

Algorithms and Data Structures Coursework

Questions 4-6

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1 Question 4

a) $f(x) + g(x)$ is $o(f(x) \times g(x))$

For a function to be little-o of another function,

$$\lim_{x \rightarrow \infty} \frac{a(x)}{b(x)} = 0$$

In our case,

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{f(x) + g(x)}{f(x)g(x)} \\ &= \lim_{x \rightarrow \infty} \left(\frac{f(x)}{f(x)g(x)} + \frac{g(x)}{f(x)g(x)} \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{1}{g(x)} + \frac{1}{f(x)} \right) \end{aligned}$$

This is not always equal to zero, for instance when $f(x) = 1$ it is equal to 1.

\therefore The statement is false.

b) $2^x \times x^2$ is $o(2.1^x)$

Considering the limit as x tends to infinity

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{2^x \times x^2}{2.1^x} \\ &= \lim_{x \rightarrow \infty} \left(\frac{2}{2.1} \right)^x \times x^2 \\ &= 0 \end{aligned}$$

\therefore The statement is true.

c) $x^2 \times \log x$ is $O(x^2)$

For $x > 0$, $\log x < x$

$$f(x) = x^2 \times \log x \leq x^2 \times x = x^3$$

In order for the statement to be true, $x^3 \leq C \times x^2$ for a constant C and $x > k$

$$x^3 \leq C \times x^2 \Rightarrow x \leq C$$

$\therefore C$ is not a constant so the statement is false.

d) $x^2 \times \log x$ is $O(x^3)$

$$f(x) = x^2 \times \log x$$

For $x > 0$, $\log x < x$

$$\therefore f(x) = x^2 \times \log x \leq x^2 \times x = x^3$$

For the statement to be true, $x^3 \leq C \times x^2$

$C = 1, k = 0$ is a valid witness pair that makes this true.

\therefore The statement is true.

e) $7x^5$ is $O(12x^4 + 5x^3 + 8)$

If $f(x) = O(12x^4 + 5x^3 + 8)$, then there must be a valid witness pair C and k such that

$$7x^5 \leq C(12x^4 + 5x^3 + 8) \text{ for } x \geq k$$

$$\frac{7x^5}{12x^4 + 5x^3 + 8} \leq C$$

C is not a constant here so the statement is false.

2 Question 5

a) $T(n) = 64T(n/8) - n^2 \times \log n$

$$\log_b a = 2, f(n) = n^2 \times \log n$$

This is case 2 of the master theorem. $\therefore T(n) = \theta(n^2 \times \log n)$

b) $T(n) = 4T(n/2) + \frac{n}{\log n}$

$$\log_b a = 2, f(n) = \frac{n}{\log n} = n \times \log^{-1} n$$

This is case 1 of the master theorem, so $T(n) = \theta(n^2)$

c) $T(n) = 2^n T(n/2) + n^n$

The master theorem cannot be applied here since coefficient of $T(n/2)$ is not a constant.

d) $T(n) = 3T(n/4) + n \times \log n$

$$\log_b a = 0.792, f(n) = n \times \log n$$

This is looking like case 3 of the master theorem, though we need to check the regularity condition:

$$\frac{3n}{4} \times \log n / 4 \leq cn \log n$$

This is true for $c = 1$.

$$\therefore T(n) = \theta(n \times \log n)$$

e) $T(n) = 3T(n/3) + n^{\frac{1}{2}}$

$$\log_b a = 1, f(n) = n^{\frac{1}{2}}$$

This is case 1 of the master theorem, $\therefore T(n) = \theta(n)$