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The Use of Kane's Dynamical Equations in Robotics

Abstract

Extensive experience has shown that the use of general-purpose, multibody-dynamics computer programs for the numerical formulation and solution of equations of motion of robotic devices leads to slow evaluation of actuator forces and torques and slow simulation of robot motions. In this paper, it is shown how improvements in computational efficiency can be effected by using Kane's dynamical equations to formulate explicit equations of motion. To these ends, a detailed analysis of the Stanford Arm is presented in such a way that each step in the analysis serves as an illustrative example for a general method of attack on problems of robot dynamics. Simulation results are reported and are used as a basis for discussing questions of computational efficiency.

1. Introduction

Dynamical equations governing the behavior of robotic devices play an important role in the design and operation of such devices. In design, dynamical equations are employed to carry out simulations for the purpose of testing the performance of a robot. For example, the response of a device to various loading conditions, or the relative merits of competing control schemes, can be studied by means of simulations. As regards robot operations, dynamical equations frequently are used to compute the forces and torques needed to drive members of a manipulator in such a way as to achieve desired end-effector motions, a task that must be performed repeatedly and, in most cases, rapidly. Consequently, it is important to construct the most efficient possible computational algorithms. It is the purpose of this paper to show how this can be accomplished by proper use of Kane's dynamical equations (Kane, Likins, and Levinson 1983, p. 275).

During the past 10 to 15 years, it has become a widespread practice in the field of robotics to relegate to the computer not merely the task of solving differential equations of motion or of calculating forces and torques by simply performing arithmetic operations, but to use the computer, in addition, for the forming of terms that make up the requisite dynamical equations, thus freeing the user of a computer program from the burden of dynamical analysis. While this practice certainly leads to savings in analysts' time and can enable individuals not well versed in dynamics to participate in projects otherwise inaccessible to them, it inevitably entails sacrifices in computational efficiency that, at times, become so severe as to preclude satisfactory operation of a robot. As is to be shown in what follows, this difficulty can be alleviated by formulating *explicit* equations of motion for the robotic device under consideration, a task that can be performed readily, provided one uses a suitable method to generate the equations.

The two formalisms most frequently employed to date in robotics, namely Lagrange and Newton-Euler equations, suffer from serious weaknesses. The Lagrange method tends to lead to computational algorithms involving large numbers of unnecessary arithmetic operations, and the Newton-Euler approach can force one to perform unnecessary calculations associated with the elimination of certain forces and torques of interaction between elements of a robot, particularly when such elements form closed loops. Both methods are quite laborious, and, when one attempts to save hand labor by resorting to the use of a symbol-manipulation computer program, one finds frequently that intermediate computations produce expressions so large that their storage requirements ex-

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ceed the capacities of the largest available computers, even when the robot being analyzed possesses only a modest number of elements. What is needed, therefore, is a method that is minimally laborious and leads directly to the simplest possible computational algorithms.

In connection with problems of spacecraft dynamics, it has been shown (Kane and Levinson 1980) that Kane's dynamical equations enable one to work systematically with dependent variables that are especially well suited to a given problem, to eliminate effortlessly forces and torques that are of no interest, and to produce straightforwardly explicit equations of motion having a computationally sound form. To aid the robotics community in gaining access to the same benefits, we undertake a detailed analysis of the Stanford Arm (Paul 1982), a six-element, six-degree-of-freedom manipulator, to illustrate specific facets of a general method for the formulation of equations of motion of any manipulator. Our goal is to assist readers who follow our analysis step by step, with pencil and paper in hand, subsequently to apply the same methodology in connection with their own problems.

The sequel begins with a detailed description of the Stanford Arm. Next, the basic ingredients needed for the construction of Kane's dynamical equations, namely, generalized speeds, partial angular velocities, partial velocities, generalized inertia forces, and generalized active forces, in general terms and are worked out for the Stanford Arm. Thereafter, it is shown how one uses the two kinds of generalized forces to obtain equations of motion in algorithmic form. Finally, simulation results are reported, and the matter of computational efficiency is discussed.

2. Analysis

In Fig. 1, the five control torques and the control force that come into play in connection with a particular maneuver of the Stanford Arm are plotted as functions of time for an interval of 10 s. The curves were generated by means of a computer program based on explicit equations of motion. Let us examine the process leading to such a computer program.

A schematic representation of a robot is an invaluable aid in the analysis of this sort of device. Figure 2 provides such a representation of the Stanford Arm, which consists of six bodies, designated $A, \ldots, F$. Body A can be rotated in a Newtonian reference frame N about a vertical axis fixed in both N and A, and A supports B, which can be made to rotate relative to A about a horizontal axis fixed in both A and B. B, in turn, supports C, and C can be made to perform purely translational motions relative to B while carrying D, which can be made to rotate relative to C about an axis fixed in both C and D. Finally, E is connected to D, and F to E, in such a way that each can be made to rotate, relative to the body that supports it, about an axis fixed in both the supporting and in the supported body, as indicated in Fig. 2.

To characterize the instantaneous configuration of the arm, we employ generalized coordinates $q_1, \ldots, q_6$, the first five of which are the radian measures of rotation angles, while q_6 is the distance between the mass centers, B^* and C^*, of B and C, respectively. For the configuration depicted in Fig. 2, $q_1, \ldots, q_5$ are regarded as being equal to zero. Dimensions of interest are designated $L_1, \ldots, L_6$ in Fig. 2, where $A^*, \ldots, F^*$ are the mass centers of $A, \ldots, F$, respectively.

Just as important as generalized coordinates are *generalized speeds,* these being quantities intimately associated with the motion of a system, rather than merely with its configuration. Formally, generalized speeds $u_1, \ldots, u_n$, where n is the number of generalized coordinates, always can be introduced as

$$u_r \triangleq \sum_{s=1}^{n} A_{rs}\dot{q}_s + B_r \qquad (r = 1, \ldots, n), \qquad (1)$$

where A_{rs} and B_r are functions of $q_1, \ldots, q_n$, and the time t; $\dot{q}_s$ denotes dq_s/dt; and A_{rs} and $B_r (r, s = 1, \ldots, n)$ are chosen such that (Eq. 1) can be solved uniquely for $\dot{q}_1, \ldots, \dot{q}_n$. The generalized speeds $u_1, \ldots, u_n$ serve as dependent variables on an equal footing with the generalized coordinates $q_1, \ldots, q_n$. Their introduction can enable one to take advantage of special features of a given physical system to bring equations of motion into a particularly simple form. Generally, this is accomplished by taking u_r to be an angular velocity measure number, a velocity measure number, or simply $\dot{q}_r$. Consider, for

Fig. 1. Control torques and control force.

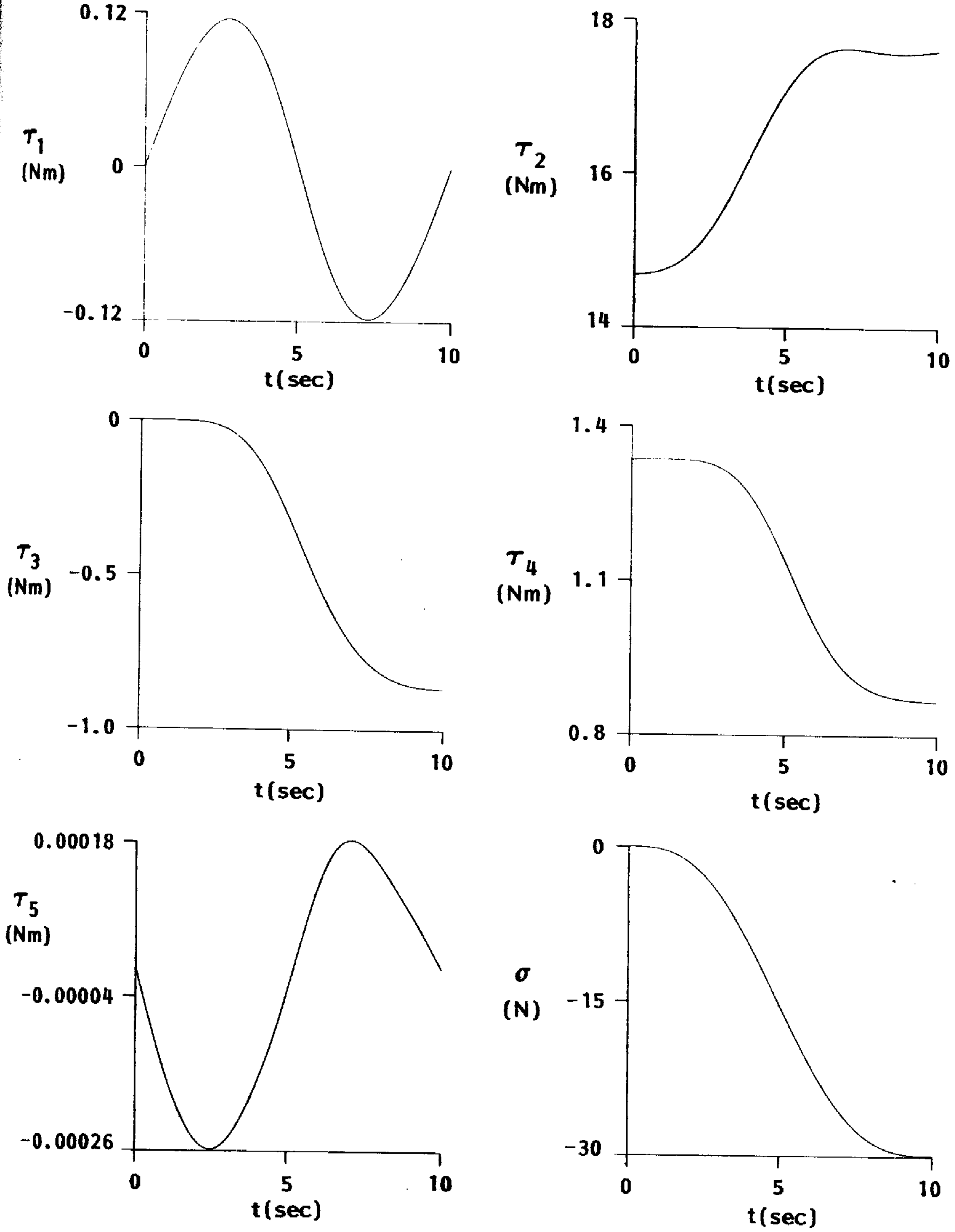
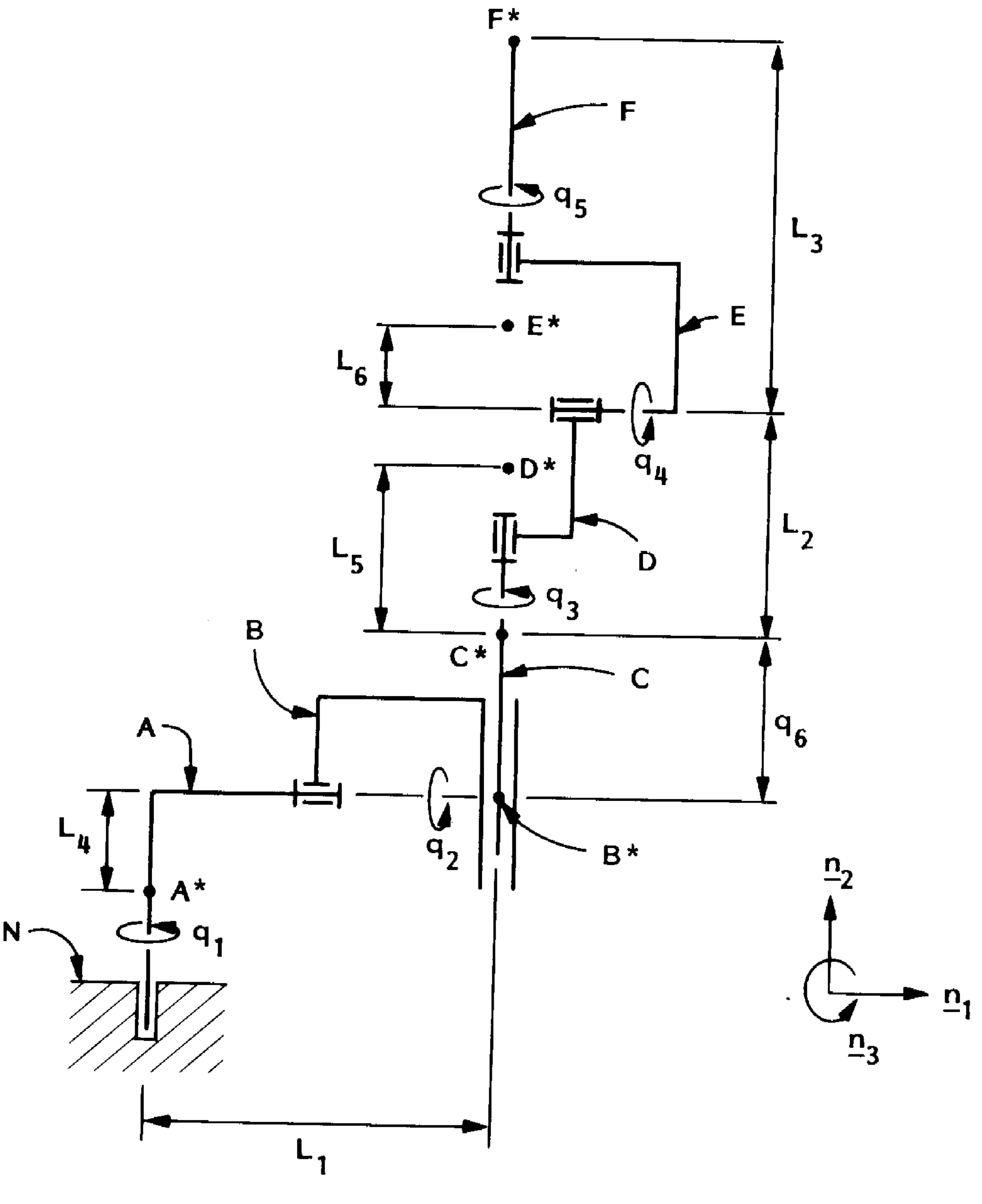


Fig. 2. Schematic representation of Stanford Arm.



example, the Stanford Arm. Let $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ form a dextral set of mutually perpendicular unit vectors fixed in the Newtonian reference frame N as shown in Fig. 2, and let $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ be a set of unit vectors fixed in D in such a way that $\mathbf{d}_i = \mathbf{n}_i$ $(i = 1, 2, 3)$ when $q_1 = q_2 = q_3 = 0$. Now, suppose it is known that the particular Stanford Arm being analyzed is not to be operated in the vicinity of $q_2 = 0$ or $q_2 = 180°$. Then one can introduce $u_1, \ldots, u_6$ as

$$u_i \triangleq \omega^D \cdot \mathbf{d}_i \qquad (i = 1, 2, 3) \tag{2}$$

and

$$u_i \triangleq \dot{q}_i \qquad (i = 4, 5, 6), \tag{3}$$

here ω^D denotes the angular velocity of D in N. To e that (Eqs. 2 and 3) imply relationships having the rm of (Eq. 1), one may first verify that

$$\omega^D = (\dot{q}_1 s_2 s_3 + \dot{q}_2 c_3)\mathbf{d}_1 + (\dot{q}_1 c_2 + \dot{q}_3)\mathbf{d}_2 + (-\dot{q}_1 s_2 c_3 + \dot{q}_2 s_3)\mathbf{d}_3, \quad (4)$$

here s_i and c_i denote $\sin q_i$ and $\cos q_i$ $(i = 1, 2, 3)$, espectively. Substitution into (Eq. 2) then yields

$$u_1 = \dot{q}_1 s_2 s_3 + \dot{q}_2 c_3, \quad (5)$$

$$u_2 = \dot{q}_1 c_2 + \dot{q}_3, \quad (6)$$

nd

$$u_3 = -\dot{q}_1 s_2 c_3 + \dot{q}_2 s_3, \quad (7)$$

which shows that (Eq. 5) corresponds to (Eq. 1) with $r = 1$ if one lets $A_{11} \triangleq s_2 s_3, A_{12} \triangleq c_3, A_{13} \triangleq A_{14} \triangleq A_{15} \triangleq A_{16} \triangleq 0$, and $B_1 \triangleq 0$. Similarly, for $r = 2$, one may let $A_{21} \triangleq c_2, A_{22} \triangleq 0, A_{23} \triangleq 1, A_{24} \triangleq A_{25} \triangleq A_{26} \triangleq 0$ and $B_2 = 0$; and for $r = 3$ we have $A_{31} \triangleq -s_2 c_3, A_{32} \triangleq s_3, A_{33} \triangleq \ldots \triangleq A_{36} \triangleq 0$, and $B_3 \triangleq 0$. As for (Eq. 3), simple inspection reveals that they have the form of (Eq. 1). Moreover, (Eqs. 5–7 and 3) can be solved uniquely for $\dot{q}_1, \ldots, \dot{q}_6$. Specifically, (Eqs. 5–7) yield

$$\dot{q}_1 = (u_1 s_3 - u_3 c_3)/s_2, \quad (8)$$

$$\dot{q}_2 = u_1 c_3 + u_3 s_3, \quad (9)$$

and

$$\dot{q}_3 = u_2 + (u_3 c_3 - u_1 s_3)c_2/s_2, \quad (10)$$

(with the singularities at $q_2 = 0$ and $q_2 = 180°$ posing no problem), and (Eq. 3) obviously permit one to write

$$\dot{q}_i = u_i \qquad (i = 4, 5, 6). \quad (11)$$

Thus, $u_1, \ldots, u_6$ as defined in (Eqs. 2 and 3) form a set of generalized speeds for the Stanford Arm.

Before leaving (Eqs. 8–10), we begin a process that is central to the method at hand, namely the introducing of quantities $Z_1, \ldots, Z_{196}$ for the dual purpose of minimizing writing and of moving most expeditiously toward an efficient computer code. The first three such quantities, Z_1, Z_2, Z_3, are defined as the right-hand members of (Eqs. 8, 9, 10), respectively, which makes it possible to replace (Eqs. 8–10) with

$$\dot{q}_i = Z_i \qquad (i = 1, 2, 3). \quad (12)$$

Expressions for Z_1, Z_2, Z_3 are recorded in Appendix 1.

The next task to be undertaken is that of expressing the angular velocity of each of $A, \ldots, F$ in N in two forms, one involving the generalized speeds $u_1, \ldots, u_6$ implicitly, the other explicitly. In both cases, we use a vector basis fixed in the body under consideration. (The reason for doing this will be set forth shortly.) For example, letting $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be a set of unit vectors fixed in A in such a way that $\mathbf{a}_i = \mathbf{n}_i$ $(i = 1, 2, 3)$ when $q_1 = q_2 = q_3 = 0$, one can express the angular velocity of A in N as

$$\omega^A = \dot{q}_1 \mathbf{a}_2 \quad (13)$$

or, in view of (Eq. 12), in the implicit form

$$\omega^A = Z_1 \mathbf{a}_2. \quad (14)$$

Alternatively, referring to the definition of Z_1 in Appendix 1, and introducing Z_4 and Z_5 as in Appendix 1, we can write ω^A in the explicit form

$$\omega^A = (Z_4 u_1 + Z_5 u_3)\mathbf{a}_2. \quad (15)$$

Similarly, with $Z_6, \ldots, Z_{11}$ defined as in Appendix 1, one can verify that ω^B is given both by

$$\omega^B = Z_2 \mathbf{b}_1 + Z_{10} \mathbf{b}_2 + Z_{11} \mathbf{b}_3 \quad (16)$$

and

$$\omega^B = (c_3 u_1 + s_3 u_3)\mathbf{b}_1 + (Z_6 u_1 + Z_7 u_3)\mathbf{b}_2 + (Z_8 u_1 + Z_9 u_3)\mathbf{b}_3. \quad (17)$$

Moreover, since C has the same rotational motion as B, so that $\omega^C = \omega^B$, (Eqs. 16 and 17) apply also when B is replaced with C, and $\mathbf{b}_i$ with $\mathbf{c}_i$ $(i = 1, 2, 3)$. As for D, it follows directly from (Eq. 2) that the explicit form of ω^D is

$$\omega^D = u_1 \mathbf{d}_1 + u_2 \mathbf{d}_2 + u_3 \mathbf{d}_3. \quad (18)$$

Since this cannot be simplified further, we shall use it also when, otherwise, the implicit form would be called for. Finally, to deal with E and F, one may introduce $Z_{12}, \ldots, Z_{21}$ as in Appendix 1 in order to express $\boldsymbol{\omega}^E$ as

$$\boldsymbol{\omega}^E = Z_{12}\mathbf{e}_1 + Z_{13}\mathbf{e}_2 + Z_{14}\mathbf{e}_3 \tag{19}$$

and

$$\boldsymbol{\omega}^E = (u_1 + u_4)\mathbf{e}_1 + (c_4u_2 + s_4u_3)\mathbf{e}_2 + (-s_4u_2 + c_4u_3)\mathbf{e}_3, \tag{20}$$

and $\boldsymbol{\omega}^F$ as

$$\boldsymbol{\omega}^F = Z_{19}\mathbf{f}_1 + Z_{20}\mathbf{f}_2 + Z_{21}\mathbf{f}_3 \tag{21}$$

and

$$\begin{aligned}\boldsymbol{\omega}^F = {} & (c_5u_1 + Z_{15}u_2 + Z_{16}u_3 + c_5u_4)\mathbf{f}_1 \\ & + (c_4u_2 + s_4u_3 + u_5)\mathbf{f}_2 \\ & + (s_5u_1 + Z_{17}u_2 + Z_{18}u_3 + s_5u_4)\mathbf{f}_3.\end{aligned} \tag{22}$$

It can be shown that, once generalized speeds have been introduced, the angular velocity $\boldsymbol{\omega}^R$ of *any* rigid body R in a Newtonian reference frame N can be expressed *uniquely* as

$$\boldsymbol{\omega}^R = \sum_{r=1}^{n} \boldsymbol{\omega}_r^R u_r + \boldsymbol{\omega}_t^R, \tag{23}$$

where $\boldsymbol{\omega}_r^R$ and $\boldsymbol{\omega}_t^R$ are functions of $q_1, \ldots, q_n$, and t. One can, therefore, obtain an expression for $\boldsymbol{\omega}_r^R$, called the rth *partial angular velocity* of R in N, simply by inspecting the coefficients of $u_1, \ldots, u_n$ in an expression for $\boldsymbol{\omega}^R$ that has the form of the right-hand member of (Eq. 23). (Partial angular velocities and their counterparts, partial velocities, to be introduced subsequently, play central roles in the construction of Kane's dynamical equations.) For example, (Eq. 15) permits one to identify $\boldsymbol{\omega}_1^A, \ldots, \boldsymbol{\omega}_6^A$ as

$$\boldsymbol{\omega}_1^A = Z_4\mathbf{a}_2, \quad \boldsymbol{\omega}_2^A = 0, \quad \boldsymbol{\omega}_3^A = Z_5\mathbf{a}_2 \tag{24}$$

and

$$\boldsymbol{\omega}_4^A = \boldsymbol{\omega}_5^A = \boldsymbol{\omega}_6^A = 0, \tag{25}$$

while (Eq. 17) shows that

$$\boldsymbol{\omega}_1^B = c_3\mathbf{b}_1 + Z_6\mathbf{b}_2 + Z_8\mathbf{b}_3, \quad \boldsymbol{\omega}_2^B = 0 \tag{26}$$

and

$$\boldsymbol{\omega}_3^B = s_3\mathbf{b}_1 + Z_7\mathbf{b}_2 + Z_9\mathbf{b}_3, \quad \boldsymbol{\omega}_r^B = 0 \quad (r = 4, 5, 6). \tag{27}$$

Similarly, expressions for the partial angular velocities, $\boldsymbol{\omega}_r^D$, $\boldsymbol{\omega}_r^E$, and $\boldsymbol{\omega}_r^F$ ($r = 1, \ldots, 6$) can be formed by inspecting (Eqs. 18, 20, and 22), respectively.

As has already been noted, we have expressed $\boldsymbol{\omega}^A$ in terms of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ (see [Eqs. 14 and 15]), $\boldsymbol{\omega}^B$ in terms of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ (see [Eq. 16 and 17]), and so forth. The reason for doing this is that it leads automatically to expressions for $\boldsymbol{\omega}_r^A$ ($r = 1, \ldots, 6$) in terms of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, $\boldsymbol{\omega}_r^B$ ($r = 1, \ldots, 6$) in terms of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, and so forth, and this will facilitate later work, where we shall assume that the central principal axes of inertia of A are parallel to $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, those of B to $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, and so forth. When it comes to dealing with the velocities of $A^*, \ldots, F^*$, the mass centers of $A, \ldots, F$, which we shall do next, it is not necessarily advantageous to work with $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in the case of A^*, with $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ for B^*, and so forth. Instead, it is best to use whatever vector basis permits one to write the simplest expression. However, it is necessary once again to construct for each velocity both an explicit and an implicit expression as regards $u_1, \ldots, u_6$.

Beginning with the velocity of A^* in N, we note that, since A^* is fixed in N,

$$\mathbf{v}^{A^*} = 0. \tag{28}$$

Next, the velocity of B^* in N is given by (see [Eq. 8])

$$\mathbf{v}^{B^*} = -L_1\dot{q}_1\mathbf{a}_3 = [-L_1(u_1s_3 - u_3c_3)/s_2]\mathbf{a}_3. \tag{29}$$

Consequently, with Z_{22}, Z_{23}, Z_{24} defined as in Appendix 1, one has

$$\mathbf{v}^{B^*} = (Z_{22}u_1 + Z_{23}u_3)\mathbf{a}_3 \tag{30}$$

and

$$\mathbf{v}^{B^*} = Z_{24}\mathbf{a}_3. \tag{31}$$

Similarly, the introduction of $Z_{25}, \ldots, Z_{33}$ as in Appendix 1 leads to

$$\mathbf{v}^{C^*} = (Z_{25}u_1 + Z_{26}u_3)\mathbf{c}_1 + (Z_{27}u_1 + Z_{28}u_3 + u_6)\mathbf{c}_2 + (Z_{29}u_1 + Z_{30}u_3)\mathbf{c}_3 \quad (32)$$

and

$$\mathbf{v}^{C^*} = Z_{31}\mathbf{c}_1 + Z_{32}\mathbf{c}_2 + Z_{33}\mathbf{c}_3. \quad (33)$$

Moreover, to obtain the desired expressions for $\mathbf{v}^{D^*}$, one need only to replace q_6 with $q_6 + L_5$ in $\mathbf{v}^{C^*}$. That is, with $Z_{34}, \ldots, Z_{39}$ as in Appendix 1, one can write

$$\mathbf{v}^{D^*} = (Z_{34}u_1 + Z_{35}u_3)\mathbf{c}_1 + (Z_{27}u_1 + Z_{28}u_3 + u_6)\mathbf{c}_2 + (Z_{36}u_1 + Z_{37}u_3)\mathbf{c}_3 \quad (34)$$

and

$$\mathbf{v}^{D^*} = Z_{38}\mathbf{c}_1 + Z_{32}\mathbf{c}_2 + Z_{39}\mathbf{c}_3. \quad (35)$$

Finally, to deal with E^* and F^*, it is helpful to introduce C_{ij} as

$$C_{ij} \triangleq \mathbf{c}_i \cdot \mathbf{e}_j \qquad (i, j = 1, 2, 3), \quad (36)$$

which means that

$$C_{11} = c_3, \qquad C_{12} = s_3 s_4, \qquad C_{13} = s_3 c_4, \quad (37)$$

$$C_{21} = 0, \qquad C_{22} = c_4, \qquad C_{23} = -s_4, \quad (38)$$

and

$$C_{31} = -s_3, \qquad C_{32} = c_3 s_4, \qquad C_{33} = c_3 c_4. \quad (39)$$

After defining $Z_{40}, \ldots, Z_{62}$ as in Appendix 1, one then finds that

$$\mathbf{v}^{E^*} = (Z_{44}u_1 + Z_{50}u_2 + Z_{51}u_3)\mathbf{e}_1 + (Z_{46}u_1 + Z_{47}u_3 + C_{22}u_6)\mathbf{e}_2 + (Z_{52}u_1 + Z_{49}u_3 + L_6u_4 + C_{23}u_6)\mathbf{e}_3, \quad (40)$$

and that

$$\mathbf{v}^{E^*} = Z_{54}\mathbf{e}_1 + Z_{55}\mathbf{e}_2 + Z_{57}\mathbf{e}_3, \quad (41)$$

while

$$\mathbf{v}^{F^*} = (Z_{44}u_1 + Z_{58}u_2 + Z_{59}u_3)\mathbf{e}_1 + (Z_{46}u_1 + Z_{47}u_3 + C_{22}u_6)\mathbf{e}_2 + (Z_{60}u_1 + Z_{49}u_3 + L_3u_4 + C_{23}u_6)\mathbf{e}_3 \quad (42)$$

and

$$\mathbf{v}^{F^*} = Z_{61}\mathbf{e}_1 + Z_{55}\mathbf{e}_2 + Z_{62}\mathbf{e}_3. \quad (43)$$

Equations (28, 30, 32, 34, 40, and 42) illustrate the fact that, in general, $\mathbf{v}^P$, the velocity in N of any point P, can be expressed *uniquely* as

$$\mathbf{v}^P = \sum_{r=1}^{n} \mathbf{v}_r{}^P u_r + \mathbf{v}_t{}^P, \quad (44)$$

where $\mathbf{v}_r{}^P$ and $\mathbf{v}_t{}^P$ are functions of $q_1, \ldots, q_n$, and t. The quantity $\mathbf{v}_r{}^P$, called the rth *partial velocity* of P in N, thus can be found simply by inspecting the coefficients of $u_1, \ldots, u_n$ in an expression for $\mathbf{v}^P$ that has the form of the right-hand member of (Eq. 44). For example, (Eqs. 28 and 30) reveal, respectively, that

$$\mathbf{v}_r{}^{A^*} = 0 \qquad (r = 1, \ldots, 6) \quad (45)$$

and that

$$\mathbf{v}_r{}^{B^*} = Z_{22}\mathbf{a}_3, \ \ \mathbf{v}_2{}^{B^*} = 0, \ \ \mathbf{v}_3{}^{B^*} = Z_{23}\mathbf{a}_3 \quad (46)$$

and

$$\mathbf{v}_r{}^{B^*} = 0 \qquad (r = 4, 5, 6). \quad (47)$$

Similarly, expressions for the partial velocities $\mathbf{v}_r{}^{C^*}, \ldots, \mathbf{v}_r{}^{F^*}$ $(r = 1, \ldots, 6)$ can be formed by inspecting (Eqs. 32, 34, 40, and 42).

In addition to partial angular velocities and partial velocities, we shall need the angular accelerations of $A, \ldots, F$ in N and the accelerations of $A^*, \ldots, F^*$ in N, all expressed in forms bringing $\dot{u}_1, \ldots, \dot{u}_6$ into evidence explicitly. To obtain the necessary expressions, one can differentiate available angular velocity and velocity formulas with respect to t. For example, to find α^A, the angular acceleration of A in N, we refer to (Eq. 15) to write

$$\alpha^A = (\dot{Z}_4 u_1 + Z_4 \dot{u}_1 + \dot{Z}_5 u_3 + Z_5 \dot{u}_3)\mathbf{a}_2. \quad (48)$$

Similarly, (Eq. 17) implies that

$$\alpha^B = \alpha^C = (\dot{c}_3 u_1 + c_3 \dot{u}_1 + \dot{s}_3 u_3 + s_3 \dot{u}_3)\mathbf{b}_1 + (\dot{Z}_6 u_1 + Z_6 \dot{u}_1 + \dot{Z}_7 u_3 + Z_7 \dot{u}_3)\mathbf{b}_2 + (\dot{Z}_8 u_1 + Z_8 \dot{u}_1 + \dot{Z}_9 u_3 + Z_9 \dot{u}_3)\mathbf{b}_3, \quad (49)$$

and (Eqs. 18, 20, and 22) lead to analogous expressions for α^D, α^E, and α^F, respectively. However, to express $\alpha^A, \ldots, \alpha^F$ in an ultimately fully useful form, one must provide for the evaluation of $\dot{Z}_4$ and $\dot{Z}_5$ in (Eq. 48), $\dot{Z}_6, \ldots, \dot{Z}_9$ in (Eq. 49), and $\dot{Z}_{15}, \ldots, \dot{Z}_{18}$ in connection with (Eq. 22). Consequently, we use Z_4 as given in Appendix 1 to write

$$\dot{Z}_4 = (s_2\dot{s}_3 - s_3\dot{s}_2)/s_2{}^2 = (s_2 c_3 \dot{q}_3 - s_3 c_2 \dot{q}_2)/s_2{}^2, \quad (50)$$

or, in view of (Eq. 12),

$$\dot{Z}_4 = (s_2 c_3 Z_3 - s_3 c_2 Z_2)/s_2{}^2. \quad (51)$$

Moreover, we define Z_{63} as in Appendix 1, so that

$$\dot{Z}_4 = Z_{63}. \quad (52)$$

Similarly, after forming $\dot{Z}_5$ with the aid of Z_5 and (Eq. 12), and introducing Z_{64} as in Appendix 1, one has

$$\dot{Z}_5 = Z_{64}, \quad (53)$$

and analogous steps associated with $\dot{Z}_6, \ldots, \dot{Z}_9$ and $\dot{Z}_{15}, \ldots, \dot{Z}_{18}$ lead to

$$\dot{Z}_6 = Z_{67}, \quad \dot{Z}_7 = Z_{68}, \quad \dot{Z}_8 = Z_{69}, \quad \dot{Z}_9 = Z_{70}, \quad (54)$$

and

$$\dot{Z}_{15} = Z_{77}, \quad \dot{Z}_{16} = Z_{78}, \quad \dot{Z}_{17} = Z_{79}, \quad \dot{Z}_{18} = Z_{80}. \quad (55)$$

With these relationships in hand, we can replace (Eq. 48) with (see [Eqs. 52 and 53], and Z_{71} in Appendix 1)

$$\alpha^A = (\dot{u}_1 Z_4 + \dot{u}_3 Z_5 + Z_{71})\mathbf{a}_2. \quad (56)$$

Similarly, (Eq. 49) gives way to

$$\alpha^B = \alpha^C = (\dot{u}_1 c_3 + \dot{u}_3 s_3 + Z_{72})\mathbf{b}_1 + (\dot{u}_1 Z_6 + \dot{u}_3 Z_7 + Z_{73})\mathbf{b}_2 + (\dot{u}_1 Z_8 + \dot{u}_3 Z_9 + Z_{74})\mathbf{b}_3, \quad (57)$$

while $\alpha^D, \ldots, \alpha^F$ become

$$\alpha^D = \dot{u}_1 \mathbf{d}_1 + \dot{u}_2 \mathbf{d}_2 + \dot{u}_3 \mathbf{d}_3, \quad (58)$$

$$\alpha^E = (\dot{u}_1 + \dot{u}_4)\mathbf{e}_1 + (\dot{u}_2 c_4 + \dot{u}_3 s_4 + Z_{75})\mathbf{e}_2 + (-\dot{u}_2 s_4 + \dot{u}_3 c_4 + Z_{76})\mathbf{e}_3, \quad (59)$$

and

$$\alpha^F = (\dot{u}_1 c_5 + \dot{u}_2 Z_{15} + \dot{u}_3 Z_{16} + \dot{u}_4 c_5 + Z_{81})\mathbf{f}_1 + (\dot{u}_2 c_4 + \dot{u}_3 s_4 + \dot{u}_5 + Z_{82})\mathbf{f}_2 + (\dot{u}_1 s_5 + \dot{u}_2 Z_{17} + \dot{u}_3 Z_{18} + \dot{u}_4 s_5 + Z_{83})\mathbf{f}_3. \quad (60)$$

The next task that must be undertaken is that of forming expressions for the accelerations of $A^*, \ldots, F^*$, which is accomplished by differentiating (Eqs. 28, 30, 32, 34, 40, and 42) with respect to t in N, these particular equations being used because, once again, $\dot{u}_1, \ldots, \dot{u}_6$ are to be kept in evidence explicitly. Since expressions for $\dot{C}_{ij}$ will be required in connection with the accelerations of E^* and F^*, we use (Eqs. 37–39) together with (Eqs. 11 and 12) to write

$$\dot{C}_{11} = -\dot{q}_3 s_3 = -Z_3 s_3, \quad \dot{C}_{12} = \dot{q}_3 c_3 s_4 + \dot{q}_4 s_3 c_4 = Z_3 c_3 s_4 + u_4 s_3 c_4, \quad (61)$$

and so forth. Moreover, after $Z_{84}, \ldots Z_{91}$ have been defined in accordance with Appendix 1, then (Eq. 61) and the remaining equations for $\dot{C}_{ij}$ can be replaced with

$$\dot{C}_{11} = Z_{84}, \quad \dot{C}_{12} = Z_{88}, \quad \dot{C}_{13} = Z_{89}, \quad (62)$$

$$\dot{C}_{21} = 0, \quad \dot{C}_{22} = Z_{86}, \quad \dot{C}_{23} = Z_{87}, \quad (63)$$

and

$$\dot{C}_{31} = Z_{85}, \quad \dot{C}_{32} = Z_{90}, \quad \dot{C}_{33} = Z_{91}. \quad (64)$$

It follows immediately from (Eq. 28) that

$$\mathbf{a}^{A^*} = 0, \quad (65)$$

and (Eq. 30) shows that $\dot{Z}_{22}$ and $\dot{Z}_{23}$ are required in connection with $\mathbf{a}^{B^*}$. Now (see Appendix 1),

$$\dot{Z}_{22} = -L_1 \dot{Z}_4, \quad \dot{Z}_{23} = -L_1 \dot{Z}_5, \quad (66)$$

or, in view of (Eqs. 52 and 53),

$$\dot{Z}_{22} = -L_1\dot{Z}_{63}, \; \dot{Z}_{23} = -L_1\dot{Z}_{64}. \tag{67}$$

Hence, with the aid of Z_{92} and Z_{93}, one finds that

$$\dot{Z}_{22} = Z_{92}, \; \dot{Z}_{23} = Z_{93}, \tag{68}$$

and time-differentiation of (Eq. 30) in N produces

$$\mathbf{a}^{B^*} = (Z_{92}u_1 + Z_{22}\dot{u}_1 + Z_{93}u_3 + Z_{23}\dot{u}_3)\mathbf{a}_3 + \boldsymbol{\omega}^A \times \mathbf{v}^{B^*}. \tag{69}$$

Furthermore, it follows from (Eqs. 14 and 31) that

$$\boldsymbol{\omega}^A \times \mathbf{v}^{B^*} = Z_1 Z_{24}\mathbf{a}_1, \tag{70}$$

which, by the way, shows why it was desirable to express $\boldsymbol{\omega}^A$ and $\mathbf{v}^{B^*}$ in what we have called the implicit form. To bring $\mathbf{a}^{B^*}$ into final form, all that remains to be done is to introduce Z_{94} and Z_{95} as in Appendix 1, whereupon one can write

$$\mathbf{a}^{B^*} = Z_{94}\mathbf{a}_1 + (Z_{22}\dot{u}_1 + Z_{23}\dot{u}_3 + Z_{95})\mathbf{a}_3. \tag{71}$$

The derivation of analogous expressions for $\mathbf{a}^{C^*}, \ldots, \mathbf{a}^{F^*}$ proceeds similarly and involves $Z_{96}, \ldots, Z_{136}$, defined in Appendix 1. The accelerations in question are given by

$$\mathbf{a}^{C^*} = (Z_{25}\dot{u}_1 + Z_{26}\dot{u}_3 + Z_{105})\mathbf{c}_1 + (Z_{27}\dot{u}_1 + Z_{28}\dot{u}_3 + \dot{u}_6 + Z_{106})\mathbf{c}_2 + (Z_{29}\dot{u}_1 + Z_{30}\dot{u}_3 + Z_{107})\mathbf{c}_3, \tag{72}$$

$$\mathbf{a}^{D^*} = (Z_{34}\dot{u}_1 + Z_{35}\dot{u}_2 + Z_{112})\mathbf{c}_1 + (Z_{27}\dot{u}_1 + Z_{28}\dot{u}_3 + \dot{u}_6 + Z_{113})\mathbf{c}_2 + (Z_{36}\dot{u}_1 + Z_{37}\dot{u}_3 + Z_{114})\mathbf{c}_3, \tag{73}$$

$$\mathbf{a}^{E^*} = (Z_{44}\dot{u}_1 + Z_{50}\dot{u}_2 + Z_{51}\dot{u}_3 + Z_{128})\mathbf{e}_1 + Z_{46}\dot{u}_1 + Z_{47}\dot{u}_3 + C_{22}\dot{u}_6 + Z_{129})\mathbf{e}_2 + (Z_{52}\dot{u}_1 + Z_{49}\dot{u}_3 + L_6\dot{u}_4 + C_{23}\dot{u}_6 + Z_{130})\mathbf{e}_3, \tag{74}$$

and

$$\mathbf{a}^{F^*} = (Z_{44}\dot{u}_1 + Z_{58}\dot{u}_2 + Z_{59}\dot{u}_3 + Z_{134})\mathbf{e}_1 + (Z_{46}\dot{u}_1 + Z_{47}\dot{u}_3 + C_{22}\dot{u}_6 + Z_{135})\mathbf{e}_2 + (Z_{60}\dot{u}_1 + Z_{49}\dot{u}_3 + L_3\dot{u}_4 + C_{23}\dot{u}_6 + Z_{136})\mathbf{e}_3. \tag{75}$$

The kinematical analysis performed so far provides the ingredients needed for the construction of expressions for generalized inertia forces, which we undertake next. To explain what is meant by *generalized inertia forces,* we consider a system S formed by ν particles $P_1, \ldots, P_\nu$ having masses $m_1, \ldots m_\nu$, respectively; suppose that n generalized speeds have been introduced; let $\mathbf{v}_r^{P_i}$ and $\mathbf{a}^{P_i}$ denote, respectively, the rth partial velocity of P_i and the acceleration of P_i in a Newtonian reference frame N; define $\mathbf{R}_i^*$, called the inertia force for P_i, as

$$\mathbf{R}_i^* \triangleq -m_i\mathbf{a}^{P_i}. \tag{76}$$

The quantities $K_1^*, \ldots, K_n^*$, defined as

$$K_r^* \triangleq \sum_{i=1}^{\nu} \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i^* \qquad (r = 1, \ldots, n), \tag{77}$$

are called generalized inertia forces for S. The contribution to K_r^* made by the particles of a rigid body R belonging to S, denoted by $(K_r^*)_R$, can be shown to be given by (Kane, Likins, and Levinson 1983, p. 259)

$$(K_r^*)_R = \boldsymbol{\omega}_r \cdot \mathbf{T}^* + \mathbf{v}_r \cdot \mathbf{R}^* \qquad (r = 1, \ldots, n), \tag{78}$$

where $\boldsymbol{\omega}_r$ and $\mathbf{v}_r$ are, respectively, the rth partial angular velocity of R in N and the rth partial velocity of the mass center of R in N, while $\mathbf{T}^*$ and $\mathbf{R}^*$ depend on the angular velocity $\boldsymbol{\omega}$ of $\mathbf{R}$ in N, the angular acceleration $\boldsymbol{\alpha}$ of R in N, the central inertia dyadic $\mathbf{I}$ of R, the mass M of R, and the acceleration $\mathbf{a}^*$ of the mass center of R in N. $\mathbf{T}^*$ and $\mathbf{R}^*$ are given by (Kane, Likins, and Levinson 1982, p. 259)

$$\mathbf{T}^* = -\boldsymbol{\alpha} \cdot \mathbf{I} - \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega}, \; \mathbf{R}^* = -M\mathbf{a}^* \tag{79}$$

and are called, respectively, the inertia torque for B and the inertia force for B. Let us turn to an example.

We assume that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are parallel to central principal axes of inertia of A, so that $\mathbf{I}^A$, the central inertia dyadic of A, can be expressed as

$$\mathbf{I}^A = A_1\mathbf{a}_1\mathbf{a}_1 + A_2\mathbf{a}_2\mathbf{a}_2 + A_3\mathbf{a}_3\mathbf{a}_3, \tag{80}$$

where A_1, A_2, A_3 denote the central principal moments of inertia of A. In accordance with (Eq. 79), $\mathbf{T}_A^*$, the inertia torque of A, can be written

$$\mathbf{T}_A^* = -\boldsymbol{\alpha}^A \cdot \mathbf{I}^A - \boldsymbol{\omega}^A \times \mathbf{I}^A \cdot \boldsymbol{\omega}^A, \tag{81}$$

where $\boldsymbol{\alpha}^A$ and $\boldsymbol{\omega}^A$ are available in (Eqs. 48 and 15), respectively. Hence,

$$\mathbf{T}_A^* = -A_2(Z_4\dot{u}_1 + Z_5\dot{u}_3 + Z_{71})\mathbf{a}_2. \tag{82}$$

As for $\mathbf{R}_A^*$, the inertia force for A, (Eq. 79) indicates that this is given by

$$\mathbf{R}_A^* = -m_A\mathbf{a}^{A^*}, \tag{83}$$

where m_A is the mass of A. Referring to (Eq. 65), one sees that

$$\mathbf{R}_A^* = 0, \tag{84}$$

and now one is in position to use (Eq. 78) to determine $(K_r^*)_A$, the contribution of A to the generalized inertia force K_r^*, by substituting into

$$(K_r^*)_A = \boldsymbol{\omega}_r^A \cdot \mathbf{T}_A^* + \mathbf{v}_r^{A^*} \cdot \mathbf{R}_A^* \qquad (r = 1, \ldots, 6), \tag{85}$$

where $\boldsymbol{\omega}_r^A$ and $\mathbf{v}_r^{A^*}$ are available in (Eqs. 24, 25, and 45), while $\mathbf{T}_A^*$ and $\mathbf{R}_A^*$ are given by (Eqs. 82 and 83), respectively. With $r = 1$, one thus finds that

$$(K_1^*)_A = -Z_4A_2(Z_4\dot{u}_1 + Z_5\dot{u}_3 + Z_{71}), \tag{86}$$

while for $r = 2, \ldots, 6$, it turns out that $(K_r^*)_A$ vanishes.

After forming $(K_r^*)_B, \ldots, (K_r^*)_F$ by proceeding in like manner, we find K_r^* by substituting into

$$K_r^* = (K_r^*)_A + \ldots + (K_r^*)_F \qquad (r = 1, \ldots, 6). \tag{87}$$

Here, we assume that $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are parallel to central principal axes of inertia of B, and similarly for $C, \ldots, F$. Also, we introduce $Z_{137}, \ldots, Z_{187}$ as in Appendix 1, and define X_{rs} $(r, s = 1, \ldots, 6)$ as in Appendix 2. With the aid of (Eq. 87), we can thereupon express K_r^* as

$$K_r^* = \sum_{s=1}^{6} X_{rs}\dot{u}_s + Z_{181+r} \qquad (r = 1, \ldots, 6). \tag{88}$$

In addition to generalized inertia forces, Kane's dynamical equations involve generalized active forces. To explain what these are, we consider once again a system S of ν particles, let $\mathbf{R}_i$ be the resultant of all contact and body forces acting on a generic particle P_i of S, and define K_r as

$$K_r \triangleq \sum_{i=1}^{\nu} \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i \qquad (r = 1, \ldots, n). \tag{89}$$

K_r is called the rth *generalized active force* for S.

The task of constructing expressions for $K_1, \ldots, K_n$ frequently is facilitated by the following facts. Many forces that contribute to $\mathbf{R}_i$ make no contributions to K_r. For example, if R is a rigid body belonging to S, the total contribution to K_r of all gravitational forces exerted by particles of R on each other is equal to zero. Furthermore, if a set of contact and/or body forces acting on R is equivalent to a couple of torque $\mathbf{T}$ together with a force $\mathbf{R}$ applied at a point Q of R, then $(K_r)_R$, the contribution of this set of forces to K_r, is given by (Kane, Likins, and Levinson 1983, p. 248)

$$(K_r)_R = \boldsymbol{\omega}_r \cdot \mathbf{T} + \mathbf{v}_r^Q \cdot \mathbf{R}, \tag{90}$$

where $\boldsymbol{\omega}_r$ and $\mathbf{v}_r^Q$ are, respectively, the rth partial angular velocity of R in N and the rth partial velocity of Q in N.

In the case of the Stanford Arm, there are two kinds of forces that contribute to the generalized active forces $K_1, \ldots, K_6$, namely, contact forces applied in order to drive $A, \ldots, F$, and gravitational forces exerted on $A, \ldots, F$ by the Earth. Considering, first, the contact forces, we replace the set of such forces transmitted from N to A (through bearings and by means of a motor) with a couple of torque $\mathbf{T}^{N/A}$ together with a force $\mathbf{R}^{N/A}$ applied to A at A^*. Similarly, the set of contact forces transmitted from B to A is replaced with a couple of torque $\mathbf{T}^{B/A}$ together with a force $\mathbf{R}^{B/A}$ applied to A at B^* (which is a point fixed in A). The law of action and reaction then guarantees that the set of contact forces transmitted from A to B is equivalent to a couple of torque $-\mathbf{T}^{B/A}$ together with the force $-\mathbf{R}^{B/A}$ applied to B at B^*. Next, the set of contact forces exerted on B by C is replaced with a couple of torque $\mathbf{T}^{C/B}$ together with a force $\mathbf{R}^{C/B}$ applied to B at $\bar{B}$, the point of B instantaneously coinciding with C^*, and the set of forces exerted by B

on C is, therefore, equivalent to a couple of torque $-\mathbf{T}^{C/B}$ together with the force $-\mathbf{R}^{C/B}$ applied to C at C^*. Similarly, torques $\mathbf{T}^{D/C}$, $\mathbf{T}^{E/D}$, $\mathbf{T}^{F/E}$ and forces $\mathbf{R}^{D/C}$, $\mathbf{R}^{E/D}$, $\mathbf{R}^{F/E}$ come into play in connection with the interactions of C and D, and so forth. As for gravitational forces exerted on $A, \ldots, F$ by the Earth, these are denoted by $\mathbf{G}_A, \ldots, \mathbf{G}_F$, respectively, and can be expressed as

$$\mathbf{G}_A = -g\, m_A \mathbf{n}_2, \quad \mathbf{G}_B = -g\, m_B\, \mathbf{a}_2, \tag{91}$$

$$\mathbf{G}_C = -g\, m_C\, (c_2\mathbf{c}_2 - s_2\mathbf{c}_3), \qquad \mathbf{G}_D = -g\, m_D\, (c_2\mathbf{c}_2 - s_2\mathbf{c}_3), \tag{92}$$

and so forth. (The reason for replacing $\mathbf{n}_2$ with $\mathbf{a}_2$ in connection with $\mathbf{G}_B$, and with $c_2\mathbf{c}_2 - s_2\mathbf{c}_3$ in connection with $\mathbf{G}_C$ and $\mathbf{G}_D$, is that $\mathbf{G}_B$ is soon to be dot-multiplied with the partial velocities $\mathbf{v}_r^{B^*}$ $(r = 1, \ldots, 6)$, which have been expressed in terms of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in [Eqs. 46 and 47], whereas $\mathbf{G}_C$ and $\mathbf{G}_D$ will be dot-multiplied with $\mathbf{v}_r^{C^*}$ and $\mathbf{v}_r^{D^*}$ $(r = 1, \ldots, 6)$, respectively, and these vectors can be written most easily in terms of $\mathbf{c}_1$, $\mathbf{c}_2, \mathbf{c}_3$, the basis vectors in terms of which the velocities $\mathbf{v}^{C^*}$ and $\mathbf{v}^{D^*}$ have been recorded in [Eqs. 32 and 34], respectively.)

Referring to (Eq. 90), one can express $(K_r)_A$, the contribution to the generalized active force K_r of all forces acting on particles of body A, as

$$(K_r)_A = \boldsymbol{\omega}_r^{A} \cdot (\mathbf{T}^{N/A} + \mathbf{T}^{B/A}) + \mathbf{v}_r^{A^*} \cdot (\mathbf{R}^{N/A} + \mathbf{G}_A) + \mathbf{v}_r^{B^*} \cdot \mathbf{R}^{B/A} \qquad (r = 1, \ldots, 6). \tag{93}$$

Similarly,

$$(K_r)_B = \boldsymbol{\omega}_r^{B} \cdot (-\mathbf{T}^{B/A} + \mathbf{T}^{C/B}) + \mathbf{v}_r^{B^*} \cdot (-\mathbf{R}^{B/A} + \mathbf{G}_B) + v_r^{\bar{B}} \cdot \mathbf{R}^{C/B} \qquad (r = 1, \ldots, 6), \tag{94}$$

$$(K_r)_C = \boldsymbol{\omega}_r^{C} \cdot (-\mathbf{T}^{C/B} + \mathbf{T}^{D/C}) + \mathbf{v}_r^{C^*} \cdot (-\mathbf{R}^{C/B} + \mathbf{G}_C) + \mathbf{v}_r^{D^*} \cdot \mathbf{R}^{D/C} \qquad (r = 1, \ldots, 6), \tag{95}$$

and so forth for $D, \ldots, F$. When K_r is formed by summing these contributions, all terms involving $\mathbf{R}^{B/A}$, $\mathbf{R}^{D/C}$, $\mathbf{R}^{E/D}$, and $\mathbf{R}^{F/E}$ cancel, but $\mathbf{R}^{C/B}$, the gravitational forces $\mathbf{G}_A, \ldots, \mathbf{G}_F$, and all the torque vectors remain in evidence. The gravitational forces already have been expressed (see [Eqs. 91 and 92]) in forms convenient for carrying out the dot-multiplications indicated in (Eqs. 93–95). What should be done regarding $\mathbf{R}^{C/B}$ and $\mathbf{T}^{N/A}$, $\mathbf{T}^{B/A}$, $\mathbf{T}^{C/B}, \ldots, \mathbf{T}^{F/D}$ depends on the partial velocities and partial angular velocities that come into play. In connection with $\mathbf{R}^{C/B}$, these are $\mathbf{v}_r^{\bar{B}}$ in (Eq. 94) and $\mathbf{v}_r^{C^*}$ in (Eq. 95). Now $\mathbf{v}^{\bar{B}}$, the velocity of $\bar{B}$ in N, is given by (see Fig. 2, [Eq. 3)], and the definition of $\bar{B}$)

$$\mathbf{v}^{\bar{B}} = \mathbf{v}^{C^*} - u_6\mathbf{c}_2 \tag{96}$$

or, in view of (Eq. 32), by

$$\mathbf{v}^{\bar{B}} = (Z_{25}u_1 + Z_{26}u_3)\mathbf{c}_1 + (Z_{27}u_1 + Z_{28}u_3)\mathbf{c}_2 + (Z_{29}u_1 + Z_{30}u_3)\mathbf{c}_3, \tag{97}$$

so that the partial velocities of $\bar{B}$ differ from those of $\mathbf{v}^{C^*}$ only for $r = 6$; that is,

$$\mathbf{v}_r^{\bar{B}} = \mathbf{v}_r^{C^*} \qquad (r = 1, \ldots, 5) \tag{98}$$

and (see [Eqs. 97 and 32])

$$\mathbf{v}_6^{\bar{B}} = 0, \quad \mathbf{v}_6^{C^*} = \mathbf{c}_2. \tag{99}$$

Consequently, the terms in (Eqs. 94 and 95) that involve $\mathbf{R}^{C/B}$ contribute to $K_1, \ldots, K_6$ as follows:

$$(\mathbf{v}_r^{\bar{B}} - \mathbf{v}_r^{C^*}) \cdot \mathbf{R}^{C/B} = 0 \qquad (r = 1, \ldots, 5) \tag{100}$$

and

$$(\mathbf{v}_6^{\bar{B}} - \mathbf{v}_6^{C^*}) \cdot \mathbf{R}^{C/B} = -\mathbf{c}_2 \cdot \mathbf{R}^{C/B}. \tag{101}$$

Hence, the only thing of interest in connection with $\mathbf{R}^{C/B}$ is precisely the dot-product appearing in (Eq. 101), to which we, therefore, assign a symbol by letting

$$\sigma \triangleq \mathbf{c}_2 \cdot \mathbf{R}^{C/B}. \tag{102}$$

Examination of (Eq. 93) together with (Eqs. 24 and 25) reveals that the only nonvanishing contribution $\mathbf{T}^{N/A}$ can make to $K_1, \ldots, K_6$ arises via $\mathbf{a}_2 \cdot \mathbf{T}^{N/A}$. Accordingly, we introduce τ_1 as

$$\tau_1 \triangleq \mathbf{a}_2 \cdot \mathbf{T}^{N/A}. \tag{103}$$

As a final example, we consider $\mathbf{T}^{B/A}$. This torque occurs both in (Eqs. 93 and 94) and thus contributes to K_r the quantity $(\boldsymbol{\omega}_r^A - \boldsymbol{\omega}_r^B) \cdot \mathbf{T}^{B/A}$. But,

$$\boldsymbol{\omega}^A - \boldsymbol{\omega}^B = -\dot{q}_2 \mathbf{b}_1 \tag{104}$$

or, in view of (Eq. 9),

$$\boldsymbol{\omega}^A - \boldsymbol{\omega}^B = -(u_1 c_3 + u_3 s_3)\mathbf{b}_1 . \tag{105}$$

Consequently,

$$\boldsymbol{\omega}_1^A - \boldsymbol{\omega}_1^B = -c_3 \mathbf{b}_1,\ \boldsymbol{\omega}_2^A - \boldsymbol{\omega}_2^B = 0,\ \boldsymbol{\omega}_3^A - \boldsymbol{\omega}_r^B = -s_3 \mathbf{b}_1, \tag{106}$$

and

$$\boldsymbol{\omega}_r^A - \boldsymbol{\omega}_r^B = 0 \qquad (r = 4, 5, 6), \tag{107}$$

and all we need to know about $\mathbf{T}^{B/A}$ is its dot-product with $\mathbf{b}_1$. Hence, we introduce τ_2 as

$$\tau_2 \triangleq \mathbf{b}_1 \cdot \mathbf{T}^{B/A}. \tag{108}$$

Similar reasoning motivates the definitions

$$\tau_3 \triangleq \mathbf{c}_2 \cdot \mathbf{T}^{D/C},\ \tau_4 \triangleq \mathbf{d}_1 \cdot \mathbf{T}^{E/D},\ \tau_5 \triangleq \mathbf{e}_2 \cdot \mathbf{T}^{F/E}, \tag{109}$$

and expressions for $K_1, \ldots, K_6$ now can be formulated by substituting into

$$K_r = (K_r)_A + \ldots + (K_r)_F \qquad (r = 1, \ldots, 6) \tag{110}$$

and using $Z_{192}, \ldots, Z_{196}$ from Appendix 1, which yields

$$K_1 = \tau_1 Z_4 - \tau_2 c_3 + \tau_3 Z_6 + Z_{192}, \tag{111}$$

$$K_2 = -\tau_3 + Z_{193}, \tag{112}$$

$$K_3 = \tau_1 Z_5 - \tau_2 s_3 + \tau_3 Z_7 + Z_{194}, \tag{113}$$

$$K_4 = -\tau_4 + Z_{195}, \tag{114}$$

$$K_5 = -\tau_5, \tag{115}$$

and

$$K_6 = -\sigma + Z_{196}. \tag{116}$$

The denouement toward which we have been moving is now within easy reach. To arrive at the dynamical equations governing the Stanford Arm, all that remains to be done is to substitute from (Eqs. 88 and 111–116) into Kane's dynamical equations, namely,

$$K_r^* + K_r = 0 \qquad (r = 1, \ldots, 6). \tag{117}$$

Thus, one obtains

$$\sum_{s=1}^{6} X_{1s}\dot{u}_s = -\tau_1 Z_4 + \tau_2 c_3 - \tau_3 Z_6 - Z_{192} - Z_{182}, \tag{118}$$

$$\sum_{s=1}^{6} X_{2s}\dot{u}_s = \tau_3 - Z_{193} - Z_{183}, \tag{119}$$

$$\sum_{s=1}^{6} X_{3s}\dot{u}_s = -\tau_1 Z_5 + \tau_2 s_3 - \tau_3 Z_7 - Z_{194} - Z_{184}, \tag{120}$$

$$\sum_{s=1}^{6} X_{4s}\dot{u}_s = \tau_4 - Z_{195} - Z_{185}, \tag{121}$$

$$\sum_{s=1}^{6} X_{5s}\dot{u}_s = \tau_5 - Z_{186}, \tag{122}$$

and

$$\sum_{s=1}^{6} X_{6s}\dot{u}_s = \sigma - Z_{196} - Z_{187}. \tag{123}$$

These equations participate preeminently in the solution of two classes of problems we shall discuss separately.

Suppose that a desired motion of the end effector F has been specified for a time interval $0 \leqslant t \leqslant T$ in such a way that the dependence of $q_1, \ldots, q_6$ on t can be established explicitly. Then, to compute the values of $\tau_1, \ldots, \tau_5$ and σ for $0 \leqslant t \leqslant T$, one proceeds as follows. Differentiate with respect to t the expressions given for q_1, q_2, q_3 to obtain corresponding expressions given for $\dot{q}_1, \dot{q}_2, \dot{q}_3$; use these in conjunction with (Eqs. 5–7 and 3) both to evaluate $u_1, \ldots, u_6$ for each value of t of interest and to form, by differentiation with respect to t, expressions for $\dot{u}_1, \ldots, \dot{u}_6$; and evaluate $\dot{u}_1, \ldots, \dot{u}_6$, $Z_1, \ldots, Z_{196}$, and $X_{11}, \ldots, X_{66}$, and use these to find τ_3, τ_4, τ_5, and σ by reference to (Eqs. 119, 121, 122, and 123), respectively. Finally, with Y_1 and Y_2 dened as

$$Y_1 \triangleq \sum_{s=1}^{6} X_{1s}\dot{u}_s + \tau_3 Z_6 + Z_{192} + Z_{182} \qquad (124)$$

nd

$$Y_2 \triangleq \sum_{s=1}^{6} X_{3s}\dot{u}_s + \tau_3 Z_7 + Z_{194} + Z_{184}, \qquad (125)$$

valuate τ_1 and τ_2 as

$$\tau_1 = s_2(-Y_1 s_3 + Y_2 c_3),\ \tau_2 = Y_1 c_3 + Y_2 s_3, \qquad (126)$$

hich relationships follow directly from (Eqs. 118, 20, 124, 125) and Appendix 1.

The curves displayed in Fig. 1 are plots of $, \ldots, \tau_5$, and σ constructed by carrying out this rocedure with $q_1, \ldots, q_6$ arbitrarily specified as

$$= q_r(0) + \frac{q_r(T) - q_r(0)}{T}\left(t - \frac{T}{2\pi}\sin\frac{2\pi t}{T}\right) \qquad (r = 1, \ldots, 6) \quad (127)$$

nd with $T = 10$ s, $q_1(0) = 0$, $q_2(0) = 90°$, $(0) = \ldots = q_6(0) = 0$, $q_r(T) = 60°$ $= 1, \ldots, 5)$ and, $q_6(T) = 0.1$ m. The values of the rameters $L_1, \ldots, L_6, m_A, \ldots, m_F$, and $, \ldots, F_i\ (i = 1, 2, 3)$ used in this connection are corded in Appendix 3.

To perform rapid simulations of motions of the nford Arm, which is a task belonging to the second oblem area of interest, one must first specify $, \ldots, \tau_5$, and σ as explicit functions of $q_1, \ldots, q_6$, $, \ldots, u_6$, and/or t. Also, initial values must be igned to $q_1, \ldots, q_6$ and to $u_1, \ldots, u_6$, the latter which can be found by using (Eqs. 5–7 and 3), e initial values have been specified for $\dot{q}_1, \ldots, \dot{q}_6$. r the purpose of determining $q_1, \ldots, q_6$ as funcns of t for $t > 0$, one then performs a simultaneous, merical integration of (Eqs. 8–11 and 118–123). se calculations involve numerically solving (Eqs. –123) for $\dot{u}_1, \ldots, \dot{u}_6$ at each call of the integrator ng employed, a process that can be carried out ciently by means of a program specifically designed this purpose.

As an illustrative example, we consider the following task: an arm characterized by the parameters used in connection with Fig. 1 (see Appendix 3) is to be brought from the state of rest considered previously, that is, $q_r(0) = \dot{q}_r(0) = 0$ $(r = 1, \ldots, 6)$, to a final state of rest such that $q_1, \ldots, q_6$ have the values $q_1^*, \ldots, q_6^*$, respectively. This, it turns out, can be accomplished in 10 s, by using feedback control such that

$$\tau_1 = k_1(q_1^* - q_1) - k_2\dot{q}_1, \qquad (128)$$

$$\begin{aligned}\tau_2 = k_3(q_2 - q_2^*) + k_4\dot{q}_2 + g\ \{&[(m_C + m_D)q_6 + m_D L_5]s_2 \\ &+ (m_E L_6 + m_F L_3)(c_4 s_2 + c_3 s_4 c_2) \\ &+ (m_E + m_F)(q_6 + L_2)s_2\},\end{aligned} \qquad (129)$$

$$\tau_3 = k_5(q_3 - q_3^*) + k_6\dot{q}_3 - g(m_E L_6 + m_F L_3)s_2 s_3 s_4, \qquad (130)$$

$$\tau_4 = k_7(q_4 - q_4^*) + k_8\dot{q}_4 + g(m_E L_6 + m_F L_3)(c_2 s_4 + s_2 c_3 c_4) \qquad (131)$$

$$\tau_5 = k_9(q_5 - q_5^*) + k_{10}\dot{q}_5, \qquad (132)$$

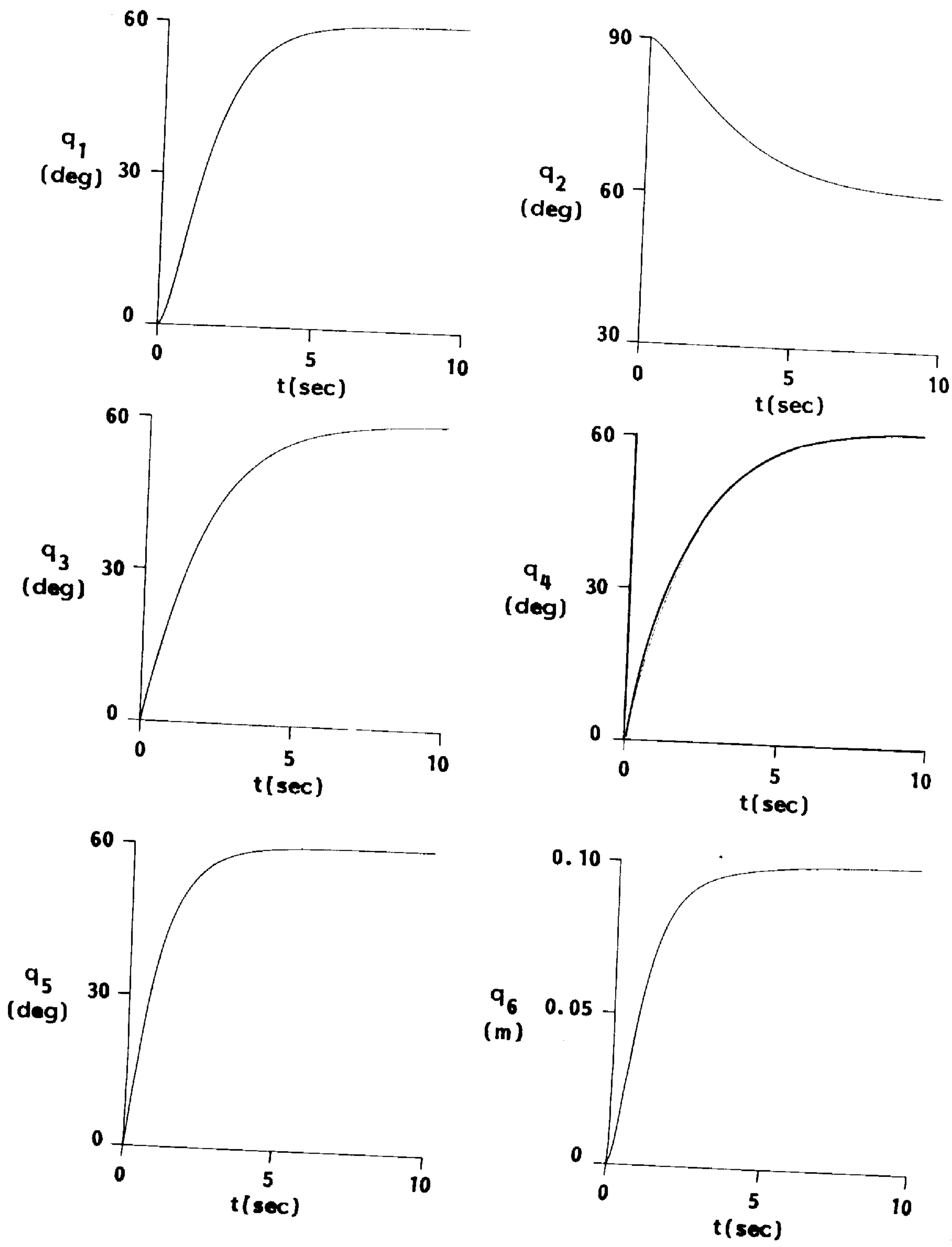
and

$$\sigma = k_{11}(q_6 - q_6^*) + k_{12}\dot{q}_6 - g(m_C + m_D + m_E + m_F)c_2, \qquad (133)$$

where $k_1, \ldots, k_{12}$ are constant "gains." (The terms in [Eqs. 129–131 and 133] involving g serve to counteract gravitational effects.) For, with $k_1 = 3.0$ N m, $k_2 = 5.0$ N m s, $k_3 = 1.0$ N m, $k_4 = 3.0$ N m s, $k_5 = 0.30$ N m, $k_6 = 0.60$ N m s, $k_7 = 0.30$ N m, $k_8 = 0.60$ N m s, $k_9 = 0.25$ N m, $k_{10} = 0.25$ N m s, $k_{11} = 30$ N m, $k_{12} = 41$ N s and $q_r^* = \pi/3$ rad $(r = 1, \ldots, 5)$ while $q_6^* = 0.1$ m, the procedure described above leads to the q_r $(r = 1, \ldots, 6)$ versus t plots shown in Fig. 3.

Aside from producing results of interest in their own right, a simulation program can furnish valuable tests of the validity of the underlying equations, and thus of torque and/or force calculations based on these equations. This is the case whenever one can construct an expression for one or more "constants of motion," that is, quantities known to remain constant throughout a motion of the system under consideration, such

Fig. 3. Generalized coordinates versus time.



as an angular momentum measure number or the total mechanical energy. In that event, since evaluation of such a quantity Q involves $q_1, \ldots, q_6$ and $u_1, \ldots, u_6$ as found by using the kinematical and dynamical equations in question, failure to obtain very nearly the same value each time Q is calculated during a simulation is a clear indication that something is amiss, whereas success of this endeavor is a strong indication, if not conclusive proof, that all is well.

When τ_1, vanishes throughout a given time interval during a motion of the Stanford Arm, then, throughout this time interval, $\mathbf{H} \cdot \mathbf{n}_2$ remains constant if $\mathbf{H}$ is the sum of the angular momenta in N of $A, \ldots, F$ with respect to point A^*. Now, $\mathbf{H}$ may be written

$$\mathbf{H} = \mathbf{I}^A \cdot \omega^A + \ldots + \mathbf{I}^F \cdot \omega^F + m_B \mathbf{p}^{B^*} \times \mathbf{v}^{B^*} + \ldots + m_F \mathbf{p}^{F^*} \times \mathbf{v}^{F^*}, \quad (134)$$

where $\mathbf{p}^{B^*}, \ldots, \mathbf{p}^{F^*}$ are the position vectors from A^* to $B^*, \ldots, F^*$, respectively; that is,

$$\mathbf{p}^{B^*} = L_1 \mathbf{a}_1 + L_4 \mathbf{a}_2, \; \mathbf{p}^{C^*} = \mathbf{p}^{B^*} + q_6 \mathbf{c}_2, \quad (135)$$

$$\mathbf{p}^{D^*} = \mathbf{p}^{C^*} + L_5 \mathbf{c}_2, \; \mathbf{p}^{E^*} = \mathbf{p}^{C^*} + L_2 \mathbf{d}_2 + L_5 \mathbf{e}_2, \quad (136)$$

and

$$\mathbf{p}^{F^*} = \mathbf{p}^{E^*} + L_2 \mathbf{d}_2 + L_3 \mathbf{e}_2. \quad (137)$$

Hence, once $W_1, \ldots, W_{31}$ have been defined as in Appendix 4, substitution from (Eqs. 135–137) into (Eq. 134) leads, with the aid of already available expressions for $\mathbf{I}^A$, ω^A, and so forth, to

$$\mathbf{H} \cdot \mathbf{n}_2 = W_{31}. \quad (138)$$

Therefore, W_{31} must remain constant throughout a simulation of a motion of the Stanford Arm so long as τ_1 vanishes. Performing a simulation differing from the one used to generate Fig. 3 only in that (Eq. 128) is replaced with $\tau_1 = 0$, we find, gratifyingly, that W_{31} takes on values differing by less than 10^{-9} from zero, the value it should have, since $\mathbf{H} \cdot \mathbf{n}_2$ is equal to zero initially.

Another check one can make both on the analyses underlying the two programs and on the programs themselves is to run the program that led to Fig. 1, but use, in place of (Eq. 127), the time histories plotted in Fig. 3. This must produce $\tau_1, \ldots, \tau_5$, and σ time-histories compatible with (Eqs. 128–133). We find that this is precisely what happens, which indicates that both our analysis and computer programs are likely to be correct.

3. Conclusion

Computational efficiency has long been one of the principal concerns of workers in the field of manipulator dynamics. Most recently (Silver 1982), this topic was addressed in an article showing that "a proper choice of the Lagrangian formulation" is "equivalent to the Newton-Euler formulation," an observation that is of interest in connection with this paper, for Silver's "proper choice" involves ideas akin to those underlying Kane's dynamical equations. Other articles (Hollerbach 1980; Luh, Walker, and Paul 1980; Thomas and Tesar 1982), containing extensive lists of references, provide the basis for comparing our approach with those taken heretofore. The conclusion that emerges from such comparisons is that the process proposed here may be expected to lead to computational algorithms involving fewer arithmetic operations than algorithms generated by employing the best available Lagrangian and Newton-Euler approaches. Specifically, according to Hollerbach (1980), when one resorts to the Lagrangian approach to determine quantities equivalent to our $\tau_1, \ldots, \tau_5$, and σ for an instant at which $q_1, \ldots, q_6$, $\dot{q}_1, \ldots, \dot{q}_6, \ddot{q}_1, \ldots, \ddot{q}_6$ are specified, one must perform 2195 multiplications and 1719 additions. These numbers are reduced to 1541 and 1196, respectively, if a Newton-Euler method reported by Walker and Orin (1982) is employed, and they become 852 and 738, respectively, when either the Newton-Euler technique discussed by Hollerbach or Silver's Lagrangian formulation is used.[1] With the algorithm set forth

1. Hollerbach mentions that a modified Lagrangian approach, involving the use of tables stored in the computer, requires fewer operations than either of these two methods. However, problems associated with storage space, interpolation, and so forth render this alternative impracticable.

in this paper, one needs but 646 multiplications and 394 additions to accomplish the same task. Moreover, these numbers can be significantly reduced further if one is willing to produce separate formulations to deal with torque and force calculations, on the one hand, and simulation of motions, on the other hand, rather than to work with a single set of equations, namely (Eqs. 118–123), in both cases. The reason for this is that all steps taken to bring $\dot{u}_1, \ldots, \dot{u}_6$ into evidence explicitly in (Eqs. 118–123) are unnecessary in connection with torque and force calculations.

A few more words can be said about $Z_1, \ldots, Z_{196}$ or, rather, about the role played by them in the formulation of dynamical equations. The use of these quantities is a central feature of the process advocated here for the formulation of dynamical equations, for it fulfills the important functions of saving analysts a great deal of writing and of obviating repetitious calculations. The fact that, generally speaking, the Z's have no special physical significance is irrelevant to the problem at hand.

Finally, the matter of closed loops deserves attention. The presence of closed loops in a robotic device can complicate the task of formulating equations of motion when one employs either the Lagrangian or Newton-Euler method. In the former case, one must perform unnecessary labor in introducing and subsequently eliminating Lagrange multipliers, quantities that may be of no interest in their own right; in the latter case, one must bring into evidence and then eliminate nonworking interbody constraint forces and torques. The use of Kane's equations in conjunction with *nonholonomic* partial angular velocities and nonholonomic partial velocities leads directly to explicit equations of motion for robots possessing closed loops of elements. Detailed exposition of this technique will be the subject of a future publication.

Appendixes

Appendix 1

$Z_1 \triangleq (u_1 s_3 - u_3 c_3)/s_2,\ Z_2 \triangleq u_1 c_3 + u_3 s_3,$
$\quad Z_3 \triangleq u_2 - Z_1 c_2$
$Z_4 \triangleq s_3/s_2,\ Z_5 \triangleq -c_3/s_2,\ Z_6 \triangleq Z_4 c_2,\ Z_7 \triangleq Z_5 c_2,$
$\quad Z_8 \triangleq -Z_4 s_2$
$Z_9 \triangleq -Z_5 s_2,\ Z_{10} \triangleq Z_6 u_1 + Z_7 u_3,\ Z_{11} \triangleq Z_8 u_1 + Z_9 u_3,$
$\quad Z_{12} \triangleq u_1 + u_4$
$Z_{13} \triangleq u_2 c_4 + u_3 s_4,\ Z_{14} \triangleq u_3 c_4 - u_2 s_4,\ Z_{15} \triangleq s_4 s_5,$
$\quad Z_{16} \triangleq -c_4 s_5$
$Z_{17} \triangleq -s_4 c_5,\ Z_{18} \triangleq c_4 c_5,$
$\quad Z_{19} \triangleq u_1 c_5 + Z_{15} u_2 + Z_{16} u_3 + u_4 c_5$
$Z_{20} \triangleq u_2 c_4 + u_3 s_4 + u_5,$
$\quad Z_{21} \triangleq u_1 s_5 + Z_{17} u_2 + Z_{18} u_3 + u_4 s_5$
$Z_{22} \triangleq -L_1 Z_4,\ Z_{23} \triangleq -L_1 Z_5,\ Z_{24} \triangleq Z_{22} u_1 + Z_{23} u_3,$
$\quad Z_{25} \triangleq -Z_8 q_6$
$Z_{26} \triangleq -Z_9 q_6,\ Z_{27} \triangleq Z_{22} s_2,\ Z_{28} \triangleq Z_{23} s_2,$
$\quad Z_{29} \triangleq Z_{22} c_2 + q_6 c_3$
$Z_{30} \triangleq Z_{23} c_2 + q_6 s_3,\ Z_{31} \triangleq Z_{25} u_1 + Z_{26} u_3,$
$\quad Z_{32} \triangleq Z_{27} u_1 + Z_{28} u_3 + u_6$
$Z_{33} \triangleq Z_{29} u_1 + Z_{30} u_3,\ Z_{34} \triangleq Z_{25} - L_5 Z_8,$
$\quad Z_{35} \triangleq Z_{26} - L_5 Z_9$
$Z_{36} \triangleq Z_{29} + L_5 c_3,\ Z_{37} \triangleq Z_{30} + L_5 s_3,$
$\quad Z_{38} \triangleq Z_{34} u_1 + Z_{35} u_3$
$Z_{39} \triangleq Z_{36} u_1 + Z_{37} u_3,\ Z_{40} \triangleq Z_{25} - L_2 Z_8,$
$\quad Z_{41} \triangleq Z_{26} - L_2 Z_9$
$Z_{42} \triangleq Z_{29} + L_2 c_3,\ Z_{43} \triangleq Z_{30} + L_2 s_3,$
$\quad Z_{44} \triangleq Z_{40} C_{11} + Z_{42} C_{31}$
$Z_{45} \triangleq Z_{41} C_{11} + Z_{43} C_{31},$
$\quad Z_{46} \triangleq Z_{40} C_{12} + Z_{27} C_{22} + Z_{42} C_{32}$
$Z_{47} \triangleq Z_{41} C_{12} + Z_{28} C_{22} + Z_{43} C_{32},$
$\quad Z_{48} \triangleq Z_{40} C_{13} + Z_{27} C_{23} + Z_{42} C_{33}$
$Z_{49} \triangleq Z_{41} C_{13} + Z_{28} C_{23} + Z_{43} C_{33},\ Z_{50} \triangleq L_6 s_4,$
$\quad Z_{51} \triangleq -L_6 c_4 + Z_{45}$
$Z_{52} \triangleq L_6 + Z_{48},\ Z_{53} \triangleq Z_{44} u_1,$
$\quad Z_{54} \triangleq Z_{53} + Z_{50} u_2 + Z_{51} u_3$
$Z_{55} \triangleq Z_{46} u_1 + Z_{47} u_3 + C_{22} u_6,\ Z_{56} \triangleq Z_{49} u_3 + C_{23} u_6$
$Z_{57} \triangleq Z_{56} + Z_{52} u_1 + L_6 u_4;\ Z_{58} \triangleq L_3 s_4,$
$\quad Z_{59} \triangleq -L_3 c_4 + Z_{45}$
$Z_{60} \triangleq L_3 + Z_{48},\ Z_{61} \triangleq Z_{53} + Z_{58} u_2 + Z_{59} u_3$
$Z_{62} \triangleq Z_{56} + Z_{60} u_1 + L_3 u_4,$
$\quad Z_{63} \triangleq (s_2 c_3 Z_3 - s_3 c_2 Z_2)/s_2^2,$
$Z_{64} \triangleq (s_2 s_3 Z_3 + c_3 c_2 Z_2)/s_2^2,\ Z_{65} \triangleq s_2 Z_2,\ Z_{66} \triangleq c_2 Z_2$
$Z_{67} \triangleq Z_{63} c_2 - Z_4 Z_{65},\ Z_{68} \triangleq Z_{64} c_2 - Z_5 Z_{65}$
$Z_{69} \triangleq -(Z_{63} s_2 + Z_4 Z_{66}),\ Z_{70} \triangleq -(Z_{64} s_2 + Z_5 Z_{66})$
$Z_{71} \triangleq Z_{63} u_1 + Z_{64} u_3,\ Z_{72} \triangleq (c_3 u_3 - s_3 u_1) Z_3$
$Z_{73} \triangleq Z_{67} u_1 + Z_{68} u_3,\ Z_{74} \triangleq Z_{69} u_1 + Z_{70} u_3$
$Z_{75} \triangleq (u_3 c_4 - u_2 s_4) u_4,\ Z_{76} \triangleq -(u_2 c_4 + u_3 s_4) u_4$
$Z_{77} \triangleq c_4 u_4 s_5 + s_4 c_5 u_5,\ Z_{78} \triangleq s_4 u_4 s_5 - c_4 c_5 u_5$
$Z_{79} \triangleq -c_4 u_4 c_5 + s_4 s_5 u_5,\ Z_{80} \triangleq -(s_4 u_4 c_5 + c_4 s_5 u_5)$
$Z_{81} \triangleq -Z_{12} s_5 u_5 + Z_{77} u_2 + Z_{78} u_3,$
$\quad Z_{82} \triangleq (u_3 c_4 - u_2 s_4) u_4$

$$Z_{83} \triangleq Z_{12}c_5u_5 + Z_{79}u_2 + Z_{80}u_3,\ Z_{84} \triangleq -Z_3s_3,\ Z_{85} \triangleq -Z_3c_3$$
$$Z_{86} \triangleq -s_4u_4,\ Z_{87} \triangleq -c_4u_4,\ Z_{88} \triangleq -Z_{87}s_3 - Z_{85}s_4$$
$$Z_{89} \triangleq Z_{86}s_3 - Z_{85}c_4,\ Z_{90} \triangleq -Z_{87}c_5 + Z_{84}s_4,\ Z_{91} \triangleq Z_{86}c_3 + Z_{84}c_4$$
$$Z_{92} \triangleq -L_1Z_{63},\ Z_{92} \triangleq -L_1Z_{64},\ Z_{94} \triangleq Z_1Z_{24},\ Z_{95} \triangleq Z_{92}u_1 + Z_{93}u_3$$
$$Z_{96} \triangleq -u_6Z_8 - q_6Z_{69},\ Z_{97} \triangleq -u_6Z_9 - q_6Z_{70},\ Z_{98} \triangleq c_2Z_2$$
$$Z_{99} \triangleq Z_{92}s_2 + Z_{22}Z_{98},\ Z_{100} \triangleq Z_{93}s_2 + Z_{23}Z_{98},\ Z_{101} \triangleq s_2Z_2$$
$$Z_{102} \triangleq q_6Z_3,\ Z_{103} \triangleq Z_{92}c_2 - Z_{22}Z_{101} + u_6c_3 - Z_{102}s_3$$
$$Z_{104} \triangleq Z_{93}c_2 - Z_{23}Z_{101} + u_6s_3 + Z_{102}c_3$$
$$Z_{105} \triangleq Z_{96}u_1 + Z_{97}u_3 + Z_{10}Z_{33} - Z_{11}Z_{32}$$
$$Z_{106} \triangleq Z_{99}u_1 + Z_{100}u_3 + Z_{11}Z_{31} - Z_2Z_{33}$$
$$Z_{107} \triangleq Z_{103}u_1 + Z_{104}u_3 + Z_2Z_{32} - Z_{10}Z_{31}$$
$$Z_{108} \triangleq Z_{96} - L_5Z_{69},\ Z_{109} \triangleq Z_{97} - L_5Z_{70}$$
$$Z_{110} \triangleq Z_{103} - L_5s_3Z_3,\ Z_{111} \triangleq Z_{104} + L_5c_3Z_3$$
$$Z_{112} \triangleq Z_{108}u_1 + Z_{109}u_3 + Z_{10}Z_{39} - Z_{11}Z_{32}$$
$$Z_{113} \triangleq Z_{99}u_1 + Z_{100}u_3 + Z_{11}Z_{38} - Z_2Z_{39}$$
$$Z_{114} \triangleq Z_{110}u_1 + Z_{111}u_3 + Z_2Z_{32} - Z_{10}Z_{38}$$
$$Z_{115} \triangleq Z_{96} - L_2Z_{69},\ Z_{116} \triangleq Z_{97} - L_2Z_{70}$$
$$Z_{117} \triangleq Z_{103} - L_2s_3Z_3,\ Z_{118} \triangleq Z_{104} + L_2c_3Z_3$$
$$Z_{119} \triangleq Z_{115}C_{11} + Z_{40}Z_{84} + Z_{117}C_{31} + Z_{42}Z_{85}$$
$$Z_{120} \triangleq Z_{116}C_{11} + Z_{41}Z_{84} + Z_{118}C_{31} + Z_{43}Z_{85}$$
$$Z_{121} \triangleq Z_{115}C_{12} + Z_{40}Z_{88} + Z_{99}C_{22} + Z_{27}Z_{86} + Z_{117}C_{32} + Z_{42}Z_{90}$$
$$Z_{122} \triangleq Z_{116}C_{12} + Z_{41}Z_{88} + Z_{100}C_{22} + Z_{28}Z_{86} + Z_{118}C_{32} + Z_{43}Z_{90}$$
$$Z_{123} \triangleq Z_{115}C_{13} + Z_{40}Z_{89} + Z_{99}C_{23} + Z_{27}Z_{87} + Z_{117}C_{33} + Z_{42}Z_{91}$$
$$Z_{124} \triangleq Z_{116}C_{13} + Z_{41}Z_{89} + Z_{100}C_{23} + Z_{28}Z_{87} + Z_{118}C_{33} + Z_{43}Z_{91}$$
$$Z_{125} \triangleq L_6u_4,\ Z_{126} \triangleq Z_{125}c_4,\ Z_{127} \triangleq Z_{125}s_4 + Z_{120}$$
$$Z_{128} \triangleq Z_{119}u_1 + Z_{126}u_2 + Z_{127}u_3 + Z_{13}Z_{57} - Z_{14}Z_{55}$$
$$Z_{129} \triangleq Z_{121}u_1 + Z_{122}u_3 + Z_{86}u_6 + Z_{14}Z_{54} - Z_{12}Z_{57}$$
$$Z_{130} \triangleq Z_{123}u_1 + Z_{124}u_3 + Z_{87}u_6 + Z_{12}Z_{55} - Z_{13}Z_{54}$$
$$Z_{131} \triangleq L_3u_4,\ Z_{132} \triangleq Z_{131}c_4,\ Z_{133} \triangleq Z_{131}s_4 + Z_{120}$$
$$Z_{134} \triangleq Z_{119}u_1 + Z_{132}u_2 + Z_{133}u_3 + Z_{13}Z_{62} - Z_{14}Z_{55}$$
$$Z_{135} \triangleq Z_{121}u_1 + Z_{122}u_3 + Z_{86}u_6 + Z_{14}Z_{61} - Z_{12}Z_{62}$$
$$Z_{136} \triangleq Z_{123}u_1 + Z_{124}u_3 + Z_{87}u_6 + Z_{12}Z_{55} - Z_{13}Z_{61}$$
$$Z_{137} \triangleq -A_2Z_4,\ Z_{138} \triangleq -A_2Z_5,\ Z_{139} \triangleq A_2Z_{71}$$
$$Z_{140} \triangleq Z_{10}Z_{11},\ Z_{141} \triangleq Z_{11}Z_2,\ Z_{142} \triangleq Z_2Z_{10}$$
$$Z_{143} \triangleq Z_{140}(B_2 - B_3) - Z_{72}B_1,\ Z_{144} \triangleq -c_3B_1$$
$$Z_{145} \triangleq -s_3B_1,\ Z_{146} \triangleq Z_{141}(B_3 - B_1) - Z_{73}B_2$$
$$Z_{147} \triangleq -Z_6B_2,\ Z_{148} \triangleq -Z_7B_2,\ Z_{149} \triangleq Z_{142}(B_1 - B_2) - Z_{74}B_3$$
$$Z_{150} \triangleq -Z_8B_3,\ Z_{151} \triangleq -Z_9B_3,\ Z_{152} \triangleq Z_{140}(C_2 - C_3) - Z_{72}C_1$$
$$Z_{153} \triangleq -c_3C_1,\ Z_{154} \triangleq -s_3C_1,\ Z_{155} \triangleq Z_{141}(C_3 - C_1) - Z_{73}C_2$$
$$Z_{156} \triangleq -Z_6C_2,\ Z_{157} \triangleq -Z_7C_2,\ Z_{158} \triangleq Z_{142}(C_1 - C_2) - Z_{74}C_3$$
$$Z_{159} \triangleq -Z_8C_3,\ Z_{160} \triangleq -Z_9C_3,\ Z_{161} \triangleq u_2u_3(D_2 - D_3)$$
$$Z_{162} \triangleq u_3u_1(D_3 - D_1),\ Z_{163} \triangleq u_1u_2(D_1 - D_2)$$
$$Z_{164} \triangleq Z_{13}Z_{14}(E_2 - E_3),\ Z_{165} \triangleq Z_{14}Z_{12}(E_3 - E_1) - Z_{75}E_2$$
$$Z_{166} \triangleq -c_4E_2,\ Z_{167} \triangleq -s_4E_2,\ Z_{168} \triangleq Z_{12}Z_{13}(E_1 - E_2) - Z_{76}E_3$$
$$Z_{169} \triangleq s_4E_3,\ Z_{170} \triangleq -c_4E_3,\ Z_{171} \triangleq Z_{20}Z_{21}(F_2 - F_3) - Z_{81}F_1$$
$$Z_{172} \triangleq -c_5F_1,\ Z_{173} \triangleq -Z_{15}F_1,\ Z_{174} \triangleq -Z_{16}F_1$$
$$Z_{175} \triangleq Z_{21}Z_{19}(F_3 - F_1) - Z_{82}F_2,\ Z_{176} \triangleq -c_4F_2,\ Z_{177} \triangleq -s_4F_2$$
$$Z_{178} \triangleq Z_{19}Z_{20}(F_1 - F_2) - Z_{83}F_3,\ Z_{179} \triangleq -s_5F_3,\ Z_{180} \triangleq -Z_{17}F_3$$
$$Z_{181} \triangleq -Z_{18}F_3$$
$$\begin{aligned} Z_{182} \triangleq &-m_BZ_{22}Z_{95} \\ &- m_C(Z_{25}Z_{105} + Z_{27}Z_{106} + Z_{29}Z_{107}) \\ &- m_D(Z_{34}Z_{112} + Z_{27}Z_{113} + Z_{36}Z_{114}) \\ &- m_E(Z_{44}Z_{128} + Z_{46}Z_{129} + Z_{52}Z_{130}) \\ &- m_F(Z_{44}Z_{134} + Z_{46}Z_{135} + Z_{60}Z_{136}) + Z_4Z_{139} \\ &+ c_3(Z_{143} + Z_{152}) + Z_6(Z_{146} + Z_{155}) \\ &+ Z_8(Z_{149} + Z_{158}) + Z_{161} + Z_{164} + c_5Z_{171} \\ &+ s_5Z_{178} \end{aligned}$$
$$\begin{aligned} Z_{183} \triangleq &-m_EZ_{50}Z_{128} - m_FZ_{58}Z_{134} + Z_{162} \\ &+ c_4(Z_{165} + Z_{175}) - s_4Z_{168} + Z_{15}Z_{171} + Z_{17}Z_{178} \end{aligned}$$
$$\begin{aligned} Z_{184} \triangleq &-m_BZ_{23}Z_{95} \\ &- m_C(Z_{26}Z_{105} + Z_{28}Z_{106} + Z_{30}Z_{107}) \\ &- m_D(Z_{35}Z_{112} + Z_{28}Z_{113} + Z_{37}Z_{114}) \\ &- m_E(Z_{51}Z_{128} + Z_{47}Z_{129} + Z_{49}Z_{130}) \\ &- m_F(Z_{59}Z_{134} + Z_{47}Z_{135} + Z_{49}Z_{136}) + Z_5Z_{139} \\ &+ s_3(Z_{143} + Z_{152}) + Z_7(Z_{146} + Z_{155}) \\ &+ Z_9(Z_{149} + Z_{158}) + Z_{163} + c_4Z_{168} + Z_{16}Z_{171} \\ &+ Z_{18}Z_{178} + s_4(Z_{175} + Z_{165}) \end{aligned}$$
$$Z_{185} \triangleq -m_EL_6Z_{130} - m_FL_3Z_{136} + Z_{164} + c_5Z_{171} + s_5Z_{178},$$
$$Z_{186} \triangleq Z_{175}$$
$$\begin{aligned} Z_{187} \triangleq &-m_CZ_{106} - m_DZ_{113} - m_E(C_{22}Z_{129} + C_{23}Z_{130}) \\ &- m_F(C_{22}Z_{135} + C_{23}Z_{136}) \end{aligned}$$
$$Z_{188} \triangleq s_2s_3,\ Z_{189} = s_2c_3,\ Z_{190} \triangleq c_2c_4 - s_4Z_{189}$$
$$\begin{aligned} Z_{191} \triangleq &-(c_2s_4 + Z_{189}c_4),\ Z_{192} \triangleq -g[(m_C + m_D)c_2Z_{27} \\ &- s_2(m_CZ_{29} + m_DZ_{36}) \\ &+ (m_E + m_F)(Z_{44}Z_{188} + Z_{46}Z_{190}) \\ &+ Z_{191}(m_EZ_{52} + m_FZ_{60})] \end{aligned}$$

$$Z_{193} \triangleq -g\, Z_{118}(m_E Z_{50} + m_F Z_{58})$$
$$Z_{194} \triangleq -g[(m_C + m_D)Z_{28}c_2 - s_2(m_C Z_{30} + m_D Z_{37}) + (m_E + m_F)(Z_{47}Z_{190} + Z_{49}Z_{191}) + Z_{188}(m_E Z_{51} + m_F Z_{59})]$$
$$Z_{195} \triangleq -g\, Z_{191}(m_E L_6 + m_F L_3)$$
$$Z_{196} \triangleq -g[(m_C + m_D)c_2 + (m_E + m_F)(C_{22}Z_{190} + C_{23}Z_{191})]$$

Appendix 2

$$X_{11} \triangleq -m_B Z_{22}{}^2 - m_C(Z_{25}{}^2 + Z_{27}{}^2 + Z_{29}{}^2) - m_D(Z_{34}{}^2 + Z_{27}{}^2 + Z_{36}{}^2) - m_E(Z_{44}{}^2 + Z_{46}{}^2 + Z_{52}{}^2) - m_F(Z_{44}{}^2 + Z_{46}{}^2 + Z_{60}{}^2) + Z_4 Z_{137} + c_3(Z_{144} + Z_{153}) + Z_6(Z_{147} + Z_{156}) + Z_8(Z_{159} + Z_{150}) - D_1 - E_1 + c_5 Z_{172} + s_5 Z_{179}$$
$$X_{12} \triangleq X_{21} \triangleq -Z_{44}(m_E Z_{50} + m_F Z_{58}) + c_5 Z_{173} + s_5 Z_{180}$$
$$X_{13} \triangleq X_{31} \triangleq -m_B Z_{22} Z_{23} - m_C(Z_{25}Z_{26} + Z_{27}Z_{28} + Z_{29}Z_{30}) - m_D(Z_{34}Z_{35} + Z_{27}Z_{28} + Z_{36}Z_{37}) - m_E(Z_{44}Z_{51} + Z_{46}Z_{47} + Z_{52}Z_{49}) - m_F(Z_{44}Z_{59} + Z_{46}Z_{47} + Z_{60}Z_{49}) + Z_4 Z_{138} + c_3(Z_{145} + Z_{154}) + Z_6(Z_{148} + Z_{157}) + Z_8(Z_{151} + Z_{160}) + c_5 Z_{174} + s_5 Z_{181}$$
$$X_{14} \triangleq X_{41} \triangleq -m_E Z_{52} L_6 - m_F Z_{60} L_3 - E_1 + c_5 Z_{172} + s_5 Z_{179},\ X_{15} = X_{51} = 0$$
$$X_{16} \triangleq X_{61} \triangleq -(m_C + m_D)Z_{27} - (m_E + m_F)Z_{46}C_{22} - (m_E Z_{52} + m_F Z_{60})C_{23}$$
$$X_{22} \triangleq -m_E Z_{50}{}^2 - m_F Z_{58}{}^2 - D_2 + c_4(Z_{166} + Z_{176}) - s_4 Z_{169} + Z_{15}Z_{173} + Z_{17}Z_{180}$$
$$X_{23} \triangleq X_{32} \triangleq -m_E Z_{50}Z_{51} - m_F Z_{58}Z_{59} + c_4(Z_{167} + Z_{177}) - s_4 Z_{170} + Z_{15}Z_{174} + Z_{17}Z_{181}$$
$$X_{24} \triangleq X_{42} \triangleq Z_{15}Z_{172} + Z_{17}Z_{179},\ X_{25} \triangleq X_{52} \triangleq -c_4 F_2,\ X_{26} = X_{62} = 0$$
$$X_{33} \triangleq -m_B Z_{23}{}^2 - m_C(Z_{26}{}^2 + Z_{28}{}^2 + Z_{30}{}^2) - m_D(Z_{35}{}^2 + Z_{28}{}^2 + Z_{37}{}^2) - m_E(Z_{51}{}^2 + Z_{47}{}^2 + Z_{49}{}^2) - m_F(Z_{59}{}^2 + Z_{47}{}^2 + Z_{49}{}^2) + Z_5 Z_{138} + s_3(Z_{145} + Z_{154}) + Z_7(Z_{148} + Z_{157}) + Z_9(Z_{151} + Z_{160}) - D_3 + s_4(Z_{167} + Z_{177}) + c_4 Z_{170} + Z_{16}Z_{174} + Z_{18}Z_{181}$$
$$X_{34} \triangleq X_{43} \triangleq -(m_E L_6 + m_F L_3)Z_{49} + Z_{16}Z_{172} + Z_{18}Z_{179},\ X_{35} \triangleq X_{53} \triangleq -s_4 F_2$$
$$X_{36} \triangleq X_{63} \triangleq -(m_C + m_D)Z_{28} - (m_E + m_F)(Z_{47}C_{22} + Z_{49}C_{23})$$
$$X_{44} \triangleq -m_E L_6{}^2 - m_F L_3{}^2 - E_1 + c_5 Z_{172} + s_5 Z_{179},\ X_{45} \triangleq X_{54} \triangleq 0$$
$$X_{46} \triangleq X_{64} \triangleq -(m_E L_6 + m_F L_3)C_{23},\ X_{55} \triangleq -F_2,\ X_{56} \triangleq X_{65} \triangleq 0$$
$$X_{66} \triangleq -(m_C + m_D + m_E + m_F)$$

Appendix 3

Quantity	Value	Units
L_1	0.1	m
L_2	0.6	m
L_3	0.2	m
L_4	0.1	m
L_5	0.7	m
L_6	0.06	m
m_A	9	kg
m_B	6	kg
m_C	4	kg
m_D	1	kg
m_E	0.6	kg
m_F	0.5	kg
A_1	0.01	kg m^2
A_2	0.02	kg m^2
A_3	0.01	kg m^2
B_1	0.06	kg m^2
B_2	0.01	kg m^2
B_3	0.05	kg m^2
C_1	0.4	kg m^2
C_2	0.01	kg m^2
C_3	0.4	kg m^2
D_1	0.0005	kg m^2
D_2	0.001	kg m^2
D_3	0.001	kg m^2
E_1	0.0005	kg m^2
E_2	0.0002	kg m^2
E_3	0.0005	kg m^2
F_1	0.001	kg m^2
F_2	0.002	kg m^2
F_3	0.003	kg m^2

Appendix 4

$$W_1 \triangleq L_4 c_2 + q_6,\ W_2 \triangleq -L_4 s_2,\ W_3 \triangleq W_1 + L_5,\ W_4 \triangleq W_1 + L_2$$

$$\triangleq L_1c_3 - W_2s_3,\ W_6 \triangleq L_1s_3 + W_2c_3,$$
$$W_7 \triangleq W_4c_4 + W_6s_4$$
$$\triangleq -W_4s_4 + W_6c_4,\ W_9 \triangleq W_7 + L_6,$$
$$W_{10} \triangleq W_7 + L_3,\ W_{11} \triangleq L_4Z_{24}$$
$$\triangleq -L_1Z_{24},\ W_{13} \triangleq W_1Z_{33} - W_2Z_{32},$$
$$W_{14} \triangleq W_2Z_{31} - L_1Z_{33}$$
$$\triangleq L_1Z_{32} - W_1Z_{31},\ W_{16} \triangleq W_3Z_{39} - W_2Z_{32},$$
$$W_{17} \triangleq W_2Z_{38} - L_1Z_{39}$$
$$\triangleq L_1Z_{32} - W_3W_{38},\ W_{19} \triangleq W_9Z_{57} - W_8Z_{55},$$
$$W_{20} \triangleq W_8Z_{54} - W_5Z_{57}$$
$$\triangleq W_5Z_{55} - W_9Z_{54},\ W_{22} \triangleq W_{10}Z_{62} - W_8Z_{55},$$
$$W_{23} \triangleq W_8Z_{61} - W_5Z_{62}$$
$$\triangleq W_5Z_{55} - W_{10}Z_{61},\ W_{25} \triangleq s_2s_3,\ W_{26} \triangleq -s_2c_3,$$
$$W_{27} \triangleq c_2c_4 + W_{26}s_4$$
$$\triangleq -c_2s_4 + W_{26}c_4,\ W_{29} \triangleq W_{25}c_5 - W_{28}s_5,$$
$$W_{30} \triangleq W_{25}s_5 + W_{28}c_5$$
$$\begin{aligned}\triangleq\ & A_2Z_1 + c_2B_2Z_{10} - s_2B_3Z_{11} + c_2C_2Z_{10} - s_2C_3Z_{11} \\ & + W_{25}D_1u_1 + c_2D_2u_2 + W_{26}D_3u_3 + W_{25}E_1Z_{12} \\ & + W_{27}E_2Z_{13} + W_{28}E_3Z_{14} + W_{29}F_1Z_{19} \\ & + W_{27}F_2Z_{20} + W_{30}F_3Z_{21} + m_BW_{12} \\ & + m_C(c_2W_{14} - s_2W_{15}) + m_D(c_2W_{17} - s_2W_{18}) \\ & + m_E(W_{25}W_{19} + W_{27}W_{20} + W_{28}W_{21}) \\ & + m_F(W_{25}W_{22} + W_{27}W_{23} + W_{28}W_{24})\end{aligned}$$

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