Communications

Direct Calculation of Minimum Set of Inertial Parameters of Serial Robots

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Abstract—**The determination of the minimum set of inertial parameters of robots contributes to the reduction of the computational cost of the dynamic models and simplifies the identification of the inertial parameters. These parameters can be obtained from the classical inertial parameters by eliminating those that have no effect on the dynamic model and by regrouping some others.**

This paper presents a direct method of determining the minimum set of inertial parameters of serial robots. The given method permits determination of most of the regrouped parameters by means of closed-form relations.

I. Introduction

In order to increase the dynamic performance of robots, many control schemes based on the dynamic model have been presented [1]–[4]. Two problems have to be solved for real time implementation of these control laws:

1) The numerical values of the inertial parameters must be known. The solution of this problem has been investigated by the use of identification procedures based on a dynamic model linear in the inertial parameters [5]–[9]. The classification and the determination of the identifiable parameters (the minimum inertial parameters) increases the robustness of the identification process [8].

2) The computational cost of the dynamic model must be reduced. The solution of this problem has been carried out by the use of the customized Newton–Euler method [10], [11]. To increase the number of parameters that are equal to zero and, consequently, to reduce the number of operations of the dynamic model, the use of the minimum set of inertial parameters in a customized Newton–Euler method has been proposed [12], [13].

The determination of the minimum set of inertial parameters has been investigated in [12]–[15]:

Khosla [14] has used a symbolic Newton–Euler algorithm. Some results concerning some parameters in the case of robots whose successive axes are either parallel or perpendicular are given. The detection of most of the combined parameters requires a long examination of the symbolic Newton–Euler model on a case-by-case basis.

Khalil *et al.* [13], [15] have determined the minimum set of inertial parameters by the examination of the symbolic expressions of the inertia matrix and the gravity forces of a Lagrangian dynamic model. Although these expressions are generated automatically by computer [16], the detection of the combined inertial parameters may take several hours.

The aim of this paper is to present a general and direct method to determine the minimum set of inertial parameters of serial robots. Most of these parameters (or in most cases, all of them) will be determined by the use of the closed-form solution.

Coincidently with our work [17], similar results concerning the special case of rotational robots whose successive axes are perpendicular or parallel have been given by Mayeda *et al.* [18].

Manuscript received April 20, 1988; revised June 9, 1989.

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IEEE Log Number 8933674.

II. Description of the Robot

The system to be considered is an open-loop mechanism of $n + 1$ links, where link 0 is the base, whereas link n is the terminal link. The description of the system will be carried out by the use of the modified Denavit–Hartenberg notation [19], [20]. Thanks to this description, the closed-form solution of regrouping the inertial parameters has been obtained. The coordinate frame j is assigned fixed with respect to link j. The z_j axis is along the axis of joint j, and the x_j axis is along the common perpendicular of z_j and z_{j+1}. The frame j is defined with respect to frame $j-1$ as a function of the parameters $(\alpha_j, d_j, \theta_j, r_j)$. The type of joint j will be defined by σ_i, where $\sigma_j = 0$ for j rotational, $\sigma_j = 1$ for j translational, and $\bar{\sigma}_j = (1 - \sigma_j)$.

III. The Minimum Inertial Parameters

The minimum inertial parameters are defined as the minimum set of constant inertial parameters that do not contain the zero element and are sufficient to calculate the dynamic model of the robot. They can be obtained from the classical inertial parameters by eliminating those that have no effect on the dynamic model and by regrouping some other parameters.

A. The Classical Inertial Parameters

The inertial parameters of link j are given by the vector X^j, which is denoted as

$$X^j = [XX_j, XY_j, XZ_j, YY_j, YZ_j, ZZ_j, mX_j, mY_j, mZ_j, m_j]^T \quad (1)$$

where $(XX_j, \cdots, ZZ_j)$ are the elements of the inertia matrix ${}^j\boldsymbol{J}_j$, which defines the inertia of link j around the origin of frame j, (mX_j, mY_j, mZ_j) are the elements of ${}^j\boldsymbol{mS}_j$, which defines the first moments of link j, and m_j is the mass of link j.

Using this set of parameters and denoting E as the kinetic energy and U as the potential energy of the robot, we deduce that

$$E = \sum_{i=1}^{m} \frac{\partial E}{\partial X_i} X_i = \sum_{i=1}^{m} DE_i X_i \quad (2)$$

$$U = \sum_{i=1}^{m} \frac{\partial U}{\partial X_i} X_i = \sum_{i=1}^{m} (dU_i + W_i) X_i = \sum_{i=1}^{m} DU_i X_i \quad (3)$$

where m is equal to $10n$, X_i is an inertial parameter, and W_i is constant.

B. Parameters Having No Effect on the Dynamic Model

Using the Lagrange equation, one can derive that if

$$DE_i = 0 \quad \text{and} \quad dU_i = 0 \quad (4)$$

the corresponding inertial parameter X_i has no effect on the dynamic model. Thus, X_i is not an element of the minimum inertial parameters.

The inertial parameters satisfying these conditions belong to the links near the base side. They can be determined easily by hand or automatically by computer [16]. Supposing that r_1 is the first rotational joint and r_2 the first succeeding rotational joint not parallel to r_1, one can derive most of the parameters having no effect by the following rules:

1) If joint j is translational and $j < r_1$, then $XX_j, XY_j, XZ_j, YY_j, YZ_j, ZZ_j, mX_j, mY_j, mZ_j$ have no effect on the dy-

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namic model due to the fact that the angular velocity of link j is equal to zero.

2) If the axis of joint r_1 is parallel to the direction of gravity and the axes of joints $j < r_1$ are parallel to the axes of joint r_1, or if $r_1 = 1$, then mX_{r1}, mY_{r1} have no effect on the dynamic model. Moreover, in the case of $r_1 = 1$, m_1 has no effect on the dynamic model.

3) For a link j such that $r_1 \le j < r_2$ and the axis of link j is parallel to that of r_1, the parameters $XX_j, XY_j, XZ_j, YY_j, YZ_j, mZ_j$ have no effect on the dynamic model. The elimination of mX_j, mY_j, m_j must be studied on a case-by-case basis. The parameters to be eliminated if the axis of joint j is perpendicular to that of r_1 can be deduced easily.

4) For the links $j \ge r_2$, all the parameters of the inertia tensor have an effect on the dynamic model. Therefore, if the parameters mX_k, mY_k, mZ_k appear in the model, then all the parameters of links $j > k$ will appear in the dynamic model.

C. Conditions for Regrouping the Inertial Parameters

An inertial parameter X_i can be regrouped to some other parameters $X_{i1}, \cdots, X_{ir}$ if

$$DE_i = \alpha_{i1} DE_{i1} + \cdots + \alpha_{ir} DE_{ir} = \sum_{k=1}^{r} \alpha_{ik} DE_{ik} \tag{5a}$$

and

$$dU_i = \alpha_{i1}\, dU_{i1} + \cdots + \alpha_{ir}\, dU_{ir} = \sum_{k=1}^{r} \alpha_{ik}\, dU_{ik} \tag{5b}$$

with α_{ik} = constant; therefore, we get the same value of E and $\partial U/\partial q$ (and consequently, the dynamic model) if we use the parameters $X_i, X_{i1}, \cdots, X_{ir}$ or if we put X_i equal to zero and the parameters X_{ik} equal to XR_{ik} where

$$XR_{ik} = X_{ik} + \alpha_{ik} X_i. \tag{6}$$

Relation (5) is similar to that given in [13]. Its use by calculating DE_i and dU_i is time consuming and error prone, especially for robots having $n \ge 3$. In the following section, we present a method that overcomes this difficulty.

IV. Closed-Form Solution of Regrouping the Inertial Parameters

Let X_i^j be the ith inertial parameter of link j, with $i = 1, \cdots, 10$, and $j = 1, \cdots, n$. Denoting $\boldsymbol{DE}^j$ the vector of the components $\partial E/\partial X_i^j$ and $\boldsymbol{DU}^j$ the vector of the components $\partial U/\partial X_i^j$

$$\boldsymbol{DE}^j = \left[\frac{\partial E}{\partial X_1^j} \frac{\partial E}{\partial X_2^j} \cdots \frac{\partial E}{\partial X_{10}^j}\right]^T \tag{7}$$

$$\boldsymbol{DU}^j = \left[\frac{\partial U}{\partial X_1^j} \frac{\partial U}{\partial X_2^j} \cdots \frac{\partial U}{\partial X_{10}^j}\right]^T. \tag{8}$$

The main idea is to calculate $\boldsymbol{DE}^j$ and $\boldsymbol{DU}^j$ as a function of $\boldsymbol{DE}^{j-1}$ and $\boldsymbol{DU}^{j-1}$, respectively.

A. Recursive Relations of $\boldsymbol{DE}^j$ and $\boldsymbol{DU}^j$

The recursive relations of $\boldsymbol{DE}^j$ and $\boldsymbol{DU}^j$ can be written as

$$\boldsymbol{DE}^j = \boldsymbol{\lambda}^j \boldsymbol{DE}^{j-1} + \dot{q}_j \boldsymbol{f}^j(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{9}$$

and

$$\boldsymbol{DU}^j = \boldsymbol{\beta}^j \boldsymbol{DU}^{j-1} \tag{10}$$

or in the expanded form, we have

$$\frac{\partial E}{\partial X_i^j} = \sum_{k=1}^{10} \lambda_{i,k}^j \frac{\partial E}{\partial X_k^{j-1}} + \dot{q}_j f_i^j(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{11}$$

$$\frac{\partial U}{\partial X_i^j} = \sum_{k=1}^{10} \beta_{i,k}^j \frac{\partial U}{\partial X_k^{j-1}}. \tag{12}$$

As $\partial U/\partial X_k^{j-1} = 0$ for $k = 1, \cdots, 6$, and taking into account that (see the Appendix) $\beta_{i,k}^j = \lambda_{i,k}^j$ $(k = 7, \cdots, 10)$, (12) can be written as

$$\frac{\partial U}{\partial X_i^j} = \sum_{k=7}^{10} \lambda_{i,k}^j \frac{\partial U}{\partial X_k^{j-1}} = \sum_{k=1}^{10} \lambda_{i,k}^j \frac{\partial U}{\partial X_k^{j-1}}. \tag{13}$$

The expressions of $\boldsymbol{\lambda}^j$ and $\boldsymbol{f}^j$ are given in the Appendix.

B. General Regrouping Relations of Inertial Parameters

Let us examine the regrouping of the inertial parameters of link j on those of link $j-1$ by the use of the recursive relations of $\boldsymbol{\lambda}^j$ and $\boldsymbol{f}^j$ given in the Appendix. We consider the following two cases:

1) If joint j is rotational, we see that

a) the coefficients $\lambda_{10,k}^j$ are constant for $k = 1, \cdots, 10$, whereas $f_{10}^j(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$. Thus, DE_{10}^j can be obtained as linear combinations of DE_k^{j-1}, i.e., DE_{10}^j verifies the condition (5a). Taking into account (13), (5b) is also verified. Thus, the tenth parameter m_j can be regrouped to the inertial parameters of link $j-1$. Using (6) and the expressions of $\lambda_{10,k}^j$ given in the Appendix, we get

$$\begin{aligned}
XXR_{j-1} &= XX_{j-1} + r_j^2 m_j \\
XYR_{j-1} &= XY_{j-1} + d_j r_j S\alpha_j m_j \\
XZR_{j-1} &= XZ_{j-1} - d_j r_j C\alpha_j m_j \\
YYR_{j-1} &= YY_{j-1} + (d_j^2 + r_j^2 C\alpha_j^2) m_j \\
YZR_{j-1} &= YZ_{j-1} + r_j^2 C\alpha_j S\alpha_j m_j \\
ZZR_{j-1} &= ZZ_{j-1} + (d_j^2 + r_j^2 S\alpha_j^2) m_j \\
mXR_{j-1} &= mX_{j-1} + d_j m_j \\
mYR_{j-1} &= mY_{j-1} - r_j S\alpha_j m_j \\
mZR_{j-1} &= mZ_{j-1} + r_j C\alpha_j m_j \\
mR_{j-1} &= m_{j-1} + m_j.
\end{aligned} \tag{14}$$

b) The coefficients $\lambda_{9,k}^j$ are constant for $k = 1, \cdots, 10$, whereas $f_9^j(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$. Similar to the foregoing case, the parameter mZ_j can be regrouped to the inertial parameters of link $j-1$.

c) The sum of DE_1^j and DE_4^j, corresponding to the parameters XX_j and YY_j, respectively, can be expressed through constant coefficients as a function of DE_k^{j-1}, $k = 1, \cdots, 10$. Thus, XX_j or YY_j can be regrouped to the parameters of link $j-1$ and YY_j or XX_j. We will always choose to regroup YY_j.

As a conclusion, if joint j is rotational, the parameters (YY_j, mZ_j, m_j) can be eliminated, whereas the combined parameters of links j and $j-1$ are given as

$$\begin{aligned}
XXR_j &= XX_j - YY_j \\
XXR_{j-1} &= XX_{j-1} + YY_j + 2r_j mZ_j + r_j^2 m_j \\
XYR_{j-1} &= XY_{j-1} + d_j S\alpha_j mZ_j + d_j r_j S\alpha_j m_j \\
XZR_{j-1} &= XZ_{j-1} - d_j C\alpha_j mZ_j - d_j r_j C\alpha_j m_j \\
YYR_{j-1} &= YY_{j-1} + C\alpha_j^2 YY_j + 2r_j C\alpha_j^2 mZ_j + (d_j^2 + r_j^2 C\alpha_j^2) m_j \\
YZR_{j-1} &= YZ_{j-1} + C\alpha_j S\alpha_j YY_j + 2r_j C\alpha_j S\alpha_j mZ_j \\
&\quad + r_j^2 C\alpha_j S\alpha_j m_j \\
ZZR_{j-1} &= ZZ_{j-1} + S\alpha_j^2 YY_j + 2r_j S\alpha_j^2 mZ_j + (d_j^2 + r_j^2 S\alpha_j^2) m_j \\
mXR_{j-1} &= mX_{j-1} + d_j m_j \\
mYR_{j-1} &= mY_{j-1} - S\alpha_j mZ_j - r_j S\alpha_j m_j \\
mZR_{j-1} &= mZ_{j-1} + C\alpha_j mZ_j + r_j C\alpha_j m_j \\
mR_{j-1} &= m_{j-1} + m_j.
\end{aligned} \tag{15}$$

2) If joint j is translational, we see that $\lambda_{i,k}^{j}$ are constant, whereas $f_i^j(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$ for $i = 1, \cdots, 6$ and $k = 1, \cdots, 10$. Thus, the parameters $XX_j, XY_j, XZ_j, YY_j, YZ_j, ZZ_j$ can be regrouped to the inertial parameters of link $j-1$ by the use of the following relations:

$$XXR_{j-1} = XX_{j-1} + C\theta_j^2 XX_j - 2CS\theta_j XY_j + S\theta_j^2 YY_j$$

$$\begin{aligned} XYR_{j-1} = XY_{j-1} &+ CS\theta_j C\alpha_j XX_j + (C\theta_j^2 - S\theta_j^2)C\alpha_j XY_j \\ &- C\theta_j S\alpha_j XZ_j - CS\theta_j C\alpha_j YY_j + S\theta_j S\alpha_j YZ_j \end{aligned}$$

$$\begin{aligned} XZR_{j-1} = XZ_{j-1} &+ CS\theta_j S\alpha_j XX_j + (C\theta_j^2 - S\theta_j^2)S\alpha_j XY_j \\ &+ C\theta_j C\alpha_j XZ_j - CS\theta_j S\alpha_j YY_j - S\theta_j C\alpha_j YZ_j \end{aligned}$$

$$\begin{aligned} YYR_{j-1} = YY_{j-1} &+ S\theta_j^2 C\alpha_j^2 XX_j + 2CS\theta_j C\alpha_j^2 XY_j \\ &- 2S\theta_j CS\alpha_j XZ_j + C\theta_j^2 C\alpha_j^2 YY_j \\ &- 2C\theta_j CS\alpha_j YZ_j + S\alpha_j^2 ZZ_j \end{aligned}$$

$$\begin{aligned} YZR_{j-1} = YZ_{j-1} &+ S\theta_j^2 CS\alpha_j XX_j + 2CS\theta_j CS\alpha_j XY_j \\ &+ S\theta_j(C\alpha_j^2 - S\alpha_j^2)XZ_j + C\theta_j^2 CS\alpha_j YY_j \\ &+ C\theta_j(C\alpha_j^2 - S\alpha_j^2)YZ_j - CS\alpha_j ZZ_j \end{aligned}$$

$$\begin{aligned} ZZR_{j-1} = ZZ_{j-1} &+ S\theta_j^2 S\alpha_j^2 XX_j + 2CS\theta_j S\alpha_j^2 XY_j \\ &+ 2S\theta_j CS\alpha_j XZ_j + C\theta_j^2 S\alpha_j^2 YY_j + 2C\theta_j CS\alpha_j YZ_j + C\alpha_j^2 ZZ \end{aligned} \tag{16}$$

where $CS(\cdot)$ is equal to $\cos(\cdot)\sin(\cdot)$.

Remarks: 1) We note that (16) corresponds to the following matrix equation:

$${}^{j-1}\boldsymbol{JR}_{j-1} = {}^{j-1}\boldsymbol{J}_{j-1} + {}^{j-1}\boldsymbol{A}_j {}^j\boldsymbol{J}_j {}^j\boldsymbol{A}_{j-1} \tag{17}$$

where ${}^{j-1}\boldsymbol{A}_j$ is the (3×3) matrix defining the orientation of frame j with respect to frame $j-1$.

This relation can be obtained directly by noting that the angular velocity of link j is equal to that of link $j-1$ if joint j is translational.

2) From the general regrouped results and taking into account the eliminated parameters of link 1, we deduce that the number of minimum inertial parameters is equal or less than

$$7n_r + 4n_t - 3 - \bar{\sigma}_1 \tag{18}$$

where n_r is the number of rotational joints $= \Sigma\bar{\sigma}_j$ and n_t is the number of translational joints $= \Sigma\sigma_j$.

C. Particular Regrouping of the Inertial Parameters

As we have seen in Section III-B, some of the inertial parameters of the first links may have no effect on the dynamic model; therefore, particular regrouping, other than that denoted in Section IV-B, may take place. The detection of these particular cases have to be studied on a case-by-case basis using (5). We find that the particular regrouping will concern only the parameters mX, mY, and mZ of the translational links lying between the joints r_1 and r_2 defined in Section III-B. For the general case, it is not easy to calculate this regrouping by a closed-form solution. If we consider that the joints between r_1 and r_2 are either parallel or perpendicular such that the angular velocity of the links referred to the link frame will have only one component different than zero, then the particular cases will take place if the axis of the joint is parallel to r_1; in this case, the parameter mZ_j has no effect on the model, whereas the parameters mX_j and mY_j will be regrouped to the parameters of link $j-1$. Two cases are considered.

1) If $S\alpha_j = 0$, then mZ_{j-1} has no effect on the dynamic model, and

$$ZZR_{j-1} = ZZ_{j-1} + 2d_j C\theta_j mX_j - 2d_j S\theta_j mY_j$$

$$mXR_{j-1} = mX_{j-1} + C\theta_j mX_j - S\theta_j mY_j$$

$$mYR_{j-1} = mY_{j-1} + S\theta_j C\alpha_j mX_j + C\theta_j C\alpha_j mY_j. \tag{19}$$

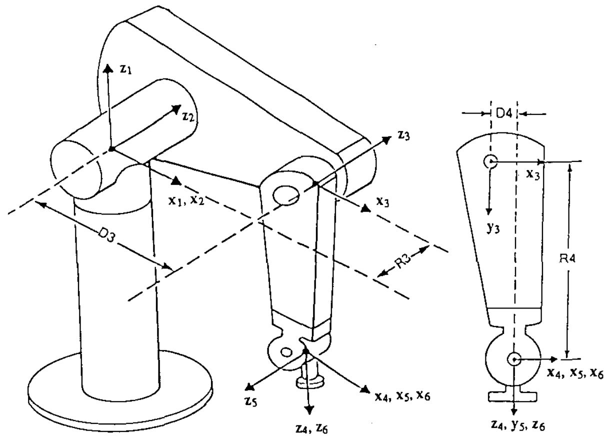

Fig. 1. PUMA robot.

TABLE I
GEOMETRIC PARAMETERS OF THE PUMA ROBOT

j	σ_j	α_j	d_j	θ_j	r_j
1	0	0	0	θ_1	0
2	0	-90	0	θ_2	0
3	0	0	D3	θ_3	R3
4	0	-90	D4	θ_4	R4
5	0	90	0	θ_5	0
6	0	-90	0	θ_6	0

2) If $C\alpha_j = 0$, then mY_{j-1} has no effect on the dynamic model, and

$$YYR_{j-1} = YY_{j-1} + 2d_j C\theta_j mX_j - 2d_j S\theta_j mY_j$$

$$mXR_{j-1} = MX_{j-1} + C\theta_j mX_j - S\theta_j mY_j$$

$$mZR_{j-1} = MZ_{j-1} + S\alpha_j S\theta_j mX_j + S\alpha_j C\theta_j mY_j. \tag{20}$$

V. Practical Utilization of the Regrouping Relations

The proposed algorithm of the detection of the set of minimum parameters will be carried out as follows:

1) Find the parameters having no effect on the dynamic model, as is given in Section III-B.

2) Apply (15) and (16) from link n to link 1.

3) Apply (5) to the parameters of translational links lying between r_1 and r_2 to detect the existence of supplementary regrouping. If these joints are parallel or perpendicular, we can use (19) and (20).

Example: In the following, we get the minimum inertial parameters of the six rotational joints of the well-known 560 PUMA robot (Fig. 1). The geometric parameters of the robot are given in Table I.

Step a): Using rules 2) and 3) given in Section III-B, the following parameters have no effect on the dynamic model:

$$XX_1, XY_1, XZ_1, YY_1, YZ_1, mX_1, mY_1, mZ_1, \text{ and } m_1.$$

Step b): Use (15) for links $6, \cdots, 1$:
link 6:

$$XXR_6 = XX_6 - YY_6$$

$$XXR_5 = XX_5 + YY_6$$

$$ZZR_5 = ZZ_5 + YY_6$$

$$mYR_5 = mY_5 + mZ_6$$

$$mR_5 = m_5 + m_6.$$

The minimum parameters of link 6 are $XXR_6, XY_6, XZ_6, YZ_6, ZZ_6, mX_6, mY_6$.

link 5:

$$XXR_5 = XX_5 + YY_6 - YY_5$$
$$XXR_4 = XX_4 + YY_5$$
$$ZZR_4 = ZZ_4 + YY_5$$
$$mYR_4 = mY_4 - mZ_5$$
$$mR_4 = m_4 + mR_5 = m_4 + m_5 + m_6.$$

The minimum parameters of link 5 are $XXR_5, XY_5, XZ_5, YZ_5, ZZR_5, mX_5, mYR_5$.

link 4:

$$XXR_4 = XX_4 + YY_5 - YY_4$$
$$XXR_3 = XX_3 + YY_4 + 2R4mZ_4 + R4^2(m_4 + m_5 + m_6)$$
$$XYR_3 = XY_3 - D4mZ_4 - D4R4(m_4 + m_5 + m_6)$$
$$YYR_3 = YY_3 + D4^2(m_4 + m_5 + m_6)$$
$$ZZR_3 = ZZ_3 + YY_4 + 2R4mZ_4 + (D4^2 + R4^2)(m_4 + m_5 + m_6)$$
$$mXR_3 = mX_3 + D4(m_4 + m_5 + m_6)$$
$$mYR_3 = mY_3 + mZ_4 + R_4(m_4 + m_5 + m_6)$$
$$mR_3 = m_3 + m_4 + m_5 + m_6.$$

The minimum parameters of link 4 are $XXR_4, XY_4, XZ_4, YZ_4, ZZR_4, mX_4, mYR_4$.

link 3:

$$XXR_3 = XX_3 + YY_4 + 2R4mZ_4 + (R4^2 - D4^2) \cdot (m_4 + m_5 + m_6) - YY_3$$
$$XXR_2 = XX_2 + YY_3 + 2R3mZ_3 + R3^2(m_3 + m_4 + m_5 + m_6) + D4^2(m_4 + m_5 + m_6)$$
$$XZR_2 = XZ_2 - D3mZ_3 - D3R3(m_3 + m_4 + m_5 + m_6)$$
$$YYR_2 = YY_2 + YY_3 + D4^2(m_4 + m_5 + m_6) + 2R3mZ_3 + (D3^2 + R3^2)(m_3 + m_4 + m_5 + m_6)$$
$$ZZR_2 = ZZ_2 + D3^2(m_3 + m_4 + m_5 + m_6)$$
$$mXR_2 = mX_2 + D3(m_3 + m_4 + m_5 + m_6)$$
$$mZR2 = mZ_2 + mZ_3 + R3(m_3 + m_4 + m_5 + m_6)$$
$$mR2 = m_2 + m_3 + m_4 + m_5 + m_6.$$

The minimum parameters of link 3 are $XXR_3, XYR_3, XZ_3, YZ_3, ZZR_3, mXR_3, mYR_3$.

link 2:

$$XXR_2 = XX_2 - YY_2 - D3^2(m_3 + m_4 + m_5 + m_6)$$
$$ZZR_1 = ZZ_1 + YY_2 + YYR3 + 2R3mZ_3 + (R3^2 + D3^2)(m_3 + m_4 + m_5 + m_6).$$

The regrouping relations on the parameters XX_1, XY_1, $XZ_1, YY_1, YZ_1, mX_1, mY_1, mZ_1, m_1$ are not written because these parameters have no effect on the dynamic model. The minimum parameters of link 2 are $XXR_2, XY_2, XZR_2, YZ_2, ZZR_2, mXR_2, mY_2$.

link 1:

The minimum parameter of link 1 is ZZR_1.

Since mZ_2 and m_2 do not appear on ZZR_1, they have no effect on the dynamic model.

Step c): This step does not have to be calculated since the robot has only rotational joints. The final results are summarized as follows:

1) The parameters having no effect on the dynamic model are

$$XX_1, XY_1, XZ_1, YY_1, YZ_1, mX_1, mY_1, mZ_1, m_1, mZ_2, m_2.$$

2) The parameters to be eliminated by regrouping are

$$YY_2, YY_3, mZ_3, m_3, YY_4, mZ_4, m_4, YY_5, mZ_5, m_5, YY_6, mZ_6, m_6.$$

VI. Application to the Identification and Control

The minimum set of inertial parameters are the only identifiable parameters by the use of the dynamic model [8], and they are only needed for the calculation of the control law. They represent, in the case of the given example, 36 parameters, i.e. 24 parameters less than the classical parameters. This will greatly facilitate the identification process. The regrouped relations may not be important in this case; only the knowledge of the parameters to be eliminated either by regrouping or because they have no effect is needed. The identification process would give the values of the identifiable parameters directly.

In many control schemes, such as computed torque, we need to calculate the dynamic model at the servo rate. The use of the minimum set of inertial parameters in an algorithm of customized Newton–Euler will contribute to reducing the cost of calculating the dynamic model [13].

VII. Conclusion

This paper presents a direct method of determining the minimum set of inertial parameters of serial robots. The given method permits determination of most of the regrouped parameters by means of closed-form relation function of the geometric parameters of the robot. The parameters that may need particular study concern the first moments of the translation links between r_1 and r_2. If the joints between r_1 and r_2 are parallel or perpendicular, we can obtain all of the minimum inertial parameters directly without calculating the dynamic model or the energy. The method is integrated to our software package of automatic symbolic modeling of robots SYMORO [16]. Generalization of this symbolic method to tree-structured robots and to robots with parallelogram closed loops are given in [22] and [23], respectively. Numerical approaches to determine the minimum inertial parameters are given in [24] and [25].

Appendix

Recursive Relations of the Derivatives of the Energy

In this Appendix, we present directly the results of the calculation of the derivatives of the energy with respect to the inertial parameters of link j in terms of the derivatives of the energy w.r.t. the inertial parameters of link $j-1$. The details can be found in [21].

A. Recursive Relation of $\boldsymbol{DE^j}$

We have the matrix relation

$$\boldsymbol{DE}^j = \boldsymbol{\lambda}^j \boldsymbol{DE}^{j-1} + \dot{q}_j \boldsymbol{f}^j \tag{A1}$$

where $\boldsymbol{\lambda}^j$ is a (10×10) matrix, and $\boldsymbol{f}^j$ is a (10×1) column matrix.

1. Expressions of λ^j: The matrix λ^j is given as

$$\lambda^j = \begin{bmatrix} \boldsymbol{DJ} & \boldsymbol{0}_{6,3} & \boldsymbol{0}_{6,1} \\ \boldsymbol{DS} & {}^j\boldsymbol{A}_{j-1} & \boldsymbol{0}_{3,1} \\ \boldsymbol{Dm} & {}^{j-1}\boldsymbol{P}_j^T & 1 \end{bmatrix} \tag{A2}$$

where

$\boldsymbol{0}_{m,n}$ is the $(m \times n)$ zero matrix:

${}^{i-1}\boldsymbol{A}_i$ is the (3×3) matrix defining the orientation of frame i with respect to frame $i-1$;

${}^{i-1}\boldsymbol{P}_i$ is the (3×1) vector defining the position of the origin of frame i with respect to frame $i-1$;

$\boldsymbol{DJ}, \boldsymbol{DS}, \boldsymbol{Dm}$, ${}^{j}A_{j-1}$, and ${}^{j-1}\boldsymbol{P}_j$ are functions of the geometric parameters $d_j, r_j, \theta_j, \alpha_j$. They are given as follows (the index j has been omitted to simplify the writing):

$$\boldsymbol{DJ} = \begin{bmatrix} CC\theta & CS\theta C\alpha & CS\theta S\alpha & SS\theta CC\alpha & SS\theta CS\alpha & SS\theta SS\alpha \\ -2CS\theta & (CC\theta - SS\theta)C\alpha & (CC\theta - SS\theta)S\alpha & 2CS\theta CC\alpha & 2CS\theta CS\alpha & 2CS\theta SS\alpha \\ 0 & -C\theta S\alpha & C\theta C\alpha & -2S\theta CS\alpha & S\theta(CC\alpha - SS\alpha) & 2S\theta CS\alpha \\ SS\theta & -CS\theta C\alpha & -CS\theta S\alpha & CC\theta CC\alpha & CC\theta CS\alpha & CC\theta SS\alpha \\ 0 & S\theta S\alpha & -S\theta C\alpha & -2C\theta CS\alpha & C\theta(CC\alpha - SS\alpha) & 2C\theta CS\alpha \\ 0 & 0 & 0 & SS\alpha & -CS\alpha & CC\alpha \end{bmatrix} \quad \text{(A3)}$$

where $CS(\cdot)$ means $\cos(\cdot)\sin(\cdot)$, $CC(\cdot)$ means $\cos^2(\cdot)$, and $SS(\cdot)$ means $\sin^2(\cdot)$:

$$\boldsymbol{DS}^T = \begin{bmatrix} 0 & 0 & 2r \\ -dS\theta C\alpha + rC\theta S\alpha & -dC\theta C\alpha - rS\theta S\alpha & dS\alpha \\ -dS\theta S\alpha - rC\theta C\alpha & -dC\theta S\alpha + rS\theta C\alpha & -dC\alpha \\ 2(dC\theta + rS\theta CS\alpha) & 2(-dS\theta + rC\theta CS\alpha) & 2rCC\alpha \\ rS\theta(SS\alpha - CC\alpha) & rC\theta(SS\alpha - CC\alpha) & 2rCS\alpha \\ 2(dC\theta - rS\theta CS\alpha) & 2(-dS\theta - rC\theta CS\alpha) & 2rSS\alpha \end{bmatrix} \quad \text{(A4)}$$

$$\boldsymbol{Dm} = [r^2 \quad drS\alpha \quad -drC\alpha \quad d^2 + r^2CC\alpha \quad r^2CS\alpha \quad d^2 + r^2SS\alpha]. \quad \text{(A5)}$$

$${}^{j-1}\boldsymbol{P}_j^T = [d \quad -rs\alpha \quad rc\alpha] \quad \text{(A6)}$$

2) Expression of f^j: Two cases are considered:

a) For j rotational

$${}^{j}A_{j-1} = \begin{bmatrix} c\theta & c\alpha s\theta & s\alpha s\theta \\ -s\theta & c\alpha c\theta & s\alpha c\theta \\ 0 & -s\alpha & c\alpha \end{bmatrix} \quad \text{(A7)}$$

$$f^j = [0 \quad 0 \quad \omega_{1,j} \quad 0 \quad \omega_{2,j} \quad \left(\omega_{3,j} - \tfrac{1}{2}\dot{q}_j\right) \quad V_{2,j} - V_{1,j} \quad 0 \quad 0]^T. \quad \text{(A8)}$$

b) for j translational

$$f^j = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -\omega_{2,j} \quad \omega_{1,j} \quad 0 \quad \left(V_{3,j} - \tfrac{1}{2}\dot{q}_j\right)]^T \quad \text{(A9)}$$

where $\dot{q}_j$ is the velocity of the joint variable j, whereas $\omega_{k,j}$, and $V_{k,j}$ denotes the kth component of ${}^{j}\omega_j$ and ${}^{j}V_j$, respectively, and where ${}^{j}V_j$ is the velocity of the origin of the link j fixed frame, referred to frame j, and ${}^{j}\omega_j$ is the angular velocity of link j, referred to frame j.

3) Initialization of the recursive relations of $\boldsymbol{DE}^j$: Since link 0 is fixed, then we initialize the calculation by

$$\boldsymbol{DE}^0 = [0 \quad 0 \quad \cdots \quad 0]^T. \quad \text{(A10)}$$

B. Recursive Relations of $\boldsymbol{DU}^j$

We have the relation

$$\boldsymbol{DU}^j = \begin{bmatrix} \boldsymbol{0}_{6,6} & \boldsymbol{0}_{6,3} & \boldsymbol{0}_{6,1} \\ \boldsymbol{0}_{3,6} & {}^{j}\boldsymbol{A}_{j-1} & \boldsymbol{0}_{3,1} \\ \boldsymbol{0}_{1,6} & {}^{j-1}\boldsymbol{p}_j^T & 1 \end{bmatrix} \boldsymbol{DU}^{j-1}. \quad \text{(A11)}$$

We initialize the recursive relation by

$$\boldsymbol{DU}^0 = [\boldsymbol{0}_{1,6} \quad -\boldsymbol{g}^T \quad 0]^T \quad \text{(A12)}$$

where $\boldsymbol{g}$ is the acceleration of gravity.

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Robustness Analysis of Nonlinear Decoupling for Elastic-Joint Robots

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Abstract—**In this communication, a robustness analysis of feedback linearization or nonlinear decoupling is proposed for a class of elastic-joint robot manipulator control loops. It is shown that by splitting the modeling-error nonlinearities of incomplete feedback linearization or nonlinear decoupling by a suitable transformation, a robustness analysis becomes feasible under weak assumptions without the necessity of estimating bounds on derivatives of robot nonlinearities.**

I. Introduction

The robustness analysis for *rigid* robot manipulators, which is outlined in [1]–[5], is based on feedback linearization [6] or nonlinear decoupling [7], where the rigid robot model is always feedback linearizable. This may not be the case for *elastic-joint* robot models in general [8]; however, under assumptions outlined in [9] or [10], the conditions for feedback linearization, or equivalently, for the existence of a completely observable nonlinear decoupling, can always be fulfilled. Then, the globally linearized elastic-joint robot model can be represented by a chain of four integrators. In [11], the robustness of such nonlinear-decoupled robot control loops is discussed. It turns out that the robust design method in [1]–[5] for rigid robot models cannot be carried over to the elastic-joint case without major difficulties when the system description of the globally linearized system is used. Either bounds on derivatives of robot nonlinearities have to be estimated, as is also suggested in [9], or a loss of performance has to be tolerated when, for example, a compensation of gravitational forces is abandoned [10].

In this contribution, the results in [10] are modified and developed further. The idea is *not* to transform a given nth-order nonlinear system into a chain of n integrators for the robustness analysis but to compensate nonlinearities "in between" the integrators exactly where they physically appear. This means that the summation point of modeling error terms that contain derivatives is shifted beyond integrators such that derivatives in these terms vanish.

Feedback linearization or nonlinear decoupling applied to rigid robot models, which is equivalent to the well-known *computed-torque* method, yield control laws that contain no derivatives of robot nonlinearities. Thus, it is obviously not necessary to apply such a nonlinear transformation to the case of rigid robots; however, it may be advantageous for any system that has a control law derived from feedback linearization or nonlinear decoupling that contains derivatives of nonlinearities. As a result, the integrator chain will be partitioned into groups, where nonlinear decoupling is accomplished in stages. This transformation does not change the nature of the control law, but it is attractive for a robustness analysis, and it facilitates the controller design.

It is shown in this communication that a robustness analysis of feedback linearization, or equivalently, of nonlinear decoupling, becomes feasible under weak assumptions. The features of the transformed globally linearized system are twofold: First, derivatives of the control-loop nonlinearities due to model uncertainties vanish. Second, measurements of the external-joint accelerations and jerks that would be needed for control [9] are reconstructed by the proposed transformation and are not used for control. However, robust observers as proposed in [12]–[13] can still be used to overcome measurement difficulties [9].

This communication is organized as follows: In Section II, the control law of feedback linearization or nonlinear decoupling is stated for a class of elastic-joint robot models. The globally linearized system is then transformed in Section III for the robustness analysis, which is sketched in Section IV. The robustness condition to be used is an extension of those previously obtained for the "rigid case" in [4]. Finally, some remarks in Section V conclude this communication.

II. Nonlinear Decoupling for a Class of Elastic-Joint Robot Models

A class of robot manipulators that can be described by a set of linear and decoupled and a set of nonlinear and coupled differential equations is considered according to [9]–[11]:

$$u_M = N_g^{-1} M_G \ddot{q}_a + M_G \Delta\ddot{q} + D\Delta\dot{q} + C\Delta q \tag{1}$$

$$0 = M_R(q_a)\ddot{q}_a + c(q_a, \dot{q}_a) + g(q_a) - N_g^{-1}[D\Delta\dot{q} + C\Delta q] \tag{2}$$

where

u_M	vector of applied torques at the motor shaft
N_g	diagonal matrix with gear ratios $n_{gi} < 1$ $(i = 1, \cdots, n)$
M_G	diagonal matrix with constant motor inertia
q_a	external joint position vector
Δq	$= q_M - N_g^{-1} q_a$ vector of angular displacements at the motor shaft, where q_M is the vector of motor shaft positions
C and D	diagonal matrices of spring stiffness and damping coefficients, respectively, measured at the motor shaft
$M_R(q_a)$	nonlinear and coupled robot inertia matrix without any entries of motor mass
$c(q_a, \dot{q}_a)$	vector of centripetal, Coriolis, and frictional forces
$g(q_a)$	vector of gravitational forces.

The model description of (1) and (2) can be shown to include the rigid robot model as $C \to \infty$ or $D \to \infty$. It represents a class of elastic-joint robots under the assumptions stated in [9]–[10]. A major assumption is that the kinetic energy of the ith motor mass due to the motion of all jth robot links, where $j < i$, can be neglected $(i, j = 1, \cdots, n)$. As is shown by Spong [9], this assumption is easy to justify for a large class of robots that have high gear transmission rates. Furthermore, the gear elasticities of industrial robots are commonly weakly damped [14], and $D = 0$ is considered in the sequel.

Applying feedback linearization [6] or nonlinear decoupling [7] to the system of (1) and (2) for $D = 0$ leads to the control law

$$u_M = \hat{H}[q_d^{(4)} + v] + \hat{h} \tag{3}$$

where

$$\hat{H} = \hat{M}_G \hat{C}^{-1} N_g \hat{M}_R \tag{4}$$

$$\hat{h} = -\hat{H}\hat{\Psi} - \hat{P}(\hat{c} + \hat{g}) + (\hat{P} N_g^{-1} + I)\hat{C}\Delta q \tag{5}$$

Manuscript received December 19, 1988; revised September 18, 1989. This work was supported by the Volkswagen-Stiftung (1/61 394).

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IEEE Log Number 8933040.