Beam Resonance Problem

Question:

A special 2 m long beam is composed of three materials; Aluminum bronze (E = 120 GPa, v = 0.316, density = 7640 kg/m^3), Gold (E = 74 GPa, v = 0.415, density = 19320 kg/m^3) and Tin (E = 47 GPa, v = 0.36, density = 7310 kg/m^3) all comprising 1/3rd of the beams thickness. The beam is 0.4 m wide and 0.2 m thick. The beam is suspended such that its left end is supported by a pin whereas its right end is fixed. This only allows for rotation about an axis parallel to the width direction axis.

a) Determine the flexural rigidity of the beam for bending in the direction about an axis parallel to the width direction axis.
b) Use Maple to determine the first three resonant frequencies in the thickness direction. Make a plot of what the mode shapes (displacement vs. x-position) look like
c) Use FlexPDE to find *any one* of the first three resonant modes & frequencies this beam has in the thickness direction. Show a deformed shape plot and state the resonant frequency you found correct to the nearest 1 rad/s

Solution:

a) We start off our question by defining the thickness varying material properties as piecewise functions in Maple. We can then proceed to find Zbar and in turn our flexural rigidity EI.

Maple Code:

```
restart:
L:=2: B:=.4: h:=.2:
E:=piecewise(z<h*0.3333, 120e9, z<h*0.6666, 74e9, z>h*0.6666, 47e9):
rho:=piecewise(z<h*0.3333, 7640, z<h*0.6666, 19320, z>h*0.6666, 7310):
mu:= int(rho, z=0..h)*B;
Zbar:=int(int(z*E, z=0..h), y=0..B)/int(int(E, z=0..h),y=0..B);
EI:=int(int(E*(z-Zbar)^2, z=0..h), y=0..B);
```

$$\mu := 913.833760$$

$$Zbar := 0.07980541398$$

$$EI := 1.955172042\ 10^{7}$$

b) Using the dynamic beam equation, ,$\frac{\partial^2 \mathrm{v}}{\partial t^2} = -\frac{\mathrm{EI}}{\mu}\frac{\partial^4 \mathrm{v}}{\partial z^4}$ and knowing that $v(z,t) = \hat{v}(z)\ cos(\omega t + \phi)$, we can obtain the simplified equation $\omega^2\hat{\mathrm{v}}(\mathrm{z}) = \frac{\mathrm{EI}}{\mu}\ \hat{\mathrm{v}}''''(\mathrm{z})$. The solution to this $4^{th} order$ differential equation is $\hat{v}(z) = c1cos\beta\ z + c2sin\beta\ z + c3cosh\beta\ z + c4\ sinh\beta\ z$, where $\omega = \beta^2\sqrt{\frac{EI}{\mu}}$. Since the beam is supported by a pin at z=0, the BC's will be: $\hat{v}(0) = 0$, $\hat{v}''(0) = 0$ and $\hat{v}(Lz) = 0$, $\hat{v}''(Lz) = 0$. We know that by solving this, we will obtain an infinite number of values for beta that satisfy the equation and each value corresponds to a different mode. We need to find the first three resonant frequencies in the thickness direction and plot their mode shapes. We first solve for our first three beta values and then by substituting the beta values found previously into the original differential equation, we can plot all the modes.

Maple Code:

```
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rho:=piecewise(z<h*0.3333, 7640, z<h*0.6666, 19320, z>h*0.6666, 7310):
mu:=int(rho, z=0..h)*B;
Zbar:=int(int(z*E, z=0..h), y=0..B)/int(int(E, z=0..h),y=0..B);
EI:=int(int(E*(z-Zbar)^2, z=0..h), y=0..B);
bv:=c1*cos(beta*z)+c2*sin(beta*z)+c3*cosh(beta*z)+c4*sinh(beta*z);
bvp:=diff(bv,z);
bvpp:=diff(bvp,z);
bvppp:=diff(bvpp,z);
c1:=solve(simplify(subs(z=0, bv)), c1);
simplify(subs(z=0, bvpp));
c3:=0:
bv;
c4:=solve(subs(z=L, bv), c4);
bv;
c2:=1:
GenFn:=simplify(subs(z=L, bvp/beta));
plot(GenFn, beta=0..8);
beta1:=fsolve(GenFn, beta=1..3);
beta2:=fsolve(GenFn, beta=3..5);
beta3:=fsolve(GenFn, beta=5..7);
bv1:=subs(beta=beta1, bv):
bv2:=subs(beta=beta2, bv):
bv3:=subs(beta=beta3, bv):
plot([bv1, bv2, bv3], z=0..L, legend=['FirstMode', 'SecondMode', 'ThirdMode']);
omega1:=beta1^2*sqrt(EI/mu);
omega2:=beta2^2*sqrt(EI/mu);
omega3:=beta3^2*sqrt(EI/mu);
```

$$bv := c1\cos(\beta z) + c2\sin(\beta z) + c3\cosh(\beta z) + c4\sinh(\beta z)$$

$$c1 := -c3$$

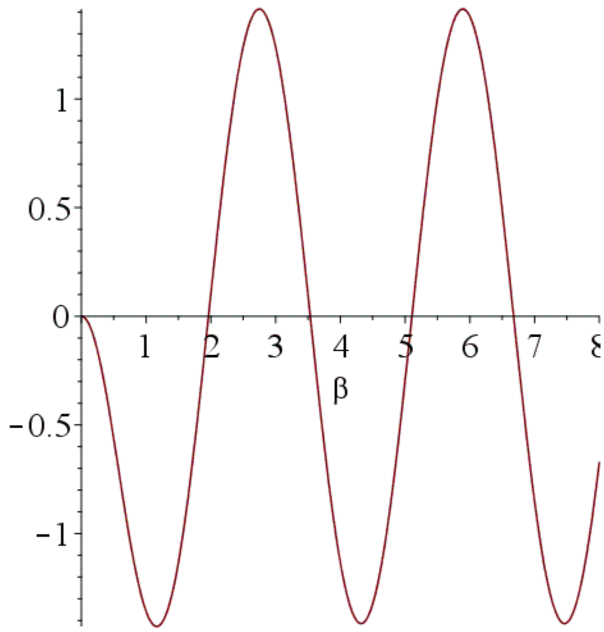
$$2\, c3\, \beta^2$$

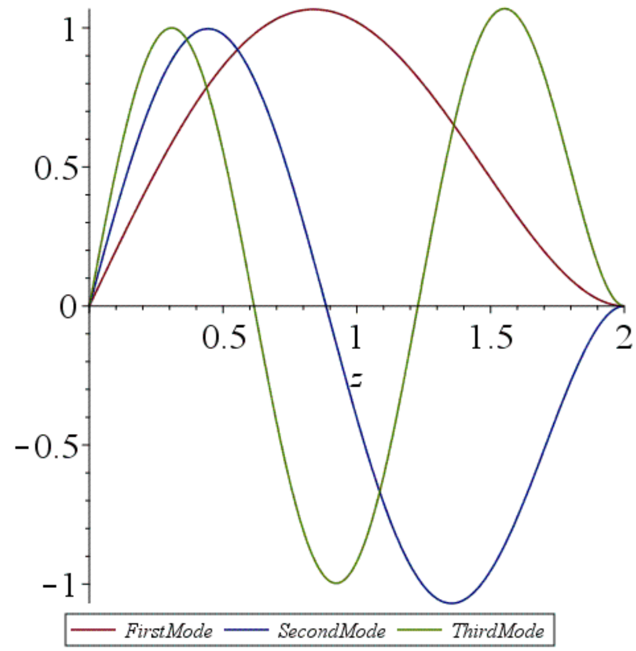
$$c2\sin(\beta z) + c4\sinh(\beta z)$$

$$c4 := -\frac{c2 \sin(2\beta)}{\sinh(2\beta)}$$

$$c2 \sin(\beta z) - \frac{c2 \sin(2\beta) \sinh(\beta z)}{\sinh(2\beta)}$$

$$GenFn := \frac{1}{\sinh(2\beta)} (\cos(2\beta) \sinh(2\beta) - \cosh(2\beta) \sin(2\beta))$$





$\beta 1 := 1.963301156$

$\beta 2 := 3.534291373$

$\beta 3 := 5.105088061$

$\omega 1 := 563.8099486$

$\omega 2 := 1827.105359$

$\omega 3 := 3812.109492$

c) We will now use FlexPDE to find our first resonant mode and its plot along with its corresponding angular frequency to confirm our answer obtained with Maple

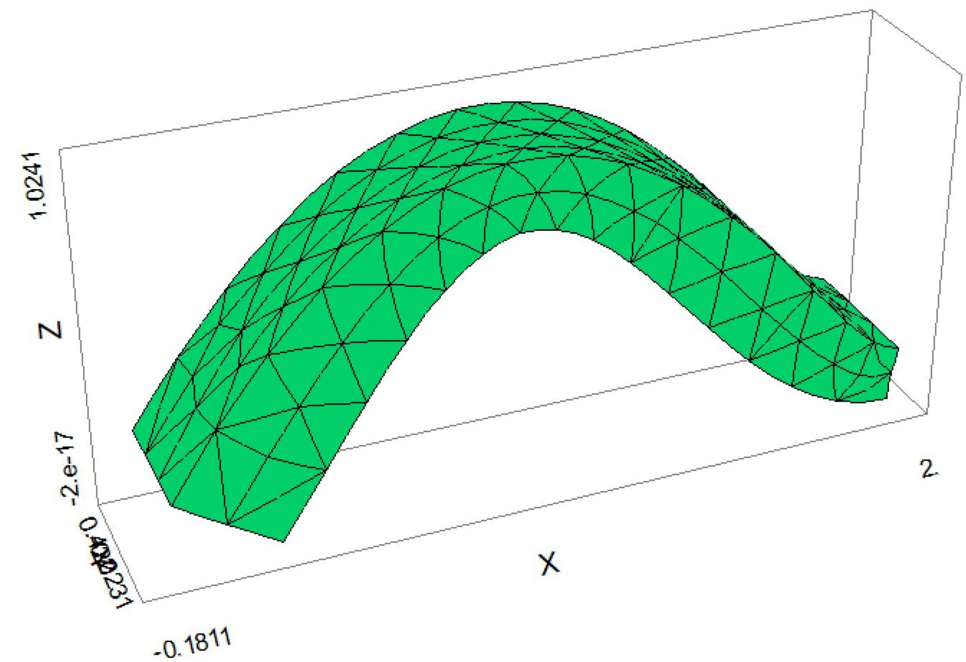
FlexPDE code:

```
TITLE
'H10.2 hussam43'
SELECT
errlim=1e-4
ngrid=15
spectral_colors
stages = 10
COORDINATES
cartesian3
VARIABLES
u       !Displacement in x
v !Displacement in y
w !Displacement in z
DEFINITIONS
mag=1000
Lx=2
```

```
Ly=.4

Lz=.2

E=if z<0.3333*Lz then 120e9 else if z<0.6666*Lz then 74e9 else 47e9

nu=if z<0.3333*Lz then 0.316 else if z<0.6666*Lz then 0.415 else 0.36

rho=if z<0.3333*Lz then 7640 else if z<0.6666*Lz then 19320 else 7310

G=E/(2*(1+nu))


omega = 565+0.1*stage


C11 =E*(1-nu)/(1+nu)/(1-2*nu)

C22 = C11

C33 = C11


C12 = E*nu/(1+nu)/(1-2*nu)

C13 = C12

C21 = C12

C23 = C12

C31 = C12

C32 = C12


!! Strain

!Axial Strain

ex=dx(u)

ey=dy(v)

ez=dz(w)

!Engineering Shear Strain

gyz = dz(v) + dy(w)

gxz = dz(u) + dx(w)

gxy = dy(u) + dx(v)


!!Stress via Hooke's law

!Axial Stress

sx = C11*ex+C12*ey+C13*ez

sy = C21*ex+C22*ey+C23*ez

sz = C31*ex+C32*ey+C33*ez

!Shear stress

sxy=G*gxy

sxz=G*gxz

syz=G*gyz


EQUATIONS

!FNet = 0

u:	dx(sx)+dy(sxy)+dz(sxz)= -rho*omega^2*u

v:	dx(sxy)+dy(sy)+dz(syz) =-rho*omega^2*v

w:	dx(sxz)+dy(syz)+dz(sz) -rho*9.81=-rho*omega^2*w !gravity in thickness direction is really just to get it started


EXTRUSION

surface 'bottom' z=0

surface 'top' z=Lz


BOUNDARIES

surface 'bottom'

		load(u)=0

		load(v)=0

		load(w)=0

surface 'top'

		load(u)=0

		load(v)=0

		load(w)=0


	REGION 1

	START(0,0) !y=0 surface:

		load(u)=0

		load(v)=0

		load(w)=0

	LINE TO (Lx,0) !x=Lx surface

		value(u)=0

		load(v)=0

		value(w)=0

		LINE TO (Lx,Ly) !y=Ly surface

		load(u)=0

		load(v)=0

		load(w)=0

		LINE TO (0,Ly) !x=0 surface

		load(u)=0

		load(v)=0

		value(w)=0

		LINE TO CLOSE


MONITORS

!contour(u) painted on x=0

!contour(v) painted on x=0

!contour(w) painted on x=0
```

```
!grid(x+u,y+v,z+w)


PLOTS

	grid(x+u*mag, y+v*mag, z+w*mag)

	contour(sxz) on surface z=Lz

	elevation(sx,sy,sz,syz,sxz,sxy) from (0,0,0) to (0,0,Lz)

	elevation(w) from (0,Ly/2,Lz/2) to(Lx,Ly/2,Lz/2)

	history(w) at (Lx/2,Ly/2,Lz/2) !the middle will show a lot of displacement


	summary

		report val(u,Lx,Ly,Lz)

		report val(v,Lx,Ly,Lz)

		report val(w,Lx,Ly,Lz)

		report val(w,Lx/2,Ly/2,Lz)

end
```

The approximate value for omega obtained from FlexPDE is 565 rad/s compared to 563.8 obtained from Maple which is extremely accurate. The depiction of the beam and the plot of the first mode is:



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