13. 15. 17. 18. 20

13解

(1). $f(x,y)=\begin{cases}\frac{21}{4}x^2y & x^2\le y\le 1\\ 0 & 其他\end{cases}$

$f_Y(y)=\begin{cases}\int_{-\infty}^{+\infty}f(x,y)\,dx\\ 0 & 其他\end{cases}=\begin{cases}\int_{-\sqrt{y}}^{\sqrt{y}}\frac{21}{4}x^2y\,dx\\ 0 & 其他\end{cases}=\begin{cases}\frac{7}{2}y^{\frac{5}{2}} & 0\le y\le 1\\ 0 & 其他\end{cases}$

则 $f_{X|Y}(x|y)=\begin{cases}\dfrac{f(x,y)}{f_Y(y)} & 0\le y\le 1.\\ 0 & 其他\end{cases}$

$=\begin{cases}\frac{3}{2}x^2y^{-\frac{3}{2}} & x^2\le y\le 1.\\ 0 & 其他\end{cases}$

$Y=\frac{1}{2}$ 时.

$f_{X|Y}(x|y=\frac{1}{2})=\begin{cases}3\sqrt{2}\,x^2 & x^2\le\frac{1}{2}\\ 0 & 其他.\end{cases}$

(2). $f_X(x)=\begin{cases}\int_{x^2}^{1}\frac{21}{4}x^2y\,dy\\ 0 & 其他\end{cases}=\begin{cases}\frac{21}{8}x^2(1-x^2) & x^2\le 1\\ 0 & 其他\end{cases}$

则 $f_{Y|X}(y|x)=\begin{cases}\dfrac{f(x,y)}{f_X(x)}\\ 0\end{cases}=\begin{cases}\dfrac{2y}{1-x^4} & x^2\le y\le 1\\ 0 & 其他\end{cases}$

$X=\frac{1}{3}$ 时, $f_{Y|X}(y|x=\frac{1}{3})=\begin{cases}\frac{81}{40}y & \frac{1}{9}\le y\le 1\\ 0 & 其他\end{cases}$

$X=\frac{1}{2}$ 时. $f_{Y|X}(y|x=\frac{1}{2})=\begin{cases}\frac{32}{15}y & \frac{1}{4}\le y\le 1\\ 0 & 其他\end{cases}$

(3). $P\{Y \geq \frac{1}{4} \mid X = \frac{1}{2}\}$

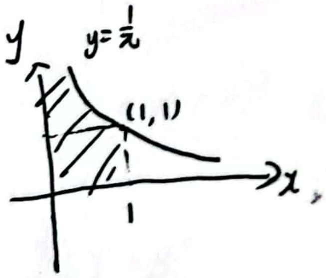
$$= \int_{\frac{1}{4}}^{+\infty} f_{Y|X}(y|x=\tfrac{1}{2})\,dy = \int_{\frac{1}{4}}^{1} \frac{32}{15} y \, dy = 1$$

$P\{Y \geq \frac{3}{4} \mid X = \frac{1}{2}\}$

$$= \int_{\frac{3}{4}}^{+\infty} f_{Y|X}(y|x=\tfrac{1}{2})\,dy = \int_{\frac{3}{4}}^{1} \frac{32}{15} y \, dy = \frac{7}{15}$$

15. (1). $f_X(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{其他} \end{cases}$

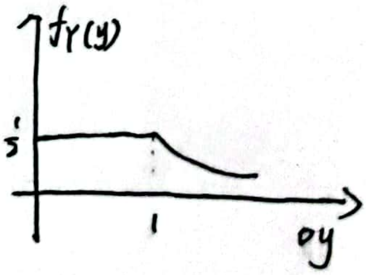
$$f(x,y) = \begin{cases} f_X(x) \cdot f_{Y|X}(y|x) & 0 < y < \frac{1}{x},\ 0 < x < 1 \\ 0 & \text{其他} \end{cases}$$



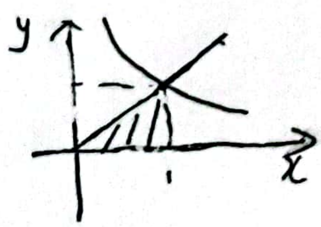
即: $f(x,y) = \begin{cases} x & 0 < y < \frac{1}{x},\ 0 < x < 1 \\ 0 & \text{其他} \end{cases}$

~~或. $f_Y(y) = \begin{cases} \int_{-\infty}^{+\infty} f(x,y)\,dx \\ 0 \quad 其他 \end{cases} = \begin{cases} \int_0^1 x \, dx \\ 0 \quad 其他 \end{cases} = \begin{cases} \frac{1}{2} \\ 0 \quad 其他 \end{cases}$~~

(2). $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \begin{cases} \int_0^1 x \, dx = \frac{1}{2} & 0 < y < 1 \\ \int_0^{\frac{1}{y}} x \, dx = \frac{1}{2y^2} & 1 \leq y < \infty \\ 0 & \text{其他} \end{cases}$



(3). $P\{X > Y\} = \int_0^1 dy \int_y^1 x \, dx = \int_0^1 \frac{1}{2}(1-y^2)\,dy = \frac{1}{3}$



17.解:

(1) $F_X(x) = F(x,+\infty) = \begin{cases} 1-e^{-\alpha x} & x \geq 0 \\ 0 & 其他 \end{cases}$

$F_Y(y) = F(+\infty, y) = \begin{cases} y & 0 \leq y \leq 1 \\ 1 & y > 1 \\ 0 & 其他 \end{cases}$

因为对于 $\forall x, y$，都有 $F(x,y) = F_X(x) \cdot F_Y(y)$.

故 X, Y 相互独立

(2) $P\{X=x\} = \sum_{y=1}^{\infty} p^2(1-p)^{x+y-2} = p^2(1-p)^{x-1} \sum_{y=1}^{\infty} (1-p)^{y-1}$

$= p^2(1-p)^{x-1} \frac{1}{1-(1-p)} = p(1-p)^{x-1}$ $\quad 0<p<1$, x 为正整数

$P\{Y=y\} = \sum_{x=1}^{\infty} p^2(1-p)^{x+y-2} = p^2(1-p)^{y-1} \sum_{x=1}^{\infty} (1-p)^{x-1}$

$= p^2(1-p)^{y-1} \frac{1}{1-(1-p)} = p(1-p)^{y-1}$ $\quad 0<p<1$ y为正整数

则对于 $\forall x, y$. 有 $P\{X=x, Y=y\} = P\{X=x\} \cdot P\{Y=y\}$

18. 解.

(1). $f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 其他 \end{cases}$

由于X和Y是两个相互独立的随机变量

$f(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} \frac{1}{2} e^{-y/2} & 0<x<1,\ y>0 \\ 0 & 其他 \end{cases}$

(2). $\Delta = (2X)^2 - 4Y > 0$

即: $X^2 - Y > 0$

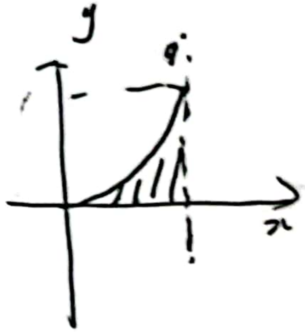
$P\{X^2 > Y\} = \int_0^1 dx \int_0^{x^2} \frac{1}{2} e^{-\frac{y}{2}} dy$

$= \int_0^1 1 - e^{-\frac{x^2}{2}} dx$

$= 1 - \sqrt{2\pi} \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$= 1 - \sqrt{2\pi} (\Phi(1) - \Phi(0))$

$= 0.1445$



20、解.

(1) X和Y是相互独立的随机变量.

则 $f(x,y)=f_X(x)\cdot f_Y(y)=\begin{cases}\lambda\mu\cdot e^{-\lambda x-\mu y} & x>0,\ y>0\\ 0 & 其他.\end{cases}$ $(\lambda>0,\mu>0)$

$f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_Y(y)}=f_X(x)=\begin{cases}\lambda e^{-\lambda x} & x>0\\ 0 & 其他\end{cases}$

(2). $P\{X\le Y\}=\int_0^{+\infty}dy\int_0^{y}\lambda\mu\cdot e^{-\lambda x-\mu y}dx$

$=\int_0^{+\infty}\mu e^{-\mu y}-\mu e^{(-\lambda-\mu)y}dy$

$=-e^{-\mu y}+\dfrac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)y}\Big|_0^{+\infty}$

$=\dfrac{\lambda}{\lambda+\mu}$

则 $P\{X>Y\}=1-P\{X\le Y\}=\dfrac{\mu}{\lambda+\mu}$

Z的分布律为:

Z	0	1
P	$\dfrac{\mu}{\lambda+\mu}$	$\dfrac{\lambda}{\lambda+\mu}$

Z的分布函数为:

$F_Z(z)=\begin{cases}0 & z<0\\ \dfrac{\mu}{\lambda+\mu} & 0\le z<1\\ 1 & z\ge1\end{cases}$