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Integrated Sciences and Mathematics (iSciM)

Quantum Mechanics Springs 2023
Example Sheet 5
Due : 16 April 2023

Problem : Fourier Transforms and Expectation Values (15 points)

Function $f(x)$ and its Fourier transform $\tilde{f}(k)$ are related by,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \tilde{f}(k)$$
$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} f(x)$$

By definition, the expectation is defined as :

$$\langle x \rangle = \int dx P(x) x = \int dx |\psi(x)|^2 x$$

Using the definition of expectation values of (powers of) the momentum operator,

$$\langle \hat{p}^n \rangle = \int dx \psi^*(x) \hat{p}^n \psi(x)$$

the form of the momentum operator,

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

and the definition of the Fourier transform,

1. **(5 points) :** Show that

$$\langle \hat{p} \rangle = \int \frac{dk}{2\pi} |\tilde{\psi}(k)|^2 k$$

2. **(5 points) :** and that

$$\langle \hat{p}^2 \rangle = \int \frac{dk}{2\pi} |\tilde{\psi}(k)|^2 k^2,$$

3. **(5 points) :** and, in general, that

$$\langle f(\hat{p}) \rangle = \int \frac{dk}{2\pi} |\tilde{\psi}(k)|^2 f(\hbar k).$$

The relation $P(k) = |\tilde{\psi}(k)|^2$, as discussed in lecture, thus follows from the Born relation,

$$P(x) = |\psi(x)|^2.$$

Problem : Superposition in the Infinite Well (20 points)

Verify the results for the eigenvalues of the energy operator for the infinite potential well of width L,

$$V(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{else} \end{cases}$$

for which the energy eigenvalues are,

$$E_n = \frac{2k_n^2}{2m} = \frac{2(n+1)^2\pi^2}{2mL^2}$$

Now suppose that at $t = 0$ we place a particle in an infinite well in the state

$$\psi_A(x,0) = \frac{1}{\sqrt{6}}\varphi_0(x) + \frac{1}{\sqrt{3}}\varphi_1(x) + \frac{1}{\sqrt{2}}\varphi_2(x)$$

Note : Each step below requires relatively little computation. You will not need the functional form of the energy eigenfunctions $\varphi_n(x)$ to complete them, only the energy eigenvalues.

1. **(2 points) :** How does ψ_A evolve with time ? Write down the expression for $\psi_A(x,t)$.
2. **(3 points) :** Calculate the expectation value of the energy, $< \hat{E} >$, for the particle described by $\psi_A(x,t)$. Write your answer in terms of E_0. Does this quantity change with time ?
3. **(2 points) :** What is the probability of measuring the energy to equal $< \hat{E} >$ as a result of a single measurement at $t = 0$? At a later time, $t = t_1$?
4. **(2 points) :** What energy values will be observed due to a single measurement at $t = 0$ and with what probabilities ? How do these probabilities change with time ?
5. **(2 points) :** The energy of the particle is found to be E_2 as a result of a single measurement at $t = t_1$. Write down the wave function $\psi_A(x,t)$, which describes the state of the particle for $t > t_1$. What energy values will be observed and with what probabilities at a time $t_2 > t_1$?
6. **(2 points) :** Construct another normalized wave function $\psi_B(x,0)$ which is linearly independent of $\psi_A(x,0)$ but yields the same value of $< \hat{E} >$ as well as the same set of measured energies with the same probabilities.
7. **(2 points) :** Construct another normalized wave function $\psi_C(x,0)$ which is linearly independent of $\psi_A(x,0)$, yields the same value of $< \hat{E} >$, but allows a different set of measured energies (which may include some but not all of E_0, E_1 and E_2, plus others).

Problem 3 : A Hard Wall [5 points]

A particle of mass m is moving in one dimension, subject to the potential $V(x)$:

$$V(x) = \begin{cases} 0, & \text{for } x > 0, \\ \infty, & \text{for } x \leq 0. \end{cases}$$

Find the stationary states and their energies. These states cannot be normalized.

Problem 4 : A Step Up on the Infinite Line [10 points]

A particle of mass m is moving in one dimension, subject to the potential $V(x)$:

$$V(x) = \begin{cases} V_0, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

Find the stationary states for energies ($0 < E < V_0$).

Problem 5 : A Wall and Half of a Finite Well [10 points]

A particle of mass m is moving in one dimension, subject to the potential $V(x)$:

$$V(x) = \begin{cases} \infty, & \text{for } x < 0, \\ -V_0, & \text{for } 0 < x < a, \quad (V_0 > 0) \\ 0, & \text{for } x > a. \end{cases}$$

In this case, find the stationary states corresponding to bound states ($E < 0$). Is there always a bound state ? Find the minimum value of z_0 given by

$$z_0^2 = \frac{2ma^2V_0}{\hbar^2},$$

For which there are three bound states. Explain the precise relation of this problem to the problem of the finite square well of width $2a$.

Problem 6 : Evaluate the following integrals :

1. $$\int_{-3}^{1} (x^3 - 3x^2 + 2x - 1)\delta(x+2)dx$$

2. $$\int_{0}^{\infty} [\cos(3x) + 2]\delta(x - \pi)dx$$

3. $$\int_{-1}^{1} \exp(|x| + 3)\delta(x - 2)dx$$

Problem 8 : Delta Potential

Consider the double delta-function potential

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)],$$

where α and a are positive constants.

1. Sketch this potential.
2. How many bound states does it posses? Find the allowed energies for $\alpha = \hbar^2/ma$ and $\alpha = \hbar^2/4ma$ and sketch the wavefunctions.