1) given the output formulation, it can be deduced that H is circulant, (as it is defined by the modulo operation)

$$H = \begin{pmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{2} & 0 & \dots & 0 & -\frac{1}{12} & \frac{4}{5} \\ \frac{4}{5} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{2} & 0 & \dots & 0 & -\frac{1}{12} \\ -\frac{1}{12} & \frac{4}{5} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{2} & 0 & \dots & 0 \\ \vdots & & & \ddots & & & & \vdots \\ -\frac{1}{12} & 0 & \dots & & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} \\ \frac{4}{3} & -\frac{1}{12} & 0 & \dots & & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} \end{pmatrix}$$

b. We saw in the lecture that for circulant Matrixes:

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{m-1} \end{pmatrix} = \sqrt{m}\ DFT^* \begin{pmatrix} h_0 \\ h_{m-1} \\ \vdots \\ h_1 \end{pmatrix}$$

Denoting λ_k as the k^{th} eigen value, and $w = \exp\left(\frac{-i2\pi}{M}\right)$

we get $\lambda_k = h_0 \cdot w^{0k} + \sum_{a=1}^{M-1} w^{ka} \cdot h_{n-a}$

$$= -\frac{5}{2} + \frac{4}{3} w^k - \frac{1}{12} w^{2k} - \frac{1}{12} w^{k(n-2)} + \frac{4}{3} w^{k(n-1)}$$

$$\lambda_k = -\frac{5}{2} + \frac{8}{3}\cos\left(\frac{2\pi k}{M}\right) - \frac{1}{6}\cos\left(\frac{4\pi k}{M}\right)$$

t.f. the pseudo inverse filter of a circulant Matrix is circulant as well where its eigen values are:

$$\lambda_k = \begin{cases} \frac{1}{\lambda_k} & \lambda_k \neq 0 \\ 0 & \lambda_k = 0 \quad \text{i.e: } k = M\left(n \pm \frac{1}{2\pi \cos(?)}\right) \end{cases} \quad n \in \mathbb{Z}$$

C. In order to fully recover data, the condition is, $\forall k \in [0, m-1]$, $\lambda_k \neq 0$

But $-\frac{5}{2} + \frac{8}{3}\cos(0) - \frac{1}{6}\cos(0) = 0$

t.f. there are signals we can not fully reciver, i.e. any signal with a fourier decomposition $\varphi_0^F \neq 0$ would not be able to be restored.

2. We will compute the expected value of the ith element of φ, which we will denote as φ_i:

$$E(\varphi_i) = E(\varphi_i \mid K = i)\cdot P(k=i) + E(\varphi_i \mid K \neq i)P(k\neq i)$$

$$= E(M+L)\cdot\frac{1}{N} + E(M)\cdot\frac{N-1}{N} = \cancel{\frac{E(M)}{N}} + \frac{E(L)}{N} + \cancel{\frac{E(M)}{N}N-1}$$

K is a uniform random variable

M and L are independant

$E(M) = 0$

$$= \frac{E(L_1)}{N}\cdot P\left(k \leq \frac{N}{2}\right) + \frac{E(L_2)}{N}\cdot P\left(k > \frac{N}{2}\right)$$

$$= 0\cdot P\left(k \leq \frac{N}{2}\right) + 0\cdot P\left(k > \frac{N}{2}\right) = \underline{\underline{0}}$$

$E(L_1) = E(L_2) = 0$

We have found that $\forall i \in [1,N]\ E(\varphi_i) = 0 \Rightarrow E(\varphi) = 0$

2.b. $$R_\varphi = \begin{pmatrix} a+c & c & c & \cdots & c \\ c & c & \ddots & & \vdots \\ \vdots & \vdots & \ddots & & \vdots \\ c & c \cdots & a+c & c \cdots & c \\ c & & b+c & & c \\ \vdots & & & \ddots & \vdots \\ c & c & \cdots & \cdots & b+c \end{pmatrix}$$

almost circulant.

using

$$\mathrm{Var}(A_{i,i}) = \mathrm{Var}(M+L) = E\left[(M+L)^2\right] - E^2(M+L)$$

$$= E(M^2 + 2ML + L^2) - (\underbrace{E(\cancel{M})}_{0} + \underbrace{E(\cancel{L})}_{0})^2$$

$$= E(M^2) + 2\underbrace{\cancel{E(ML)}}_{0} + E(L^2) = C + \frac{\cancel{N}}{\cancel{N}}a \quad \text{or} \quad C + \frac{\cancel{N}}{\cancel{N}}b$$

c. As show in tutorial, satisify the condition that

R_φ is circulant, t.f. diagnolizable by DFT^*.

$\Rightarrow$ $a + c = b + c \Rightarrow a = b$

As show in the lecture $\begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \sqrt{n}\, DFT^* \begin{pmatrix} h_0 \\ h_{n-1} \\ \vdots \\ h_1 \end{pmatrix}$

t.f. for λ_k,

$$\lambda_k = (a+c)w^0 + c\sum_{z=1}^{n-1} w^{z \cdot k} = \begin{cases} a + N \cdot C & k = 0 \\ a + c + c \cdot \frac{w^k(1-w^k)^N}{1-w^k} = a & k \neq 0 \end{cases}$$

$\Rightarrow$ the λ's are ordered, i.e.

$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_k$ and DFT^* is the PCA

Matrix

$R_{\varphi_{ii}}$ (R_φ at i row i column) $= E(\varphi_i^2) =$

(like before) $E(\varphi_i^2 \mid k = i)P(k = i) + E(\varphi_i^2 \mid k \neq i)P(k \neq i)$

$$= E(M^2) \cdot \frac{N-1}{N} + E(M + L)^2 \cdot \frac{1}{N}$$

$$= \frac{cN-1}{N} + E(M^2)\frac{1}{N} + \underbrace{\cancel{E(2ML)}}_{0}\frac{1}{N} + E(L^2)\frac{1}{N}$$

$E(k^2)\cdot\frac{1}{n}$

if $i \leq \frac{N}{2}$: $= c\,\frac{N-1}{N} + a + \frac{c}{N} = a + \frac{cN}{N} = a + c$

$i > \frac{N}{2}$ $= c\,\frac{N-1}{N} + \frac{c}{N} + b = b + c$

$R\varphi_{i,j}$: $i \neq j$: $= E(\varphi_i \varphi_j)$

$= E(\varphi_i \varphi_j \mid k = i)\,P(k = i) + E(\varphi_i \varphi_j \mid k = i)\,P(k = j)$

$+ E(\varphi_i \varphi_j \mid k = i)\,P(k \neq j, i)$

$= E[(M + L)(M)]\,\frac{1}{N}\cdot 2 + E[M \cdot M]\cdot\frac{N-1}{N}$

$= \frac{E(M^2)2}{N} + \frac{E(ML)2}{N} + E(M^2)\,\frac{N-2}{N}$

$= \frac{2C}{N} + 0 + C\cdot\frac{N-2}{N} = C$

3 $e = [e_1, \dots, e_N]^T$, N is even.

$e = [M, \dots, M, M+L, M, \dots, M, M+L, M, \dots, M]^T$

$M+L$ is in the k-th and $(k+\frac{N}{2})$th elements.

k is a uniform random variable over $\{1, \dots \frac{N}{2}\}$

M satisfies $E(M)=0$, $E(M^2)=c$

L satisfies $E(L)=0$, $E(L^2)=\frac{N}{2}(1-c)$, c=const $\in (0,1)^{\mathbb{R}}$

a Compute R_e. Is it circulant?

$R_e = E(e e^*)$

denote: $r_{ij} = E(e_i e_j)$, A: $i \neq k$ and $i \neq k+\frac{N}{2}$

$B = \bar{A}$: $i = k$ or $i = k+\frac{N}{2}$

* $i=j$:

$$E(e_i e_i) = E(e_i^2) = E(M^2)\cdot P(A) + E((M+L)^2)\cdot P(B)$$

$$= c \cdot \frac{N-2}{N} + E(M^2+2ML+L^2)\cdot \frac{2}{N}$$

$$= c \cdot \frac{N-2}{N} + (E(M^2) + E(2ML) + E(L^2))\cdot \frac{2}{N}$$

$$= c \cdot \frac{N-2}{N} + \left(c + \underbrace{2E(M)E(L)}_{0} + \frac{N}{2}(1-c)\right)\frac{2}{N}$$

$$= \cancel{c} - \cancel{\frac{2c}{N}} + \cancel{\frac{2c}{N}} + 1 - \cancel{c} = 1$$

* $i \neq j$:

* $|i-j| = \frac{N}{2}$

$$E(e_i e_j) = E(M^2)\cdot P(i,j \neq k, k+\tfrac{N}{2})$$
$$+ E((M+L)^2)\cdot [P(i=k, j=k+\tfrac{N}{2}) + P(j=k, i=k+\tfrac{N}{2})]$$

$$= c\cdot\left(\cancel{2}\cdot\left(\frac{N-2}{\frac{N}{\cancel{2}}}\right)\right) + \left(c+\frac{N}{2}(1-c)\right)\left(\frac{1}{N}+\frac{1}{N}\right)$$

$$= \cancel{c} - \cancel{\frac{2c}{N}} + \cancel{\frac{2c}{N}} + 1 - \cancel{c} = 1$$

$* \; |i-j| \neq \frac{N}{2}$

$$E(e_i e_j) = E(M^2)\cdot P(\overbrace{i,j \neq k, k+\tfrac{N}{2}}^{\text{event } G}) +$$

$$E(M(M+L))\underbrace{\left[P(i\in\{k,k+\tfrac{N}{2}\} \text{ and } j\notin\{k,k+\tfrac{N}{2}\}) + P(j\in\{k,k+\tfrac{N}{2}\} \text{ and } i\notin\{k,k+\tfrac{N}{2}\})\right]}_{\text{This is the complementary event as } |i-j|\neq\frac{N}{2}}$$

$$\Rightarrow \quad = 1 - P(i,j \neq k, k+\tfrac{N}{2})$$

$$= c\cdot P(G) + E(M^2+LM)\cdot P(\bar{G})$$

$$= cP(G) + \left[E(M^2) + \underbrace{E(L)E(M)}_{0}\right]P(\bar{G})$$

$$= c\left(P(G)+P(\bar{G})\right) = c$$

$$\Rightarrow R_e = \begin{array}{r c} & \begin{matrix} & & & & 1+\frac{N}{2} \downarrow & & \end{matrix} \\ \begin{matrix} \\ \\ \\ 1+\frac{N}{2}\rightarrow \\ \\ \\ \end{matrix} & \begin{pmatrix} 1 & c & \cdots & c & 1 & c & \cdots & c \\ c & 1 & & & & \ddots & & \vdots \\ \vdots & & \ddots & & & & \ddots & c \\ c & & & \ddots & & & & 1 \\ 1 & \ddots & & & \ddots & & & \\ c & & \ddots & & & \ddots & & c \\ \vdots & & & \ddots & & & 1 & \\ c & \cdots & c & 1 & & c & & 1 \end{pmatrix} \end{array}$$

$\Rightarrow$ Re is a circulant matrix (sum of circulants)

b As Re is a circulant matrix, it is diagonalisable in the Fourier basis, thus the eigen values of Re can be computed by: $\vec{\lambda} = \sqrt{N}\cdot DFT^{*}\,\vec{c}$ where $\vec{c}$ is the transpose of Re's first row. (/first col.–)

4.a. $R_\varphi = E(\bar{\varphi}^*, \bar{\varphi}^T)$ definition

$$\Rightarrow R_{\varphi_{i,j}} = E(\bar{\varphi}^*, \bar{\varphi}^{*T}) = E\left((H\bar{\varphi}^* + n)(H\bar{\varphi}^* + n)^T\right)$$

$$= E\left(H\bar{\varphi}^*\bar{\varphi}^{*T}H^T + H\bar{\varphi}^* n^T + n\bar{\varphi}^{*T}H^T + nn^T\right)$$

$$= H E(\bar{\varphi}^*\bar{\varphi}^{*T})H^T + H E(\bar{\varphi}^* n^T) + E(n\bar{\varphi}^T)H^T + E(nn^T)$$

linearity

$$= H R_\varphi H^T + H \underbrace{\cancel{E(\bar{\varphi}^*)E(n^T)}}_{0} + \underbrace{\cancel{E(n)E(\bar{\varphi}^T)}}_{0}H^T + R_n$$

n and φ are independent, $E(n) = 0$

$$= H R_\varphi H^T + R_n$$

b. Since we dont assume that the signal has zero mean we will use the weiner filter as shown the tutorial, but use the covariance matrix or $R_\varphi - E(\varphi)E(\varphi^T)$

t.f.

$$\varphi_{opt} = \left(R_\varphi - \underbrace{E(\varphi)E(\varphi^T)}\right)H^T\left(H R_\varphi H^T + \sigma_n^2 I\right)^{-1}\varphi$$

this will reduce to zero if mean is zero, giving us the weiner filter of the tutorial.

C. we will Denote the k^{th} column of the $[DFT]^*$ Matrix as d_k:

$$A = [DFT]^* \Lambda_A [DFT] = \sum_{k=1}^{N} \lambda_k d_k d_k^*$$

multiplying a circulant matrix by a scalar results in a circulant matrix.

summing crculant matrixs results in a circulant matrix.

t.f. if we prove that $d_k d_k^*$ is circulant we have proven that A is circulant.

denote: $W_N = \exp\left(\frac{2\pi i}{N}\right)$

$$d_k = \begin{bmatrix} W_N^{0 \cdot k-1} \\ \vdots \\ W_N^{(N-1)(k-1)} \end{bmatrix} = \begin{bmatrix} e^0 \\ \\ \exp\left(\frac{2\pi i}{N}\left((N-1)(k-1)\right)\right) \end{bmatrix}$$

denote the element $\Rightarrow (i,j)^{th}$ of $d_k d_k^*$ as $(d_k d_k^*)_{ij}$

$$= e^{i\frac{2\pi}{N}(i-1)(k-1)} \cdot e^{-\frac{2\pi i}{N}(j-1)(k-1)} = e^{\frac{i2\pi}{N}(i-j)(k-1)}$$

$$\Rightarrow e^{i\frac{2\pi}{N}\underbrace{(i-j) \bmod N}_{\text{period of } e^i}(k-1)}$$

we see that coordinate are dependant on $(i-j) \bmod N$ - much like the first quertion we can conclude that $d_k d_k^*$ is circulant.

$\Rightarrow A$ is circulant.

d. No using a simple example, where $H = I$, $\sigma_n = I$ and even a signal that has zero mean and thus we do not need to take into accout the covariance matrix, and an autocorrelation matrix

$$R_\varphi = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\upsilon = R_\varphi H^T (H R_\varphi H^T + \sigma_n^2 I)^{-1}$$

$$= R_\varphi I (I R_\varphi I + I I)^{-1}$$

$$= R_\varphi (R_\varphi + I)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & \\ & 6/7 \end{pmatrix}$$

which is <u>not circulant</u>

f. F - not shift invariant.

e. in the general case

$$W = \left(R_{\varphi} - E(\varphi)E(\varphi^T)\right)H^*\left(H R_{\varphi} H^* + \sigma_n^2 I\right)^{-1}$$

the condition that W will be shift invariant is that W be circulant.

$\sigma^2 I$ is circulant and t.f no condition is on σ.

we need t.f that $\left(R_\varphi - E(\varphi)E(\varphi^T)\right)$ and H be circulant.

or R_φ, $E(\varphi)E(\varphi^T)$ and H be circulant

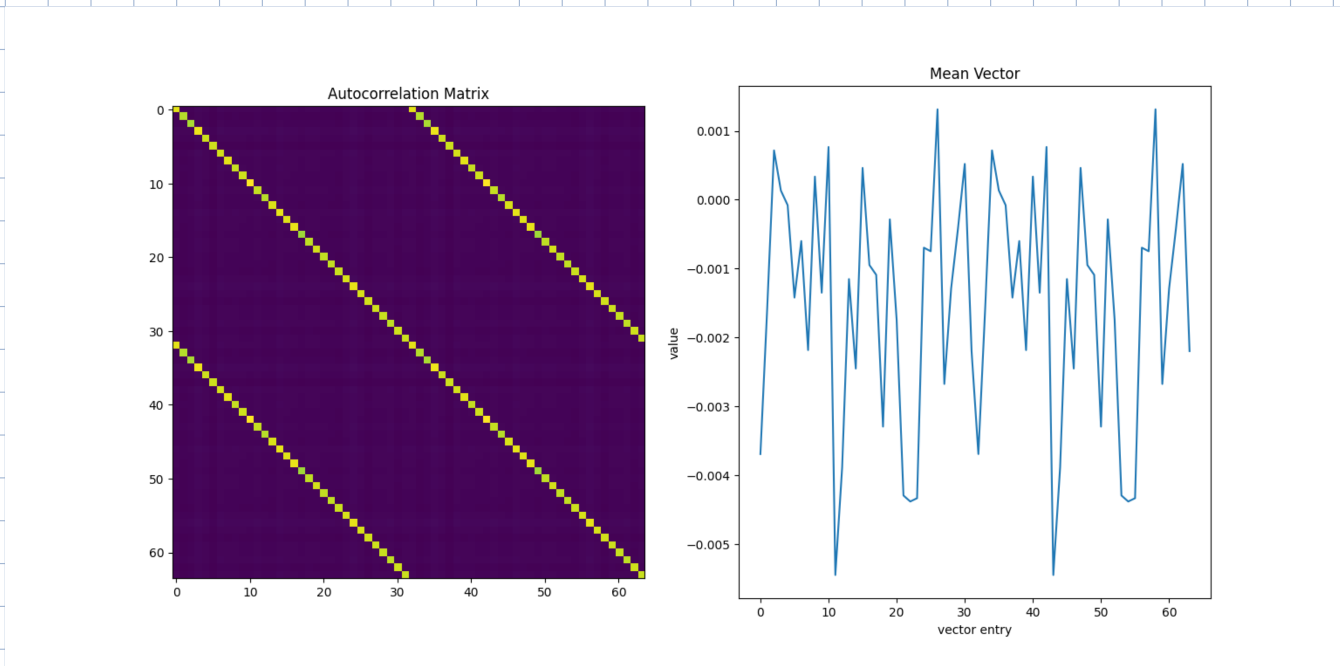
This is not a necessary condition

Part 2- Implementation

$N=64,\ c=0.6,\ M\sim Norm(0,0.6),\ L\sim Norm(0,12.8),$

$k\sim uni(1,\frac{N}{2})$

a



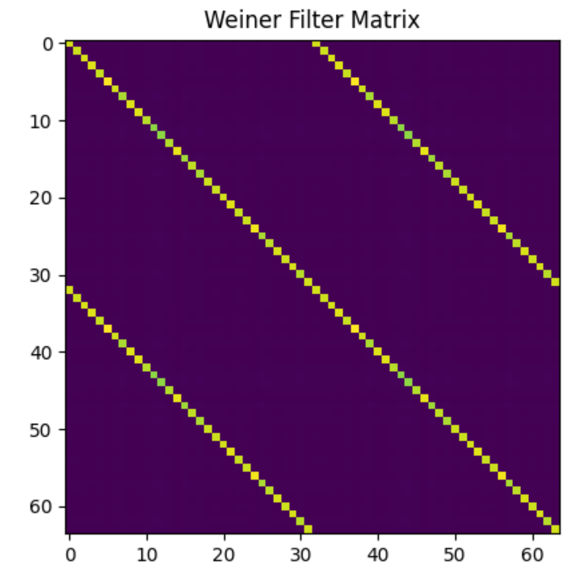
As seen in the above figure, the empirical approximation is very close to our analytical resusts.

① In the Autocorrelation matrix, we can clearly see the diagonal vaues we expected as well as the circulant nature of the matrix.

② In the mean vector, we can see that the values are realy close to zero ($\in[-0.005, 0.0002]$).

* The above result is achived with $\sim 10^5$ realization for extra accuracy, but could also be achived from $\sim 10^4$ realizations

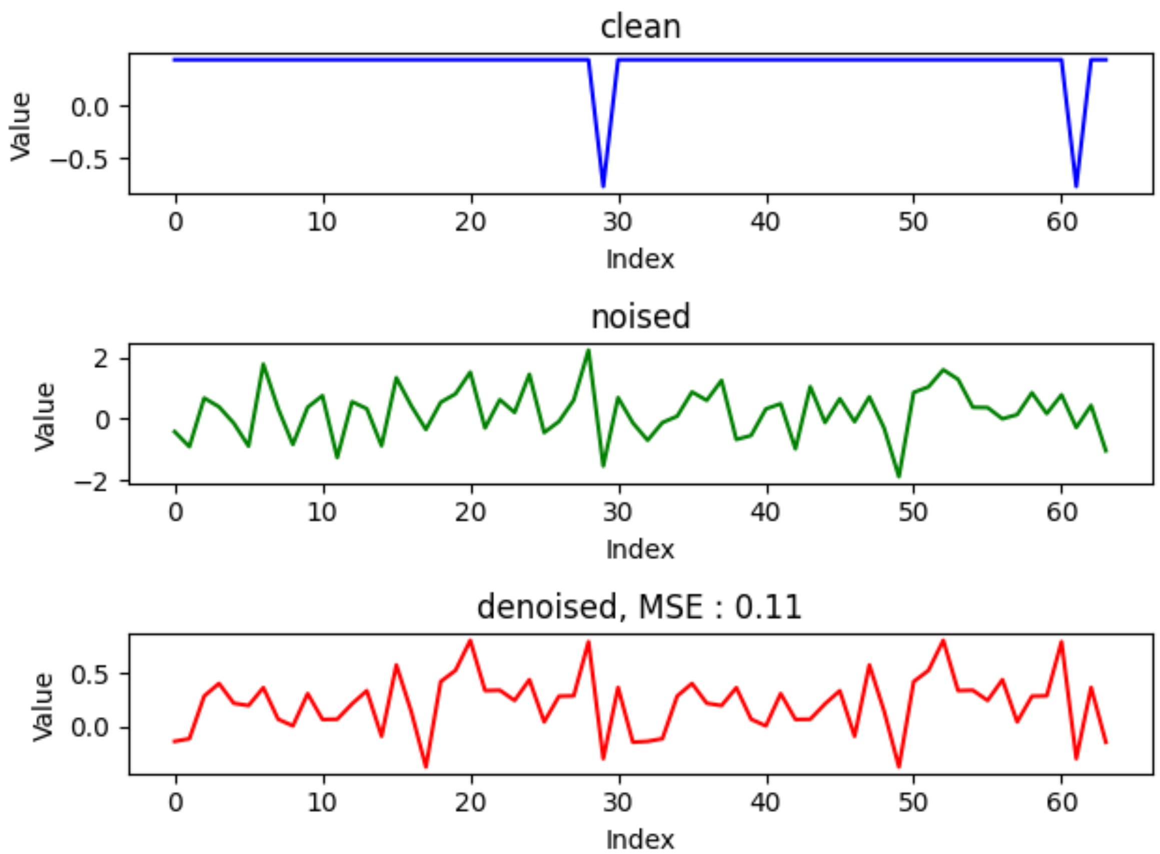
b $\varphi^* = \varphi + n$, $R_n = \sigma_n^2 I$, $\sigma_n = 1$



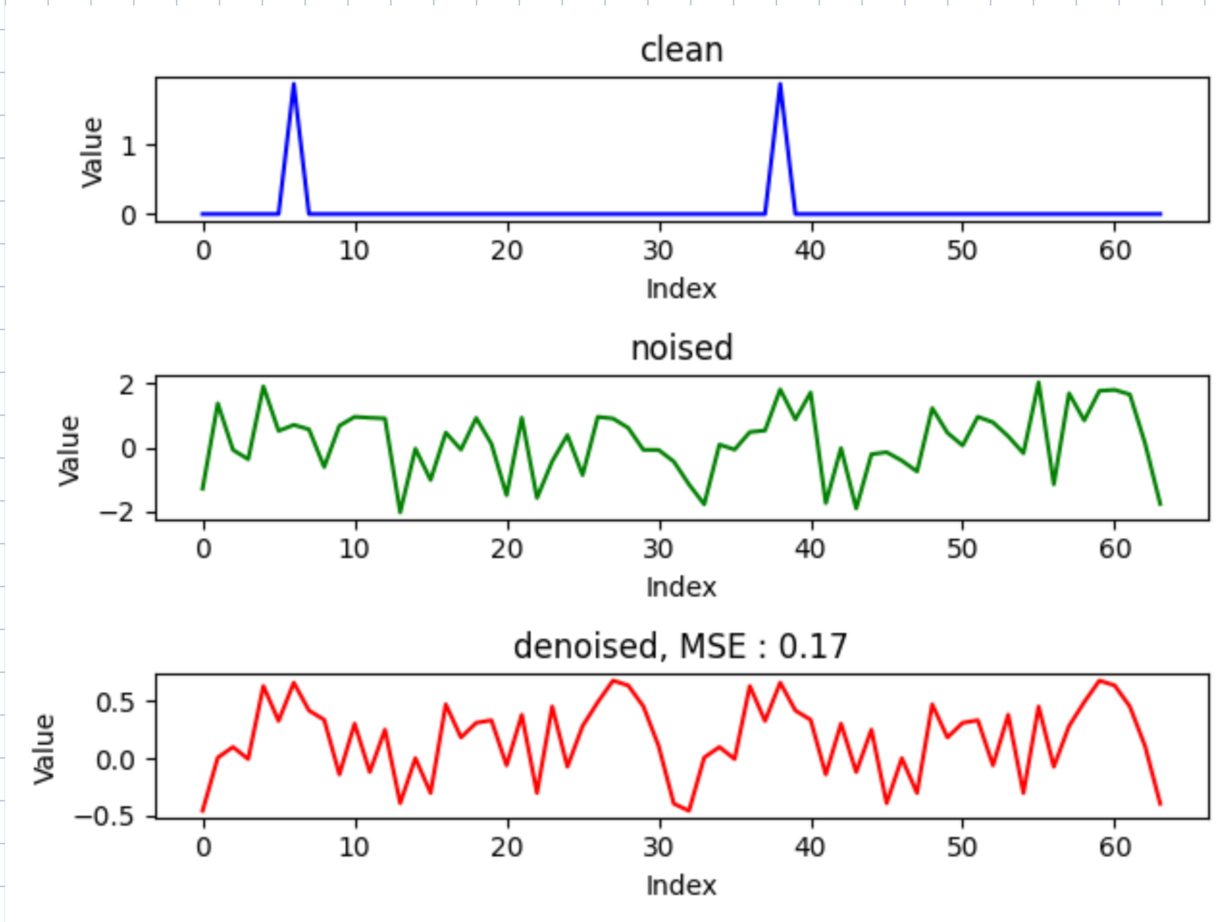
As $H = I$, we can clearly see the Wiener filter is heavily similar to R_φ. Thus, we can also observe its circulant structure.

Clean vs noised vs denoised examples:

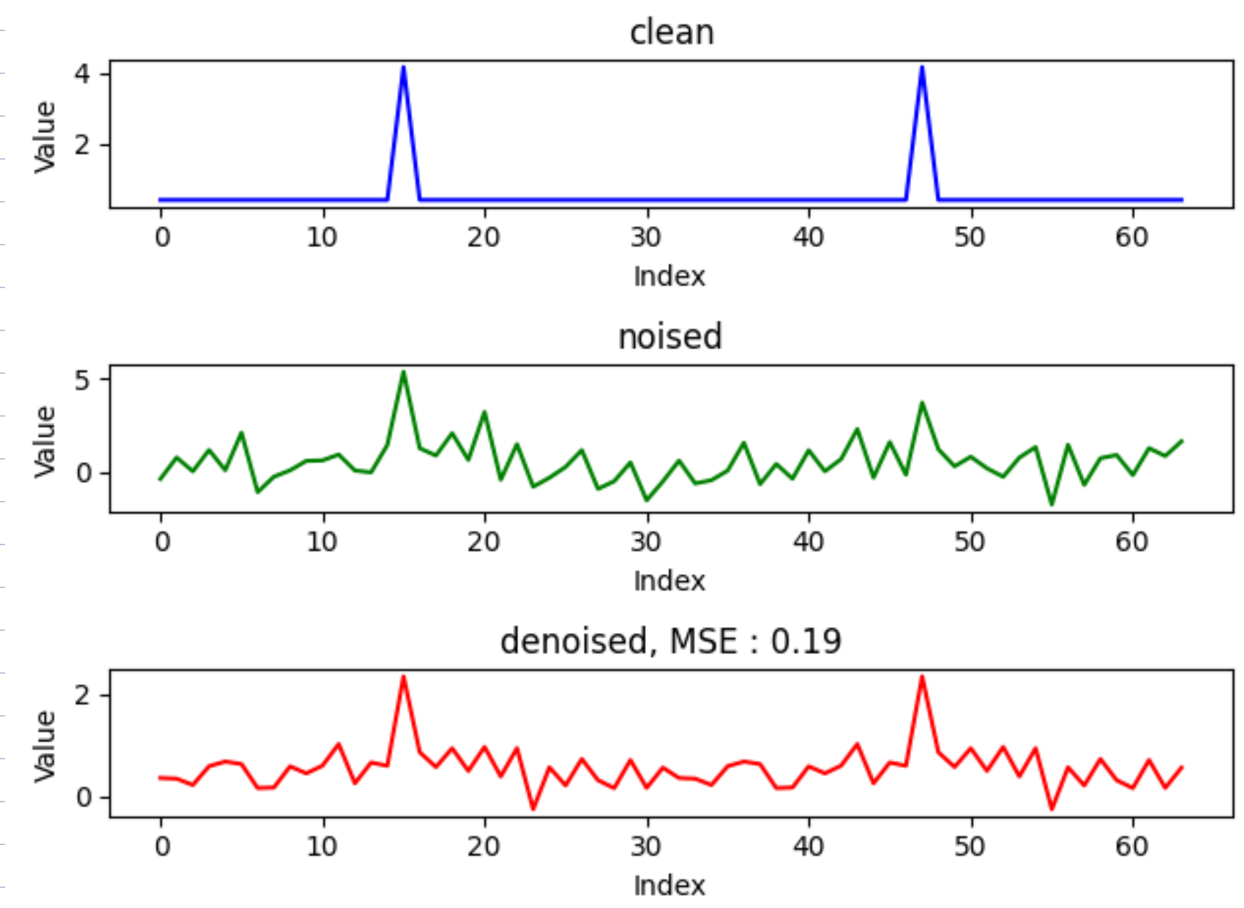
①



②



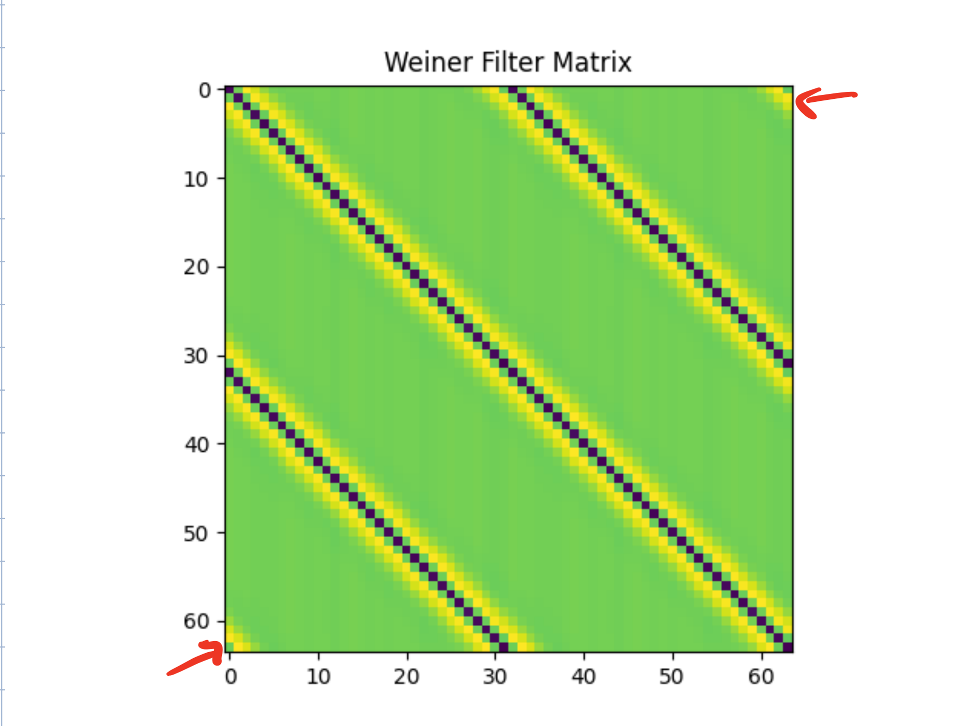
③



The total Avg of the MSE's is ~ 0.23008

c $e^* = He + n$

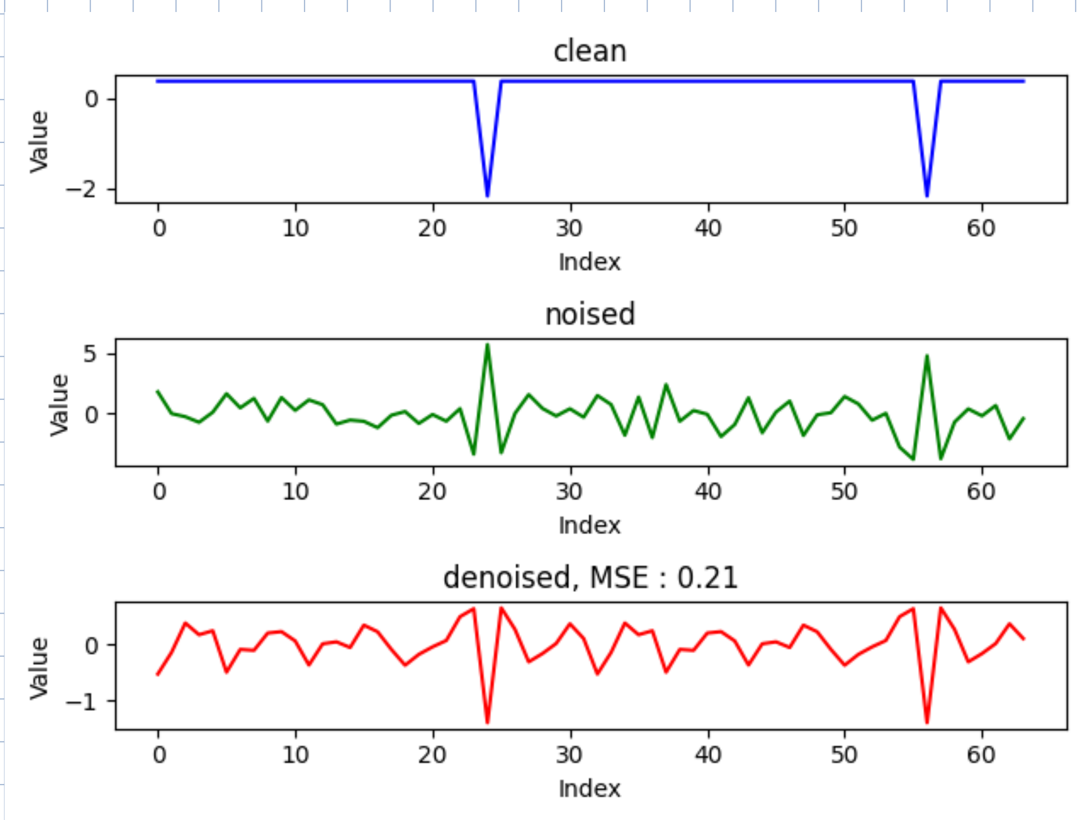
$$He_j = -\frac{1}{12} e_{j-2}[M] + \frac{4}{3} e_{j-1}[M] - \frac{5}{2} e_j[M] + \frac{4}{3} e_{j+1}[M] - \frac{1}{12} e_{j+2}[M]$$



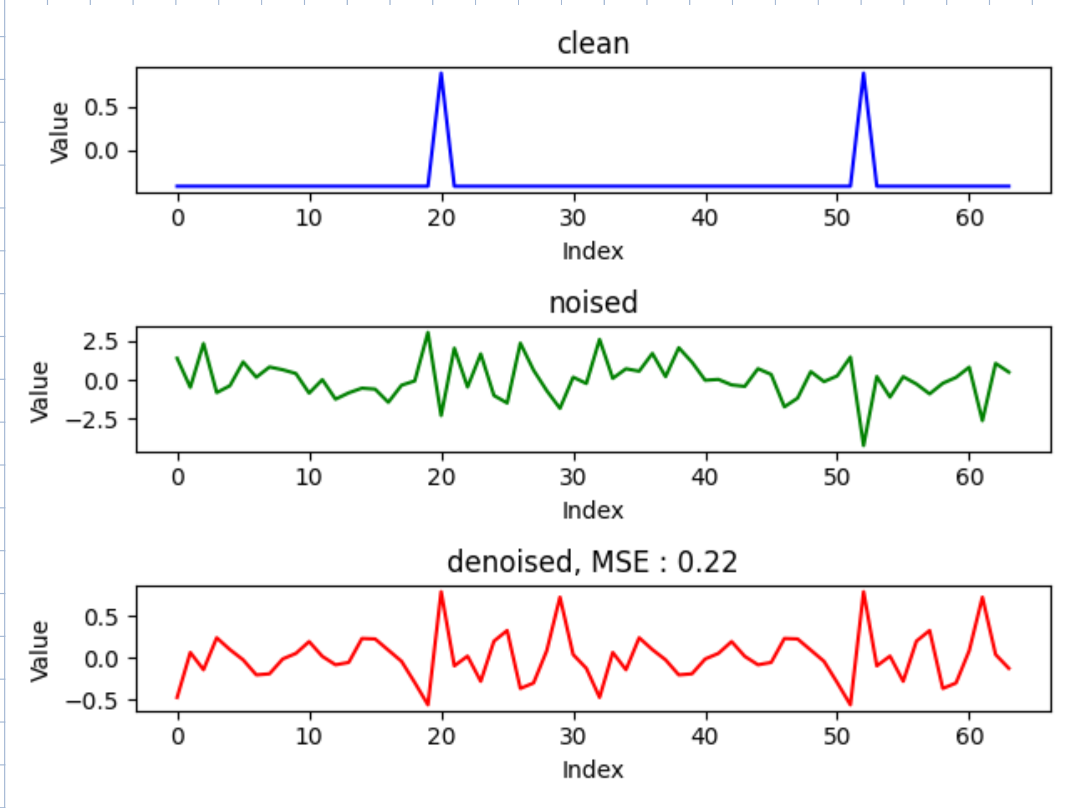
Still a toepliz matrix, different than Re. The distinct differences are around the diagonals and in the corners.

Clean vs noised vs denoised examples:

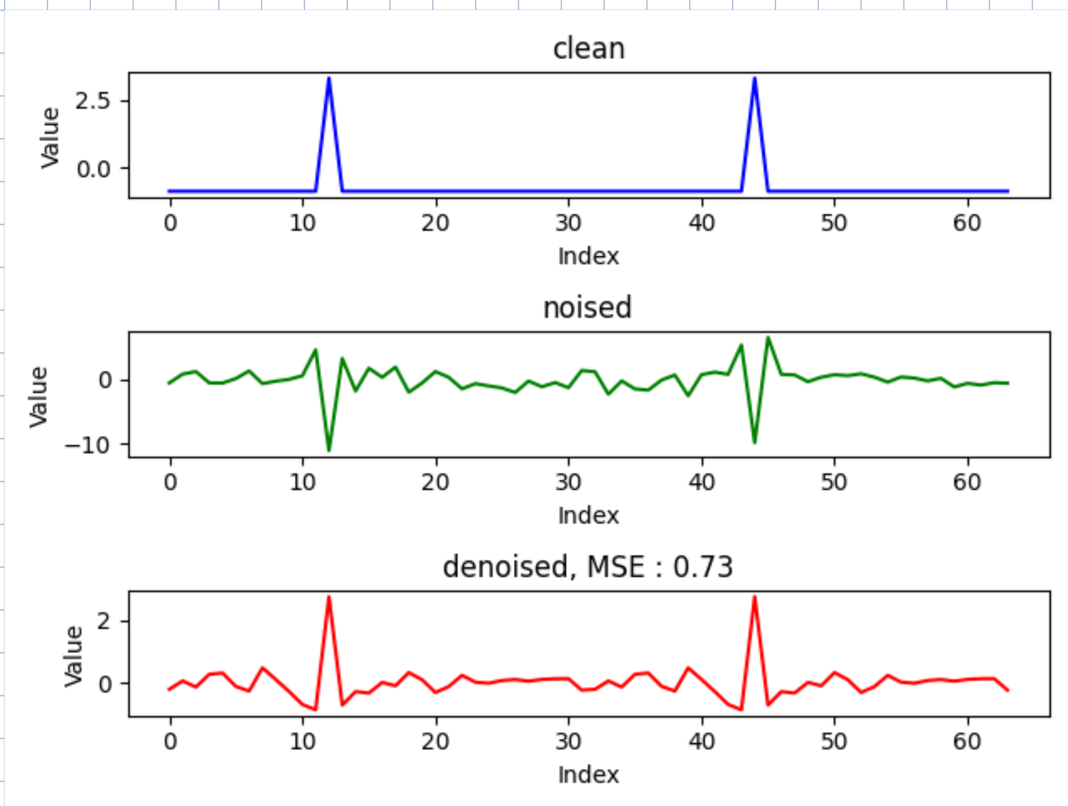
①



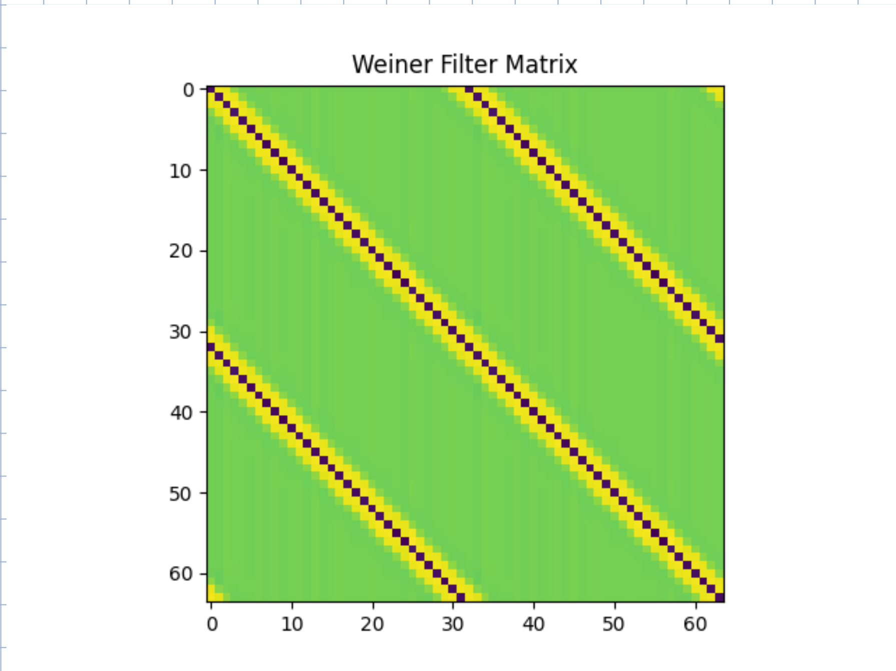
2



3

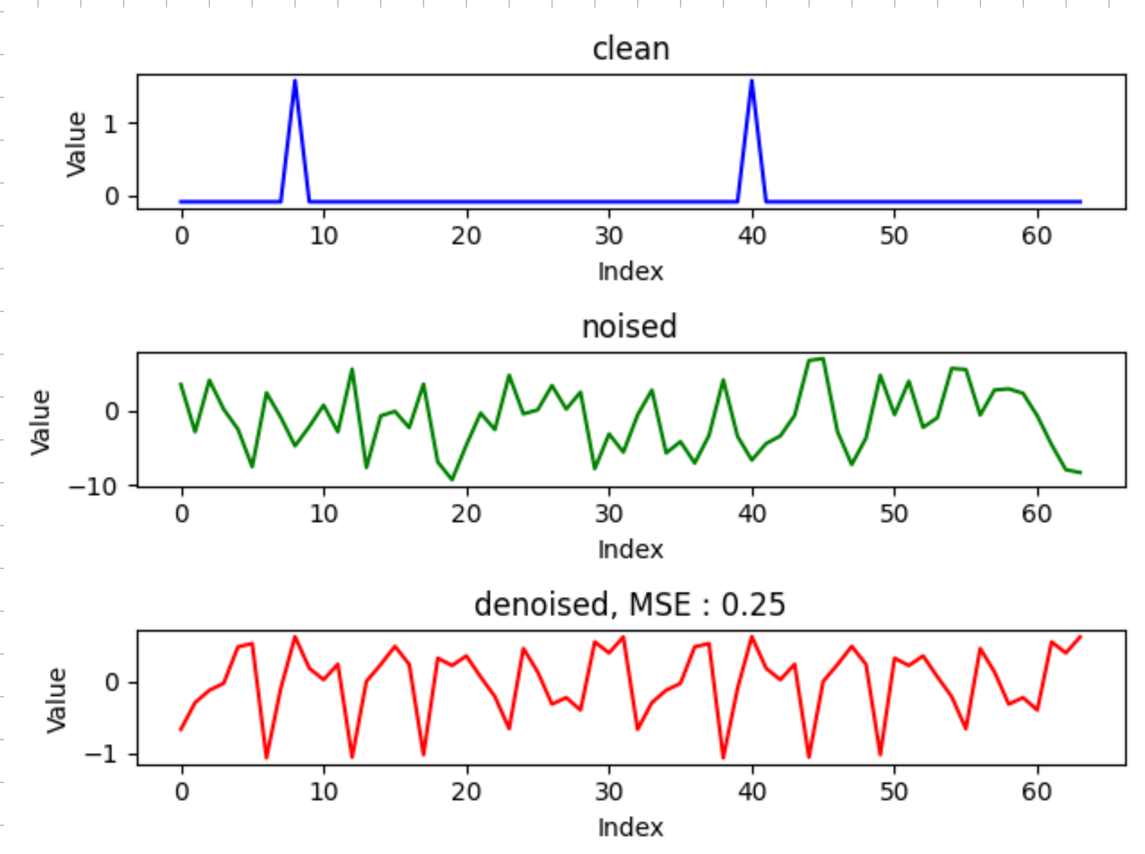


Total MSE Avg is ~0.754 (higher than the previous) d Same as c, var = 58

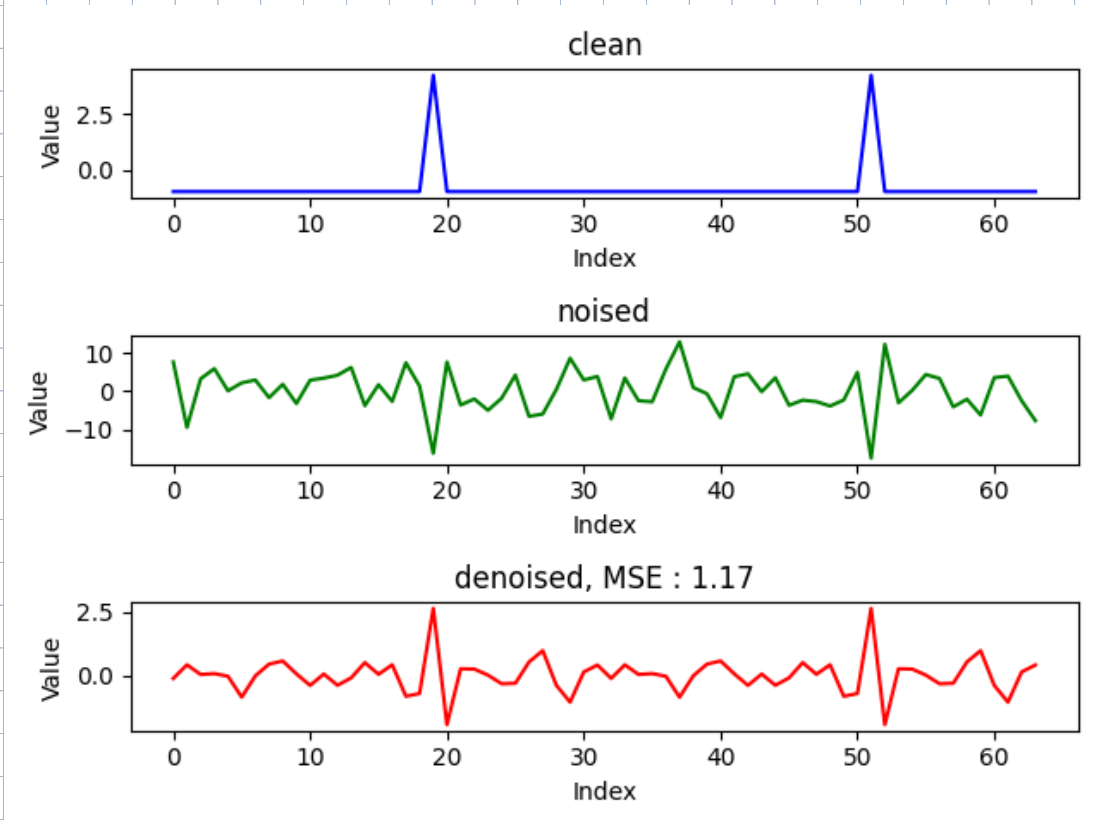


In general, same as before. Slight difference in the range of the values is seen in the plot (around the diagonals & corners).

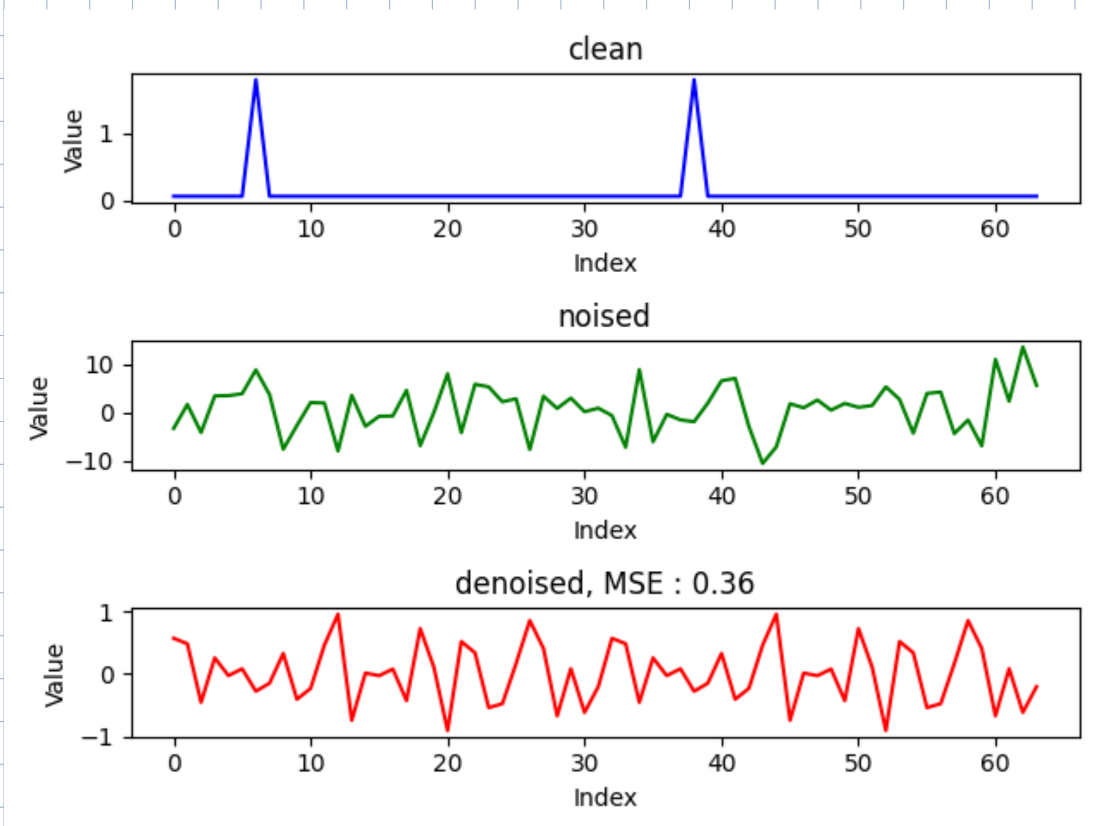
Clean vs noised vs denoised examples:

① 

②

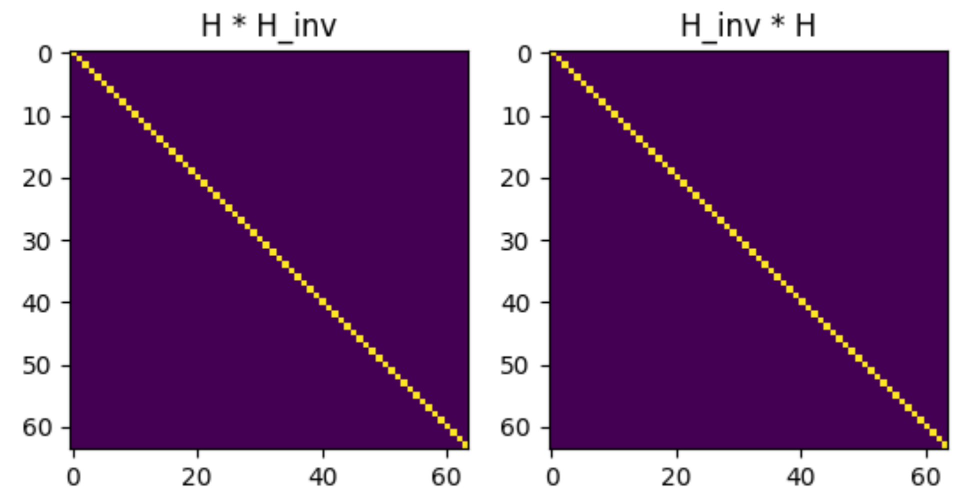


③



Total MSE Avg is ~1.055. higher than before.

e $H \cdot H^+ = H^+ H = I$ as seen in these plots:



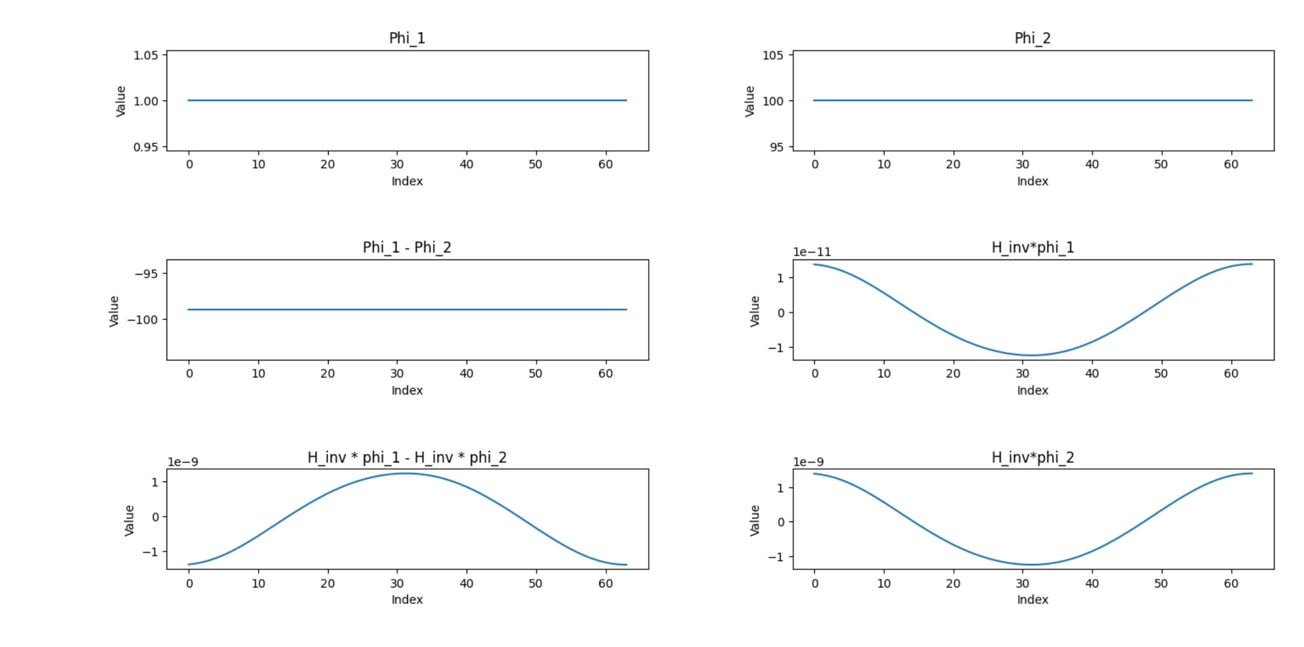
as proved in the theoretical part, 0 is an eigen val of both H and H^+. As the first column of the DFT* (which diagonalizes H) is $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

as the vectors $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 100 \\ \vdots \\ 100 \end{pmatrix}$ are corresponding eigenvectors to 0, we get:

$$H^+ \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 100 \\ \vdots \\ 100 \end{pmatrix} = H^+ \begin{pmatrix} 100 \\ \vdots \\ 100 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

But: $\left\| \begin{pmatrix} 100 \\ \vdots \\ 100 \end{pmatrix} - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 99 \\ \vdots \\ 99 \end{pmatrix} \right\| = 64 \cdot 99^2 > 256$

Plots8


Phi_1
Phi_2
Phi_1 - Phi_2
H_inv*phi_1
H_inv * phi_1 - H_inv * phi_2
H_inv*phi_2
Value
Index
1e−11
1e−9