

421 HW 9 Group

change this to your names

NOTE: Unless stated otherwise, G is a (multiplicative) group with identity element e .

Hint: Some of these problems are related.

Problem (8.1.21). Let H and K , each of prime order p , be subgroups of a group G . If $H \neq K$, prove that $H \cap K = \langle e \rangle$.

Solution.

Problem (8.1.36). Let H and K be subgroups of a finite group G such that $[G : H] = p$ and $[G : K] = q$, with p and q distinct primes. Prove that pq divides $[G : H \cap K]$.

Solution.

Problem (8.1.37). Let G be an abelian group of order n and let k be a positive integer. If $(k, n) = 1$, prove that the function $f : G \rightarrow G$ given by $f(a) = a^k$ is an isomorphism. [Hint: To show f is a bijection, find a formula for f^{-1} .]

Solution.

Problem (8.1.40). If a prime p divides the order of a finite group G , prove that the number of elements of order p in G is a multiple of $p - 1$. [Cauchy's Theorem says that the number of elements of order p is positive. However, you do not need to prove that or apply it. That is, for this exercise, the proof will work fine if the number of elements of order p is $0(p - 1) = 0$.]

Solution.

Outline:

Let $Y = \{a \in G \mid |a| = p\}$

Let \sim be an equivalence relation on Y such that $a \sim b \Leftrightarrow b \in \langle a \rangle$ (conceptually, this should be equivalent to $a \sim b \Leftrightarrow \langle a \rangle = \langle b \rangle$, but this is not proven).

Each equivalence class is of size $p - 1$ ($[a] = \langle a \rangle \setminus \{e\}$).

Y is the union of these disjoint equivalence classes.

$|Y|$ must be a multiple of $p - 1$.

This depends upon the theorem that any group of size p , where p is prime, is isomorphic to \mathbb{Z}_p .

Proof:

To prove $[a] \subseteq \langle a \rangle \setminus \{e\}$

$$b \in [a]$$

$$b \in \langle a \rangle$$

$$[a] \subseteq Y$$

$$|b| = p$$

$$b \neq e$$

$$b \in \langle a \rangle \setminus \{e\}$$

$$\text{To prove } \langle a \rangle \setminus \{e\} \subseteq [a]$$

$$b \in \langle a \rangle \setminus \{e\}$$

$$|b| = p$$

$$b \in Y$$

$$b \in \langle a \rangle$$

$$b \in [a]$$

$$\text{Therefore, } \langle a \rangle \setminus \{e\} = [a].$$

$$\text{To prove } \forall a \in Y \quad \langle a \rangle \cong \mathbb{Z}_p$$

\sim is reflexive: yeah

\sim is symmetric:

$$b \in \langle a \rangle$$

$$\exists k \in \mathbb{Z} \quad b = a^k$$

$$b^{-k} = a$$

$$a \in \langle b \rangle$$

\sim is transitive:

$$a \sim b$$

$$b \sim c$$

$$\exists x, y \in \mathbb{Z} \quad b = a^x, c = a^y$$

$$c = a^{xy}$$

x cannot be a multiple of p , since, if it were, $b = e$. y can similarly not be a multiple of p . By Euclid's lemma, $p \nmid xy$. Thus, $a \sim c$.

Each equivalence class is of order $p - 1$:

$$e \notin [a], \text{ so } [a][e] = \langle a \rangle.$$

By closure in G , $\langle a \rangle \leq G$.

Since $|a| = p$, $|\langle a \rangle| = p$, so $\langle a \rangle \cong \mathbb{Z}_p$.

Problem (8.1.41). Prove that a group of order 33 contains an element of order 3. [Of course you are not allowed to apply Cauchy's Theorem.]

Solution.