A9: Statistics

Sheet 4 — HT 2024

(Lectures 12–16, Notes section 4)

1. Suppose $X_1, \ldots, X_n$ are independent, each having a geometric distribution with probability mass function $f(x \mid \theta) = (1-\theta)^x\theta$ for $x = 0, 1, \ldots$. Suppose that the prior for θ is a Beta(a, b) density. Find the posterior distribution of θ.

2. Let $\theta > 0$ be an unknown parameter and let $c > 0$ be a known constant. Conditional on θ, suppose $X_1, \ldots, X_n$ are independent each with probability density function

$$f(x|\theta) = \theta c^\theta x^{-(\theta+1)}, \quad x \geqslant c$$

and suppose the prior for θ is a Gamma(α, β) density. Find the posterior distribution of θ.

3. Let $r \geqslant 1$ be a known integer and let $\theta \in [0, 1]$ be an unknown parameter. The negative binomial distribution with index r and parameter θ has probability mass function

$$f(x \mid \theta) = \binom{x+r-1}{x}(1-\theta)^x\theta^r \quad \text{for } x = 0, 1, \ldots.$$

Let θ have a Beta(a, b) prior density and suppose, given θ, that $X_1, \ldots, X_n$ are independent each with the above negative binomial distribution.

   (a) Show that the posterior density is also a Beta density.

   (b) Explain how to construct a $100(1-\alpha)\%$ equal-tailed credible interval for θ. Will this interval be a highest posterior density interval?

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4. Suppose that X has a $N(\theta, \phi)$ distribution, where ϕ is known, Suppose also that the prior distribution for θ is $N(\theta_0, \phi_0)$, where θ_0 and ϕ_0 are known.

   (a) Find the posterior distribution of θ given $X = x$.

   (b) Show that the posterior mean of θ always lies between the prior mean and the observed value x.

   (c) Construct a $100(1-\alpha)\%$ highest posterior density interval for θ.

   (d) Let $\phi = 2$, $\theta_0 = 0$ and $\phi_0 = 2$.

      (i) Suppose the observed value is $x = 4$. What are the mean and variance of the resulting posterior distribution? Sketch the prior, likelihood, and posterior on a single set of coordinate axes.

      (ii) Repeat (i) assuming $\phi_0 = 18$. Explain any resulting differences. Which of these two priors would likely have more appeal for a frequentist statistician?

5. Let X be the number of heads when a coin with probability θ of heads is flipped n times.

   (a) When the prior is $\pi(\theta)$, the prior predictive distribution for X (the predictive distribution before observing any data) is given by

   $$P(X = k) = \int_0^1 P(X = k|\theta)\pi(\theta)\, d\theta, \quad k = 0, 1, \ldots, n.$$

   Find the prior predictive distribution when $\pi(\theta)$ is uniform on $(0, 1)$.

   (b) Suppose you assign a Beta(a, b) prior for θ, and then you observe x heads out of n flips. Show that the posterior mean of θ is always lies between your prior mean, $a/(a + b)$, and the observed relative frequency of heads, x/n.

   (c) Show that, if the prior distribution on θ is uniform, then the posterior variance is always less than the prior variance.

   (d) Give an example of a Beta(a, b) prior distribution and values of x, n for which the posterior variance is larger than the prior variance. (Try $x = n = 1$.)

6. A coin, with probability θ of heads, is flipped n times and r heads are observed.

   (a) If the prior for θ is a uniform distribution on $(0, 1)$, what is the probability that the next flip is a head?

   (b) Can you generalise to the case where θ has a Beta(a, b) prior and where we wish to find the probability of getting k heads from m further flips?

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7. (a) Let $X \sim N(\theta, \sigma_0^2)$, where σ_0^2 is known. Find the Jeffreys' prior for θ.

   (b) Let $X \sim N(\mu_0, \sigma^2)$, where μ_0 is known. Find the Jeffreys' prior for σ.

   (c) Let X be Poisson with parameter λ. Find the Jeffreys' prior for λ. Check that the posterior distribution of λ given $X = x$ is proper, but that the Jeffreys' prior is not.

8. Suppose X is the number of successes in a binomial experiment with n trials and probability of success θ. Either $H_0 : \theta = \frac{1}{2}$ or $H_1 : \theta = \frac{3}{4}$ is true. Show that the posterior probability that H_0 is true is greater than the prior probability for H_0 if and only if

$$x \log 3 < n \log 2.$$

9. Let $X \sim \text{Binomial}(n, \theta)$, where the prior for θ is uniform on $(0, 1)$. Suppose that we wish to compare the hypotheses $H_0 : \theta \leqslant \frac{1}{2}$ and $H_1 : \theta > \frac{1}{2}$.

   What are the prior odds of H_0 relative to H_1?

   Find an expression for the posterior odds of H_0 relative to H_1.

   If we observe $X = n$, find the Bayes factor B of H_0 relative to H_1.

   Check that $B \to 0$ as $n \to \infty$. Why is this expected?

10. Suppose we have a random sample $X_1, \ldots, X_n$ from a Poisson distribution with mean θ. Suppose we wish to test the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ and that, under H_1, the prior distribution $\pi(\theta | H_1)$ for θ is given by

$$\pi(\theta | H_1) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha - 1} e^{-\beta\theta}, \quad \theta > 0.$$

   Calculate the Bayes factor of H_0 relative to H_1.

   When $n = 6$, $\sum x_i = 19$, $\theta_0 = 2$, find the numerical value of the Bayes factor (i) when $\alpha = 4$ and $\beta = \frac{2}{3}$, and (ii) when $\alpha = 36$ and $\beta = 6$. Compare and interpret the values of the Bayes factor in cases (i) and (ii).

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