Equations Of Motion Of a Wheeled Inverted Pendulum

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1 Assumptions and Definitions of Terms

Assumption: The pose of the body is fixed and only the wheels and the first link of the robot (i.e. the base of Krang) are free to move.

Figure 1 shows a simplified representation of side view the Krang robot. Define:

- O is the mid-point of the line connecting wheel-centers
- G is the center of gravity of the robot body
- $\dot{x}$ is the heading speed of the robot
- $\dot{\phi}$ is the rotation speed of G about the wheel axis
- $\dot{\psi}$ is the spin speed of the robot or the rate of change of heading direction
- L is the distance between wheel centers
- R is the radius of the wheel
- τ_R, τ_L is the torque applied by right and left wheel motors respectively
- $x_0y_0z_0$ is the frame of reference with origin at O, x_0 in the heading direction, z_0 always vertically upwards and y_0 pointing from right wheel to the left wheel
- $x_1y_1z_1$ is the frame of reference fixed to the body with origin at O, x_1 opposite to y_0 (from left to right wheel center), y_1 along the base of Krang (the first link above the wheels at angle q_{imu} from z_0). When body pose is fixed $\dot{q}_{imu} = -\dot{\phi}$ i.e. G rotates with the same speed as the base link. They increase/decrease in opposite direction though.
- m_B, m_w are the masses of robot body and one wheel respectively
- $I_w = \begin{bmatrix} \mathbf{XX}_w & 0 & 0 \\ 0 & \mathbf{YY}_w & 0 \\ 0 & 0 & \mathbf{ZZ}_w \end{bmatrix}$ is the inertia matrix of the wheel wrt frame $x_0y_0z_0$
- $\mathbf{MS}_B = \begin{bmatrix} \mathbf{MX}_B & \mathbf{MY}_B & \mathbf{MZ}_B \end{bmatrix}^T$ is the mass times center of mass of the body expressed in frame $x_1y_1z_1$
- $I_B = \begin{bmatrix} \mathbf{XX}_B & \mathbf{XY}_B & \mathbf{XZ}_B \\ \mathbf{XY}_B & \mathbf{YY}_B & \mathbf{YZ}_B \\ \mathbf{XZ}_B & \mathbf{YZ}_B & \mathbf{ZZ}_B \end{bmatrix}$ is the inertia matrix of the body wrt frame $x_1y_1z_1$

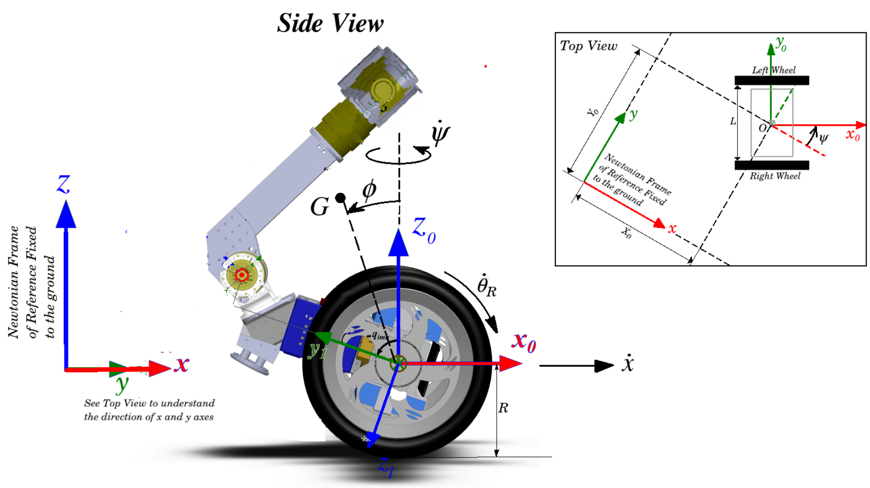

Figure 1: Frames of references on the robot

2 Generalized Coordinates

We select five generalized co-ordinates: $\{q\} = \{\ X_0,\ Y_0,\ \theta_L,\ \theta_R$ and $\phi\}$. Where:

X_0, Y_0 are the position coordintes of O wrt a Newtonian reference frame xyz fixed on ground

θ_L, θ_R are the rotation angles of the left and right wheels respectively

Given this Newtonian frame of reference we can define ψ as the heading direction (x_0) measured as an angle from the x-axis of the Newtonian frame.

3 Constraint Equations

Let $\bar{i}_0$, $\bar{j}_0$ and $\bar{k}_0$ be the unit vectors in frame $x_0y_0z_0$ and $\bar{I}$, $\bar{J}$ and $\bar{K}$ be the unit vectors in frame xyz.

Under the assumption of no slipping/skidding, we have two constraint equations:

$$\bar{v}_L = R\dot{\theta}_L\bar{i}_0 \tag{1}$$
$$\bar{v}_R = R\dot{\theta}_R\bar{i}_0 \tag{2}$$

Since

$$\begin{aligned}
\bar{v}_L &= \bar{v}_0 + \bar{\omega}_0 \times \bar{r}_{L/O} \\
&= \dot{X}_0\bar{I} + \dot{Y}_0\bar{J} + \dot{\psi}\bar{K} \times (\frac{L}{2}\bar{j}_0) \\
&= \dot{X}_0(cos\psi\bar{i}_0 - sin\psi\bar{j}_0) + \dot{Y}_0(sin\psi\bar{i}_0 + cos\psi\bar{k}_0) - \frac{L}{2}\dot{\psi}\bar{i}_0 \\
&= (\dot{X}_0cos\psi + \dot{Y}_0sin\psi - \frac{L}{2}\dot{\psi})\bar{i}_0 - (\dot{X}_0sin\psi - \dot{Y}_0cos\psi)\bar{j}_0
\end{aligned}$$

And similarly,

$$\bar{v}_R = (\dot{X}_0 cos\psi + \dot{Y}_0 sin\psi + \frac{L}{2}\dot{\psi})\bar{i}_0 - (\dot{X}_0 sin\psi - \dot{Y}_0 cos\psi)\bar{j}_0$$

Comparing the coefficients of $\bar{i}_0$ in eqs. 1-2 gives:

$$\dot{X}_0 cos\psi + \dot{Y}_0 sin\psi - \frac{L}{2}\dot{\psi} = R\dot{\theta}_L \tag{3}$$

$$\dot{X}_0 cos\psi + \dot{Y}_0 sin\psi + \frac{L}{2}\dot{\psi} = R\dot{\theta}_R \tag{4}$$

Subtraction and addition of the two equations gives us:

$$\dot{\psi} = \frac{R}{L}(\dot{\theta}_R - \dot{\theta}_L) \tag{5}$$

$$\dot{X}_0 cos\psi + \dot{Y}_0 sin\psi = \frac{R}{2}(\dot{\theta}_L + \dot{\theta}_R) \tag{6}$$

Eq. 5 can be integrated to give $\psi = \frac{R}{L}(\theta_R - \theta_L)$ which can be substituted in eq. 6 to give our first constraint equation relating all generalized co-ordinates.

Comparing the coefficients of $\bar{j}_0$ in eqs. 1-2 gives us our second constraint equation:

$$\dot{X}_0 sin\psi - \dot{Y}_0 cos\psi = 0 \tag{7}$$

Eqs. 6-7 give the two equations relating our generalized velocities as a result of the nonholonomic constrainsts. Five generalized coordinates with two constraint equations leads to three degrees of freedom.

4 Defining Generalized Velocities

It is easier to derive the dynamic model of the system in terms of the generalized velocities: $\{\dot{q}\} = \{\, \dot{x},\, \dot{\phi},\, \dot{\psi}\}$. These three velocities can take arbitrary values all of whom will be kinematically admissible. In other words, they represent our three degrees of freedom. Here $\dot{x}$ should be refered to as a quasi-velocity as this velocity has meaning only as a veolocity but its corresponding position variable x does not give any physical meaning. Although, the infinitesimal change of position $\delta x = \dot{x}dt$ (sometimes refered to as virtual displacement) has physical meaning.

The older generalized velocities can be calculated from the new ones as follows:

$$\dot{X}_0 = \dot{x} cos\psi \tag{8}$$

$$\dot{Y}_0 = \dot{x} sin\psi \tag{9}$$

$$\dot{\theta}_L = \frac{1}{R}\dot{x} - \frac{L}{2R}\dot{\psi} \tag{10}$$

$$\dot{\theta}_R = \frac{1}{R}\dot{x} + \frac{L}{2R}\dot{\psi} \tag{11}$$

$$\dot{\phi} = \dot{\phi} \tag{12}$$

Where the first two relationships are derived by comparing the coefficients in $\dot{X}_0\bar{I} + \dot{Y}_0\bar{J} = \bar{v}_0 = \dot{x}\bar{i}_0 = \dot{x}(cos\psi\bar{I} + sin\psi\bar{J})$. And the next two relationships are derived by substituting the the first two relationships in eqs. 3-4. When these relationships (8-12) are substituted in our constraint equations 6-7, both sides of the equations vanish. Indicating that these three degrees of freedom are not bound by any constraint. We will now derive three dynamic equations in terms of our new generalized velocities. Those equations in conjunction with these relationships can solve for all five generalized coordinates.

5 Dynamic Equations

Since we are dealing with quasi-velocities, we will use Kane's method to derive the dynamic equations.

5.1 Introduction to Kane's formulation

The Kane's formulation is as follows:

$$\sum_k \left[m_k \bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j + \left(\frac{d\bar{H}_{Gk}}{dt} \right) \cdot (\bar{\omega}_k)_j \right] = \sum_n \bar{F}_n \cdot (\bar{v}_n)_j + \sum_n \bar{M}_m \cdot (\bar{\omega}_m)_j \quad j = 1...K \tag{13}$$

where

- j is the unique number identifying each generalized co-ordinate in the system
- k is the unique number identifying each rigid body in the system
- n is the unique number identifying each external force acting on the system
- m is the unique number identifying each external torque acting on the system
- m_k is the mass of the kth body
- $\bar{a}_{Gk}$ is the acceleration of the center of mass of kth body
- $\bar{v}_{Gk}$ is the velocity of the center of mass of the kth body
- $\bar{H}_{Gk}$ is the angular momentum of body k about its center of mass
- $\bar{\omega}_k$ is the angular velocity of the body k
- F_n is the nth external force
- M_m is the mth external moment
- $\bar{v}_n$ is the velocity of the point at which external Force F_n is acting
- $\bar{\omega}_m$ is the angular velocity of the body on which torque is acting relative to the actuator applying the torque
- $()_j$ $= \frac{\partial ()}{\partial \dot{q}_j}$ the partial derivative of the quantity in brackets () with respect to the generalized velocity $\dot{q}_j$

5.2 Kane's Left-Hand Side

The left hand side of the Kane's equation containes a sum whose range is equal to the number of bodies in the system. We have three bodies: Left-wheel (L), right wheel (R) and the body of robot (B). Each term in the sum consists of the acceleration ($\bar{a}_{Gk}$), velocity ($\bar{v}_{Gk}$), angular momentum ($\bar{H}_{Gk}$) of the center of mass and the body's angular velocity ($\bar{\omega}_k$). And then some partieal derivatives wrt to the generalized coordinates ($(\bar{\omega}_k)_j = \frac{\partial \bar{\omega}_k}{\partial \dot{q}_j}$ and $(\bar{v}_{Gk})_j = \frac{\partial \bar{v}_{Gk}}{\partial \dot{q}_j}$). We will have three equations corresponding to each generalized coordinate $\{\dot{q}_j\} = \{\ \dot{x}, \dot{\phi}, \dot{\psi}\}$.

5.2.1 Left Wheel

This evaluation takes place in the $x_L y_L z_L$ frame fixed to the left wheel such that it is parallel to frame $x_0 y_0 z_0$ when $\theta_L = 0$. So $\bar{i}_0 = cos\theta_L \bar{i}_L + sin\theta_L \bar{k}_L$, $\bar{j}_0 = \bar{j}_L$ and $\bar{k}_0 = -sin\theta_L \bar{i}_L + cos\theta_L \bar{k}_L$. Angular velocity:

$$\begin{aligned}
\bar{\omega}_L &= \dot{\psi}\bar{k}_0 + \dot{\theta}_L \bar{j}_0 \\
&= \dot{\psi}\bar{k}_0 + \left(\frac{1}{R}\dot{x} - \frac{L}{2R}\dot{\psi}\right)\bar{j}_0 \\
&= -\dot{\psi} sin\theta_L \bar{i}_L + \left(\frac{1}{R}\dot{x} - \frac{L}{2R}\dot{\psi}\right)\bar{j}_L + \dot{\psi} cos\theta_L \bar{k}_L
\end{aligned} \tag{14}$$

The terms that follow are also similarly to be expressed in frame $x_L y_L z_L$ but that step is skipped for brevity. Velocity:

$$\begin{aligned}
\bar{v}_{GL} &= \bar{v}_0 + \bar{\omega}_0 \times \bar{r}_{L/O} \\
&= \dot{x}\bar{i}_0 + \dot{\psi}\bar{k}_0 \times \frac{L}{2}\bar{j}_0 \\
&= \left(\dot{x} - \frac{L}{2}\dot{\psi}\right)\bar{i}_0
\end{aligned} \tag{15}$$

Linear acceleration:

$$\begin{aligned}
\bar{a}_{GL} &= \bar{a}_0 + \bar{\alpha}_0 \times \bar{r}_{L/O} + \bar{\omega}_0 \times \left(\omega_0 \times \bar{r}_{L/O}\right) \\
&= \ddot{x}\bar{i}_0 + \dot{x}\left(\dot{\psi}\bar{k}_0 \times \bar{i}_0\right) + \ddot{\psi}\bar{k}_0 \times \frac{L}{2}\bar{j}_0 + \dot{\psi}\bar{k}_0 \times \left(\dot{\psi}\bar{k}_0 \times \frac{L}{2}\bar{j}_0\right) \\
&= \left(\ddot{x} - \frac{L}{2}\ddot{\psi}\right)\bar{i}_0 + \left(\dot{x}\dot{\psi} - \frac{L}{2}\dot{\psi}^2\right)\bar{j}_0
\end{aligned} \tag{16}$$

Angular momentum and its derivative:

$$\begin{aligned}
\bar{H}_{GL} &= I_w \bar{\omega}_L \\
\frac{d\bar{H}_{GL}}{dt} &= \frac{\partial \bar{H}_{GL}}{\partial t} + \bar{\omega}_L \times \bar{H}_{GL}
\end{aligned} \tag{17}$$

where $I_w = \begin{bmatrix} \mathbf{ZZ}_w & 0 & 0 \\ 0 & \mathbf{YY}_w & 0 \\ 0 & 0 & \mathbf{ZZ}_w \end{bmatrix}$. Due to symmetry the off-diogonal terms in the inertia matrix vanish, and the inertia about x_L-axis and z_L-axis are both equal (signified by $\mathbf{ZZ}_w$).

5.2.2 Right Wheel

This evaluation takes place in the $x_R y_R z_R$ frame fixed to the right wheel such that it is parallel to frame $x_0 y_0 z_0$ when $\theta_R = 0$. So $\bar{i}_0 = cos\theta_R \bar{i}_R + sin\theta_R \bar{k}_R$, $\bar{j}_0 = \bar{j}_R$ and $\bar{k}_0 = -sin\theta_R \bar{i}_R + cos\theta_R \bar{k}_R$. Angular velocity:

$$\begin{aligned}\bar{\omega}_R &= \dot{\psi}\bar{k}_0 + \dot{\theta}_R \bar{j}_0 \\ &= \dot{\psi}\bar{k}_0 + \left(\frac{1}{R}\dot{x} + \frac{L}{2R}\dot{\psi}\right)\bar{j}_0 \\ &= -\dot{\psi} sin\theta_L \bar{i}_R + \left(\frac{1}{R}\dot{x} + \frac{L}{2R}\dot{\psi}\right)\bar{j}_R + \dot{\psi} cos\theta_L \bar{k}_R \end{aligned} \tag{18}$$

The terms that follow are also similarly to be expressed in frame $x_R y_R z_R$ but that step is skipped for brevity. Velocity:

$$\begin{aligned}\bar{v}_{GR} &= \bar{v}_0 + \bar{\omega}_0 \times \bar{r}_{R/O} \\ &= \dot{x}\bar{i}_0 + \dot{\psi}\bar{k}_0 \times \left(-\frac{L}{2}\bar{j}_0\right) \\ &= \left(\dot{x} + \frac{L}{2}\dot{\psi}\right)\bar{i}_0 \end{aligned} \tag{19}$$

Linear acceleration:

$$\begin{aligned}\bar{a}_{GR} &= \bar{a}_0 + \bar{\alpha}_0 \times \bar{r}_{R/O} + \bar{\omega}_0 \times \left(\omega_0 \times \bar{r}_{R/O}\right) \\ &= \ddot{x}\bar{i}_0 + \dot{x}\left(\dot{\psi}\bar{k}_0 \times \bar{i}_0\right) + \ddot{\psi}\bar{k}_0 \times \left(-\frac{L}{2}\bar{j}_0\right) + \dot{\psi}\bar{k}_0 \times \left(\dot{\psi}\bar{k}_0 \times \left(-\frac{L}{2}\bar{j}_0\right)\right) \\ &= \left(\ddot{x} + \frac{L}{2}\ddot{\psi}\right)\bar{i}_0 + \left(\dot{x}\dot{\psi} + \frac{L}{2}\dot{\psi}^2\right)\bar{j}_0 \end{aligned} \tag{20}$$

Angular momentum and its derivative:

$$\begin{aligned}\bar{H}_{GR} &= I_w \bar{\omega}_R \\ \frac{d\bar{H}_{GR}}{dt} &= \frac{\partial \bar{H}_{GR}}{\partial t} + \bar{\omega}_0 \times \bar{H}_{GR} \end{aligned} \tag{21}$$

5.2.3 Body

We will evaluate the quantities in frame $x_1 y_1 z_1$. It is useful to know that $\bar{i}_0 = sinq_{imu}\bar{j}_1 - cosq_{imu}\bar{k}_1$, $\bar{j}_0 = -\bar{i}_1$ and $\bar{k}_0 = cosq_{imu}\bar{j}_1 + sinq_{imu}\bar{k}_1$, For brevity we will leave the expressions in closed forms.
Angular velocity:

$$\bar{\omega}_B = \dot{\psi}\bar{k}_0 + \dot{\phi}\bar{i}_1 \tag{22}$$

Expressing $\bar{k}_0$ in frame $x_1 y_1 z_1$ gives us the expression in this frame.
Velocity:

$$\begin{aligned}\bar{v}_{GB} &= \bar{v}_0 + \bar{\omega}_B \times \bar{r}_{B/O} \\ &= \dot{x}\bar{i}_0 + \bar{\omega}_B \times \frac{1}{m_B}\mathbf{MS}_B \end{aligned} \tag{23}$$

isa Linear acceleration:

$$\bar{a}_{GB} = \bar{a}_0 + \bar{\alpha}_B \times \bar{r}_{B/O} + \bar{\omega}_B \times \left(\omega_B \times \bar{r}_{B/O}\right) \tag{24}$$

where

$$\begin{aligned}
\bar{a}_0 &= \frac{d\bar{v}_0}{dt} = \frac{d\left(\dot{x}\bar{i}_0\right)}{dt} = \ddot{x}\bar{i}_0 + \dot{x}\left(\dot{\psi}\bar{k}_0 \times \bar{i}_0\right) \\
\bar{\alpha}_B &= \frac{d\bar{\omega}_B}{dt} = \frac{d\left(\dot{\psi}\bar{k}_0 + \dot{\phi}\bar{i}_1\right)}{dt} = \ddot{\psi}\bar{k}_0 + \ddot{\phi}\bar{i}_1 + \dot{\phi}\left(\bar{\omega}_B \times \bar{i}_1\right)
\end{aligned} \tag{25}$$

Angular momentum and its derivative:

$$\begin{aligned}
\bar{H}_{GB} &= I_B\bar{\omega}_B \\
\frac{d\bar{H}_{GB}}{dt} &= \frac{\partial \bar{H}_{GB}}{\partial t} + \bar{\omega}_B \times \bar{H}_{GB}
\end{aligned} \tag{26}$$

5.2.4 Left-Hand Side Final Evaluation

The Kanes' formulation (Eq. 13) are in fact three equation each corresponding to a different generalized velocity $\dot{q}_j = \dot{x}, \dot{\psi}, \dot{\phi}$. Each of these three equation is a summation of the given expression evaluated for each body $k = L, R, B$ i.e. Left-wheel, right-wheel and body. When the expressions that we evaluated in eqs. 14-26 are substituted in Kanes' equations (Eq. 13) and the results evaluted (using MATLAB code listed in the Appendix section A.1), we get the following expression:

$$\mathbf{A}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}}$$

where

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} m_B + 2m_w + \frac{2\mathbf{YY}_w}{R^2} & \mathbf{MX}_b & -\mathbb{M}_1 \\ \mathbf{MX}_b & \frac{m_w L^2}{2} + \frac{\mathbf{YY}_w L^2}{2R^2} + 2\mathbf{ZZ}_w + \frac{\mathbb{M}_2^2}{m_b} + \mathbb{M}_3 + \frac{\mathbf{MX}_b^2}{m_B} & \mathbb{M}_4 \\ -\mathbb{M}_1 & \mathbb{M}_4 & \mathbb{M}_6 + \mathbf{XX}_b \end{bmatrix} \\
\mathbf{C} &= \begin{bmatrix} 0 & \mathbb{M}_2\dot{\psi} & \mathbb{M}_2\dot{\phi} \\ -\frac{1}{2}\mathbb{M}_2\dot{\psi} & -\frac{1}{2}\mathbb{M}_2\dot{x} + \frac{1}{2}\mathbb{M}_7\dot{\phi} & \mathbb{M}_5\dot{\phi} + \frac{1}{2}\mathbb{M}_7\dot{\psi} \\ 0 & \mathbb{M}_8 - \frac{1}{m_b}\mathbb{M}_1\mathbb{M}_2 & 0 \end{bmatrix}
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\mathbb{M}_1 &= \left(\mathbf{MY}_b cosq_{imu} + \mathbf{MZ}_b sinq_{imu}\right) \\
\mathbb{M}_2 &= \mathbf{MZ}_b cosq_{imu} - \mathbf{MY}_b sinq_{imu} \\
\mathbb{M}_3 &= \mathbf{ZZ}_b sin^2 q_{imu} + \mathbf{YY}_b cos^2 q_{imu} + \mathbf{YZ}_b cosq_{imu} sinq_{imu} \\
\mathbb{M}_4 &= \mathbf{XZ}_b sinq_{imu} + \mathbf{XY}_b cosq_{imu} + \frac{\mathbf{MX}_b}{m_B}\mathbb{M}_1 \\
\mathbb{M}_5 &= -\mathbf{XZ}_b cosq_{imu} + \mathbf{XY}_b sinq_{imu} + \frac{\mathbf{MX}_b}{m_B}\mathbb{M}_2 \\
\mathbb{M}_6 &= \frac{1}{m_B}\left(\mathbf{MY}_b^2 + \mathbf{MZ}_b^2\right) \\
\mathbb{M}_7 &= \frac{1}{m_B}\left(\left(\mathbf{MZ}_b^2 - \mathbf{MY}_b^2\right) sin2q_{imu} + 2\mathbf{MX}_b\mathbf{MZ}_b cos2q_{imu}\right) - 2\mathbb{M}_8 \\
\mathbb{M}_8 &= \mathbf{YZ}_b cos2q_{imu} + \left(\mathbf{ZZ}_b - \mathbf{YY}_b\right) cosq_{imu} sinq_{imu}
\end{aligned}$$

The matrices **A** amd **C** are saved in the symbolic variables `AA` and `CC` generated by the code.

5.2.5 Comparing with [1]

Deriving Expressions in [1]
A similar dynamic model has been derived by [1]. From the closed form expressions of their paper they have directly stated thier final expressions while skipping the details of derivations. In order to ascertain the correctness of their work, we derive the expressions on our own by substituting the velocities, accelerations etc they derived in the closed form expressions. This is done using MATLAB code listed in appendix (section A.2). These are the resulting expressions for the left-hand sides of the threee equations:

$$
\begin{aligned}
F_1 &= -\frac{1}{R^2}(2R^2\ddot{x}m_C + 2I_{w3}\ddot{x} + R^2\ddot{x}m_S - R^2d\ddot{\phi}m_S cos\phi + R^2d\dot{\phi}^2m_S sin\phi + R^2d\dot{\psi}^2m_S sin\phi)\\
&= -2\ddot{x}m_C - \frac{2I_{w3}}{R^2}\ddot{x} - \ddot{x}m_S + d\ddot{\phi}m_S cos\phi - d\dot{\phi}^2m_S sin\phi - d\dot{\psi}^2m_S sin\phi\\
&= -(m_S + 2m_C + \frac{2I_{w3}}{R^2})\ddot{x} + m_S dcos\phi\ddot{\phi} - m_S dsin\phi(\dot{\phi}^2 + \dot{\psi}^2)\\
F_2 &= -\frac{1}{R^2}(2I_{w3}L^2\ddot{\psi} + I_2R^2\ddot{\psi} + 2I_{w2}R^2\ddot{\psi} + 2L^2R^2\ddot{\psi}m_C + R^2d^2\ddot{\psi}m_S sin\phi^2\\
&\qquad + \frac{1}{2}R^2d^2\dot{\phi}\dot{\psi}m_S sin(2\phi))\\
&= -\frac{2I_{w3}L^2}{R^2}\ddot{\psi} - I_2\ddot{\psi} - 2I_{w2}\ddot{\psi} - 2L^2\ddot{\psi}m_C - d^2\ddot{\psi}m_S sin\phi^2 - \frac{1}{2}d^2\dot{\phi}\dot{\psi}m_S sin(2\phi)\\
&= -(2m_C L^2 + \frac{2I_{w3}L^2}{R^2} + 2I_{w2} + m_S d^2 sin\phi^2 + I_2)\ddot{\psi} - m_S d^2 sin\phi cos\phi\dot{\phi}\dot{\psi}\\
F_3 &= -d^2\ddot{\phi}m_S - I_3\ddot{\phi} + d\ddot{x}m_S cos\phi + \frac{1}{2}(d^2\dot{\psi}^2m_S sin(2\phi))\\
&= -(m_S d^2 + I_3)\ddot{\phi} + m_S dcos\phi\ddot{x} + m_S d^2 sin\phi cos\phi\dot{\psi}^2
\end{aligned}
$$

The expressions match exactly as are there in the paper if $I_{w3} = \frac{m_C R^2}{2}$ and $I_{w2} = \frac{m_C}{4R^2}$. The paper has made a typo in the last equation using $\dot{\phi}^2$ instead of $\dot{\psi}^2$.

We can write down the A and C matrices using the above equations so that it becomes easier to match it term by term with our derived equations. So we have:

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} -(m_S + 2m_C + \frac{2I_{w3}}{R^2}) & 0 & m_S dcos\phi \\ 0 & -(2m_C L^2 + \frac{2I_{w3}L^2}{R^2} + 2I_{w2} + m_S d^2 sin\phi^2 + I_2) & 0 \\ m_S dcos\phi & 0 & -(m_S d^2 + I_3) \end{bmatrix}\\
\mathbf{C} &= \begin{bmatrix} 0 & -m_S dsin\phi\dot{\psi} & -m_S dsin\phi\dot{\phi} \\ 0 & -m_S d^2 sin\phi cos\phi\dot{\phi} & 0 \\ 0 & m_S d^2 sin\phi cos\phi\dot{\psi} & 0 \end{bmatrix}
\end{aligned}
\tag{28}
$$

Substituting [1]'s assumptions, symbols and equivalent expressions in place of ours
In order to compare our work with [1] we re-write our derived expression in terms of the symbols used by [1]. In this subsection we will write down definitions of symbols used by us in terms of equivalent expressions consisting of the symbols used by [1].

The symbols they used for the masses of the body and the wheel are m_S and m_C respectively. The distance between wheel centers they assumed is $2L$. So:

$$m_B = m_S \qquad m_w = m_C \qquad L = 2L \tag{29}$$

The inertia terms for the wheel are as follows:

$$I_w = \begin{bmatrix} \mathbf{XX}_w & 0 & 0 \\ 0 & \mathbf{YY}_w & 0 \\ 0 & 0 & \mathbf{ZZ}_w \end{bmatrix} = \begin{bmatrix} I_{w2} & 0 & 0 \\ 0 & I_{w3} & 0 \\ 0 & 0 & I_{w2} \end{bmatrix} \tag{30}$$

The radial distance from O to center of mass G is assumed in the paper to be d. This means our mass times COM parameter becomes:

$$\mathbf{MS}_b = \begin{bmatrix} \mathbf{MX}_b & \mathbf{MY}_b & \mathbf{MZ}_b \end{bmatrix}^T = \begin{bmatrix} 0 & m_S d cos\kappa & m_S d sin\kappa \end{bmatrix}^T \tag{31}$$

where $\kappa = q_{imu} + \phi$ is the angle from y_1-axis to $\overline{OG}$ (see Fig. 2A).

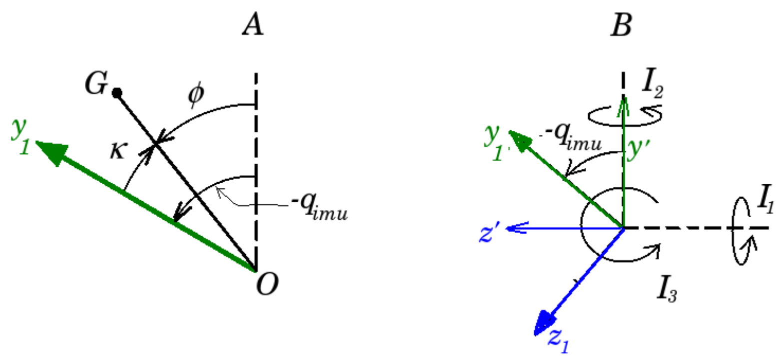

Figure 2: A. Understanding the definition of angles in our analysis. Follow the arrows and you get: $-q_{imu} + \kappa = \phi$ or $\kappa = q_{imu} + \phi$ B. Inertias as defined in [1]

Finally the inertia for body in [1] is assumed to be I_3 about the wheel-axis, I_2 about the fixed vertical and I_1 about the forward direction (see Fig. 2B). In order to find the equivalent expressions for the inertia parameters of the body we defined for our analysis we will have to perform rotation transformation of the inertia as defined in [1] to express it in the frame we defined in our analysis i.e. frame $x_1y_1z_1$. The formula for rotation transformation is:

$$I' = R\, I\, R^T$$

where I is the inertia matrix with respect to (say) frame 1 and I' is the same in frame 2 and R is the rotation transformation that transforms a vector ω expressed in frame 1 to ω' in frame 2.

$$\omega' = R\, \omega$$

In our case we have an inertia matrix $\begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_1 \end{bmatrix}$ defined in frame $x'y'z'$ (see Fig. 2B) and we need to rotate it into frame $x_1y_1z_1$. The rotation transformation will be:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cosq_{imu} & -sinq_{imu} \\ 0 & sinq_{imu} & cosq_{imu} \end{bmatrix}$$

Applying these concepts we find that:

$$\begin{bmatrix} \mathbf{XX}_B & \mathbf{XY}_B & \mathbf{XZ}_B \\ \mathbf{XY}_B & \mathbf{YY}_B & \mathbf{YZ}_B \\ \mathbf{XZ}_B & \mathbf{YZ}_B & \mathbf{ZZ}_B \end{bmatrix} = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_1 sin^2 q_{imu} + I_2 cos^2 q_{imu} & (I_2 - I_1)\, cosq_{imu} sinq_{imu} \\ 0 & (I_2 - I_1)\, cosq_{imu} sinq_{imu} & I_1 cos^2 q_{imu} + I_2 sin^2 q_{imu} \end{bmatrix} \tag{32}$$

Result of Substitution
Applying the substitutions described in eqs. 29-32 on the matrices **A** and **C** (eq. 27), we get the following results. These results are evaluated in the MATLAB code (section A.1) and saved in symbolic vairables `Acheck` and `Ccheck`.

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} m_S + 2m_C + \frac{2I_{w3}}{R^2} & 0 & -m_S d cos\phi \\ 0 & 2m_C L^2 + \frac{2I_{w3}L^2}{R^2} + 2I_{w2} + m_S d^2 sin\phi^2 + I_2 & 0 \\ -m_S d cos\phi & 0 & m_S d^2 + I_3 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} 0 & m_S d sin\phi\dot{\psi} & m_S d sin\phi\dot{\phi} \\ -\frac{1}{2} m_S d sin\phi\dot{\psi} & -\frac{1}{2} m_S d sin\phi\dot{x} + \frac{1}{2} m_S d^2 sin2\phi\dot{\phi} & \frac{1}{2} m_S d^2 sin2\phi\dot{\psi} \\ 0 & -\frac{1}{2} m_S d^2 sin2\phi\dot{\psi} & 0 \end{bmatrix} \end{aligned} \tag{33}$$

Comparing with the matrices derived in [1] shown in eq. 28 we see that all the terms are identical except that the signs of all terms are opposite which only indicates that the expressions evaluated from [1] are on the other side of the equation. One other exception is in row 2 of the **C** matrix where we see some additional terms in our analysis that are not present in [1]. To understand the differences, let's looks at the final expression, which we can find by multiplying **C** with $\dot{\mathbf{q}}$ and look at the form of the second element of the resulting vector. For [1], we get:

$$[\mathbf{C}\dot{\mathbf{q}}]_2 = -m_S d^2 sin\phi cos\phi\dot{\phi}\dot{\psi}$$

For our analysis, we get:

$$\begin{aligned} [\mathbf{C}\dot{\mathbf{q}}]_2 &= -\frac{1}{2} m_S d sin\phi\dot{\psi}\dot{x} - \frac{1}{2} m_S d sin\phi\dot{x}\dot{\psi} + \frac{1}{2} m_S d^2 sin2\phi\dot{\phi}\dot{\psi} + \frac{1}{2} m_S d^2 sin2\phi\dot{\psi}\dot{\phi} \\ &= -m_S d sin\phi\dot{\psi}\dot{x} + m_S d^2 sin2\phi\dot{\phi}\dot{\psi} \\ &= -m_S d sin\phi\dot{\psi}\dot{x} + 2m_S d^2 sin\phi cos\phi\dot{\phi}\dot{\psi} \end{aligned}$$

So there is an additional $-m_S d sin\phi\dot{\psi}\dot{x}$ and there is a coefficient 2 with the matching term $m_S d^2 sin\phi cos\phi\dot{\phi}\dot{\psi}$ that was not present in the expression from [1].

Both these differences are explained by pointing out errors in the derivation of [1]. They did not evaluate the linear and angular acceleration of the body correctly. They forgot to take the derivative of the unit vector when they evaluated $\frac{d^F \mathbf{v}^{S_C}}{dt}$ ([1, eq. 10]) and ${}^F\alpha^S$([1, eq. 9]). These terms are labeled as $\bar{a}_0$ and $\bar{\alpha}_B$ in our analysis and evaluated in eq.25. The reason why derivative of the unit vectors is important is because these unit vectors are not fixed. When the body rotates the unit vectors $\bar{i}_0$ and $\bar{i}_1$ rotate with angular speed $\dot{\psi}\bar{k}_0$. Using the symbols of [1] we will say unit vectors $\mathbf{n}_1$ and $\mathbf{n}_3$ rotate with a speed of $\dot{\psi}\mathbf{n}_2$ or $u_2\mathbf{n}_2$. This would mean that the correct evaluation of the two terms in [1] is as follows:

$$\begin{aligned}
\frac{d^F\mathbf{v}^{S_C}}{dt} &= \frac{d\left(u_1\mathbf{n}_1\right)}{dt} = \dot{u}_1\mathbf{n}_1 + u_1\left(u_2\mathbf{n}_2 \times \mathbf{n}_1\right)\\
&= \dot{u}_1\mathbf{n}_1 - u_1u_2\mathbf{n}_3\\
{}^F\alpha^S &= \frac{d\left({}^F\omega^S\right)}{dt} = \frac{d\left(u_2\mathbf{n}_2 + u_3\mathbf{n}_3\right)}{dt} = \dot{u}_2\mathbf{n}_2 + \dot{u}_3\mathbf{n}_3 + u_3\left(u_2\mathbf{n}_2 \times \mathbf{n}_3\right)\\
&= \dot{u}_2\mathbf{n}_2 + \dot{u}_3\mathbf{n}_3 + u_2u_3\mathbf{n}_1
\end{aligned}$$

When these new expressions are used for the evaluation of body's linear acceleration ${}^F\mathbf{a}^{S^*}$ ([1, eq. 10]), we get:

$$\begin{aligned}
{}^F\mathbf{a}^{S^*} &= \frac{d^F\mathbf{v}^{S_C}}{dt} + {}^F\alpha^S \times \overline{S^CS^*} + {}^F\omega^S \times \left({}^F\omega^S \times \overline{S^CS^*}\right)\\
&= \begin{bmatrix}\dot{u}_1\\0\\-u_1u_2\end{bmatrix} + \begin{bmatrix}u_2u_3\\\dot{u}_2\\\dot{u}_3\end{bmatrix} \times \begin{bmatrix}-dsin\phi\\dcos\phi\\0\end{bmatrix} + \begin{bmatrix}0\\u_2\\u_3\end{bmatrix} \times \left(\begin{bmatrix}0\\u_2\\u_3\end{bmatrix} \times \begin{bmatrix}-dsin\phi\\dcos\phi\\0\end{bmatrix}\right)\\
&= \begin{array}{c}\left(\dot{u}_1 - \dot{u}_3dcos\phi + (u_2^2+u_3^2)dsin\phi\right)\mathbf{n}_1\\ +\left(-\dot{u}_3dsin\phi - u_3^2dcos\phi\right)\quad \mathbf{n}_2\\ +\left(\dot{u}_2dsin\phi + 2u_2u_3dcos\phi - u_1u_2\right)\mathbf{n}_3\end{array}
\end{aligned} \tag{34}$$

This expressions matches the one derived in [1, eq. 10] except that the last component has an additional $-u_1u_2$ and coefficient 2 for the terms $u_2u_3dcos\phi$. Implementing these changes gives us the dynamic equations that match exactly with the matrices that resulted from substitution at the beginning of this section (eq.33). This is implemented in the MATLAB code in the appendix section A.3.

5.3 Kane's Right Hand Side

The right hand side of the Kane's forumulation 13 is the sum of some dot product terms. Each term is either the dot product of:

- force applied on the system $\bar{F}_n$
- the linear velocity $\bar{v}_n$ of the point differentiated partially wrt the the unique gerneralized speed $\dot{q}_j$ corresponding to each equation i.e. $\frac{\partial \bar{v}_n}{\partial \dot{q}_j}$

or the dot product of:

- torque applied on the system $\bar{\tau}_n$
- the angular velocity $\bar{\omega}_n$ of the body differentiated partially wrt the the unique gerneralized speed $\dot{q}_j$ corresponding to each equation i.e. $\frac{\partial \bar{\omega}_n}{\partial \dot{q}_j}$

So, in order to analyse the right-hand side of the equation, we need to list down all the forces and torques applied to the system and the points at which they are being applied. They are as follows:

$\bar{\tau}_L, \bar{\tau}_R$ Torques applied by wheel motors (in the body) on the right wheel and left wheel at points R and L on the wheels

$-\bar{\tau}_L, -\bar{\tau}_R$ Reaction torques applied on body at points R and L on the body

$\bar{F}_g$ $= m_B g$ the weight of the body applied at point G

$\bar{F}_{el}, \bar{\tau}_{el}$ Force and torque applied by the environment on the left hand end-effector at point E_l

$\bar{F}_{er}, \bar{\tau}_{er}$ Force and torque applied by the environment on the right hand end-effector at point E_r

So there are nine forces/torques so far. Three can be further added to the list by analysing the forces $\bar{F}_g$, $\bar{F}_{el}$ and $\bar{F}_{er}$ wrt point O so each of them is broken into a force-torque couple acting at point O. So:

$\bar{F}_g$ at G gives the effect of force $\bar{F}_g$ and a torque $\bar{r}_{G/O} \times \bar{F}_g$ wrt O

$\bar{F}_{el}$ at E_l gives the effect of force $\bar{F}_{el}$ and a torque $\bar{r}_{E_l/O} \times \bar{F}_{el}$ wrt O

$\bar{F}_{er}$ at E_r gives the effect of force $\bar{F}_{er}$ and a torque $\bar{r}_{E_r/O} \times \bar{F}_{er}$ wrt O

Let $\bar{F}_{el} = \begin{bmatrix} F_{lx} & F_{ly} & F_{lz} \end{bmatrix}^T$, $\bar{F}_{er} = \begin{bmatrix} F_{rx} & F_{ry} & F_{rz} \end{bmatrix}^T$, $\bar{\tau}_{el} = \begin{bmatrix} \tau_{lx} & \tau_{ly} & \tau_{lz} \end{bmatrix}^T$ and $\bar{\tau}_{er} = \begin{bmatrix} \tau_{rx} & \tau_{ry} & \tau_{rz} \end{bmatrix}^T$ be the components of forces/torques defined in the frame $x_0 y_0 z_0$. Also $\bar{r}_{E_l/O} = \begin{bmatrix} r_{lx} & r_{ly} & r_{lz} \end{bmatrix}^T$ and $\bar{r}_{E_r/O} = \begin{bmatrix} r_{rx} & r_{ry} & r_{rz} \end{bmatrix}^T$ be the position coordinates of the end effectors again presented in the frame $x_0 y_0 z_0$. Also $\bar{F}_g = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$. Also $\bar{r}_{G/O}$ is to be expressed in the $x_0 y_0 z_0$ which will be $\bar{r}_{G/O} = \frac{1}{m_B} \begin{bmatrix} 0 & sq_{imu} & -cq_{imu} \\ -1 & 0 & 0 \\ 0 & cq_{imu} & sq_{imu} \end{bmatrix} \begin{bmatrix} \mathbf{MX}_B \\ \mathbf{MY}_B \\ \mathbf{MZ}_B \end{bmatrix}$.

We now give closed form expressions of terms contributed on the RHS by these forces and torques:

1. Forces at point O will contribute the following terms:

$$\left(\bar{F}_{er} + \bar{F}_{el} + \bar{F}_g \right) \cdot \frac{\partial \bar{v}_0}{\partial \dot{q}_j} \tag{35}$$

   where $\bar{v}_0 = \begin{bmatrix} \dot{x} & 0 & 0 \end{bmatrix}^T$

2. Torques at point O will contribute the following terms:

$$\left(\bar{r}_{G/O} \times \bar{F}_g + \bar{r}_{E_l/O} \times \bar{F}_{el} + \bar{r}_{E_r/O} \times \bar{F}_{er} - \left(\bar{\tau}_L + \bar{\tau}_R \right) \bar{j}_0 + \bar{\tau}_{er} + \bar{\tau}_{el} \right) \cdot \frac{\partial \bar{\omega}_B}{\partial \dot{q}_j} \tag{36}$$

   where $\bar{\omega}_B = -\dot{\phi} \bar{j}_0 + \dot{\psi} \bar{k}_0$

3. Torque on the wheel at L will contribute the following terms

$$\tau_L \bar{j}_0 \frac{\partial \bar{\omega}_L}{\partial \dot{q}_j} \tag{37}$$

   where $\bar{\omega}_L = \dot{\psi} \bar{k}_0 + \left(\frac{1}{R} \dot{x} - \frac{L}{2R} \dot{\psi} \right) \bar{j}_0$

4. Torque on the wheel at R will contribute the following terms

$$\tau_R \bar{j}_0 \cdot \frac{\partial \bar{\omega}_R}{\partial \dot{q}_j} \tag{38}$$

where $\bar{\omega}_R = \dot{\psi}\bar{k}_0 + \left(\frac{1}{R}\dot{x} + \frac{L}{2R}\dot{\psi}\right)\bar{j}_0$

The MATLAB script `kanesRHS.m` is used to evaluate the above closed-form expressions to give the final terms. Vectors `a`, `b`, `c` and `d` store the three terms produced by each of the four closed form expressions listed above. The result of the evaluation fives the following right hand sides:

eq1: $\frac{\tau_L}{R} + \frac{\tau_R}{R} + F_{lx} + F_{rx}$

eq2: $-\frac{L}{2R}\tau_L + \frac{L}{2R}\tau_R + \tau_{lz} + \tau_{rz} + r_{lx}F_{ly} - r_{ly}F_{lx} + r_{rx}F_{ry} - E_{ry}F_{rx}$

eq3: $\tau_L + \tau_R + g\left(\mathbf{MZ}_B cos q_i mu - \mathbf{MY}_B sin q_{imu}\right) - \tau_{ry} - \tau_{ly} + r_{lx}F_l z - r_{lz}F_{lx} + r_{rx}F_{rz} - r_{rz}F_{rx}$

A MATLAB code for Evaluation of Kane's Equaiton

A.1 Kane's LHS

This is the code listing for the evaluation of Kane's equation LHS (Eq. 13) for our robot:

```
%% Definitions:

% define frames:
% x0y0z0: Origin wheels mid, z0 always up, x0 always along heading
% x1y1z1: Origin on wheels mid, x1 along wheels connect (L->R), y1 along
% the base (at angle -q_imu from the x0 axis)

clear all
syms x psii phi dpsi dphi dx ddpsi ddphi ddx L g mw mb real
syms XXw XYw XZw YYw YZw ZZw real
syms XXb XYb XZb YYb YZb ZZb real
syms tau_R tau_L R qimu_plus_phi qimu real
syms MXb MYb MZb real

syms t X(t) PSI(t) PHI(t) dX dPSI dPHI ddX ddPSI ddPHI real
dX=diff(X,t);dPSI=diff(PSI,t);dPHI=diff(PHI,t);
ddX=diff(dX,t);ddPSI=diff(dPSI,t);ddPHI=diff(dPHI,t);
q = [x psii phi]'; dq = [dx dpsi dphi]'; ddq = [ddx ddpsi ddphi]';

mydiff = @(H) formula(subs(diff(symfun(subs(H,...
    [x,psii,phi,dx,dpsi,dphi,ddx,ddpsi,ddphi],...
    [X, PSI,PHI,dX,dPSI,dPHI,ddX,ddPSI,ddPHI]),t),t),...
    [X, PSI,PHI,dX,dPSI,dPHI,ddX,ddPSI,ddPHI],...
    [x,psii,phi,dx,dpsi,dphi,ddx,ddpsi,ddphi]));

q_imu = -phi + qimu_plus_phi; % qimu_plus_phi is a constant if pose fixed

%% Left Wheel


thetaL = x/R - psii*L/(2*R);
dthetaL = dx/R - dpsi*L/(2*R);
iL = [1 0 0]'; jL = [0 1 0]'; kL = [0 0 1]';
i0 = [cos(thetaL) 0 sin(thetaL)]'; j0 = [0 1 0]'; k0 = [-sin(thetaL) 0 cos(thetaL)]';
w0 = dpsi*k0;
v0 = dx*i0;
alpha0 = ddpsi*k0;
a0 = ddx*i0 + dx*(cross(dpsi*k0,i0));
r0L = (L/2)*j0;
Iw=[ZZw 0 0;0 YYw 0;0 0 ZZw];
wL = w0 + dthetaL*j0;
vGL = v0 + cross(w0, r0L);
aGL = a0 + cross(alpha0, r0L) + cross(w0, cross(w0, r0L));
HGL = Iw*wL;
p = mydiff(HGL);
dHGL = p + cross(wL,HGL);

% %% Right Wheel
```

```
thetaR = x/R + psii*L/(2*R);
dthetaR = dx/R + dpsi*L/(2*R);
iR = [1 0 0]'; jR = [0 1 0]'; kR = [0 0 1]';
i0 = [cos(thetaR) 0 sin(thetaR)]'; j0 = [0 1 0]'; k0 = [-sin(thetaR) 0 cos(thetaR)]';
w0 = dpsi*k0;
v0 = dx*i0;
alpha0 = ddpsi*k0;
a0 = ddx*i0 + dx*(cross(dpsi*k0,i0));
r0R = -(L/2)*j0;
wR = w0 + dthetaR*j0;
vGR = v0 + cross(w0, r0R);
aGR = a0 + cross(alpha0, r0R) + cross(w0, cross(w0, r0R));
HGR = Iw*wR;
dHGR = mydiff(HGR)+cross(wR,HGR);

%% Body

i1 = [1 0 0]'; j1 = [0 1 0]'; k1 = [0 0 1]';
i0 = [0 sin(q_imu) -cos(q_imu)]'; j0 = [-1 0 0]'; k0 = [0 cos(q_imu) sin(q_imu)]';
w0 = dpsi*k0;
v0 = dx*i0;
a0 = ddx*i0 + dx*(cross(dpsi*k0,i0));
IB=[XXb XYb XZb;XYb YYb YZb; XZb YZb ZZb];
wB = w0 + dphi*i1;
alphaB = ddpsi*k0 + ddphi*i1 + dphi*cross(wB, i1);
r0B = [MXb MYb MZb]'/mb;
vGB = v0 + cross(wB,r0B);
aGB = a0 + cross(alphaB, r0B) + cross(wB, cross(wB, r0B));
HGB = IB*wB;
dHGB = mydiff(HGB)+cross(wB,HGB);

%% Kanes LHS

KL = sym(zeros(3,1)); KR = sym(zeros(3,1)); KB = sym(zeros(3,1));
for i=1:3
    KL(i)=mw*aGL'*diff(vGL,dq(i))+dHGL'*diff(wL,dq(i));
    KR(i)=mw*aGR'*diff(vGR,dq(i))+dHGR'*diff(wR,dq(i));
    KB(i)=mb*aGB'*diff(vGB,dq(i))+dHGB'*diff(wB,dq(i));
end
Kw = KL + KR;
K = Kw + KB;

%% Potential Energy
gVec = [0 -g*cos(q_imu) -g*sin(q_imu)]';
V = gVec'*[MXb MYb MZb]';

%% Virtual Work

i0 = [1 0 0]'; j0 = [0 1 0]'; k0 = [0 0 1]';
dW=tau_L*j0'*(wL-dphi*j0)+tau_R*j0'*(wR-dphi*j0);

%% Substituting (qimu + phi) in place of qimu_plus_phi for simplification.
% We couldn't have done it before because we wanted q_imu to be
% differentiated properly because it is a function of phi

K = subs(K,qimu_plus_phi, qimu+phi);
V = subs(V,qimu_plus_phi, qimu+phi);
dW = subs(dW,qimu_plus_phi, qimu+phi);

%% Equations

AA = sym(zeros(3,3)); CC = sym(zeros(3,3));
QQ=sym(zeros(3,1)); Gamma=sym(zeros(3,1));
for i=1:3
    for j=1:3
        AA(i,j)=getcoeff(K(i),ddq(j),1);
        % This divides the coefficient of (dqj)(dqk) equally in all column
        % j and k
        CC(i,j)=getcoeff(K(i), dq(j),2)*dq(j);
        ccc = getcoeff(K(i),dq(j),1);
        CC(i,j) = CC(i,j)+ccc;
        for k=1:3
            CC(i,j) = CC(i,j) - 0.5*(getcoeff(ccc,dq(k),1))*dq(k);
        end
    end
    QQ(i) = diff(V,q(i));
    Gamma(i) = diff(dW,dq(i));
end

AA=simplify(AA);
CC=simplify(CC);

K=K-Gamma+QQ;

%% Comparing with kim

syms mS mC Iw1 Iw2 Iw3 I1 I2 I3 d real

Acheck=simplify(subs(AA,...
    [mb,mw,L,XXw,YYw,ZZw,MXb,MYb,MZb,XXb,YYb,ZZb,XYb,XZb,YZb],...
    [mS,mC,2*L,Iw2,Iw3,Iw2,0,mS*d*cos(qimu+phi),mS*d*sin(qimu+phi),I3,...
    I2*cos(qimu)^2 + I1*sin(qimu)^2,I1*cos(qimu)^2 + I2*sin(qimu)^2,0,0,...
```

```
    I2*cos(qimu)*sin(qimu)-I1*cos(qimu)*sin(qimu)]));

Ccheck=simplify(subs(CC,...
    [mb,mw,L,XXw,YYw,ZZw,MXb,MYb,MZb,XXb,YYb,ZZb,XYb,XZb,YZb],...
    [mS,mC,2*L,Iw2,Iw3,Iw2,0,mS*d*cos(qimu+phi),mS*d*sin(qimu+phi),I3,...
    I2*cos(qimu)^2 + I1*sin(qimu)^2,I1*cos(qimu)^2 + I2*sin(qimu)^2,0,0,...
    I2*cos(qimu)*sin(qimu)-I1*cos(qimu)*sin(qimu)]));
```

A.2 Evaluation of Expressions in [1]

This code is used to derive the final expressions from the closed form expressions in [1].

```
syms L R d phi dx dpsi dphi ddx ddpsi ddphi real
syms m_S m_C I_1 I_2 I_3 Iw_1 Iw_2 Iw_3 real

u_1 = dx; u_2 = dpsi; u_3 = dphi;
du_1 = ddx; du_2 = ddpsi; du_3 = ddphi;

w_S  = [0; u_2; u_3];
w_C1 = [0; u_2; -(1/R)*u_1+(L/R)*u_2];
w_C2 = [0; u_2; -(1/R)*u_1-(L/R)*u_2];

v_S  = [u_1-u_3*d*cos(phi); -u_3*d*sin(phi); u_2*d*sin(phi)];
v_C1 = [u_1-u_2*L; 0; 0];
v_C2 = [u_1+u_2*L; 0; 0];

alpha_S  = [0; du_2; du_3];
alpha_C1 = [-(1/R)*u_1*u_2+(L/R)*u_2^2; du_2; -(1/R)*du_1+(L/R)*du_2];
alpha_C2 = [-(1/R)*u_1*u_2-(L/R)*u_2^2; du_2; -(1/R)*du_1-(L/R)*du_2];

a_S = [du_1-du_3*d*cos(phi)+(u_2^2+u_3^2)*d*sin(phi); ...
    -du_3*d*sin(phi)-u_3^2*d*cos(phi); ...
    du_2*d*sin(phi)+u_2*u_3*d*cos(phi)];
a_C1 = [du_1-L*du_2; 0;  L*u_2^2];
a_C2 = [du_1+L*du_2; 0; -L*u_2^2];

I_S = [I_1 0 0; 0 I_2 0; 0 0 I_3];
I_C1 = [Iw_1 0 0; 0 Iw_2 0; 0 0 Iw_3];
I_C2 = I_C1;

T_S = -I_S*alpha_S-cross(w_S, I_S*w_S);
T_C1 = -I_C1*alpha_C1-cross(w_C1, I_C1*w_C1);
T_C2 = -I_C2*alpha_C2-cross(w_C2, I_C2*w_C2);

R_S = -m_S*a_S;
R_C1 = -m_C*a_C1;
R_C2 = -m_C*a_C2;


F_1 = diff(w_S,  u_1)'*T_S  + diff(v_S,  u_1)'*R_S ...
    + diff(w_C1, u_1)'*T_C1 + diff(v_C1, u_1)'*R_C1 ...
    + diff(w_C2, u_1)'*T_C2 + diff(v_C2, u_1)'*R_C2;
F_2 = diff(w_S,  u_2)'*T_S  + diff(v_S,  u_2)'*R_S ...
    + diff(w_C1, u_2)'*T_C1 + diff(v_C1, u_2)'*R_C1 ...
    + diff(w_C2, u_2)'*T_C2 + diff(v_C2, u_2)'*R_C2;
F_3 = diff(w_S,  u_3)'*T_S  + diff(v_S,  u_3)'*R_S ...
    + diff(w_C1, u_3)'*T_C1 + diff(v_C1, u_3)'*R_C1 ...
    + diff(w_C2, u_3)'*T_C2 + diff(v_C2, u_3)'*R_C2;
```

A.3 Evaluation of Expressions in [1] with Corrections

This code is used to derive the final expressions from the closed form expressions in [1] with corrected expression for ${}^{F}\mathbf{a}^{S^*}$ as mentioned in eq. 34.

```
syms L R d phi dx dpsi dphi ddx ddpsi ddphi real
syms m_S m_C I_1 I_2 I_3 Iw_1 Iw_2 Iw_3 real

u_1 = dx; u_2 = dpsi; u_3 = dphi;
du_1 = ddx; du_2 = ddpsi; du_3 = ddphi;

w_S  = [0; u_2; u_3];
w_C1 = [0; u_2; -(1/R)*u_1+(L/R)*u_2];
w_C2 = [0; u_2; -(1/R)*u_1-(L/R)*u_2];

v_S  = [u_1-u_3*d*cos(phi); -u_3*d*sin(phi); u_2*d*sin(phi)];
v_C1 = [u_1-u_2*L; 0; 0];
v_C2 = [u_1+u_2*L; 0; 0];

alpha_S  = [0; du_2; du_3];
alpha_C1 = [-(1/R)*u_1*u_2+(L/R)*u_2^2; du_2; -(1/R)*du_1+(L/R)*du_2];
alpha_C2 = [-(1/R)*u_1*u_2-(L/R)*u_2^2; du_2; -(1/R)*du_1-(L/R)*du_2];

% Correction done here
a_S = [du_1-du_3*d*cos(phi)+(u_2^2+u_3^2)*d*sin(phi); ...
    -du_3*d*sin(phi)-u_3^2*d*cos(phi); ...
```

```
    du_2*d*sin(phi)+2*u_2*u_3*d*cos(phi) - u_1*u_2]; %Here specifically
a_C1 = [du_1-L*du_2; 0;  L*u_2^2];
a_C2 = [du_1+L*du_2; 0; -L*u_2^2];

I_S = [I_1 0 0; 0 I_2 0; 0 0 I_3];
I_C1 = [Iw_1 0 0; 0 Iw_2 0; 0 0 Iw_3];
I_C2 = I_C1;

T_S = -I_S*alpha_S-cross(w_S, I_S*w_S);
T_C1 = -I_C1*alpha_C1-cross(w_C1, I_C1*w_C1);
T_C2 = -I_C2*alpha_C2-cross(w_C2, I_C2*w_C2);

R_S = -m_S*a_S;
R_C1 = -m_C*a_C1;
R_C2 = -m_C*a_C2;


F_1 = diff(w_S,  u_1)'*T_S  + diff(v_S,  u_1)'*R_S ...
    + diff(w_C1, u_1)'*T_C1 + diff(v_C1, u_1)'*R_C1 ...
    + diff(w_C2, u_1)'*T_C2 + diff(v_C2, u_1)'*R_C2;
F_2 = diff(w_S,  u_2)'*T_S  + diff(v_S,  u_2)'*R_S ...
    + diff(w_C1, u_2)'*T_C1 + diff(v_C1, u_2)'*R_C1 ...
    + diff(w_C2, u_2)'*T_C2 + diff(v_C2, u_2)'*R_C2;
F_3 = diff(w_S,  u_3)'*T_S  + diff(v_S,  u_3)'*R_S ...
    + diff(w_C1, u_3)'*T_C1 + diff(v_C1, u_3)'*R_C1 ...
    + diff(w_C2, u_3)'*T_C2 + diff(v_C2, u_3)'*R_C2;
```

A.4 Kane's RHS

This is the code listing for the evaluation of Kane's equation RHS (Eq. 13) for our robot:

```
syms Frx Fry Frz Flx Fly Flz mB g dx dpsi dphi MXb MYb MZb real
syms Erx Ery Erz Elx Ely Elz tau_l tau_r R L qimu real
syms Trx Try Trz Tlx Tly Tlz real

dq = [dx dpsi dphi]';
a = sym(zeros(3,1)); b = sym(zeros(3,1));
c = sym(zeros(3,1)); d = sym(zeros(3,1));

%Forces at point O
i0 = [1 0 0]'; j0 = [0 1 0]'; k0 = [0 0 1]';
Fr = [Frx Fry Frz]'; Fl = [Flx Fly Flz]';
Fg = mB*[0 0 -g]';
v0 = [dx 0 0]';

for i=1:3
    a(i) = (Fr + Fl + Fg)'*diff(v0, dq(i));
end

%Torques at point O
i1=[0 -1 0]'; j1=[sin(qimu) 0 cos(qimu)]'; k1=[-cos(qimu) 0 sin(qimu)]';
rOG = (MXb*i1 + MYb*j1 + MZb*k1)/mB;
Er = [Erx Ery Erz]'; El = [Elx Ely Elz]';
wB = [0 -dphi dpsi]';
Tau_r = [0 tau_r 0]'; %this applies to the wheel. sign to be reversed for point O becuse it is on the body
Tau_l = [0 tau_l 0]'; %this applies to the wheel. sign to be reversed for point O becuse it is on the body
Tr = [Trx Try Trz]';
Tl = [Tlx Tly Tlz]';

for i=1:3
    b(i) = (cross(rOG,Fg) + cross(Er,Fr) + cross(El,Fl) ...
       - Tau_r - Tau_l + Tl + Tr)'*diff(wB,dq(i));
end

%Torques at point R

wR = dpsi*k0 + (dx/R + (L*dpsi)/(2*R))*j0;

for i=1:3
    c(i) = Tau_r'*diff(wR,dq(i));
end

%Torques at point L
wL = dpsi*k0 + (dx/R - (L*dpsi)/(2*R))*j0;

for i=1:3
    d(i) = Tau_l'*diff(wL,dq(i));
end
```

References

[1] Yeonhoon Kim, Soo Hyun Kim, and Yoon Keun Kwak. Dynamic analysis of a nonholonomic two-wheeled inverted pendulum robot. *Journal of Intelligent*

and Robotic Systems, 44(1):25–46, 2005.