4 (1)　5.　8 (1), (2)　10　13　15

4.(1) 解:

(i) $\bar{x} = \dfrac{x_1 + x_2 + x_3}{3} = \dfrac{4}{3}$

$\mu_1 = \theta^2 + 4\theta(1-\theta) + 3(1-\theta)^2 = 3-2\theta$

则 $\hat{\theta} = \frac{1}{2} \times (3 - \mu_1)$

故 $\hat{\theta} = \dfrac{3-\bar{x}}{2} = \dfrac{5}{6}$

(ii). $L(\theta) = \prod_{i=1}^{3} P\{X_i = x_i\} = P\{X_1=1\} \cdot P\{X_2=2\} \cdot P\{X_3=1\}$

$= \theta^2 \cdot 2\theta(1-\theta) \cdot \theta^2$

$= 2\theta^5 - 2\theta^6 = 2\theta^5(1-\theta)$

$\ln L(\theta) = \ln[2\theta^5(1-\theta)] = \ln 2 + 5\ln\theta + \ln(1-\theta)$

$\dfrac{d}{d\theta} L(\theta) = \dfrac{5}{\theta} - \dfrac{1}{1-\theta}$ 单减.

令 $\dfrac{d}{d\theta} L(\theta) = 0$. 得 $\theta = \dfrac{5}{6}$

此时 $L(\theta)$ 有最大值.

则 $\hat{\theta} = \dfrac{5}{6}$

5. 解:

$L(\theta, c) = L(x_1, x_2, \cdots x_n; \theta, c)$

$$= \begin{cases} \prod_{i=1}^{n} \frac{1}{\theta} e^{-(x_i - c)/\theta} & x_i \geq c \\ 0 & \text{其他} \end{cases}$$

由于 $x_1 \leq x_2 \leq \cdots \leq x_n$.

则 $c \leq x_1 \leq x_2 \leq \cdots \leq x_n$

则 $L(\theta, c) = \begin{cases} \frac{1}{\theta^n} e^{-(\sum_{i=1}^{n} x_i - nc)/\theta} & x_i \geq c \\ 0 & \text{其他.} \end{cases}$

则 L 关于 c 单增. 当 $c = x_1$ 时, $L(\theta, c)$ 最大值.

则 $\hat{c} = x_1$

~~L 关于 θ 单减 θ = x~~

$c > x_1$ 时, $\ln L(\theta, c) = -n \ln \theta - \frac{1}{\theta}(\sum_{i=1}^{n} x_i - nc)$

$$\frac{\partial}{\partial \theta} \ln L(\theta, c) = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i - nc}{\theta^2}$$

令 $\frac{\partial}{\partial \theta} \ln(\theta, c) = 0$. 则 $\theta = \bar{x} - c$

故 $\hat{\theta} = \bar{x} - c = \bar{x} - x_1$

即 θ, c 的最大似然估计为

$$\begin{cases} \hat{\theta} = \bar{x} - x_1 \\ \hat{c} = x_1 \end{cases}$$

(2). $\mu_1 = \int_{-\infty}^{+\infty} t f(t)\, dt = \int_c^{\infty} \frac{t}{\theta} e^{-(t-c)/\theta} dt$

令 $u = \frac{t-c}{\theta}$. 则 $\mu_1 = \int_0^{\infty} (u\theta + c) e^{-u} du = c+\theta$

$\mu_2 = \int_{-\infty}^{+\infty} t^2 f(t)\, dt = \int_c^{\infty} \frac{t^2}{\theta} e^{-(t-c)/\theta} dt$

令 $u = \frac{t-c}{\theta}$ 则 $\mu_2 = \int_0^{\infty} (u\theta+c)^2 e^{-u} du = \theta^2\Gamma(3) + 2c\theta\Gamma(2) + c^2$

$= 2\theta^2 + 2c\theta + c^2$

$= (c+\theta)^2 + \theta^2$

则 $\theta = \sqrt{\mu_2 - \mu_1^2}$ $\quad c = \mu_1 - \sqrt{\mu_2 - \mu_1^2}$

令 $\mu_1 = A_1 = \bar{x}$

$\mu_2 = A_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

则 $A_2 - A_1^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

故 $\hat{\theta} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}$ $\quad \hat{c} = \bar{x} - \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}$

8. 解:

(1). $L(\theta) = \prod_{i=1}^{n} P\{X_i = x_i\} = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$

$\ln L(\theta) = n\ln\theta + (\theta-1)\ln\left(\prod_{i=1}^{n} x_i\right)$

$\frac{d}{d\theta}\ln L(\theta) = \frac{n}{\theta} + \ln\left(\prod_{i=1}^{n} x_i\right) = \frac{n}{\theta} + \sum_{i=1}^{n}\ln x_i$ 关θ单调减

故令 $\frac{d}{d\theta}\ln L(\theta) = 0$, $\theta = -\frac{n}{\sum_{i=1}^{n}\ln x_i}$ 此时 $L(\theta)$ 最大

故 $\hat{\theta} = -\frac{n}{\sum_{i=1}^{n}\ln x_i}$

$U = e^{-\frac{1}{\theta}}$ ~~关于θ单调递~~ 可逆.

则 $\hat{U} = e^{-1/\hat{\theta}}$

9). $\hat{\mu} = \bar{x}$ $X \sim N(\mu, 1)$

$\theta = P\{X > 2\} = 1 - P\{X \le 2\} = 1 - P\{X - \mu \le 2 - \mu\}$

$= 1 - \Phi(2 - \mu)$

则 $\hat{\theta} = 1 - \Phi(2 - \hat{\mu})$ 。

即 $\hat{\theta} = 1 - \Phi(2 - \bar{x})$

10. 解

(1). $\sigma^2 = E\left(c \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2\right)$

$= c \sum_{i=1}^{n-1} E[(X_{i+1} - X_i)^2]$

$= c \sum_{i=1}^{n-1} \{D(X_{i+1} - X_i) + [E(X_{i+1} - X_i)]^2\}$

$= c \sum_{i=1}^{n} [D(X_{i+1}) + D(X_i)]$

$= c \cdot 2\sigma^2 (n-1)$

故 $c = \frac{1}{2(n-1)}$

(2). $\mu^2 = E[(\bar{x})^2 - cS^2] = E(\bar{x}^2) - cE(S^2)$

$= (E\bar{x})^2 + D(\bar{x}) - cE(S^2)$

$= \mu^2 + \frac{\sigma^2}{n} - c\sigma^2$

故 $c = \frac{1}{n}$

13. 解.

(1) 证明：$\hat{\theta}$是θ的无偏估计

~~若$\hat{\theta^2}=(\hat{\theta})^2$是$\theta^2$的无偏估计~~ 则 $E(\hat{\theta})=\theta$

$$E[(\hat{\theta})^2] = D(\hat{\theta}) + [E(\hat{\theta})]^2$$

$$= D(\hat{\theta}) + \theta^2$$

由于 $D(\hat{\theta})>0$

故 $E[(\hat{\theta})^2] \neq \theta^2$

则 $(\hat{\theta})^2$ 不是 θ^2 的无偏估计

(2). $L(\theta) = \prod_{i=1}^{n} P\{X_i=x_i\} = \begin{cases} \frac{1}{\theta^n} & 0<x_i\le\theta \\ 0 & 其他 \end{cases}$

$L(\theta)$ 单关于 θ 单减

故 当θ取最小值时，$L(\theta)$最大

则 $\hat{\theta} = x_{(n)} = \max\{x_1, \cdots, x_n\}$

~~$E(\hat{\theta}) = E(x_{(n)})$~~

求 $X_{(n)}$ 的分布

$F_z(z) = P\{x_{(n)}\le z\} = \prod_{i=1}^{n} P\{x_i\le z\} = [F(z)]^n$

$f_z(z) = n[F(z)]^{n-1}\cdot f_z = \begin{cases} n\frac{z^{n-1}}{\theta^n} & 0\le z\le\theta \\ 0 & 其他 \end{cases}$

$E x_{(n)} = \int_0^\theta z\, n\frac{z^{n-1}}{\theta^n}\, dz = \frac{n}{n+1}\theta$

$E x_{(n)} \neq \theta$

故 $\hat{\theta}=x_{(n)}$ 不是 θ 的最大似然估计

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15. 解.

若 $\hat{\theta}$ 是 θ 的无偏估计

则 $\theta = E(\hat{\theta}) = E(\sum_{i=1}^{k} a_i X_i)$

$= \sum_{i=1}^{k} E(a_i X_i) = \sum_{i=1}^{k} a_i E(X_i) = \sum_{i=1}^{k} a_i \theta$

故 $\sum_{i=1}^{k} a_i = 1$

$D(X_i) = \sigma_i^2 \quad i=1,2,\cdots,k$，且 $X_1, X_2, \cdots, X_k$ 相互独立.

$D(\hat{\theta}) = D(\sum_{i=1}^{k} a_i X_i) = \sum_{i=1}^{k} D(a_i X_i)$

$= \sum_{i=1}^{k} a_i^2 D(X_i) = \sum_{i=1}^{k} a_i^2 \sigma_i^2$

$= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \cdots + a_k^2 \sigma_k^2$

要使 $D(\hat{\theta})$ 最小，且满足 $\sum_{i=1}^{k} a_i = 1$

设 $g(a_1, a_2, \cdots, a_k, \lambda) = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \cdots a_k^2 \sigma_k^2 + \lambda(a_1 + a_2 + \cdots + a_k - 1)$

则令 $\begin{cases} \dfrac{\partial g}{\partial a_1} = 2 a_1 \sigma_1^2 + \lambda = 0 \\ \vdots \\ \dfrac{\partial g}{\partial a_k} = 2 a_k \sigma_k^2 + \lambda = 0 \\ \dfrac{\partial g}{\partial \lambda} = a_1 + a_2 + \cdots + a_k - 1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -\dfrac{\lambda}{2\sigma_1^2} \\ \vdots \\ a_k = -\dfrac{\lambda}{2\sigma_k^2} \\ \lambda = \dfrac{-2}{\sum_{i=1}^{k} \frac{1}{\sigma_i^2}} \end{cases}$

设 $\sum_{i=1}^{k} \dfrac{1}{\sigma_i^2} = \dfrac{1}{\sigma_0^2} \Rightarrow \lambda = -2\sigma_0^2$

则 $a_1 = \dfrac{\sigma_0^2}{\sigma_1^2}, \cdots, a_k = \dfrac{\sigma_0^2}{\sigma_k^2}$

即当 $a_i = \dfrac{\sigma_0^2}{\sigma_i^2}, i=1,2,\cdots,k$ 时，

用 $\hat{\theta} = \sum_{i=1}^{k} a_i X_i$ 估计 θ 时，$\hat{\theta}$ 无偏且 $D(\hat{\theta})$ 最小.