Equations Of Motion Of a Wheeled Inverted Pendulum

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In an earlier report [3] we had derived using Newton-Euler method the equations of motion of a wheeled inverted pendulum. But this model did not take into account the spin motion of the robot. Before attempting to derive the new equations including the spin, we will look at the equations derived in the existing literature.

1 Using the equation from [1]

The equations derived in [1] for the motion of two-wheeled inverted pendulum robot are:

$$3(m_c + m_s)\ddot{x} - m_s d cos\phi\ddot{\phi} + m_s d sin\phi(\dot{\phi}^2 + \dot{\psi}^2) = -\frac{\alpha_3 + \beta_3}{R} \quad (1)$$

$$\left\{(3L^2 + 1/2R^2)m_c + m_s d^2 sin^2\phi + I_2\right\}\ddot{\psi} + m_s d^2 sin\phi cos\phi\dot{\psi}\dot{\phi} = \frac{L}{R}(\alpha_3 - \beta_3) \quad (2)$$

$$m_s d cos\phi\ddot{x} + (-m_s d^2 - I_3)\ddot{\phi} + m_s d^2 sin\phi cos\phi\dot{\phi}^2 + m_s g d sin\phi = \alpha_3 + \beta_3 \quad (3)$$

where,
$\dot{x}$ is the heading speed of the robot
ϕ is the rotation of the C.G. about n_3-directional
ψ is the heading angle (angle between n_1 and the world frame)
α_3 is the torque of the left wheel
β_3 is the torque of the right wheel
m_c is the mass of the wheel
m_s is the mass of the body
d is the distance between wheel axis to C.G.
R is the radius of the wheel
L is the half distance between wheels
I_2 is the n_2-directional rotational inertia of the body
I_3 is the n_3-directional rotational inertia of the body
n_2 is the unit vector pointing vertically upwards
n_3 is the unit vector pointing from the left wheel to the right wheel

The equations that we had derived in our earlier report [3] did not include the rotation about the vertical axis of the robot. One of the equations above represent that motion. We will try to first match the equations we had derived with the equations listed above as a sanity check. Then we will write down the thrid equation using the variables that we had used in our earlier report. Then we will attempt to derive the equation using Newton-Euler method. Equations from our earlier report are listed here:

$$[(m+M)r + I_w/r + \eta^2 I_m/r]\ddot{x} + (mrlcos\theta - \eta^2 I_m)\ddot{\theta} = K_f u - \tau_f + F_{ext} r cos\theta + mrl\dot{\theta}^2 sin\theta \tag{4}$$

$$(mlcos\theta - \eta^2 I_m/r)\ddot{x} + (ml^2 + I + \eta^2 I_m)\ddot{\theta} = -K_f u + \tau_f + F_{ext} l + mglsin\theta \tag{5}$$

where,
m is the mass of the body
M is the mass of the wheels
r is the radius of the wheel
I_w is the inertia of the wheel
η is the gear ratio of the motor
I_m is the motor inertia
$\dot{x}$ is the heading speed
l is the distance between wheel axis and the C.G.
θ is the rotation of the C.G. about the wheel axis
K_f is the torque to current ratio of the motor
τ_f is the frictional torque on the wheels
F_{ext} is the external force being applied at the C.G. perpendicular to l
I is the inertia of the robot about the wheel axis

Using the symbols used in equations 4-5, we re-write the equations 1-3:

$$3(M+m)\ddot{x} - mlcos\theta\ddot{\theta} + mlsin\theta(\dot{\theta}^2 + \dot{\psi}^2) = -\frac{K_f(u_1+u_2)}{r} \tag{6}$$

$$\left\{(3L^2 + 1/2r^2)M + ml^2 sin^2\theta + I_2\right\}\ddot{\psi} + ml^2 sin\theta cos\theta\dot{\psi}\dot{\theta} = \frac{L}{r}K_f(u_1 - u_2) \tag{7}$$

$$mlcos\theta\ddot{x} + (-ml^2 - I)\ddot{\theta} + ml^2 sin\theta cos\theta\dot{\theta}^2 + mglsin\theta = K_f(u_1+u_2) \tag{8}$$

where, we have retained the symbols for quantities that do not appear in equations 4-5 i.e. ψ and I_2 which represent the heading direction and the inertia about the vertical axis respectively.

In equations 6-8, we make the following observations:

(i) Equations 6 and 8 are the equivalents of the equations 4 and 5 respectively

(ii) The equations 6 and 8 ignore the effects of wheel inertia I_w, the motor inertia I_m, frictional torque τ_f at the wheel motor and the external force F_{ext}, all of which were considered in equations 4 and 5

(iii) The terms containing $\ddot{x}$ in equations 4 and 5 appear with opposite signs in equations 6 and 8

(iv) The term $mrlsin\theta\dot{\theta}^2$ of equation 4 appears as $mrlsin\theta(\dot{\theta}^2+\dot{\psi}^2)$ in equation 6

(v) Equation 4 does not contain the coefficient 3 with the $\ddot{x}$ term which appears in equation 6

(vi) Equation 5 does not contain the term $ml^2 sin\theta cos\theta\dot{\theta}^2$ which appears in equation 8

Point number (iii) may be explained by assuming that the two derivations assumed x increases in different directions. Point number (iv) may be explained by the fact that equations 4 and 5 assume constant heading direction i.e. $\dot{\psi} = 0$ But the last two points are not easy to explain. An interesting fact regarding point regarding (v) is that the paper [1] changes the term from $3(m_c + m_s)$ in the original equation that we cited aboce to $3m_c + m_s$ in the later equations of the same paper. It appears that the latter expression is more accurate and basically $3m_c$ represents the mass of three wheels of equal mass, one of which is a supporting wheel. The paper discusses supporting wheels at length, so it won't be surprise. That solves the mystery of the second last point. What remains now is to discuss the very last point. Since we see that there is a typo done in the earlier equation, we can expect this term to have a typo, in that it is missing a $\dot{\psi}$ term. If this was true we will safely assume that the reason this term isn't present in our earlier analysis (i.e. equations 4 and 5) is because we assumed constant heading direction i.e. $\dot{\psi} = 0$.

2 Using equations from [2]

In the book [2] following equations of motion are derived:

$$\left(M + 2M_w + m + 2\frac{I_w}{r^2}\right)\dot{v} + ml\ddot{\alpha}cos\alpha - ml\dot{\alpha}^2 sin\alpha = \frac{\tau_l}{r} + \frac{\tau_r}{r} + d_l + d_r \tag{9}$$

$$\left(I_p + 2\left(M_w + \frac{I_w}{r^2}\right)d^2\right)\dot{\omega} = 2d\left(\frac{\tau_l}{r} - \frac{\tau_r}{r} + d_l - d_r\right) \tag{10}$$

$$ml\dot{v}cos\alpha + \left(ml^2 + I_M\right)\ddot{\alpha} - mglsin\alpha = 0 \tag{11}$$

where,
M is the mass of the platform
M_w is the mass of one wheel
m is the mass of the robot
I_w is the inertia of one wheel
r is the radius of one wheel
v is the heading speed of the wheel
l is the distance from C.G. to wheel axis
α is the rotation of C.G. about the wheel axis
τ_l is the torque applied by left wheel motor
τ_r is the torque applied by right wheem motor
d_l is the external force acting on the left wheel
d_r is the external force acting on the right wheel
Now, replacing these variables with the ones we had used in [3], equations, 9-11 become:

$$\left(M_p + M + m + \frac{I_w}{r^2}\right)\ddot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta = \frac{K_f u_1}{r} + \frac{K_f u_2}{r} + d_l + d_r \tag{12}$$

$$\left(I_z + \left(M + \frac{I_w}{r^2}\right)L^2\right)\ddot{\psi} = 2L\left(\frac{K_f u_1}{r} - \frac{K_f u_2}{r} + d_l - d_r\right) \tag{13}$$

$$ml\ddot{x}cos\theta + \left(ml^2 + I\right)\ddot{\theta} - mglsin\theta = 0 \tag{14}$$

Following observation are made:

(i) Equations 12 and 14 are the equivalents of equations 4 and 5 respectively

(ii) Equation 13 represents spin

(iii) There is no difference between equation 12 and 4 except that (a) equation 4 considers motor inertia I_m while 12 does not and (b) eq 12 consider platform as different from the pendulum while krang has no such thing as a platform so the additional mass term M_p in equation 12 is not there in 4

(iv) There is no difference between equation 12 and 4 except that (a) equation 4 considers motor inertia I_m while 12 does not and (b) the effect of counter torque on the pendulum is not considered in the eq 12 as the pendulum does not experience the countertorque due to it being on a platform and not directly attached to the motor so we don't see any term on the right hand side of eq 12

(v) Equation 13 when compared with eq 7 that represented spin in the previous section we see that the coefficient of $\ddot{\psi}$ was a funtion of θ there but here it is not. The reason is that I_z term in eq 13 is actually a function of θ. That function is written over there but not here.

(vi) Also there is a $\dot{\psi}$ term in equation 7 that is not there in eq 13. It seems like equation 7 makes more sense as a non-zero $\dot{\theta}$ will introduce a corriolis force in the system that is apparently not taken into account by equation 13

3 Comparing [1] and [2]

It appears that the analysis done by [1] is more correct with regards to understanding of the dynamics, only that it seems to have introduced some typos and thus can't be trusted blindly. The analysis in [2] on the other hand is very cleanly explained and does not contain typos, it is weak in representation of all dynamic effects in the system. The way we will move forward is by using the expressions for velocities that are more completely derived in [1] and use the detailed procedures explained in [2] to come up with expressions of dynamics that are useful for our purposes.

4 Deriving the Dynamic Model for our robot

In [1] the expressions we have for the kinematics of the system are as follows:

Angular velocity of the body and the velocity at the center of gravity in the body are governed as follows:

$$\begin{aligned}
{}^F\omega^S &= u_2\mathbf{n}_2 + u_3\mathbf{n}_3 \\
{}^F\mathbf{v}^{S^C} &= u_1\mathbf{n}_1 \\
{}^F\mathbf{v}^{S^*} &= {}^F\mathbf{v}^{S^C} + {}^F\omega^S \times \mathbf{d} \\
&= (u_1 - u_3 d cos\phi)\mathbf{n}_1 - u_3 d sin\phi\mathbf{n}_2 + u_2 d sin\phi\mathbf{n}_3
\end{aligned} \tag{15}$$

The angular velocities of each wheel and the velocity at the center of each wheel are governed as follows:

$$\begin{aligned}
{}^F\omega^{C_1} &= \left(-\frac{1}{R}u_1 + \frac{L}{R}u_2\right)\mathbf{n}_3 + u_2\mathbf{n}_2 \\
{}^F\mathbf{v}^{C_1^*} &= (u_1 - u_2 L)\mathbf{n}_1
\end{aligned} \tag{16}$$

$$\begin{aligned}
{}^F\omega^{C_2} &= \left(-\frac{1}{R}u_1 - \frac{L}{R}u_2\right)\mathbf{n}_3 + u_2\mathbf{n}_2 \\
{}^F\mathbf{v}^{C_2^*} &= (u_1 - u_2 L)\mathbf{n}_1
\end{aligned} \tag{17}$$

The angular accelerations for each left and right wheel are governed as follows:

$$\begin{aligned}
{}^F\alpha^S &= \dot{u}_2\mathbf{n}_2 + \dot{u}_3\mathbf{n}_3 \\
{}^F\alpha^{C_1} &= \left(-\frac{1}{R}u_1u_2 + \frac{L}{R}u_2^2\right)\mathbf{n}_1 + \dot{u}_2\mathbf{n}_2 + \left(-\frac{1}{R}\dot{u}_1 + \frac{L}{R}\dot{u}_2\right)\mathbf{n}_3 \\
{}^F\alpha^{C_2} &= \left(-\frac{1}{R}u_1u_2 - \frac{L}{R}u_2^2\right)\mathbf{n}_1 + \dot{u}_2\mathbf{n}_2 + \left(-\frac{1}{R}\dot{u}_1 - \frac{L}{R}\dot{u}_2\right)\mathbf{n}_3
\end{aligned} \tag{18}$$

The acceleration of the robot's body and the acceleration at the center of each wheel are governed as follows:

$$
\begin{aligned}
{}^{F}\mathbf{a}^{S^*} &= \frac{d^F\mathbf{v}^{S^C}}{dt} + {}^{F}\alpha^{S} \times \overline{S^C S^*} + {}^{F}\omega^{S} \times \left({}^{F}\omega^{S} \times \overline{S^C S^*}\right) \\
&= \begin{cases} \dot{u}_1 - \dot{u}_3 d cos\phi + \left(u_2^2 + u_3^2\right) d sin\phi & \mathbf{n}_1 \\ -\dot{u}_3 d sin\phi - u_3^2 d cos\phi & \mathbf{n}_2 \\ +\dot{u}_2 d sin\phi + u_2 u_3 d cos\phi & \mathbf{n}_3 \end{cases} \\
{}^{F}\mathbf{a}^{C_1^*} &= \frac{d^F\mathbf{v}^{S^C}}{dt} + {}^{F}\alpha^{S_C} \times \overline{S^C C_1^*} + {}^{F}\omega^{S_C} \times \left({}^{F}\omega^{S_C} \times \overline{S^C C_1^*}\right) \\
&= \begin{cases} \dot{u}_1 - L\dot{u}_2 & \mathbf{n}_1 \\ +0 & \mathbf{n}_2 \\ +Lu_2^2 & \mathbf{n}_3 \end{cases} \\
{}^{F}\mathbf{a}^{C_2^*} &= \frac{d^F\mathbf{v}^{S^C}}{dt} + {}^{F}\alpha^{S_C} \times \overline{S^C C_2^*} + {}^{F}\omega^{S_C} \times \left({}^{F}\omega^{S_C} \times \overline{S^C C_2^*}\right) \\
&= \begin{cases} \dot{u}_1 + L\dot{u}_2 & \mathbf{n}_1 \\ +0 & \mathbf{n}_2 \\ -Lu_2^2 & \mathbf{n}_3 \end{cases}
\end{aligned}
\tag{19}
$$

where, ${}^{F}\omega^{S}$ u_1, u_2, u_3 $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ ${}^{F}\mathbf{v}^{S^C}$ ${}^{F}\mathbf{v}^{S^*}$ $\mathbf{d}$ ϕ ${}^{F}\omega^{C_1}$ L R ${}^{F}\mathbf{v}^{C_1^*}$ ${}^{F}\omega^{C_2}$ ${}^{F}\mathbf{v}^{C_2^*}$ ${}^{F}\alpha^{S}$ ${}^{F}\alpha^{C_1}$ ${}^{F}\alpha^{C_2}$ ${}^{F}\mathbf{a}^{S^*}$ ${}^{F}\mathbf{a}^{C_1^*}$ ${}^{F}\mathbf{a}^{C_2^*}$

References

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