421 HW 8 Group

change this to your names

NOTE: Unless stated otherwise, G is a (multiplicative) group with identity element e.

Problem (7.4.26)**.** If $G = \langle a \rangle$ is a cyclic group and $f : G \to H$ is a surjective homomorphism of groups, show that $f(a)$ is a generator of H, that is, H is the cyclic group $\langle f(a) \rangle$. [Hint: Exercise 7.4.15.]

Solution. By 7.4.15, $f(a)^n = f(a^n)$.

Since f is surjective, $\forall h \in H \quad \exists g \in G \quad f(g) = h$.

Since $g \in G$, $g = a^n$ for some n, and thus

$$\begin{aligned} f(g) &= h \\ f(a^n) &= h \\ f(a)^n &= h \end{aligned}$$

Every member of H is some power of $f(a)$.

Problem (7.4.30)**.** Let $f : G \to H$ be a homomorphism of groups and let J be a subgroup of H. Prove that the set $L = \{a \in G \mid f(a) \in J\}$ is a subgroup of G. [Note: The set L is the *inverse image* of J. You may have seen the notation $L = f^{-1}(J)$. Exercise 7.4.33 is the special case $J = \{e_H\}$. That is, $K_f = f^{-1}(\{e_H\})$, the *kernel* of f. I changed some of the letters in this exercise so as not to conflict with Exercise 7.4.33.]

Solution.

Problem (7.4.34)**.** The function $f : \mathbb{Z} \to \mathbb{Z}_5$ given by $f(x) = [x]$ is a homomorphism by Example 13. Find K_f and justify your finding. (Notation as in Exercise 7.4.33.)

Solution.

Problem (7.4.53)**.** Let $f : G \to H$ be an isomorphism of groups. Let $g : H \to G$ be the inverse function of f as defined in Appendix B. Prove that g is also an isomorphism of groups. (You may assume that the inverse of a bijection is a bijection, since you hopefully saw this in MTH 331.) [Hint: To show that $g(ab) = g(a)g(b)$, consider the images of the left-and right-hand sides under f and use the facts that f is a homomorphism and $f \circ g$ is the identity map.]

Solution.

Problem (7.4.DK2)**.** Let $\cong$ represent isomorphism between groups. Prove that $\cong$ is an equivalence relation. [Hint: See 7.4 Example 8, Individual Exercise 7.4.DK1, and Exercise 7.4.53]

Solution.