Bayes Theorem

Monday, 31 January 2022 11:56

Independence

Events A & B ; They are ~~independent~~ if

$P(A \cap B) = P(A) . P(B)$.

1. If $P(A) \neq 0$ Then A & B are independent if $P(A|B) = P(A)$.

2. If $P(B) \neq 0$ Then A & B are independent if $P(B|A) = P(B)$.

Ex. Toss a fair Coin 3times.

H_1 = 'head on first toss'

A = 'two heads in total'

Are these events independent?

$H_1 = \{HHH, HHT, HTH, HTT\}$

$A = \{HHT, HTH, THH\}$.

$P(A) = 3/8$

$P(A|H_1) = 2/4 = 1/2$.

$P(A|H_1) \neq P(A) \Rightarrow A$ & H_1 are not independent.

Ex: Draw one card from standard deck of cards.

A = 'Card is an ace'

H = 'The card is a heart'

R = 'The card is red'

$P(A) = 4/52 = 1/13.$

$P(H) = 13/52 = 1/4$

$P(R) = 1/2$

$P(A|H) = 1/13 = P(A$

$\boxed{\text{A is independent of H}}$

(b) $P(A|R) = 2/26 = 1/13. = P(A)$

$\boxed{\text{A is independent of R}}$

(c) $P(H) = 1/4$ $\quad P(R) = 1/2$,

$P(H|R) = 1/2$

$\boxed{\text{H \& R are not independent}}$

Bayes Theorem :

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

Ex: Toss a coin 5 times.

H_1 = 'first toss is head'

H_A = all 5 tosses are head.

$P(H_1|H_A) = 1$

$P(H_A|H_1) = 1/16$

$$P(H_1|H_A) = \frac{P(H_A|H_1) \cdot P(H_1)}{P(H_A)}$$

$P(H_1) = 1/2.$

$P(H_A) = 1/32$

$$P(H_1 | H_A) = \frac{(1/16)\cdot(1/2)}{(1/32)} = 1.$$

The Monty Hall Problem

$N = 1000$ boxes.

only one box contains a prize.

999 boxes are empty.

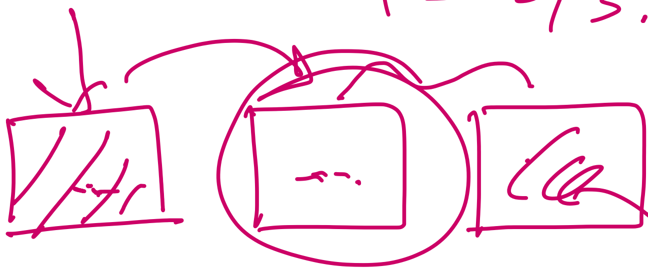
You choose a box at random

Prob. that it contains the prize is $1/1000$.

The probability that other 999 boxes contains the prize is $\frac{999}{1000}$

$N = 3$

$P = 2/3.$



P(each box contains the prize) = $1/3$.

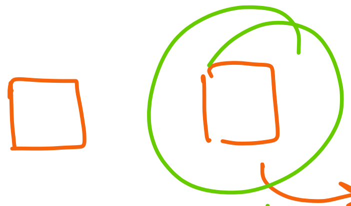
opens → empty box.

I chose This on

999

Another person who "knows which box contains the prize"

he opens 998 empty boxes.



998 are opened.

empty box.

which I chose.

$P(\text{prize}) = \frac{1}{1000}$

what is the prob. that this one contains the prize.

$$\text{Prob} = \frac{999}{1000}.$$

H_i is the hypothesis that the prize is in the ith box.

$$P(H_i \mid I) = 1/N$$

→ Page 69-71

Statistics, Data Mania book.

background information.

The data that $N-2$ boxes are empty is d_k $(k>1)$

d_k says that the k^{th} box remains closed.

$$P(H_1|d_k, I) = \frac{P(d_k|H_1, I)P(H|I)}{P(d_k|I)}$$

Prob that the k^{th} box remain unopened given that H_1 is true $P(d_k|H_1 I)$.

$$= 1/_{N-1}$$

$$P(d_k|I) = \sum_{i=1}^{N} P(d_k|H_i, I)P(H_i|I)$$

The prob that the k^{th} box stays unopened given that H_1 is true is

$$P(d_k|H_i, I) = \begin{cases} 1 & \text{for } k=i \\ 0 & \text{otherwise.} \end{cases}$$

except when $i=1$.

$P(d_k|I) = P(d_k|H_1 I)P(H_1|I) +$

$$P(d_k|I) = P(d_k|\ldots$$

$$P(d_k|H_k,I)P(H_k|I)$$

$$= \frac{1}{(N-1)N} + \frac{1}{N} = \frac{1}{N-1}$$

$$P(H_1|d_k I) = \frac{(1/N-1)\cdot 1/N}{1/(N-1)} = \frac{1}{N}$$

$$\boxed{P(H_k|d_k,I) = \frac{N-1}{N}} \rightarrow \frac{999}{1000}$$

$$\boxed{P(H_1|d_k I) = P(H_1|I)}$$

Ex: 2x2 Contingency Rule

Medical test. T → negative (0)
→ Positive (1).

Disease : D. → have disease (1)
→ No disease.

Sample Space. $\{(D, T)\}$.

(a) $T=0, D=0$ (c) $T=1, D=0$

(b) $T=0, D=1$ (d) $T=1, D=1$

D \ T	0	1
0	$1-\epsilon_{fP}$	ϵ_{fP}
1	ϵ_{fN}	$1-\epsilon_{fN}$

$P(T|D)$ -

$P(T=1|D=0) = \epsilon_{fp}$ 'false positive'

$P(T=0|D=0) = 1 - P(T=1|D=0) = 1 - \epsilon_{fp}$

$P(T=0|D=1) = \epsilon_{fN}$ — false negative.

$P(T=1|D=1) = 1 - \epsilon_{fN}$.

$P(D=1) = \epsilon_D$.

A patient took the test and it came out positive $(T=1)$

what $P(D=1|T=1)$?

$$P(D=1|T=1) = \frac{P(T=1|D=1)P(D=1)}{P(T=1|D=0)P(D=0) + P(T=1|D=1)P}$$

$$= \frac{(1-\epsilon_{fN})\epsilon_D}{\epsilon_{fp}(1-\epsilon_D) + (1-\epsilon_{fN}).\epsilon_D}.$$

$$= \frac{\epsilon_D - \epsilon_D \epsilon_{fN}}{\epsilon_{fp} - \epsilon_{fp}\epsilon_D + \epsilon_D - \epsilon_D \epsilon_{fN}}$$

neglect

$$P(D=1|T=1) = \frac{\epsilon_D}{\epsilon_D + \epsilon_{fp}}.$$

$$\boxed{P(D=1|T=1) \sim 1}$$

if $\epsilon_{fp} \ll \epsilon_D$.

if $\epsilon_{fp} \gg \epsilon_D$ $\quad P(D=1|T=1) \sim \frac{\epsilon_D}{\epsilon_{fp}} \ll 1$

Important to ask the test's false Positive Rat