is a conformal mapping at z_0.

Proof Suppose that f is analytic in a domain D containing z_0, and that $f'(z_0) \neq 0$. Let C_1 and C_2 be two smooth curves in D parametrized by $z_1(t)$ and $z_2(t)$, respectively, with $z_1(t_0) = z_2(t_0) = z_0$. In addition, assume that $w = f(z)$ maps the curves C_1 and C_2 onto the curves C'_1 and C'_2. We wish to show that the angle θ between C_1 and C_2 at z_0 is equal to the angle ϕ between C'_1 and C'_2 at $f(z_0)$ in both magnitude and sense. We may assume, by renumbering C_1 and C_2 if necessary, that $z'_1 = z'_1(t_0)$ can be rotated counterclockwise about 0 through the angle θ onto $z'_2 = z'_2(t_0)$. Thus, by (1), the angle θ is the unique value of $\arg(z'_2) - \arg(z'_1)$ in the interval $[0, \pi]$. From (11) of Section 2.2, C'_1 and C'_2 are parametrized by $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$. In order to compute the tangent vectors w'_1 and w'_2 to C'_1 and C'_2 at $f(z_0) = f(z_1(t_0)) = f(z_2(t_0))$ we use the chain rule

$$w'_1 = w'_1(t_0) = f'(z_1(t_0)) \cdot z'_1(t_0) = f'(z_0) \cdot z'_1,$$

and

$$w'_2 = w'_2(t_0) = f'(z_2(t_0)) \cdot z'_2(t_0) = f'(z_0) \cdot z'_2.$$

Since C_1 and C_2 are smooth, both z_1' and z_2' are nonzero. Furthermore, by our hypothesis, we have $f'(z_0) \neq 0$. Therefore, both w_1' and w_2' are nonzero, and the angle ϕ between C_1' and C_2' at $f(z_0)$ is a value of

$$\arg(w_2') - \arg(w_1') = \arg(f'(z_0) \cdot z_2') - \arg(f'(z_0) \cdot z_1').$$

Now by two applications of (8) from Section 1.3 we obtain:

$$\begin{aligned}\arg(f'(z_0) \cdot z_2') - \arg(f'(z_0) \cdot z_1') &= \arg(f'(z_0)) + \arg(z_2') - [\arg(f'(z_0)) + \arg(z_1')] \\ &= \arg(z_2') - \arg(z_1').\end{aligned}$$

This expression has a unique value in $[0, \pi]$, namely θ. Therefore, $\theta = \phi$ in both magnitude and sense, and consequently the $w = f(z)$ is a conformal mapping at z_0. ✎

In light of Theorem 7.1 it is relatively easy to determine where an analytic function is a conformal mapping.

EXAMPLE 2 Conformal Mappings

(**a**) By Theorem 7.1 the entire function $f(z) = e^z$ is conformal at every point in the complex plane since $f'(z) = e^z \neq 0$ for all z in $\mathbf{C}$.

(**b**) By Theorem 7.1 the entire function $g(z) = z^2$ is conformal at all points z, $z \neq 0$, since $g'(z) = 2z \neq 0$.

Critical Points

The function $g(z) = z^2$ in part (b) of Example 2 is not a conformal mapping at $z_0 = 0$. The reason for this is that $g'(0) = 0$. In general, if a complex function f is analytic at a point z_0 and if $f'(z_0) = 0$, then z_0 is called a **critical point** of f. Although it does not follow from Theorem 7.1, it is true that analytic functions are not conformal at critical points. More specifically, we can show that the following magnification of angles occurs at a critical point.

Theorem 7.2 Angle Magnification at a Critical Point

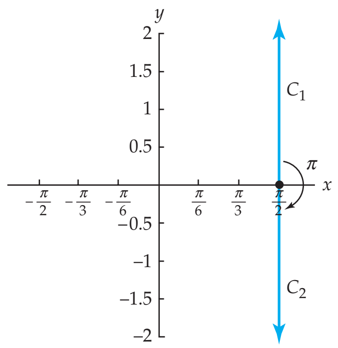
Let f be analytic at the critical point z_0. If $n > 1$ is an integer such that $f'(z_0) = f''(z_0) = \ldots = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$, then the angle between any two smooth curves intersecting at z_0 is increased by a factor of n by the complex mapping $w = f(z)$. In particular, $w = f(z)$ is not a conformal mapping at z_0.

A proof of Theorem 7.2 is sketched in Problem 22 of Exercises 7.1.

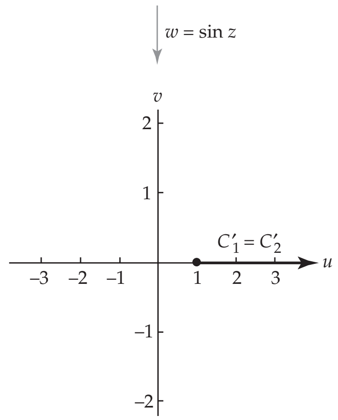
EXAMPLE 3 Conformal Mappings

Find all points where the mapping $f(z) = \sin z$ is conformal.

Solution The function $f(z) = \sin z$ is entire, and from Section 4.3 we have that $f'(z) = \cos z$. In (21) of Section 4.3 we found that $\cos z = 0$ if and only if $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$, and so each of these points is a critical point of f. Therefore, by Theorem 7.1, $w = \sin z$ is a conformal mapping at z for all $z \neq (2n+1)\pi/2,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$. Furthermore, by Theorem 7.2, $w = \sin z$ is not a conformal mapping at z if $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$. Because $f''(z) = -\sin z = \pm 1$ at the critical points of f, Theorem 7.2 also indicates that angles at these points are increased by a factor of 2.



(a) The angle between the vertical rays in the z-plane is π



(b) The angle between the images of the rays in (a) is 2π or 0

Figure 7.4 The mapping $w = \sin z$

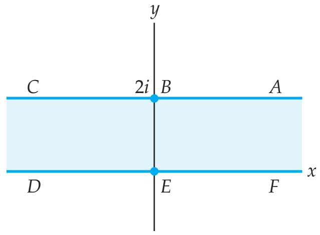
The angle magnification at a critical point of the complex mapping $w = \sin z$ in Example 3 can be seen directly. For example, consider the critical point $z = \pi/2$. Under $w = \sin z$, the vertical ray C_1 in the z-plane emanating from $z = \pi/2$ and given by $z = \pi/2 + iy$, $y \geq 0$, is mapped onto the set in the w-plane given by $w = \sin(\pi/2)\cosh y + i\cos(\pi/2)\sinh y$, $y \geq 0$. Because $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, the image can be rewritten as $w = \cosh y$, $y \geq 0$. In words, the image C_1' is a ray in the w-plane emanating from $w = 1$ and containing the point $w = 2$. A similar analysis reveals that the image C_2' of the vertical ray C_2 given by $z = \pi/2 + iy$, $y \leq 0$, is also the ray emanating from $w = 1$ and containing the point $w = 2$. That is, $C_1' = C_2'$. The angle between the rays C_1 and C_2 in the z-plane is π, and so Theorem 7.2 implies that the angle between their images in the w-plane is increased to 2π, or, equivalently, 0. This agrees with the observation that $C_1' = C_2'$. See Figure 7.4.

Conformal Mappings Using Tables

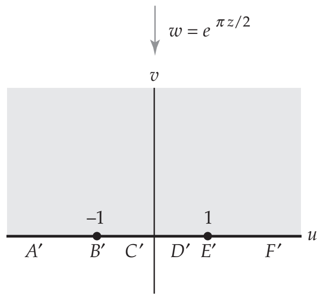
In Section 4.5 we introduced a method of solving a particular type of boundary-value problem using complex mappings. Specifically, we saw that a Dirichlet problem in a complicated domain D can be solved by finding an *analytic* mapping of D onto a simpler domain D' in which the associated Dirichlet problem has already been solved. At the end of this chapter we will see a similar application of *conformal* mappings to a generalized type of Dirichlet problem. In these applications our method for producing a solution in a domain D will first require that we find a *conformal* mapping of D onto a simpler domain D' in which the associated boundary-value problem has a solution. An important aid in this task is the table of conformal mappings given in Appendix III.

The mappings in Appendix III have been categorized as elementary mappings (E-1 to E-9), mappings of half-planes (H-1 to H-6), mappings onto circular regions (C-1 to C-5), and miscellaneous mappings (M-1 to M-10). Many properties of the mappings appearing in this table have been derived in Chapter 2 and Chapter 4, whereas other properties will be derived in the coming sections. When using the table, bear in mind that in some cases the desired mapping will appear as a single entry in the table, whereas in other cases one or more successive mappings from the table may be required. You

Note ☞ should also note that the mappings in Appendix III are, in general, only con-



(a) The horizontal strip $0 \le y \le 2$



(b) Image of the strip in (a)

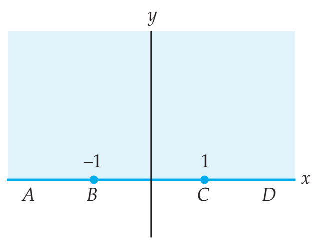
Figure 7.5 Figure for Example 4

formal mappings of the interiors of the regions shown. For example, it is clear that the complex mapping shown in Entry E-4 is not conformal at $B = 0$. As a general rule, when we refer to a conformal mapping of a region R onto a region R' we are requiring only that the mapping be conformal at the points in the interior of R.

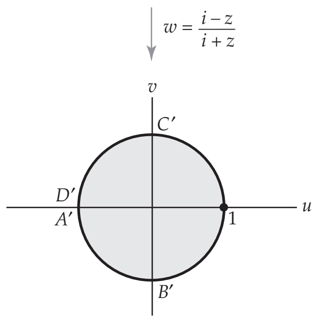
EXAMPLE 4 Using a Table of Conformal Mappings

Use Appendix III to find a conformal mapping from the infinite horizontal strip $0 \le y \le 2$, $-\infty < x < \infty$, onto the upper half-plane $v \ge 0$. Under this mapping, what is the image of the negative x-axis?

Solution Entry H-2 in Appendix III gives a mapping from an infinite horizontal strip onto the upper half-plane. Setting $a = 2$, we obtain the desired mapping $w = e^{\pi z/2}$. From H-2 we also see that the points labeled D and $E = 0$ on the negative x-axis in the z-plane are mapped onto the points D' and $E' = 1$ on the positive u-axis in the w-plane. Noting the relative positions of these points, we conclude that the negative x-axis is mapped onto the interval $(0,\ 1]$ in the u-axis by $w = e^{\pi z/2}$. See Figure 7.5. This observation can also be verified using parametrizations.



(a) Image of the strip $0 \le y \le 2$ under $w = e^{\pi z/2}$



(b) Image of the half-plane in (a) under $w = \dfrac{i - z}{i + z}$

Figure 7.6 Figure for Example 5

EXAMPLE 5 Using a Table of Conformal Mappings

Use Appendix III to find a conformal mapping from the infinite horizontal strip $0 \le y \le 2$, $-\infty < x < \infty$, onto the unit disk $|w| \le 1$. Under this mapping, what is the image of the negative x-axis?

Solution Appendix III does not have an entry that maps an infinite horizontal strip onto the unit disk. Therefore, we construct the a conformal mapping that does this by composing two mappings in the table. In Example 4 we found that the infinite horizontal strip $0 \le y \le 2$, $-\infty < x < \infty$, is mapped onto the upper half-plane by $f(z) = e^{\pi z/2}$. In addition, from entry C-4 we see that the upper half-plane is mapped onto the unit disk by $g(z) = \dfrac{i - z}{i + z}$. The composition of these two functions

$$w = g(f(z)) = \frac{i - e^{\pi z/2}}{i + e^{\pi z/2}}$$

therefore maps the strip $0 \le y \le 2$, $-\infty < x < \infty$, onto the unit disk $|w| \le 1$. Under the first of these successive mappings, the negative real axis is mapped onto the interval $(0,\ 1]$ in the real axis as was noted in Example 4. Inspection of entry C-4 (or Figure 7.6) reveals that the interval from 0 to $C = 1$ is mapped onto the circular arc from 1 to $C' = i$ on the unit circle $|w| = 1$. Therefore, we conclude that the negative real axis is mapped onto the circular arc from 1 to i on the unit circle under $w = \dfrac{i - e^{\pi z/2}}{i + e^{\pi z/2}}$.

Remarks

In the foregoing discussion regarding conformal mappings using tables we alluded to the fact that in many applications one needs to find a conformal mapping of a domain D onto a simpler domain D'. A natural question to ask is whether such a mapping always exists. That is, given domains D and D', does there exist a conformal mapping of D onto D'? An answer to this question was given by the mathematician Bernhard Riemann (1826–1866). Although there was a gap in Riemann's original proof (which was subsequently filled), this amazing theorem still bears his name.

> ***The Riemann Mapping Theorem*** *Let D be a simply connected domain in the z-plane such that D is not all of* $\mathbf{C}$. *Then there exists a one-to-one conformal mapping $w = f(z)$ from D onto the open unit disk $|w| < 1$ in the w-plane.*

It is not immediately clear that this theorem answers our question of the existence of a mapping from D onto D'. To see that it does, we first use the theorem to find a conformal mapping f from D onto the open unit disk $|w| < 1$. We then apply the theorem a second time to obtain a mapping g from D' onto the open unit disk $|w| < 1$. Since the theorem ensures that g is one-to-one, it has a well defined inverse function g^{-1} that maps the open unit disk onto D'. The desired mapping from D onto D' is then given by the composition $w = g^{-1} \circ f(z)$.

Riemann's theorem is of critical theoretical importance, but its proof is not *constructive*. This means that the theorem establishes the *existence* of the mapping f but offers no method of actually finding a formula for f. A proof of the Riemann mapping theorem is well beyond the scope of this text. The interested reader is encouraged to refer to the text *Complex Analysis* by Lars V. Alfors, McGraw-Hill, 1979.

EXERCISES 7.1

Answers to selected odd-numbered problems begin on page ANS-21.

In Problems 1–6, determine where the complex mapping $w = f(z)$ is conformal.

1. $f(z) = z^3 - 3z + 1$

2. $f(z) = z^2 + 2iz - 3$

3. $f(z) = z - e^{-z} + 1 - i$

4. $f(z) = ze^{z^2-2}$

5. $f(z) = \tan z$

6. $f(z) = z - \mathrm{Ln}(z + i)$

In Problems 7–10, proceed as in Example 1 to show that the given function f is not conformal at the indicated point.

7. $f(z) = (z - i)^3$; $z_0 = i$

8. $f(z) = (iz - 3)^2$; $z_0 = -3i$

9. $f(z) = e^{z^2}$; $z_0 = 0$

10. the principle square root function $f(z) = z^{1/2}$; $z_0 = 0$

In Problems 11–16, use Appendix III to find a conformal mapping of the region R shown in color onto the region R' shown in gray. Then find the image of the curve from A to B.

11.

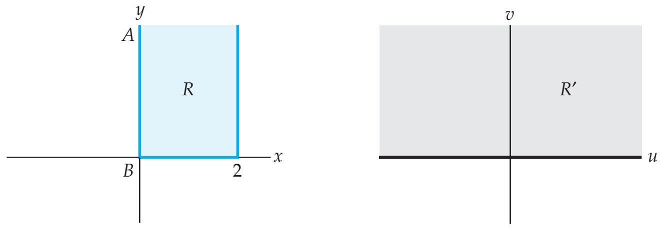


Figure 7.7 Figure for Problem 11

12.

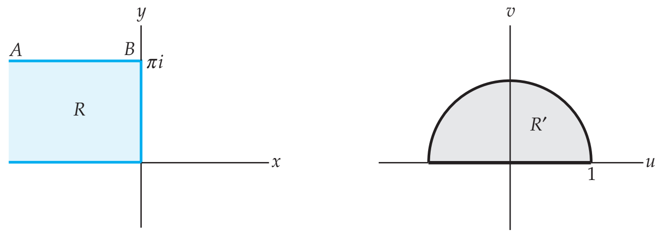


Figure 7.8 Figure for Problem 12

13.

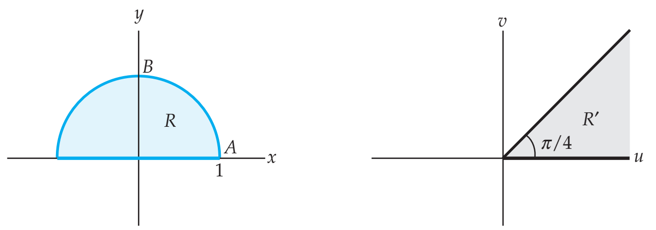


Figure 7.9 Figure for Problem 13

14.

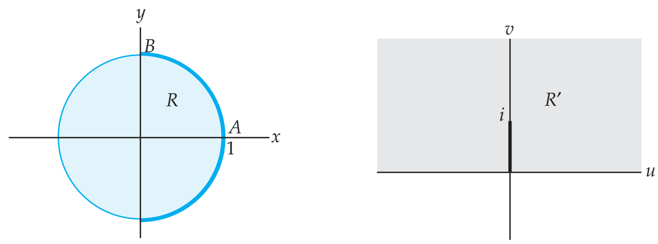


Figure 7.10 Figure for Problem 14

15.

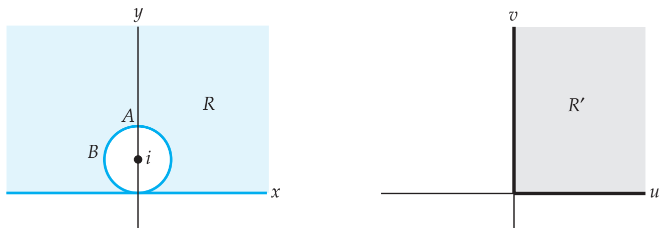


Figure 7.11 Figure for Problem 15

16.

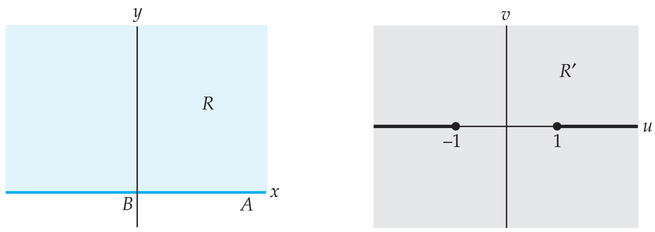


Figure 7.12 Figure for Problem 16

Focus on Concepts

17. Where is the mapping $w = \bar{z}$ conformal? Justify your answer.

18. Suppose $w = f(z)$ is a conformal mapping at every point in the complex plane. Where is the mapping $w = \overline{f(\bar{z})}$ conformal? Justify your answer.

19. Suppose that $w = f(z)$ is a conformal mapping at every point in the complex plane. Where is the mapping $w = e^{f(z)}$ conformal?

20. This problem concerns determining the angle between two curves C_1 and C_2 at a point where one (or both) of the curves has a zero tangent vector.

(**a**) Assume that two curves C_1 and C_2 are parametrized by $z_1(t)$ and $z_2(t)$, respectively, and that the curves intersect at $z_1(t_0) = z_2(t_0) = z_0$. Assume further that both z_1 and z_2 are differentiable functions of t, and let $z_1' = z_1'(t_0)$ and $z_2' = z_2'(t_0)$. Explain why $\arg(z_2') - \arg(z_1')$ *does not* represent the angle between C_1 and C_2 if either z_1' or z_2' is zero.

(**b**) Explain why $\lim_{t\to t_0}[\arg(z_2(t) - z_0)] - \lim_{t\to t_0}[\arg(z_1(t) - z_0)]$ does represent the angle between C_1 and C_2 regardless of whether z_1' or z_2' is zero.

(**c**) Use part (b) to determine the angle between the curves parametrized by $z_1(t) = t + it^2$ and $z_2(t) = t^2 + it^2$, $-1 \le t \le 1$, at $z_0 = 0$. Does this computation match your intuition?

21. On page 393 we showed that the function $f(z) = z^2$ was not conformal at $z_0 = 0$ because the angle between the positive x- and y-axes was doubled. In this problem you will show that *every* pair of smooth curves intersecting at $z_0 = 0$ has the angle between them doubled by $f(z) = z^2$. This is a very specific case of Theorem 7.2.

(**a**) Suppose that the smooth curves C_1 and C_2 are parametrized by $z_1(t)$ and $z_2(t)$ with $z_1(t_0) = z_2(t_0) = 0$. If $z_1' = z_1'(t_0)$ and $z_2' = z_2'(t_0)$ are both nonzero, then the angle θ between C_1 and C_2 is given by (1). Explain why $\phi = \arg(f'(0) \cdot z_2') - \arg(f'(0) \cdot z_1')$ *does not* represent the angle between the images C_1' and C_2' of C_1 and C_2 under the mapping $w = f(z) = z^2$, respectively.

(**b**) Use Problem 20 to write down an expression involving arguments that does represent the angle ϕ between C_1' and C_2'. [*Hint*: C_1' and C_2' are parametrized by $w_1(t) = f(z_1(t)) = [z_1(t)]^2$ and $w_2(t) = f(z_2(t)) = [z_2(t)]^2$.]

(**c**) Use (8) of Section 1.3 to show that your expression for ϕ from (b) is equal to 2θ.

22. In this problem you will prove Theorem 7.2. Let f be an analytic function at the point z_0 such that $f'(z_0) = f''(z_0) = ... = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$ for some $n > 1$.

(**a**) Explain why f can be written as

$$f(z) = f(z_0) + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n (1 + g(z)),$$

where g is an analytic function at z_0 and $g(z_0) = 0$.

(**b**) Use (a) and Problem 20 to show that the angle between two smooth curves intersecting at z_0 is increased by a factor of n by the mapping $w = f(z)$.

7.2 Linear Fractional Transformations

In many applications that involve boundary-value problems associated with Laplace's equation, it is necessary to find a conformal mapping that maps a disk onto the half-plane $v \geq 0$. Such a mapping would have to map the circular boundary of the disk to the boundary line of the half-plane. An important class of elementary conformal mappings that map circles to lines (and vice versa) are the linear fractional transformations. In this section we will define and study this special class of mappings.

Linear Fractional Transformations In Section 2.3 we examined complex linear mappings $w = az + b$ where a and b are complex constants and $a \neq 0$. Recall that such mappings act by rotating, magnifying, and translating points in the complex plane. We then discussed the complex reciprocal mapping $w = 1/z$ in Section 2.5. An important property of the reciprocal mapping, when defined on the extended complex plane, is that it maps certain lines to circles and certain circles to lines. A more general type of mapping that has similar properties is a linear fractional transformation defined next.

Definition 7.2 Linear Fractional Transformation

If a, b, c, and d are complex constants with $ad-bc \neq 0$, then the complex function defined by:

$$T(z) = \frac{az+b}{cz+d} \tag{1}$$

is called a **linear fractional transformation**.

Linear fractional transformations are also called **Möbius transformations** or **bilinear transformations**. If $c = 0$, then the transformation T given by (1) is a linear mapping, and so a linear mapping is a special case of a linear fractional transformation. If $c \neq 0$, then we can write

$$T(z) = \frac{az+b}{cz+d} = \frac{bc-ad}{c}\frac{1}{cz+d} + \frac{a}{c}. \tag{2}$$

Setting $A = \dfrac{bc-ad}{c}$ and $B = \dfrac{a}{c}$, we see that the linear transformation T in (2) can be written as the composition $T(z) = f \circ g \circ h(z)$, where $f(z) = Az+B$ and $h(z) = cz+d$ are linear functions and $g(z) = 1/z$ is the reciprocal function.

The domain of a linear fractional transformation T given by (1) is the set of all complex z such that $z \neq -d/c$. Furthermore, since

$$T'(z) = \frac{ad-bc}{(cz+d)^2}$$

it follows from Theorem 7.1 and the requirement that $ad-bc \neq 0$ that linear fractional transformations are conformal on their domains. The requirement that $ad-bc \neq 0$ also ensures that the T is a one-to-one function on its domain. See Problem 27 in Exercises 7.2.

Observe that if $c \neq 0$, then (1) can be written as

$$T(z) = \frac{az+b}{cz+d} = \frac{(a/c)\,(z+b/a)}{z+d/c} = \frac{\phi(z)}{z-(-d/c)},$$

where $\phi(z) = (a/c)\,(z+b/a)$. Because $ad-bc \neq 0$, we have that $\phi\,(-d/c) \neq 0$, and so from Theorem 6.12 of Section 6.4 it follows that the point $z = -d/c$ is a simple pole of T.

When $c \neq 0$, that is, when T is not a linear function, it is often helpful to view T as a mapping of the extended complex plane. Since T is defined for all points in the extended plane except the pole $z = -d/c$ and the ideal point ∞, we need only extend the definition of T to include these points. We make this definition by considering the limit of T as z tends to the pole and as z tends to the ideal point. Because

$$\lim_{z\to -d/c} \frac{cz+d}{az+b} = \frac{0}{a\,(-d/c)+b} = \frac{0}{-ad+bc} = 0,$$

it follows from (25) of Section 2.6 that

$$\lim_{z\to -d/c} \frac{az+b}{cz+d} = \infty.$$

Moreover, from (24) of Section 2.6 we have that

$$\lim_{z\to\infty}\frac{az+b}{cz+d}=\lim_{z\to 0}\frac{a/z+b}{c/z+d}=\lim_{z\to 0}\frac{a+zb}{c+zd}=\frac{a}{c}.$$

The values of these two limits indicate how to extend the definition of T. In particular, if $c\neq 0$, then we regard T as a one-to-one mapping of the extended complex plane defined by:

$$T(z)=\begin{cases}\dfrac{az+b}{cz+d}, & z\neq -\dfrac{d}{c}, z\neq\infty\\ \infty, & z=-\dfrac{d}{c}\\ \dfrac{a}{c}, & z=\infty\,.\end{cases}\tag{3}$$

A special case of (3) corresponding to $a=0$, $b=1$, $c=1$, and $d=0$ is the reciprocal function defined on the extended complex plane. Refer to Definition 2.7.

EXAMPLE 1 A Linear Fractional Transformation

Find the images of the points 0, $1+i$, i, and ∞ under the linear fractional transformation $T(z)=(2z+1)/(z-i)$.

Solution For $z=0$ and $z=1+i$ we have:

$$T(0)=\frac{2(0)+1}{0-i}=\frac{1}{-i}=i\quad\text{and}\quad T(1+i)=\frac{2(1+i)+1}{(1+i)-i}=\frac{3+2i}{1}=3+2i.$$

Identifying $a=2$, $b=1$, $c=1$, and $d=-i$ in (3), we also have:

$$T(i)=T\left(-\frac{d}{c}\right)=\infty\quad\text{and}\quad T(\infty)=\frac{a}{c}=2.$$

Circle-Preserving Property In the discussion preceding Example 1 we indicated that the reciprocal function $1/z$ is a special case of a linear fractional transformation. We saw two interesting properties of the reciprocal mapping in Section 2.7. First, the image of a circle centered at the pole $z=0$ of $1/z$ is a circle, and second, the image of a circle with center on the x- or y-axis and containing the pole $z=0$ is a vertical or horizontal line. Linear fractional transformations have a similar mapping property. This is the content of the following theorem.

Theorem 7.3 Circle-Preserving Property

If C is a circle in the z-plane and if T is a linear fractional transformation given by (3), then the image of C under T is either a circle or a line in the extended w-plane. The image is a line if and only if $c \neq 0$ and the pole $z = -d/c$ is on the circle C.

Proof When $c = 0$, T is a linear function, and we saw in Section 2.3 that linear functions map circles onto circles. It remains to be seen that the theorem still holds for $c \neq 0$. Assume then that $c \neq 0$. From (2) we have that $T(z) = f \circ g \circ h(z)$, where $f(z) = Az + B$ and $h(z) = cz + d$ are linear functions and $g(z) = 1/z$ is the reciprocal function. Observe that since h is a linear mapping, the image C' of the circle C under h is a circle. We now examine two cases:

Case 1 Assume that the origin $w = 0$ is on the circle C'. This occurs if and only if the pole $z = -d/c$ is on the circle C. From the Remarks in Section 2.5, if $w = 0$ is on C', then the image of C' under $g(z) = 1/z$ is either a horizontal or vertical line L. Furthermore, because f is a linear function, the image of the line L under f is also a line. Thus, we have shown that if the pole $z = -d/c$ is on the circle C, then the image of C under T is a line.

Case 2 Assume that the point $w = 0$ is not on C'. That is, the pole $z = -d/c$ is not on the circle C. Let C' be the circle given by $|w - w_0| = \rho$. If we set $\xi = f(w) = 1/w$ and $\xi_0 = f(w_0) = 1/w_0$, then for any point w on C' we have

$$|\xi - \xi_0| = \left|\frac{1}{w} - \frac{1}{w_0}\right| = \frac{|w - w_0|}{|w|\;|w_0|} = \rho|\xi_0||\xi|. \tag{4}$$

It can be shown that the set of points satisfying the equation

$$|\xi - a| = \lambda|\xi - b| \tag{5}$$

is a line if $\lambda = 1$ and is a circle if $\lambda > 0$ and $\lambda \neq 1$. See Problem 28 in Exercises 7.2. Thus, with the identifications $a = \xi_0$, $b = 0$, and $\lambda = \rho|\xi_0|$ we see that (4) can be put into the form (5). Since $w = 0$ is not on C', we have $|w_0| \neq \rho$, or, equivalently, $\lambda = \rho|\xi_0| \neq 1$. This implies that the set of points given by (4) is a circle. Finally, since f is a linear function, the image of this circle under f is again a circle, and so we conclude that the image of C under T is a circle. ✎

The key observation in the foregoing proof was that a linear fractional transformation (1) can be written as a composition of the reciprocal function and two linear functions as shown in (2). In Problem 27 of Exercises 2.5 you were asked to show that the image of any line L under the reciprocal mapping

$w = 1/z$ is a line or a circle. Therefore, using similar reasoning, we can also show:

Mapping Lines to Circles with $T(z)$

If T is a linear fractional transformation given by (3), then the image of a line L under T is either a line or a circle. The image is a circle if and only if $c \neq 0$ and the pole $z = -d/c$ is not on the line L.

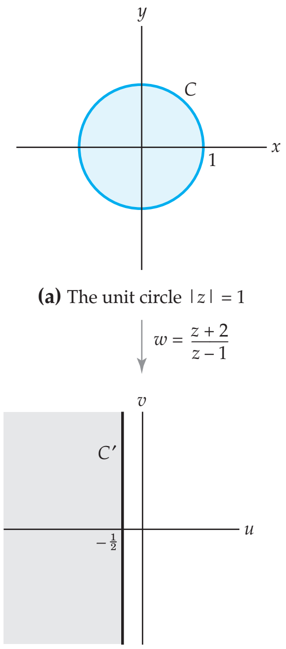


Figure 7.13 The linear fractional transformation $T(z) = (z+2)/(z-1)$

EXAMPLE 2 Image of a Circle

Find the image of the unit circle $|z| = 1$ under the linear fractional transformation $T(z) = (z+2)/(z-1)$. What is the image of the interior $|z| < 1$ of this circle?

Solution The pole of T is $z = 1$ and this point is on the unit circle $|z| = 1$. Thus, from Theorem 7.3 we conclude that the image of the unit circle is a line. Since the image is a line, it is determined by any two points. Because $T(-1) = -\frac{1}{2}$ and $T(i) = -\frac{1}{2} - \frac{3}{2}i$, we see that the image is the line $u = -\frac{1}{2}$. To answer the second question we first note that a linear fractional transformation is a rational function, and so it is continuous on its domain. As a consequence, the image of the interior $|z| < 1$ of the unit circle is either the half-plane $u < -\frac{1}{2}$ or the half-plane $u > -\frac{1}{2}$. Using $z = 0$ as a **test point**, we find that $T(0) = -2$, which is to the left of the line $u = -\frac{1}{2}$, and so the image is the half-plane $u < -\frac{1}{2}$. This mapping is illustrated in Figure 7.13. The circle $|z| = 1$ is shown in color in Figure 7.13(a) and its image $u = -\frac{1}{2}$ is shown in black in Figure 7.13(b).

EXAMPLE 3 Image of a Circle

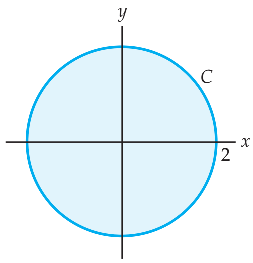
Find the image of the unit circle $|z| = 2$ under the linear fractional transformation $T(z) = (z+2)/(z-1)$. What is the image of the disk $|z| \leq 2$ under T?

Solution In this example the pole $z = 1$ does not lie on the circle $|z| = 2$, and so Theorem 7.3 indicates that the image of $|z| = 2$ is a circle C'. To find an algebraic description of C', we first note that the circle $|z| = 2$ is symmetric with respect to the x-axis. That is, if z is on the circle $|z| = 2$, then so is $\bar{z}$. Furthermore, we observe that for all z,

$$T(\bar{z}) = \frac{\bar{z}+2}{\bar{z}-1} = \frac{\overline{z+2}}{\overline{z-1}} = \overline{\left(\frac{z+2}{z-1}\right)} = \overline{T(z)}.$$

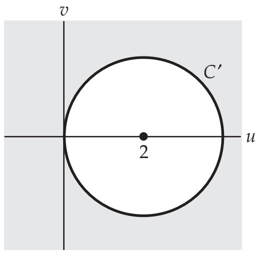
Hence, if z and $\bar{z}$ are on the circle $|z| = 2$, then we must have that both $w = T(z)$ and $\bar{w} = \overline{T(z)} = T(\bar{z})$ are on the circle C'. It follows that C'

is symmetric with respect to the u-axis. Since $z = 2$ and -2 are on the circle $|z| = 2$, the two points $T(2) = 4$ and $T(-2) = 0$ are on C'. The symmetry of C' implies that 0 and 4 are endpoints of a diameter, and so C' is the circle $|w - 2| = 2$. Using $z = 0$ as a test point, we find that $w = T(0) = -2$, which is outside the circle $|w - 2| = 2$. Therefore, the image of the interior of the circle $|z| = 2$ is the exterior of the circle $|w - 2| = 2$. In summary, the disk $|z| \leq 2$ shown in color in Figure 7.14(a) is mapped onto the region $|w - 2| \geq 2$ shown in gray in Figure 7.14(b) by the linear fractional transformation $T(z) = (z + 2)/(z - 1)$.



(a) The circle $|z| = 2$

$w = \dfrac{z+2}{z-1}$



(b) The image of the circle in (a)

Figure 7.14 The linear fractional transformation $T(z) = (z + 2)/(z - 1)$

Linear Fractional Transformations as Matrices

Matrices can be used to simplify many of the computations associated with linear fractional transformations. In order to do so, we associate the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6}$$

with the linear fractional transformation

$$T(z) = \frac{az + b}{cz + d} \tag{7}$$

The assignment in (6) is not unique because if e is a nonzero complex number, then the linear fractional transformation $T(z) = (az + b)/(cz + d)$ is also given by $T(z) = (eaz + eb)/(ecz + ed)$. However, if $e \neq 1$, then the two matrices

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} ea & eb \\ ec & ed \end{pmatrix} = e\mathbf{A} \tag{8}$$

are not equal even though they represent the same linear fractional transformation.

It is easy to verify that the composition $T_2 \circ T_1$ of two linear fractional transformations

$$T_1(z) = (a_1 z + b_1)/(c_1 z + d_1) \quad \text{and} \quad T_2(z) = (a_2 z + b_2)/(c_2 z + d_2)$$

is represented by the product of matrices

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{pmatrix}. \tag{9}$$

In Problem 27 of Exercises 7.2 you are asked to find the formula for $T^{-1}(z)$ by solving the equation $w = T(z)$ for z. The formula for the inverse function $T^{-1}(z)$ of a linear fractional transformation T of (7) is represented by the inverse of the matrix $\mathbf{A}$ in (6)

$$\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

By identifying $e = \dfrac{1}{ad - bc}$ in (8) we can also represent $T^{-1}(z)$ by the matrix

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.^{*} \tag{10}$$

EXAMPLE 4 Using Matrices

Suppose $S(z) = (z - i)/(iz - 1)$ and $T(z) = (2z - 1)/(z + 2)$. Use matrices to find $S^{-1}(T(z))$.

Solution We represent the linear fractional transformations S and T by the matrices

$$\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix},$$

respectively. By (10), the transformation S^{-1} is given by

$$\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix},$$

and so, from (9), the composition $S^{-1} \circ T$ is given by

$$\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 + i & 1 + 2i \\ 1 - 2i & 2 + i \end{pmatrix}$$

Therefore,

$$S^{-1}(T(z)) = \frac{(-2 + i)\, z + 1 + 2i}{(1 - 2i)\, z + 2 + i}.$$

Cross-Ratio In applications we often need to find a conformal mapping from a domain D that is bounded by circles onto a domain D' that is bounded by lines. Linear fractional transformations are particularly well-suited for such applications. However, in order to use them, we must determine a general

*You may recall that this matrix is called the **adjoint matrix** of **A**.

Recall, a circle is uniquely determined by three noncolinear points.

method to construct a linear fractional transformation $w = T(z)$, which maps three given distinct points z_1, z_2, and z_3 on the boundary of D to three given distinct points w_1, w_2, and w_3 on the boundary of D'. This is accomplished using the **cross-ratio**, which is defined as follows.

Definition 7.3 Cross-Ratio

The **cross-ratio** of the complex numbers z, z_1, z_2, and z_3 is the complex number

$$\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}. \tag{11}$$

When computing a cross-ratio, we must be careful with the order of the complex numbers. For example, you should verify that the cross-ratio of 0, 1, i, and 2 is $\frac{3}{4}+\frac{1}{4}i$, whereas the cross-ratio of 0, i, 1, and 2 is $\frac{1}{4}-\frac{1}{4}i$.

We extend the concept of the cross-ratio to include points in the extended complex plane by using the limit formula (24) from the Remarks in Section 2.6. For example, the cross-ratio of, say, ∞, z_1, z_2, and z_3 is given by the limit

$$\lim_{z\to\infty}\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}.$$

The following theorem illustrates the importance of cross-ratios in the study of linear fractional transformations. In particular, we prove that the cross-ratio is invariant under a linear fractional transformation.

Theorem 7.4 Cross-Ratios and Linear Fractional Transformations

If $w = T(z)$ is a linear fractional transformation that maps the distinct points z_1, z_2, and z_3 onto the distinct points w_1, w_2, and w_3, respectively, then

$$\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}=\frac{w-w_1}{w-w_3}\frac{w_2-w_3}{w_2-w_1} \tag{12}$$

for all z.

Proof Let R be the linear fractional transformation

$$R(z)=\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}, \tag{13}$$

and note that $R(z_1)=0$, $R(z_2)=1$, and $R(z_3)=\infty$. Consider also the linear fractional transformation

$$S(z)=\frac{z-w_1}{z-w_3}\frac{w_2-w_3}{w_2-w_1}. \tag{14}$$

For the transformation S, we have $S(w_1) = 0$, $S(w_2) = 1$, and $S(w_3) = \infty$. Therefore, the points z_1, z_2, and z_3 are mapped onto the points w_1, w_2, and w_3, respectively, by the linear fractional transformation $S^{-1}(R(z))$. From this it follows that 0, 1, and ∞ are mapped onto 0, 1, and ∞, respectively, by the composition $T^{-1}(S^{-1}(R(z)))$. Now it is a straightforward exercise to verify that the only linear fractional transformation that maps 0, 1, and ∞ onto 0, 1, and ∞ is the identity mapping. See Problem 30 in Exercises 7.2. From this we conclude that $T^{-1}(S^{-1}(R(z))) = z$, or, equivalently, that $R(z) = S(T(z))$. Identifying $w = T(z)$, we have shown that $R(z) = S(w)$. Therefore, from (13) and (14) we have

$$\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1} = \frac{w-w_1}{w-w_3}\frac{w_2-w_3}{w_2-w_1}.$$

✎

EXAMPLE 5 Constructing a Linear Fractional Transformation

Construct a linear fractional transformation that maps the points 1, i, and -1 on the unit circle $|z| = 1$ onto the points -1, 0, 1 on the real axis. Determine the image of the interior $|z| < 1$ under this transformation.

Solution Identifying $z_1 = 1$, $z_2 = i$, $z_3 = -1$, $w_1 = -1$, $w_2 = 0$, and $w_3 = 1$, in (12) we see from Theorem 7.4 that the desired mapping $w = T(z)$ must satisfy

$$\frac{z-1}{z-(-1)}\frac{i-(-1)}{i-1} = \frac{w-(-1)}{w-1}\frac{0-1}{0-(-1)}.$$

After solving for w and simplifying we obtain

Note: A linear fractional transformation can have many equivalent forms. ☞

$$w = T(z) = \frac{z-i}{iz-1}.$$

Using the test point $z = 0$, we obtain $T(0) = i$. Therefore, the image of the interior $|z| < 1$ is the upper half-plane $v > 0$.

EXAMPLE 6 Constructing a Linear Fractional Transformation

Construct a linear fractional transformation that maps the points $-i$, 1, and ∞ on the line $y = x-1$ onto the points 1, i, and -1 on the unit circle $|w| = 1$.

Solution We proceed as in Example 5. Using (24) of Section 2.6, we find that the cross-ratio of z, $z_1 = -i$, $z_2 = 1$, and $z_3 = \infty$ is

$$\lim_{z_3\to\infty}\frac{z+i}{z-z_3}\frac{1-z_3}{1+i} = \lim_{z_3\to 0}\frac{z+i}{z-1/z_3}\frac{1-1/z_3}{1+i} = \lim_{z_3\to 0}\frac{z+i}{zz_3-1}\frac{z_3-1}{1+i} = \frac{z+i}{1+i}.$$

Now from (12) of Theorem 7.4 with $w_1 = 1$, $w_2 = i$, and $w_3 = -1$, the desired mapping $w = T(z)$ must satisfy

$$\frac{z+i}{1+i} = \frac{w-1}{w+1}\frac{i+1}{i-1}.$$

After solving for w and simplifying we obtain

$$w = T(z) = \frac{z+1}{-z+1-2i}.$$

EXERCISES 7.2

Answers to selected odd-numbered problems begin on page ANS-22.

In Problems 1–4, find the images of the points 0, 1, i, and ∞ under the given linear fractional transformation T.

1. $T(z) = \dfrac{i}{z}$

2. $T(z) = \dfrac{2}{z-i}$

3. $T(z) = \dfrac{z+i}{z-i}$

4. $T(z) = \dfrac{z-1}{z}$

In Problems 5–8, find the image of the disks $|z| \le 1$ and $|z - i| \le 1$ under the given linear fractional transformation T.

5. T is the mapping in Problem 1

6. T is the mapping in Problem 2

7. T is the mapping in Problem 3

8. T is the mapping in Problem 4

In Problems 9–12, find the image of the half-planes $x \ge 0$ and $y \le 1$ under the given linear fractional transformation T.

9. T is the mapping in Problem 1

10. T is the mapping in Problem 2

11. T is the mapping in Problem 3

12. T is the mapping in Problem 4

In Problems 13–16, find the image of the region shown in color under the given linear fractional transformation.

13. $T(z) = \dfrac{z}{z-2}$

14. $T(z) = \dfrac{z-i}{z+1}$

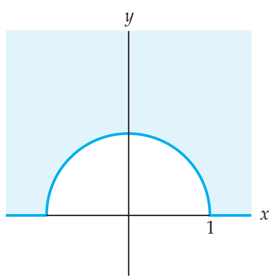


Figure 7.15 Figure for Problem 13

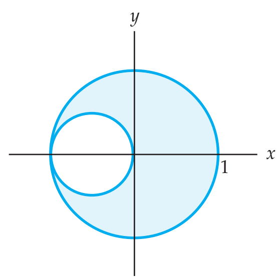


Figure 7.16 Figure for Problem 14

15. $T(z) = \dfrac{z+1}{z-2}$

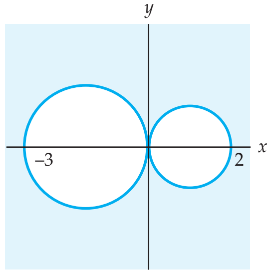


Figure 7.17 Figure for Problem 15

16. $T(z) = \dfrac{-z-1+i}{z-1+i}$

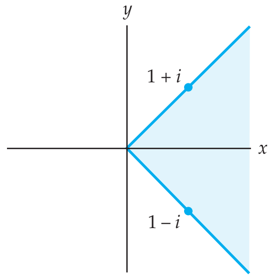


Figure 7.18 Figure for Problem 16

In Problems 17–20, use matrices to find (**a**) $S^{-1}(z)$ and (**b**) $S^{-1}(T(z))$.

17. $T(z) = \dfrac{z}{iz-1}$, $S(z) = \dfrac{iz+1}{z-1}$

18. $T(z) = \dfrac{iz}{z-2i}$, $S(z) = \dfrac{2z+1}{z+1}$

19. $T(z) = \dfrac{2z-3}{z-3}$, $S(z) = \dfrac{z-2}{z-1}$

20. $T(z) = \dfrac{z-1+i}{iz-2}$, $S(z) = \dfrac{(2-i)z}{z-1-i}$

In Problems 21–26, construct a linear fractional transformation that takes the given points z_1, z_2, and z_3 onto the given points w_1, w_2, and w_3, respectively.

21. $z_1 = -1,\ z_2 = 0,\ z_3 = 2;\ w_1 = 0,\ w_2 = 1,\ w_3 = \infty$

22. $z_1 = i,\ z_2 = 0,\ z_3 = -i;\ w_1 = 0,\ w_2 = 1,\ w_3 = \infty$

23. $z_1 = 0,\ z_2 = i,\ z_3 = \infty;\ w_1 = 0,\ w_2 = 1,\ w_3 = 2$

24. $z_1 = -1,\ z_2 = 0,\ z_3 = 1;\ w_1 = i,\ w_2 = 0,\ w_3 = \infty$

25. $z_1 = 1,\ z_2 = i,\ z_3 = -i;\ w_1 = -1,\ w_2 = 0,\ w_3 = 3$

26. $z_1 = 1,\ z_2 = i,\ z_3 = -i;\ w_1 = -i,\ w_2 = i,\ w_3 = \infty$

Focus on Concepts

27. Let a, b, c, and d be complex numbers such that $ad - bc \neq 0$.

(**a**) Solve the equation $w = \dfrac{az+b}{cz+d}$ for z.

(**b**) Explain why (a) implies that the linear fractional transformation $T(z) = (az+b)/(cz+d)$ is a one-to-one function.

28. Consider the equation

$$|z-a| = \lambda|z-b| \tag{15}$$

where λ is a positive real constant.

(**a**) Show that the set of points satisfying (15) is a line if $\lambda = 1$.

(**b**) Show that the set of points satisfying (15) is a circle if $\lambda \neq 1$.

29. Let $T(z) = (az+b)/(cz+d)$ be a linear fractional transformation.

(**a**) If $T(0) = 0$, then what, if anything, can be said about the coefficients a, b, c, and d?

(b) If $T(1) = 1$, then what, if anything, can be said about the coefficients a, b, c, and d?

(c) If $T(\infty) = \infty$, then what, if anything, can be said about the coefficients a, b, c, and d?

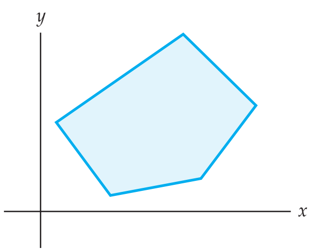
30. Use Problem 29 to show that if T is a linear fractional transformation and $T(0) = 0$, $T(1) = 1$, and $T(\infty) = \infty$, then T must be the identity function. That is, $T(z) = z$.

31. Use Theorem 7.4 to derive the mapping in entry H-1 in Appendix III.

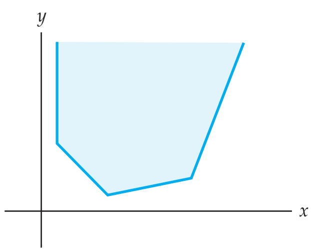
32. Use Theorem 7.4 to derive the mapping in entry H-3 in Appendix III.

7.3 Schwarz-Christoffel Transformations

One problem that arises frequently in the study of fluid flow is that of constructing the flow of an ideal fluid that remains inside a polygonal domain D'. We will see in Section 7.5 that this problem can be solved by finding a one-to-one complex mapping of the half-plane $y \geq 0$ onto the polygonal region that is a conformal mapping in the domain $y > 0$. The existence of such a mapping is guaranteed by the Riemann mapping theorem discussed in the Remarks at the end of Section 7.1. However, even though the Riemann mapping theorem does assert the existence of a mapping, it gives no practical means of finding a formula for the mapping. In this section we present the Schwarz-Christoffel formula, which provides an explicit formula for the derivative of a conformal mapping from the upper half-plane onto a polygonal region.



(a) A bounded polygonal region



(b) An unbounded polygonal region

Figure 7.19 Polygonal regions

Polygonal Regions

A **polygonal region** in the complex plane is a region that is bounded by a simple, connected, piecewise smooth curve consisting of a finite number of line segments. The boundary curve of a polygonal region is called a **polygon** and the endpoints of the line segments in the polygon are called **vertices** of the polygon. If a polygon is a closed curve, then the region enclosed by the polygon is called a **bounded polygonal region**, and a polygonal region that is not bounded is called an **unbounded polygonal region**. See Figure 7.19. In the case of an unbounded polygonal region, the ideal point ∞ is also called a vertex of the polygon.

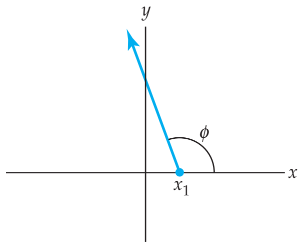
Simple examples of polygonal regions include the region bounded by the triangle with vertices 0, 1, and i, which is an example of a bounded polygonal region, and the region defined by $0 \leq x \leq 1$, $0 \leq y < \infty$, which is an example of an unbounded polygonal region whose vertices are 0, 1, and ∞.

Special Cases

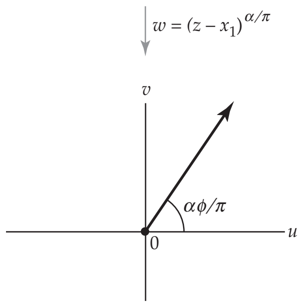
In order to motivate a general formula for a conformal mapping of the upper half-plane $y \geq 0$ onto a polygonal region, we first examine the complex mapping

$$w = f(z) = (z - x_1)^{\alpha/\pi}, \tag{1}$$

where x_1 and α are a real numbers and $0 < \alpha < 2\pi$. The mapping in (1) is the composition of a translation $T(z) = z - x_1$ followed by the real power

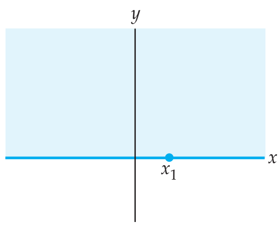


(a) A ray emanating from x_1

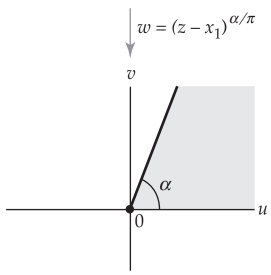


(b) Image of the ray in (a)

Figure 7.20 The mapping $w = (z - x_1)^{\alpha/\pi}$



(a) The half-plane $y \geq 0$



(b) The image of the half-plane

Figure 7.21 The mapping $w = (z - x_1)^{\alpha/\pi}$

function $F(z) = z^{\alpha/\pi}$. Because x_1 is real, T translates in a direction parallel to the real axis. Under this translation the x-axis is mapped onto the u-axis with the point $z = x_1$ mapping onto the point $w = 0$. In order to understand the power function F as a complex mapping we replace the symbol z with the exponential notation $re^{i\theta}$ to obtain:

$$F(z) = \left(re^{i\theta}\right)^{\alpha/\pi} = r^{\alpha/\pi} e^{i(\alpha\theta/\pi)}. \tag{2}$$

From (2) we see that the complex mapping $w = z^{\alpha/\pi}$ can be visualized as the process of magnifying or contracting the modulus r of z to the modulus $r^{\alpha/\pi}$ of w, and rotating z through α/π radians about the origin to increase or decrease an argument θ of z to an argument $\alpha\theta/\pi$ of w. Thus, under the composition $w = F(T(z)) = (z - x_1)^{\alpha/\pi}$, a ray emanating from x_1 and making an angle of ϕ radians with the real axis is mapped onto a ray emanating from the origin and making an angle of $\alpha\phi/\pi$ radians with the real axis. See Figure 7.20.

Now consider the mapping (1) on the half-plane $y \geq 0$. Since this set consists of the point $z = x_1$ together with the set of rays $\arg(z - x_1) = \phi$, $0 \leq \phi \leq \pi$, the image under $w = (z - x_1)^{\alpha/\pi}$ consists of the point $w = 0$ together with the set of rays $\arg(w) = \alpha\phi/\pi$, $0 \leq \alpha\phi/\pi \leq \alpha$. Put another way, the image of the half-plane $y \geq 0$ is the point $w = 0$ together with the wedge $0 \leq \arg(w) \leq \alpha$. See Figure 7.21.

The function f given by (1), which maps the half-plane $y \geq 0$ onto an unbounded polygonal region with a single vertex, has derivative:

$$f'(z) = \frac{\alpha}{\pi}(z - x_1)^{(\alpha/\pi)-1}. \tag{3}$$

Since $f'(z) \neq 0$ if $z = x + iy$ and $y > 0$, it follows that $w = f(z)$ is a conformal mapping at any point z with $y > 0$. In general, we will use the derivative f', not f, to describe a conformal mapping of the upper half-plane $y \geq 0$ onto an arbitrary polygonal region. With this in mind, we will now present a generalization of the mapping in (1) based on its derivative in (3).

Consider a new function f, which is analytic in the domain $y > 0$ and whose derivative is:

$$f'(z) = A(z - x_1)^{(\alpha_1/\pi)-1}(z - x_2)^{(\alpha_2/\pi)-1}, \tag{4}$$

where x_1, x_2, α_1, and α_2 are real, $x_1 < x_2$, and A is a complex constant. A useful fact that will help us determine the image of the half-plane $y \geq 0$ under f is that a parametrization $w(t)$, $a < t < b$, gives a line segment if and only if there is a constant value of $\arg(w'(t))$ for all t in the interval $a < t < b$. We now use this fact to determine the images of the intervals $(-\infty,\ x_1)$, $(x_1,\ x_2)$, and $(x_2,\ \infty)$ on the real axis under the complex mapping $w = f(z)$. If we parametrize the interval $(-\infty,\ x_1)$ by $z(t) = t$, $-\infty < t < x_1$, then by (11) of Section 2.2 the image under $w = f(z)$ is parametrized by $w(t) = f(z(t)) = f(t)$, $-\infty < t < x_1$. From (4) with the identification $z = t$, we obtain:

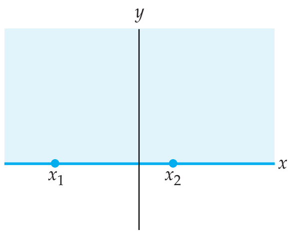
$$w'(t) = f'(t) = A(t - x_1)^{(\alpha_1/\pi)-1}(t - x_2)^{(\alpha_2/\pi)-1}.$$

An argument of $w'(t)$ is then given by:

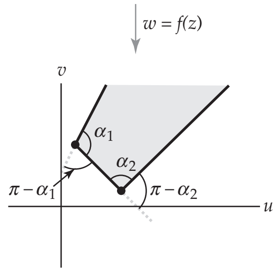
$$\operatorname{Arg}(A) + \left(\frac{\alpha_1}{\pi} - 1\right)\operatorname{Arg}(t - x_1) + \left(\frac{\alpha_2}{\pi} - 1\right)\operatorname{Arg}(t - x_2). \qquad (5)$$

Because $-\infty < t < x_1$, we have that $t - x_1$ is a negative real number, and so $\operatorname{Arg}(t - x_1) = \pi$. In addition, since $x_1 < x_2$, we also have that $t - x_2$ is a negative real number, and thus $\operatorname{Arg}(t - x_2) = \pi$. By substituting these values into (5) we find that $\operatorname{Arg}(A) + \alpha_1 + \alpha_2 - 2\pi$ is a constant value of $\arg(w'(t))$ for all t in the interval $(-\infty, x_1)$. Therefore, we conclude that the interval $(-\infty, x_1)$ is mapped onto a line segment by $w = f(z)$.

By similar reasoning we determine that the intervals (x_1, x_2) and (x_2, ∞) also map onto line segments. A value of the argument of w' for each interval is summarized in the following table. The change in the value of the argument is also listed.



(a) The upper half-plane $y \geq 0$



(b) The image of the region in (a)

Figure 7.22 The mapping associated with (4)

Interval	An Argument of w'	Change in Argument
$(-\infty, x_1)$	$\operatorname{Arg}(A) + \alpha_1 + \alpha_2 - 2\pi$	0
(x_1, x_2)	$\operatorname{Arg}(A) + \alpha_2 - \pi$	$\pi - \alpha_1$
(x_2, ∞)	$\operatorname{Arg}(A)$	$\pi - \alpha_2$

Table 7.1 Arguments of w'

Since f is an analytic (and, hence, continuous) mapping, we conclude that the image of the half-plane $y \geq 0$ is an unbounded polygonal region. By Table 7.1 we see that the exterior angles between successive sides of the polygonal boundary are given by the change in argument of w' from one interval to the next. Therefore, the interior angles of the polygon are α_1 and α_2. See Figure 7.22.

Schwarz-Christoffel Formula The foregoing discussion can be generalized to produce a formula for the derivative f' of a function f that maps the half-plane $y \geq 0$ onto a polygonal region with any number of sides. This formula, given in the following theorem, is called the **Schwarz-Christoffel formula**.

Theorem 7.5 Schwarz-Christoffel Formula

Let f be a function that is analytic in the domain $y > 0$ and has the derivative

$$f'(z) = A\,(z - x_1)^{(\alpha_1/\pi)-1}\,(z - x_2)^{(\alpha_2/\pi)-1}\cdots(z - x_n)^{(\alpha_n/\pi)-1}, \qquad (6)$$

where $x_1 < x_2 < \cdots < x_n$, $0 < \alpha_i < 2\pi$ for $1 \leq i \leq n$, and A is a complex constant. Then the upper half-plane $y \geq 0$ is mapped by $w = f(z)$ onto an unbounded polygonal region with interior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$.

It follows from Theorem 7.1 of Section 7.1 that the function given by the Schwarz-Christoffel formula (6) is a conformal mapping in the domain $y > 0$. For the sake of brevity, we will, henceforth, refer to a mapping obtained from (6) as a *conformal mapping from the upper half-plane* onto a polygonal region. It should be kept in mind that although such a mapping is defined on the upper half-plane $y \geq 0$, it is only conformal in the domain $y > 0$.

Note ☞

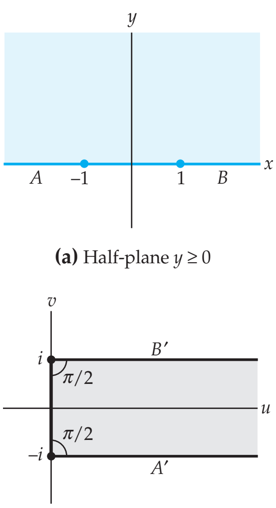
Before investigating some examples of the Schwarz-Christoffel formula, we need to point out three things. First, in practice we usually have some freedom in the selection of the points x_k on the x-axis. A judicious choice can simplify the computation of $f(z)$. Second, Theorem 7.4 provides a formula only for the derivative of f. A general formula for f is given by an integral

$$f(z) = A \int (z - x_1)^{(\alpha_1/\pi)-1} (z - x_2)^{(\alpha_2/\pi)-1} \cdots (z - x_n)^{(\alpha_n/\pi)-1} \, dz + B,$$

where A and B are complex constants. Thus, f is the composition of the function

$$g(z) = \int (z - x_1)^{(\alpha_1/\pi)-1} (z - x_2)^{(\alpha_2/\pi)-1} \cdots (z - x_n)^{(\alpha_n/\pi)-1} \, dz$$

and the linear mapping $h(z) = Az + B$. As described in Section 2.3, the linear mapping h allows us to rotate, magnify (or contract), and translate the polygonal region produced by g. Third, although it is not stated in Theorem 7.4, the Schwarz-Christoffel formula (6) can also be used to construct a mapping of the upper half-plane $y \geq 0$ onto a bounded polygonal region. To do so, we apply (6) using only $n - 1$ of the n interior angles of the bounded polygonal region.[†] We illustrate these ideas in the following examples.



(a) Half-plane $y \geq 0$

(b) Semi-infinite strip

Figure 7.23 Figure for Example 1

EXAMPLE 1 Using the Schwarz-Christoffel Formula

Use the Schwarz-Christoffel formula (6) to construct a conformal mapping from the upper half-plane onto the polygonal region defined by $u \geq 0$, $-1 \leq v \leq 1$.

Solution Observe that the polygonal region defined by $u \geq 0$, $-1 \leq v \leq 1$, is the semi-infinite strip shown in gray in Figure 7.23(b). The interior angles of this unbounded polygonal region are $\alpha_1 = \alpha_2 = \pi/2$, and the vertices are $w_1 = -i$ and $w_2 = i$. To find the desired mapping, we apply Theorem 7.4 with $x_1 = -1$ and $x_2 = 1$. With these identifications, (6) gives

$$f'(z) = A\,(z+1)^{-1/2}\,(z-1)^{-1/2}. \tag{7}$$

From Theorem 7.4, $w = f(z)$ is a conformal mapping from the half-plane $y \geq 0$ onto the polygonal region $u \geq 0$, $-1 \leq v \leq 1$. A formula for $f(z)$ is found by integrating (7). Since z is in the upper half-plane $y \geq 0$, we first use

[†]For a bounded polygon in the plane, any $n-1$ of its interior angles uniquely determine the remaining one.

the principal square root to rewrite (7) as

$$f'(z) = \frac{A}{(z^2-1)^{1/2}}.$$

Furthermore, since the principal value of $(-1)^{1/2} = i$, we have

$$f'(z) = \frac{A}{(z^2-1)^{1/2}} = \frac{A}{[-1\,(1-z^2)]^{1/2}} = \frac{A}{i}\frac{1}{(1-z^2)^{1/2}} = -Ai\frac{1}{(1-z^2)^{1/2}}. \tag{8}$$

From (7) of Section 4.4 we recognize that an antiderivative of (8) is given by

$$f(z) = -Ai\sin^{-1} z + B, \tag{9}$$

where $\sin^{-1} z$ is the single-valued function obtained by using the principal square root and principal value of the logarithm and where A and B are complex constants. If we choose $f(-1) = -i$ and $f(1) = i$, then the constants A and B must satisfy the system of equations

$$\begin{aligned} -Ai\sin^{-1}(-1) + B &= Ai\frac{\pi}{2} + B = -i \\ -Ai\sin^{-1}(1) + B &= -Ai\frac{\pi}{2} + B = i. \end{aligned}$$

By adding these two equations we see that $2B = 0$, or, $B = 0$. Now by substituting $B = 0$ into either the first or second equation we obtain $A = -2/\pi$. Therefore, the desired mapping is given by

$$f(z) = i\frac{2}{\pi}\sin^{-1} z.$$

This mapping is shown in Figure 7.23. The line segments labeled A and B shown in color in Figure 7.23(a) are mapped by $w = i\frac{2}{\pi}\sin^{-1} z$ onto the line segments labeled A' and B' shown in black in Figure 7.23(b).

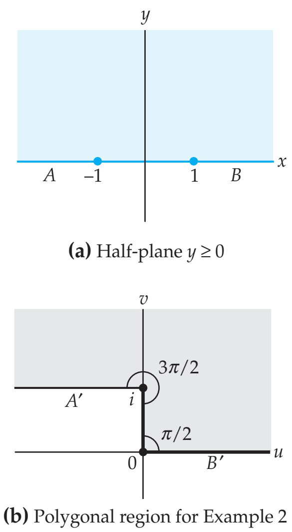


Figure 7.24 Figure for Example 2

EXAMPLE 2 Using the Schwarz-Christoffel Formula

Use the Schwarz-Christoffel formula (6) to construct a conformal mapping from the upper half-plane onto the polygonal region shown in gray in Figure 7.24(b).

Solution We proceed as in Example 1. The region shown in gray in Figure 7.24(b) is an unbounded polygonal region with interior angles $\alpha_1 = 3\pi/2$ and $\alpha_1 = \pi/2$ at the vertices $w_1 = i$ and $w_2 = 0$, respectively. If we select $x_1 = -1$ and $x_2 = 1$ to map onto w_1 and w_2, respectively, then (6) gives

$$f'(z) = A\,(z+1)^{1/2}\,(z-1)^{-1/2}. \tag{10}$$

Since

$$(z+1)^{1/2}(z-1)^{-1/2} = \left(\frac{z+1}{z-1}\right)^{1/2}\left(\frac{z+1}{z+1}\right)^{1/2} = \frac{z+1}{\left(z^2-1\right)^{1/2}},$$

we can rewrite (10) as

$$f'(z) = A\left[\frac{z}{\left(z^2-1\right)^{1/2}} + \frac{1}{\left(z^2-1\right)^{1/2}}\right]. \tag{11}$$

An antiderivative of (11) is given by

$$f(z) = A\left[\left(z^2-1\right)^{1/2} + \cosh^{-1} z\right] + B,$$

where A and B are complex constants, and where $\left(z^2-1\right)^{1/2}$ and $\cosh^{-1} z$ represent branches of the square root and inverse hyperbolic cosine functions defined on the domain $y > 0$. Because $f(-1) = i$ and $f(1) = 0$, the constants A and B must satisfy the system of equations

$$\begin{aligned} A\left(0+\cosh^{-1}(-1)\right) + B &= A\pi i + B = i \\ A\left(0+\cosh^{-1} 1\right) + B &= B = 0. \end{aligned}$$

Therefore, $A = 1/\pi$, $B = 0$, and the desired mapping is given by

$$f(z) = \frac{1}{\pi}\left[\left(z^2-1\right)^{1/2} + \cosh^{-1} z\right].$$

The mapping is illustrated in Figure 7.24. The line segments labeled A and B shown in color in Figure 7.24(a) are mapped by $w = f(z)$ onto the line segments labeled A' and B' shown in black in Figure 7.24(b).

When using the Schwarz-Christoffel formula, it is not always possible to express $f(z)$ in terms of elementary functions. In such cases, however, numerical techniques can be used to approximate f with great accuracy. The following example illustrates that even relatively simple polygonal regions can lead to integrals that cannot be expressed in terms of elementary functions.

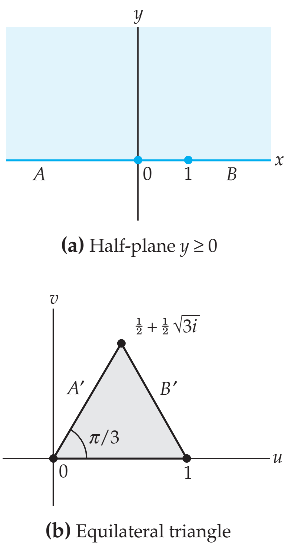


Figure 7.25 Figure for Example 3

EXAMPLE 3 Using the Schwarz-Christoffel Formula

Use the Schwarz-Christoffel formula (6) to construct a conformal mapping from the upper half-plane onto the polygonal region bounded by the equilateral triangle with vertices $w_1 = 0$, $w_2 = 1$, and $w_3 = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$. See Figure 7.25.

Solution The region bounded by the equilateral triangle is a bounded polygonal region with interior angles $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$. As mentioned on page 413, we can find a desired mapping by using the Schwarz-Christoffel formula (6) with $n-1=2$ of the interior angles. After selecting $x_1=0$ and $x_2=1$, (6) gives

$$f'(z)=Az^{-2/3}(z-1)^{-2/3}. \tag{12}$$

There is no antiderivative of the function in (12) that can be expressed in terms of elementary functions. However, f' is analytic in the simply connected domain $y>0$, and so, from Theorem 5.8 of Section 5.4, an antiderivative f does exist in this domain. The antiderivative is given by the integral formula

$$f(z)=A\int_0^z \frac{1}{s^{2/3}(s-1)^{2/3}}ds+B, \tag{13}$$

where A and B are complex constants. Requiring that $f(0)=0$ allows us to solve for the constant B. Since $\int_0^0 \;=0$, we have

$$f(0)=A\int_0^0 \frac{1}{s^{2/3}(s-1)^{2/3}}ds+B=0+B=B,$$

and so $f(0)=0$ implies that $B=0$. If we also require that $f(1)=1$, then

$$f(1)=A\int_0^1 \frac{1}{s^{2/3}(s-1)^{2/3}}ds=1.$$

Let Γ denote value of the integral

$$\Gamma=\int_0^1 \frac{1}{s^{2/3}(s-1)^{2/3}}ds.$$

Then $A=1/\Gamma$ and f can be written as

$$f(z)=\frac{1}{\Gamma}\int_0^z \frac{1}{s^{2/3}(s-1)^{2/3}}ds.$$

Values of f can be approximated using a CAS. For example, using the **NIntegrate** command in *Mathematica* we find that

$$f(i)\approx 0.4244+0.3323i \quad \text{and} \quad f(1+i)\approx 0.5756+0.3323i.$$

The Schwarz-Christoffel formula can also sometimes be used to find mappings onto nonpolygonal regions. Such mappings are often needed in the study of ideal fluid flows. The Schwarz-Christoffel formula can be used when the desired nonpolygonal region can be obtained as a “limit” of a sequence of polygonal regions. The following example illustrates this technique.

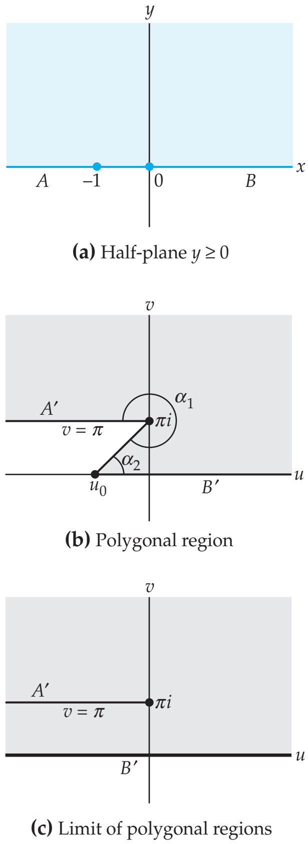


Figure 7.26 Figure for Example 4

EXAMPLE 4 Using the Schwarz-Christoffel Formula

Use the Schwarz-Christoffel formula (6) to construct a conformal mapping from the upper half-plane onto the nonpolygonal region defined by $v \geq 0$, with the horizontal half-line $v = \pi$, $-\infty < u \leq 0$, deleted. See Figure 7.26(c).

Solution Let u_0 be a point on the nonpositive u-axis in the w-plane. We can approximate the non-polygonal region defined by $v \geq 0$, with the half-line $v = \pi$, $-\infty < u \leq 0$, deleted by the polygonal region whose boundary consists of the horizontal half-line $v = \pi$, $-\infty < u \leq 0$, the line segment from πi to u_0, and the horizontal half-line $v = 0$, $u_0 \leq u \leq \infty$. The vertices of this polygonal region are $w_1 = \pi i$ and $w_2 = u_0$, with corresponding interior angles α_1 and α_2. See Figure 7.26(b). If we choose the points $z_1 = -1$ and $z_2 = 0$ to map onto the vertices $w_1 = \pi i$ and $w_2 = u_0$, respectively, then (6) gives the derivative

$$A\,(z+1)^{(\alpha_1/\pi)-1}\,z^{(\alpha_2/\pi)-1}. \tag{14}$$

Observe in Figure 7.26(b) that as u_0 approaches $-\infty$ along the u-axis, the interior angle α_1 approaches 2π and the interior angle α_2 approaches 0. With these limiting values, (14) suggests that our desired mapping f has derivative

$$f'(z) = A\,(z+1)^1\,z^{-1} = A\left(1 + \frac{1}{z}\right). \tag{15}$$

An antiderivative of the function in (15) is given by

$$f(z) = A\,(z + \operatorname{Ln} z) + B, \tag{16}$$

where A and B are complex constants.

In order to determine the appropriate values of the constants A and B, we first consider the mapping $g(z) = z + \operatorname{Ln} z$ on the upper half-plane $y \geq 0$. The function g has a point of discontinuity at $z = 0$; thus, we will consider separately the boundary half-lines $y = 0$, $-\infty < x < 0$, and $y = 0$, $0 < x < \infty$, of the half-plane $y \geq 0$. If $z = x + 0i$ is on the half-line $y = 0$, $-\infty < x < 0$, then $\operatorname{Arg}(z) = \pi$, and so $g(z) = x + \log_e |x| + i\pi$.

When $x < 0$, $x + \log_e |x|$ takes on all values from $-\infty$ to -1. Thus, the image of the negative x-axis under g is the horizontal half-line $v = \pi$, $-\infty < u < -1$. On the other hand, if $z = x + 0i$ is on the half-line $y = 0$, $0 < x < \infty$, then $\operatorname{Arg}(z) = 0$, and so $g(z) = x + \log_e |x|$. When $x > 0$, $x + \log_e |x|$ takes on all values from $-\infty$ to ∞. Therefore, the image of the positive x-axis under g is the u-axis. It follows that the image of the half-plane $y \geq 0$ under $g(z) = z + \operatorname{Ln} z$ is the region defined by $v \geq 0$, with the horizontal half-line $v = \pi$, $-\infty < u < -1$ deleted. In order to obtain the region shown in Figure 7.26(c), we should compose g with a translation by 1. Therefore, the desired mapping is given by

$$f(z) = z + \operatorname{Ln}(z) + 1.$$