FA1_Part2

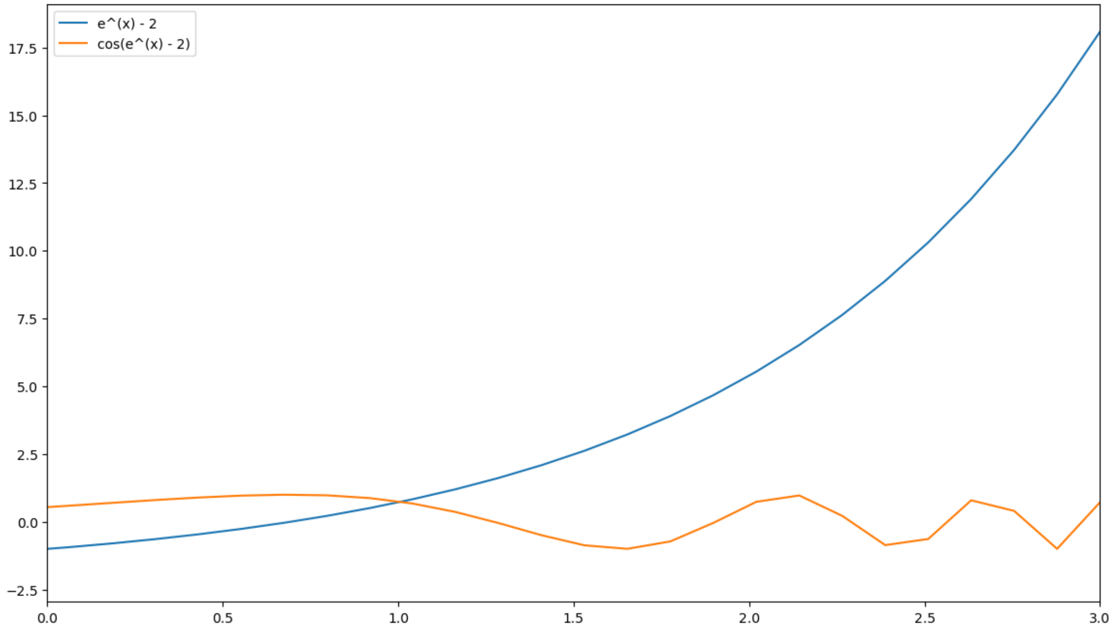
February 1, 2025

[1]:
```
import matplotlib.pyplot as plt
import numpy as np
import polars as pl
```

0.1 1.

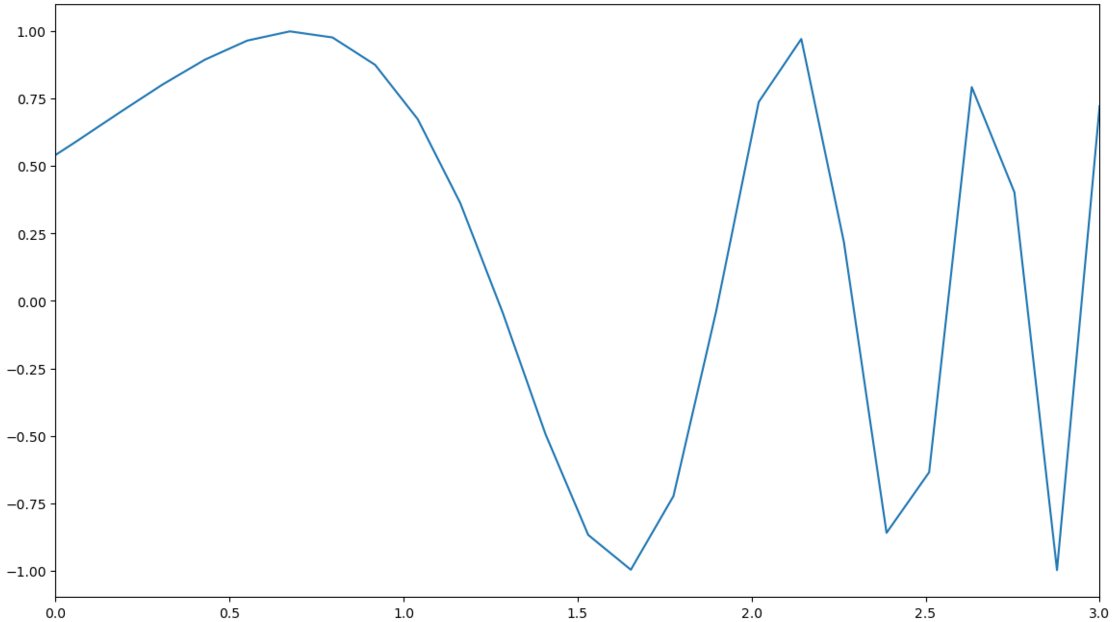
$$y = e^x - 2, y = cos(e^x - 2)$$

[2]:
```
# Creating vector for plotting
x = np.linspace(-3, 3, 50)

fig = plt.figure(figsize = (14, 8))

y1 = np.exp(x)-2
plt.plot(x, y1, label ='e^(x) - 2')

y2 = np.cos(np.exp(x)-2)
plt.plot(x, y2, label ='cos(e^(x) - 2)')

# Add features to our figure
plt.legend()
plt.xlim([0, 3])

# show plot
plt.show()
```



[3]:

```
# plotting e^(x)-2 = cos( e^(x)-2 )

# Creating vector for plotting
x = np.linspace(-3, 3, 50)

fig_next = plt.figure(figsize = (14, 8))

y3 = (np.exp(x)-2)-np.cos(np.exp(x)-2)

plt.plot(x, y2, label ='e^(x) - 2 = cos(e^(x) - 2)')
plt.xlim([0,3])

plt.show()
```



let $a_n = 0.75, b_n = 1.25$

from the theorem:

$$|p - p_n| \leq \frac{b_n - a_n}{2^n}$$

We can calculate n from:

$$\frac{1.25 - 0.75}{2^n} < 10^{-5}$$
$$\Rightarrow 2^{-n} < \frac{10^{-5}}{0.5}$$
$$\Rightarrow -n = \frac{-5 - \log 0.5}{\log 2}$$

Solving for how many iterations, or n, is needed to achieve an approximation with an error within 10^{-5} yields:

[4]:
```
n_approx = -(-5 - np.log10(0.5))/np.log10(2)
print(n_approx)
```

```
15.609640474436812
```

Then, we can set n to 16

[5]:
```
# set n to 16, a1 = 0.75, b1=1.25
iterations = 16
an = 0.75
bn = 1.25
```

[6]:
```
def bisecting_iter_num1(iter, an, bn):

    # create dictionary to store iteration data
    val_table_dict = {'n': [], 'an': [], 'bn': [], 'pn': [],
                      'f(pn)': []}

    # starting loop
    i = 0

    while i <= iter:


        # setting n and pn for each iteration
        n = i+1
        pn = (an+bn)/2

        fpn = (np.exp(pn)-2)-np.cos(np.exp(pn)-2)

        val_table_dict['n'].append(n)
        val_table_dict['an'].append(an)
        val_table_dict['bn'].append(bn)
        val_table_dict['pn'].append(pn)
        val_table_dict['f(pn)'].append(fpn)

        # break the loop if solution found earlier than expected:

        if fpn == 0:
            break

        # compare signs

        fan = (np.exp(an)-2)-np.cos(np.exp(an)-2)
        fbn = (np.exp(bn)-2)-np.cos(np.exp(bn)-2)

        if (fan>0 and fpn>0) or (fan<0 and fpn<0):
            an = pn
        elif (fbn>0 and fpn>0) or (fbn<0 and fpn<0):
            bn = pn

        i=i+1

    return val_table_dict
```

[7]:
```
df_val_table = pl.DataFrame(bisecting_iter_num1(iterations, an, bn))

with pl.Config(tbl_rows = 17):
    print(df_val_table)
```

```
shape: (17, 5)

 n    an        bn         pn         f(pn)
 ---  ---       ---        ---        ---
 i64  f64       f64        f64        f64

 1    0.75      1.25       1.0        -0.034656
 2    1.0       1.25       1.125      0.60908
 3    1.0       1.125      1.0625     0.266982
 4    1.0       1.0625     1.03125    0.111148
 5    1.0       1.03125    1.015625   0.037003
 6    1.0       1.015625   1.0078125  0.000864
 7    1.0       1.0078125  1.003906   -0.016973
 8    1.003906  1.0078125  1.005859   -0.008073
 9    1.005859  1.0078125  1.006836   -0.003609
 10   1.006836  1.0078125  1.007324   -0.001374
 11   1.007324  1.0078125  1.007568   -0.000255
 12   1.007568  1.0078125  1.00769    0.000305
 13   1.007568  1.00769    1.007629   0.000025
 14   1.007568  1.007629   1.007599   -0.000115
 15   1.007599  1.007629   1.007614   -0.000045
 16   1.007614  1.007629   1.007622   -0.00001
 17   1.007622  1.007629   1.007626   0.000007
```

[8]:
```
df_val_table.filter(pl.col('n') == 16).select([ 'pn', 'f(pn)'])
```

[8]:
```
shape: (1, 2)

 pn        f(pn)
 ---       ---
 f64       f64

 1.007622  -0.00001
```

$p_{16} = 1.007622$ yielded $f(p_{16}) = -0.00001$, an approximation with an accuracy within 10^{-5}

0.2 2.

$$x = \frac{5}{x^2} + 2$$

Earlier, it was established that the interval to be used is [2.25,3] as well as that the number of iterations needed to achieve an approximation accurate to within 10^{-5} is 17.

We let $p_0 = 2.25$

[9]:
```
# set preliminary parameters
```

```
iter_num2 = 17
p0 = 2.25
```

[10]:
```
# create function for iteration

def fixed_point_iter(iter, pn ):

    # create dictionary to store iteration data
    val_table_dict = {'n': [], 'pn': []}

    # starting loop
    i = 0

    while i <= iter:

        # get n, and pn+1
        n = i+1
        p = (5/(pn**2))+2

        # append
        val_table_dict['n'].append(n)
        val_table_dict['pn'].append(p)

        if (abs(p-pn))==0:
            break

        pn = p

        i=i+1

    return val_table_dict
```

[11]:
```
df_val_table_2 = pl.DataFrame(fixed_point_iter(iter_num2, p0))
with pl.Config(tbl_rows = 18):
    print(df_val_table_2)
```

```
shape: (18, 2)

 n     pn
 ---   ---
 i64   f64

 1     2.987654
 2     2.560156
 3     2.762846
 4     2.655023
 5     2.709306
```

```
6      2.681168
7      2.69554
8      2.688143
9      2.691935
10     2.689987
11     2.690987
12     2.690473
13     2.690737
14     2.690602
15     2.690671
16     2.690635
17     2.690654
18     2.690644
```

at n=17, $p_17 = 2.690654$. From our prior, calculations that 17 iterations are needed to get an approximation within 10^{-5}. As we have achieved that, we can be sure that $p_17 = 2.690654$ is an approximation of the fixed-point solution that is accurate within 10^{-5}.

1 3.

1.1 $\ln(x-1) + \cos(x-1) = 0$

Find approximations of the solution within an error of 10^{-5} using Newton's method, secant method, and false position method, for the interval [1.3, 2].

let f(x) = $\ln(x-1) + \cos(x-1)$, and let $f'(x) = \frac{1}{x-1} - \sin(x-1)$

[12]:
```python
# Define initial parameters

max_n = 50

p0_3 = 1.3
p1_3 = 2

# We'll only use p1_3 for the secant and false position method
```

[13]:
```python
def newton_iter(iter, pn):

    # create dictionary for table of values:

    val_table_dict = {'n': [], 'pn-1': [], 'pn': [], 'f(pn)': [], 'pn - pn-1':
 ↪[]}

    # starting loop
    i = 0

    while i <= iter:
```

```
        n = i+1
        p = pn - (np.log(abs(pn-1)) + np.cos(pn-1) )/ ( (1/(pn-1)) - np.
↪sin(pn-1) )

        fp = np.log(abs(p-1)) + np.cos(p-1)

        err = abs(p-pn)

        val_table_dict['n'].append(n)
        val_table_dict['pn'].append(p)
        val_table_dict['pn-1'].append(pn)
        val_table_dict['f(pn)'].append(fp)
        val_table_dict['pn - pn-1'].append(err)

        if fp == 0:
            break
        elif err < 10**(-5):
            break

        pn = p

        i=i+1

    return val_table_dict
```

[14]:
```
df_val_newton = pl.DataFrame(newton_iter(max_n, p0_3))
with pl.Config(tbl_rows = 4):
    print(df_val_newton)
```

```
shape: (4, 5)

 n     pn-1       pn         f(pn)        pn - pn-1
 ---   ---        ---        ---          ---
 i64   f64        f64        f64          f64

 1     1.3        1.381847   -0.034757    0.081847
 2     1.381847   1.397321   -0.00091     0.015474
 3     1.397321   1.397748   -6.6247e-7   0.000427
 4     1.397748   1.397748   -3.5116e-13  3.1148e-7
```

[15]:
```
# pn1 = p_{n-1}, pn2 = p_{n-2}, that is pn1 is the bigger number, or rightmost
def secant_iter(iter, pn1, pn2):

    # create dictionary for table of values:
```

```
    val_table_dict = {'n': [], 'pn-1': [], 'pn-2': [], 'pn': [], 'f(pn)': [],␣
↪'pn - pn-1': []}

    # starting loop
    i = 0

    while i <= iter:

        n = i+1

        fpn1 = np.log(abs(pn1-1)) + np.cos(pn1-1)
        fpn2 = np.log(abs(pn2-1)) + np.cos(pn2-1)

        pn = pn1 - (fpn1 * (pn1 - pn2))/(fpn1 - fpn2)

        fpn = np.log(abs(pn-1)) + np.cos(pn-1)

        err = abs(pn-pn1)

        val_table_dict['n'].append(n)
        val_table_dict['pn'].append(pn)
        val_table_dict['f(pn)'].append(fpn)
        val_table_dict['pn - pn-1'].append(err)
        val_table_dict['pn-1'].append(pn1)
        val_table_dict['pn-2'].append(pn2)

        if fpn == 0:
            break
        elif err < 10**(-5):
            break

        pn2 = pn1
        pn1 = pn

        i=i+1

    return val_table_dict
```

[16]:
```
df_val_secant = pl.DataFrame(secant_iter(max_n, p1_3, p0_3), strict=False)
with pl.Config(tbl_rows = 8):
    print(df_val_secant)
```

```
shape: (8, 6)

 n     pn-1        pn-2        pn          f(pn)          pn - pn-1
 ---   ---         ---         ---         ---            ---
 i64   f64         f64         f64         f64            f64
```

1	2.0	1.3	1.520607	0.214758	0.479393
2	1.520607	2.0	1.204358	-0.608692	0.316249
3	1.204358	1.520607	1.438128	0.080304	0.233771
4	1.438128	1.204358	1.410882	0.02732	0.027247
5	1.410882	1.438128	1.396833	-0.00195	0.014049
6	1.396833	1.410882	1.397769	0.000044	0.000936
7	1.397769	1.396833	1.397749	6.8013e-8	0.00002
8	1.397749	1.397769	1.397748	-2.3767e-12	3.1980e-8

[17]:
```
# pn1 = p_{n-1}, pn2 = p_{n-2}, that is pn1 is the bigger number, or rightmost
def false_position_iter(iter, pn1, pn2, pn3=0):

    # create dictionary for table of values:

    val_table_dict = {'n': [], 'pn-1': [], 'pn-2': [], 'pn-3': [], 'pn': [],␣
↪'f(pn)': [], 'pn - pn-1': []}

    # starting loop
    i = 0

    while i <= iter:

        n = i+1

        fpn1 = np.log(abs(pn1-1)) + np.cos(pn1-1)
        fpn2 = np.log(abs(pn2-1)) + np.cos(pn2-1)
        fpn3 = np.log(abs(pn3-1)) + np.cos(pn3-1)

        if (fpn1 * fpn2) < 0:
            pn = pn1 - (fpn1 * (pn1 - pn2))/(fpn1 - fpn2)
            checker = 0

        else:
            if (fpn1 * fpn3) < 0:
                pn = pn1 - (fpn1 * (pn1 - pn3))/(fpn1 - fpn3)
                checker = 1

        fpn = np.log(abs(pn-1)) + np.cos(pn-1)

        err = abs(pn-pn1)

        val_table_dict['n'].append(n)
        val_table_dict['pn'].append(pn)
        val_table_dict['f(pn)'].append(fpn)
        val_table_dict['pn - pn-1'].append(err)
        val_table_dict['pn-1'].append(pn1)
```

```
        val_table_dict['pn-2'].append(pn2)
        val_table_dict['pn-3'].append(pn3)

        if fpn == 0:
            break
        elif err < 10**(-5):
            break

        # pn3 becomes pn2, and pn2 becomes pn3, and pn3 = pn2 for next iter

        if checker == 0:
            pn3 = pn2
            pn2 = pn1
            pn1 = pn

        elif checker == 1:
            # pn3 = pn3
            # pn3 becomes pn2, and pn2 becomes pn3, and pn3 = pn2 for next iter
            pn2 = pn1
            pn1 = pn

        i=i+1

    return val_table_dict
```

[18]:
```
df_val_false_position = pl.DataFrame( false_position_iter(max_n, p1_3, p0_3),␣
 ↪strict=False)
with pl.Config(tbl_rows = 20):
    print(df_val_false_position)
```

```
shape: (8, 7)

 n     pn-1       pn-2       pn-3   pn         f(pn)       pn - pn-1
 ---   ---        ---        ---    ---        ---         ---
 i64   f64        f64        f64    f64        f64         f64

 1     2.0        1.3        0.0    1.520607   0.214758    0.479393
 2     1.520607   2.0        1.3    1.418368   0.042359    0.102239
 3     1.418368   1.520607   1.3    1.401138   0.007166    0.01723
 4     1.401138   1.418368   1.3    1.398304   0.001181    0.002833
 5     1.398304   1.401138   1.3    1.39784    0.000194    0.000465
 6     1.39784    1.398304   1.3    1.397763   0.000032    0.000076
 7     1.397763   1.39784    1.3    1.397751   0.000005    0.000012
 8     1.397751   1.397763   1.3    1.397749   8.5220e-7   0.000002
```

From the 3 attempts at approximating $\ln(x-1)+\cos(x-1)=0$, we get that \$p_n = 1.397749 \$, or $pn = 1.397749$, is an approximation of the root finding problem that has an accuracy within

10^{-5}.

Newton's method got the approximation at the shortest iterations (4). The Secant and False Position achieved the same answer, albeit at a slower 8 iterations for both methods.