

## 421 HW 8 Group

**change this to your names**

**NOTE:** Unless stated otherwise,  $G$  is a (multiplicative) group with identity element  $e$ .

**Problem (7.4.26).** If  $G = \langle a \rangle$  is a cyclic group and  $f : G \rightarrow H$  is a surjective homomorphism of groups, show that  $f(a)$  is a generator of  $H$ , that is,  $H$  is the cyclic group  $\langle f(a) \rangle$ . [Hint: Exercise 7.4.15.]

**Solution.** By 7.4.15,  $f(a)^n = f(a^n)$ .

Since  $f$  is surjective,  $\forall h \in H \quad \exists g \in G \quad f(g) = h$ .

Since  $g \in G$ ,  $g = a^n$  for some  $n$ , and thus

$$\begin{aligned} f(g) &= h \\ f(a^n) &= h \\ f(a)^n &= h \end{aligned}$$

Every member of  $H$  is some power of  $f(a)$ .

**Problem (7.4.30).** Let  $f : G \rightarrow H$  be a homomorphism of groups and let  $J$  be a subgroup of  $H$ . Prove that the set  $L = \{a \in G \mid f(a) \in J\}$  is a subgroup of  $G$ . [Note: The set  $L$  is the *inverse image* of  $J$ . You may have seen the notation  $L = f^{-1}(J)$ . Exercise 7.4.33 is the special case  $J = \{e_H\}$ . That is,  $K_f = f^{-1}(\{e_H\})$ , the *kernel* of  $f$ . I changed some of the letters in this exercise so as not to conflict with Exercise 7.4.33.]

**Solution.** To prove:  $L$  is nonempty

$J$  has  $e_H$  since it's a subgroup of  $H$ .

$f(e_G)$  must be  $e_H$  as a result of  $f$  being a homomorphism.

Since  $f(e_G) \in L$ ,  $L$  is nonempty.

To prove:  $L$  is closed under multiplication

Let  $a, b \in L$ . Since  $f$  is a homomorphism,  $f(a)f(b) = f(ab)$ . By the definition of  $L$ ,  $f(a), f(b) \in J$ . Since  $J$  is a group,  $f(ab) \in J$ , and  $ab \in L$ .

To prove:  $L$  is closed under inverses

Let  $a \in L$ . By the definition of  $L$ ,  $f(a) \in J$ . Since  $J$  is a group,  $f(a)^{-1} \in J$ . As a result of  $f$  being a homomorphism,  $f(a^{-1}) \in J$ . By the definition of  $L$ ,  $a^{-1} \in L$ .

**Problem (7.4.34).** The function  $f : \mathbb{Z} \rightarrow \mathbb{Z}_5$  given by  $f(x) = [x]$  is a homomorphism by Example 13. Find  $K_f$  and justify your finding. (Notation as in Exercise 7.4.33.)

**Solution.**

**Problem** (7.4.53). Let  $f : G \rightarrow H$  be an isomorphism of groups. Let  $g : H \rightarrow G$  be the inverse function of  $f$  as defined in Appendix B. Prove that  $g$  is also an isomorphism of groups. (You may assume that the inverse of a bijection is a bijection, since you hopefully saw this in MTH 331.) [Hint: To show that  $g(ab) = g(a)g(b)$ , consider the images of the left-and right-hand sides under  $f$  and use the facts that  $f$  is a homomorphism and  $f \circ g$  is the identity map.]

**Solution.**

**Problem** (7.4.DK2). Let  $\cong$  represent isomorphism between groups. Prove that  $\cong$  is an equivalence relation. [Hint: See 7.4 Example 8, Individual Exercise 7.4.DK1, and Exercise 7.4.53]

**Solution.**