Habib University | Dr. Muntazir Abidi
Dhanani School of Science and Engineering | Integrated Sciences and Mathematics (iSciM)

Quantum Mechanics Spring 2023
Exercise Sheet 4
Issued : 07 Feb 2023 Due : 14 Feb 2023

Note : *Please submit your scanned solutions directly on canvas before the deadline.*

Problem 1 (15 marks)

Let's consider one spatial dimension and time. Remember, we have defined the position and momentum operators in the class, $\hat{x}$ and $\hat{p}$, respectively. The momentum operator is the derivative operator defined as :

$$\hat{p} = -i\hbar \frac{d}{dx} \tag{1}$$

where $\hbar$ is the modified Planck's constant. You now know that if $\psi(x,t)$ is the wavefunction of the quantum system, then $|\psi(x,t)|^2$ is the probability distribution function. For any operator $\hat{Q}$, the expectation of it is defined by the following equation :

$$\langle \hat{Q} \rangle = \int dx \psi(x,t)^* \hat{Q} \psi(x,t) \tag{2}$$

1. Calculate the expectations of the position and momentum operators, $\langle \hat{x} \rangle$ and $\langle \hat{p} \rangle$.
2. Calculate the time derivative of the position expectation, $d\langle \hat{x} \rangle / dt$.
3. We define the time derivative of the position expectation by $\hat{v}$. Show that $\langle p \rangle = m\langle v \rangle$.
4. The kinetic energy is defined as $T = \frac{p}{2m}$. Define the kinetic energy operator and calculate the expectation value of the kinetic energy operator $\langle \hat{T} \rangle$.
5. Show that
$$\frac{\langle \hat{p} \rangle}{dt} = \langle -\frac{dV}{dx} \rangle \tag{3}$$
This is known as **Ehrenfest's Theorem**, which shows that the expectation values obey Newton's Second law.

Problem 2 (15 marks)

In class, we derived the Probability conservation law in QM. The probability conservation tells us that the particle is conserved "locally" and is stable. Suppose you want to describe an "unstable" particle that spontaneously disintegrates with a lifetime of τ. In that case, the total probability of finding the particle somewhere should not be constant but decrease exponentially :

$$P(t) \equiv \int dx |\psi(x,t)|^2 = e^{-t/\tau} \tag{4}$$

In our derivation, we used the fact that the potential energy V is **real**. This leads to the conservation of probability. What if we assign to V an imaginary part :

$$V = V_0 - i\Gamma \tag{5}$$

where V_0 is the true potential energy and Γ is the positive real constant.

1. Calculate $\frac{dP}{dt}$.
2. Solve for $P(t)$ and find the lifetime of the particle in terms of Γ.