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General Elasticity Problem

Question:

A 10 cm cube is made of a material with the following stiffness matrix:

$$\begin{bmatrix} 20 & -9 & 19 & 1 & -10 & 12 \\ -9 & 52 & 0 & -9 & 7 & 0 \\ 19 & 0 & 91 & 0 & 0 & 2 \\ 1 & -9 & 0 & 17 & 0 & -12 \\ -10 & 7 & 0 & 0 & 21 & 5 \\ 12 & 0 & 2 & -12 & 5 & 48 \end{bmatrix} \text{GPa}$$

a) What do each of the -9 GPa components do?

b) If the material is constrained to have no displacement in the y-direction anywhere but is free to move in the x and z directions, and it's two x-surfaces are each given a 5 MPa compressive stress, determine all the resulting axial and shear strains and the y-stress that arises in the centre of the material using Maple

c) Repeat part b using FlexPDE

d) Should the stress and strain be uniform? Why or why not?

Solution:

a) Using Voigt notation, we can set up our hooks law matrix to determine what each component in the stiffness matrix does. Since the -9 components are in the first row it will correspond to the axial stress in x and y in respect to the axial strain in x and y. C12 dot product with ε_{yy} will produce a component in σ_{xx} and C21 dot product ε_{xx} will produce a component in σ_{yy}.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$

b) The first step is to produce the stiffness matrix in maple

Maple Code:

```
restart;
with(LinearAlgebra):
C:=Matrix([
[20,-9,19,1,-10,12],
[-9,52,0,-9,7,0],
[19,0,91,0,0,2],
[1,-9,0,17,0,-12],
```

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```
[-10,7,0,0,21,5],
[12,0,2,-12,5,48]]);
```

$$C := \begin{bmatrix} 20 & -9 & 19 & 1 & -10 & 12 \\ -9 & 52 & 0 & -9 & 7 & 0 \\ 19 & 0 & 91 & 0 & 0 & 2 \\ 1 & -9 & 0 & 17 & 0 & -12 \\ -10 & 7 & 0 & 0 & 21 & 5 \\ 12 & 0 & 2 & -12 & 5 & 48 \end{bmatrix}$$

We can then use the initial conditions for stress and strain given in the question to set up our hooks law matrix It is key to remember the units of strain.

```
S:=MatrixInverse(C): evalf(%):
Stress:=<-.005,sy,0,0,0,0>;
Strain:=<ex,0,ez,gyz,gxz,gxy>;
HookesLaw:=Stress=C.Strain;
solve([seq(lhs(HookesLaw)[n]=rhs(HookesLaw)[n],n=1..6)]);
```

$$Stress := \begin{bmatrix} -0.005 \\ sy \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Strain := \begin{bmatrix} ex \\ 0 \\ ez \\ gyz \\ gxz \\ gxy \end{bmatrix}$$

$$HookesLaw := \begin{bmatrix} -0.005 \\ sy \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 20\,ex + 19\,ez + gyz - 10\,gxz + 12\,gxy \\ -9\,ex - 9\,gyz + 7\,gxz \\ 19\,ex + 91\,ez + 2\,gxy \\ ex + 17\,gyz - 12\,gxy \\ -10\,ex + 21\,gxz + 5\,gxy \\ 12\,ex + 2\,ez - 12\,gyz + 5\,gxz + 48\,gxy \end{bmatrix}$$

Solving the left and right side of the matrix equation will give us the solution of all unknowns

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$$\{ex = -0.0008939676447,\ ez = 0.0001791263715,\ gxy = 0.0003424427197,\ gxz = -0.0005072328593,\ gyz = 0.0002943106048,\ sy = 0.001846283344\}$$

c) Now that maple has a solution to all stress and strain value we can use FlexPDE to confirm this. Since the matrix is symmetric we can assign the top right triangle of the matrix the constant values and equation all complementary values of the bottom triangle to the top triangle.

```
C11 =20e9
C12 = -9e9
C13 = 19e9
C14 =1e9
C15 = -10e9
C16 = 12e9
C22 = 52e9
C23 = 0
C24 = -9e9
C25 = 7e9
C26 = 0
C33 = 91e9
C34 = 0
C35 = 0
C36 = 2e9
C44 = 17e9
C45 = 0
C46 = -12e9
C55 = 21e9
C56 = 5e9
C66 = 48e9
C21=C12 !matrix is symmetric
C31=C13
C32=C23
C41=C14
C42=C24
C43=C34
C51=C15
C52=C25
C53=C35
C61=C16
C62=C26
C63=C36
C54=C45
C64=C46
C65=C56
```

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FlexPDE Code:

```
TITLE 'H11.2 hussam43' { the problem
identification }
COORDINATES cartesian3 { coordinate
system, 1D,2D,3D, etc }
VARIABLES { system variables }
 u
v
w
SELECT { method controls }
ngrid = 5
DEFINITIONS { parameter definitions
}
mag = 200
Lx = .1
Ly = .1
Lz = .1
sapplied=-5e6
!Matrix components
C11 =20e9
C12 = -9e9
C13 = 19e9
C14 =1e9
C15 = -10e9
C16 = 12e9
C22 = 52e9
C23 = 0
C24 = -9e9
C25 = 7e9
C26 = 0
C33 = 91e9
C34 = 0
C35 = 0
C36 = 2e9
C44 = 17e9
C45 = 0
C46 = -12e9
C55 = 21e9
C56 = 5e9
C66 = 48e9
C21=C12 !matrix is symmetric
C31=C13
C32=C23
C41=C14
C42=C24
C43=C34
C51=C15
C52=C25
C53=C35
C61=C16
C62=C26
C63=C36
C54=C45
C64=C46
C65=C56
!Strain definitions from
displacements
ex = dx(u)
ey = dy(v)
ez = dz(w)
gyz = dy(w) + dz(v)
gxz = dx(w) + dz(u)
gxy = dx(v) + dy(u)
!Hookes Law
sx = C11*ex + C12*ey + C13*ez +
C14*gyz + C15*gxz + C16*gxy
sy = C21*ex + C22*ey + C23*ez +
C24*gyz + C25*gxz + C26*gxy
sz = C31*ex + C32*ey + C33*ez +
C34*gyz + C35*gxz + C36*gxy
syz = C41*ex + C42*ey + C43*ez +
C44*gyz + C45*gxz + C46*gxy
sxz = C51*ex + C52*ey + C53*ez +
C54*gyz + C55*gxz + C56*gxy
sxy = C61*ex + C62*ey + C63*ez +
C64*gyz + C65*gxz + C66*gxy
EQUATIONS { PDE's, one for each
variable }
u: dx(sx) + dy(sxy) + dz(sxz) = 0
v: dx(sxy) + dy(sy) + dz(syz) = 0
w: dx(sxz) + dy(syz) + dz(sz) = 0
EXTRUSION
surface 'bottom' z = 0
surface 'top' z = Lz
BOUNDARIES { The domain definition }
surface 'bottom'
load(u) =0
value(v) = 0
load(w) = 0
surface 'top'
value(v) = 0
 REGION 1 { For each material region
}
 START(0,0) !y=0
load(u) =0
value(v) = 0
load(w) = 0
 LINE TO (Lx,0) !x = Lx
load(u) =sapplied
```

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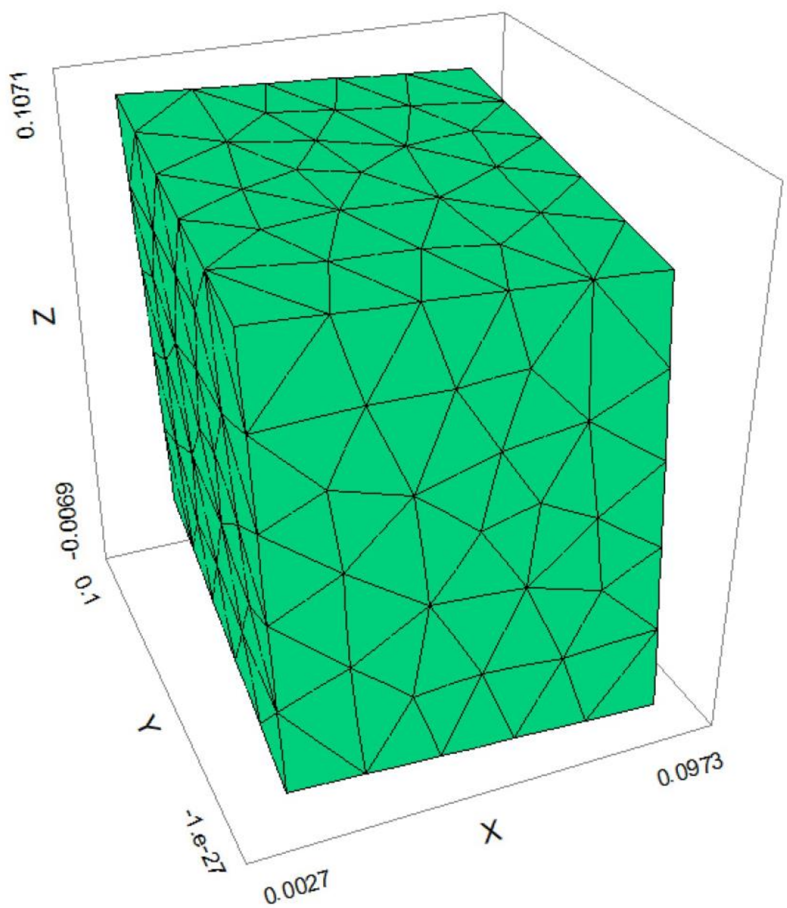
```
LINE TO (Lx,Ly) !y = Ly
load(u) =0
LINE TO (0,Ly) !x = 0
load(u) =-sapplied
!value(u) = 0
LINE TO CLOSE
PLOTS { save result displays }
grid(x+mag*u, y+mag*v, z+mag*w)
CONTOUR(sxz) on x = Lx/3
CONTOUR(v) on x = Lx/3
CONTOUR(u) on x = Lx/3
CONTOUR(ey) on x = Lx/3
SUMMARY
report val(sy, Lx/2,Ly/2,Lz/2)
report val(ex, Lx/2,Ly/2,Lz/2)
report val(ey, Lx/2,Ly/2,Lz/2)
report val(ez, Lx/2,Ly/2,Lz/2)
report val(gyz, Lx/2,Ly/2,Lz/2)
report val(gxz, Lx/2,Ly/2,Lz/2)
report val(gxy, Lx/2,Ly/2,Lz/2)
report val(sy, Lx*.8, Ly*.8, Lz*.8)
report val(ex, Lx*.8,Ly*.8,Lz*.8)
report val(ey, Lx*.8,Ly*.8,Lz*.8)
report val(ez, Lx*.8,Ly*.8,Lz*.8)
report val(gyz, Lx*.8,Ly*.8,Lz*.8)
report val(gxz, Lx*.8,Ly*.8,Lz*.8)
report val(gxy, Lx*.8,Ly*.8,Lz*.8)
report val(sapplied/ex, Lx,Ly,Lz) as
'Effective Stiffness in x'
report val(sapplied/ex,
Lx/2,Ly/2,Lz/2) as 'Effective
Stiffness'
report val(sapplied/ex,
Lx/2,.8*Ly,.8*Lz) as 'Effective
Stiffness'
report val(sapplied/ex,
Lx/2,.8*Ly,.4*Lz/2) as 'Effective
Stiffness'
END
```



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```
SUMMARY

val(sy, Lx/2,Ly/2,Lz/2)=  1846283.
val(ex, Lx/2,Ly/2,Lz/2)= -8.939677e-4
val(ey, Lx/2,Ly/2,Lz/2)= -2.752959e-12
val(ez, Lx/2,Ly/2,Lz/2)=  1.791264e-4
val(gyz, Lx/2,Ly/2,Lz/2)=  2.943106e-4
val(gxz, Lx/2,Ly/2,Lz/2)= -5.072329e-4
val(gxy, Lx/2,Ly/2,Lz/2)=  3.424427e-4
val(sy, Lx*.8, Ly*.8, Lz*.8)=  1846283.
val(ex, Lx*.8,Ly*.8,Lz*.8)= -8.939676e-4
val(ey, Lx*.8,Ly*.8,Lz*.8)= -3.621262e-13
val(ez, Lx*.8,Ly*.8,Lz*.8)=  1.791264e-4
val(gyz, Lx*.8,Ly*.8,Lz*.8)=  2.943106e-4
val(gxz, Lx*.8,Ly*.8,Lz*.8)= -5.072329e-4
val(gxy, Lx*.8,Ly*.8,Lz*.8)=  3.424427e-4
Effective Stiffness in x=  5.593044e+9
Effective Stiffness=  5.593043e+9
Effective Stiffness=  5.593043e+9
Effective Stiffness=  5.593043e+9
```

The output of Flex agrees with maple therefore we know steps in maple were correct.

d) Since the material is fixed in y and there is strain in x, these initial conditions allow for the shape to deform uniformly. This can also be seen using the contour plots produced in FlexPDE.