Quantum Mechanics Spring 2023
Exercise Sheet 3 - Solutions

Problem 1 (15 marks)

Consider the wave function :

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t} \tag{1}$$

where A, λ and ω are positive real numbers.

1. Normalise Ψ
2. Determine the expectation values of x and x^2
3. Find the standard deviation of x.Sketch the graph of Ψ as a function of x, and mark the points $(\langle x\rangle + \sigma)$ and $(\langle x\rangle - \sigma)$ to illustrate the sense in which σ represent the "spread" of x. What is the probability that the particle would be found outside this range ?

Solution

Consider the wave function :

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t} \tag{2}$$

where A, λ and ω are positive real numbers.

1. Normalise Ψ

   The normalisation condition is given by

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx &= 1 \\
\Rightarrow \int_{-\infty}^{\infty} |Ae^{-\lambda|x|}e^{-i\omega t}|^2 dx &= 1 \\
\Rightarrow \int_{-\infty}^{\infty} A^2 e^{-2\lambda|x|} dx &= 1 \\
\Rightarrow A^2 \int_{-\infty}^{0} e^{2\lambda x} dx + A^2 \int_{0}^{\infty} e^{-2\lambda x} dx &= 1 \\
\Rightarrow A^2 \left[\frac{1}{2\lambda}e^{2\lambda x}\right] -\infty^0 + A^2 \left[-\frac{1}{2\lambda}e^{-2\lambda x}\right] 0^\infty &= 1 \\
\Rightarrow A^2 \left(\frac{1}{2\lambda} + \frac{1}{2\lambda}\right) &= 1 \\
\Rightarrow A^2 &= \frac{\lambda}{2} \\
\Rightarrow A &= \sqrt{\frac{\lambda}{2}}
\end{aligned}$$

   Therefore, the normalised wave function is

$$\Psi(x,t) = \sqrt{\frac{\lambda}{2}} e^{-\lambda|x|}e^{-i\omega t}$$

2. Determine the expectation values of x and x^2 To determine the expectation value of x, we use the formula :

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx$$

We first need to find $|\Psi(x,t)|^2$:

$$\begin{aligned}
|\Psi(x,t)|^2 &= |Ae^{-\lambda|x|}e^{-i\omega t}|^2 \\
&= |A|^2|e^{-\lambda|x|}|^2|e^{-i\omega t}|^2 \\
&= |A|^2 e^{-2\lambda|x|}
\end{aligned}$$

Substituting this into the formula for the expectation value, we get :

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx \\
&= \int_{-\infty}^{\infty} x|A|^2 e^{-2\lambda|x|} dx \\
&= 2\int_{0}^{\infty} x|A|^2 e^{-2\lambda x} dx \quad \text{(since the integrand is even)} \\
&= \frac{1}{\lambda}
\end{aligned}$$

To determine the expectation value of x^2, we use the formula :

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2|\Psi(x,t)|^2 dx$$

Substituting $|\Psi(x,t)|^2$ from above, we get :

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty} x^2|A|^2 e^{-2\lambda|x|} dx \\
&= 2\int_{0}^{\infty} x^2|A|^2 e^{-2\lambda x} dx \quad \text{(since the integrand is even)} \\
&= \frac{1}{2\lambda^2}
\end{aligned}$$

3. To find the standard deviation of x, we use the formula :

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

Substituting the expectation values we just found, we get :

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{1}{2\lambda^2} - \frac{1}{\lambda^2}} \\
&= \sqrt{\frac{1}{2\lambda^2}} \\
&= \frac{1}{\sqrt{2}} \frac{1}{\lambda}
\end{aligned}
$$

To sketch the graph of Ψ as a function of x, we can use the fact that $|\Psi(x,t)|^2$ is proportional to $e^{-2\lambda|x|}$. This exponential function decays rapidly as $|x|$ increases. The graph will look like a "spike" centred at $x = 0$, with the height of the spike decreasing rapidly as $|x|$ increases. The points $(\langle x \rangle + \sigma)$ and $(\langle x \rangle - \sigma)$ are located at $\frac{1}{\sqrt{2}} \frac{2}{\lambda}$ and $\frac{1}{\sqrt{2}} \frac{2}{\lambda}$, respectively. These points mark the locations where the probability of finding the particle is significantly smaller than at $x = 0$. The standard deviation σ represents the "spread" of x, or how far away from the mean value $\langle x \rangle$ we can expect to find the particle. The probability that the particle would be found outside this range is given by the integral of $|\Psi(x,t)|^2$ over the region $[\langle x \rangle - \sigma, \langle x \rangle + \sigma]$. This probability can be calculated by evaluating the integral :

$$
P = \int_{\langle x \rangle - \sigma}^{\langle x \rangle + \sigma} |\Psi(x,t)|^2 dx
$$

Which gives the probability of finding the particle within the range $[\langle x \rangle - \sigma, \langle x \rangle + \sigma]$. The probability of finding the particle outside this range is $1 - P$.

Problem 2 (15 marks)

At the time $t = 0$, the particle waver function is represented by :

$$\Psi(x,0) = \begin{cases} Ax/a & \text{if } 0 \leq x \leq a \\ A(b-x)/(b-a) & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where A, a, and b are constants.

1. Normalise Ψ, that is A in terms of a and b.
2. Sketch $\Psi(x,0)$ as a function of x.
3. Where is the particle most likely to be found at $t = 0$?
4. what is the probability of finding the particle to the left of a ? Check your results in the limiting cases when $b = a$ and $b = 2a$.
5. What is the expectation value of x ?

solution

1. To normalize Ψ, we need to find the value of A that satisfies the normalization condition :

$$\begin{aligned} \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx &= 1 \\ \int_0^a |Ax/a|^2 dx + \int_a^b |A(b-x)/(b-a)|^2 dx &= 1 \\ \frac{A^2}{a^2}\int_0^a x^2 dx + \frac{A^2}{(b-a)^2}\int_a^b (b-x)^2 dx &= 1 \\ \frac{A^2 b}{3} &= 1 \end{aligned}$$

$$\Longrightarrow A = \sqrt{\frac{3}{b}}$$

2. Fig 1
3. The particle is most likely found at $x = a$
4. The probability of finding the particle to the left of a is given by :

$$\begin{aligned} P &= \int_{-\infty}^{a} |\Psi(x,0)|^2 dx \\ &= \int_0^a |Ax/a|^2 dx \\ &= \frac{A^2}{a^2}\int_0^a x^2 dx \\ &= \frac{aA^2}{3} = \frac{a}{b} \end{aligned}$$

In the limiting case when $b = a$, we have $P = 1$, which means the particle will be found to the left of a. In the limiting case when $b = 2a$, we have $P = 1/2$, which means the particle is equally likely to be found to the left or right of a.

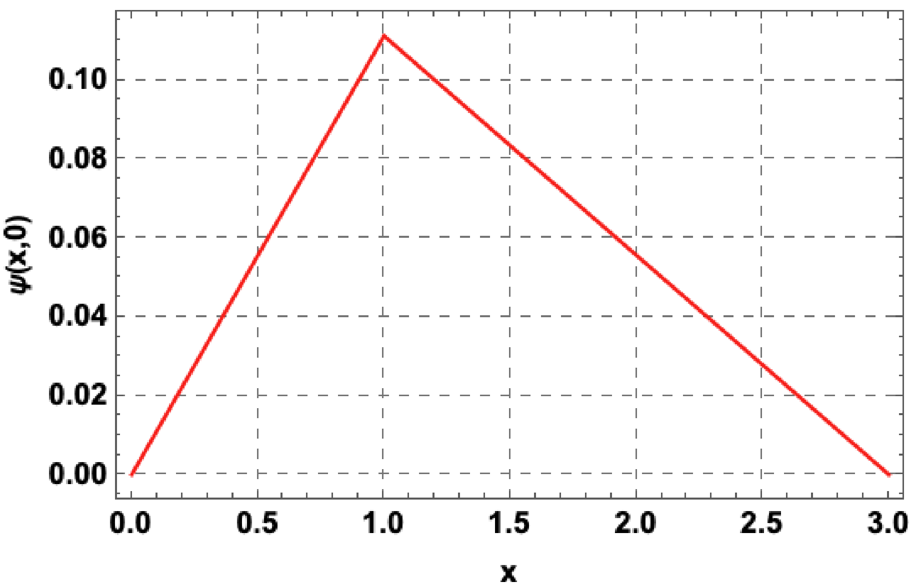

FIGURE 1 – Here $a = 1$ and $b = 3$

5. The expectation value of x is given by :

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x|\Psi(x,0)|^2 dx \\
&= \int_0^a x|Ax/a|^2 dx + \int_a^b x|A(b-x)/(b-a)|^2 dx \\
&= \frac{A^2}{a^2}\int_0^a x^3 dx + \frac{A^2}{(b-a)^2}\int_a^b x(b-x)^2 dx \\
&= \frac{2a+b}{4}
\end{aligned}$$