Problems solving

Monday, 7 March 2022 11:51

Two players playing the coin toss game

E_1 : A's n coins have more heads than B's n coins.

E_2 : A's n coins have equal # of heads than B's n coins

E_3 : A's n coins have fewer # of heads than B.

$P(E_1) = P(E_3) = x$

$P(E_2) = y$

Total law of Prob : $\sum P(\omega) = 1 \Rightarrow 2x + y = 1$

Now we consider the $(n+1)^{th}$ coin.

$P(E_2) \to 0.5y$

$$x + 0.5y = x + 0.5(1 - 2x) = 0.5$$

Q2 Ω is the set of all sequence of n birthdays.

$\to \omega = (b_1 b_2, \dots b_n)$ sample space.

(a) There are 365^n sequences of n birthdays.

$$P(\omega) = \frac{1}{365^n}.$$

(b) A : Some one in the group shares your birthday

B : Some two people in the group share a birthday

C : Some three people in the group share a birthday

Suppose my birthday is on b

"an outcome ω is in A" is equivalent to.

"b is in the sequence ω"

$b = b_k \quad k = 1, 2, \ldots n.$

B: "an outcome ω is in B" is equivalent to "two of the entries in ω are the same" if and only if $b_j = b_k \quad j, 2\ k\ (1, \ldots n)$

C: an outcome ω is in C if and only if $b_j = b_k = b_l$ (distinct) indices j, k, l.

$(1, 2, 3 \ldots n)$

$P(A)$. $P(A^c)$ There are 364^n outcomes in A^c.

$$P(A) = 1 - P(A^c) = 1 - \frac{364^n}{365^n}$$

what is the value of n such that $P(A) > 0.5$.

$$1 - \frac{364^n}{365^n} > 0.5 \Rightarrow \left(\frac{364}{365}\right)^n < 0.5.$$

Taking a natural log on both sides.

$$\ln\left(\frac{364}{365}\right)^n = \ln(0.5)$$

$$n \ln(364/365) = \ln(0.5)$$

$$n = \frac{\ln(0.5)}{\ln(364/365)} \simeq 252.65.$$

$P(A) > 0.5$

if n is at least 253 — More likely that one of them share your birthday.

that one of them shares yo...

While 365/2 different birthdays would have a 50% chance of matching your birthday.

- 365/2 people probably don't all have different birthdays. so they have less than 50% chance of matching.

P(B)

n	P(B)
20	0.4123
30	0.707
40	0.8913, 0.8867, 0.892 ...
41	0.9005, 0.9003 ....

answers.

$P(B) > 0.9$ if $n = 41$

Its easier to calculate $P(B^c)$

there are 365 choices for the first birthday.

" " 364 choices for the second birthday.

$$P(B) = 1 - P(B^c) = 1 - \frac{365 . 364 . \cdots (365 - n + 1)}{365^n}$$

$$P(B) = 1 - \frac{365!}{(365-n)! 365^n}$$

$(365-n)!\,365)$

Q3 Unfair Coin:

if there are two events A & B.

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|A^c)\, P(A^c)}$$

Information:
- 1000 Coins.
- 1 Coin is unfair (heads both sides)
- 999 Coins are fair.

$P(\text{choosing unfair Coin}) \equiv P(A) = 1/1000$

$P(A^c) = 999/1000$

B : 10 heads Consequ...
A : unfair Coin.

$P(B|A) = 1$

$P(B|A^c) = (1/2)^{10} = 1/1204$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)}$$

$$= \frac{1 \cdot 1/1000}{1 \cdot 1/1000 + (1/1024)\frac{999}{1000}} \simeq 0.5.$$

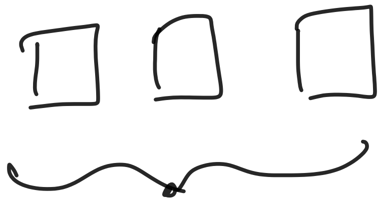
$$\boxed{P(A|B) = 0.5}$$

Ex 4 Dice Order:

We throw 3 dice one by one. Prob. that we obtain 3 points in strictly increasing order.

Probability of strictly increasing order is

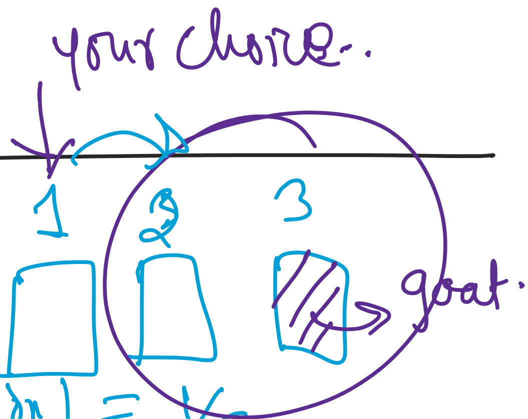
$$1/3! = 1/6$$



$$P = P(\text{different numbers in all three throws}) \times P(\text{increasing order} \mid 3 \text{ different numbers})$$

$$= \left(\underbrace{1 \times \frac{5}{6} \times \frac{4}{6}}\right) \times \frac{1}{6} = 5/54.$$

Ex 5 Monty hall problem:



$$P(\text{winning a car} \mid \text{No information}) = 1/3$$

$P(\text{one are e have a car}) = 2/2$

$P(1 \text{ has a car}) = 1/3$

~~P(~~ 3 does not have a car. ← after this information

$P(2 \text{ will have a car}) = 2/3$.

Ex 6 The Base Rate fallacy:

- freq of the disease in The population (Base rate) = 0.5%
- Accuracy of the test is 5%
- false negative : 10%

You have two variables (T, D)

D=0 : you don't have the disease

D=1 : you have the disease

T=0 : test is negative.

T=1 : test is positive.

$(T=0, D=0)$, $(T=0, D=1)$, $(T=1, D=0)$

$(T=1, D=1)$

$P(D=1) = 0.005$ (0.5%)

$$P(D=0) = 1 - P(D=1)$$
$$= 0.995.$$

$P(\text{false positive}) = P(T=1 \mid D=0) = 0.05$

$P(\text{false negative}) = P(T=0 \mid D=1) = 0.1$

$P(T=0 \mid D=0) = 1 - P(T=1 \mid D=0) = 0.9$

$P(T=1 \mid D=1) = 1 - P(T=0 \mid D=1) = 0.9.$

$$P(D=1 \mid T=1) = \frac{P(T=1 \mid D=1) \cdot P(D=1)}{P(T=1)}$$

$P(T=1) = 0.995 \times 0.05 + 0.005 \times 0.9 = 0.054$

Probability that test is positive

$$P(D=1 \mid T=1) = \frac{0.9 \times 0.005}{0.05425} = 0.0829 \approx 8.3\%$$

"95% of all tests are accurate does not imply 95% of positive tests are accurate"

Ex 7

X	x :	1	2	3	4
pmf	$P_X(x)$:	$1/10$	$2/10$	$3/10$	$4/10$.

Y	y :	1	2	3	4	5
pmf	$P_Y(y)$:	$1/15$	$2/15$	$3/15$	$4/15$	$5/15$