Equations Of Motion of Krang on Fixed Wheels

Munzir Zafar

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In this report we attempt to find the dynamic model of Golem Krang with its wheels fixed. So it is reduced to a serial robot with a tree-structure (due to two arms branching out). Figure 1 shows the frames of references we will be using to determine the transforms and the coordinates on the robot. We denote these frames using symbol R_i where $i \in \mathbb{F} = \{0, 1, 2, 3, 4l, 5l, 6l, 7l, 8l, 9l, 10l, 4r, 5r, 6r, 7r, 8r, 9r, 10r\}$. R_0 is the world frame fixed in the middle of the two wheels. R_1, R_2, R_3 are fixed on the base, spine and torso with their rotations represented by q_{imu}, q_w and q_{torso} respectively. Frames $R_{4l}, ...R_{10l}$ are frames fixed on the links left 7-DOF arm with their motion represented by $q_{1l}, ...q_{7l}$. Similarly, frames $R_{4r}, ...R_{10r}$ are frames fixed on the links right 7-DOF arm with their motion represented by $q_{1r}, ...q_{7r}$. All equations in the following text that do not show r or l in the subscript where they are supposed to, will mean that the respective equations are valid for both subscripts.

We will be using the Kane's formulation. This is done so that our current analysis can easily be merged with the dynamic modelling of wheeled inverted pendulum which is found in terms of quais-velocities, that prohibit the use of Lagrange for analytical modelling of the robot. Kane's method however is applicable for this problem.

1 Introduction to Kane's Formulation

$$\sum_k \left[m_k \bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j + \left(\frac{d\bar{H}_{Gk}}{dt} \right) \cdot (\bar{\omega}_k)_j \right] = \sum_n \bar{F}_n \cdot (\bar{v}_n)_j + \sum_n \bar{M}_m \cdot (\bar{\omega}_m)_j \quad j = 1...K \tag{1}$$

where

j is the unique number identifyin each generalized co-ordinate in the system

k is the unique number identifying each rigid body in the system

n is the unique number identifying each external force acting on the system

m is the unique number identifying each external torque acting on the system

m_k is the mass of the kth body

$\bar{a}_{Gk}$ is the acceleration of the center of mass of kth body

$\bar{v}_{Gk}$ is the velocity of the center of mass of the kth body

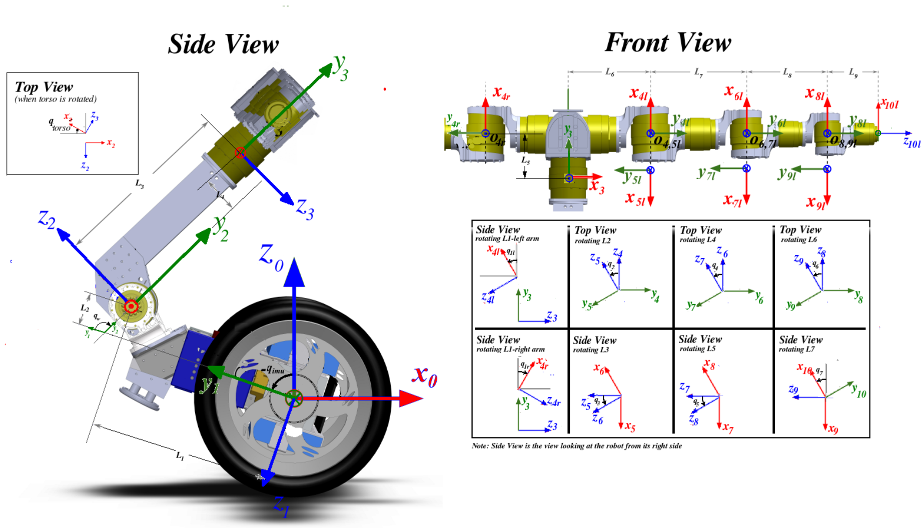

Figure 1: Frames of references on the robot

$\bar{H}_{Gk}$ is the angular momentum of body k about its center of mass

$\bar{\omega}_k$ is the angular velocity of the body k

F_n is the nth external force

M_m is the mth external moment

$\bar{v}_n$ is the velocity of the point at which external Force F_n is acting

$\bar{\omega}_m$ is the angular velocity of the body on which torque is acting relative to the actuator applying the torque

$()_j$ $= \frac{\partial()}{\partial \dot{q}_j}$ the partial derivative of the quantity in brackets () with respect to the generalized velocity $\dot{q}_j$

2 Finding Dynamic Model for our robot

In this section we determine the symbolic expression for the total kinetic energy E of the robot.

2.1 Transformations

The transformation of frame R_i into frame R_j is represented by the homogeneous transformation matrix ${}^iT_j$ such that.

$$ {}^iT_j = \begin{bmatrix} {}^is_j & {}^in_j & {}^ia_j & {}^iP_j \end{bmatrix} = \begin{bmatrix} & {}^iA_j & & {}^iP_j \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & n_x & a_x & P_x \\ s_y & n_y & a_y & P_y \\ s_z & n_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2} $$

where ${}^is_j$, ${}^in_j$ and ${}^ia_j$ contain the components of the unit vectors along the x_j, y_j and z_j axes respectively expressed in frame R_i, and where ${}^iP_j$ is the vector representing the coordinates of the origin of frame R_j expressed in frame R_i.

The transformation matrix ${}^iT_j$ can be interpreted as: (a) the transformation from frame R_i to frame R_j and (b) the representation of frame R_j with respect to frame R_i. Using figure 1, we can write down these transformation matrices for our system as follows:

$$
{}^0T_1 = \begin{bmatrix} 0 & sq_{imu} & -cq_{imu} & 0 \\ -1 & 0 & 0 & 0 \\ 0 & cq_{imu} & sq_{imu} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
{}^1T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cq_w & sq_w & L_1 \\ 0 & -sq_w & cq_w & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},
{}^2T_3 = \begin{bmatrix} -cq_{torso} & 0 & sq_{torso} & 0 \\ 0 & 1 & 0 & L_3 \\ -sq_{torso} & 0 & -cq_{torso} & L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^3T_{4l} = \begin{bmatrix} 0 & 1 & 0 & L_6 \\ cq_{1l} & 0 & -sq_{1l} & L_5 \\ -sq_{1l} & 0 & -cq_{1l} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
{}^3T_{4r} = \begin{bmatrix} 0 & -1 & 0 & -L_6 \\ cq_{1r} & 0 & -sq_{1r} & L_5 \\ sq_{1r} & 0 & cq_{1r} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
{}^4T_5 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -cq_2 & -sq_2 & 0 \\ 0 & -sq_2 & cq_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^5T_6 = \begin{bmatrix} -cq_3 & 0 & sq_3 & 0 \\ 0 & -1 & 0 & -L_7 \\ sq_3 & 0 & cq_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
{}^6T_7 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -cq_4 & -sq_4 & 0 \\ 0 & -sq_4 & cq_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
{}^7T_8 = \begin{bmatrix} -cq_5 & 0 & sq_5 & 0 \\ 0 & -1 & 0 & -L_8 \\ sq_5 & 0 & cq_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^8T_9 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -cq_6 & -sq_6 & 0 \\ 0 & -sq_6 & cq_6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
{}^9T_{10} = \begin{bmatrix} -cq_7 & -sq_7 & 0 & 0 \\ 0 & 0 & -1 & -L_9 \\ sq_7 & -cq_7 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

2.2 Velocities and Accelerations of Frames

The angular and linear velocities of the frames can be calculated using the recursive formulation:

$$
\begin{aligned}
{}^j\omega_j &= {}^jA_i {}^i\omega_i + \dot{q}_j \, {}^je_j && (3)\\
{}^j\alpha_j &= {}^jA_i {}^i\alpha_i + \ddot{q}_j \, {}^je_j + \dot{q}_j \, ({}^j\omega_j \times {}^je_j) && (4)\\
{}^jV_j &= {}^jA_i \left({}^iV_i + {}^i\omega_i \times {}^iP_j\right) && (5)\\
{}^ja_j &= {}^jA_i \left({}^ia_i + {}^i\alpha_i \times {}^iP_j + {}^i\omega_i \times ({}^i\omega_i \times {}^iP_j)\right) && (6)
\end{aligned}
$$

where ${}^i\omega_j$, ${}^i\alpha_j$, ${}^ia_j$ and ${}^iV_j$ denote the angular velocity, linear velocity, angular acceleration and linear acceleration repectively of frame j measured with respect to the world frame and represented in frame i. ${}^je_j$ denotes the direction of local angular velocity of frame j represented in frame j. $i, j \in \mathbb{F}$ identify the frames and i identifies the antecedent frame of j. So, the rotation ${}^jA_i$ and the translation ${}^jP_i$ that appear in these equations can not be directly deduced from the transformations listed in the previous section, as the they all represent ${}^iT_j$ (note the position of i and j). Rather, we need to use following expressions to deduce our matrices:

$$
\begin{aligned}
{}^jA_i &= {}^iA_j^T \\
{}^jP_i &= -{}^iA_j^T \, {}^iP_j
\end{aligned}
$$

Since frame R_0 is fixed ${}^0\omega_0$ and ${}^0V_0$ are both $\begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$. We can deduce directions of local angular velocities of the frames using figure 1 as follows.

$$
\begin{gathered}
{}^1e_1 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^2e_2 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^3e_3 = \begin{bmatrix}0 & -1 & 0\end{bmatrix}^T, {}^4e_4 = \begin{bmatrix}0 & -1 & 0\end{bmatrix}^T, \\
{}^5e_5 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^6e_6 = \begin{bmatrix}0 & -1 & 0\end{bmatrix}^T, {}^7e_7 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^8e_8 = \begin{bmatrix}0 & -1 & 0\end{bmatrix}^T, \\
{}^9e_9 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^{10}e_{10} = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^T
\end{gathered}
$$

This information can now be used to derive expressions for the velocities and accelerations of the frames.

2.3 Inertial Forces

$$
\begin{aligned}
\bar{v}_{Gk} &= \bar{v}_k + \bar{\omega}_k \times \bar{S}_k \\
\bar{a}_{Gk} &= \bar{a}_k + \bar{\alpha}_k \times \bar{S}_k + \bar{\omega}_k \times (\bar{\omega}_k \times \bar{S}_k) \\
\bar{H}_{Gk} &= \mathbf{J}_{Gk}\bar{\omega}_k \\
\frac{d\bar{H}_{Gk}}{dt} &= \mathbf{J}_{Gk}\bar{\alpha}_k + \bar{\omega}_k \times \mathbf{J}_{Gk}\bar{\omega}_k \\
&= (\mathbf{J}_k + m_k\mathbf{S}_k^\times\mathbf{S}_k^\times)\bar{\alpha}_k + \bar{\omega}_k \times (\mathbf{J}_k + m_k\mathbf{S}_k^\times\mathbf{S}_k^\times)\bar{\omega}_k \\
&= \mathbf{J}_k\bar{\alpha}_k + \bar{\omega}_k \times \mathbf{J}_k\bar{\omega}_k + m_k\mathbf{S}_k^\times\mathbf{S}_k^\times\bar{\alpha}_k + \bar{\omega}_k \times (m_k\mathbf{S}_k^\times\mathbf{S}_k^\times\bar{\omega}_k) \\
m_k\bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j &= m_k\bar{a}_{Gk} \cdot \left(\bar{v}_k + \bar{\omega}_k \times \bar{S}_k\right)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j + m_k\bar{a}_{Gk} \cdot \left(\bar{\omega}_k \times \bar{S}_k\right)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j - m_k\left(\bar{a}_k + \bar{\alpha}_k \times \bar{S}_k + \bar{\omega}_k \times (\bar{\omega}_k \times \bar{S}_k)\right) \cdot \left(\bar{S}_k \times \bar{\omega}_k\right)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j - m_k\left(\bar{a}_k - \bar{S}_k \times \bar{\alpha}_k - \bar{\omega}_k \times (\bar{S}_k \times \bar{\omega}_k)\right) \cdot \left(\bar{S}_k \times (\bar{\omega}_k)_j\right) \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j - m_k\left(\bar{a}_k - \mathbf{S}_k^\times\bar{\alpha}_k - \omega_k^\times\mathbf{S}_k^\times\bar{\omega}_k\right)^T\mathbf{S}_k^\times(\bar{\omega}_k)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j - m_k\left(\mathbf{S}_k^{\times T}\bar{a}_k - \mathbf{S}_k^{\times T}\mathbf{S}_k^\times\bar{\alpha}_k - \mathbf{S}_k^{\times T}\omega_k^\times\mathbf{S}_k^\times\bar{\omega}_k\right)^T(\bar{\omega}_k)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j + m_k\left(\mathbf{S}_k^\times\bar{a}_k - \mathbf{S}_k^\times\mathbf{S}_k^\times\bar{\alpha}_k - \mathbf{S}_k^\times\omega_k^\times\mathbf{S}_k^\times\bar{\omega}_k\right)^T(\bar{\omega}_k)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j + m_k\left(\bar{S}_k \times \bar{a}_k - \bar{S}_k \times \bar{S}_k \times \bar{\alpha}_k - \bar{S}_k \times (\bar{\omega}_k \times (\bar{S}_k \times \bar{\omega}_k))\right) \cdot (\bar{\omega}_k)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j + m_k\left(\bar{S}_k \times \bar{a}_k - \bar{S}_k \times \bar{S}_k \times \bar{\alpha}_k + \bar{\omega}_k \times ((\bar{S}_k \times \bar{\omega}_k) \times \bar{S}_k) + (\bar{S}_k \times \bar{\omega}_k) \times (\bar{S}_k \times \bar{\omega}_k)\right) \cdot (\bar{\omega}_k)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j + m_k\left(\bar{S}_k \times \bar{a}_k - \bar{S}_k \times \bar{S}_k \times \bar{\alpha}_k - \bar{\omega}_k \times (\bar{S}_k \times (\bar{S}_k \times \bar{\omega}_k)) + 0\right) \cdot (\bar{\omega}_k)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j + \left(m_k\bar{S}_k \times \bar{a}_k - m_k\mathbf{S}_k^\times\mathbf{S}_k^\times\bar{\alpha}_k - \bar{\omega}_k \times (m_k\mathbf{S}_k^\times\mathbf{S}_k^\times\bar{\omega}_k)\right) \cdot (\bar{\omega}_k)_j \\
m_k\bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j &+ \left(\frac{d\bar{H}_{Gk}}{dt}\right) \cdot (\bar{\omega}_k)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j + \left(m_k\bar{S}_k \times \bar{a}_k - m_k\mathbf{S}_k^\times\mathbf{S}_k^\times\bar{\alpha}_k - \bar{\omega}_k \times (m_k\mathbf{S}_k^\times\mathbf{S}_k^\times\bar{\omega}_k)\right) \cdot (\bar{\omega}_k)_j \\
&\qquad + \left(\mathbf{J}_k\bar{\alpha}_k + \bar{\omega}_k \times \mathbf{J}_k\bar{\omega}_k + m_k\mathbf{S}_k^\times\mathbf{S}_k^\times\bar{\alpha}_k + \bar{\omega}_k \times (m_k\mathbf{S}_k^\times\mathbf{S}_k^\times\bar{\omega}_k)\right) \cdot (\bar{\omega}_k)_j \\
&= m_k\bar{a}_{Gk} \cdot (\bar{v}_k)_j + \left(m_k\bar{S}_k \times \bar{a}_k + \mathbf{J}_k\bar{\alpha}_k + \bar{\omega}_k \times \mathbf{J}_k\bar{\omega}_k\right) \cdot (\bar{\omega}_k)_j
\end{aligned}
$$

2.4 Potential Energy

The total potential energy U of the robot is given by:

$$U = \sum_{j \in \mathbb{F}} U_j = \sum_{j \in \mathbb{F}} -M_j\mathbf{g}^T(L_{0,j} + S_j) \tag{7}$$

where $L_{0,j}$ is the position vector from the origin O_0 to O_j and $\mathbf{g}$ is the gravitatonal acceleration. Projecting the vectors appearing in 7 into frame R_0, we obtain:

$$
\begin{aligned}
U_j &= -M_j\, {}^0\mathbf{g}^T({}^0P_j + {}^0A_j\, {}^jS_j) && (8)\\
&= -{}^0\mathbf{g}^T(M_j\, {}^0P_j + {}^0A_j\, {}^j\mathbf{MS}_j) && (9)\\
&= -\begin{bmatrix} {}^0\mathbf{g}^T & 0 \end{bmatrix} {}^0T_j \begin{bmatrix} {}^j\mathbf{MS}_j \\ M_j \end{bmatrix} && (10)
\end{aligned}
$$

Given the frames defined in figure 1, ${}^0\mathbf{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$.

References

A Expression for Kinetic Energy

We show here how the equation **??** was derived from **??**. Equation **??** is:

$$E_j = \frac{1}{2}(\omega_j^T I_{Gj}\omega_j + M_j V_{Gj}^T V_{Gj}) \tag{11}$$

where the velocity of the center of mass can be expressed as:

$$V_{Gj} = V_j + \omega_j \times S_j$$

and since:

$$J_j = I_{Gj} - M_j \hat{S}_j \hat{S}_j^T$$

So equation 11 becomes:

$$
\begin{aligned}
E_j &= \frac{1}{2}(\omega_j^T(J_j + M_j\hat{S}_j\hat{S}_j)\omega_j + M_j(V_j + \omega_j \times S_j)^T(V_j + \omega_j \times S_j))\\
E_j &= \frac{1}{2}(\omega_j^T J_j\omega_j + M_j V_j^T V_j + \omega_j^T M_j\hat{S}_j\hat{S}_j\omega_j + M_j V_j^T(\omega_j \times S_j)\\
&\qquad + M_j(\omega_j \times S_j)^T V_j + M_j(\omega_j \times S_j)^T(\omega_j \times S_j))
\end{aligned}
$$

Noting that the last term:

$$
\begin{aligned}
M_j(\omega_j \times S_j)^T(\omega_j \times S_j) &= (-)(-)M_j(S_j \times \omega_j)^T(S_j \times \omega_j)\\
&= M_j(\hat{S}_j\omega_j)^T(\hat{S}_j\omega_j)\\
&= M_j\omega_j^T\hat{S}_j^T\hat{S}_j\omega_j\\
&= -M_j\omega_j^T\hat{S}_j\hat{S}_j\omega_j
\end{aligned}
$$

cancels out the third term. And noting that the fourth and fifth terms are equal, we are left with:

$$E_j = \frac{1}{2}(\omega_j^T J_j\omega_j + M_j V_j^T V_j + 2M_j(\omega_j \times S_j)^T V_j)$$

The last term in the above expression can be simplified as follows:

$$\begin{aligned}
M_j(\omega_j \times S_j)^T V_j &= M_j(\hat{\omega}_j S_j)^T V_j \\
&= M_j S_j^T \hat{\omega}_j^T V_j \\
&= -M_j S_j^T \hat{\omega}_j V_j \\
&= -M_j S_j^T (\omega_j \times V_j) \\
&= \mathbf{MS}_j^T (V_j \times \omega_j)
\end{aligned}$$

so we end up with:

$$E_j = \frac{1}{2}(\omega_j^T J_{Gj}\omega_j + M_j V_j^T V_j + 2\mathbf{MS}_j^T (V_j \times \omega_j))$$

B Comparison with Lagrange

When we compare the result of Kane's method to that of Lagrange's method it turns out that there are significant differences. Since the two systems of equations involve hundreds of variables that makes it too complex to compare the two results, we only perform dynamic analysis on a two-link robot. The resulting systems of equation are as follows:

B.1 Lagrange

$$\mathbf{A} = \begin{bmatrix} \mathbf{XX}_1 + \mathbf{XX}_2 + 2\mathbf{MY}_2\mathbb{M}_4 + 2\mathbf{MZ}_2\mathbb{M}_3 + m_2\mathbb{M}_4^2 + m_2\mathbb{M}_3^2 & \mathbb{M}_1 \\ \mathbb{M}_1 & \mathbf{XX}_2 \end{bmatrix} \tag{12}$$

$$\mathbf{C} = \begin{bmatrix} \dot{q}_w\mathbb{M}_2 & \dot{q}_w\mathbb{M}_2 + \dot{q}_{imu}\mathbb{M}_2 \\ -\dot{q}_{imu}\mathbb{M}_2 & 0 \end{bmatrix} \tag{13}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{MZ}_1 g cosq_{imu} - \mathbf{MY}_1 g sinq_{imu} + \mathbf{MZ}_2 g\mathbb{M}_5 - \mathbf{MY}_2 g\mathbb{M}_6 - gm_2(L_2 cosq_{imu} + L_1 sinq_{imu}) \\ \mathbf{MZ}_2 g\mathbb{M}_5 - \mathbf{MY}_2 g\mathbb{M}_6 \end{bmatrix} \tag{14}$$

$$\tag{15}$$

where

$$\begin{aligned}
\mathbb{M}_1 &= \mathbf{XX}_2 + \mathbf{MY}_2\mathbb{M}_4 + \mathbf{MZ}_2\mathbb{M}_3 \\
\mathbb{M}_2 &= \mathbf{MZ}_2\mathbb{M}_4 - \mathbf{MY}_2\mathbb{M}_3 \\
\mathbb{M}_3 &= -L_2 cos(q_w) + L_1 sin(q_w) \\
\mathbb{M}_4 &= L_1 cos(q_w) + L_2 sin(q_w) \\
\mathbb{M}_5 &= cosq_{imu} cosq_w - sinq_{imu} sinq_w) \\
\mathbb{M}_6 &= cosq_{imu} sinq_w + cosq_w sinq_{imu})
\end{aligned}$$

B.2 Kane's

$$\mathbf{A} = \begin{bmatrix} \mathbb{XX}_1 + \mathbb{XX}_2 + 2\mathbf{MY}_2\mathbb{M}_4 + 2\mathbf{MZ}_2\mathbb{M}_3 + m_2\mathbb{M}_4^2 + m_2\mathbb{M}_3^2 & \mathbb{M}_1 \\ \mathbb{M}_1 & \mathbb{XX}_2 \end{bmatrix} \quad (16)$$

$$\mathbf{C} = \begin{bmatrix} \dot{q}_w\mathbb{M}_2 & \dot{q}_w\mathbb{M}_2 + \dot{q}_{imu}\mathbb{M}_2 \\ -\dot{q}_{imu}\mathbb{M}_2 & 0 \end{bmatrix} \quad (17)$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{MZ}_1 g cosq_{imu} - \mathbf{MY}_1 g sinq_{imu} + \mathbf{MZ}_2 g\mathbb{M}_5 - \mathbf{MY}_2 g\mathbb{M}_6 - gm_2(L_2 cosq_{imu} + L_1 sinq_{imu}) \\ \mathbf{MZ}_2 g\mathbb{M}_5 - \mathbf{MY}_2 g\mathbb{M}_6 \end{bmatrix} \quad (18)$$

where

$$\begin{aligned}
\mathbb{XX}_1 &= \mathbf{XX}_1 + \tfrac{\mathbf{MY}_1^2}{m_1} + \tfrac{\mathbf{MZ}_1^2}{m_1} \\
\mathbb{XX}_2 &= \mathbf{XX}_2 + \tfrac{\mathbf{MY}_2^2}{m_2} + \tfrac{\mathbf{MZ}_2^2}{m_2} \\
\mathbb{M}_1 &= \mathbb{XX}_2 + \mathbf{MY}_2\mathbb{M}_4 + \mathbf{MZ}_2\mathbb{M}_3 \\
\mathbb{M}_2 &= \mathbf{MZ}_2\mathbb{M}_4 - \mathbf{MY}_2\mathbb{M}_3 \\
\mathbb{M}_3 &= -L_2 cosq_w + L_1 sinq_w \\
\mathbb{M}_4 &= L_1 cosq_w + L_2 sinq_w \\
\mathbb{M}_5 &= cosq_{imu} cosq_w - sinq_{imu} sinq_w \\
\mathbb{M}_6 &= cosq_{imu} sinq_w + cosq_w sinq_{imu}
\end{aligned}$$