Equations Of Motion Of a Wheeled Inverted Pendulum

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In an earlier report [3] we had derived using Newton-Euler method the equations of motion of a wheeled inverted pendulum. But this model did not take into account the spin motion of the robot. Before attempting to derive the new equations including the spin, we will look at the equations derived in the existing literature.

1 Using the equation from [1]

The equations derived in [1] for the motion of two-wheeled inverted pendulum robot are:

$$3(m_c + m_s)\ddot{x} - m_s d cos\phi\ddot{\phi} + m_s d sin\phi(\dot{\phi}^2 + \dot{\psi}^2) = -\frac{\alpha_3 + \beta_3}{R} \tag{1}$$

$$\left\{(3L^2 + 1/2R^2)m_c + m_s d^2 sin^2\phi + I_2\right\}\ddot{\psi} + m_s d^2 sin\phi cos\phi\dot{\psi}\dot{\phi} = \frac{L}{R}(\alpha_3 - \beta_3) \tag{2}$$

$$m_s d cos\phi\ddot{x} + (-m_s d^2 - I_3)\ddot{\phi} + m_s d^2 sin\phi cos\phi\dot{\phi}^2 + m_s g d sin\phi = \alpha_3 + \beta_3 \tag{3}$$

where,
$\dot{x}$ is the heading speed of the robot
ϕ is the rotation of the C.G. about n_3-directional
ψ is the heading angle (angle between n_1 and the world frame)
α_3 is the torque of the left wheel
β_3 is the torque of the right wheel
m_c is the mass of the wheel
m_s is the mass of the body
d is the distance between wheel axis to C.G.
R is the radius of the wheel
L is the half distance between wheels
I_2 is the n_2-directional rotational inertia of the body
I_3 is the n_3-directional rotational inertia of the body
n_2 is the unit vector pointing vertically upwards
n_3 is the unit vector pointing from the left wheel to the right wheel

The equations that we had derived in our earlier report [3] did not include the rotation about the vertical axis of the robot. One of the equations above represent that motion. We will try to first match the equations we had derived with the equations listed above as a sanity check. Then we will write down the thrid equation using the variables that we had used in our earlier report. Then we will attempt to derive the equation using Newton-Euler method. Equations from our earlier report are listed here:

$$[(m+M)r + I_w/r + \eta^2 I_m/r]\ddot{x} + (mrlcos\theta - \eta^2 I_m)\ddot{\theta} = K_f u - \tau_f + F_{ext} rcos\theta + mrl\dot{\theta}^2 sin\theta \tag{4}$$

$$(mlcos\theta - \eta^2 I_m/r)\ddot{x} + (ml^2 + I + \eta^2 I_m)\ddot{\theta} = -K_f u + \tau_f + F_{ext} l + mglsin\theta \tag{5}$$

where,

m is the mass of the body
M is the mass of the wheels
r is the radius of the wheel
I_w is the inertia of the wheel
η is the gear ratio of the motor
I_m is the motor inertia
$\dot{x}$ is the heading speed
l is the distance between wheel axis and the C.G.
θ is the rotation of the C.G. about the wheel axis
K_f is the torque to current ratio of the motor
τ_f is the frictional torque on the wheels
F_{ext} is the external force being applied at the C.G. perpendicular to l
I is the inertia of the robot about the wheel axis

Using the symbols used in equations 4-5, we re-write the equations 1-3:

$$3(M+m)\ddot{x} - mlcos\theta\ddot{\theta} + mlsin\theta(\dot{\theta}^2 + \dot{\psi}^2) = -\frac{K_f(u_1+u_2)}{r} \tag{6}$$

$$\left\{(3L^2 + 1/2r^2)M + ml^2 sin^2\theta + I_2\right\}\ddot{\psi} + ml^2 sin\theta cos\theta\dot{\psi}\dot{\theta} = \frac{L}{r}K_f(u_1 - u_2) \tag{7}$$

$$mlcos\theta\ddot{x} + (-ml^2 - I)\ddot{\theta} + ml^2 sin\theta cos\theta\dot{\theta}^2 + mglsin\theta = K_f(u_1 + u_2) \tag{8}$$

where, we have retained the symbols for quantities that do not appear in equations 4-5 i.e. ψ and I_2 which represent the heading direction and the inertia about the vertical axis respectively.

In equations 6-8, we make the following observations:

(i) Equations 6 and 8 are the equivalents of the equations 4 and 5 respectively

(ii) The equations 6 and 8 ignore the effects of wheel inertia I_w, the motor inertia I_m, frictional torque τ_f at the wheel motor and the external force F_{ext}, all of which were considered in equations 4 and 5

(iii) The terms containing $\ddot{x}$ in equations 4 and 5 appear with opposite signs in equations 6 and 8

(iv) The term $mrlsin\theta\dot{\theta}^2$ of equation 4 appears as $mrlsin\theta(\dot{\theta}^2+\dot{\psi}^2)$ in equation 6

(v) Equation 4 does not contain the coefficient 3 with the $\ddot{x}$ term which appears in equation 6

(vi) Equation 5 does not contain the term $ml^2 sin\theta cos\theta\dot{\theta}^2$ which appears in equation 8

Point number (iii) may be explained by assuming that the two derivations assumed x increases in different directions. Point number (iv) may be explained by the fact that equations 4 and 5 assume constant heading direction i.e. $\dot{\psi} = 0$ But the last two points are not easy to explain. An interesting fact regarding point regarding (v) is that the paper [1] changes the term from $3(m_c + m_s)$ in the original equation that we cited aboce to $3m_c + m_s$ in the later equations of the same paper. It appears that the latter expression is more accurate and basically $3m_c$ represents the mass of three wheels of equal mass, one of which is a supporting wheel. The paper discusses supporting wheels at length, so it won't be surprise. That solves the mystery of the second last point. What remains now is to discuss the very last point. Since we see that there is a typo done in the earlier equation, we can expect this term to have a typo, in that it is missing a $\dot{\psi}$ term. If this was true we will safely assume that the reason this term isn't present in our earlier analysis (i.e. equations 4 and 5) is because we assumed constant heading direction i.e. $\dot{\psi} = 0$.

2 Using equations from [2]

In the book [2] following equations of motion are derived:

$$\left(M + 2M_w + m + 2\frac{I_w}{r^2}\right)\dot{v} + ml\ddot{\alpha}cos\alpha - ml\dot{\alpha}^2 sin\alpha = \frac{\tau_l}{r} + \frac{\tau_r}{r} + d_l + d_r \tag{9}$$

$$\left(I_p + 2\left(M_w + \frac{I_w}{r^2}\right)d^2\right)\dot{\omega} = 2d\left(\frac{\tau_l}{r} - \frac{\tau_r}{r} + d_l - d_r\right) \tag{10}$$

$$ml\dot{v}cos\alpha + \left(ml^2 + I_M\right)\ddot{\alpha} - mglsin\alpha = 0 \tag{11}$$

where,
M is the mass of the platform
M_w is the mass of one wheel
m is the mass of the robot
I_w is the inertia of one wheel
r is the radius of one wheel
v is the heading speed of the wheel
l is the distance from C.G. to wheel axis
α is the rotation of C.G. about the wheel axis
τ_l is the torque applied by left wheel motor
τ_r is the torque applied by right wheem motor
d_l is the external force acting on the left wheel
d_r is the external force acting on the right wheel
Now, replacing these variables with the ones we had used in [3], equations, 9-11 become:

$$\left(M_p + M + m + \frac{I_w}{r^2}\right)\ddot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta = \frac{K_f u_1}{r} + \frac{K_f u_2}{r} + d_l + d_r \tag{12}$$

$$\left(I_z + \left(M + \frac{I_w}{r^2}\right)L^2\right)\ddot{\psi} = 2L\left(\frac{K_f u_1}{r} - \frac{K_f u_2}{r} + d_l - d_r\right) \tag{13}$$

$$ml\ddot{x}cos\theta + \left(ml^2 + I\right)\ddot{\theta} - mglsin\theta = 0 \tag{14}$$

Following observation are made:

(i) Equations 12 and 14 are the equivalents of equations 4 and 5 respectively

(ii) Equation 13 represents spin

(iii) There is no difference between equation 12 and 4 except that (a) equation 4 considers motor inertia I_m while 12 does not and (b) eq 12 consider platform as different from the pendulum while krang has no such thing as a platform so the additional mass term M_p in equation 12 is not there in 4

(iv) There is no difference between equation 12 and 4 except that (a) equation 4 considers motor inertia I_m while 12 does not and (b) the effect of counter torque on the pendulum is not considered in the eq 12 as the pendulum does not experience the countertorque due to it being on a platform and not directly attached to the motor so we don't see any term on the right hand side of eq 12

(v) Equation 13 when compared with eq 7 that represented spin in the previous section we see that the coefficient of $\ddot{\psi}$ was a funtion of θ there but here it is not. The reason is that I_z term in eq 13 is actually a function of θ. That function is written over there but not here.

(vi) Also there is a $\dot{\psi}$ term in equation 7 that is not there in eq 13. It seems like equation 7 makes more sense as a non-zero $\dot{\theta}$ will introduce a corriolis force in the system that is apparently not taken into account by equation 13

3 Comparing [1] and [2]

It appears that the analysis done by [1] is more correct with regards to understanding of the dynamics, only that it seems to have introduced some typos and thus can't be trusted blindly. The analysis in [2] on the other hand is very cleanly explained and does not contain typos, it is weak in representation of all dynamic effects in the system. The way we will move forward is by using the expressions for velocities that are more completely derived in [1] and use the detailed procedures explained in [2] to come up with expressions of dynamics that are useful for our purposes.

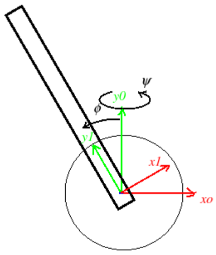

Figure 1: Frames of references on the robot

4 Deriving the Dynamic Model for our robot

4.1 Kinetic and Potential Energy of the Body

$$
\begin{aligned}
{}^{0}V_{0} &= \begin{bmatrix} \dot{x} & 0 & 0 \end{bmatrix}^{T} \\
{}^{0}\omega_{0} &= \begin{bmatrix} 0 & \dot{\psi} & 0 \end{bmatrix}^{T} \\
{}^{0}\mathbf{g} &= \begin{bmatrix} 0 & -g & 0 \end{bmatrix}^{T} \\
{}^{0}T_{1} &= \begin{bmatrix} & {}^{0}A_{1} & & {}^{0}P_{1} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{0}\mathbf{x}_{1} & {}^{0}\mathbf{y}_{1} & 0^{\mathbf{z}}_{1} & {}^{0}P_{1} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} cos\phi & -sin\phi & 0 & 0 \\ sin\phi & cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^{1}e_{1} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{T} \\
\dot{q}_{1} &= \dot{\phi}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
{}^{1}\omega_{1} &= {}^{1}A_{0}{}^{0}\omega_{0} + {}^{1}e_{1}\dot{q}_{1} \\
{}^{1}V_{1} &= {}^{1}A_{0}{}^{0}\left({}^{0}V_{0} + {}^{0}\omega_{0} \times {}^{0}P_{1}\right)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
E_{1} &= \frac{1}{2}\left(\omega_{1}^{T} J_{G1}\omega_{1} + M_{1}V_{1}^{T}V_{1} + 2\mathbf{MS}_{1}^{T}(V_{1} \times \omega_{1})\right) \\
U_{1} &= -\begin{bmatrix} {}^{0}\mathbf{g}^{T} & 0 \end{bmatrix} {}^{0}T_{1} \begin{bmatrix} \mathbf{MS}_{1} \\ M_{1} \end{bmatrix}
\end{aligned}
\tag{17}
$$

4.2 Kinetic and Potential Energy of the Wheels

$$
\begin{aligned}
{}^{L}\omega_{L} &= \begin{bmatrix} 0 & \dot{\psi} & -\frac{1}{R}\dot{x} + \frac{L}{2R}\dot{\psi} \end{bmatrix}^{T} \\
{}^{L}V_{L} &= \begin{bmatrix} \dot{x} - \dot{\psi}\frac{L}{2} & 0 & 0 \end{bmatrix}^{T} \\
E_{L} &= \frac{1}{2}\left(\omega_{L}^{T} J_{Gw}\omega_{L} + M_{w}V_{L}^{T}V_{L} + 2\mathbf{MS}_{w}^{T}(V_{L} \times \omega_{L})\right)
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
{}^{R}\omega_R &= \begin{bmatrix} 0 & \dot{\psi} & -\frac{1}{R}\dot{x} - \frac{L}{2R}\dot{\psi} \end{bmatrix}^T \\
{}^{R}V_R &= \begin{bmatrix} \dot{x} + \dot{\psi}\frac{L}{2} & 0 & 0 \end{bmatrix}^T \\
E_R &= \frac{1}{2}\left(\omega_R^T J_{Gw}\omega_R + M_w V_R^T V_R + 2\mathbf{MS}_w^T(V_R \times \omega_R)\right)
\end{aligned} \tag{19}
$$

4.3 Applying Lagrange Method to find matrices A, C and Q

The Lagrange formulation describes the behavior of a dynamic system in terms of work and energy stored in the system. The Lagrange equations are commonly written in the form:

$$\Gamma_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \quad for\ i \in \mathbb{F} \tag{20}$$

where L is the Lagrangian of the robot defined as the difference berween the kinetic energy E and potential energy U of the system:

$$L = E - U$$

4.3.1 General Form of the Dynamic Equations

The kinetic energy of the system is a quadratic function in the joint velocities such that:

$$E = \frac{1}{2}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{A}\dot{\mathbf{q}} \tag{21}$$

where $\mathbf{A}$ is the $n \times n$ symmetric and positive definite *inertia matrix* of the robot. Its elements are functions of the joint positions. The (i, j) element of $\mathbf{A}$ is denoted A_{ij}. Since the potential energy is a function of the joint positions, equation 20 leads to:

$$\mathbf{\Gamma} = \mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{Q}(\mathbf{q}) \tag{22}$$

where:

- $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ is the $n \times 1$ vector of Coriolis and centrifugal torques, such that: $\mathbf{C}\dot{\mathbf{q}} = \dot{\mathbf{A}}\dot{\mathbf{q}} - \frac{\partial E}{\partial \mathbf{q}}$
- $\mathbf{Q} = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_{4l} & ... & Q_{10l} & Q_{4r} & ... & Q_{10r} \end{bmatrix}^T$ is the vector of gravity torques

So the dynamic model of a tree-structured robot is described by n coupled and nonlinear second order differential equations. The elements of $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{Q}$ are functions of geometric and inertial parameters of the robot.

4.3.2 Computation of the elements of A, C and Q

To compute the elements of $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{Q}$, we begin by symbollically computing the expressions of the kinetic and potential energies of all the links of the robot. Then we proceed as follows:

- the elements A_{ij} is equal to the coefficient of $\left(\frac{\dot{q}_i^2}{2}\right)$ in the expression of the kinetic energy, while A_{ij}, for $i \neq j$, is equal to the coefficient of $\dot{q}_i \dot{q}_j$
- for calculating the elements of $\mathbf{C}$, there exist several forms of the vector $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$. Using the *Christoffell symbols* $c_{i,jk}$, the (i,j) elements of the matrix $\mathbf{C}$ can be written as:

$$\begin{cases} C_{ij} = \sum\limits_{k=1}^{n} c_{i,jk}\dot{q}_k \\ c_{i,jk} = \frac{1}{2}\left[\frac{\partial A_{ij}}{\partial q_k} + \frac{\partial A_{ik}}{\partial q_j} - \frac{\partial A_{jk}}{\partial q_i}\right] \end{cases} \tag{23}$$

- The Q_i element of the vector $\mathbf{Q}$ is calculated according to:

$$Q_i = \frac{\partial U}{\partial q_i} \tag{24}$$

4.3.3 Resulting Matrices

$$\mathbf{A} = \begin{bmatrix} M_1 + 2M_w + 2\frac{\mathbf{ZZ}_w}{R^2} & \mathbf{MZ}_1 - 2\frac{\mathbf{YZ}_w}{R} & -\mathbf{MY}_1 cos\phi - \mathbf{MX}_1 sin\phi \\ \mathbf{MZ}_1 - 2\frac{\mathbf{YZ}_w}{R} & \mathbf{XX}_1 + 2\mathbf{YY}_w + 2M_w L^2 - (\mathbf{XX}_1 - \mathbf{YY}_1)cos^2\phi + \mathbf{XY}_1 sin(2\phi) + \frac{2L^2\mathbf{ZZ}_w}{R^2} & \mathbf{YZ}_1 cos\phi + \mathbf{XZ}_1 sin\phi \\ -\mathbf{MY}_1 cos\phi - \mathbf{MX}_1 sin\phi & \mathbf{YZ}_1 cos\phi + \mathbf{XZ}_1 sin\phi & \mathbf{ZZ}_1 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 0 & 0 & -\dot{\phi}\,(\mathbf{MX}_1 cos\phi - \mathbf{MY}_1 sin\phi) \\ 0 & \frac{\dot{\phi}}{2}\,(2\mathbf{XY}_1 cos(2\phi) + (\mathbf{XX}_1 - \mathbf{YY}_1)sin(2\phi)) & \dot{\psi}\,(\mathbf{XY}_1 cos(2\phi) + (\mathbf{XX}_1 - \mathbf{YY}_1)cos\phi sin\phi) - \dot{\phi}\,(\mathbf{YZ}_1 sin\phi - \mathbf{XZ}_1 cos\phi) \\ 0 & -\frac{\dot{\psi}}{2}\,(2\mathbf{XY}_1 cos(2\phi) + (\mathbf{XX}_1 - \mathbf{YY}_1)sin(2\phi)) & 0 \end{bmatrix}$$
$$\mathbf{Q} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{MX}_1 g cos\phi - \mathbf{MY}_1 g sin\phi \end{bmatrix} \tag{25}$$

4.3.4 Comparing to our previous Work [3]

As a sanity check let's compare this result to our earlier work in [3] i.e. equations 4-5. To do so, we are going to make $\dot{\psi} = \ddot{\psi} = 0$. This reduces the LHS of our new equations to:

$$\mathbf{A}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{Q} =$$
$$\begin{bmatrix} M_1 + 2M_w + 2\frac{\mathbf{ZZ}_w}{R^2} & -\mathbf{MY}_1 cos\phi - \mathbf{MX}_1 sin\phi \\ -\mathbf{MY}_1 cos\phi - \mathbf{MX}_1 sin\phi & \mathbf{ZZ}_1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\phi}\,(\mathbf{MX}_1 cos\phi - \mathbf{MY}_1 sin\phi) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{MX}_1 g cos\phi - \mathbf{MY}_1 g sin\phi \end{bmatrix} \tag{26}$$

Observations:

(i) The two equations are equivalent to the LHS of equations 4-5 except for the rotor inertia term I_m that is missing in our new equations. That is only because we did not consider the effect of rotor inertia in our current analysis

(ii) To see the equivalence more clearly, the following substitution will be helpful:

- $M_1 = m$, $2M_w = M$, $2\mathbf{ZZ}_w = I_w$, $\phi = -\theta$, $\dot{\phi} = -\dot{\theta}$, $\ddot{\phi} = -\ddot{\theta}$ that have to with the definition of the variables in the two analyses
- $\mathbf{MY}_1 = ml$ and $\mathbf{MX}_1 = 0$ since the center of mass of the pendulum in the earlier analysis was at $(0, l, 0)$ if placed in the frame R_1 in the current analysis

- $\mathbf{ZZ}_1 = ml^2 + I$ as the inertia of body I was defined about the wheel axis in our earlier analysis but it is defined around the center of mass in this analysis. Parallel axis theorem on inertia gives us this relationship.

This reduces the equation 26 to:

$$\begin{aligned}
&\mathbf{A}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{Q} \\
=&\begin{bmatrix} m + M + \frac{I_w}{R^2} & -mlcos\theta \\ -mlcos\theta & ml^2 + I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ -\ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & \dot{\theta}mlsin\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ -\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ mlgsin\theta \end{bmatrix} \\
=&\begin{bmatrix} \left(m + M + \frac{I_w}{R^2}\right)\ddot{x} + mlcos\theta\ddot{\theta} \\ -mlcos\theta\ddot{x} - (ml^2 + I)\ddot{\theta} \end{bmatrix} + \begin{bmatrix} -mlsin\theta\dot{\theta}^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ mlgsin\theta \end{bmatrix} \\
=&\begin{bmatrix} \left(m + M + \frac{I_w}{R^2}\right)\ddot{x} + mlcos\theta\ddot{\theta} - mlsin\theta\dot{\theta}^2 \\ -mlcos\theta\ddot{x} - (ml^2 + I)\ddot{\theta} + mlgsin\theta \end{bmatrix}
\end{aligned} \tag{27}$$

This expression for LHS matches exactly with the LHS of equations 4-5.

4.3.5 Comparing the spin motion equations to those in [1]

- First look at the added terms due to spin in the two equations we discussed in the last section. The first equation contains the additional term:

$$\left(\mathbf{MZ}_1 - 2\frac{\mathbf{YZ}_w}{R}\right)\ddot{\psi}$$

If the body is symmetrical about the X_0Y_0 plane (which was assumed true in our [1]) then $\mathbf{MZ}_1$ will be zero. As for the $\mathbf{YZ}_w$ it is zero if the wheel is symmetrical about its YZ plane parallel to the Y_0Z_0 plane passing through its center. And wheels indeed have this characteristic. So this term reduces to zero. And so it is no surprise that such a term doesn't appear in equation 1. Though we have a term in equation one which is:

$$m_s d sin\phi\dot{\psi}^2$$

and we have no such term in our equation. The only way we could have a $\dot{\psi}^2$ in our results would be if $\mathbf{C}_{12}$ was non-zero. Now using equations 23, we have:

$$\begin{aligned}
C_{12} &= c_{1,21}\dot{x} + c_{1,22}\dot{\psi} + c_{1,23}\dot{\phi} \\
c_{1,21} &= \frac{1}{2}\left[\frac{\partial A_{12}}{\partial x} + \frac{\partial A_{11}}{\partial \psi} - \frac{\partial A_{21}}{\partial x}\right] \\
c_{1,22} &= \frac{1}{2}\left[\frac{\partial A_{12}}{\partial \psi} + \frac{\partial A_{12}}{\partial \psi} - \frac{\partial A_{22}}{\partial x}\right] \\
c_{1,23} &= \frac{1}{2}\left[\frac{\partial A_{12}}{\partial \phi} + \frac{\partial A_{13}}{\partial \psi} - \frac{\partial A_{23}}{\partial x}\right]
\end{aligned} \tag{28}$$

Since $\frac{\partial A_{ij}}{\partial x} = \frac{\partial A_{ij}}{\partial \psi} = 0$ as none of the elements of A have a dependency on x or ψ the expression for C_{12} reduces to:

$$C_{12} = \frac{1}{2}\frac{\partial A_{12}}{\partial \phi}\dot{\phi}$$

The only way we could have a non-zero C_{12} would be if $\mathbf{A}_{12}$ was a function of ϕ. As we can see the A matrix, that is not the case. And hence there is no way using our method to have this term in our equations. After this analysis, it seemed probable that the $\dot{\psi}^2$ term in the first equation is an error. So we checked the derivation in the paper. This derivation can be found in the appendix. It turns out that this term is indeed present. Now, we will question our own methodology to see where the problem may be. We will follow the derivation of $\dot{\phi}^2$ term in ouenergy expressionr first equation assuming a similar term needs to be present in the original energy expression for $\dot{\psi}^2$ and is somehow missed in our analysis. The coefficient of ϕ^2 is deduced from the element C_{13}. The expression for C_{13} is:

$$\begin{aligned}
C_{13} &= c_{1,31}\dot{x} + c_{1,32}\dot{\psi} + c_{1,33}\dot{\phi} \\
c_{1,31} &= \frac{1}{2}\left[\frac{\partial A_{13}}{\partial x} + \frac{\partial A_{11}}{\partial \phi} - \frac{\partial A_{31}}{\partial x}\right] \\
c_{1,32} &= \frac{1}{2}\left[\frac{\partial A_{13}}{\partial \psi} + \frac{\partial A_{12}}{\partial \phi} - \frac{\partial A_{32}}{\partial x}\right] \\
c_{1,33} &= \frac{1}{2}\left[\frac{\partial A_{13}}{\partial \phi} + \frac{\partial A_{13}}{\partial \phi} - \frac{\partial A_{33}}{\partial x}\right]
\end{aligned} \tag{29}$$

Again $\frac{\partial A_{ij}}{\partial x} = \frac{\partial A_{ij}}{\partial \psi} = 0$ as none of the elements of A have a dependency on x or ψ. Also, $\frac{\partial A_{11}}{\partial \phi} = \frac{\partial A_{12}}{\partial \phi} = 0$ as both A_{11} and A_{12} are independent of ϕ. So, the expression for C_{13} reduces to:

$$C_{13} = \frac{\partial A_{13}}{\partial \phi}\dot{\phi}$$

The question now is, what is the origin of the term A_{13}. This could give clue regarding a similar term missing in the energy expression evaluation leading to a term at A_{12} similat to A_{13} this giving us eventually the required $\dot{\psi}^2$ in the final equations. As we know, A_{13} is the coefficient of $\dot{x}\dot{\phi}$ in the energy expression. Investigating further, if we trace back the derivation of the $\dot{\psi}^2$ term in [1] we have:

$$\begin{aligned}
&\frac{\partial v_S}{\partial \dot{x}} \cdot (m_S\,\omega_S \times (\omega_S \times r_S)) \\
&\quad = \frac{\partial}{\partial \dot{x}}\left(\begin{bmatrix} \dot{x} - \dot{\phi} d cos\phi \\ -\dot{\phi} d sin\phi \\ \dot{\psi} d sin\phi \end{bmatrix}\right) \cdot \left(m_S \begin{bmatrix} 0 \\ \dot{\psi} \\ \dot{\phi} \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ \dot{\psi} \\ \dot{\phi} \end{bmatrix} \times \begin{bmatrix} -d sin\phi \\ d cos\phi \\ 0 \end{bmatrix}\right)\right) \\
&\quad = m_S \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ \dot{\psi} \\ \dot{\phi} \end{bmatrix} \times \begin{bmatrix} -d cos\phi\dot{\phi} \\ -d sin\phi\dot{\phi} \\ d sin\phi\dot{\psi} \end{bmatrix}\right) \\
&\quad = m_S \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} d sin\phi(\dot{\phi}^2 + \dot{\psi}^2) \\ -d cos\phi\dot{\phi}^2 \\ d cos\phi\dot{\phi}\dot{\psi} \end{bmatrix} \\
&\quad = m_S d sin\phi(\dot{\phi}^2 + \dot{\psi}^2)
\end{aligned} \tag{30}$$

- As for the additional terms in the third equation, we have:

$$(\mathbf{YZ}_1 cos\phi + \mathbf{XZ}_1 sin\phi)\ddot{\psi}$$

No such term appears in equation 3. This could be because of certain assumptions on symmetry of the body that makes the product inertia terms equal to zero (although that's not entirely possible to achieve for $\mathbf{XZ}_1$). We also have:

$$-\frac{\dot{\psi}^2}{2}\left(2\mathbf{XY}_1 cos(2\phi) + (\mathbf{XX}_1 - \mathbf{YY}_1)sin(2\phi)\right)$$

We have a term with $\dot{\phi}^2$ in equation 3. And that seems to be just the result of a typo. As their coefficient also contains a $sin(2\phi)$ term (although it is expressed as $sin\phi cos\phi$ which is equivalent).

- Now we proceed to compare the third equation introduced due to spin. Let us write down the two equations and then we will make comparisons. Equation from [1] has LHS =:

$$\left\{(3L^2 + 1/2R^2)m_c + m_s d^2 sin^2\phi + I_2\right\}\ddot{\psi} + m_s d^2 sin\phi cos\phi\dot{\psi}\dot{\phi}$$

The equation we derived has the LHS =:

$$\begin{aligned}&\left(\mathbf{MZ}_1 - 2\frac{\mathbf{YZ}_w}{R}\right)\ddot{x} + \left(\mathbf{XX}_1 + 2\mathbf{YY}_w + 2M_wL^2 - (\mathbf{XX}_1 + \mathbf{YY}_1)cos^2\phi + \mathbf{XY}_1 sin(2\phi) + \frac{2L^2\mathbf{ZZ}_w}{R^2}\right)\ddot{\psi}\\&+(\mathbf{YZ}_1 cos\phi + \mathbf{XZ}_1 sin\phi)\ddot{\phi} + \left(\frac{\dot{\phi}}{2}\left(2\mathbf{XY}_1 cos(2\phi) + (\mathbf{XX}_1 - \mathbf{YY}_1)sin(2\phi)\right)\right)\dot{\psi}\\&+\left(\dot{\psi}\left(\mathbf{XY}_1 cos(2\phi) + (\mathbf{XX}_1 - \mathbf{YY}_1)cos\phi sin\phi\right) - \dot{\phi}\left(\mathbf{YZ}_1 sin\phi + \mathbf{XZ}_1 cos\phi\right)\right)\dot{\phi}\end{aligned} \tag{31}$$

We can see the equivalence of these expression by noticing that under the assumptions of symmetry a number of terms were ommited from the expression in [1]. This includes $\mathbf{MZ}_1 = \mathbf{YZ}_w = \mathbf{XY}_1 = \mathbf{XZ}_1 = \mathbf{YZ}_1 = 0$ that makes the first and third terms $= 0$ in our expression. Also the last two can be added together. The resulting simplified expression looks like:

$$\left(2M_wL^2 + \frac{2L^2\mathbf{ZZ}_w}{R^2} + \mathbf{XX}_1 sin^2\phi - \mathbf{YY}_1 cos^2\phi + 2\mathbf{YY}_w\right)\ddot{\psi} + (\mathbf{XX}_1 - \mathbf{YY}_1)cos\phi sin\phi\dot{\psi}\dot{\phi} \tag{32}$$

5 Nonholonomic

$$\mathbf{q} = \begin{bmatrix} x_0 & y_0 & \theta_l & \theta_r & \phi \end{bmatrix}^T$$

Nonholonomic constraints:

$$\begin{aligned}\dot{x}_0 sin\psi - \dot{y}_0 cos\psi &= 0\\ \dot{x}_0 sin\psi + \dot{y}_0 cos\psi &= \frac{R}{2}(\dot{\theta}_l + \dot{\theta}_r)\end{aligned} \tag{33}$$

Substituting $\psi = \frac{R}{L}(\theta_l - \theta_r)$ and $\dot{x} = \dot{x}_0 cos\psi + \dot{y}_0 sin\psi$ in the derivation for the dynamics in preceding sections the resulting matrices as follows:

$$
\begin{aligned}
&\mathbf{A(q)\ddot{q} + C(q, \dot{q})\dot{q} + Q(q) = \Gamma + G}[\lambda_1 \quad \lambda_2]^{\mathbf{T}} \qquad (34)\\
&\mathbf{A} = \begin{bmatrix} \mathbb{M}cos^2\psi & \frac{1}{2}\mathbb{M}sin2\psi & 0 & 0 & -m_S d cos\phi cos\psi \\ \frac{1}{2}\mathbb{M}sin2\psi & \mathbb{M}sin^2\psi & 0 & 0 & -m_S d cos\phi sin\psi \\ 0 & 0 & \mathbb{I} & -\mathbb{I} & 0 \\ 0 & 0 & -\mathbb{I} & \mathbb{I} & 0 \\ -m_S d cos\phi cos\psi & -m_S d cos\phi sin\psi & 0 & 0 & I_3 + m_S d^2 \end{bmatrix}\\
&\mathbf{C} = \begin{bmatrix} \frac{1}{2}\mathbb{M}s2\psi\dot{\psi} & -\frac{1}{2}\mathbb{M}c2\psi\dot{\psi} & \mathbb{S}_1 & -\mathbb{S}_1 & \frac{1}{2}m_S d(2c\psi s\phi\dot{\phi} - s\psi c\phi\dot{\psi}) \\ -\frac{1}{2}\mathbb{M}c2\psi\dot{\psi} & -\frac{1}{2}\mathbb{M}s2\psi\dot{\psi} & -\mathbb{S}_2 & \mathbb{S}_2 & \frac{1}{2}m_S d(2s\psi s\phi\dot{\phi} + c\psi c\phi\dot{\psi}) \\ -\mathbb{S}_1 & \mathbb{S}_2 & \frac{R}{L}\mathbb{N}s2\phi\dot{\phi} & -\frac{R}{L}\mathbb{N}s2\phi\dot{\phi} & \mathbb{Z} \\ \mathbb{S}_1 & -\mathbb{S}_2 & -\frac{R}{L}\mathbb{N}s2\phi\dot{\phi} & \frac{R}{L}\mathbb{N}s2\phi\dot{\phi} & -\mathbb{Z} \\ -\frac{1}{2}m_S d s\psi c\phi\dot{\psi} & \frac{1}{2}m_S d c\psi c\phi\dot{\psi} & -\mathbb{Z} & \mathbb{Z} & 0 \end{bmatrix}\\
&\mathbf{Q} = [0 \quad 0 \quad 0 \quad 0 \quad -m_S d g sin\phi]^T\\
&\mathbf{\Gamma} = [0 \quad 0 \quad \tau_l \quad \tau_r \quad \tau_l + \tau_r]^T\\
&\mathbf{G} = \begin{bmatrix} sin\psi & -cos\psi & 0 & 0 & 0 \\ cos\psi & sin\psi & -\frac{R}{2} & -\frac{R}{2} & 0 \end{bmatrix}^T
\end{aligned} \qquad (35)
$$

where λ_1 and λ_2 are Lagrange multipliers and

$$
\begin{aligned}
\mathbb{M} &= \frac{2I_{w3}}{R^2} + 2m_C + m_S\\
\mathbb{I} &= 2I_{w3} + 2R^2 m_C + \frac{R^2}{L^2}\left(2I_{w2} + (I_1 + m_S d^2)sin^2\phi + I_2 cos^2\phi\right)\\
\mathbb{N} &= \frac{R}{2L}(m_S d^2 + I_1 - I_2)\\
\mathbb{S}_1 &= \frac{R}{2L}(\mathbb{M}(\dot{x}_0 s2\psi - \dot{y}_0 c2\psi) - m_S d s\psi c\phi\dot{\phi})\\
\mathbb{S}_2 &= \frac{R}{2L}(\mathbb{M}(\dot{x}_0 c2\psi + \dot{y}_0 s2\psi) - m_S d c\psi c\phi\dot{\phi})\\
\mathbb{Z} &= \mathbb{N}s2\phi\dot{\psi} + \frac{R}{2L}m_S d c\phi(\dot{x}_0 s\psi - \dot{y}_0 c\psi)
\end{aligned}
$$

Eliminating Lagrange multipliers gives us the exact same equations that we had in our earlier derivation. The code for this is found at stableForceInteraction/Implementation/1-ForceControlWhileBalancing/1-ControlProblem1/1-DynamicModeling/1-DynamicModelOfWheeledPendulum matlab/kim/nonholonomic.

6 Energy expression in the world frame

Define:

Frame 0: The world frame fixed to the ground

Frame 1: The frame with origin of on the middle of the the two wheels but all axes always parallel to those of Frame 0

Frame 2: The frame with z-axis along the z-axis of frame 1 while the x-axis along the heading direction of the robot

Frame 3: The frame fixed to the body of the robot with z-axis connecting the mid-point between wheels to the center of mass of the robot

$$^0T_1 = \begin{bmatrix} & ^0A_1 & & ^0P_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ^0\mathbf{x}_1 & ^0\mathbf{y}_1 & ^0\mathbf{z}_1 & ^0P_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{36}$$

$$^1T_2 = \begin{bmatrix} & ^1A_2 & & ^1P_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ^1\mathbf{x}_2 & ^1\mathbf{y}_2 & ^1\mathbf{z}_2 & ^1P_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} cos\psi & -sin\psi & 0 & 0 \\ sin\psi & cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{37}$$

$$^2T_3 = \begin{bmatrix} & ^2A_3 & & ^2P_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ^2\mathbf{x}_3 & ^2\mathbf{y}_3 & ^2\mathbf{z}_3 & ^2P_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} cos\phi & 0 & -sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ sin\phi & 0 & cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{38}$$

$$^0V_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \tag{39}$$

$$^0\omega_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \tag{40}$$

$$\dot{q}_1 = \dot{x} \tag{41}$$

$$\dot{q}_2 = \dot{\psi} \tag{42}$$

$$\dot{q}_3 = \dot{\phi} \tag{43}$$

$$^1e_1 = \begin{bmatrix} cos\psi & sin\psi & 0 \end{bmatrix}^T \tag{44}$$

$$^2e_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \tag{45}$$

$$^3e_3 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T \tag{46}$$

$$^1\omega_1 = {}^1\mathbf{A}_0{}^0\omega_0 \tag{47}$$

$$^2\omega_2 = {}^2\mathbf{A}_1{}^1\omega_1 + \dot{q}_2{}^2e_2 \tag{48}$$

$$^3\omega_3 = {}^3\mathbf{A}_2{}^2\omega_2 + \dot{q}_3{}^3e_3 \tag{49}$$

$$^1V_1 = {}^1A_0({}^0V_0 + {}^0\omega_0 \times {}^0P_1) + \dot{q}_1{}^1e_1 \tag{50}$$

$$^2V_3 = {}^2A_1({}^1V_1 + {}^1\omega_1 \times {}^1P_2) \tag{51}$$

$$^3V_3 = {}^3A_2({}^2V_2 + {}^2\omega_2 \times {}^2P_3) \tag{52}$$

$$\tag{53}$$

7 Newton-Euler

7.1 Kinematics

Let **i**, **j** and **k** be the unite vectors of the frame xyz whose origin is at the center of the the two wheel, its z-axis is always pointing upwards and its x-axis is along the heading direction of the robot.

$$
\begin{align}
\omega_B &= \dot{\phi}\mathbf{j} + \dot{\psi}\mathbf{k} \tag{54} \\
v_B &= v_{xyz} + \omega_B \times d \tag{55} \\
&= (\dot{x} + \dot{\phi} d cos\phi)\mathbf{i} + \dot{\psi} d sin\phi \mathbf{j} - \dot{\phi} d sin\phi \mathbf{k} \tag{56} \\
\omega_L &= \left(\frac{1}{R}\dot{x} - \frac{L}{R}\dot{\psi}\right)\mathbf{j} + \dot{\psi}\mathbf{k} \tag{57} \\
v_L &= (\dot{x} - \dot{\psi} L)\mathbf{i} \tag{58} \\
\omega_R &= \left(\frac{1}{R}\dot{x} + \frac{L}{R}\dot{\psi}\right)\mathbf{j} + \dot{\psi}\mathbf{k} \tag{59} \\
v_R &= (\dot{x} + \dot{\psi} L)\mathbf{i} \tag{60} \\
\alpha_B &= \ddot{\psi}\mathbf{k} + \ddot{\phi}\mathbf{j} \tag{61} \\
\alpha_L &= \left(-\frac{1}{R}\dot{x}\dot{\psi} + \frac{L}{R}\dot{\psi}^2\right)\mathbf{i} + \ddot{\psi}\mathbf{k} + \left(\frac{1}{R}\ddot{x} - \frac{L}{R}\ddot{\psi}\right)\mathbf{j} \tag{62} \\
\alpha_R &= \left(-\frac{1}{R}\dot{x}\dot{\psi} - \frac{L}{R}\dot{\psi}^2\right)\mathbf{i} + \ddot{\psi}\mathbf{k} + \left(\frac{1}{R}\ddot{x} + \frac{L}{R}\ddot{\psi}\right)\mathbf{j} \tag{63} \\
a_B &= \frac{dv_{xyz}}{dt} + \alpha_B \times \mathbf{d} + \omega_B \times (\omega_B \times \mathbf{d}) \tag{64} \\
&= \left(\ddot{x} + \ddot{\phi} d cos\phi - (\dot{\psi}^2 + \dot{\phi}^2) d sin\phi\right)\mathbf{i} + \left(-\ddot{\phi} d sin\phi - \dot{\phi}^2 d cos\phi\right)\mathbf{k} + \left(\ddot{\psi} d sin\phi + \dot{\psi}\dot{\phi} d cos\phi\right)\mathbf{j} \tag{65} \\
a_L &= \frac{dv_{xyz}}{dt} + \alpha_{xyz} \times \overline{O_{xyz}L} + \omega_{xyz} \times (\omega_{xyz} \times \overline{O_{xyz}L}) \tag{66} \\
&= (\ddot{x} - L\ddot{\psi})\mathbf{i} - (L\dot{\psi}^2)\mathbf{j} \tag{67} \\
a_R &= \frac{dv_{xyz}}{dt} + \alpha_{xyz} \times \overline{O_{xyz}R} + \omega_{xyz} \times (\omega_{xyz} \times \overline{O_{xyz}R}) \tag{68} \\
&= (\ddot{x} + L\ddot{\psi})\mathbf{i} + (L\dot{\psi}^2)\mathbf{j} \tag{69}
\end{align}
$$

8 Appendix

8.1 Deriving the equation in [1]

We take the expressions for velocities and accelerations derived in the paper and we evaluate the closed form expression in the paper for generalized inertia forces. The resulting expression should match the one derived in the paper.

```
syms L R d phi dx dpsi dphi ddx ddpsi ddphi real
syms m_S m_C I_1 I_2 I_3 Iw_1 Iw_2 Iw_3 real

u_1 = dx; u_2 = dpsi; u_3 = dphi;
du_1 = ddx; du_2 = ddpsi; du_3 = ddphi;

w_S  = [0; u_2; u_3];
w_C1 = [0; u_2; -(1/R)*u_1+(L/R)*u_2];
w_C2 = [0; u_2; -(1/R)*u_1-(L/R)*u_2];

v_S  = [u_1-u_3*d*cos(phi); -u_3*d*sin(phi); u_2*d*sin(phi)];
v_C1 = [u_1-u_2*L; 0; 0];
v_C2 = [u_1+u_2*L; 0; 0];

alpha_S  = [0; du_2; du_3];
alpha_C1 = [-(1/R)*u_1*u_2+(L/R)*u_2^2; du_2; -(1/R)*du_1+(L/R)*du_2];
alpha_C2 = [-(1/R)*u_1*u_2-(L/R)*u_2^2; du_2; -(1/R)*du_1-(L/R)*du_2];

a_S = [du_1-du_3*d*cos(phi)+(u_2^2+u_3^2)*d*sin(phi); ...
    -du_3*d*sin(phi)-u_3^2*d*cos(phi); ...
```

```
    du_2*d*sin(phi)+u_2*u_3*d*cos(phi)];
a_C1 = [du_1-L*du_2; 0;  L*u_2^2];
a_C2 = [du_1+L*du_2; 0; -L*u_2^2];

I_S = [I_1 0 0; 0 I_2 0; 0 0 I_3];
I_C1 = [Iw_1 0 0; 0 Iw_2 0; 0 0 Iw_3];
I_C2 = I_C1;

T_S = -I_S*alpha_S-cross(w_S, I_S*w_S);
T_C1 = -I_C1*alpha_C1-cross(w_C1, I_C1*w_C1);
T_C2 = -I_C2*alpha_C2-cross(w_C2, I_C2*w_C2);

R_S = -m_S*a_S;
R_C1 = -m_C*a_C1;
R_C2 = -m_C*a_C2;


F_1 = diff(w_S,  u_1)'*T_S  + diff(v_S,  u_1)'*R_S ...
    + diff(w_C1, u_1)'*T_C1 + diff(v_C1, u_1)'*R_C1 ...
    + diff(w_C2, u_1)'*T_C2 + diff(v_C2, u_1)'*R_C2;
F_2 = diff(w_S,  u_2)'*T_S  + diff(v_S,  u_2)'*R_S ...
    + diff(w_C1, u_2)'*T_C1 + diff(v_C1, u_2)'*R_C1 ...
    + diff(w_C2, u_2)'*T_C2 + diff(v_C2, u_2)'*R_C2;
F_3 = diff(w_S,  u_3)'*T_S  + diff(v_S,  u_3)'*R_S ...
    + diff(w_C1, u_3)'*T_C1 + diff(v_C1, u_3)'*R_C1 ...
    + diff(w_C2, u_3)'*T_C2 + diff(v_C2, u_3)'*R_C2;
```

The resulting expressions are:

$$
\begin{aligned}
F_1 &= -\frac{1}{R^2}(2R^2\ddot{x}m_C + 2I_{w3}\ddot{x} + R^2\ddot{x}m_S - R^2d\ddot{\phi}m_S cos\phi + R^2d\dot{\phi}^2m_S sin\phi + R^2d\dot{\psi}^2m_S sin\phi) \quad = -2\ddot{x}m_C - \frac{2I_{w3}}{R^2}\ddot{x} - \ddot{x}m_S + d\ddot{\phi}m_S cos\phi \\
&= -(m_S + 2m_C + \frac{2I_{w3}}{R^2})\ddot{x} + m_S dcos\phi\ddot{\phi} - m_S dsin\phi(\dot{\phi}^2 + \dot{\psi}^2) \\
F_2 &= -\frac{1}{R^2}(2I_{w3}L^2\ddot{\psi} + I_2R^2\ddot{\psi} + 2I_{w2}R^2\ddot{\psi} + 2L^2R^2\ddot{\psi}m_C + R^2d^2\ddot{\psi}m_S sin\phi^2 \\
&\qquad + \frac{1}{2}R^2d^2\dot{\phi}\dot{\psi}m_S sin(2\phi)) \\
&= -\frac{2I_{w3}L^2}{R^2}\ddot{\psi} - I_2\ddot{\psi} - 2I_{w2}\ddot{\psi} - 2L^2\ddot{\psi}m_C - d^2\ddot{\psi}m_S sin\phi^2 - \frac{1}{2}d^2\dot{\phi}\dot{\psi}m_S sin(2\phi) \\
&= -(2m_CL^2 + \frac{2I_{w3}L^2}{R^2} + 2I_{w2} + m_Sd^2sin\phi^2 + I_2)\ddot{\psi} - m_Sd^2sin\phi cos\phi\dot{\phi}\dot{\psi} \\
F_3 &= -d^2\ddot{\phi}m_S - I_3\ddot{\phi} + d\ddot{x}m_S cos\phi + \frac{1}{2}(d^2\dot{\psi}^2m_S sin(2\phi)) \\
&= -(m_Sd^2 + I_3)\ddot{\phi} + m_S dcos\phi\ddot{x} + m_Sd^2sin\phi cos\phi\dot{\psi}^2
\end{aligned}
$$

The expressions match exactly as are there in the paper if $I_{w3} = \frac{m_C R^2}{2}$ and $I_{w2} = \frac{m_C}{4R^2}$. The paper has made a typo in the last equation using $\dot{\phi}^2$ instead of $\dot{\psi}^2$.

We can write down the A and C matrices using the above equations so that it becomes easier to match it term by term with our derived equations. So we have:

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} -(m_S + 2m_C + \frac{2I_{w3}}{R^2}) & 0 & m_S dcos\phi \\ 0 & -(2m_CL^2 + \frac{2I_{w3}L^2}{R^2} + 2I_{w2} + m_Sd^2sin\phi^2 + I_2) & 0 \\ m_S dcos\phi & 0 & -(m_Sd^2 + I_3) \end{bmatrix} \\
\mathbf{C} &= \begin{bmatrix} 0 & -m_S dsin\phi\dot{\psi} & -m_S dsin\phi\dot{\phi} \\ 0 & -m_Sd^2sin\phi cos\phi\dot{\phi} & 0 \\ 0 & m_Sd^2sin\phi cos\phi\dot{\psi} & 0 \end{bmatrix}
\end{aligned} \tag{70}
$$

$M_1 = m_S$, $M_w = m_C$, $\begin{bmatrix} \mathbf{MX}_1 \\ \mathbf{MY}_1 \\ \mathbf{MZ}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ m_S d \\ 0 \end{bmatrix}$, $\begin{bmatrix} \mathbf{XX}_w & \mathbf{XY}_w & \mathbf{XZ}_w \\ \mathbf{XY}_w & \mathbf{YY}_w & \mathbf{YZ}_w \\ \mathbf{XZ}_w & \mathbf{YZ}_w & \mathbf{ZZ}_w \end{bmatrix} = \begin{bmatrix} I_{w1} & 0 & 0 \\ 0 & I_{w2} & 0 \\ 0 & 0 & I_{w3} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{XX}_1 & \mathbf{XY}_1 & \mathbf{XZ}_1 \\ \mathbf{XY}_1 & \mathbf{YY}_1 & \mathbf{YZ}_1 \\ \mathbf{XZ}_1 & \mathbf{YZ}_1 & \mathbf{ZZ}_1 \end{bmatrix} = \begin{bmatrix} I_1 + m_S d^2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 + m_S d^2 \end{bmatrix}$

Mismatched elements: A_{22}, C_{12}, C_{22}, C_{23} and C_{32}.
Resulting difference in the equations:

- Equation 1: $m_S dsin\phi\dot{\psi}^2$
- Equation 2: $(I_1 - I_2)sin^2\phi\ddot{\psi} + (2(I_1 - I_2) + m_S d^2)sin\phi cos\phi\dot{\phi}\dot{\psi}$
- Equation 3: $-(I_1 - I_2)sin\phi cos\phi\dot{\psi}^2$

Element	Derived Expression	Equivalent form in [1]	Expression in [1]
A_{11}	$M_1 + 2M_w + 2\frac{\mathbf{ZZ}_w}{R^2}$	$m_S + 2m_C + 2\frac{I_{w3}}{R^2}$	$-(m_S + 2m_C + \frac{2I_{w3}}{R^2})$
A_{12}	$\mathbf{MZ}_1 - 2\frac{\mathbf{YZ}_w}{R}$	$0 - 2\frac{0}{R} = 0$	0
A_{13}	$-\mathbf{MY}_1 cos\phi - \mathbf{MX}_1 sin\phi$	$-m_S dcos\phi - 0sin\phi = -m_S dcos\phi$	$m_S dcos\phi$
A_{21}	$\mathbf{MZ}_1 - 2\frac{\mathbf{YZ}_w}{R}$	$0 - 2\frac{0}{R} = 0$	0
A_{22}	$\mathbf{XX}_1 + 2\mathbf{YY}_w + 2M_w L^2 - (\mathbf{XX}_1 - \mathbf{YY}_1)cos^2\phi$ $+\mathbf{XY}_1 sin(2\phi) + \frac{2L^2\mathbf{ZZ}_w}{R^2}$	$I_1 + m_S d^2 + 2I_{w2} + 2m_C L^2 - (I_1 + m_S d^2 - I_2)cos^2\phi + 0sin(2\phi) + \frac{2L^2 I_{w3}}{R^2}$ $= 2m_C L^2 + \frac{2I_{w3}L^2}{R^2} + 2I_{w2} + (I_1 + m_S d^2)sin^2\phi + I_2 cos^2\phi$	$-(2m_C L^2 + \frac{2I_{w3}L^2}{R^2} + 2I_{w2} + m_S d^2 sin\phi^2 + I_2)$
A_{23}	$\mathbf{YZ}_1 cos\phi + \mathbf{XZ}_1 sin\phi$	$0cos\phi + 0sin\phi = 0$	0
A_{31}	$-\mathbf{MY}_1 cos\phi - \mathbf{MX}_1 sin\phi$	$-m_S dcos\phi - 0sin\phi = -m_S dcos\phi$	$m_S dcos\phi$
A_{32}	$\mathbf{YZ}_1 cos\phi + \mathbf{XZ}_1 sin\phi$	$0cos\phi + 0sin\phi = 0$	0
A_{33}	$\mathbf{ZZ}_1$	$m_S d^2 + I_3$	$-(m_S d^2 + I_3)$
C_{11}	0	0	0
C_{12}	0	0	$-m_S dsin\phi\dot{\psi}$
C_{13}	$-\dot{\phi}\,(\mathbf{MX}_1 cos\phi - \mathbf{MY}_1 sin\phi)$	$-\dot{\phi}\,(0cos\phi - m_S dsin\phi) = m_S dsin\phi\dot{\phi}$	$-m_S dsin\phi\dot{\phi}$
C_{21}	0	0	0
C_{22}	$\frac{\dot{\phi}}{2}\,(2\mathbf{XY}_1 cos(2\phi) + (\mathbf{XX}_1 - \mathbf{YY}_1)sin(2\phi))$	$\frac{\dot{\phi}}{2}\left(2 \times 0cos(2\phi) + (I_1 + m_S d^2 - I_2)sin(2\phi)\right)$ $= (I_1 + m_S d^2 - I_2)sin\phi cos\phi\dot{\phi}$	$-m_S d^2 sin\phi cos\phi\dot{\phi}$
C_{23}	$\dot{\psi}\,(\mathbf{XY}_1 cos(2\phi) + (\mathbf{XX}_1 - \mathbf{YY}_1)cos\phi sin\phi)$ $-\dot{\phi}\,(\mathbf{YZ}_1 sin\phi - \mathbf{XZ}_1 cos\phi)$	$\dot{\psi}\left(0cos(2\phi) + (I_1 + m_S d^2 - I_2)cos\phi sin\phi\right) - \dot{\phi}\,(0sin\phi - 0cos\phi)$ $= (I_1 + m_S d^2 - I_2)cos\phi sin\phi\dot{\psi}$	0
C_{31}	0	0	0
C_{32}	$-\frac{\dot{\psi}}{2}\,(2\mathbf{XY}_1 cos(2\phi) + (\mathbf{XX}_1 - \mathbf{YY}_1)sin(2\phi))$	$-\frac{\dot{\psi}}{2}\,(2 \times 0cos(2\phi) + (I_1 - I_2)sin(2\phi))$ $= -(I_1 + m_S d^2 - I_2)sin\phi cos\phi\dot{\psi}$	$m_S d^2 sin\phi cos\phi\dot{\psi}$
C_{33}	0	0	0

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