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AstroStatistics Spring 2022
Exercise Sheet 2
Issued : 3 March 2022 Due : 10 March 2022

1 Normal Distribution

Recall that the normal distribution $N(\mu, \sigma^2)$ has pdf

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(- \frac{(x-\mu)^2}{2\sigma^2}\Big) \tag{1}$$

The standard normal distribution $N(0,1)$ has mean 0 (by symmetry), variance 1, and pdf $\phi(z)$ given by setting $= 0$ and $\sigma = 1$ above. The cdf is denoted $\Phi(z)$ and does not have a nice formula. In this problem, we'll show that scaling and shifting a normal random variable gives a normal random variable. Suppose $Z \sim N(0,1)$ and $X = aZ + b$.

1. Compute the mean μ and variance σ^2 of X.
2. Express the cdf $P(X \leq x)$ of X in terms of Φ and then use the chain rule to find the pdf $p(x)$ of X.
3. Use (b) to show that X follows the $N(b, a^2)$ distribution.
4. Use (a) and (c) to conclude that the $N(\mu, \sigma^2)$ distribution has mean μ and variance σ^2.
5. Lets assume that $a = 3$ and $b = 1$. Find $P(-1 \leq X \leq 1)$
6. The probability that Z is within one standard deviation of its mean is approximately 68%. What is the probability that X is within one standard deviation of its mean.

2 Transformation of Random variables

Let $X_1, X_2, \ldots, X_n$ independent random variables drawn from $N(0,1)$. Let $Y_n = X_1^2 + \cdots + X_n^2$

1. Use the formula $Var(X_j) = E(X_j^2) - E(X_j)^2$ to show that $E(X_j^2) = 1$.
2. Set up an integral in x for computing $E(x_j^4)$. Solve the integral using the integration by parts (or you may use Python to solve the integration). If you use Python to solve the integration write the Python code below.
3. Deduce from (1) and (2) that $Var(X_j^2) = 2$
4. use the central limit theorem to approximate $P(Y_{100} > 110)$

If the probability density distribution is of a random variable x is given by $p(x)$, what is the probability density function $p(y)$ where $y = \exp(x)$?

3 Central Limit Theorem

Qs : The average IQ in a population is 100 with standard deviation 15 (by definition, IQ is normalized so this is the case). What is the probability that a randomly selected group of 100 people has an average IQ above 115 ?

4 Data

The following data is from a random sample : 5, 1, 3, 3, 8. Compute the sample mean, sample standard deviation and sample median.

5 Plots

Use Python to plots the following distributions :

1. Gaussian distribution for $(\mu = 0, \sigma = 0.5)$, $(\mu = 0, \sigma = 1.0)$ and $(\mu = 0, \sigma = 2.0)$.
2. Binomial distribution for $(b = 0.2, n = 20)$, $(b = 0.6, n = 20)$, and $(b = 0.8, n = 40)$, where b is the probability of success and n is the total number of trials.
3. The Poisson distribution for $\mu = 1$,$\mu = 5$, and $\mu = 15$.

6 Covariances

Let X and Y are two random variables with the joint probability density function given as $p(x_i, y_j)$. In the lectures we discussed the properties of the covariances of the two random variables. Using those properties show that

$$\mathrm{Cov}(X,Y) = \sum_i \sum_j p(x_i, y_j)(x_i - \mu_X)(y_j - \mu_X) - \mu_X \mu_Y. \tag{2}$$

Similarly for the continuous case :

$$\mathrm{Cov}(X,Y) = \int_a^b \int_c^d f(x,y)(x - \mu_X)(y - \mu_X) - \mu_X \mu_Y \tag{3}$$

where $X \in [a, b]$ and $Y \in [c, d]$.