Equations Of Motion of Krang on Fixed Wheels

Munzir Zafar

December 10, 2014

In this report we attempt to find the dynamic model of Golem Krang with its wheels fixed. So it is reduced to a serial robot with a tree-structure (due to two arms branching out). Figure 1 shows the frames of references we will be using to determine the transforms and the coordinates on the robot. We denote these frames using symbol R_i where $i \in \mathbb{F} = \{0, 1, 2, 3, 4l, 5l, 6l, 7l, 8l, 9l, 10l, 4r, 5r, 6r, 7r, 8r, 9r, 10r\}$. R_0 is the world frame fixed in the middle of the two wheels. R_1, R_2, R_3 are fixed on the base, spine and torso with their rotations represented by q_{imu}, q_w and q_{torso} respectively. Frames $R_{4l}, ...R_{10l}$ are frames fixed on the links left 7-DOF arm with their motion represented by $q_{1l}, ...q_{7l}$. Similarly, frames $R_{4r}, ...R_{10r}$ are frames fixed on the links right 7-DOF arm with their motion represented by $q_{1r}, ...q_{7r}$. All equations in the following text that do not show r or l in the subscript where they are supposed to, will mean that the respective equations are valid for both subscripts.

We will be using the Lagrange formulation with a systematic approach presented in [1] to derive the equations of motion.

1 Introduction to Lagrange Formulation

The Lagrange formulation describes the behavior of a dynamic system in terms of work and energy stored in the system. The Lagrange equations are commonly written in the form:

$$\Gamma_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \quad for i \in \mathbb{F} \tag{1}$$

where L is the Lagrangian of the robot defined as the difference berween the kinetic energy E and potential energy U of the system:

$$L = E - U$$

1.1 General Form of the Dynamic Equations

The kinetic energy of the system is a quadratic function in the joint velocities such that:

$$E = \frac{1}{2}\dot{\mathbf{q}}^{\mathbf{T}}\mathbf{A}\dot{\mathbf{q}} \tag{2}$$

where $\mathbf{A}$ is the $n \times n$ symmetric and positive definite *inertia matrix* of the robot. Its elements are functions of the joint positions. The (i, j) element of $\mathbf{A}$

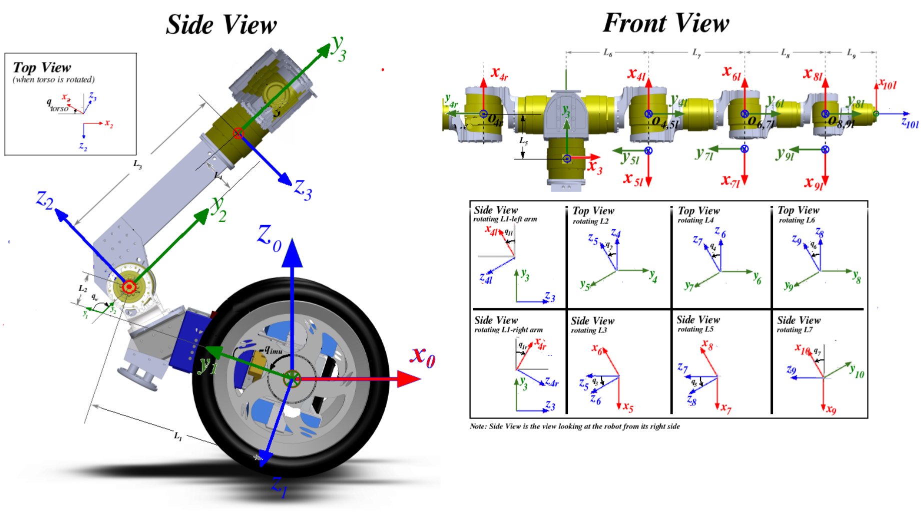

Figure 1: Frames of references on the robot

is denoted A_{ij}. Since the potential energy is a function of the joint positions, equation 1 leads to:

$$\mathbf{\Gamma} = \mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{Q}(\mathbf{q}) \tag{3}$$

where:

- $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ is the $n \times 1$ vector of Coriolis and centrifugal torques, such that: $\mathbf{C}\dot{\mathbf{q}} = \dot{\mathbf{A}}\dot{\mathbf{q}} - \frac{\partial E}{\partial \mathbf{q}}$
- $\mathbf{Q} = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_{4l} & \dots & Q_{10l} & Q_{4r} & \dots & Q_{10r} \end{bmatrix}^T$ is the vector of gravity torques

So the dynamic model of a tree-structured robot is described by n coupled and nonlinear second order differential equations. The elements of $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{Q}$ are functions of geometric and inertial parameters of the robot.

1.2 Computation of the elements of A, C and Q

To compute the elements of $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{Q}$, we begin by symbollically computing the expressions of the kinetic and potential energies of all the links of the robot. Then we proceed as follows:

- the elements A_{ij} is equal to the coefficient of $\left(\frac{\dot{q}_i^2}{2}\right)$ in the expression of the kinetic energy, while A_{ij}, for $i \neq j$, is equal to the coefficient of $\dot{q}_i \dot{q}_j$
- for calculating the elements of $\mathbf{C}$, there exist several forms of the vector $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$. Using the *Christoffell symbols* $c_{i,jk}$, the (i, j) elements of the matrix $\mathbf{C}$ can be written as:

$$\begin{cases} C_{ij} = \sum\limits_{k=1}^{n} c_{i,jk}\dot{q}_k \\ c_{i,jk} = \frac{1}{2}\left[\frac{\partial A_{ij}}{\partial q_k} + \frac{\partial A_{ik}}{\partial q_j} - \frac{\partial A_{jk}}{\partial q_i}\right] \end{cases} \tag{4}$$

- The Q_i element of the vector $\mathbf{Q}$ is calculated according to:

$$Q_i = \frac{\partial U}{\partial q_i} \tag{5}$$

2 Finding A, C and Q for our robot

In this section we determine the symbolic expression for the total kinetic energy E of the robot.

2.1 Transformations

The transformation of frame R_i into frame R_j is represented by the homogeneous transformation matrix ${}^iT_j$ such that.

$${}^iT_j = \begin{bmatrix} {}^is_j & {}^in_j & {}^ia_j & {}^iP_j \end{bmatrix} = \begin{bmatrix} & {}^iA_j & & {}^iP_j \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & n_x & a_x & P_x \\ s_y & n_y & a_y & P_y \\ s_z & n_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

where ${}^is_j$, ${}^in_j$ and ${}^ia_j$ contain the components of the unit vectors along the x_j, y_j and z_j axes respectively expressed in frame R_i, and where ${}^iP_j$ is the vector representing the coordinates of the origin of frame R_j expressed in frame R_i.

The transformation matrix ${}^iT_j$ can be interpreted as: (a) the transformation from frame R_i to frame R_j and (b) the representation of frame R_j with respect to frame R_i. Using figure 1, we can write down these transformation matrices for our system as follows:

$${}^0T_1 = \begin{bmatrix} 0 & sq_{imu} & -cq_{imu} & 0 \\ -1 & 0 & 0 & 0 \\ 0 & cq_{imu} & sq_{imu} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^1T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cq_w & sq_w & L_1 \\ 0 & -sq_w & cq_w & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^2T_3 = \begin{bmatrix} -cq_{torso} & 0 & sq_{torso} & 0 \\ 0 & 1 & 0 & L_3 \\ -sq_{torso} & 0 & -cq_{torso} & L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$${}^3T_{4l} = \begin{bmatrix} 0 & 1 & 0 & L_6 \\ cq_{1l} & 0 & -sq_{1l} & L_5 \\ -sq_{1l} & 0 & -cq_{1l} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^3T_{4r} = \begin{bmatrix} 0 & -1 & 0 & -L_6 \\ cq_{1r} & 0 & -sq_{1r} & L_5 \\ sq_{1r} & 0 & cq_{1r} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^4T_5 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -cq_2 & -sq_2 & 0 \\ 0 & -sq_2 & cq_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$${}^5T_6 = \begin{bmatrix} -cq_3 & 0 & sq_3 & 0 \\ 0 & -1 & 0 & -L_7 \\ sq_3 & 0 & cq_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^6T_7 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -cq_4 & -sq_4 & 0 \\ 0 & -sq_4 & cq_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^7T_8 = \begin{bmatrix} -cq_5 & 0 & sq_5 & 0 \\ 0 & -1 & 0 & -L_8 \\ sq_5 & 0 & cq_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$${}^8T_9 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -cq_6 & -sq_6 & 0 \\ 0 & -sq_6 & cq_6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, {}^9T_{10} = \begin{bmatrix} -cq_7 & -sq_7 & 0 & 0 \\ 0 & 0 & -1 & -L_9 \\ sq_7 & -cq_7 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.2 Angular and Linear Velocities of Frames

The angular and linear velocities of the frames can be calculated using the recursive formulation:

$$ {}^j\omega_j = {}^jA_i{}^i\omega_i + {}^je_j\dot{q}_j \tag{7} $$

$$ {}^jV_j = {}^jA_i\left({}^iV_i + {}^i\omega_i \times {}^iP_j\right) \tag{8} $$

where ${}^i\omega_j$ and ${}^iV_j$ denote the angular and linear velocities repectively of frame j measured with respect to the world frame and represented in frame i. ${}^je_j$ denotes the direction of local angular velocity of frame j represented in frame j. $i, j \in \mathbb{F}$ identify the frames and i identifies the antecedent frame of j. So, the rotation ${}^jA_i$ and the translation ${}^jP_i$ that appear in these equations can not be directly deduced from the transformations listed in the previous section, as the they all represent ${}^iT_j$ (note the position of i and j). Rather, we need to use following expressions to deduce our matrices:

$$ {}^jA_i = {}^iA_j^T $$

$$ {}^jP_i = -{}^iA_j^T\,{}^iP_j $$

Since frame R_0 is fixed ${}^0\omega_0$ and ${}^0V_0$ are both $\begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$. We can deduce directions of local angular velocities of the frames using figure 1 as follows.

$$ {}^1e_1 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^2e_2 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^3e_3 = \begin{bmatrix}0 & -1 & 0\end{bmatrix}^T, {}^4e_4 = \begin{bmatrix}0 & -1 & 0\end{bmatrix}^T, $$

$$ {}^5e_5 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^6e_6 = \begin{bmatrix}0 & -1 & 0\end{bmatrix}^T, {}^7e_7 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^8e_8 = \begin{bmatrix}0 & -1 & 0\end{bmatrix}^T, $$

$$ {}^9e_9 = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T, {}^{10}e_{10} = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^T $$

This information can now be used to derive expressions for the angular and linear velocities of the frames.

2.3 Kinetic Energy

The kinetic energy of the robot is given as:

$$ E = \sum_{j\in\mathbb{F}} E_j \tag{9} $$

where E_j denotes the kinetic energy of link j, which can be computed by

$$ E_j = \frac{1}{2}(\omega_j^T I_{Gj}\omega_j + M_jV_{Gj}^TV_{Gj}) \tag{10} $$

where the velocity of the center of mass can be expressed as:

$$ V_{Gj} = V_j + \omega_j \times S_j $$

and since:

$$ J_j = I_{Gj} - M_j\hat{S}_j\hat{S}_j $$

equation 10 becomes:

$$ E_j = \frac{1}{2}(\omega_j^T J_{Gj}\omega_j + M_jV_j^TV_j + 2\mathbf{MS}_j^T(V_j \times \omega_j)) \tag{11} $$

See section A in the appendix to know the details of the derivation.

2.4 Potential Energy

The total potential energy U of the robot is given by:

$$U = \sum_{j \in \mathbb{F}} U_j = \sum_{j \in \mathbb{F}} -M_j \mathbf{g}^T (L_{0,j} + S_j) \tag{12}$$

where $L_{0,j}$ is the position vector from the origin O_0 to O_j and $\mathbf{g}$ is the gravitatonal acceleration. Projecting the vectors appearing in 12 into frame R_0, we obtain:

$$U_j = -M_j \, {}^0\mathbf{g}^T ({}^0P_j + {}^0A_j \, {}^jS_j) \tag{13}$$

$$= -{}^0\mathbf{g}^T (M_j \, {}^0P_j + {}^0A_j \, {}^j\mathbf{MS}_j) \tag{14}$$

$$= -\begin{bmatrix} {}^0\mathbf{g}^T & 0 \end{bmatrix} {}^0T_j \begin{bmatrix} {}^j\mathbf{MS}_j \\ M_j \end{bmatrix} \tag{15}$$

Given the frames defined in figure 1, ${}^0\mathbf{g} = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$.

3 Effects of forces and torques on the end effectors

The well known relationship between joint torques and end effector forces on a simple serial robot is:

$$\mathbf{\Gamma} = \mathbb{J}_n^T \mathbb{f}_{e@n}$$

where

- $\mathbf{\Gamma}$ is the vector of torques of the individual joints in the chain
- $\mathbb{f}_{e@n} = \begin{bmatrix} f_{e@n} \\ \tau_{e@n} \end{bmatrix}$ is the wrench applied by the robot at the origin of the nth frame (i.e. the last link in the chain which has the end-effector mounted on it). This wrench is usually represented in frame R_n or in the world frame R_0 denoted as ${}^n\mathbb{f}_{e@n}$ or ${}^0\mathbb{f}_{e@n}$ respectively.
- $\mathbb{J}_n$ is $6 \times n$ Jacobian matrix of the robot calculated using:

$$\mathbb{J}_n = \begin{bmatrix} e_1 \times L_{1,n} & \ldots & e_n \times L_{n,n} \\ e_1 & \ldots & e_n \end{bmatrix}$$

  where e_j denotes the unit vectors along the local angular velocities of the frame j and $L_{j,n}$ is the position vector from O_j to O_n. These vectors are expressed in the same frame as the wrench $\mathbb{f}_{e@n}$. So for ${}^0\mathbb{f}_{e@n}$ all vectors in the Jacobian matrix will be expressed in frame 0 and the Jacobian will be denoted as ${}^0\mathbb{J}_n$. Similarly for ${}^n\mathbb{f}_{e@n}$ the Jacobian will be denoted ${}^n\mathbb{J}_n$.

3.1 Jacobians for the two-armed robot

For the case of krang, we will have two wrenches $\mathbb{f}_{el@10l}$ and $\mathbb{f}_{er@10r}$ applied at two end-effectors on the right and the left arms respectively. As previously el and er are identifying the wrench and $10l$ and $10r$ are idenfying the frames at whose origin the wrenches are being applied. The joint torques will now be calculated using the equation:

$$\mathbf{\Gamma} = \mathbb{J}_{10l}^T \mathbb{f}_{el@10l} + \mathbb{J}_{10r}^T \mathbb{f}_{er@10r} \tag{16}$$

where

- $\mathbf{\Gamma} = \begin{bmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_{4l} & ... & \tau_{10l} & \tau_{4r} & ... & \tau_{10r} \end{bmatrix}^T$
- $\mathbb{J}_{10l} = \begin{bmatrix} e_1 \times L_{1,10l} & e_2 \times L_{2,10l} & e_3 \times L_{3,10l} & e_{4l} \times L_{4l,10l} & ... & e_{10l} \times L_{10l,10l} & O_{3\times7} \\ e_1 & e_2 & e_3 & e_{4l} & ... & e_{10l} & O_{3\times7} \end{bmatrix}$
- $\mathbb{J}_{10r} = \begin{bmatrix} e_1 \times L_{1,10r} & e_2 \times L_{2,10r} & e_3 \times L_{3,10r} & O_{3\times7} & e_{4r} \times L_{4r,10r} & ... & e_{10r} \times L_{10r,10r} \\ e_1 & e_2 & e_3 & O_{3\times7} & e_{4r} & ... & e_{10r} \end{bmatrix}$

4 Other terms in the Lagrange Equations

4.1 Considering Friction

The most often employed model for friction is composed of Coulomb friction together with viscous friction. Therefor, the friction torque at joint i is written as:

$$\Gamma_{fi} = F_{ci} sign(\dot{q}_i) + F_{vi}\dot{q}_i$$

To take into account the friction in the dynamic model of a robot we add the vector $\mathbf{\Gamma}_f$ to the right side of the Lagrange equation (i.e. the vector of generalized forces), such that:

$$\mathbf{\Gamma}_f = \mathbf{diag}(\dot{\mathbf{q}})\mathbf{F_v} + \mathbf{diag}[\mathbf{sign}(\dot{\mathbf{q}}]\mathbf{F_c} \tag{17}$$

where

- $\mathbf{F}_v = \begin{bmatrix} F_{v1} & F_{v2} & F_{v3} & F_{v4l} & ... & F_{v10l} & F_{v4r} & ... & F_{v10r} \end{bmatrix}^T$
- $\mathbf{F}_c = \begin{bmatrix} F_{c1} & F_{c2} & F_{c3} & F_{c4l} & ... & F_{c10l} & F_{c4r} & ... & F_{c10r} \end{bmatrix}^T$
- $\mathbf{diag}(\dot{\mathbf{q}}) is the diagonal matrix whose elements are the components of \dot{\mathbf{q}}$

4.2 Considering rotor inertia

The kinetic energy of the rotor (and transmission system) and actuator j, is given by the expression $\frac{1}{2} I_{aj} \dot{q}_j^2$. The inertial parameter I_{aj} denotes the equivalent inertia referred to the joint velocity. It is given by:

$$I_{aj} = N_j^2 J_{mj} \tag{18}$$

where J_{mj} is the moment of inertia of the rotor and transmissions of actuator j, N_j is the transmission ratio of the joint axis, equal to $\frac{\dot{q}_{mj}}{\dot{q}_j}$ where $\dot{q}_{mj}$ denotes the rotor velocity of actuator j. In the case of a prismatic joint, I_{aj} is an equivalent mass.

In order to consider the rotor inertia in the dynamic model of the robot, we add the inertia (or mass) I_{aj} to the A_{jj} element of the matrix $\mathbf{A}$.

References

[1] Wisama Khalil and Etienne Dombre. *Modeling, identification and control of robots*. Butterworth-Heinemann, 2004.

A Expression for Kinetic Energy

We show here how the equation 11 was derived from 10. Equation 10 is:

$$E_j = \frac{1}{2}(\omega_j^T I_{Gj}\omega_j + M_j V_{Gj}^T V_{Gj}) \tag{19}$$

where the velocity of the center of mass can be expressed as:

$$V_{Gj} = V_j + \omega_j \times S_j$$

and since:

$$J_j = I_{Gj} - M_j \hat{S}_j \hat{S}_j^T$$

So equation 19 becomes:

$$\begin{aligned} E_j &= \frac{1}{2}(\omega_j^T (J_j + M_j \hat{S}_j \hat{S}_j)\omega_j + M_j (V_j + \omega_j \times S_j)^T (V_j + \omega_j \times S_j)) \\ E_j &= \frac{1}{2}(\omega_j^T J_j \omega_j + M_j V_j^T V_j + \omega_j^T M_j \hat{S}_j \hat{S}_j \omega_j + M_j V_j^T (\omega_j \times S_j) \\ &\qquad + M_j(\omega_j \times S_j)^T V_j + M_j (\omega_j \times S_j)^T (\omega_j \times S_j)) \end{aligned}$$

Noting that the last term:

$$\begin{aligned} M_j(\omega_j \times S_j)^T(\omega_j \times S_j) &= (-)(-)M_j(S_j \times \omega_j)^T(S_j \times \omega_j) \\ &= M_j(\hat{S}_j\omega_j)^T(\hat{S}_j\omega_j) \\ &= M_j\omega_j^T\hat{S}_j^T\hat{S}_j\omega_j \\ &= -M_j\omega_j^T\hat{S}_j\hat{S}_j\omega_j \end{aligned}$$

cancels out the third term. And noting that the fourth and fifth terms are equal, we are left with:

$$E_j = \frac{1}{2}(\omega_j^T J_j \omega_j + M_j V_j^T V_j + 2M_j(\omega_j \times S_j)^T V_j)$$

The last term in the above expression can be simplified as follows:

$$\begin{aligned} M_j(\omega_j \times S_j)^T V_j &= M_j(\hat{\omega}_j S_j)^T V_j \\ &= M_j S_j^T \hat{\omega}_j^T V_j \\ &= -M_j S_j^T \hat{\omega}_j V_j \\ &= -M_j S_j^T(\omega_j \times V_j) \\ &= \mathbf{MS}_j^T(V_j \times \omega_j) \end{aligned}$$

so we end up with:

$$E_j = \frac{1}{2}(\omega_j^T J_{Gj}\omega_j + M_j V_j^T V_j + 2\mathbf{MS}_j^T(V_j \times \omega_j))$$