421 HW 5 Group

change this to your names

Note: R denotes a ring and F denotes a field and p denotes a positive prime number.

Problem (4.1.17)**.** Let R be an integral domain. Assume that the Division Algorithm always holds in $R[x]$. Prove that R is a field.

Solution. Statement of the division algorithm: given $f \in F[x]$ and $g \in F[x]$, there exist some polynomials p and r such that $f = gp + r$ and either $r = 0$ or degree of r is less than degree of g.

Suppose that a is an arbitrary nonzero element of R.

By the Division Algorithm, there exists some p and r for which $1 = pa + r$, where r is either 0 or has degree less than a. a has degree 0, so it must be that $r = 0$. Thus, $1 = pa$. a has an inverse.

Since a was an arbitrary nonzero element of R, every nonzero element of R has a multiplicative inverse. Therefore, R is a field.

Problem (4.2.14)**.** Let $f(x), g(x), h(x) \in F[x]$, with $f(x)$ and $g(x)$ relatively prime. If $f(x) \mid h(x)$ and $g(x) \mid h(x)$, prove that $f(x)g(x) \mid h(x)$.

Solution.

Problem (4.3.12)**.** Express $x^4 - 4$ as a product of irreducibles in $\mathbb{Q}[x]$, in $\mathbb{R}[x]$, and in $\mathbb{C}[x]$.

Solution.

$\mathbb{Q}[x]$	$(x^2 + 2)(x^2 - 2)$
$\mathbb{R}[x]$	$(x^2 + 2)(x - \sqrt{2})(x + \sqrt{2})$
$\mathbb{C}[x]$	$(x - i\sqrt{2})(x + i\sqrt{2})(x - \sqrt{2})(x + \sqrt{2})$

Problem (4.4.16)**.** Let $f(x), g(x) \in F[x]$ have degree $\leq n$ and let $c_0, c_1, \ldots, c_n$ be distinct elements of F. If $f(c_i) = g(c_i)$ for $i = 0, 1, \ldots, n$, prove that $f(x) = g(x)$ in $F[x]$.

Solution. For $i \in 0, 1, ..., n$, it is said that $f(c_i) = g(c_i)$. With subtraction, $f(c_i) - g(c_i) = 0$. Since the degree of f and degree of g are both $\leq n$, it must be that $f - g = 0$ or the degree of $f - g$ is $\leq n$.

If $f - g$ is nonzero, since the degree of $f - g$ is $\leq n$, then $f - g$ must have at most n roots. This is not the case, as it is said to have $n + 1$ roots.

$f - g$ must therefore be the zero polynomial.

$f(x) - g(x) = 0$, so $f(x) = g(x)$.

Problem (4.4.19)**.** We say that $a \in F$ is a multiple root of $f(x) \in F[x]$ if $(x-a)^k$ is a factor of $f(x)$ for some $k \geq 2$.
(a) Prove that $a \in \mathbb{R}$ is a multiple root of $f(x) \in \mathbb{R}[x]$ if and only if a is a root of both $f(x)$ and $f'(x)$, where $f'(x)$ is the derivative of $f(x)$.
(b) If $f(x) \in \mathbb{R}[x]$ and if $f(x)$ is relatively prime to $f'(x)$, prove that $f(x)$ has no multiple root in $\mathbb{R}$.

Solution.