421 HW 8 Group

change this to your names

NOTE: Unless stated otherwise, G is a (multiplicative) group with identity element e.

Problem (7.4.26)**.** If $G = \langle a \rangle$ is a cyclic group and $f : G \to H$ is a surjective homomorphism of groups, show that $f(a)$ is a generator of H, that is, H is the cyclic group $\langle f(a) \rangle$. [Hint: Exercise 7.4.15.]

Solution. By 7.4.15, $f(a)^n = f(a^n)$.

Since f is surjective, $\forall h \in H \quad \exists g \in G \quad f(g) = h$.

Since $g \in G$, $g = a^n$ for some n, and thus

$$\begin{aligned} f(g) &= h \\ f(a^n) &= h \\ f(a)^n &= h \end{aligned}$$

Every member of H is some power of $f(a)$.

Problem (7.4.30)**.** Let $f : G \to H$ be a homomorphism of groups and let J be a subgroup of H. Prove that the set $L = \{a \in G \mid f(a) \in J\}$ is a subgroup of G. [Note: The set L is the *inverse image* of J. You may have seen the notation $L = f^{-1}(J)$. Exercise 7.4.33 is the special case $J = \{e_H\}$. That is, $K_f = f^{-1}(\{e_H\})$, the *kernel* of f. I changed some of the letters in this exercise so as not to conflict with Exercise 7.4.33.]

Solution. To prove: L is nonempty

J has e_H since it's a subgroup of H.

$f(e_G)$ must be e_H as a result of f being a homomorphism.

Since $f(e_G) \in L$, L is nonempty.

To prove: L is closed under multiplication

Let $a, b \in L$. Since f is a homomorphism, $f(a)f(b) = f(ab)$. By the definition of L, $f(a), f(b) \in J$. Since J is a group, $f(ab) \in J$, and $ab \in L$.

To prove: L is closed under inverses

Let $a \in L$. By the definition of L, $f(a) \in J$. Since J is a group, $f(a)^{-1} \in J$. As a result of f being a homomorphism, $f(a^{-1}) \in J$. By the definition of L, $a^{-1} \in L$.

Problem (7.4.34)**.** The function $f : \mathbb{Z} \to \mathbb{Z}_5$ given by $f(x) = [x]$ is a homomorphism by Example 13. Find K_f and justify your finding. (Notation as in Exercise 7.4.33.)

Solution. Exercise 33 says $K_f = \{a \in G \mid f(a) = e_H\}$

In this case, $K_f = \{a \in \mathbb{Z} \mid [a]_5 = [0]_5\}$

K_f is the set of all integer multiples of 5.

If an integer k is a multiple of 5, then $5 \mid k - 0$, and $[k]_5 = [0]_5$. If an integer k is not a multiple of 5, then, for some $b \in \{1, 2, 3, 4\}$, $5 \mid k - b$, $[k]_5 = [b]_5$, where $[b]_5 \neq [0]_5$. By transitivity of equivalence, $[k]_5 \neq [0]_5$. Therefore, $k \in K_f$ if and only if $5 \mid k$.

Problem (7.4.53)**.** Let $f : G \to H$ be an isomorphism of groups. Let $g : H \to G$ be the inverse function of f as defined in Appendix B. Prove that g is also an isomorphism of groups. (You may assume that the inverse of a bijection is a bijection, since you hopefully saw this in MTH 331.) [Hint: To show that $g(ab) = g(a)g(b)$, consider the images of the left-and right-hand sides under f and use the facts that f is a homomorphism and $f \circ g$ is the identity map.]

Solution.

Problem (7.4.DK2)**.** Let $\cong$ represent isomorphism between groups. Prove that $\cong$ is an equivalence relation. [Hint: See 7.4 Example 8, Individual Exercise 7.4.DK1, and Exercise 7.4.53]

Solution.