CS281 Lecture19 Problem

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December 2017

1 Rejection Sampling Question

We are trying to sample from a distribution $p(x)$.

1.1 Part a

Assume for this part that $p(x)$ has finite support on $x \in [a, b]$ and known maximum M. Without a proposal distribution we choose a proposal $\tilde{x} \sim \text{Unif}(a, b)$. Describe a strategy to accept these proposals such that $\tilde{x} \sim p(x)$.

1.2 Part b

Describe two issues with the above method.

1.3 Part c

We now decide to implement the method for rejection sampling described in class. We find a proposal function $q(x)$ such that $Mq(x) > p(x)$. Describe what a good proposal functional $q(x)$ would look like.

1.4 Part d

Calculate the acceptance probability using the proposal and acceptance strategy from the rejection sampling described in class.

1.5 Part e

Interpret the results from part d, and describe how it affects an ideal choice of M.

2 Solution

2.1 Part a

We want $a(\tilde{x}) = p(x)$ where $a(x)$ is the acceptance function. Since $\tilde{x}$ is drawn uniformly over the support, we simply require that the probability of acceptance is proportional to the relative pdf values of $p(x)$. We can do this by sampling $\tilde{y} \sim \text{Unif}(0, M)$. If $\tilde{y} < p(\tilde{x})$, accept the sample.

2.2 Part b

1. This requires knowing the maximum (or at least bounding the maximum) of $p(x)$
2. Acceptance probability will be very low, we won't get many samples (imagine a very pointed distribution, we will essentially only accept samples if $\tilde{x}$ is on the point of the distribution.

2.3 Part c

We want the shape of $q(x)$ to match the shape roughly of $p(x)$. The problem with our previous method of sampling was that we were proposing samples from areas with low probability density on $p(x)$. If we can get the shape of $q(x)$ to match the shape of $p(x)$, we will have a smaller probability of proposing samples in low probability density regions of $p(x)$.

2.4 Part d

The probability of accepting a sample is the probability of proposing that sample and then accepting it: $q(x)a(x)$. We integrate over x to determine the acceptance probability:

$$\int q(x)a(x)dx = \int q(x)\frac{p(x)}{Mq(x)}dx = \frac{1}{M}\int p(x)dx = \frac{1}{M}$$

2.5 Part e

The acceptance probability is inversely proportional to our choice of M. We thus want to pick M as close to 1 as possible. Clearly, we can make M arbitarily large to satisfy our constraint that $Mq(x) > p(x)$, but larger M decreases our acceptance probability. By matching the shape of $q(x)$ to $p(x)$ and then making M as close to 1 as possible, we maximize our acceptance probability.