421 HW 4 Group

change this to your names

You may use Subring Test Theorem 3.6 for your justifications in Section 3.1.

Problem (3.2.13)**.** Let S and T be subrings of a ring R. In (a) and (b), if the answer is "yes," prove it. If the answer is "no," give a counterexample.
(a) Is $S \cap T$ a subring of R?
(b) Is $S \cup T$ a subring of R?

Solution. (a)
Let a be an element of $S \cap T$, and let b be an element of $S \cap T$.
Since $a \in S \cap T$ and $b \in S \cap T$, $a \in S$ and $a \in T$, and $b \in S$ and $b \in T$.
Since $a \in S$ and $b \in S$, $ab \in S$ and $a + b \in S$ since S is a ring. Similarly, $ab \in T$ and $a+b \in T$. Since $ab \in S$ and $ab \in T$, $ab \in S \cap T$, so $S \cap T$ is closed under multiplication. Similarly, $a + b \in S$ and $a + b \in T$, so $a + b \in S \cap T$, so $S \cap T$ is closed under addition.
Since $S \neq \emptyset$ (because S is a ring), there exists $s \in S$. Since $s \in S$, s has an additive inverse $-s$. $s + (-s) = 0_R$ (because addition in S is the same as addition in R), so, by closure of addition, $0_R \in S$. Similarly, $0_R \in T$. Therefore, $0_R \in S \cap T$, so $S \cap T$ is nonempty.
By the Subring Test Theorem, $S \cap T$ is a subring of R.
(b)
$\{0, 2, 4\}$ and $\{0, 3\}$ are both subrings of $\mathbb{Z}_6$. However, $\{0, 2, 3, 4\}$ is not closed under addition ($2 + 3 = 5$, $2 + 3 \notin \{0, 2, 3, 4\}$), so not necessarily a subring.

Problem (3.2.25)**.** Let S be a subring of a ring R with identity.
(a) If S has an identity, show by example that 1_S may not be the same as 1_R.
(b) If both R and S are integral domains, prove that $1_S = 1_R$.

Solution. (a)
$\mathbb{Z}_6$ has an identity 1, but its subring consisting of elements $\{0, 2, 4\}$ has 4 as a multiplicative identity.
(b)
Since S is a subring of R, $1_S \in R$.

Problem (3.2.31)**.** A **Boolean ring** is a ring R with identity in which $x^2 = x$ for every $x \in R$. For examples, see Exercises 19 and 44 in Section 3.1. If R is a Boolean ring, prove that
(a) $a + a = 0_R$ for every $a \in R$, which means that $a = -a$. [Hint: Expand $(a + a)^2$.]
(b) R is commutative. [Hint: Expand $(a + b)^2$.]

Solution. (a)

$$\begin{aligned}(a+a)^2 &= (a+a)^2 \\ a^2+a^2+a^2+a^2 &= a+a \\ a+a+a+a &= a+a \\ a+a &= 0 \\ a &= -a\end{aligned}$$

(b)

$$\begin{aligned}(a+b)^2 &= (a+b)^2 \\ a+b &= (a+b)(a+b) \\ a+b &= a^2+ab+ba+b^2 \\ a+b &= a+ab+ba+b \\ 0 &= ab+ba \\ -ba &= ab \\ ba &= ab\end{aligned}$$

Problem (3.2.DK1)**.** Let R be a ring with identity. Prove that if $1_R = 0_R$, then $R = \{0_R\}$. That is, R is the zero ring.

Solution.