Habib University
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Integrated Sciences and Mathematics (iSciM)

Quantum Mechanics Spring 2023
Exercise Sheet 2
Issued : 24 January 2022 **Due : 31 January 2023**

Note : *Please submit your scanned solutions directly on canvas before the deadline.*

Problem 1 (10 marks)

A star such as our Sun will eventually evolve to a "red giant" star and then to a "white dwarf" star. A typical white dwarf is approximately the size of Earth, and its surface temperature is about $2.5 \times 10^4 K$. A typical red giant has a surface temperature of $3.0 \times 10^3 K$ and a radius $100,000$ times larger than that of a white dwarf. What is the average radiated power per unit area and the total power radiated by each of these stars? How do they compare?

Solution

A simple proportion based on Stefan's law gives :

$$\frac{P_{\text{dwarf}}/A_{\text{dwarf}}}{P_{\text{giant}}/A_{\text{giant}}} = \frac{\sigma T^4_{\text{dwarf}}}{\sigma T^4_{\text{giant}}} = \left(\frac{T_{\text{dwarf}}}{T_{\text{giant}}}\right)^4 = \left(\frac{2.5 \times 10^4}{3.0 \times 10^3}\right)^4 = 4820$$

The power emitted per unit area by a white dwarf is about 5000 times that of the power emitted by a red giant. Denoting this ratio by $a = 4.8 \times 10^3$, we get :

$$\frac{P_{\text{dwarf}}}{P_{\text{giant}}} = a\frac{A_{\text{dwarf}}}{A_{\text{giant}}} = a\frac{4\pi R^2_{\text{dwarf}}}{4\pi R^2_{\text{giant}}} = a\left(\frac{R_{\text{dwarf}}}{R_{\text{giant}}}\right)^2 = 4.8 \times 10^{-7}$$

We see that the total power emitted by a white dwarf is a tiny fraction of the total power emitted by a red giant. Despite its relatively lower temperature, the overall power radiated by a red giant far exceeds that of the white dwarf because the red giant has a much larger surface area. To estimate the absolute value of the emitted power per unit area, we again use Stefan's law. For the white dwarf, we obtain :

$$P_{\text{dwarf}}/A_{\text{dwarf}} = \sigma T^4_{\text{dwarf}} = 5.67 \times 10^{-8}\text{W m}^{-2}\text{K}^{-4} \times (2.5 \times 10^4\text{K})^4 \approx 2.2 \times 10^{10}\text{W m}^{-2}$$

The analogous result for the red giant is obtained by scaling the result for a white dwarf :

$$P_{\text{giant}}/A_{\text{giant}} = 2.2 \times 10^{10}/4.82 \times 10^3\text{W m}^{-2} = 4.56 \times 10^6\text{W m}^{-2} \approx 4.6 \times 10^6\text{W m}^{-2}$$

Problem 2 (5 marks)

An iron poker is being heated. As its temperature rises, the poker begins to glow—first dull red, then bright red, then orange, and then yellow. Use either the blackbody radiation curve or Wien's law to explain these changes in the colour of the glow.

Solution

The changes in the colour of the glow of the iron poker as its temperature rises can be explained by Wien's law. Wien's law states that the peak wavelength of radiation emitted by a blackbody is inversely proportional to its temperature. As the temperature of the iron poker increases, the peak wavelength of the radiation it emits shifts to shorter wavelengths, meaning it emits more blue light. At lower temperatures, the iron poker emits radiation primarily in the infrared region, which is not visible to the human eye. As the temperature increases, the peak wavelength shifts towards the visible spectrum. At first, the iron poker appears dull red because it emits radiation at the longest wavelengths visible to the human eye. As the temperature increases, the poker emits more radiation at shorter wavelengths, causing it to appear brighter red, orange, and yellow as it approaches white light.

Therefore, the changes in the colour of the glow of the iron poker can be explained by the changing temperature, which causes the peak wavelength of the emitted radiation to shift towards shorter, visible wavelengths.

Problem 3 (5 marks)

Suppose that two stars, α and β, radiate exactly the same total power. If the radius of star α is three times that of star β, what is the ratio of the surface temperatures of these stars ? Which one is hotter ?

Solution

The total power a star radiates is proportional to its surface area and surface temperature to the fourth power (Stefan-Boltzmann law). Let's denote the surface temperature of star α by T_α and the surface temperature of star β by T_β. Let's also denote the radius of star α by R_α and the radius of star β by R_β.
Since both stars radiate the same total power, we can write :

$$\sigma\pi R_\alpha^2 T_\alpha^4 = \sigma\pi R_\beta^2 T_\beta^4$$

where σ is the Stefan-Boltzmann constant and π is the mathematical constant pi.
Dividing both sides of the equation by $\sigma\pi R_\beta^2 T_\alpha^4$, we get :

$$(R_\alpha/R_\beta)^2 = T_\beta^4/T_\alpha^4$$

Taking the square root of both sides of the equation, we get :

$$R_\alpha/R_\beta = (T_\beta/T_\alpha)^2$$

Since $R_\alpha = 3R_\beta$, we can substitute this into the equation to obtain :

$$(T_\beta/T_\alpha)^2 = 3 \implies T_\beta/T_\alpha = \sqrt{3}$$

so the star β is hotter.

Problem 4 (10 marks)

A 1.0-kg mass oscillates at the end of a spring with a spring constant of 1000 N/m. The amplitude of these oscillations is 0.10 m. Use the concept of quantization to find the energy spacing for this classical oscillator. Is the energy quantization significant for macroscopic systems, such as this oscillator ?

Solution

The energy of a classical harmonic oscillator is given by :
$E = \frac{1}{2}kA^2$
where k is the spring constant and A is the amplitude of oscillation.
In this case, $k = 1000$ N/m and $A = 0.10$ m, so the energy of the oscillator is :
$E = \frac{1}{2}(1000 \text{ N/m})(0.10 \text{ m})^2 = 5$ J
The concept of quantization tells us that the energy of a system can only take on certain discrete values, rather than any arbitrary value. For a quantum harmonic oscillator, the energy spacing between these discrete levels is given by :
$\Delta E = hf$
where h is Planck's constant and f is the oscillator's frequency. For a classical oscillator, we can use this same formula to find the energy spacing, but we need to express the frequency in terms of the amplitude and mass of the oscillator.
The frequency of a classical harmonic oscillator is given by :
$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$
where m is the mass of the oscillator. In this case, $m = 1.0$ kg, so the frequency of the oscillator is :
$f = \frac{1}{2\pi}\sqrt{\frac{1000 \text{ N/m}}{1.0 \text{ kg}}} \approx 5.0$ Hz
Using this frequency and Planck's constant $h \approx 6.63 \times 10^{-34}$ J s, we can find the energy spacing :
$\Delta E = hf \approx (6.63 \times 10^{-34} \text{ J s})(5.0 \text{ Hz}) \approx 3.3 \times 10^{-33}$ J
This energy spacing is incredibly small compared to the total energy of the oscillator, which is 5 J. Therefore, energy quantization is not significant for macroscopic systems such as this oscillator.

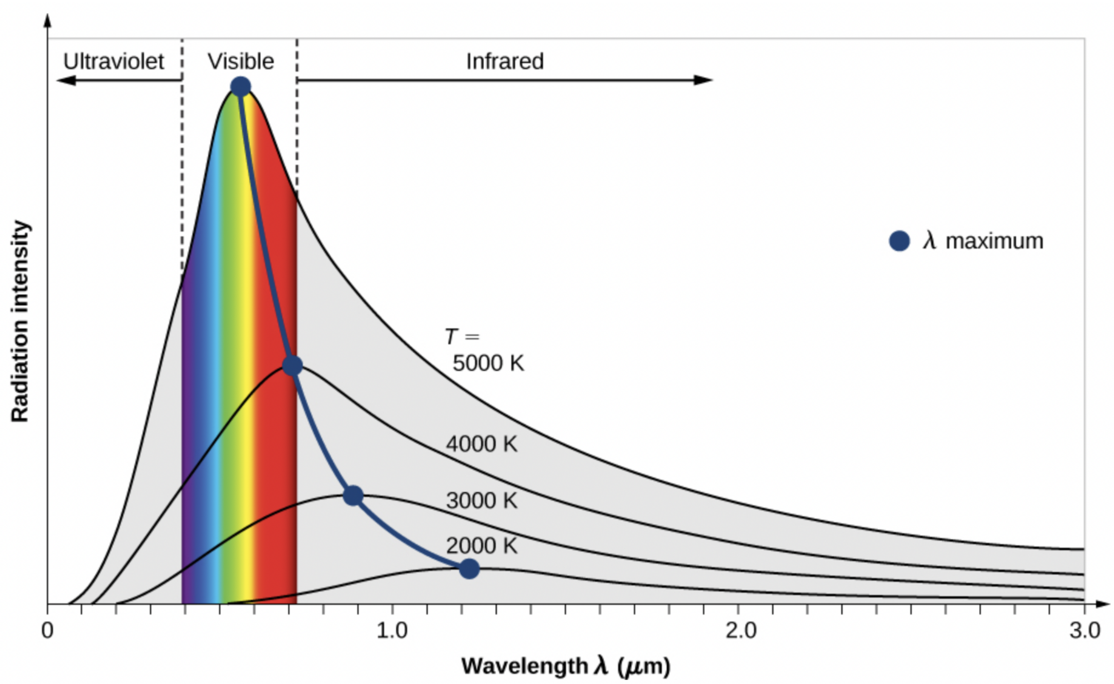

FIGURE 1 – The intensity of blackbody radiation versus the wavelength of the emitted radiation.