1.a) Using the finite O.N. family

$$\mathcal{B} = \{\beta_1, \dots \beta_n\}$$

$$F = Vec(\beta_1, \dots, \beta_n)$$

The K term approximation of f in F using $Vec(\beta_{i_1}, \dots \beta_{i_k})$

is: $$\varphi_k(t) = \sum_{i=1}^{k} < f(t), \beta_i(t) > \beta_i(t)$$

The $SE(f \to \varphi_k)$ is:

$$SE = \int_{-\infty}^{\infty} f^2(t)\, dt - \sum_{i=1}^{k} (< f(t), \beta_i(t) >)^2$$

as seen in lecture.

1.a.6)

$$SE = \int_{-\infty}^{\infty} f^2(t)\, dt - \sum_{i=1}^{k} \left(< f(t), B_i(t) > \right)^2$$

The first part of error is constant over any ordering of k, and the second part is dependant on B_i.

The best k-approximation will therefore be the one that maximizes the $< f(t), B_i(t) >^2$ s i.e $\Rightarrow$ orders the B_i s such that the k first ones maxime the SE. and so it is not unique

$B = \{ B_1, \dots B_n \}$
$\tilde{B} = \{ \tilde{B}_1, \dots \tilde{B}_n \}$

are 2 O.N. families of funtions over F.

The n- approixmation is:

$$\varphi_n(t) = \sum_{i=1}^{n} < f(t), B_i(t) > B_i(t)$$

Both B and $\tilde{B}$ are bases of F

$$\Rightarrow \sum_{i=1}^{n} < f(t), B_i(t) > B_i(t) = \sum_{i=1}^{N} < f(t), \tilde{B}_i(t) > \tilde{B}_i(t)$$

$\Rightarrow$ The induced SE is the same.

b) As there is no assumption regarding the order of the bases $B, \tilde{B}$, the k-term approximation can vary $\to$ therefore the SE can very as a result of k.

for instance if $\mathcal{B} = \{\beta_1, \dots \beta_n\}$
$\tilde{\mathcal{B}} = \{\tilde{\beta}_1, \dots \tilde{\beta}_n\}$

s.t. $\beta_1 \neq \tilde{\beta}_1$ if $f(t) = \beta_1(t)$, then the $k=1$ SE is 0 for $\mathcal{B}$ and not 0 for $\tilde{\mathcal{B}}$.

2) $\phi = a + b\cos(2\pi\ ) + c\cos^2(\pi t)$

$$\tilde{H}_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

prove that

$$\tilde{H}_4 \tilde{H}_4^* = \tilde{H}_4^* \tilde{H}_4 = U$$

$$\tilde{H}_4 \tilde{H}_4^* = \frac{1}{2}\frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U$$

$$\tilde{H}_4 \tilde{H}_4^* = \frac{1}{2}\cdot\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U$$

as show in lecture, (Both are ON bases)

$$\begin{bmatrix} \psi_1^H(t) \\ \psi_2^H(t) \\ \psi_3^H t \\ \psi_4^H(t) \end{bmatrix} = \tilde{H}^T_W \cdot \frac{1}{4} \cdot \frac{1}{2} \begin{bmatrix} \sqrt{4} \; 1_{\Delta_1}(t) \\ \sqrt{4} \; 1_{\Delta_2}(t) \\ \sqrt{4} \; 1_{\Delta_3}(t) \\ \sqrt{4} \; 1_{\Delta_4}(t) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1_{\Delta_1}(t) \\ \vdots \\ \vdots \\ 1_{\Delta_4}(t) \end{bmatrix} \Rightarrow$$

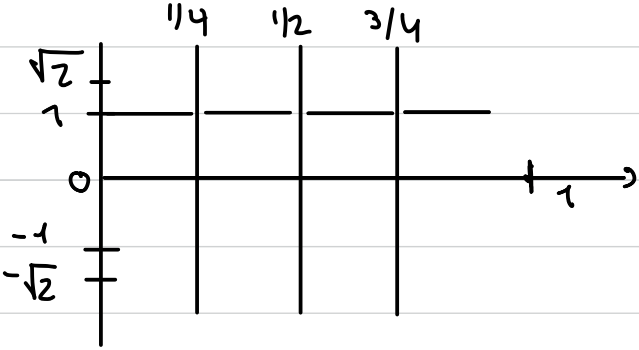
$$\psi_1^H(t) = 1_{[0,1]}(t)$$

$$\psi_2^H(t) = 1_{[0,\,1/2]}(t) - 1_{[1/2,\,1]}(t)$$

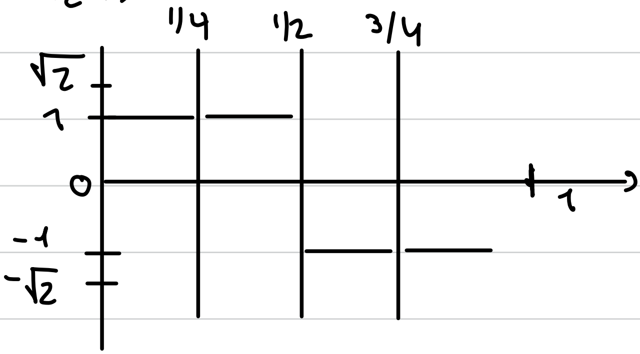
$$\psi_3^H(t) = \sqrt{2}\, 1_{[0,\,1/4]}(t) - \sqrt{2}\, 1_{[1/4,\,1/2]}(t)$$

$$\psi_4^H(t) = \sqrt{2}\, 1_{[1/2,\,3/4]}(t) - \sqrt{2}\, 1_{[3/4,\,1]}(t)$$

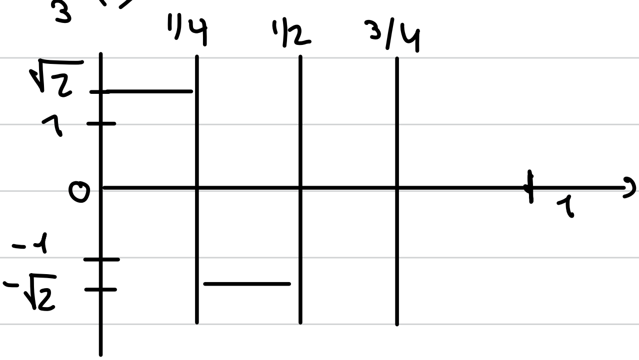
$\psi^W_1(t)$

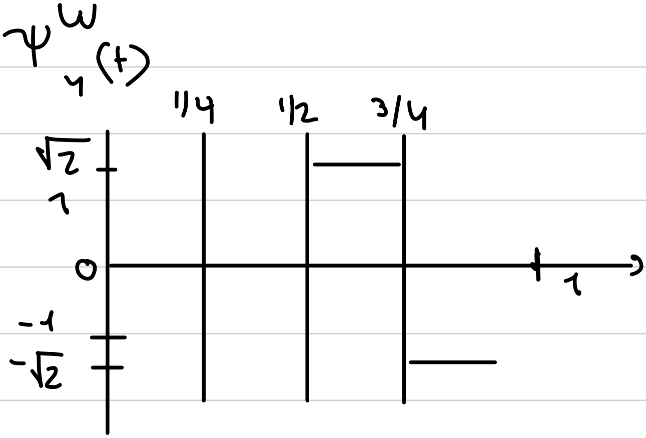


$\psi^W_2(t)$



$\psi^W_3(t)$





iii) The best approximation of ϕ using the HAAR basis was shown in the classroom by:

$$\phi_i^* = \langle \phi(t), \psi_i^H(t) \rangle = \int_0^1 \phi_i(t)\, \psi_i^H(t)\, dt$$

$$\phi_1^1 = \int_0^1 a + b\cos(2\pi t) + c\cos^2(\pi t)\, dt$$

$$= \int_0^1 a\, dt + \underbrace{\cancel{\int_0^1 b\cos(2\pi t)\, dt}}_{\to 0} + \int_0^1 c\cos^2(\pi t)\, dt$$

$$= at + c\left[\frac{\frac{\sin(2\pi t)}{2} + \pi t}{\pi\ 2}\right]_0^1 = a + \frac{c}{2}$$

$$\phi_2^* = \int_0^1 \left(a + b\cos(2\pi t) + c\cos^2(\pi t)\right)\left(1_{[0,1/2]}(t) - 1_{[1/2,1]}(t)\right) dt$$

$$= \int_0^{1/2} a + b\cos(2\pi t) + c\cos^2(\pi t)\, dt - \int_{1/2}^{1} a + b\cos(2\pi t) + \cos^2(\pi t)\, dt$$

$$= \cancel{\frac{a}{2}} + \left[\frac{t}{2} + \cancel{\frac{\sin 2\pi t}{4\pi}}\right]_0^{1/2} - \cancel{\frac{a}{2}} - \left[\frac{t}{2} + \frac{\sin(2\pi t)}{4\pi}\right]_{1/2}^{1} = 0$$

↑ computed before

$$\phi_3^* = \int_0^1 a + b\cos(2\pi t) + c\cos^2(\pi t)\left(\sqrt{2}\, 1_{[0,1/4]}(t) - \sqrt{2}\, 1_{[1/4,1/2]}(t)\right) dt$$

linearity

$$= \sqrt{2}\int_0^{1/4} a + b\cos(2\pi t) + c\cos^2(\pi t)\, dt - \sqrt{2}\int_{1/4}^{1/2} a + b\cos(2\pi t) + c\cos^2(\pi t)\, dt$$

$$= \sqrt{2}\left[\frac{a}{4} + \frac{b}{2\pi}\sin(2\pi t)\right]_0^{1/4} + c\left[1/2 + \frac{\sin 2\pi t}{4\pi}\right]_0^{1/4} - \dots$$

$$\sqrt{2}\left[\frac{a}{4} + \frac{b}{2\pi}\sin(2\pi t)\right]_{1/4}^{1/2} + c\left[1/2 + \frac{\sin 2\pi t}{4\pi}\right]_{1/4}^{1/2}$$

$$= \sqrt{2}\left[\frac{a}{4} + \frac{b}{2\pi}(1-0) + c\left(\frac{1}{8} + \frac{1}{\pi 4}\right) - \left(\frac{a}{4} + \frac{b}{2\pi}(0-1) + c\left(\frac{1}{8} - \frac{c}{4\pi}\right)\right)\right]$$

$$= \sqrt{2}\left(\frac{b}{\pi} + \frac{c}{2\pi}\right) = \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)$$

$$\phi_4^* = \int_0^1 a + b\cos(2\pi t) + c\cos^2(\pi t) \left(\sqrt{2}\, 1_{[1/2,\,3/4]}(t) - \sqrt{2}\, 1_{[3/4,\,1]}(t)\right) dt$$

like before

$$= \sqrt{2}\left[\frac{a}{4} + \frac{b}{2\pi}\sin(2\pi t)\right]_{1/2}^{3/4} + c\left[\frac{1}{2} + \frac{\sin 2\pi t}{4\pi}\right]_{1/2}^{3/4} - \dots$$

$$\sqrt{2}\left[\frac{a}{4} + \frac{b}{2\pi}\sin 2\pi t\right]_{3/4}^{1} + c\left(\frac{1}{2} + \frac{\sin 2\pi t}{4\pi}\right]_{3/4}^{1}$$

$$= \sqrt{2}\left[\frac{a}{4} + \frac{b}{2\pi}(-1+0) + c\left(\frac{3}{8} - \frac{1}{4\pi} - \frac{1}{4}\right) - \right.$$

$$\left. - \frac{a}{4} + \frac{b}{2\pi}(0+1) + c\left(1/2 - 3/8 + \frac{1}{4\pi}\right)\right]$$

$$= -\frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)$$

As seen in tutorial + lesson:

$$\tilde{\phi}(t) = \Sigma\, \phi_i^* \, \varphi_i^H(t)$$

$$\Rightarrow \tilde{\phi}(t) = a + \frac{c}{2}\cdot \psi_1^H(t) + \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\psi_3^H(t) - \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\psi_4^H(t)$$

or

$$\tilde{\phi}(t) = \begin{cases} a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right) & 0 \leq t \leq 1/4 \\ a + \frac{c}{2} - \frac{2}{\pi}\left(b + \frac{c}{2}\right) & 1/4 \leq t \leq 1/2 \\ a + \frac{c}{2} - \frac{3}{\pi}\left(b + \frac{c}{2}\right) & 1/2 \leq t \leq 3/4 \\ a + \frac{c}{2} + \frac{3}{\pi}\left(b + \frac{c}{2}\right) & 3/4 \leq t \leq 1 \end{cases}$$

as we saw in lecture:

Calculating MSE:

$$\int_0^1 (\phi(t) - \tilde{\phi}(t))^2 dt = \int_0^1 \phi^2(t) - 2\int_0^1 \phi(t)\,\tilde{\phi}(t) + \int_0^1 \tilde{\phi}^2(t)$$

$$= \int_0^1 \phi(t)^2 - \sum_{i=1}^{4} (\phi_i^*)^2 dt$$

$$\sum_{i=1}^{4} (\phi_i^*)^2 d = \left(a + \frac{c}{2}\right)^2 + \left(\frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\right)^2 + \left(\frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\right)^2$$

$$= a^2 + ac + \frac{c^2}{4} + \frac{4}{\pi^2}\left(b^2 + bc + \frac{c}{4}\right)$$

$$= a^2 + ac + c^2\left(\frac{1}{4} + \frac{1}{\pi^2}\right) + \frac{4}{\pi^2}\left(b^2 + bc\right)$$

$$\int_0^1 \phi^2(t) =$$

$$= \int_0^1 \left[a + b\cos(2\pi t) + c\cos^2(\pi t)\right]^2 dt$$

$$= \int_0^1 \left[a^2 + 2ab\cos(2\pi t) + 2ac \cdot \cos^2(\pi t) + b^2\cos^2(2\pi t)\right.$$

$$\left. + 2bc\cos(2\pi t)\cos^2(\pi t) + c^2\cos^4(\pi t)\right] dt$$

$$= a^2 \int_0^1 dt + 2ab \int_0^1 \cos(2\pi t)\,dt + 2ac \int_0^1 \cos^2(\pi t)\,dt$$

$$+ b^2 \int_0^1 \cos^2(2\pi t)\,dt + 2bc \int_0^1 \cos(2\pi t)\cos^2(\pi t) + c^2 \int_0^1 \cos^4(\pi t)\,dt$$

$$= a^2 + ac + \frac{b^2}{2} + \frac{bc}{2} + \frac{3}{8} \cdot c^2$$

$$\Rightarrow MSE = \int_0^1 \phi(t)^2 - \sum_{i=1}^{4} (\phi_i^*)^2\,dt$$

$$= a^2 + ac + \frac{b^2}{2} + \frac{bc}{2} + \frac{3}{8} \cdot c^2 -$$

$$- \left(a^2 + ac + c^2\left(\frac{1}{4} + \frac{1}{\pi^2}\right) + \frac{4}{\pi^2}(b^2 + bc)\right)$$

$$= b^2\left(\frac{1}{2} - \frac{4}{\pi}\right) + bc\left(\frac{1}{2} - \frac{4}{\pi^2}\right) + c^2\left(\frac{1}{3} - \frac{1}{\pi^2}\right) = MSE$$

(IV) The best K^{th} term is computed when the terms are ordered bys absolute val

$$|\phi_1^*| \geq \underbrace{|\phi_3^*| \geq |\phi_4^*|}_{} \geq |\phi_2^*|$$

$$a + \frac{c}{2} \geq \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right) \geq 0$$

and then the largest coefficients are used:

1st term: $\phi_1^* \psi_1^H(t) = \left(a + \frac{c}{2}\right) \cdot 1_{[0,1]}(t)$

2nd term: $\phi_1^* \psi_1^H(t) + \phi_3^* \psi_3^H(t) = \left(a + \frac{c}{2}\right) \cdot 1_{[0,1]}(t)$

$+ \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right) \cdot \sqrt{2}\, 1_{[1/4,1/2]}(t) + \frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\left(-\sqrt{2}\, 1_{[1/4,1/2]}(t)\right)$

$= \left(a + c\left(\frac{1}{2} + \frac{1}{\pi}\right) + \frac{2b}{\pi}\right) 1_{[0,1/2]}(t) + \left(a + c\left(1/2 - 1/\pi\right) - \frac{2b}{\pi}\right) 1_{[1/4,1/2]}(t)$

$+ (a + c/2)\, 1_{[1/2,1]}(t)$

3rd term: $\phi_1^* \psi_1^H(t) + \phi_3^* \psi_3^H(t) + \phi_4^* \psi_4^H(t) = \ldots$

$= \left(a + c\left(\frac{1}{2} + \frac{1}{\pi}\right) + \frac{2b}{\pi}\right) I_{[0,1/4],[3/4,1]}(t) - \left[a + c\left(\frac{1}{2} - \frac{1}{\pi}\right) - \frac{2b}{\pi}\right] I_{[1/4,3/4]}(t)$

4th term - same as above as $\phi_2^* = 0$

V) the ordering stays the same in abs value as such we only need to plug in the values

$\underline{1^{st}\text{ term}}$: $\varphi^*_1 \psi^{\#}_1(t) = \left(a + \frac{c}{2}\right) \cdot 1_{[0,1]}(t)$

$= \left(\frac{1}{\pi} + \frac{3}{4}\right) 1_{[0,1]}(t) = 1.068 \cdot 1_{[0,1]}(t)$

2nd =

$= \left(a + c\left(\frac{1}{2} + \frac{1}{\pi}\right) + \frac{2b}{\pi}\right) 1_{[0,1/2]}(t) + \left(a + c\left(1/2 - 1/\pi\right) - \frac{2b}{\pi}\right) 1_{[1/4,1/2]}(t)$

$+ (a + c/2)\, 1_{[1/2,1]}(t) = 2.182 \cdot 1_{[0,1/4]}(t) - 0.046\, 1_{[1/4,1/2]}(t)$

$+ 1.068 \cdot 1_{[1/2,1]}(t)$

3 term $= 2.182 \cdot 1_{[0,1/4][3/4,1]}(t) - 0.046\, 1_{[1/4,1/2]}(t)$

4^{th} - same

2.b

$$W_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

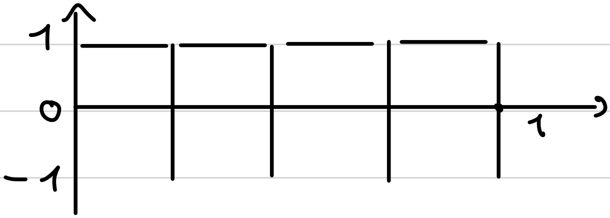
W_4 is unitary i.e. $W_4 W_4^* = W_4^* W_4 = \mathcal{U}$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} =$$

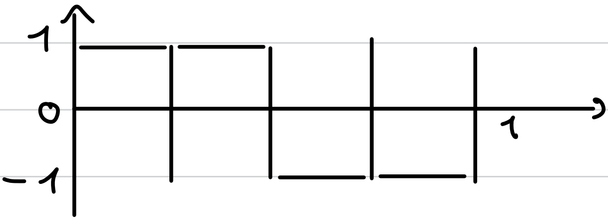
$$\frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \mathcal{U}$$

as $W_4 = W_4^*$ the calculation is the same for both sides.

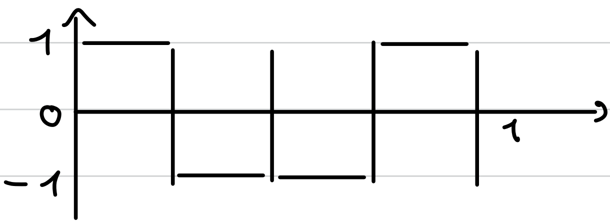
$\psi^{W}_{1}(t)$



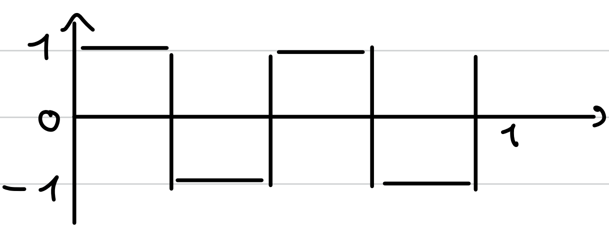
$\psi^{W}_{2}(t)$



$\psi^{W}_{3}(t)$



$\psi^{W}_{4}(t)$



In order to derive $\{\psi_i^w(t)\}_{i=1}^{4}$

We use formula shown in the lecture.

$$\begin{bmatrix} \psi_1^w(t) \\ \psi_2^w(t) \\ \psi_3^w t \\ \psi_4^w(t) \end{bmatrix} = H_4^w \begin{bmatrix} \sqrt{4}\, 1_{\Delta_i}(t) \\ \sqrt{4}\, 1_{\Delta_i}(t) \\ \sqrt{4}\, 1_{\Delta_i}(t) \\ \sqrt{4}\, 1_{\Delta_i}(t) \end{bmatrix}$$

$$= 2 \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1_{\Delta_i} t \\ \vdots \\ 1_{\Delta_i} 1 \end{bmatrix}$$

iii) using the same formula as before

$$\Phi_1^* = \int_0^1 a + b\cos(2t\pi) + c\cos^2(\pi t)\, dt$$

$$= a\int_0^1 t\, dt + b\int_0^1 \cos(2\pi t)\, dt + c\int_0^1 \cos^2(\pi t)\, dt$$

$$= a + c\left[1/2\, t + \frac{\sin(2\pi t)}{8\pi}\right]_0^1 = a + c\,1/2$$

$$\phi_2^* = \int_0^1 (a + b\cos(2t\pi) + c\cos^2(\pi t))\left(1_{\Delta[0,1/2)}(t) - 1_{\Delta[1/2,1]}(t)\right] dt$$

$$\overset{\text{linearity}}{=} \int_0^{1/2} (a + b\cos(2t\pi) + c\cos^2(\pi t))\, dt -$$

$$\int_{1/2}^1 (a + b\cos(2t\pi) + c\cos^2(\pi t))\, dt$$

$$= a\int_0^{1/2} dt + c\int_0^{1/2} \cos^2 \pi t\, dt - a\int_{1/2}^1 dt - c\int_{1/2}^1 \cos^2(\pi t)\, dt$$

$$= \frac{a}{2} + \frac{c}{4} - \frac{a}{2} - \frac{c}{4} = \underline{0}$$

$$\phi_3^* = \int_0^1 (a + b\cos(2t\pi) + c\cos^2(\pi t))\left(1_{\Delta[0,1/4],[3/4,1]}(t) - 1_{\Delta[1/4,3/4]}(t)\right] dt$$

$$= \int_0^{1/4} (a + b\cos(2t\pi) + c\cos^2(\pi t))\, dt -$$

$$\int_{1/4}^{3/4} (a + b\cos(2t\pi) + c\cos^2(\pi t))\, dt +$$

$$\int_{3/4}^1 (a + b\cos(2t\pi) + c\cos^2(\pi t))\, dt$$

$$= \frac{a}{4} + \frac{b}{2\pi}\Big[\sin(2\pi t)\Big]_0^{1/4} + c\Big[\frac{\sin 2\pi t}{4\pi} + \frac{t}{2}\Big]_0^{1/4} - \frac{a}{2}$$

$$- \frac{b}{2\pi}\Big[\sin(2\pi t)\Big]_{1/4}^{3/4} - c\Big[\frac{\sin(2\pi t)}{4\pi} + t/2\Big]_{1/4}^{3/4} + a/4$$

$$+ \frac{b}{2\pi}\Big[\sin(2\pi t)\Big]_{3/4}^{1} + c\Big[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\Big]_{3/4}^{1} = \frac{2}{\pi}\Big(b + \frac{c}{2}\Big)$$

$$\phi^*_4 = \int_0^1 \big(a + b\cos(2t\pi) + c\cos^2(\pi t)\big)\Big(1_{[0,1/4][1/2,3/4]}(t) - 1_{[1/4,1/2][3/4,1]}(t)\Big]dt$$

$$= \int_0^{1/4} \big(a + b\cos(2t\pi) + c\cos^2(\pi t)\big)\,dt \; -$$

$$\int_{1/4}^{3/4} \big(a + b\cos(2t\pi) + c\cos^2(\pi t)\big)\,dt \; +$$

$$\int_{3/4}^{1} \big(a + b\cos(2t\pi) + c\cos^2(\pi t)\big)\,dt \; -$$

$$\int_{3/4} \big(a + b\cos(2t\pi) + c\cos^2(\pi t)\big)\,dt$$

$$= \frac{a}{4} + \frac{b}{2\pi}\Big(\sin(2\pi t)\Big]_0^{1/4} + c\Big[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\Big]_0^{1/4} - \frac{a}{2}$$

$$- \frac{b}{2\pi}\Big[\sin 2\pi t\Big]_{1/4}^{1/2} - c\Big[\frac{\sin 2\pi t}{4\pi} + \frac{t}{2}\Big]_{1/4}^{1/2} + a/4 \; +$$

$$+\frac{b}{2\pi}\left(\sin(2\pi t)\right]_{1/2}^{3/4}+c\left[\frac{\sin 2\pi t}{4\pi}+\frac{t}{2}\right]_{1/2}^{3/4}$$

$$-\frac{a}{2}-\frac{b}{2c}\left[\sin 2\pi t\right]_{3/4}^{1}-c\left(\left[\frac{\sin 2\pi t}{4\pi}+\frac{t}{2}\right]_{3/4}^{1}\right.$$

$$=0$$

$$\tilde{\phi}(t)=\begin{cases} a+\frac{c}{2}+\frac{2}{\pi}(b+\frac{c}{2}) & 0\leq t\leq 1/4 \\ a+\frac{c}{2}-2/\pi(b+\frac{c}{2}) & 1/4<t\leq 3/4 \\ a+\frac{c}{2}+2/\pi(b+\frac{c}{2}) & 3/4<t\leq 1 \end{cases}$$

$$MSE=\int_0^1 \phi(t)-\tilde{\phi}(t))^2\,dt=\int_0^1 \phi^2_{(t)}\,dt+2\int_0^1 \phi(t)\tilde{\phi}(t)\,dt$$

$$+\int_0^1 \tilde{\phi}(t)^2\,dt=\int_0^1 \phi^2(t)-\sum_{i=1}^{4}\langle \phi_i^4\,]^2\,dt$$

same as before

$$=a^2+ac+\frac{b^2}{2}+\frac{bc}{2}+3c^2/8-\left(a^2+ac+c^2(1/4+1/\pi^2)\right.$$

$$\left.+\frac{4b^2}{\pi^2}+\frac{4bc}{\pi^2}\right)$$

$$=b^2\left(1/2-4/\pi^2\right)+bc\left(1/2-4/\pi^2\right)+c^2\left(1/8-1/\pi^2\right)$$

$$=MSE$$

(iv) order by absolute val

$$a + \frac{c}{2} \geq 2/\pi \, (b + c/2) \geq 0$$

$$|\phi_1^*| \geq |\phi_3^*| \geq |\phi_{2,4}^*|$$

$$1^{st} = \left(a + \frac{c}{2}\right) \cdot 1_{\Delta\,[0,1]}(t)$$

$$2^{nd} = \left(a + \frac{c}{2}\right) 1_{\Delta\,[0,1]}(t) + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\left(1_{[0,1/4]}(t) - \left(1_{\Delta\,[1/4,3/4]}(t) + 1_{[3/4,1]}(t)\right)\right)$$

$$= \left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right) 1_{\Delta\,[0,1/4] [3/4,1]}(t) +$$

$$+ \left[a + \frac{c}{2} - \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right] 1_{\Delta\,[1/4,3/4]}(t)$$

3 + 4 are the same as $\phi^*_{2,4} = 0$

(v) This time the order changes:

$$2\pi\,(1 + 3/4) = \frac{7}{2\pi} > 1/\pi + 3/4 > 0$$

$$|\phi_3^*| > |\phi_1^*| > |\phi_{2,4}^*|$$

1 term: $2\pi(b+c)\cdot \mathbb{1}_{[0,1/4][1/2,3/4]}(t) - 2\pi(b+c/2)\cdot \mathbb{1}_{[1/4,1/2][3/4,1]}(t)$

$= 1.114 \cdot \mathbb{1}_{[0,1/4][1/2,3/4]}(t) + 1.114 \cdot \mathbb{1}_{[1/4,1/2][3/4,1]}(t)$

2,3,4 term: $\left(a + \frac{c}{2} + \frac{2}{\pi}\left(b + \frac{c}{2}\right)\right)\mathbb{1}_{[0,1/4][3/4,1]}(t)$

$+ \left(a + \frac{c}{2} - \frac{2}{\pi}(b + c/2)\right)\mathbb{1}_{[1/4,3/4]}(t)$

$= 2.182\, \mathbb{1}_{[0,1/4][3/4,1]}(t) - 0.0461\, \mathbb{1}_{[1/4,3/4]}(t)$

3) Proof by induction:

We will assume that $\lambda_n = \frac{1}{\sqrt{2^n}}$

base: $i = 0$ $\quad H_0 = [1]$ symetric, real, unitary,

and

$$H_0 = \lambda_1 A \qquad \lambda_1 = \frac{1}{\sqrt{2^0}} \ , \ A_0 = [1]$$

Assume for $n-1$, and prove for n

$$H_N \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} H_{N-1} & H_{N-1} \\ H_{N-1} & -H_{N-1} \end{bmatrix}$$

$H_n = \lambda_n A_n$:

$$H_{n-1} = \lambda_{n-1} A_{n-1} = \frac{1}{\sqrt{2^{n-1}}} A_{n-1}$$

$$H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2^{n-1}}} A_{n-1} & \frac{1}{\sqrt{2^{n-1}}} A_{n-1} \\ \frac{1}{\sqrt{2^{n-1}}} A_{n-1} & -\frac{1}{\sqrt{2^{n-1}}} A_{n-1} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2^n}} \overbrace{\begin{bmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & A_{n-1} \end{bmatrix}}^{A_n}$$

as we assume that A_{n-1} is a matrix with only ± 1 as entries, A_n has only ± 1 entries as well.

symetric: H_n is a square block matrice from the way we build H.

$$H_n^{+} = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1}^{+} & H_{n-1}^{+} \\ H_{n-1}^{+} & -H_{n-1}^{+} \end{bmatrix}$$

H_{n-1} is symtric by assumption, and $H_{n-1} = H_{n-1}$

$\Rightarrow H_n$ is symetric

real. we proved that H_n is $\frac{1}{\sqrt{2^n}} \cdot A$, with A being a real matrix therefore H_n is real as well, as the product of real arguments.

Unitary: $H_n \cdot H_n^* \overset{\text{symmetric, real}}{=} H_n \cdot H_n =$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix} \cdot \frac{1}{\sqrt{2^n}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

Unitary by assumption

$$= \frac{1}{2} \cdot \begin{bmatrix} 2H_{n-1}^2 & 0 \\ 0 & 2H_{n-1}^2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2I & 0 \\ 0 & 2I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

b.i.

denoting S_i^j as the j th element in the i_{th} sequence. We will assume $|S_1| = |S_2| = k$

let us look at the two cases:

$S_1^k = S_2^1$.

$\Rightarrow$ $S(s_1 s_2) = S(s_1) + S(s_2)$ as the two sequences end and start with the same values, meaning there will be no change of sign

$S_1^k \neq S_2^k$

$\Rightarrow$ $S(s_1 s_2) = S(s_1) + S(s_2)^{+1}$ as the two sequences end and start with different values, meaning we added one change of sign.

b.ii) proof by induction:

base. $i=0$ $\quad H_{2^0} = H_1 = [1] \quad S(r_1) = 0$

$\{S(r_1)\} = \{0\}$

assume that it's true for $n-1$ and prove for n

$$H_n \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

by assumption, for H_{n-1}, $\{S(r_1), S(r_2) \ldots S(r_{2^{i-1}})\}$

$= \{0, \ldots 2^{i-1} - 1\}$

for every integer $K \in [0, 2^i - 1]$

we will find a row r_j s.t. $S(r_j) = K$

from assumption, for $\bar{K} = \lfloor \frac{K}{2} \rfloor$ there is a row

$r_{\bar{j}}$ in H_{n-1}.

from exercise a, we know that in $\underline{H_N}$

either:

$$S(r_{2j}) = S(r_j r_j) = k$$

or

$$S(r_{2j}) = S(r_j - r_j) = k+1$$

depending on if the row was concatenated to itself or to the negative matrix in the building of the matrix.

We have 2^i rows, and therefor from this we get that

$$\{S(r_1) \ldots S(r_{2^i-1})\} = \{0, 1 \ldots 2^i - 1\}$$

4) a. No, for instance $\tilde{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

which is not symetric

4.b) yes. proven in class

4.c) yes " " "

4.d) We actually showed this definition in class

$$\tilde{H}_{2^{n+1}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{H}_{2^n} \otimes (1,1) \\ I_{2^n} \otimes (1,-1) \end{pmatrix}, \quad H_1 = [1]$$

4.e) as shown $A \otimes B$

$$= \begin{pmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \dots & a_{2,n}B \\ \vdots & & & \\ a_{n,1}B & a_{n,2}B & \dots & a_{n,n}B \end{pmatrix}$$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \quad A^T = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{n,2} & \cdots & a_{n,2} \\ \vdots & & & \\ a_{1,2} & a_{2,n} & \cdots & a_{n,n} \end{pmatrix}$$

$$(A \otimes B)^T = \begin{pmatrix} a_{1,1} B & a_{1,2} B & \cdots & a_{1,n} B \\ a_{2,1} B & a_{2,2} B & \cdots & a_{2,n} B \\ \vdots & & & \\ a_{n,1} B & a_{n,2} B & \cdots & a_{n,n} B \end{pmatrix}^T$$

transpose of block Matrices

↓

$$= \begin{pmatrix} a_{1,1} B^T & a_{2,1} B^T & \cdots & a_{n,1} B^T \\ a_{1,2} B^T & a_{2,2} B^T & \cdots & a_{n,2} B^T \\ \vdots & & & \\ a_{1,n} B^T & a_{2,n} B^T & \cdots & a_{n,n} B^T \end{pmatrix}$$

$$A^T \otimes B^T = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{n,2} & \cdots & a_{n,2} \\ \vdots & & & \\ a_{1,2} & a_{2,n} & \cdots & a_{n,n} \end{pmatrix} \otimes B^T$$

$$= \begin{pmatrix} a_{1,1}B^T & a_{2,1}B^T & \cdots & a_{n,1}B^T \\ a_{1,2}B^T & a_{2,2}B^T & \cdots & a_{n,2}B^T \\ \vdots & & & \\ a_{1,n}B^T & a_{2,n}B^T & \cdots & a_{n,n}B^T \end{pmatrix} = (A \otimes B)^T$$

$$\hat{H}_N = \begin{pmatrix} \hat{H}^{+}_{N-1} \otimes [1, 1] \\ I \otimes [1, -1] \end{pmatrix}^{+}$$

4.f) $$\hat{H}_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{H}^T_{n-1} \otimes [1 \quad 1] \\ I \otimes [1 \quad -1] \end{pmatrix}^T$$

which is also able to be represented by:

$$= \frac{1}{\sqrt{2}} \left(\hat{H}_{n-1} \otimes [1 \quad 1] \quad I \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

Implementation

Question 1 – Numerical and Practical Bit Allocation for two dimensional Signals

$$\phi(x,y) = A\cos(\overbrace{2\pi\omega_x}^{\alpha} x)\sin(\overbrace{2\pi\omega_y}^{\beta} y) \quad \Big| \quad (x,y)\in \overbrace{[0,1]\times[0,1]}^{\triangleq \Delta},\ A = 2500$$

$$\omega_x = 2,\ \omega_y = 7$$

a) *Value-range $= \Phi_H - \Phi_L = 2A$

$\forall \alpha,\beta:\ -1 \le \cos(\alpha) \le 1,\ -1 \le \sin(\beta) \le 1 \Rightarrow -1 \le \cos(\alpha)\sin(\beta) \le 1$

$$\left.\begin{aligned} &\Rightarrow \Phi_H = A\cdot 1\cdot 1 = A \\ &\Phi_L = A\cdot(-1)\cdot 1 = -A \end{aligned}\right\} \Rightarrow \text{range} = A-(-A) = 2A$$

$$*\overbrace{\text{Horizontal-derivative Energy}}^{\triangleq HDE} = \iint_{\Delta}\left(\frac{\partial \phi(x,y)}{\partial x}\right)^2 dx\,dy$$

$$\frac{\partial A\cos(\alpha x)\sin(\beta y)}{\partial x} = -\alpha A\sin(\beta y)\sin(\alpha x)$$

$$\iint_{\Delta}\left(-\alpha A\sin(\beta y)\sin(\alpha x)\right)^2 dx\,dy = \iint_{\Delta}\alpha^2A^2\sin^2(\beta y)\sin^2(\alpha x)\,dx\,dy$$

$$= \alpha^2A^2\int_0^1\sin^2(\beta y)\,dy\int_0^1\sin^2(\alpha x)\,dx \overset{*}{=} \alpha^2A^2\underbrace{\int_0^1\left(\frac{1-\cos(2\beta y)}{2}\right)dy}_{*}\int_0^1\left(\frac{1-\cos(2\alpha x)}{2}\right)dx$$

$$*\sin^2(r) = \frac{1-\cos(2r)}{2}$$

$$*\int_0^1\left(\frac{1-\cos(2\beta y)}{2}\right)dy = \int_0^1\left(\frac{1}{2}-\frac{\cos(2\beta y)}{2}\right)dy = \left(\frac{y}{2}-\frac{\sin(2\beta y)}{4\beta}\right)\Bigg|_0^1 = \frac{1}{2}$$

(same for dx)

$$\Rightarrow HDE = \alpha^2 A^2 \left(\tfrac{1}{2}\right)^2 = \frac{(\cancel{2}\pi w_x)^2 \cdot A^2}{\cancel{4}} = \pi^2 w_x^2 A^2 = 246{,}740{,}110$$

$$\text{Horizontal-derivative Energy} = (\pi w_x A)^2 = 246{,}740{,}110$$

$$* \overbrace{\text{Vertical-derivative Energy}}^{\triangleq\ VDE} = \iint_{\Delta} \left(\frac{\partial \phi(x,y)}{\partial y}\right)^2 dx\,dy$$

$$\frac{\partial A\cos(\alpha x)\sin(\beta y)}{\partial y} = \beta A\cos(\alpha x)\cos(\beta y)$$

$$\Rightarrow \iint_{\Delta} \left(\beta A\cos(\alpha x)\cos(\beta y)\right)^2 dx\,dy = \iint_{\Delta} \beta^2 A^2 \cos^2(\alpha x)\cos^2(\beta y)\,dx\,dy$$

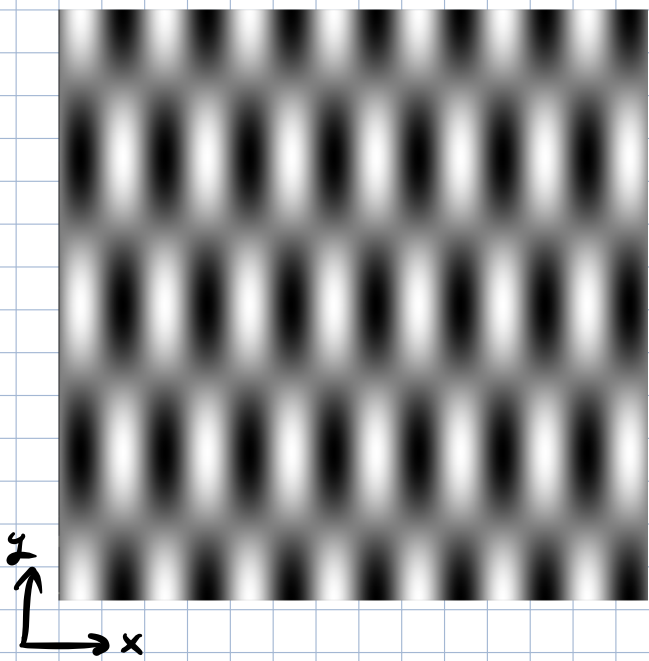
$$= \beta^2 A^2 \int_0^1 \cos^2(\alpha x)\,dx \int_0^1 \cos^2(\beta y)\,dy \overset{*}{=} \beta^2 A^2 \underbrace{\int_0^1 \left(\frac{1+\cos(2\alpha x)}{2}\right) dx}_{1/2} \underbrace{\int_0^1 \left(\frac{1+\cos(2\beta y)}{2}\right) dy}_{1/2}$$

$$* \cos^2(\gamma) = \frac{1+\cos(2\gamma)}{2}$$

* Same as previously as $\int_0^1 \cos(2\alpha x) = -\int_0^1 \cos(2\alpha x) = \underline{0}$

$$\Rightarrow VDE = \beta^2 A^2 \tfrac{1}{2}^2 = (\pi w_y A)^2 = 3{,}022{,}566{,}348$$

b) $N_x = N_y = 512$



As the signal is cyclic $[0, 2\pi]$ $(\Phi(x,y) = A\cos(2\pi\omega_x x)\sin(2\pi\omega_y y))$ we can see the pattern in each axis. Additionally, $\omega_x < \omega_y$ $\Rightarrow$ A higher frequency in the x axis.

c) Code Outputs

```
numerical range is: [ 2500.0 ,  -2500.0 ]. max_val-min_val=(5000.0)
(numerical HDE is: 246727578.93322715.) (0.005078650071465222 % deviation) from analytical
(numerical VDE is: 3008930213.7793164) (0.4511442479900056 % deviation) from analytical
```

	Analytical	Numerical	Error % of Analytical
value-range	2A	5000	0
HDE	246,740,110	≈246,727,578	≈0.005
VDE	3,022,566,348	≈3,008,930,213	≈0.451

We can see that the error $\left(\frac{|\text{analytical} - \text{numerical}|}{\text{analytical}} \cdot 100\right)$ is smaller than 0.5% for these values. We can infer the approximation is close to the real signal.

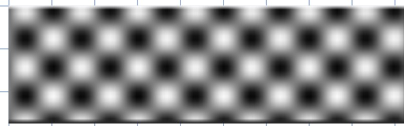
e) Results: B_low = 5.0008 $N_x = 21$, $N_y = 73$, $b = 3$

B_high = 50.0008 $N_x = 53$, $N_y = 188$, $b = 5$

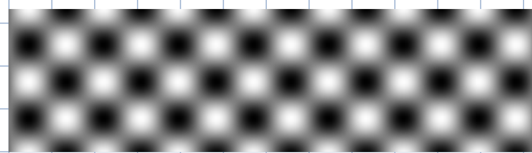
Outputs:

```
results fo B_low:
results before rounding:
Nx = 20.748117879156666
Ny = 72.45623204846109
b = 3.325948743480738
Correponding MSE =  116240.81040114893
results after rounding:
Nx = 21
Ny = 73
b = 3
Correponding MSE =  126227.5956370463
results fo B_high:
results before rounding:
Nx = 53.58219908958615
Ny = 187.11889625502502
b = 4.986913365607297
Correponding MSE =  16394.471012980135
results after rounding:
Nx = 53
Ny = 188
b = 5
Correponding MSE =  16448.45259167034
```

reconstructed B_low:



reconstructed B_high:



g) Results: B_low = 5.0008 $N_x = 21$, $N_y = 79$, $b = 3$

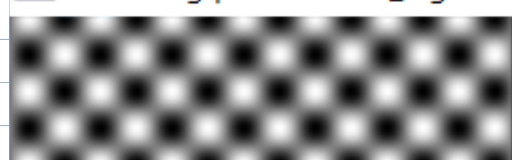
B_high = 50.0008 $N_x = 54$, $N_y = 185$, $b = 5$

Outputs:

```
section g:
results for B_low:
nx = 21
ny = 79
b = 3
Corresponding MSE=  119351.75496324532
results for B_high:
nx = 54
ny = 185
b = 5
Corresponding MSE=  16411.822041658303
```

searching procedure B_low

searching procedure B_high



Explanations

In the searching procedure we found allocation with a lower corresponding MSE (≡ A better allocation). That is because the numerical Minimizer consisted of 2 steps:

1) Finding the best allocation where values $\in \mathbb{R}$.

2) rounding the results so values $\in \mathbb{N}$.

when doing step 2, we increased the MSE without comparing it to other possible results outside the rounding range, possibly "losing" better allocations.

In terms of reconstructed pictures, the difference is barely visible to the naked eye.

h) $\phi(x,y) = A\cos(2\pi w_x x)\sin(2\pi w_y y)$

$A = 2500, \; w_x = 7, \; w_y = 2$

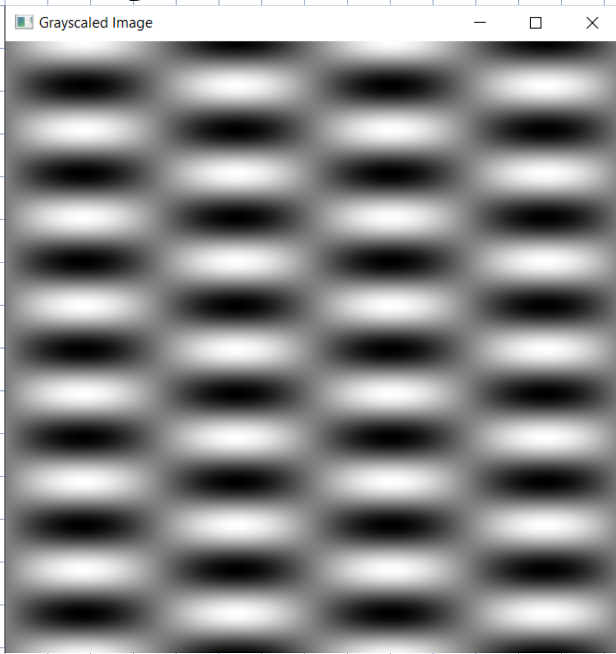
a) The range is not affected by the change as φ is still $\varphi \in [-1,1]$ thus: range $= [-A, A] \Rightarrow A-(-A) = 2A$

for HDE & VDE, the specific integration is symmetric w.r.t w_x, w_y. What changes is the eventual calculation & assignment, thus:

HDE = 3,022,566,348

VDE = 246,740,110

b) 512x512



We can see that the picture is practically the same as before with axes changes. the same observation about the frequencies is seen (reversed) as now $\omega_x > \omega_y$.

c) Outputs

```
section c:
numerical range is: [ 2500.0 ,  -2500.0 ]. max_val-min_val= 5000.0
numerical HDE is: 3020686344.9418874 . 0.06219890125345302 % deviation from analytical
numerical VDE is: 245764089.03364965 . 0.3955663983250825 % deviation from analytical
```

again, we can see that the deviation is really minor (> 0.5%) thus we infer the the approximation is good.

e)

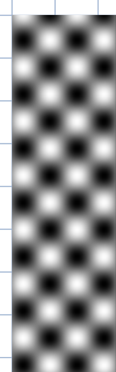
```
section e:
results for B_low:
results before rounding:
Nx = 72.59782899131287
Ny = 20.70761926182421
b = 3.3259537028489037
Correponding MSE =  116240.17740329988
results after rounding:
Nx = 73
Ny = 21
b = 3
Correponding MSE =  126229.36919285082
results for B_high:
results before rounding:
Nx = 187.48463383046936
Ny = 53.47762566060133
b = 4.986917786104998
Correponding MSE =  16394.3761015943
results after rounding:
Nx = 188
Ny = 53
b = 5
Correponding MSE =  16447.5875289508
```

B_low = 5.0008

$N_x = 73$, $N_y = 21$, $b = 3$

B_high = 50.0008

$N_x = 188$, $N_y = 53$, $b = 5$



as expected, we get the same numbers with axes switching.

g) Results: B_low = 5.0008 $N_x = 79$, $N_y = 21$, $b = 3$

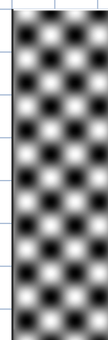
B_high = 50.0008 $N_x = 185$, $N_y = 54$, $b = 5$

Output:

```
section g:
results for B_low:
nx = 79
ny = 21
b = 3
Corresponding MSE=  119326.66405923486
results for B_high:
nx = 185
ny = 54
b = 5
Corresponding MSE=  16412.912084534863
```

searching procedure B_low

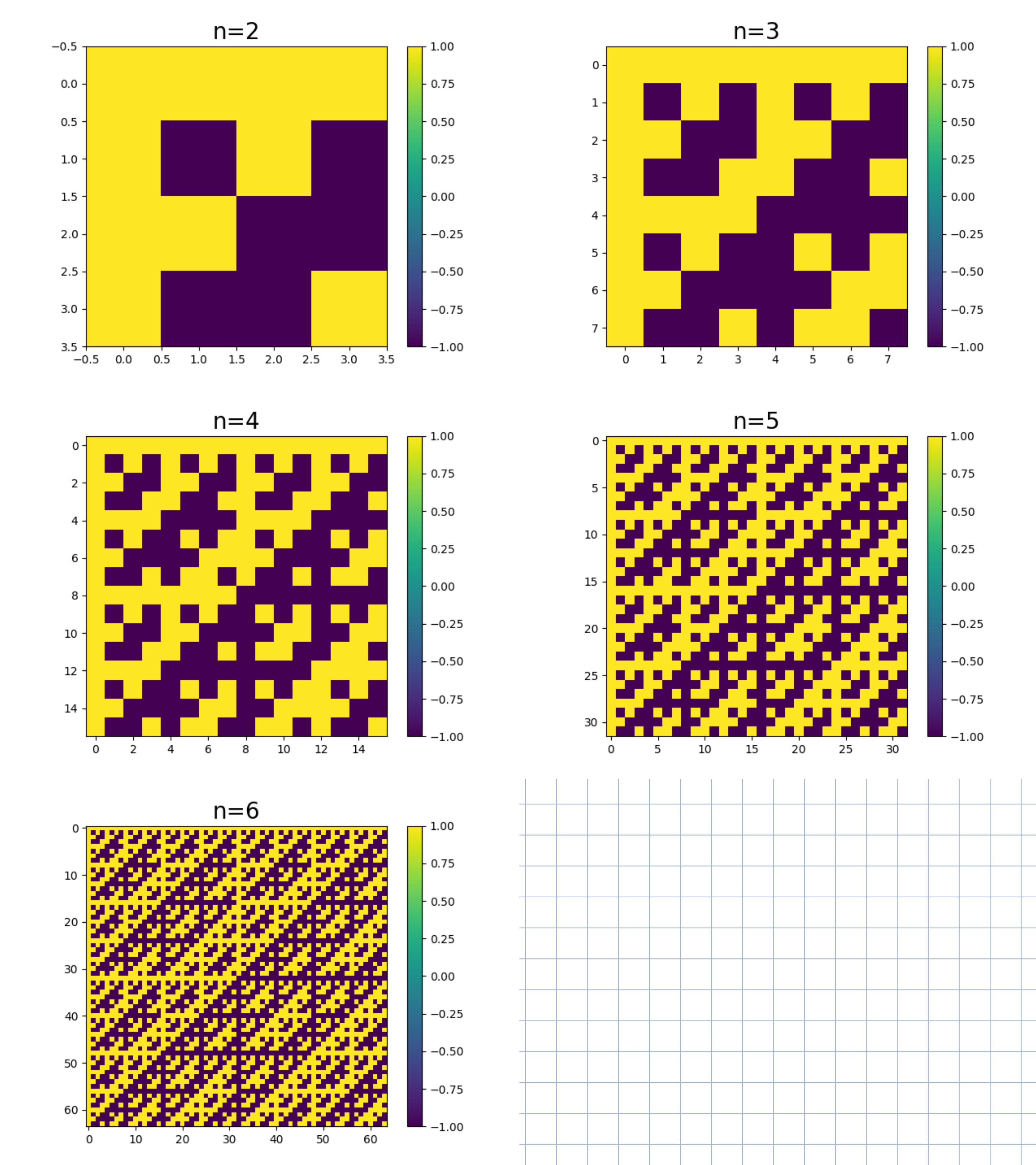
searching procedure B_high



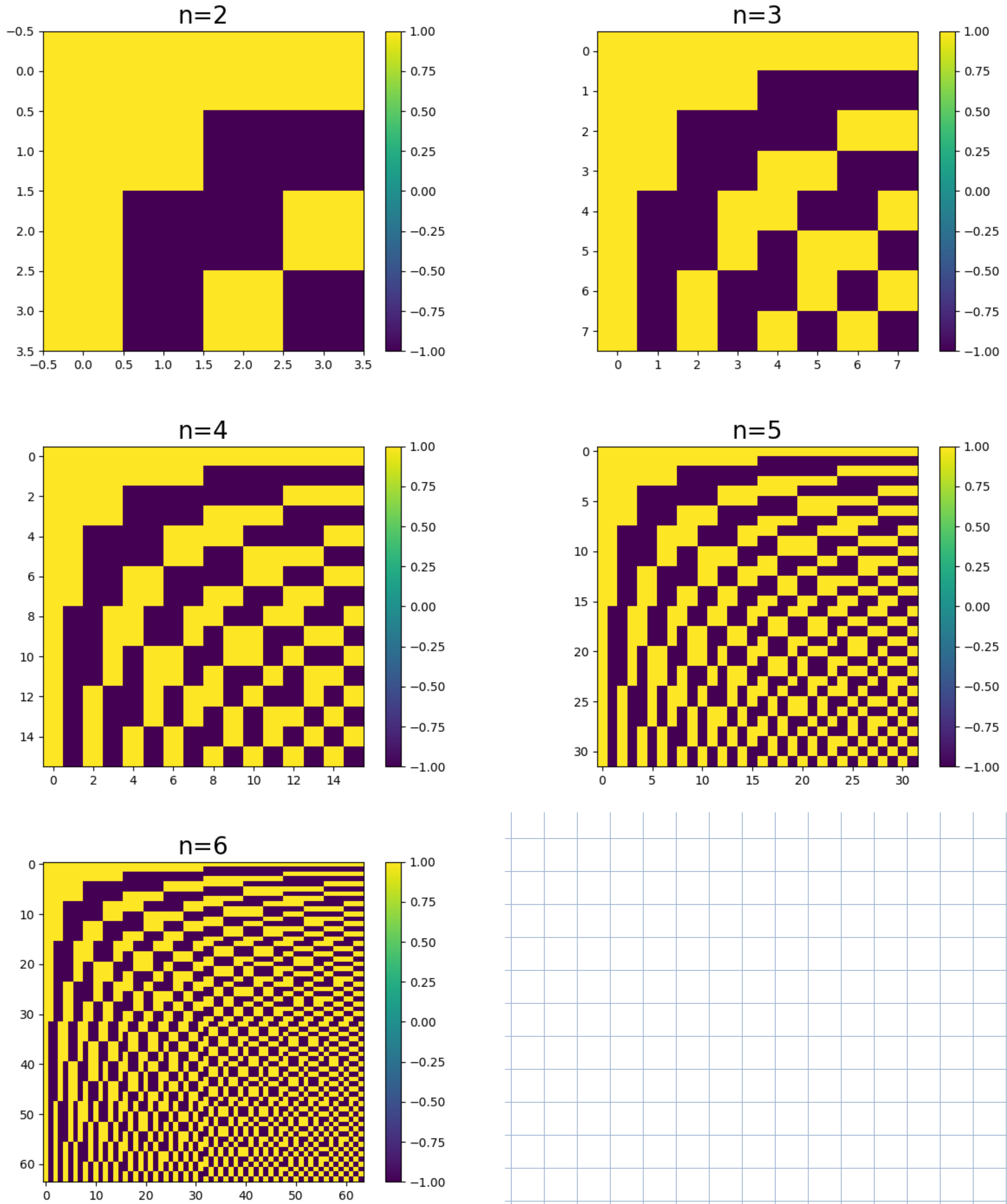
Again, as expected, we get the same results and conclusions with inverted axes.

Question 2 - Hadamard, Hadamard-Walsh & Haar matrices

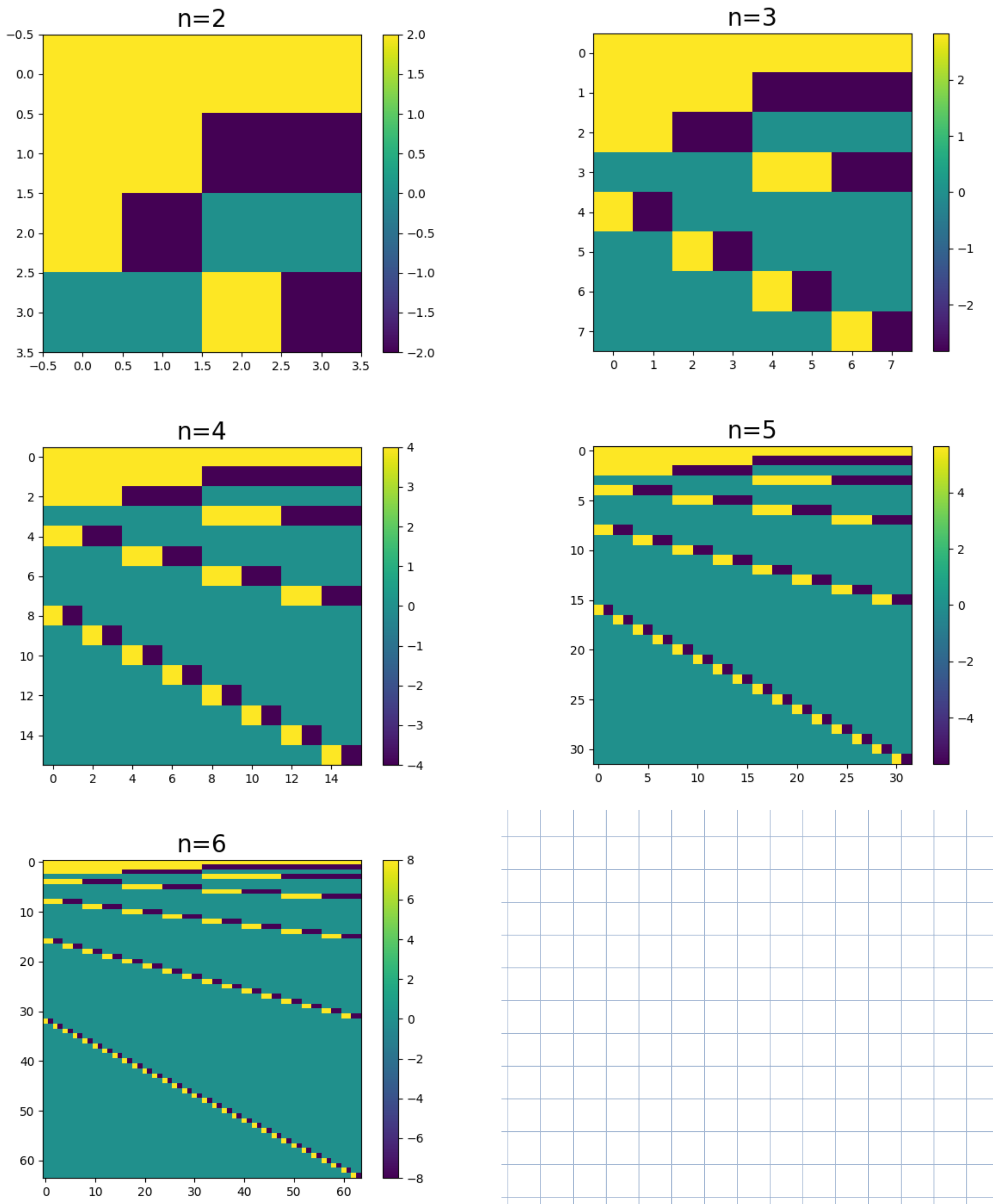
b) Hadamard:



d) Walsh-Hadamard



f) Haar



g) $t \in [-4,5]$, $\phi(t) = t \cdot e^t$, $n=2 \Rightarrow k = 1, \dots, 4$

$$* \quad \varphi_i^{\beta} = \langle \varphi, \beta_i \rangle = \int_{-4}^{5} \varphi(t)\beta_i(t)\,dt = \sum_{i=1}^{4} \int_{\Delta_i} \varphi(t)\beta_i(t)\,dt = \sum_{i=1}^{4} \beta_i(t) \underbrace{\int_{\Delta_i} \varphi(t)\,dt}_{*}$$

$$* \quad \int_{\Delta_i} \varphi(t)\,dt = \int_{\Delta} t e^t\,dt \overset{*}{=} (te^t)^{\Delta} - \int_{\Delta} 1 \cdot e^t\,dt = (te^t - e^t)^{\Delta} = \boxed{[(t-1)e^t]^{\Delta_i}}$$

$$\int v u' = vu - \int v' u$$

$$v = t, \quad u' = e^t$$

$$v' = 1, \quad u = e^t$$

Calculate numericaly.

Solution steps:

① for each basis, normalise the orthogonal vectors w.r.t the domain $[-4,5]$ so we have an ON basis w.r.t φ.

$$\left(\langle \beta_i(t), \beta_j(t) \rangle = \int_{-4} \beta_i(t) \cdot \beta_j(t)\,dt = \delta_{ij} = \begin{cases} 0 & i \neq j \ ✓ \ \text{OG} \\ 1 & i = j \ \text{Needed} \end{cases} \right)$$

② for each basis, and for each $i \in k$, find approximators φ_i^{β}

$$\varphi_i^{\beta} = \langle \varphi, \beta_i \rangle = \int_{-4}^{5} \varphi(t)\beta_i(t)\,dt \leftarrow \text{calculate numericaly.}$$

④ Sort the $\{\varphi_i^{\beta}\}$ list by abs val, reversed = True.

③ for each approx list, calculate corresponding MSE list.

$$\left(MSE = \int_I \varphi^2 - \sum_{i_j=1}^{k} \varphi_{i_j}^2 \right)$$

④ Plot + compare.